Week 8 R for 2D

$$g_T(t) = \frac{6\sqrt{3}}{\pi} \cdot \frac{1}{(3+t^2)^2}$$

$\downarrow$

according to item 20 for Student t density $k=3$ & $n=4$

$$E(T) = \int_{-\infty}^{\infty} t \, g_T(t) \, dt$$

$$= \int_{-\infty}^{\infty} t \cdot \frac{6\sqrt{3}}{\pi} \cdot \frac{1}{(3+t^2)^2} \, dt$$

$$= \frac{6\sqrt{3}}{\pi} \int \frac{t}{(3+t^2)^2} \, dt$$

let $t = \sqrt{3} \tan u$

$t^2 = 3\tan^2 u$

$3+t^2 = 3 + 3\tan^2 u$

$dt = \sqrt{3} \sec^2 u \, du$

since $t = \sqrt{3} \tan u$

$\tan u = t/\sqrt{3}$

$u = \arctan\left(\frac{t}{\sqrt{3}}\right)$

boundaries $\Rightarrow$

$t = -\infty \rightarrow u = -\pi/2$

$t = \infty \rightarrow u = \pi/2$

$$= \frac{6\sqrt{3}}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\sqrt{3}\tan u}{(3+3\tan^2 u)^2} \sqrt{3} \sec^2 u \, du$$

simplify $(3+3\tan^2 u)^2$

$$= 9\,(1+\tan^2 u)^2$$
$$= 9\,(\sec^2 u)^2$$
$$= 9\sec^4 u$$

$$= \frac{6\sqrt{3}}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\sqrt{3}\tan u}{9\sec^4 u} \cdot \sqrt{3}\sec^2 u \, du$$

$$= \frac{3}{9} \cdot \frac{6\sqrt{3}}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\tan u}{\sec^2 u} \, du$$

$$= \frac{2\sqrt{3}}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\frac{\sin u}{\cancel{\cos u}}}{\frac{1}{\cos^{\cancel{2}} u}} \, du$$

$$= \frac{2\sqrt{3}}{\pi} \underbrace{\int_{-\pi/2}^{\pi/2} \sin u \cos u \, du}_{\int \sin u \, \cancel{\cos u} \, \frac{d(\sin u)}{\cancel{\cos u}}}$$

$$= \frac{2\sqrt{3}}{\pi} \cdot \left.\frac{\sin^2 u}{2}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{2\sqrt{3}}{\pi} \cdot \left(\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right) = \frac{2\sqrt{3}}{\pi} \cdot 0 = \boxed{0 = E(T)}$$

which matches in R

$$E(T^2) = \int_{-\infty}^{\infty} t^2 \, f_T(t) \, dt$$

$$= \int_{-\infty}^{\infty} t^2 \cdot \frac{6\sqrt{3}}{\pi} \cdot \frac{1}{(3+t^2)^2} \, dt$$

$$= \frac{6\sqrt{3}}{\pi} \int_{-\infty}^{\infty} \frac{t^2}{(3+t^2)^2} \, dt$$

using same subs as earlier

$$\text{let } t = \sqrt{3} \tan u$$

$$t^2 = 3 \tan^2 u$$

$$= \frac{6\sqrt{3}}{\pi} \int_{-\pi/2}^{\pi/2} \frac{3\tan^2 u}{(3+3\tan^2 u)^2} \sqrt{3} \sec^2 u \, du$$

$$= \frac{6\sqrt{3}}{\pi} \int_{-\pi/2}^{\pi/2} \frac{3\tan^2 u}{9\sec^4 u} \sqrt{3} \cdot \sec^2 u \, du$$

$$= \frac{6\cancel{\sqrt{3}}}{\pi} \cdot \frac{\cancel{3}}{\cancel{9}} \cancel{\sqrt{3}} \int_{-\pi/2}^{\pi/2} \frac{\tan^2 u}{\sec^2 u} \, du$$

$$= \frac{6}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\frac{\sin^2 u}{\cos^2 u}}{\frac{1}{\cos^2 u}} \, du$$

$$= \frac{6}{\pi} \int_{\frac{\pi}{2}}^{\pi/2} \sin^2 u \, du$$

half angle identity.

$$= \frac{6}{\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{2}(1-\cos 2u)\,du$$

$$= \frac{1}{2}\cdot\frac{6}{\pi}\int_{-\pi/2}^{\pi/2}(1-\cos 2u)\,du$$

$$= \frac{3}{\pi}\left(u - \cancel{\cos 2}\ \frac{1}{2}\sin 2u\right)\Big]_{-\pi/2}^{\pi/2}$$

$$= \frac{3}{\pi}\left(\left(\frac{\pi}{2}+\frac{\pi}{2}\right)-(0-0)\right)$$

$$= \frac{3}{\pi}\cdot\pi = 3 = E(T^2)$$

↳ same like in R

variance $= \cancel{E(x)}\ E(T^2) - (E(T))^2$

$$\boxed{Var(T) = 3}$$ → same like in R

Week 8 for 2e.

Show w/ $c = 12$ $\int g_T(t) = 1$

$$= \int_{-\infty}^{\infty} \frac{12}{(4+t^2)^{5/2}}\, dt$$

trig substitution

$t = 2\tan u \longrightarrow u = \arctan\left(\frac{t}{2}\right)$

$dt = 2\sec^2 u\, du$

$\rightarrow t^2 = 4\tan^2 u$

$t = -\infty \longrightarrow u = \arctan\left(\frac{-\infty}{2}\right) = -\pi/2$

$t = \infty \longrightarrow u = \pi/2$

$$= \int_{-\pi/2}^{\pi/2} \underbrace{\frac{12}{(4+4\tan^2 u)^{5/2}}}\cdot 2\sec^2 u\, du$$

$$4^{5/2} \cdot (1+\tan^2 u)^{5/2}$$

$$= 4^{5/2} \cdot (\sec^2 u)^{5/2}$$

$$= 4^{5/2} \cdot \sec^5 u = 32\sec^5 u$$

$$= \int_{-\pi/2}^{\pi/2} \frac{12}{32\sec^5 u} \cdot 2\sec^2 u \, du$$

$$= \frac{3}{4}\int_{-\pi/2}^{\pi/2} \sec^{-3} u \, du$$

$$= \frac{3}{4}\left[\int_{-\pi/2}^{\pi/2} \cos^3 u \, du\right]$$

$$= \frac{3}{4}\left[\int_{-\pi/2}^{\pi/2} \cos u \cdot (1-\sin^2 u) du\right]$$

$$= \frac{3}{4}\left[\int_{-\pi/2}^{\pi/2} \cos u - \cos u \sin^2 u \, du\right]$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\int_{-\pi/2}^{\pi/2} \cos u \, du \qquad\qquad \int_{-\pi/2}^{\pi/2} \cancel{\cos u} \sin^2 u \, \frac{d(\sin u)}{\cancel{\cos u}}$$

$$= \sin u \Big]_{-\pi/2}^{\pi/2} \qquad\qquad = \frac{\sin^3 u}{3}\Big]_{-\pi/2}^{\pi/2}$$

$$= 1-(-1) = 2 \qquad\qquad = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

$$= \frac{3}{4}\left(2 - \frac{2}{3}\right)$$

$$= \frac{3}{4} \cdot \frac{4}{3} = 1$$

proven!

first moment

$$E(T) = \int_{-\infty}^{\infty} \frac{12t}{(4+t^2)^{5/2}}\, dt$$

$\downarrow$ using same substitution as before.

$$= \int_{-\pi/2}^{\pi/2} \frac{12\,(2\tan u)}{32 \sec^5 u} \cdot 2\sec^2 u\, du$$

$$= \frac{3}{2} \int_{-\pi/2}^{\pi/2} \frac{\tan u}{\sec^3 u}\, du$$

$$= \frac{3}{2} \int_{-\pi/2}^{\pi/2} \frac{\frac{\sin u}{\cos u}}{\frac{1}{\cos^3 u}}\, du$$

$$= \frac{3}{2} \int_{-\pi/2}^{\pi/2} \cancel{\sin u} \cos^2 u \, \frac{d(\cos u)}{-\cancel{\sin u}}$$

$$= -\frac{3}{2} \cdot \cos^3 u \Big]_{-\pi/2}^{\pi/2}$$

$$= 0$$

Second moment

$$E(T^2) = \int_{-\infty}^{\infty} \frac{12t^2}{(4+t^2)^{5/2}}\,dt$$

$\downarrow$ using same substitution as before

$$= \int_{-\pi/2}^{\pi/2} \frac{12(2\tan u)^2}{32\sec^5 u} \cdot 2\sec^2 u\,du$$

$$= 3\int_{-\pi/2}^{\pi/2} \frac{\tan^2 u}{\sec^3 u}\,du$$

$$= 3\int_{-\pi/2}^{\pi/2} \frac{\frac{\sin^2 u}{\cos^2 u}}{\frac{1}{\cos^3 u}}\,du$$

$$= 3\int_{-\pi/2}^{\pi/2} \sin^2 u\,\cancel{\cos u}\,\frac{d(\sin u)}{\cancel{\cos u}}$$

$$= 3 \cdot \frac{\sin^3 u}{3}\Bigg]_{-\pi/2}^{\pi/2}$$

$$= 3\left(\frac{1}{3}+\frac{1}{3}\right) = \underline{2}$$

$$Var(T) = E(T^2) - [E(T)]^2$$

$$= 2 - 0$$

$$= 2 \blacksquare$$