Lecture 12
Chaos and discrete maps

In this lecture we discuss the notion of "chaos" using a very simple model, the "logistic map". The logistic map is a discretized dynamical system that exhibits stable periodic orbits or chaotic trajectories depending on the value of a control parameter. Many of the qualitative properties of this map are also found in more complex dynamical systems like the Hénon-Heiles Hamiltonian discussed in Lecture 11.

12.1 Logistic equation

The logistic equation is a minimal model of population dynamics. If we call $x(t)$ the number of individuals in a society (e.g. humans, bacteria, insects, etc), the change of the population in a given time interval will be given by the number of births minus the number of deaths. Let us call b the birth rate, i.e. the number of births per individual, and d the death rate. For example, in 2020 in the U.S. we had $b = 12/1000$ and $d = 9/1000$. Assuming that these values do not change in time, the rate of change of the population will be:

$$\frac{dx}{dt} = (b - d)x.$$

This model is too crude to describe the dynamics of real populations. For example, it predicts that a population either goes extinct (when $b - d < 0$) or grows exponentially (when $b - d > 0$), without accounting for the possibility of an equilibrium nonzero population size.

A slightly more refined model must take into accound that resources such as food and energy are in limited supply, so larger populations tend to experience higher death rates. The next level of sophistication is therefore to assume that d is not a constant, but it depends on x. The simplest relation between d and x is linear, therefore we can set $d = cx$ with c being another constant. All three constants b, c, d are positive. The previous equation is modified as follows:

$$\frac{dx}{dt} = (b - cx)x. \tag{12.1}$$

The is the **logistic equation** of Verhulst (1838). I assume that the name "logistic" has to do with the fact that the equation describes how population dynamics is linked to available resources. In his original paper Verhulst called (12.1) "equation logistique" without explanations. This equation does not admit a simple Lagrangian (and hence Hamiltonian), so in the following we will not be using the Lagrangian or Hamiltonian formalism.

The solution of (12.1) can be found as follows:

$$\int \frac{dx}{(b - cx)x} = \int dt$$

$$\frac{c}{b}\int \frac{dx}{b - cx} + \frac{1}{b}\int \frac{dx}{x} = \int dt$$

$$\left[-\frac{1}{b}\log(b - cx) + \frac{1}{b}\log x\right]_{t_1}^{t_2} = t_2 - t_1$$

$$\frac{x}{b - cx} = \frac{x_0}{b - cx_0}e^{bt}, \qquad \text{with } x_0 = x(t = 0)$$

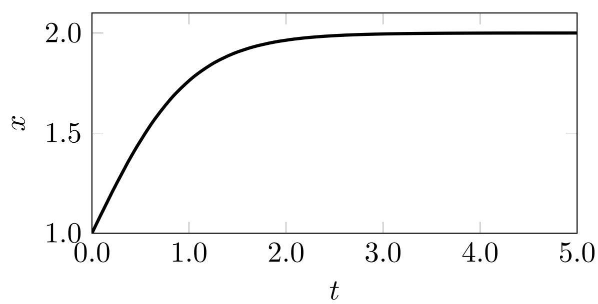
therefore we have:

$$x(t) = \frac{ab}{ac + e^{-bt}}, \qquad \text{with } a \stackrel{\text{def}}{=} \frac{x_0}{b - cx_0} \tag{12.2}$$

This solution resembles the Fermi-Dirac distribution. For $t \to -\infty$ the population tends to $x = 0$, for $t \to +\infty$ the population tends to $x = b/c$.

For example, if we set $x_0 = 1$, $b = 2$, $c = 1$, we have:

$$x(t) = \frac{2}{e^{-2t} + 1}$$



We see that the solutions of the logistic equation is a smooth function that tends towards an equilibrium population ($x = 2$ in the above plot). This behavior describes the saturation of population size due to limited availability of resources, and constitutes a standard model in ecology and evolution.

12.2 Logistic map

The situation becomes much more intricate and interesting when we attempt to integrate the logistic equation numerically [as opposed to using the closed-form solution in (12.2)]. To this aim, we evaluate the time-derivative $\dot{x}$ using the finite-difference formula:

$$\dot{x}(t) \simeq \frac{x(t + \Delta t) - x(t)}{\Delta t}, \qquad \Delta t \to 0$$

so that (12.1) can be approximated by:

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = [b - cx(t)]x(t).$$

After a few steps of rearrangement we can rewrite the last expression as:

$$\frac{c\Delta t}{1 + b\Delta t} x(t + \Delta t) = (1 + b\Delta t) \left[1 - \frac{c\Delta t}{1 + b\Delta t} x(t) \right] \frac{c\Delta t}{1 + b\Delta t} x(t)$$

Now let us introduce the definitions of the function at discrete time points as follows:

$$x_n \stackrel{\text{def}}{=} \frac{c\Delta t}{1 + b\Delta t} x(t), \qquad x_{n+1} \stackrel{\text{def}}{=} \frac{c\Delta t}{1 + b\Delta t} x(t + \Delta t),$$

and let us introduce the ratio:

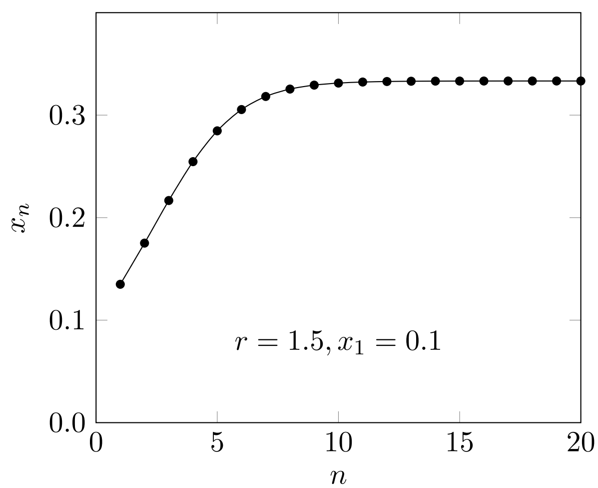
$$r \stackrel{\text{def}}{=} 1 + b\Delta t. \tag{12.3}$$

Since both b and Δt are positive, we have $r > 1$. Using these definitions we can write the time evolution as:

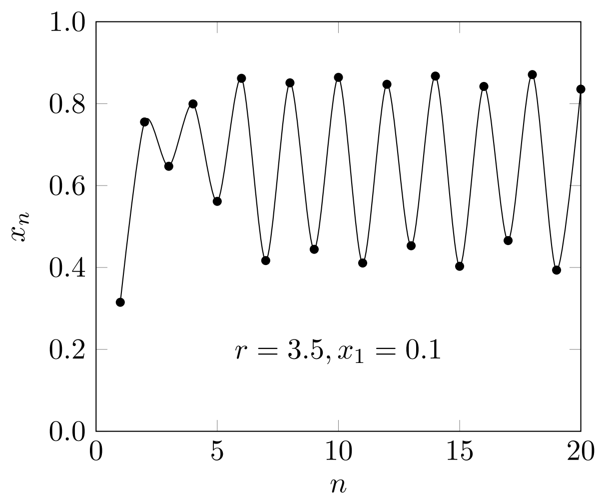
$$x_{n+1} = r\,(1 - x_n)\,x_n. \tag{12.4}$$

This relation is called the **logistic map**. We call it a "map" because it maps the value x_n into the value x_{n+1}. If we consider $n = 1, 2, 3, \cdots$, this map generates a sequence of values which should correspond to the evolution of $x(t)$ from (12.1) as a function of the time $t = n\Delta t$.

Let us verify this statement by evaluating the series $\{x_n\}$ for a parameter $r = 1.5$ and initial condition $x_1 = 0.1$.

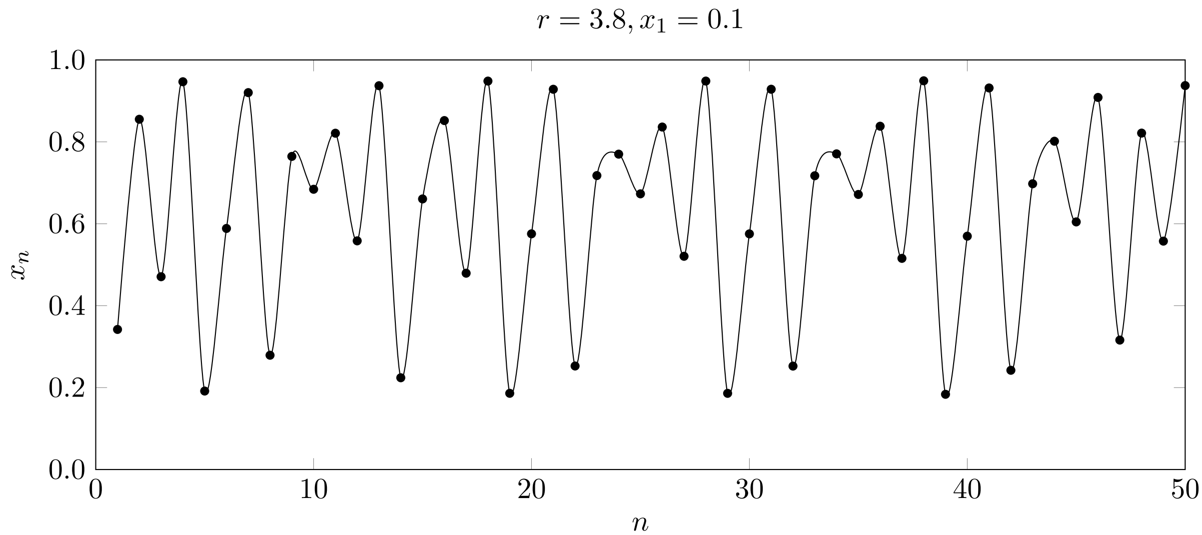


This behavior is indeed very similar to what we found at the previous page using the closed-form solution of the logistic differential equation. Now let us evaluate once more the series $\{x_n\}$, this time starting from the same initial codition $x_1 = 0.1$ but using the parameter $r = 3.5$:



Here we see that something unexpected happened: instead of finding a smooth solution that converges to a certain value, we have an oscillating solution. If we perform a Fourier analysis of this signal, we obtain a dominant frequency of oscillation corresponding approximately to $\Delta n = 2$.

This behavior is **qualitatively** different from the exact closed-form solutions obtained in (12.2). If we experiment more with the parameter r, we find that the nature of the oscillations of the solution changes significantly with this parameter. For example, if we use $r = 3.8$ we obtain the following discrete time evolution:



In this case, not only we do not find the smooth exponential solution that is expected from (12.2), but we do not even find the period-2 oscillations observed for the case $r = 3.5$. It appears that the discretized dynamical system exhibits wildly different behaviors as we tune the control parameter r. It is understandable that people came to call this weird behavior **cahotic**.

12.3 Spiderweb diagrams

In this section we want to understand what is the origin of the unexpected dynamics of the logistic map illustrated by the figures above.

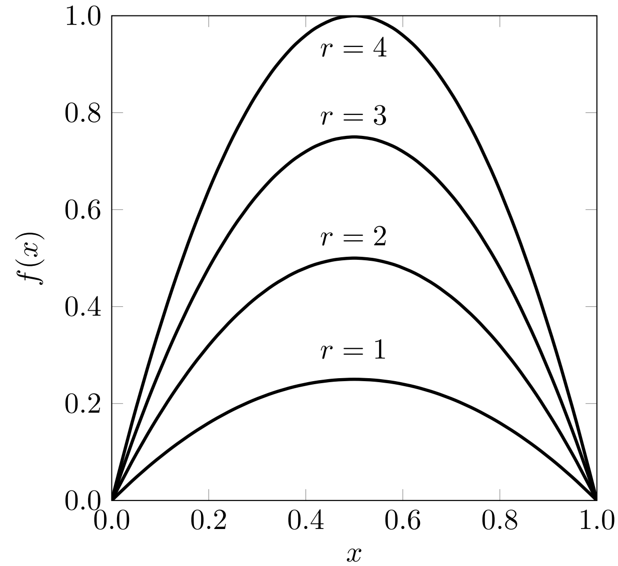
We consider the function

$$f(x; r) = rx(1 - x), \tag{12.5}$$

so that the logistic map in (12.4) can be written as:

$$x_{n+1} = f(x_n; r). \tag{12.6}$$

The function f is simply a parabola opening downward that intersects the $y = 0$ axis at $x = 0$ and $x = 1$:



The function has its maximum at $x = 1/2$, $f_{\mathrm{max}} = r/4$. For $0 \leq x \leq 1$ this function *maps* the interval $[0, 1]$ into the interval $[0, r/4] \in [0, 1]$. Usually one does not consider values $r > 4$ because the the values of f may end up outside of the interval $[0, 1]$. So we restrict ourselves to consider the "mapping" of $[0, 1]$ into $[0, 1]$ or a fraction of it.

To explore the logistic map it is common to draw "spiderweb plots". The idea of these plots is to represent the sequence $\{x_n\}$ within the same plot, via successive applications of the map:

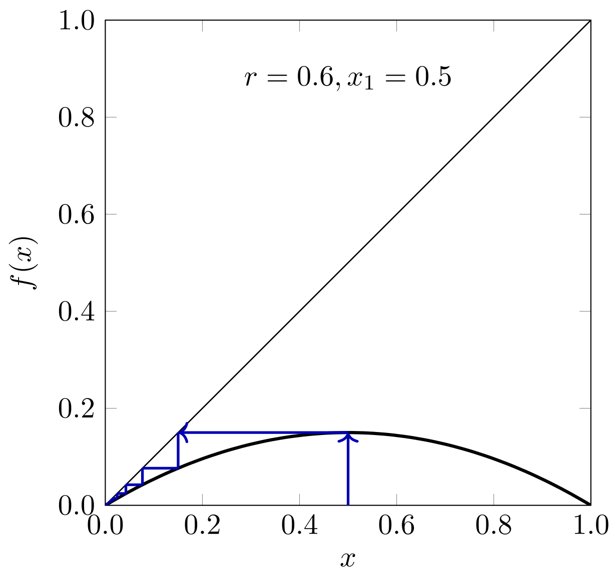
$$x_2 = f(x_1), \qquad x_3 = f(x_2) = f(f(x_1)), \qquad \cdots \qquad x_n = \underbrace{f(f(\cdots f}_{n-1 \text{ times}}(x_1)))$$

In the following we analyse spiderweb plots as a function of the parameter r.

12.3.1 Logistic map with $0 < r < 1$

To begin with, we consider the example $x_1 = 0.5$ and $r = 0.6$.

Starting at $x_1 = 0.5$, the first application of the map gives $f(x_1) = 0.15$. To represent this step, we draw a vertical line from $(0, 0.5)$ to $(0, 0.15)$. Now we move to the next point. We want to restart from $x_2 = 0.15$. To find this point in the diagram, we draw a horizontal line from the previous end-point towards the straight line at 45°. Next we move vertically until we hit the function $f(x)$. This will give $x_3 = f(x_2) = 0.0765$. As we repeat this process, the variable x_n approaches $x = 0$.



The value $x = 0$ is a **fixed point** of the map. Fixed points are encountered when $x_{n+1} = x_n$. Using (12.5) we see that the fixed points are defined by:

$$x = f(x; r) \quad \text{that is} \quad x = rx(1 - x)$$

The equality is obviously verified for $x^{(1)} = 0$. When $x \neq 0$ we have $x^{(2)} = 1 - 1/r$. Since in the present example we have chosen $r < 1$, the fixed point $x^{(2)}$ is in the negative semiaxis, and cannot be reached because the dynamical variable eventually settles at $x^{(1)} = 0$. To check this, let us consider a starting coordinate

$$x_1 = \varepsilon > 0, \qquad \varepsilon \to 0$$

If we replace this value inside (12.5) we obtain

$$x_2 = r\varepsilon(1 - \varepsilon) = r\varepsilon + \mathcal{O}(\varepsilon^2)$$

Upon further iteration of the map, we find:

$$x_3 = r[r\varepsilon + \mathcal{O}(\varepsilon^2)][1 - r\varepsilon - \mathcal{O}(\varepsilon^2)] = r^2\varepsilon + \mathcal{O}(\varepsilon^2)$$

therefore

$$x_n = r^{n-1}\varepsilon + \mathcal{O}(\varepsilon^2)$$

In the limit $n \to \infty$, r^{n-1} goes to zero whenever $0 < r < 1$, and at the same time x_n remains positive. Therefore we can say that $x^{(1)}$ is not only a **stable fixed point** but also an **attractor** of the dynamics, in the sense that points nearby tend to move toward this attractor upon iterating the map.

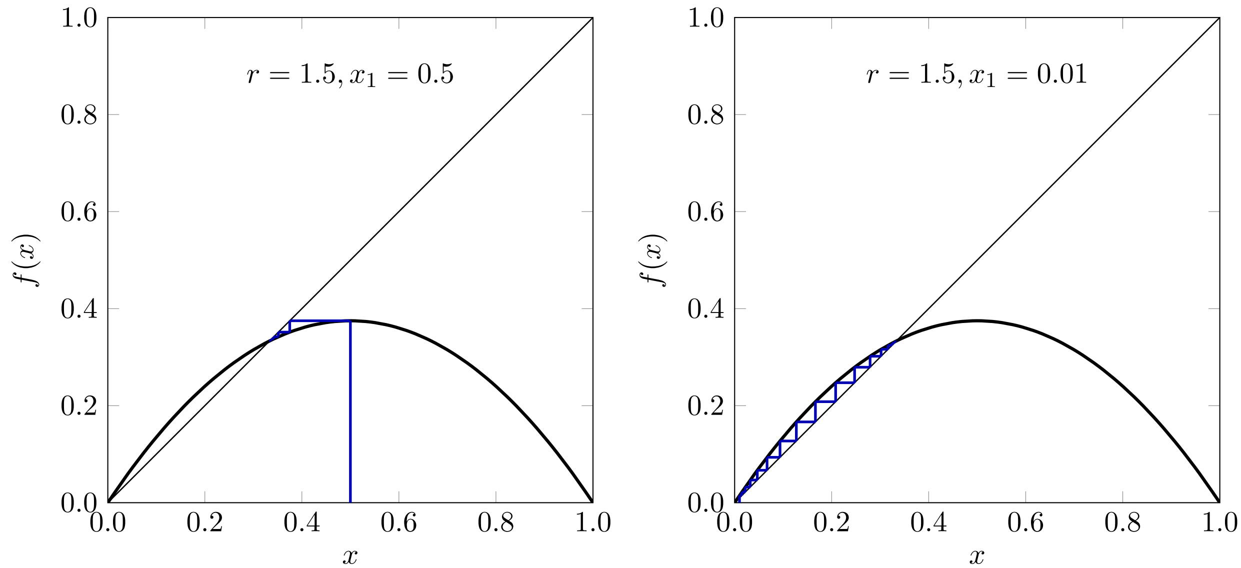
In the context of the Verhulst equation (12.1), we can say that situations with $0 < r < 1$ correspond to a population size reaching zero asymptotically, that is extinction.

12.3.2 Logistic map with $1 \leq \mathbf{r} \leq 2$

Now we increase the control parameter beyond $r = 1$. In this case, the fixed point $x^{(2)} = 1 - 1/r \geq 0$. When $r = 1$ we have that $x^{(2)} = x^{(1)} = 0$ and the two fixed points coincide. When $r > 1$ we have $x^{(2)} > x^{(1)}$ and there could be two attractors of the dynamics.

Let us check this scenario numerically by iterating the map for $r = 1.5$. In this case, the second fixed point is $x^{(2)} = 0.33$.

In the left panel below we start from the same initial condition as before, $x_1 = 0.5$. In the right panel, we start from the initial condition $x_1 = 0.01$.



The left panel shows that the point $x^{(2)}$ is indeed an attractor. Conversely, the right panel shows that the point $x^{(1)}$ is no longer a fixed point. In fact it tends to "repel" the point away from its initial location. To rationalize these observations, it is convenient to place the analysis of stability on more rigorous grounds.

Let us call x_f one fixed point of the map, so that

$$f(x_f) = x_f. \tag{12.7}$$

The question that we want to answer is: If we start with x_1 in the vicinity of x_f, and we iterate the map f, how can we know whether the point will remain in the neighborhood of x_f, be attracted to x_f, or be repelled away from x_f?

To answer this question, we consider $x_1 = x_f + \varepsilon$, where ε is a small value (either positive or negative). We can write:

$$x_3 = f(x_2) = f(f(x_1)) = f(f(x_f + \varepsilon)).$$

By performing a Taylor expansion and using the chain rule, to first order in ε this expression becomes:

$$x_3 = f\left(f(x_f) + \left.\frac{df}{dx}\right|_{x_f} \varepsilon\right) \overset{(12.7)}{=} f\left(x_f + \varepsilon \left.\frac{df}{dx}\right|_{x_f}\right) = f(x_f) + \varepsilon \left.\frac{df}{dx}\right|_{x_f} \left.\frac{df}{dx}\right|_{x_f} \overset{(12.7)}{=} x_f + \varepsilon \left(\left.\frac{df}{dx}\right|_{x_f}\right)^2$$

We can repeat the same reasoning for $x_4, x_5, \cdots$ We will find that the general rule is:

$$x_n = x_f + \varepsilon\alpha^{n-1}, \qquad \alpha \stackrel{\text{def}}{=} \left.\frac{df}{dx}\right|_{x_f}. \tag{12.8}$$

Now we can distinguish three cases:

- $|\alpha| < 1$ In this case α^{n-1} tends to zero for large n, therefore $x_n \to x_f$. The point x_f is an attractor.
- $|\alpha| = 1$ In this case α^{n-1} is either $+1$ or -1, therefore x_n keeps orbiting around x_f. The point x_f is a stable fixed point.
- $|\alpha| > 1$ In this case α^{n-1} diverges at large n, therefore x_n is repelled by the fixed point. The point x_f is an unstable fixed point.

These considerations carry general validity and can be extended to the study of any dynamical systems. For systems with many degrees of freedom, instead of the α defined above, one has to check the eigenvalues of the Jacobian matrix of partial derivatives.

Let us evaluate the derivative df/dx for the two fixed points of the logistic map:

$$\alpha^{(1)} = \left.\frac{df}{dx}\right|_{x^{(1)}} = r(1-2x)|_0 = r \tag{12.9}$$

$$\alpha^{(2)} = \left.\frac{df}{dx}\right|_{x^{(2)}} = r(1-2x)|_{1-1/r} = 2 - r \tag{12.10}$$

Since in the example that we are considering now, with $r = 1.5$, $\alpha^{(1)} > 1$ and $\alpha^{(2)} < 1$, we find that the first fixed point should be unstable and the second one should be an attractor. This is precisely what we observed in the two panels above.

12.3.3 Logistic map with $2 < r < 3$

The nature of the fixed points $x^{(1)} = 0$ and $x^{(2)} = 1 - 1/r$ remains unchanged as we increase r in the interval $[2, 3)$. In fact, (12.9) and (12.10) show that in this range we have:

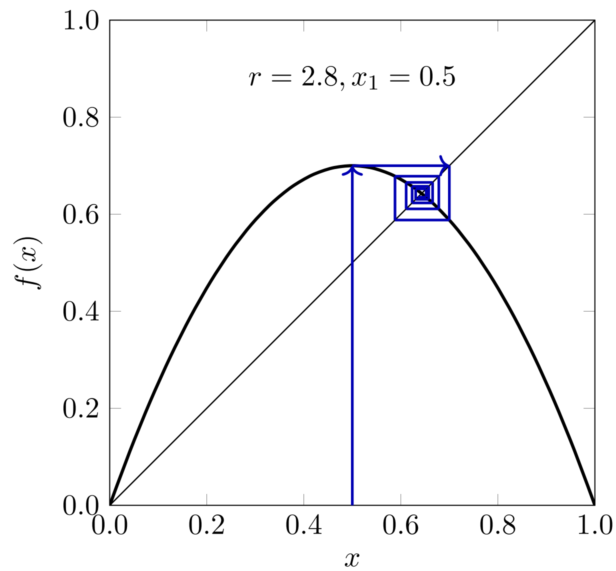
$$r \in [2,3): \quad \lambda^{(1)} = r > 1 \quad \longrightarrow \quad \text{unstable point}$$

$$r \in [2,3): \quad \lambda^{(2)} = 2 - r \in [0,1) \quad \longrightarrow \quad \text{stable point and attractor}$$

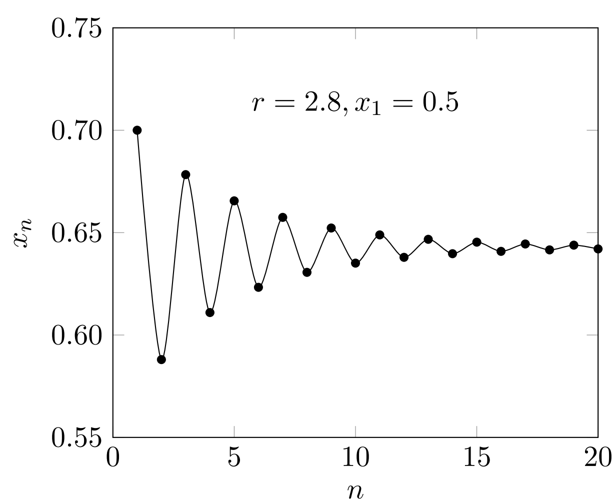
However, thee is an important change in the dynamics when $r \in [2, 3)$.

In all of the plots that we have seen for $r \leq 2$, the convergence of the variable x_n toward the fixed point was always **monotonic**. In fact, if we exclude the point x_1 from the sequence of values, we find that the series is ordered as $x_2 > x_3 > x_4 > \cdots$ or $x_2 < x_3 < x_4 < \cdots$ depending on where the initial point x_1 is located with respect to the fixed point. These findings are in agreement with the closed-form solution of the *continuous* Verhulst equation (12.1).

If we now look at the convergence toward the attractor when $r > 2$, we find a different behavior. The following plot is for $r = 2.8$, $x_1 = 0.5$:



This is called a **spiderweb diagram** because it starts looking like a spiderweb. We see that the variable x_n tends to the attractor $x^{(2)} = 1 - 1/r = 0.64$ in a **non-monotonic** fashion, oscillating between the left and the right hand sides of the attractor. If we plot this dynamics as a function of the discrete time n we obtain:



This behavior is fundamentally different from the dynamics of the continuous Verhulst equation (12.1). Here the convergence to the attractor is clearly taking place via composite dynamics that involves a component resembling exponential decay and a component resembling a **periodic** oscillation. In particular, from the figure we can guess that the period is $\Delta n = 2$.

To understand this behavior, let us consider a point in the vicinity of the fixed point x_f, $x_1 = x_f + \varepsilon > x_f$ with ε small and positive. Using (12.6) and a Taylor expansion we can write:

$$x_2 = f(x_1) = f(x_f + \varepsilon) = f(x_f) + \varepsilon \left.\frac{df}{dx}\right|_{x_f} \overset{(12.7)}{=} x_f + \varepsilon \left.\frac{df}{dx}\right|_{x_f}$$

The point x_1 is on the right on the fixed point by construction. The point x_2 will be on the left of the fixed point if:

$$\alpha = \left.\frac{df}{dx}\right|_{x_f} < 0 \tag{12.11}$$

This relation constitutes the criterion for the **onset of oscillations** around the fixed point, because at every application of the map the variable x_n will jump to the other side of the fixed point.

In the case of the logistic map, we have from (12.9) and (12.10):

$$\alpha = \left.\frac{df}{dx}\right|_{x_f} = r(1-2x_f) = \begin{cases} x^{(1)} = 0: & \alpha^{(1)} = r \\ x^{(2)} = 1 - 1/r: & \alpha^{(2)} = 2 - r \end{cases}$$

Therefore we start seeing oscillations in the dynamics when $r > 2$ because $\alpha^{(2)}$ turns negative in this range. Although these trajectories are not cahotic, we can say that these oscillations are the *precursors* of chaotic dynamics.

At this point we might ask how is that possible that the discrete logistic map departs from the smooth monotonic solution of the continuous logistic equation in (12.2).

To answer this question, let us recall the meaning of the parameter r from (12.3):

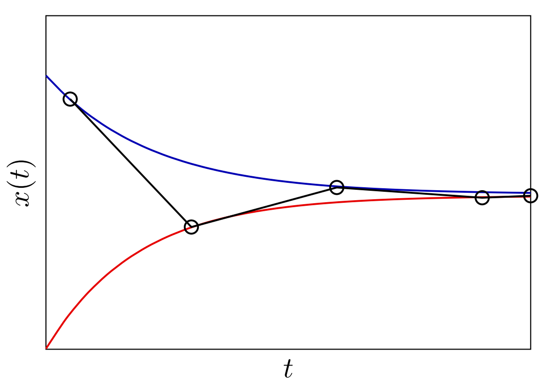
$$r \stackrel{\text{def}}{=} 1 + b\Delta t.$$

Using this relation, the condition $r > 2$ for the onset of oscillations translates into the condition

$$\Delta t > \frac{1}{b} \tag{12.12}$$

This means that the departure from monotonic solutions happens when the time-step of the discretization, Δt, exceeds the characteristic decay time of the population in the continuous Verhulst equation, $1/b$.

In practice, the use of the finite difference formula $\dot{x} \simeq [x(t+\Delta t) - x(t)]/\Delta t$ means that at each time point we linearize the solution. If the step Δt is smaller than the natural timescale of the problem, we will follow the correct trajectory. If, instead, the time step Δt is much longer than the natural timescale, we will overshoot the exact solution. At the next step, the variable will start from a different continuous solution of the problem (corresponding to a different starting point), and we will undershoot the correct solution, and so forth. A graphical representation of this effect may be as follows:



We can say that the "resonance" between the time step used in the discretization and the natural timescale of the original system conspire to introduce additional features in the discrete map that are not present in the continuous Verhulst equation.

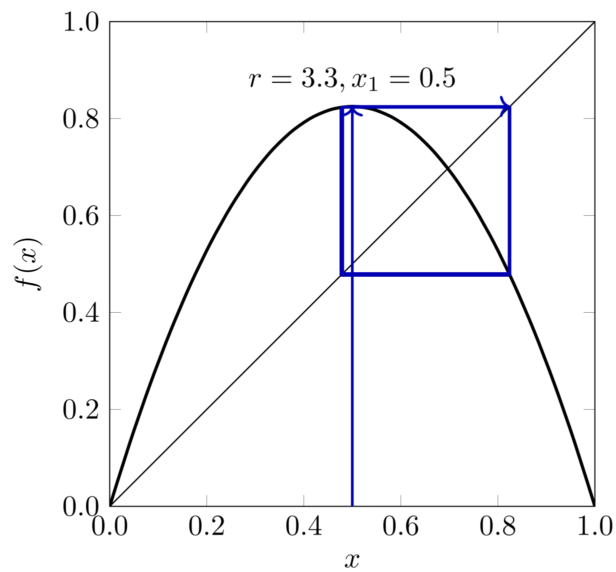
In a way we could say that the "new" oscillations that we just found are only an artifact of the discretization. In practical computer simulations, the artifact can simply be avoided by reducing the time step Δt. However, the presence of qualitatively new solutions is of interest regardless of the application of the logistic equation to population dynamics. From this point onward, we forget about the continous logistic equation and we continue the analysis of the logistic map as an interesting dynamical model. We can think of the logistic map as if representing the Poincaré map of some unspecified dynamical system.

12.3.4 Logistic map with $3 \leq \mathbf{r} < 4$

The most interesting behavior of the logistic map is found for values $r \geq 3$.

What we can expect from the above observations is that, as we keep increasing r above $r = 3$, the oscillations noted in the last plot become stronger and eventually dominate the dynamics.

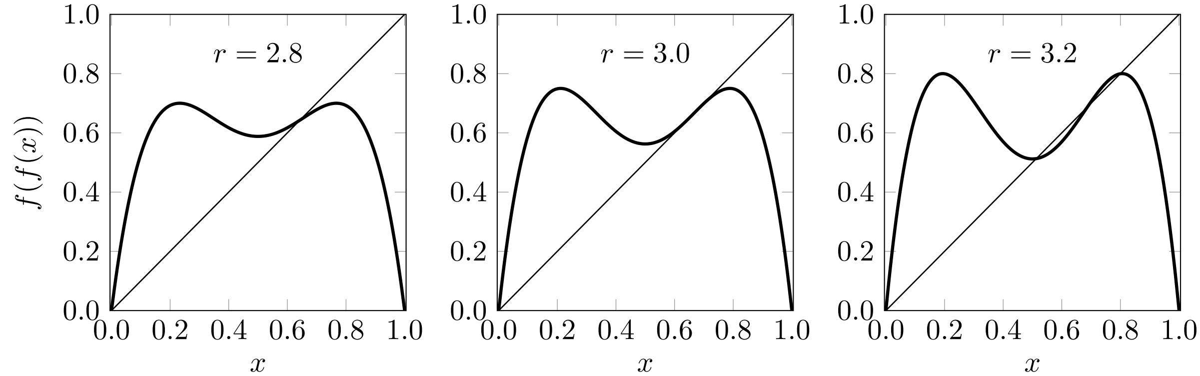
The following spiderweb diagram shows that this is indeed what happens. For $r = 3.3$ the fixed point $x^{(2)}$ is still stable, but it is no longer an attractor. Instead we observe a **limit cycle** where the variable oscillates indefinitely between two values.



To understand this behavior we can proceed as follows. In order for the dynamics to settle on a limit cycle consisting of two alternating values, two applications on the map should yield the original value:

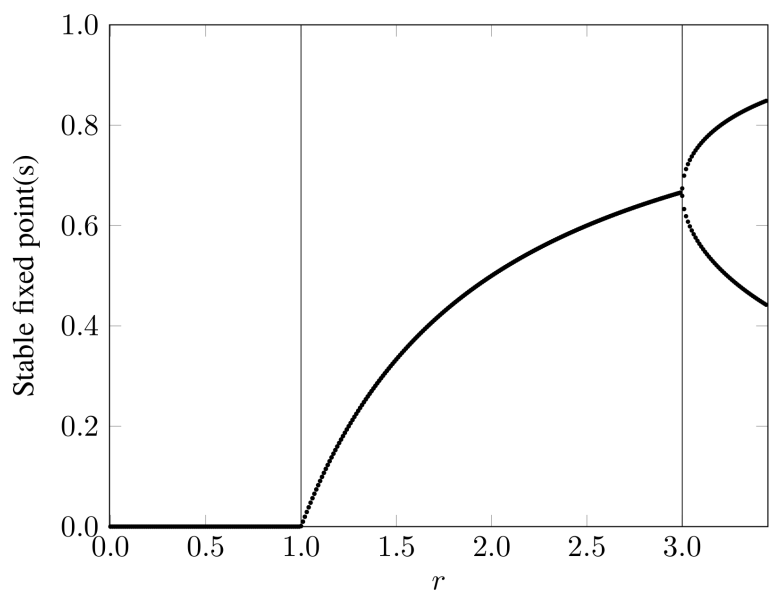
$$f(f(x)) = x \tag{12.13}$$

Below I show the function $f(f(x))$ for a few values of the parameter r:



(12.13) is satisfied when the black curve in each of these panels meets the straight line at $45°$. For $r < 3$ there is only one crossing, which coincides with the fixed point $x^{(2)}$. When $r = 3$ there is still only one crossing, but the curve $f(f(x))$ has the same slope as the straight line at the fixed point $x^{(2)}$. For values $r > 3$ we have three crossings. By performing some numerical experiments, we find that the crossing point is an unstable fixed point, and the other two crossings are stable fixed points.

The scenario illustrated in the above panels describes a **phase transition** where the asymptotic values of the dynamical variable x_n settle on qualitatively different trajectories. For those of you who have studied the Landau theory of phase transitions in Statistical Mechanics, you will recognize an analogy between this scenario and the second order paramagnetic/ferromagnetic phase transition. In fact, if we plot the attractors of the dynamics vs. r we obtain the same type of **bifurcation** that one finds in the theory of phase transitions:



In complete analogy with the logistic map, also in the case of magnetic phase transitions the trigger of the transition is the change of the roots of a cubic equation. In fact, we can rewrite (12.13) for the logistic map as:

$$r[rx(1-x)][1-rx(1-x)] = x \tag{12.14}$$

This equation has the solution $x^{(1)} = 0$ as before. For $x \neq 0$ we have a cubic equation:

$$(1-x)[1-rx(1-x)] - \frac{1}{r^2} = 0$$

The roots of this equation are (from Mathematica):

$$\begin{aligned}
x^{(2)} &= 1 - \frac{1}{r} \\
x^{(3)} &= 1 - \frac{1}{2}\left(1 - \frac{1}{r} - \sqrt{-\frac{3}{r^2} - \frac{2}{r} + 1}\right) \\
x^{(4)} &= 1 - \frac{1}{2}\left(1 - \frac{1}{r} + \sqrt{-\frac{3}{r^2} - \frac{2}{r} + 1}\right)
\end{aligned}$$

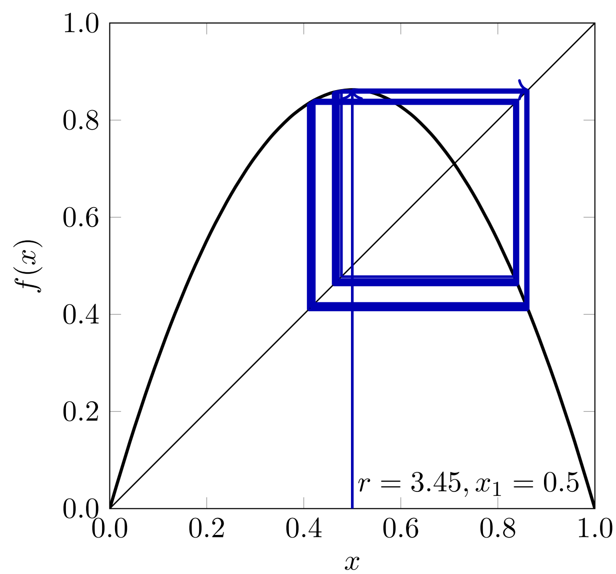
The root $x^{(2)}$ is the same fixed point that we have seen earlier. The two other roots $x^{(3)}$ and $x^{(4)}$ are real only when the argument of the square root is non-negative, therefore the critical point must satisfy the condition:

$$-\frac{3}{r^2} - \frac{2}{r} + 1 = 0 \quad \longrightarrow \quad r^2 - 2r - 3 = 0 \quad \longrightarrow \quad r = 1 \pm \sqrt{1+3} = 1 \pm 2$$

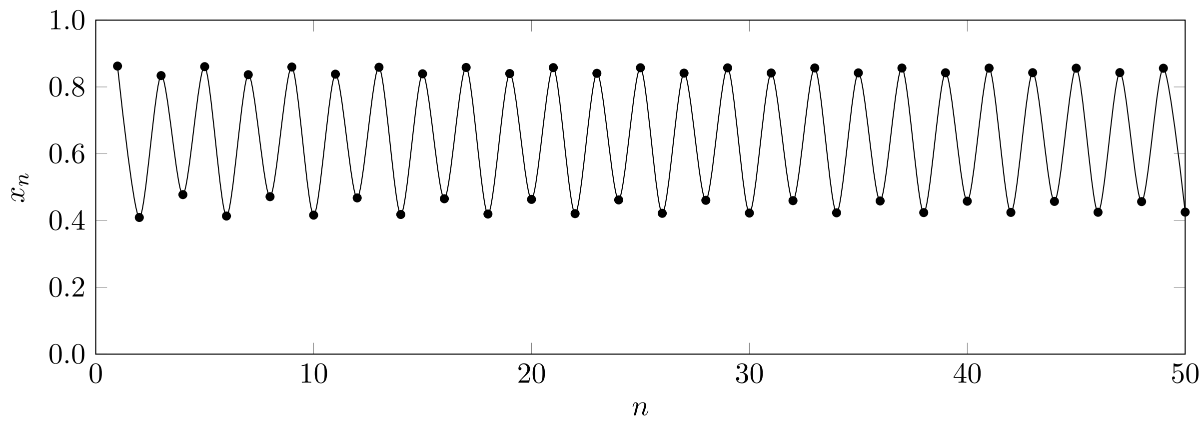
From this relation we see that, for $r < 3$ we have a pari for conjugate complex roots, for $r = 3$ we have three identical real roots, and for $r > 3$ we have three distinct real roots. This is consistent with the phase diagram shown at the previous page.

Note that the **nonlinear** nature of the function $f(x)$ is essential to observe this bifurcation effect.

If we keep increasing the value of r, we find another pair of bifurcations at $r_1 = 1 + \sqrt{6} = 3.4495$.



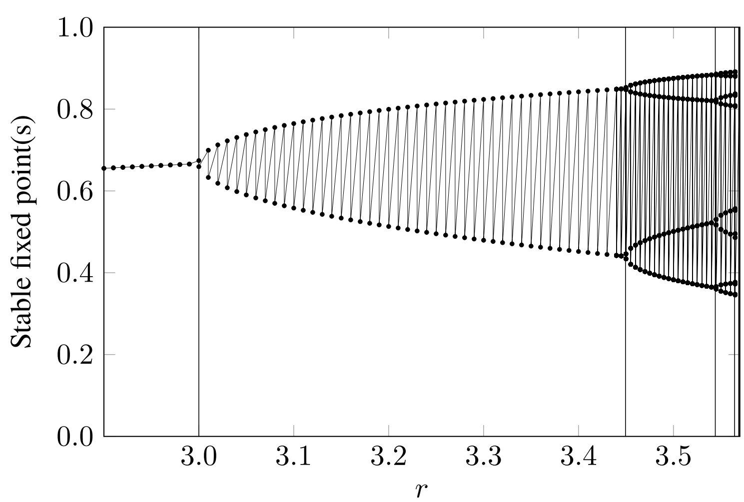
This scenario corresponds to a limit cycle with a period $\Delta n = 4$, as it can be seen from the plot of x_n vs. n below:



It is natural to imagine that, as we increase r, we will keep encountering further bifurcations. In fact, numerical experiments show that we will find the following bifurcations:

generation	r	number of stable points = period
r_1	3	2
r_2	3.44949	4
r_3	3.54409	8
r_4	3.56441	16
r_5	3.56875	32
r_6	3.56969	64
r_7	3.56989	128
r_8	3.56993	256
r_9	3.56994	512
r_{10}	3.56995	1024
r_{11}	3.56995	2048
$r_\infty = 3.569945672$	∞	

This is shown in the following plot:



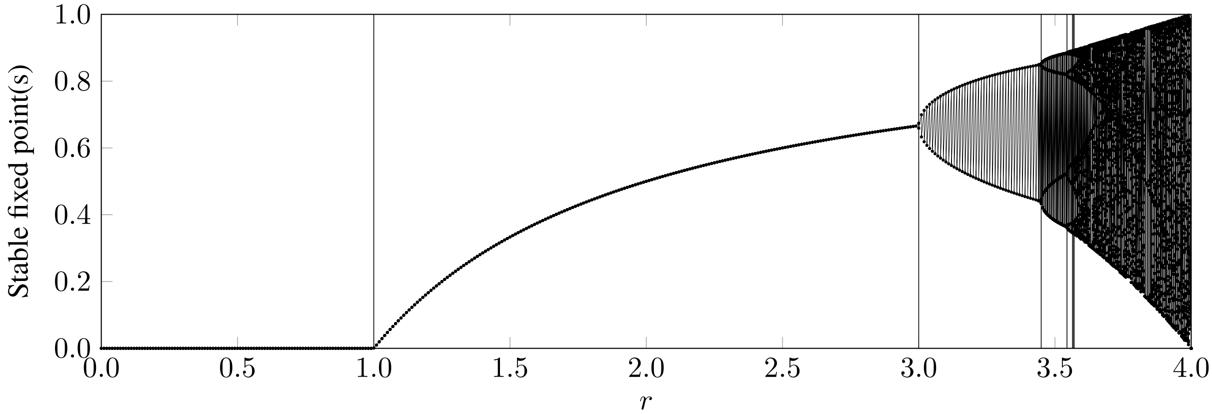
The interesting part of this plot is that we have an *infinity* of bifurcations as we approach the accumulation point $r_\infty = 3.569945672$. At this point, the period is *infinite* and we say that the orbit is **chaotic**, because it shows no regularity whatsoever.

In the literature you will find mention of the **Feigenbaum constant**, which is the limit of the ratio between the separation of successive bifurcations:

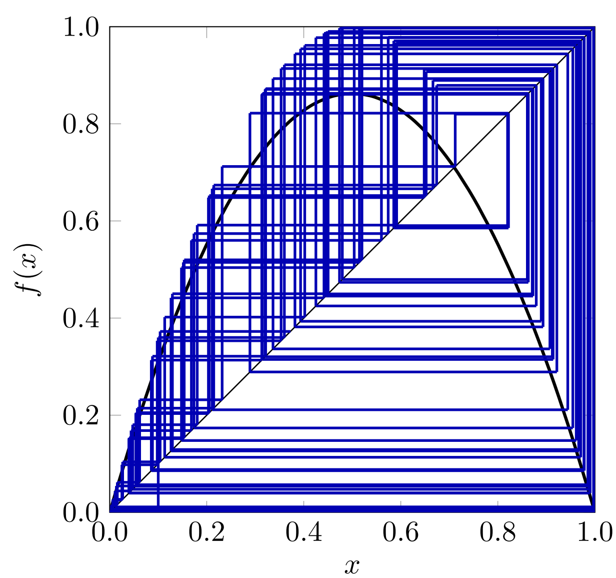
$$\delta \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{r_{n+1} - r_n}{r_{n+2} - r_{n+1}} = 4.669201609\ldots$$

This constant seems to appear in many other unrelated dynamical models, and there is a large body of literature devoted to it.

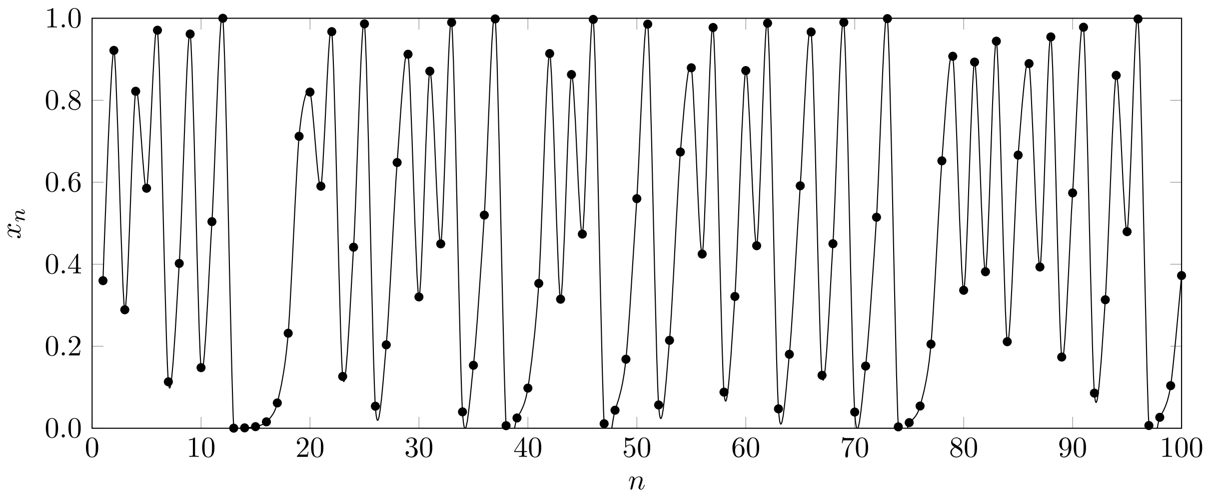
The complete map for $r \in [0, 4]$ is shown below:



For values of r between r_∞ and 4, the orbits ar no longer periodic (except for some small ranges of parameters which are called “island of stability”). The spiderweb diagram for $x_1 = 0.1$ and $r = 4$ looks as follows, for $n = 1, 2, \cdots 100$:



Here we can see that the dynamical variable x_n does not settle on a limit cycle, but keeps jumping around the interval $[0, 1]$ with no apparent regularity. A plot of the time evolution corresponding to the above spiderweb diagram is shown below:



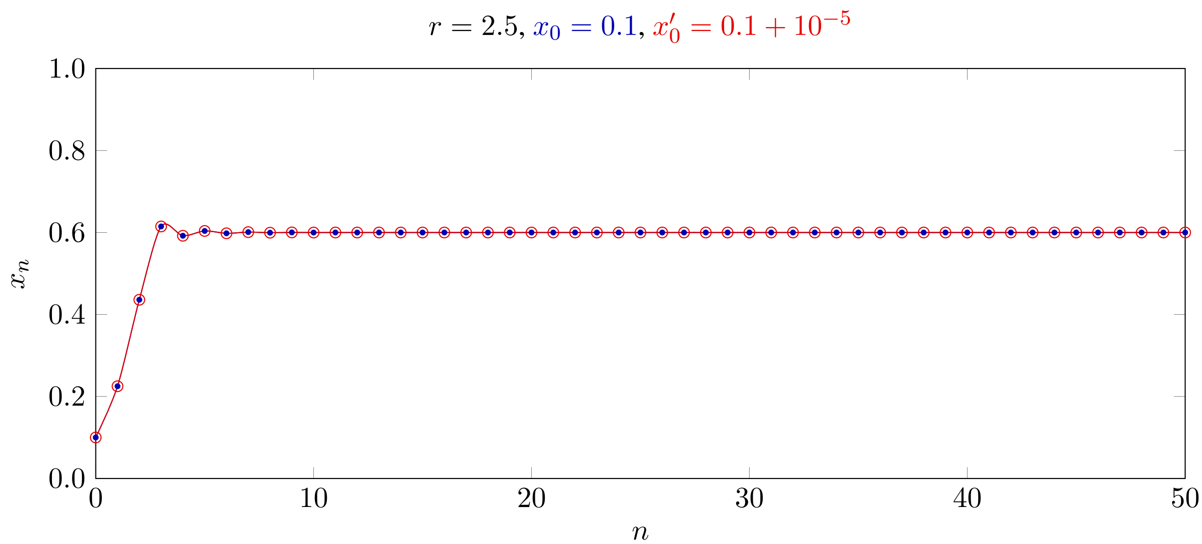
12.4 Liapunov exponents

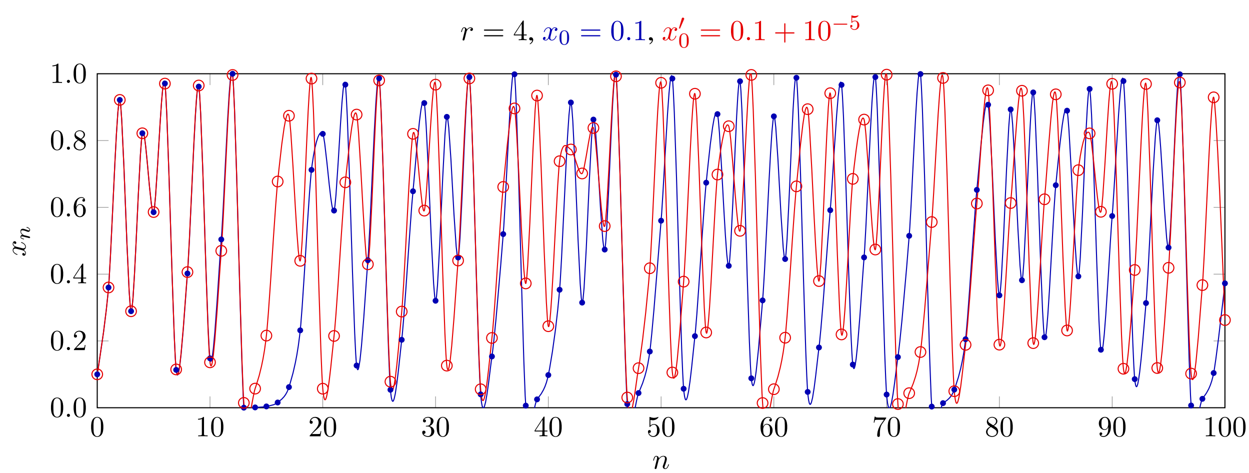
In the above analysis of the logistic map we has seen that, when the parameter r of the map is above $r = 3$, we start seeing bifurcations. For $r = r_\infty = 3.569945672$ we have an infinity of bifurcations, and beyond this point the system is chaotic. Now we want to define more carefully what is meant by "chaotic".

At an intuitive level, a "chaotic" patterns is one where we cannot find any regularity, and we have no way to predict how the system will evolve over time.

From a mathematical standpoint, this intuitive notion can be formalized by comparing how a trajectory depends on small changes of the initial conditions.

The following two plots compare the trajectories of the logistic map for two initial conditions, $x_0 = 0.1$ (blue) and $x'_0 = 0.1 + 10^{-5}$ (red), and for two cases: $r = 2.5$ (no bifurcations) and $r = 4$ (fully developed chaos):





The first of these plots show that, if we slightly alter the initial condition, the dynamics is almost unchanged for $r = 2.5$. Conversely, in the chaotic region $r = 4$, a tiny change in the initial conditions (10^{-5}) leads to completely different trajectories. We say that the system exhibits **strong sensitivity to the initial conditions**. Such a sensitivity is one of the defining characteristics of chaos.

The sensitivity to initial conditions carries both practical and philosophical implications. At a practical level, if initial conditions come from experiments, then we certainly have an error in the measurements, and this error will lead to complete loss of predictive power in simulations of chaotic systems. Furthermore, even with very accurate measurements, in current computer simulations we use finite-precision arithmetics (e.g. 16-digits precision), hence the results are no longer reliable after a few steps. At a philosophical level, the very existence of chaotic solutions challenges the classical deterministic notion that to predict the future behavior of a system we only need to know the laws of motion (e.g. Newton's laws). In chaotic systems the ability to predict the future is no longer guaranteed.

In order to quantify how fast one looses predictive power in chaotic systems, it is possible to use the so-called **Liapunov exponent** as follows.

Let us call $x(t;x_0)$ and $x(t;x_0+\varepsilon)$ two trajectories starting from slightly different initial conditions x_0 and $x_0+\varepsilon$. We define the Liapunov exponent λ as the *relative rate of change* of the deviation between these trajectories:

$$\frac{ds}{s} = \lambda\, dt, \qquad s(t;x_0) \stackrel{\text{def}}{=} x(t;x_0+\varepsilon) - x(t;x_0). \tag{12.15}$$

In this definition, the exponent λ is typically time-dependent. If we consider a short time interval for which λ is approximatly a constant, the integration of the above relation gives:

$$s(t) = s(t=0)\, e^{\lambda t} \quad \longrightarrow \quad |s(t)| = |s(t=0)|\, e^{\lambda t} \tag{12.16}$$

therefore the Liapunov exponent quantifies the **exponential** convergence or divergence rate of nearby trajectories.

In the case of a discrete map such as the logistic map, (12.16) can be rewritten as:

$$|s_n| = |s_0|\, e^{\lambda n}. \tag{12.17}$$

In practice, to eliminate the dependence on the difference between initial conditions ε and on the iteration step n, one takes the limits of small ε and large n:

$$\lambda = \lim_{n\to\infty} \lim_{\varepsilon\to 0} \frac{1}{n} \log\left|\frac{s_n}{s_0}\right|. \tag{12.18}$$

This expression can be written in a form that is easier to evaluate, using the following reasoning:

$$\begin{aligned}
\lim_{\varepsilon\to 0} s_n &= \lim_{\varepsilon\to 0}\; x_n(x_0+\varepsilon) - x_n(x_0) = \varepsilon \frac{\partial x_n}{\partial x_0} \\
&= \varepsilon \frac{\partial}{\partial x_0} \underbrace{f(f(\cdots f}_{n \text{ times}}(x_0))) \\
&= \varepsilon \prod_{i=0}^{n-1} \left.\frac{\partial f}{\partial x}\right|_{x_i}
\end{aligned}$$

Since $s_0 = \varepsilon$ by construction, we can rewrite (12.18) as:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log\left|\prod_{i=0}^{n-1} \left.\frac{\partial f}{\partial x}\right|_{x_i}\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left|\left.\frac{\partial f}{\partial x}\right|_{x_i}\right| \tag{12.19}$$

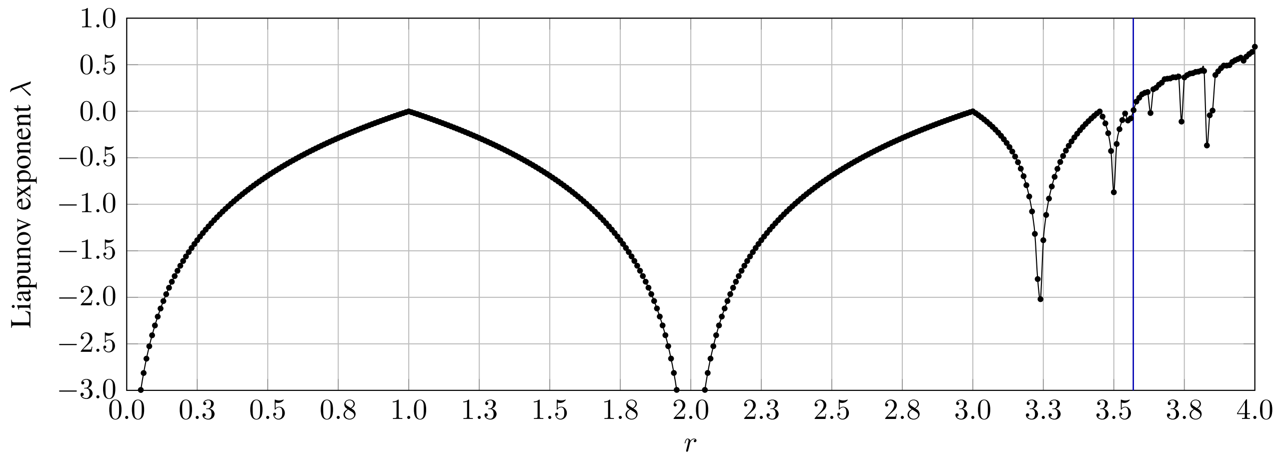
For the logistic map we know that

$$\frac{\partial f}{\partial x} = r(1-2x)$$

therefore the Liapunov exponent is:

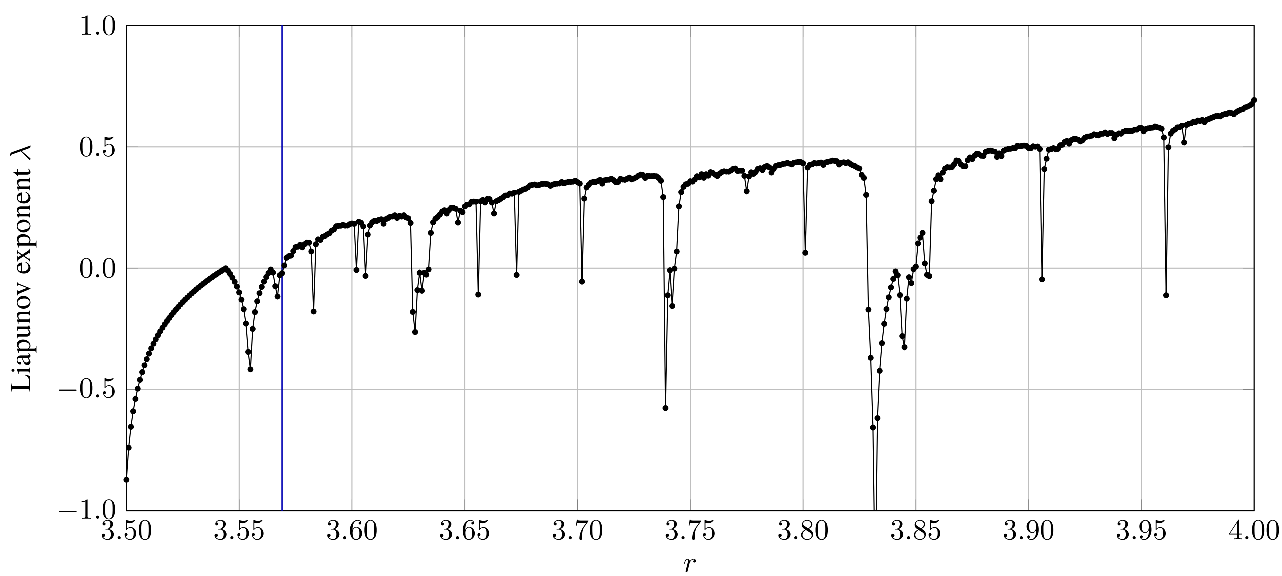
$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log[r(1-2x_i)] = \log r + \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|1-2x_i| \tag{12.20}$$

In the following plot I show the Liapunov exponent that I calculated using (12.20) and $n = 10,000$.



The exponent is negative throughout the range $r \in [0, r_\infty] = [0, 3.569945672]$, indicated by the blue line in the figure. In this range, the trajectories correspondind to slightly different initial conditions tend to coalesce into the same trajectory.

Conversely, in the chaotic region $r > r_\infty$, the Liapunov exponent becomes negative. Here we have strong sensitivity to initial conditions. A zoom in the chaotic region is shown below:



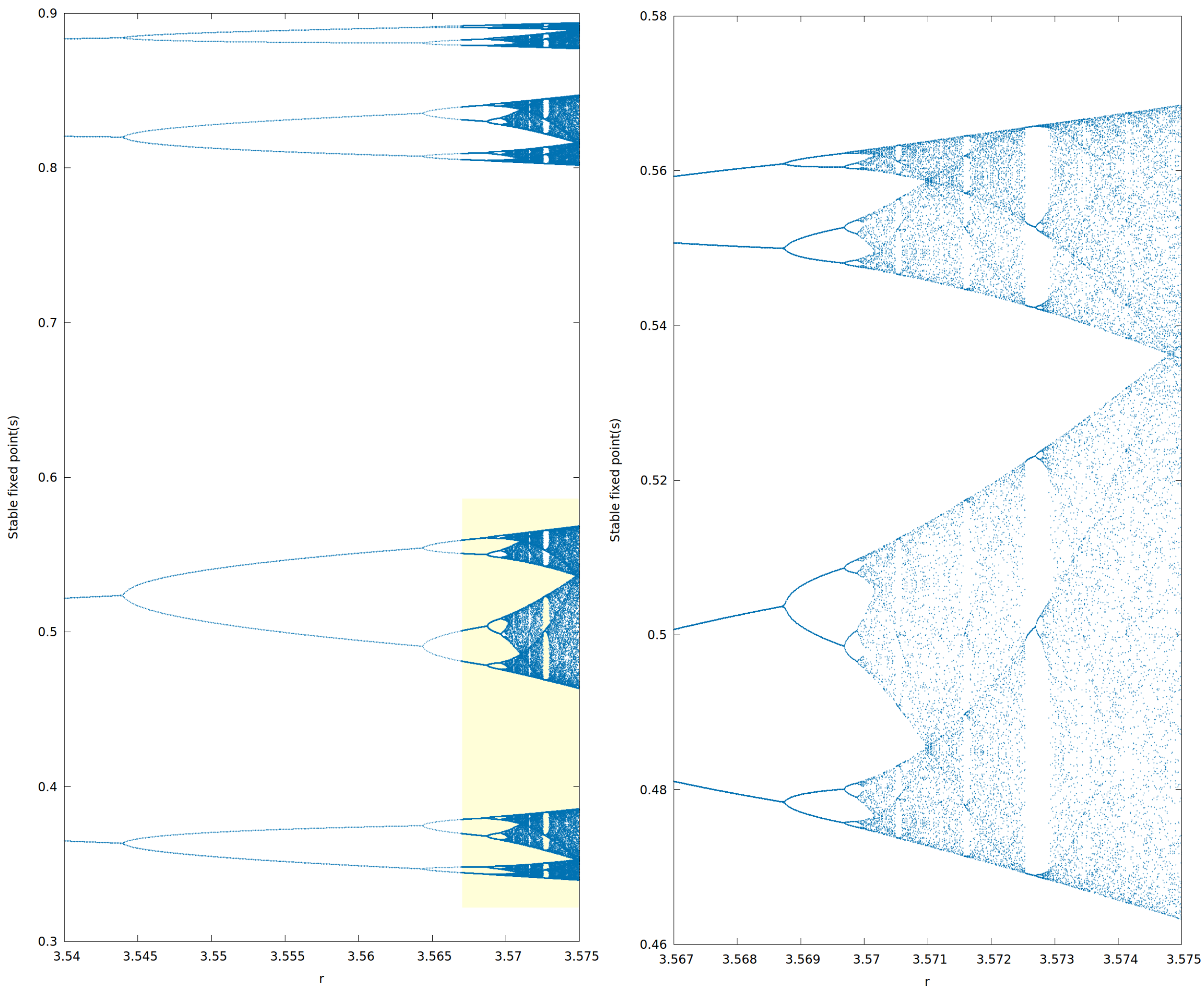
It is quite remarkable that the definition of Liapunov exponent in (12.18) provides a very accurate marker for the transition to the chaotic regime. Notice that **all** points to the left of the vertical blue bar have $\lambda \le 0$.

In this plot we also see that, in the chaotic region, we have occasional intervals where λ dips below zero. These intervals correspond to the islands of stability mentioned earlier.

12.5 Self-similarity

One last property of chaos that we want to mention is the fact that the phase diagram of a chaotic system exhibits smaller replicas of itself when we zoom in. This property is called **self-similarity**.

To see this effect, I plot the stable points of the logistic map vs. r once again, this time by running the map for a much finer mesh than I did earlier. The right panel corresponds to a magnification of the yellow portion of the left panel:



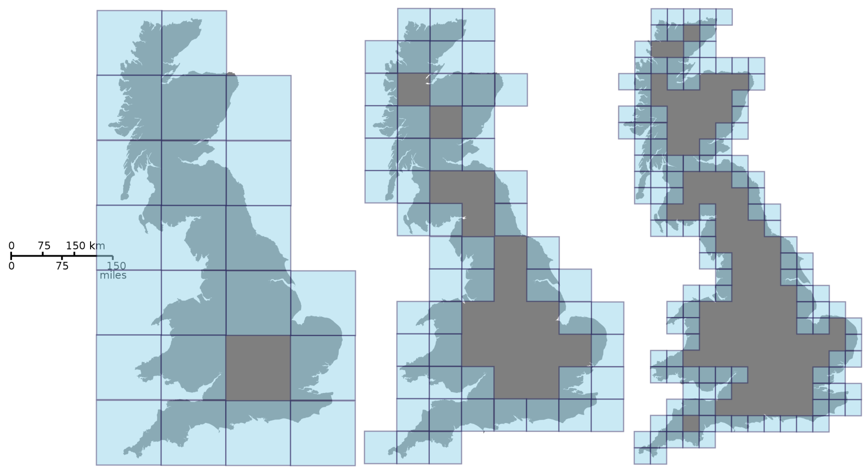
From this plot we see that the phase diagram tends to reproduce itself at smaller scales. This is just a graphical manifestation of the accumulation of bifurcations that we discussed earlier.

Self-similarity is another defining characteristic of chaotic dynamical systems. We already discussed this property for the Hénon-Heiles Hamiltonian, and now we are seeing this property for the logistic map. The property is indeed general and it is found whenever chaotic dynamics emerges.

As a curiosity, I only mention that self-similarity is linked to the **fractal** nature of chaotic attractors. Without going into the details, the notion of "fractal" nature refers to the fact that the dimensionality of the attractor is not an integer but a fractional number. For example, a point has dimension 0 in a 3-d Euclidean space, a line has dimension 1, etc. We could ask the dimension of the attractor of the logistic map, i.e. the "size" of the set of points that we find for a given r. It is obvious from the above plots that these points tend to "fill" continuous intervals, therefore we suspect that they might have the same dimension as a line, $d = 1$. However, at any iteration we have a finite number of points, therefore we might also argue that the dimension should be that of points, $d = 0$.

It turns out that a quantitative analysis of these structure requires a new definition of the concept of dimension. The standard procedure is to introduces a "fractal dimension". The fractal dimension is designed to yield the usual dimension for common objects like a point or a line, and provides a quantification of how much chaotic attractors can fill up the phase space.

There are several recipies to evaluate the fractal dimension. The most intuitive version is the **Minkowski – Bouligand dimension** or box-counting dimension. The idea can be understood by looking at the following figure (from `https://en.wikipedia.org/wiki/Minkowski%E2%80%93Bouligand_dimension`):



To find the box-counting dimension we slice the figure in a two-dimensional regular grid with spacing a. We count how many boxes contain land (from the UK in this image). Then we repeat the same operation, but for a grid with spacing $a/2$, and so on. If we call N_k the number of filled (or partially filled) boxes at the spacing a/k, with $k = 1, 2, 3, \cdots$ the box-counting dimension is defined as:

$$d \stackrel{\text{def}}{=} \lim_{k\to\infty} \frac{\log N_k}{\log k}. \tag{12.21}$$

The idea of this metric is that it takes into account how the object fills the space with increasingly higher resolution.

We can check that, in the case of a standard square, the box-counting dimension is precisely $d = 2$ as expected. In fact, the number of boxes filled by a square of size a when we use a grid with spacing a/k is:

$$N_k = k^2$$

therefore (12.21) yields:

$$d = \lim_{k\to\infty} \frac{\log k^2}{\log k} = 2.$$

In the case of the logistic map, the box-counting dimension is $d \sim 0.5$ when approaching r_∞, which is between 0 and 1, in line with our intuitive considerations.