LECTURE 3

3. Fourier Transform theory

In section (2.2) we introduced the *Discrete-time Fourier transform* and developed its most important properties. Recall that the DTFT maps a finite-energy sequence $a = \{a_n\} \in \ell^2$ to a finite-energy function $\widehat{a} \in L^2[-\frac{1}{2}, \frac{1}{2}]$, where

$$\widehat{a}(\xi) = \sum_n a_n e^{-2\pi i n \xi}.$$

For each fixed $\xi \in [-\frac{1}{2}, \frac{1}{2}]$, $\hat{a}(\xi)$ measures the degree to which a correlates with (responds to) the oscillatory sequence

$$e_\xi = \{e^{2\pi i n \xi}\}_n.$$

For this reason, we often refer to $\widehat{a}(\xi)$ as the 'Frequency Response function' of a. Following the electrical engineering convention, we sometimes use the notation $A(\xi)$ for $\widehat{a}(\xi)$.

The map $a \mapsto \widehat{a}$ is linear, preserves energy (and consequently the inner product), maps delay into modulation, and maps convolution into pointwise multiplication. It is also invertible—a may be recovered from A by the formula

$$a = \sum_n \left(\int_{-1/2}^{1/2} A(\xi) e^{2\pi i n \xi} d\xi \right) \varepsilon^{(n)}.$$

Interpretation: at position n, the value a_n of the sequence a is a 'weighted average' (over all frequencies $\xi \in [-\frac{1}{2}, \frac{1}{2}]$) of the values $e^{2\pi i n \xi}$ (= the value of e_ξ at position n). The 'weight' assigned to the value $e^{2\pi i n \xi}$ is $A(\xi)$, the response of the sequence a to the frequency ξ.

In this section, we discuss three other 'incarnations' of the Fourier transform, namely (1) *Fourier series*, (2) the *continuous Fourier transform*, and (3) the *discrete Fourier transform* (DFT). (1) is a mapping $L^2([0,1]) \to \ell^2$ (periodic functions to sequences), (2) is a mapping $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ (functions to functions), and (3) is a mapping $\ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$ (finite sequences to finite sequences).

In the previous expression, $\mathbb{Z}_N$ refers to the additive group of integers modulo N. That is, $\mathbb{Z}_N = \{0, 1, ..., N-1\}$ together with the operation $\oplus_N$ of addition modulo N, which produces as the result of adding m and n, not $m+n$ but rather the remainder of $m+n$ upon division by N (e.g. $5 \oplus_9 7 = 3$). $\ell^2(\mathbb{Z}_N)$ (hereafter just ℓ^2_N) then denotes the set of all finite complex sequences $a = \{a_0, a_1, ..., a_{N-1}\}$ of length N. Alternatively, one could think of ℓ^2_N as the space of all infinite complex sequences which are N-periodic (repeat every N terms). The energy of such a sequence, $E(a) = \sum_{n=0}^{N-1} |a_n|^2$, is necessarily finite (being a finite sum). We will show that all of these Fourier transforms are linear, energy-preserving, and invertible, map translation to modulation, and map convolution to multiplication.

The question then begs, is there some general theory which subsumes these four examples ((1)-(3) + the DTFT)? The answer is yes, although the complete details are beyond the scope of these notes. Suffice it to say the following: for most 'nice' groups G, there exists a Fourier transform. It maps finite-energy functions on G (whatever that means) to finite-energy functions on the 'dual group' $\widehat{G}$ (whatever that means) and has all the nice properties of the DTFT. Not surprisingly, the dual group of $\mathbb{Z}$ is $[0, 1]$ (the circle), the dual group of $[0, 1]$ is $\mathbb{Z}$, the dual group of $\mathbb{R}$ is itself, and likewise for $\mathbb{Z}_N$. If this made no sense to you (which is likely if you haven't done any group theory), just take away the fact that every Fourier transform we discuss is a special case of a more general theory. If, on the other hand, some of this was familiar to you and you'd like to learn more, try *Fourier Analysis on Groups* by Walter Rudin.

3.1. **Fourier Series.** Trigonometric functions, i.e., periodic wave-like functions which oscillate indefinitely without decay, are the basic building blocks used to construct Fourier series expansions. The periodic behavior is one feature distinguishing Fourier series expansions from wavelet expansions.

Recall that a function ϕ is said to be *periodic* and have *period 1* if the equality $\phi(x + 1) = \phi(x)$ holds for all x. For instance, each member of the family

$$\{e^{2\pi inx} : n \in \mathbb{Z}\}$$

is periodic and has period 1. This periodicity ensures that this family is orthonormal in $L^2([0,1])$. That is,

$$(e^{2\pi imx}, e^{2\pi inx}) = \int_0^1 e^{2\pi imx}\overline{e^{2\pi inx}}dx = \int_0^1 e^{2\pi imx}e^{-2\pi inx}dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

(exercise). Thus, the ideas of section 1 apply.

Definition 3.1.1 (Fourier coefficients, series)**.** *For each $f \in L^2([0,1])$ the coefficients*

$$\widehat{f}(n) = (f, e^{2\pi inx}) = \int_0^1 f(y)e^{-2\pi iny}dy \qquad (n \in \mathbb{Z})$$

of f with respect to the orthonormal family $\{e^{2\pi inx} : n \in \mathbb{Z}\}$ are called the ***Fourier coefficients*** *of f and the corresponding orthonormal series development*

$$S[f](x) = \sum_n \widehat{f}(n)e^{2\pi inx}$$

is called the ***Fourier series*** *of f.*

Such series expansions were introduced in 1807 by Joseph Fourier in an astoundingly original paper in which he used them to solve problems in heat flow. Initially, their introduction was very controversial because many eminent mathematicians found it hard to accept that an arbitrary function on $[0,1]$ could be represented by indefinitely differentiable functions which oscillate periodically. In part the controversy arose because the notion of function and theory of infinite series of functions had not been fully developed at that time.

Let

$$S_n[f](x) = \sum_{k=-n}^{n} \widehat{f}(k)e^{2\pi ikx}$$

be the 'nth' partial sum of the Fourier series $S[f]$ of f. Being a finite sum of period 1 functions having derivatives of all orders, each partial sum is a period 1 function having derivatives of all orders. The basic fundamental question in Fourier series from the time they were first introduced was to decide on the sense in which $S_n[f](x)$ converges to $f(x)$. The sense in which $S[f]$ does represent f only became clear with the development of inner product spaces by David Hilbert around the turn of the twentieth century and the complete answer was not given until 1964-65. The fundamental L^2-results are contained in the next theorem.

Theorem 3.1.1. *The family $\{e^{2\pi inx} : n \in \mathbb{Z}\}$ is a complete orthonormal family in $L^2([0,1])$. In particular, the equalities*

(i) $\int_0^1 |f(x)|^2 dx = \sum_n |\widehat{f}(n)|^2$ *(Plancherel)*

(ii) $\int_0^1 f(x)\overline{g(x)}dx = \sum_n \widehat{f}(n)\overline{\widehat{g}(n)}$ *(Parseval)*

hold for all $f, g \in L^2([0,1])$, and f can be recovered from $\{\widehat{f}(n)\}$ via the Fourier series

$$f(x) = S[f](x) = \sum_n \widehat{f}(n)e^{2\pi inx},$$

in the sense that $E(f - S_n[f]) \to 0$ as $n \to \infty$.

To prove theorem 3.1.1, we will need some important auxiliary results. In the interest of time, we won't prove these. Instead, we refer the reader to any good book on Fourier analysis. The first such result specifies conditions on a function under which its Fourier series converges pointwise (as opposed to with respect to energy). It shows that pointwise convergence depends on *smoothness* (i.e., existence and continuity of derivatives). It came as a great shock that there are continuous functions f for which the equation $f(x) = S[f](x)$ fails at many points (Du Bois-Reymond, 1876).

Theorem 3.1.2. *Let f be a continuous period 1 function having a continuous derivative. Then the Fourier series $S[f](x)$ converges pointwise for every x, $-\infty < x < \infty$; furthermore,*

$$\lim_{n\to\infty} S_n[f](x) = f(x) \qquad (-\infty < x < \infty).$$

We will also need a 'density' theorem.

Theorem 3.1.3. *Let $f \in L^2([0,1])$ be a finite-energy function and $\epsilon > 0$. Then there exists continuous period 1 function g having a continuous derivative, such that*

$$E^{1/2}(f - g) < \epsilon.$$

Now we proceed with the proof of theorem 3.1.1.

Proof. Let $f \in L^2([0,1])$ be a finite-energy function. We must show that $E(S[f]) = E(f)$, or what is the same,

$$\lim_{n\to\infty} E^{1/2}(f - S_n[f]) = 0.$$

So let $\epsilon > 0$. By theorem 3.1.3, there exists a continuous period 1 function g having a continuous derivative, such that

$$E^{1/2}(f-g) < \epsilon.$$

Thus

$$\begin{aligned} E^{1/2}(f - S_n[f]) &= E^{1/2}(f - g + g - S_n[g] + S_n[g] - S_n[f]) \\ &\leq E^{1/2}(f-g) + E^{1/2}(g - S_n[g]) + E^{1/2}(S_n[g-f]) \\ &\leq 2E^{1/2}(f-g) + E^{1/2}(g - S_n[g]) < 2\epsilon + E^{1/2}(g - S_n[g]), \end{aligned}$$

since Bessel's inequality ensures that

$$E^{1/2}(S_n[g-f]) \leq E^{1/2}(g-f) = E^{1/2}(f-g).$$

Thus it is enough to show that

$$\lim_{n\to\infty} E^{1/2}(g - S_n[g]) = 0.$$

But by theorem 3.1.2, the equality

$$g(x) = \sum_n \widehat{g}(k) e^{2\pi i k x} \qquad (-\infty < x < \infty)$$

holds pointwise, and so

$$E(g - S_n[g]) = \int_0^1 |g(x) - S_n[g](x)|^2 \, dx = \int_0^1 \left| \sum_{|k|>n} \widehat{g}(k) e^{2\pi i k x} \right|^2 dx = \sum_{|k|>n} |\widehat{g}(k)|^2.$$

By Bessel's inequality yet again however,

$$\sum_k |\widehat{g}(k)|^2 \leq \int_0^1 |g(x)|^2 dx,$$

and so

$$\lim_{n\to\infty} \sum_{|k|>n} |\widehat{g}(k)|^2 = 0.$$

This completes the proof. $\square$

Example 3.1.1. *To illustrate these ideas, consider the following functions:*

(i) *(Saw-tooth function – linear version). Define* f *on* $[-1/2, 1/2)$ *by*

$$f(x) = x \qquad (-1/2 \leq x < 1/2)$$

and by $f(x+1) = f(x)$ *elsewhere. Then by construction it has period 1 but it is not continuous. After integration by parts, we deduce that*

$$S[f](x) = \frac{2}{\pi}\left[\frac{1}{2}\sin(2\pi x) - \frac{1}{4}\sin(4\pi x) + ...\right] = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\sin(2\pi kx).$$

(ii) *(Saw-tooth function – quadratic version). Define* f *on* $[-1/2, 1/2)$ *by*

$$f(x) = x^2 \qquad (-1/2 \leq x < 1/2)$$

and by $f(x+1) = f(x)$ *elsewhere. Then*

$$S[f](x) = \frac{1}{12} + \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos(2\pi kx).$$

(iii) *(Square-wave function). Define* f *on* $[-1/2, 1/2)$ *by*

$$f(x) = \begin{cases} -1 & \text{if } -1/2 \leq x < 0 \\ 1 & \text{if } 0 \leq x < 1/2 \end{cases}$$

and by $f(x+1) = f(x)$ *elsewhere. Then*

$$S[f](x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin(2\pi(2k+1)x).$$

(iv) *(Jagged-edge function). Define* f *on* $[-1/2, 1/2)$ *by*

$$f(x) = 1 - 2|x| \qquad (-1/2 \leq x < 1/2)$$

and by $f(x+1) = f(x)$ *elsewhere. Then*

$$S[f](x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2\pi(2k-1)x).$$

Note that theorem 3.1.2 does not apply to the saw-tooth or square-wave functions because their graphs show they are not continuous. Nor does it apply to the jagged-edge function for it is clear from its graph that it is continuous everywhere but fails to a have a derivative at every point $\pm 1/2, \pm 3/2, \ldots$. There are generalizations of theorem 3.1.2 which explain what happens for these functions.

One final comment about Fourier series before we move on to the continuous Fourier transform. Let $f \in L^2([0,1])$. By Plancherel,

$$\sum_n |\widehat{f}(n)|^2 = \int_0^1 |f(x)|^2 dx < \infty.$$

Thus the Fourier coefficients of f go to zero at infinity. This is the Riemann-Lebesgue lemma. Even more is true. The smoother f is (the more continuous derivatives it possesses), the faster the Fourier coefficients decay at infinity. More precisely, we have the following result:

Theorem 3.1.4 (smoothness implies decay). *Let $f \in C^k([0,1])$ (i.e. f is k-times continuously differentiable on $[0,1]$). Then its Fourier coefficients satisfy*

$$|\widehat{f}(n)| \leq C|n|^{-k}$$

for some constant C (independent of n).

The proof is straightforward and we leave it as an exercise. Conversely, we have the following result:

Theorem 3.1.5 (decay implies smoothness). *Let $f \in L^2([0,1])$. If*

$$|\widehat{f}(n)| \leq C|n|^{-1-\epsilon}$$

for some constant C and some $\epsilon > 0$, then $f \in C([0,1])$. Similarly, if

$$|\widehat{f}(n)| \leq C|n|^{-1-k-\epsilon},$$

then $f \in C^k([0,1])$.

Notice that we have lost a one in the exponent here when we try to characterize smoothness from the Fourier coefficients. We can obtain better characterizations using Littlewood-Paley theory (octave bands).

3.2. **Continuous Fourier Transform.** The Fourier transform of a finite-energy function $f \in L^2(\mathbb{R})$ is an another function on $\mathbb{R}$, $\widehat{f}$, defined by the formula

$$\widehat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx \qquad (\xi \in \mathbb{R}).$$

Interpretation: in analogy with the case of the of the DTFT, $\widehat{f}(\xi)$ measures how aligned f is with the 'pure tone' $e^{2\pi i x\xi}$. Since $e^{2\pi i x\xi}$ is not a finite-energy function, a little care must be exercised. Certainly if f lives on a finite interval, say $[-M, M]$, there is no problem—the defining integral becomes

$$\widehat{f}(\xi) = \int_{-M}^{M} f(x)e^{-2\pi i x\xi}dx,$$

and both f and $e^{2\pi i x\xi}$ have finite energy on $[-M, M]$. In the general case, when f doesn't have compact support, an approximation argument is used (you replace f by its restriction to $[-M, M]$, calculate the Fourier transform, and then let $M \to \infty$).

Having defined the Fourier transform for functions in $L^2(\mathbb{R})$, we now record its main properties. These should be familiar by now.

Theorem 3.2.1. *On $L^2(\mathbb{R})$, the Fourier transform has the following properties:*

(i) $\int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 d\xi = \int_{-\infty}^{\infty} |f(x)|^2 dx$ *(Plancherel)*

(ii) $\int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$ *(Parseval)*

In particular, the Fourier transform is a unitary operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ in the sense that

$$(\widehat{f}, \widehat{g}) = (f, g) \qquad (f, g \in L^2).$$

Actually, it is onto as well. Its inverse function, the inverse Fourier transform, is given by the formula

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i x\xi}d\xi.$$

Further properties include:

Theorem 3.2.2. (i) $\mathcal{F}(f(x-a))(\xi) = e^{-2\pi i a\xi}(\mathcal{F}f)(\xi)$ *(delay $\rightarrow$ modulation)*

(ii) $\mathcal{F}(\lambda^{-1/2}f(x/\lambda))(\xi) = \lambda^{1/2}\mathcal{F}(f)(\lambda\xi)$ *(dilation $\rightarrow$ reciprocal dilation)*

(iii) $\mathcal{F}(f'(x))(\xi) = 2\pi i\xi(\mathcal{F}f)(\xi)$ *(differentiation $\rightarrow$ multiplication by ξ)*

(iv) $\mathcal{F}(xf(x))(\xi) = -\frac{1}{2\pi i}(\mathcal{F}f)'(\xi)$ *(multiplication by x $\rightarrow$ differentiation)*

(v) $\mathcal{F}(f * g)(\xi) = (\mathcal{F}f)(\xi)(\mathcal{F}g)(\xi)$ *(convolution $\rightarrow$ product)*

We will prove only a couple of these properties, including property (iii), which explains the strong connection between the Fourier transform and differential equations (namely the Fourier transform turns differential equations into algebraic equations!).

Proof. (ii) Using the change of variables $y = x/\lambda$, we obtain

$$\begin{aligned}
\mathcal{F}(\lambda^{-1/2}f(x/\lambda))(\xi) &= \int_{-\infty}^{\infty} \lambda^{-1/2} f(x/\lambda)e^{-2\pi i x\xi}dx \\
&= \int_{-\infty}^{\infty} \lambda^{1/2} f(y)e^{-2\pi i(\lambda y)\xi}dy \\
&= \lambda^{1/2}\mathcal{F}(f)(\lambda\xi).
\end{aligned}$$

(iii) Using integration by parts (and keeping in mind that $f(x) \rightarrow 0$ as $x \rightarrow \pm\infty$), we obtain

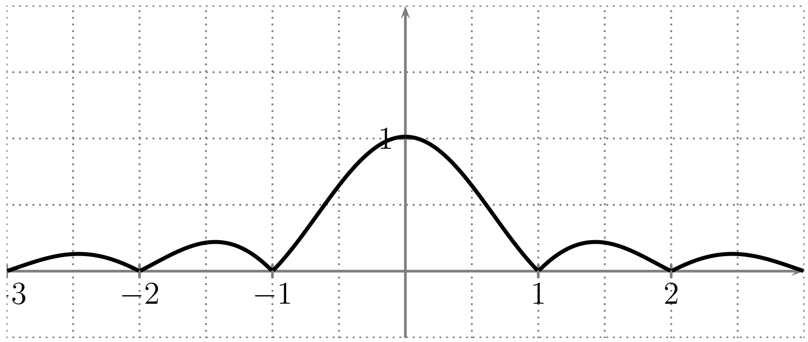
$$\begin{aligned}
\mathcal{F}(f'(x))(\xi) &= \int_{-\infty}^{\infty} f'(x)e^{-2\pi i x\xi}dx \\
&= -\int_{-\infty}^{\infty} f(x)\frac{e^{-2\pi i x\xi}}{-2\pi i\xi}dx \\
&= \frac{1}{2\pi i\xi}\mathcal{F}(f)(\xi).
\end{aligned}$$

$\square$

Two examples of Fourier transforms will be importance to us, namely those of the box and Haar functions: (i) *box function*

$$\widehat{b}(\xi) = e^{-\pi i\xi}\left(\frac{\sin \pi\xi}{\pi\xi}\right)$$

so that the graph

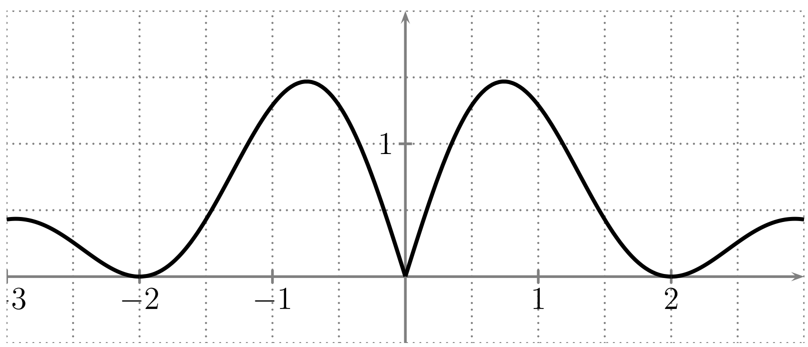


of $|\widehat{b}(\xi)|$ shows that the box function might well be thought of as an analogue version of a low-pass filter, while

(ii) *Haar function*

$$\widehat{h}(\xi) \;=\; 2i\, e^{-\pi i\xi}\, \frac{(\sin \frac{1}{2}\pi\xi)^2}{\pi\xi}$$

so that the graph



of $|\widehat{h}(\xi)|$ shows that the Haar function can be interpreted as an analogue high-pass filter. Establishing these results is left as a homework problem.

An excellent illustration of the interconnections of the Discrete and Continuous Fourier transforms is provided by the following result which is well-known in signal analysis:

Theorem 3.2.3 (Shannon sampling)**.** *Assume that f is a band-limited function; i.e., that its Fourier transform 'lives' on $[-1/2, 1/2]$. Then f is determined by its samples at the integers; i.e.,*

$$f(x) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin(\pi(x-n))}{\pi(x-n)}.$$

Before moving on to the discrete Fourier transform, we make one final comment. Another way to analyze a function $f \in L^2(\mathbb{R})$ (besides the continuous Fourier transform) is to repeatedly zoom in on finite pieces of its graph (pieces corresponding to an interval of length one in the domain, say), and on those finite pieces analyze the function using Fourier series. This idea goes by the names 'short-time Fourier transform' and 'windowed Fourier transform'. In terms of the language we have already developed, what is being said here is that the set $\{b_{km}(x) : k, m \in \mathbb{Z}\}$ is a complete orthonormal family for $L^2(\mathbb{R})$, where

$$b_{km}(x) = \begin{cases} e^{2\pi i k x} & \text{if } m \leq x < m+1 \\ 0 & \text{otherwise} \end{cases}$$

(proving this isn't hard, and is left to the exercises). Then every $g \in L^2(\mathbb{R})$ has a representation

$$g = \sum_{k,m} (g, b_{k,m}) b_{k,m},$$

where the convergence is in L^2. It is an interesting project to figure out how this representation relates to the continuous Fourier transform on $\mathbb{R}$.

3.3. **Discrete Fourier Transform.** The discrete Fourier transform (DFT) operates on sequences of length N. As with all previous Fourier transforms, it determines how an arbitrary signal decomposes in terms of pure tones. For sequences of length N, there are only N pure tones available, namely

$$\xi_k = \frac{1}{\sqrt{N}}(1, \omega^k, \omega^{2k}, ..., \omega^{(N-1)k}) \qquad (0 \leq k \leq N-1),$$

where

$$\omega = e^{2\pi i/N}$$

is an Nth root of unity ($\omega^N = 1$). Thus the DFT produces as output a sequence of length N as well. Namely, given $z = (z_0, z_1, ..., z_{N-1})$, its discrete Fourier transform is the sequence $\widehat{z} = (\widehat{z}_0, \widehat{z}_1, ..., \widehat{z}_{N-1})$ defined by

$$\widehat{z}_k = (z, \xi_k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} z_j \omega^{-jk} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} z_j \overline{\omega}^{jk}.$$

Since $\{\xi_0, \xi_1, ...\xi_{N-1}\}$ is a complete orthonormal family for ℓ^2_N (exercise) we have Plancherel:

$$\sum_{k=0}^{N-1} |\widehat{z}_k|^2 = \sum_{k=0}^{N-1} |z_k|^2,$$

Parseval:

$$\sum_{k=0}^{N-1} \widehat{z}_k \overline{\widehat{w}_k} = \sum_{k=0}^{N-1} z_k \overline{w_k},$$

and an inversion formula:

$$z = \sum_{k=0}^{N-1} \widehat{z}_k \xi_k \Rightarrow z_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \widehat{z}_k \omega^{jk}.$$

This would be the end of the story, except that since everything in sight is finite-dimensional, we have a computational tool which wasn't previously available—linear algebra! Indeed, the DFT is really just a *change of basis* operator: it takes the coefficients of a vector z with respect to the standard basis

$$\varepsilon_j = (0, ..., 0, 1_j, 0, ..., 0) \qquad (0 \leq j \leq N-1)$$

and produces its coefficients with respect to the new basis $\{\xi_0, \xi_1, ..., \xi_{N-1}\}$. As with any operator from an N-dimensional vector space into itself, the DFT can be expressed as a matrix. Indeed, the DFT corresponds to matrix multiplication on the right by

$$U = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \overline{\omega} & \dots & \overline{\omega}^{N-1} \\ 1 & \overline{\omega}^2 & \dots & \overline{\omega}^{2(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{\omega}^{N-1} & \dots & \overline{\omega}^{(N-1)(N-1)} \end{pmatrix}.$$

The inverse DFT corresponds to matrix multiplication by U^*, the adjoint of U.

Given that all real-world signals must be sampled a finite number of times before we can operate on them, the importance of the DFT cannot be understated. It is at the heart of a mind-boggling number of real-world calculations. It is therefore of great interest to find efficient ways to calculate it. The naive algorithm, matrix multiplication, takes approximately N^2 operations. One of the great achievements of the 20th century was the discovery by Cooley and Tukey in 1965 (or Runge and König in 1924, depending on who you ask) of a recursive algorithm, the fast Fourier transform (FFT), which computes the DFT using only on the order of $N\log(N)$ operations (see *Introduction to Algorithms* by Cormen, Leiserson, and Rivest for the details). This algorithm is now a standard part of any good program which calculates the DFT.

3.4. **Problems.**

Problem 1. *Let ϕ be periodic with period 1. Prove that*

$$\int_a^{a+1} \phi(x)dx = \int_0^1 \phi(x)dx$$

for all $a \in \mathbb{R}$.

Problem 2. *Prove the assertion made in section (3.1) that the family $\{e^{2\pi inx} : n \in \mathbb{Z}\}$ is orthonormal in $L^2([0,1])$. That is, show that*

$$(e^{2\pi inx}, e^{2\pi imx}) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}.$$

Using problem 1, deduce that this family is orthonormal in $L^2([a, a+1])$ for all $a \in \mathbb{R}$.

Problem 3. *Pick any two of the four example functions in section (3.1) and verify that their Fourier series are as indicated.*

Problem 4 (smoothness implies decay)**.** *Let $f \in C^k([0,1])$.*

(i) *Using integration by parts k times (and remembering that f has period 1), prove that*

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx = \left(\frac{1}{2\pi in}\right)^k \int_0^1 f^{(k)}(x)e^{-2\pi inx}dx.$$

(ii) *Deduce that there exists a constant $C > 0$ (independent of n) such that*

$$|\widehat{f}(n)| \leq Cn^{-k}$$

for all $n \in \mathbb{Z}$.

Problem 5. *Let $f, g \in L^2([0,1])$ (implicitly, f and g are periodic with period 1). Then we define a function $f * g$ on $[0,1]$ by the formula*

$$(f * g)(x) = \int_0^1 f(x-t)g(t)dt$$

for all $x \in [0,1]$. We have that $f * g \in L^2([0,1])$ (in fact, it's better than that, it's bounded). By mimicking the proof in the case of the DTFT, show that Fourier series map convolution into product. That is,

$$\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$$

for all $n \in \mathbb{Z}$.

Problem 6 (the Gaussian function)**.** *Let $g(x) = e^{-\pi x^2}$ be the Gaussian function (= the 'bell curve' = the normal distribution). Its Fourier transform is given by*

$$\widehat{g}(\xi) = \int_{-\infty}^{\infty} g(x)e^{-2\pi i x \xi}dx.$$

Prove that $\widehat{g}(\xi) = e^{-2\pi\xi^2}$ (i.e. the Gaussian is its own Fourier transform), using the following strategy:

(i) *Show that $g(x)$ satisfies the first-order linear differential equation*

$$u'(x) + 2\pi x u(x) = 0.$$

(ii) *By using the properties of the Fourier transform, show that if $u(x)$ is a solution to the above differential equation, then so is $\widehat{u}$. In particular, $\widehat{g}$ will satisfy the above differential equation.*

(iii) *Prove that $g(0) = \widehat{g}(0) = 1$.*

(iv) *By using the uniqueness theorem for solutions of first-order, linear initial value problems, deduce that $g = \widehat{g}$.*

It should be noted that a more direct proof is possible, using complex analysis to evaluate the integral which defines $\widehat{g}(\xi)$.

Problem 7 (Shannon sampling theorem)**.** *Prove the Shannon sampling theorem of section (3.2) as follows:*

(i) *Since* $\widehat{f}$ *lives on* $[-1/2, 1/2]$*, it has a Fourier series representation*

$$\widehat{f}(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n\xi},$$

valid on $[-1/2, 1/2]$*. Prove that* $c_n = f(-n)$*.* ***Hint:*** *Write down the formula for* c_n *as an integral involving* $\widehat{f}(\xi)$*, then think 'inverse Fourier transform'.*

(ii) *Show that*

$$\int_{-1/2}^{1/2} e^{2\pi i n x} e^{2\pi i n \xi} d\xi = \frac{\sin(\pi(x+n))}{\pi(x+n)}.$$

(iii) *Finish the proof of the Shannon sampling theorem by justifying the following steps:*

$$\begin{aligned}
f(x) &= \int_{-1/2}^{1/2} \widehat{f}(\xi) e^{2\pi i x\xi} d\xi \\
&= \int_{-1/2}^{1/2} \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x} e^{2\pi i x\xi} d\xi \\
&= \sum_{n\in\mathbb{Z}} f(-n) \int_{-1/2}^{1/2} e^{2\pi i n x} e^{2\pi i x\xi} d\xi \\
&= \sum_{n\in\mathbb{Z}} f(-n) \frac{\sin(\pi(x+n))}{\pi(x+n)} \\
&= \sum_{n\in\mathbb{Z}} f(n) \frac{\sin(\pi(x-n))}{\pi(x-n)}.
\end{aligned}$$

Problem 8. *Prove* $\{b_{km}(x) : k, m \in \mathbb{Z}\}$ *is a complete orthonormal family for* $L^2(\mathbb{R})$*, where*

$$b_{km}(x) = \begin{cases} e^{2\pi i k x} & \textit{if } m \le x < m+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Hint: *You will make heavy use of the fact that for each fixed* $m \in \mathbb{Z}$*,* $\{b_{km}(x) : k \in \mathbb{Z}\}$ *is a complete orthonormal family for* $L^2([m, m+1])$*, as has been shown previously.*

Problem 9. *Show that* $\{\xi_0, \xi_1, ..., \xi_{N-1}\}$ *is an orthonormal basis for* ℓ^2_N*, where*

$$\xi_k = (1, \omega^k, \omega^{2k}, ..., \omega^{(N-1)k}) \qquad (0 \le k \le N-1),$$

and

$$\omega = e^{2\pi i/N}.$$

Hint: *In computing quantities such as* (ξ_k, ξ_m)*, think geometric series.*

Problem 10. *Establish the respective Fourier transform results*

$$\widehat{b}(\xi) \;=\; e^{-\pi i\xi}\left(\frac{\sin \pi\xi}{\pi\xi}\right)$$

and

$$\widehat{h}(\xi) \;=\; 2i\, e^{-\pi i\xi}\, \frac{(\sin \frac{1}{2}\pi\xi)^2}{\pi\xi}$$

for the box and Haar functions given in the lecture.