Final Project Report

Isaiah Hollars, Kayla Pritchett, Bek Yuldashbaev, Ana Passmore
Differential Equations
Dr. Andrew Miller

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Part a

In this part, we make the assumption that the hospital is always full. That is, we assume flow in is equal to flow out. This assumption looks like

$$\Lambda = \sigma_s S + \sigma_h H + \sigma_c C$$

If flow in equals flow out, the population remains stable. Thus for all values of time:

$$N = S + C + H$$

Note that S can always be determined if we already know C and H, so we want to eliminate it from the system. We start by rewriting S in terms of the other two variables: $S = N - C - H$. We will substitute this into the equations for $\frac{dH}{dt}$ and $\frac{dC}{dt}$ wherever an S shows up. Originally, $\frac{dH}{dt}$ and $\frac{dC}{dt}$ were determined as:

$$\frac{dH}{dt} = \frac{\beta_H SH}{N} - \alpha_H H - \sigma_H H$$

$$\frac{dH}{dt} = \frac{\beta_C SC}{N} - \alpha_C C - \sigma_C C$$

Now all we do is substitute $(N - C - H)$ in for S into these two equations to simplify the system:

$$\begin{aligned} \frac{dH}{dt} &= \frac{\beta_H}{N}(N - C - H)H - (\alpha_H + \sigma_H)H \\ \frac{dC}{dt} &= \frac{\beta_C}{N}(N - C - H)C - (\alpha_C + \sigma_C)C \end{aligned} \tag{1}$$

Part b

Equilibria solutions for the system will occur when $0 = \frac{dH}{dt} = \frac{dC}{dt}$. Setting $\frac{dH}{dt} = 0$ (see equation) and factoring gives :

$$0 = H\left(\frac{\beta_H}{N}(N - C - H) - (\alpha_H + \sigma_H)\right) \tag{2}$$

Then by the zero product property, $H = 0$ or $\left(\frac{\beta_H}{N}(N - C - H) - (\alpha_H + \sigma_H)\right) = 0$.

Similarly, we set $\frac{dC}{dt} = 0$ and factor to get

$$0 = C\left(\frac{\beta_C}{N}(N - C - H) - (\alpha_C + \sigma_C)\right) \tag{3}$$

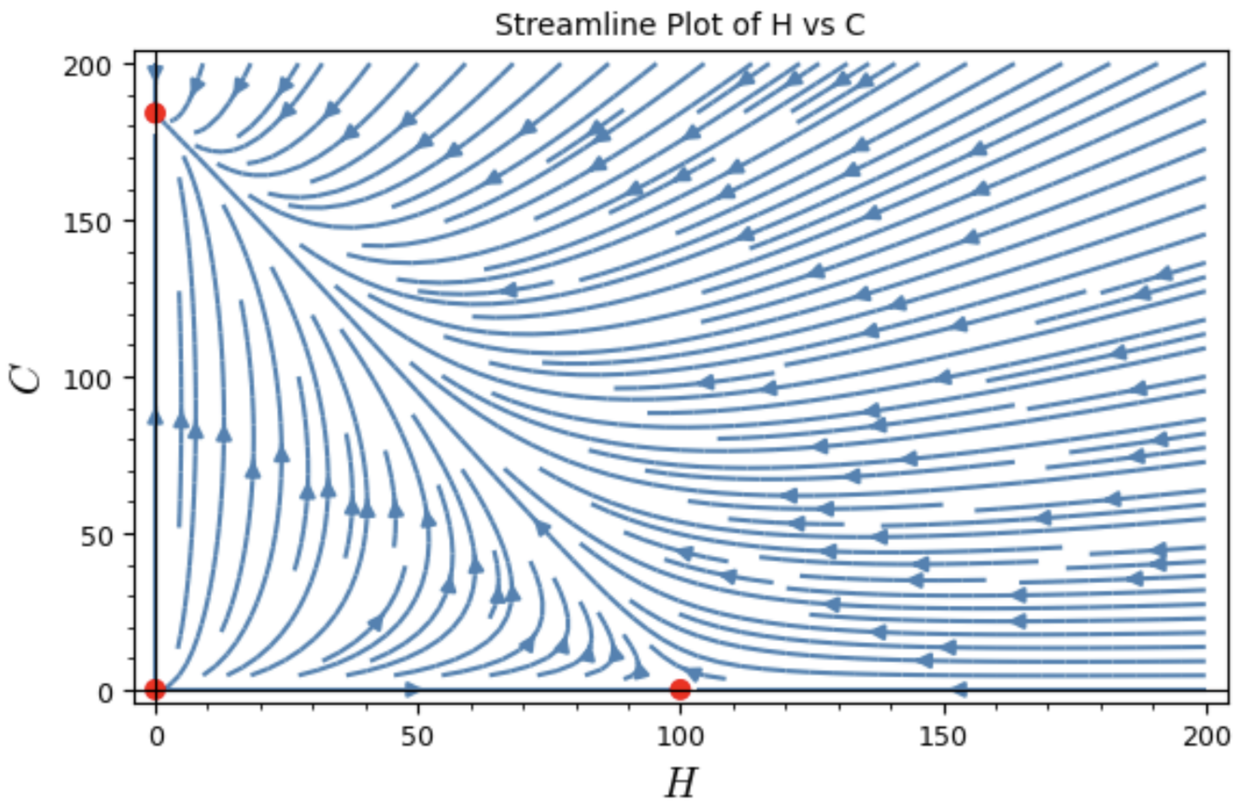
Again using the zero product property, $C = 0$ or $\left(\frac{\beta_C}{N}(N - C - H) - (\alpha_C + \sigma_C)\right) = 0$.

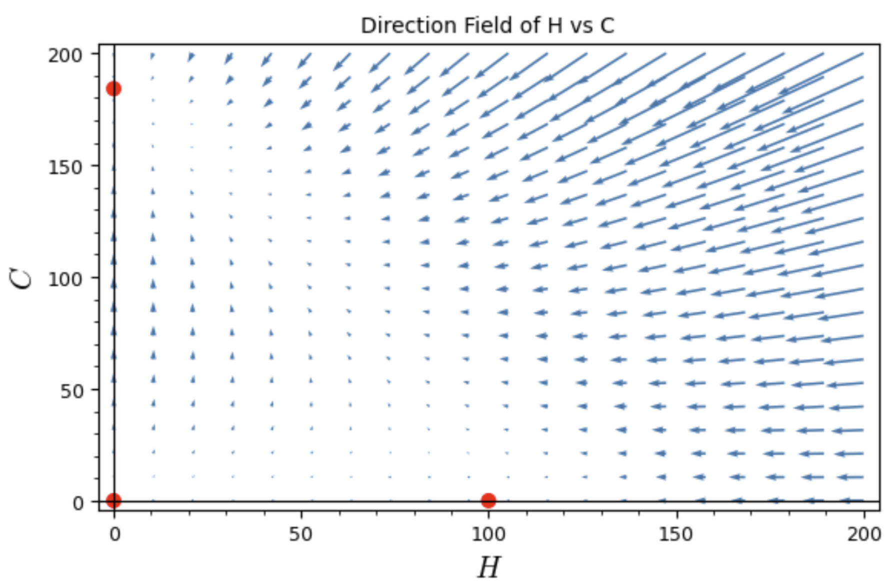
If we let $H = 0$, we get two possible solutions for C. The first is the trivial solution $\boxed{H = 0,\ C = 0}$. The second is $\boxed{H = 0,\ C = N - \frac{N}{\beta_C}(\sigma_C + \alpha_C)}$. Lastly, if instead of supposing $H = 0$, we suppose that $\left(\frac{\beta_H}{N}(N - C - H) - (\alpha_H + \sigma_H)\right) = 0$, we solve for H to get: $H = N - C - (\sigma_H + \alpha_H)\frac{N}{\beta_H}$. We then substitute this expression into equation 3. It turns out that only one solution is possible here, that is: $\boxed{H = N - (\sigma_H + \alpha_H)\frac{N}{\beta_H},\ C = 0}$.

Plugging in for our various parameters and writing the equilibria as ordered pairs (H, C), we get the equilibria

$$\begin{aligned} (0,0) \\ (100,0) \\ (0,184.13) \end{aligned} \tag{4}$$

Part c


Streamline Plot of H vs C
200
150
100
50
0
C
0
50
100
150
200
H


Direction Field of H vs C
200
150
100
50
0
C
0
50
100
150
200
H

Part d

We can first discuss what each of the 3 equilibria mean in term of the numbers susceptible patients, patients with HA-MRSA, and patients with CA-MRSA over time. The first solution $(0, 0)$ corresponds with 0 patients being infected and 400 patients remaining in the susceptible group. The second solution $(100, 0)$ means the number of HA-MRSA cases will stay at 100, CA-MRSA cases will stay at 0, and the susceptible group will remain at 300. The last equillibrium solution is also the most likely (we will explain why in short time). This solution $(0, 184)$ refers to CA-MRSA cases staying around 184, HA-MRSA staying at 0, and the susceptible population remaining at 216.

Now, we want to examine what types of equilibria solutions our model predicts. We do this by appealing to the Hartman-Grobman Theorem. In a nutshell, this theorem tells us that a linearization can be used to accurately predict the behavior of a nonlinear system in a neighborhood around its equilibria points (as long as the corresponding linear system has nonzero real part eigenvalues).

The linearization will have the Jacobian matrix as coefficients. We have that $\frac{dH}{dt} = f(H, C)$ and $\frac{dC}{dt} = g(H, C)$, where $f(H, C)$ and $g(H, C)$ are shown in the system in equation (1).

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial f}{\partial H} & \frac{\partial f}{\partial C} \\ \frac{\partial g}{\partial H} & \frac{\partial g}{\partial C} \end{bmatrix} = \begin{bmatrix} \beta_H - \frac{\beta_H C}{N} - \frac{2\beta_H}{N} H - \sigma_H - \alpha_H & -\frac{H\beta_H}{N} \\ -\frac{C\beta_C}{N} & \beta_C - \frac{\beta_C H}{N} - \frac{2C\beta_C}{N} - \sigma_C - \alpha_C \end{bmatrix}$$

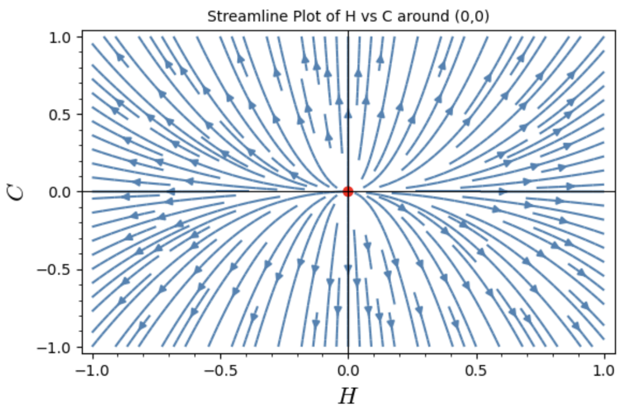
$(0, 0)$ Equilibrium

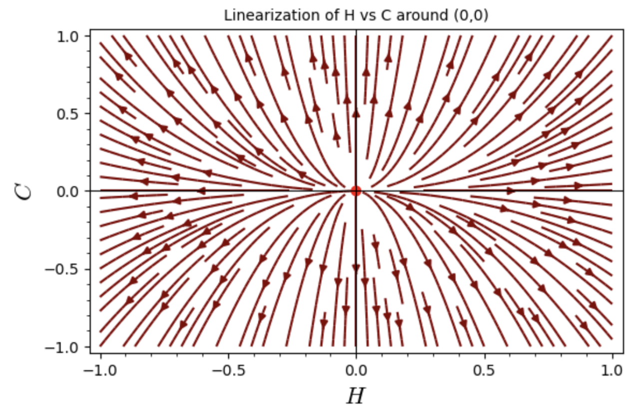
We first evaluate the Jacobian at $(0, 0)$.

$$\boldsymbol{J}\Big|_{(0,0)} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.207 \end{bmatrix}$$

Thus, the linear approximation around $(0, 0)$ looks like:

$$\frac{dH}{dt} = 0.1(H - 0) + 0(C - 0)$$
$$\frac{dC}{dt} = 0(H - 0) + 0.207(C - 0)$$





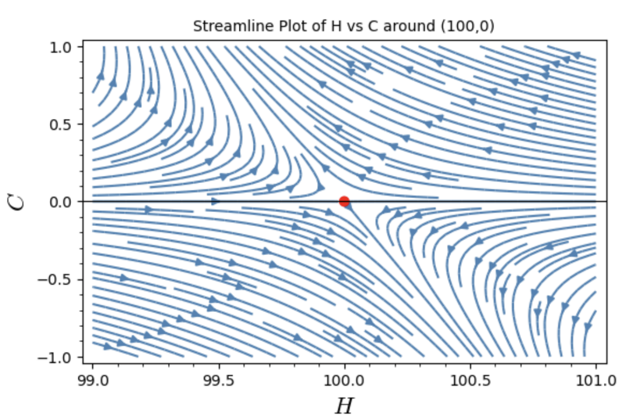
The blue graph is the original streamline plot of the nonlinear model just zoomed in. The maroon plot is the plot of the linear model above. The linear model is a great approximation close to the equilibrium point. To find out the equilibrium type, we look back to $\boldsymbol{J}\big|_{(0,0)}$. We see that $\boldsymbol{J}\big|_{(0,0)}$ has 2 real eigenvectors.

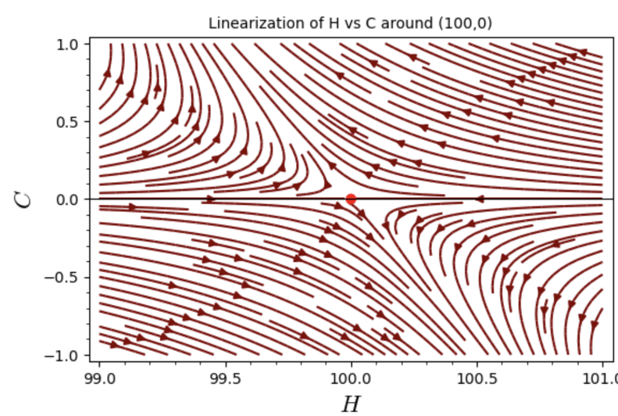
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda_1 = 0.1$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda_2 = 0.2071$$

This indicates that the equilibrium at $(0, 0)$ in the original system behaves like a **source** that bends towards the vertical eigenvector. The interpretation is that a small initial population of patients with the H-strain or the C-strain will grow exponentially in the beginning, with the C-strain population growing at a faster rate. This equilibrium is not stable!

$(100, 0)$ Equilibrium





$$\boldsymbol{J}\big|_{(100,0)} = \begin{bmatrix} -0.1 & -0.1 \\ 0 & 0.095 \end{bmatrix}$$

$$\begin{bmatrix}1\\0\end{bmatrix}, \lambda_1 = -0.1$$

$$\begin{bmatrix}-0.46\\0.89\end{bmatrix}, \lambda_2 = 0.095$$

Our Jacobian linearization has 2 real eigenvalues with different signs, so our system behaves like a **saddle** in a region around $(100, 0)$. The saddle has a separatrix along the line $C = 0$. In real life, C is non negative. If the C-strain population were to increase to at least 1 around this equilibrium, then solutions would shoot up and to the left. This corresponds with the C-strain population growing rapidly and the H-strain population shrinking.

$(0, 184.13)$ **Equilibrium**

$$\boldsymbol{J}\Big|_{(0,184.13)} = \begin{bmatrix}-0.084 & 0\\-0.207 & -0.207\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix}, \lambda_1 = -0.21$$

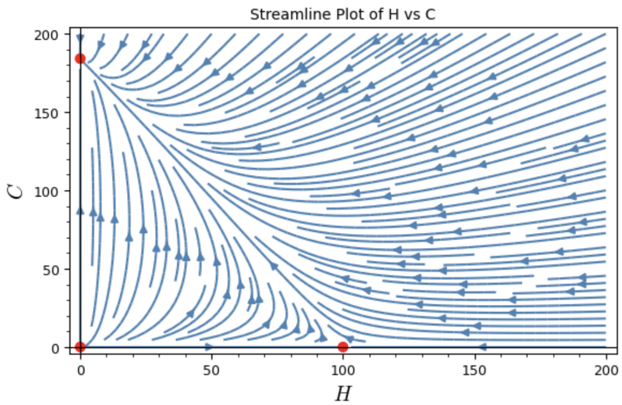
$$\begin{bmatrix}0.51\\-0.86\end{bmatrix}, \lambda_2 = -0.084$$

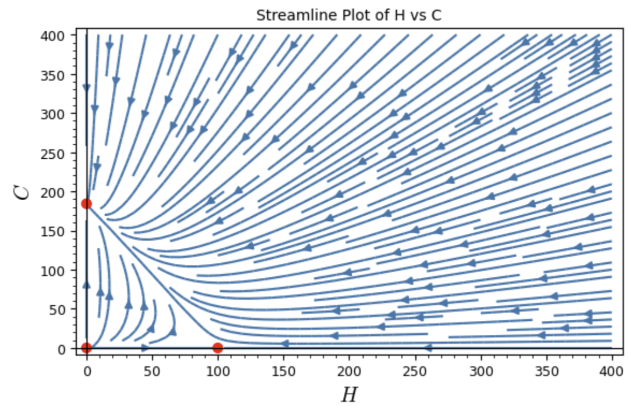
Thus, in a neighborhood around the $(0, 184.13)$ equilibrium, our system looks like a **sink**. It is important to note that this is the only stable equilibrium in the entire system.

Part e

Our conclusion is that CA-MRSA will overtake HA-MRSA in the hospital and will keep spreading until it infects roughly 46% of the hospital's population (184 people). At this point, its spread will be just enough to keep 46% of the hospital's patients infected.

We are able to make this conclusion because this is the only stable equilibrium point in the entire system and the 2 staph populations are limited in their growth.





This is easy to see when examining a streamline plot of the system. Also note that the growth of the H-strain and C-strain are limited as H and C increase. So, in the unrealistic case that the H-strain or C-strain infect more than our stable equilibrium, the infected populations will decrease back down to this stable equilibrium point.

We recommend that the hospital take immediate and drastic action if it wishes to avoid this outcome. One of the variables the hospital has some control over is the transmission rate of the C-strain. By utilizing better quarantining systems, the hospital may be able to lower this transmission rate, which will have some effect on the system's behavior. The extent of this effect is currently undetermined. In our next report, we will investigate the effect of lowering this transmission variable on the system.

References

Nagle, Saff, Snider: Fundamentals of Differential Equations (8th Edition)

Noonburg, V.W. Differential Equations: From Calculus to Dynamical Systems. MAA Press: Providence, 2010.

Repo for project code