1 2017 Exam

1. (a) Let G be a finite group with a subgroup of finite index $n > 1$. Then if $|G|$ does not divide $n!$ then G is not simple.

   Let Ω be the set of left cosets of H. Consider ϕ an action of G under left multiplication on Ω. Consider $K = \ker\phi$. Then K is a normal subgroup of G contained in H. G/K is isomorphic to a subgroup of $Sym(\Omega)$ which has order $n!$. So $|G : K|$ is finite and divides $n!$. As G does not divide $n!$, K cannot be trivial, and so G is not simple.

   (b) (Assignment 2) Let G be a group of order 400. Let $n_2(G)$ be the number of Sylow 2-groups and $n_5(G)$ be the number of of Sylow 5-groups. We know that $n_5(G)$ is 1 or 16 as $n_p(G) \cong 1 \mod p$. If $n_5(G) = 1$ then we are done, so suppose $n_5(G) = 16$.

   Suppose the intersection of Sylow 5-subgroups is always trivial. Then there are $24 * 16 = 384$ nontrivial elements, and so there must be a unique (and hence normal) Sylow 2-subgroup.

   If there are Sylow 5-subgroups P and Q such that their intersection is not trivial, then $|P \cap Q| = 5$. But $|PQ| = 125$ and $PQ \subseteq N_G(P \cap Q)$ since $P \cap Q$ is normal in each of P and Q. So $|N_G(P \cap Q)| > 125$ and is a divisor of 400, therefore $|G : N_G(P \cap Q)| < 4$. As $|G|$ does not divide $3!$, $P \cap Q \lhd G$.

   (c) We can present Q_8 as $Q_8 = \langle x, y : x^2 = y^2, (xy)^2 = y^2\rangle$. This has a unique element of order 2, so every subgroup of Q_8 of order 4 must be cyclic and have x^2 as its element of order 2. So the intersection of any subgroup of order 4 and any subgroup of order 2 is nontrivial.

2. (a) Theorem: Take a group action $(G, \Omega, \cdot)$. Then let O_α be the orbit of $\alpha \in \Omega$ under $\cdot$. Let $H = G_\alpha$ be the stabiliser of α in G. Then there exists a bijection

$$O_\alpha \leftrightarrow \{G_\alpha x : x \in G\}$$

   Proof: Define $f : O_\alpha \to \{Hx : x \in G\}$ be the following: take $\beta \in O_\alpha$ and choose $x \in G$ with $\beta = \alpha \cdot x$, and then give

$$f(\beta) = Hx$$

   First consider that if $f(\beta) = Hy$, then we can show that $Hy = Hx$, so f is well defined. Then consider

$$Hx = f(\alpha \cdot x)$$

   so f is onto. Lastly, take $f(\beta) = f(\gamma)$. Then $\beta = \alpha \cdot x$ and $\gamma = \alpha \cdot y$, so $Hx = Hy$ and thus $y = hx$ for some $h \in H$. Then

$$\gamma = \alpha \cdot y = \alpha \cdot (hx) = (\alpha \cdot h) \cdot x = \alpha \cdot x = \beta$$

   as $h \in H$. Thus f is also injective, and thus the theorem is proved.

(b) Let H and K be soluble normal subgroups of a finite group G. Then HK is a soluble normal subgroup of G. Clearly HK is a normal subgroup, so we are left with showing that it is soluble. Then $HK/H \simeq K/(K \cap H)$. As this is a factor group of a soluble group, HK/H is soluble, as is H, so HK is soluble.

(c) Let G be a finite group. Then G has a largest soluble normal subgroup. Let N be a soluble normal subgroup of G. If it is not maximal, then $\exists M$ such that $N < M < G$ and M is a soluble normal subgroup of G. But then MN is a soluble normal subgroup of G. If MN is not maximal, then we can repeat this process. As G is finite, this process must end somewhere, and hence there is a largest soluble normal subgroup. It is unique, as if M, N are both the largest soluble normal subgroups of G, then MN is also a soluble normal subgroup that is larger than both if they are not equal. Hence $M = N$.

3. (a) A group G is residually finite if, for all $x \in G \backslash \{1\}$, there exists a normal group N_x in G such that $x \notin N_x$ and $|G : N_x| < \infty$.

(b) Let $x \in \mathbb{Z}^n$ where x is not the identity. We can write $x = \{x_1, x_2, ..., x_n\}$. For each x_i with $1 \leq i \leq n$ we can take p_i such that p_i is not a divisor of x_i. Then we can take the direct product of $p_i\mathbb{Z}$ for $1 \leq i \leq n$, which is a subgroup of $\mathbb{Z}^n$ which is normal as $\mathbb{Z}^n$ is abelian. Thus $\mathbb{Z}^n$ is residually finite for

(c) For $\mathbb{Q}$ to be residually finite, it must have a proper subgroup of finite index. Let H be a subgroup of $\mathbb{Q}$, with $[\mathbb{Q} : H] = n$. Then $nq \in H$ for every $q \in \mathbb{Q}$. But then $\mathbb{Q} = H$ and so it is not a proper subgroup.

4. (a) Let F_n and F_m be isomorphic. Let $G = \langle g : g^2 = 1\rangle$. Consider a homomorphism $\phi : F_m \to G$. This is completely determined by the images of each $x_i \in F_m$ - either $x_i \mapsto g$ or $x_i \mapsto g^0 = 1$. Thus the number of nontrivial homomorphisms from F_m to G is $2^m - 1$. Then $K = \ker \phi \lhd F_m$ and $F_m/K \simeq \mathbb{Z}_2$ by the first isomorphism theorem. Every normal subgroup of index 2 is of the form $\ker \phi$ for some non-trivial ϕ. Thus F_m has $2^m - 1$ normal subgroups of index 2. Similarly, F_n has $2^n - 1$ subgroups of index 2. Thus as $F_m \simeq F_n$, $2^m - 1 = 2^n - 1$ and hence $m = n$.

(b) Let $G = \langle x, y | x^7 = y^5 = 1, [x, y] = x\rangle$. Show that G is cyclic of order 5. We can rewrite $[x, y] = x$ as $x^y = x^2$. Then, since $y^6 = y$, $x^2 = x^y = x^{y^6} = x^{2^6}$ and so $x^62 = 1$. Then the order of x in G divides both 7 and 62, and so $x = 1$. Thus, $G = \langle y\rangle$ has order dividing 5. By von Dyck's theorem, G maps onto $\mathbb{Z}_5$ via $x \mapsto 0$, $y \mapsto 1$ and so $G \cong \mathbb{Z}_5$.

(c) Assignment Q

5. (a) Suppose the elements of S_n act on at least the elements i, j, k. Let $\pi \in S_n$ such that $\pi(i) = j$. Now we can find a $\rho \in S_n$ such that

$\rho(j) = k$ but fixes every other element. But then $\rho^{-1}\pi\rho(i) = k$, so π is not in the center of S_n As π can be any non-trivial element of S_n, then $Z(S_n) = \{1\}$

(b) Let $N \lhd S_n$.

(c) We can take $\mathbb{Z}_2 \times \mathbb{Z}_2$ given as a subgroup of S_4 by the elements $\{(), (1,3)(2,4), (1,2)(3,4), (1,4)(2,3)\}$. This is all elements that are the products of two disjoint transpositions. As conjugation in S_n does not change cycle structure, this subgroup is normal in S_n.

6. (a)

   (b) First we show that G has exactly one element of order 2. Then the Sylow 2-subgroup is unique and hence normal.

   (c) i. Clearly b commutes with both a and c as they are disjoint. Thus the two commutators we care about are (1,5)(2,6)(2,6,5)(1,5)(2,6)(2,5,6)=(1,2)(5,6) and (2,6,5)(1,5)(2,6)(2,5,6)(1,5)(2,6)=(1,2)(5,6), so the derived group is the normal closure of $\{(1,2)(5,6)\}$, which is $G' = \{(), (1,2)(5,6), (1,5)(2,6), (1,6)(2,$ (a representation of the Klein 4-group). Then $G^{(2)}$ is $\{1\}$, and so we are done.

   ii. We attempt to take the lower central series of G. The third term, $G_3 = [G', G]$ must contain G', and hence cannot be a proper normal subgroup of G'. Thus, this series will never terminate and G is not nilpotent.

2 2016

Taught by Jianbei - not representative of our exam

3 2015

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