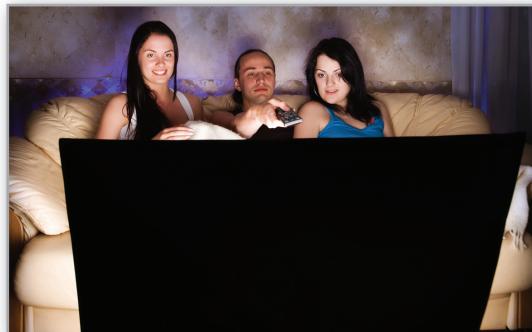


9 Linear Programming



- 9.1** Systems of Linear Inequalities
- 9.2** Linear Programming Involving Two Variables
- 9.3** The Simplex Method: Maximization
- 9.4** The Simplex Method: Minimization
- 9.5** The Simplex Method: Mixed Constraints



Product Advertising (p. 475)



Minimizing Steel Waste (p. 469)



Optimization Software (p. 453)



Restocking ATMs (p. 445)



Target Heart Rate (p. 438)

9.1 Systems of Linear Inequalities

- Sketch the graph of a linear inequality.
- Sketch the graph of a system of linear inequalities.

LINEAR INEQUALITIES AND THEIR GRAPHS

The following statements are inequalities in two variables:

$$3x - 2y < 6 \quad \text{and} \quad x + y \geq 6.$$

An ordered pair (a, b) is a **solution of an inequality** in x and y if the inequality is true when a and b are substituted for x and y , respectively. For instance, $(1, 1)$ is a solution of the inequality $3x - 2y < 6$ because

$$3(1) - 2(1) = 1 < 6.$$

The **graph of an inequality** is the collection of all solutions of the inequality. To sketch the graph of an inequality such as

$$3x - 2y < 6$$

begin by sketching the graph of the *corresponding equation*

$$3x - 2y = 6.$$

The graph of the equation separates the plane into two regions. In each region, one of the following two statements must be true.

1. All points in the region are solutions of the inequality.
2. No point in the region is a solution of the inequality.

So, you can determine whether the points in an entire region satisfy the inequality by simply testing *one* point in the region.

REMARK

When possible, use test points that are convenient to substitute into the inequality, such as $(0, 0)$.



Sketching the Graph of an Inequality in Two Variables

1. Replace the inequality sign with an equal sign, and sketch the graph of the resulting equation. (Use a dashed line for $<$ or $>$ and a solid line for \leq or \geq .)
2. Test one point in each of the regions formed by the graph in Step 1. If the point satisfies the inequality, then shade the entire region to denote that every point in the region satisfies the inequality.

In this section, you will work with **linear inequalities** of the following forms.

$$ax + by < c$$

$$ax + by \leq c$$

$$ax + by > c$$

$$ax + by \geq c$$

The graph of each of these linear inequalities is a half-plane lying on one side of the line $ax + by = c$. When the line is dashed, the points on the line *are not* solutions of the inequality; when the line is solid, the points on the line *are* solutions of the inequality. The simplest linear inequalities are those corresponding to horizontal or vertical lines, as shown in Example 1 on the next page.

EXAMPLE 1**Sketching the Graph of a Linear Inequality**

Sketch the graphs of (a) $x > -2$ and (b) $y \leq 3$.

SOLUTION

- The graph of the corresponding equation $x = -2$ is a vertical line. The points that satisfy the inequality $x > -2$ are those lying to the right of this line, as shown in Figure 9.1.
- The graph of the corresponding equation $y = 3$ is a horizontal line. The points that satisfy the inequality $y \leq 3$ are those lying below (or on) this line, as shown in Figure 9.2.

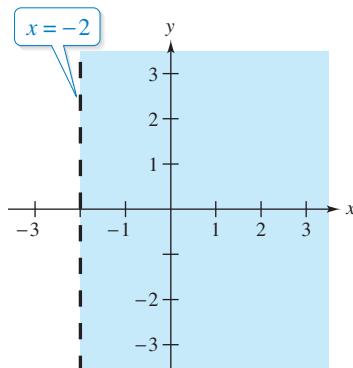


Figure 9.1

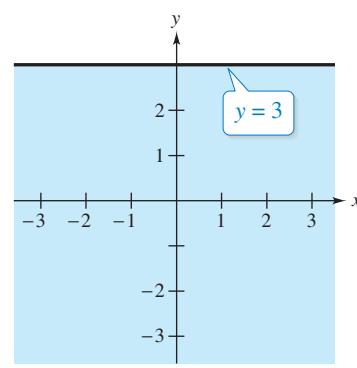


Figure 9.2

EXAMPLE 2**Sketching the Graph of a Linear Inequality**

Sketch the graph of $x - y < 2$.

SOLUTION

The graph of the corresponding equation $x - y = 2$ is a line, as shown in Figure 9.3. Because the origin $(0, 0)$ satisfies the inequality, the graph consists of the half-plane lying above the line. (Try checking a point below the line to see that it does not satisfy the inequality.)

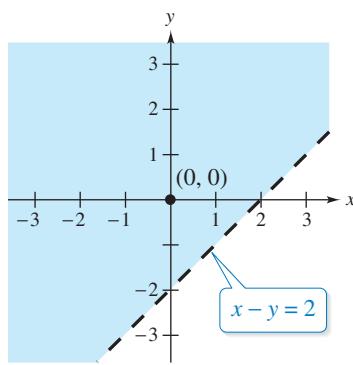


Figure 9.3

For a linear inequality in two variables, you can sometimes simplify the graphing procedure by writing the inequality in *slope-intercept* form. For instance, by writing $x - y < 2$ in the form

$$y > x - 2$$

you can see that the solution points lie *above* the line $y = x - 2$, as shown in Figure 9.3.

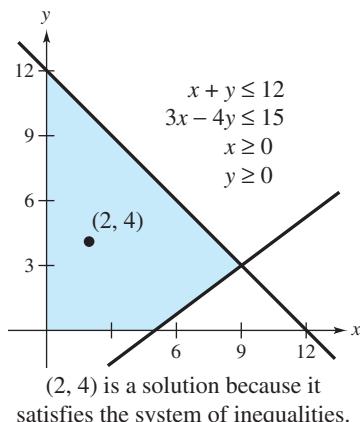


Figure 9.4

SYSTEMS OF INEQUALITIES

Many practical problems in business, science, and engineering involve systems of linear inequalities. An example of such a system is shown below.

$$\begin{aligned}x + y &\leq 12 \\3x - 4y &\leq 15 \\x &\geq 0 \\y &\geq 0\end{aligned}$$

A **solution** of a system of inequalities in x and y is a point (x, y) that satisfies each inequality in the system. For instance, $(2, 4)$ is a solution of this system because $x = 2$ and $y = 4$ satisfy each of the four inequalities in the system. The **graph** of a system of inequalities in two variables is the collection of all points that are solutions of the system. For instance, the graph of the system above is the region shown in Figure 9.4. Note that the point $(2, 4)$ lies in the shaded region because it is a solution of the system of inequalities.

To sketch the graph of a system of inequalities in two variables, first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is *common* to every graph in the system. This region represents the **solution set** of the system. For systems of linear inequalities, it is helpful to find the *vertices* of the solution region, as shown in Example 3.

EXAMPLE 3 Solving a System of Inequalities

Sketch the graph (and label the vertices) of the solution set of the system shown below.

$$\begin{aligned}x - y &< 2 \\x &> -2 \\y &\leq 3\end{aligned}$$

SOLUTION

You have already sketched the graph of each of these inequalities in Examples 1 and 2. The triangular region common to all three graphs can be found by superimposing the graphs on the same coordinate plane, as shown in Figure 9.5. To find the vertices of the region, find the points of intersection of the boundaries of the region.

Vertex A: $(-2, -4)$
Obtained by finding
the point of
intersection of
 $x - y = 2$
 $x = -2$.

Vertex B: $(5, 3)$
Obtained by finding
the point of
intersection of
 $x - y = 2$
 $y = 3$.

Vertex C: $(-2, 3)$
Obtained by finding
the point of
intersection of
 $x = -2$
 $y = 3$.

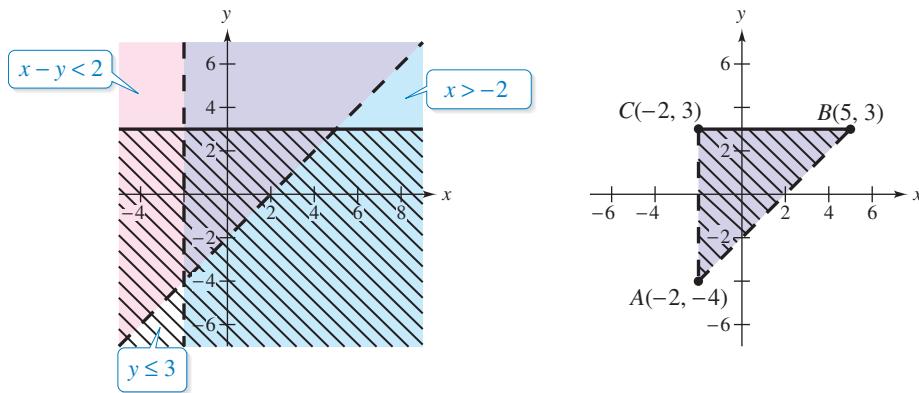
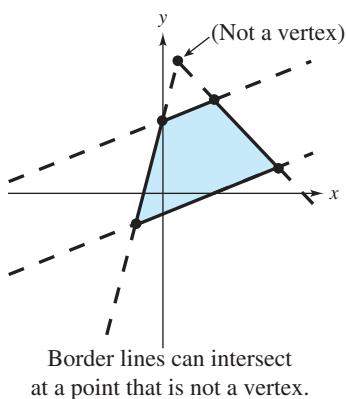


Figure 9.5

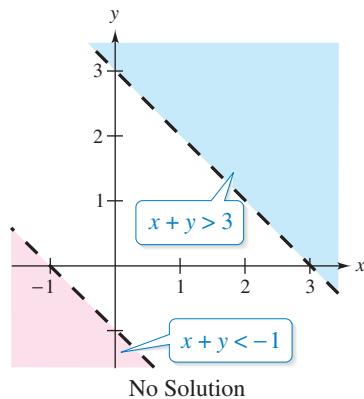
**Figure 9.6**

For the triangular region shown in Figure 9.5, each point of intersection of a pair of boundary lines corresponds to a vertex. With more complicated regions, two border lines can sometimes intersect at a point that is not a vertex of the region, as shown in Figure 9.6. To determine which points of intersection are actually vertices of the region, sketch the region and refer to your sketch as you find each point of intersection.

When solving a system of inequalities, be aware that the system might have no solution. For instance, the system

$$\begin{aligned}x + y &> 3 \\x + y &< -1\end{aligned}$$

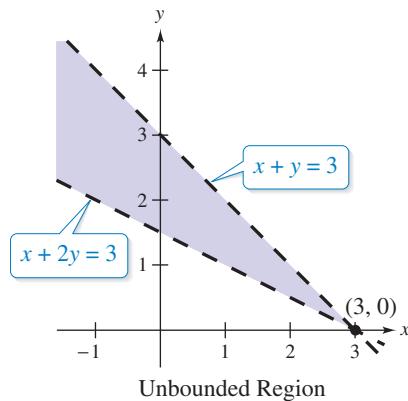
has no solution points because the quantity $(x + y)$ cannot be both less than -1 and greater than 3 , as shown in Figure 9.7.

**Figure 9.7**

Another possibility is that the solution set of a system of inequalities can be unbounded. For instance, consider the following system.

$$\begin{aligned}x + y &< 3 \\x + 2y &> 3\end{aligned}$$

The graph of the inequality $x + y < 3$ is the half-plane that lies below the line $x + y = 3$. The graph of the inequality $x + 2y > 3$ is the half-plane that lies above the line $x + 2y = 3$. The intersection of these two half-planes is an *infinite wedge* that has a vertex at $(3, 0)$, as shown in Figure 9.8. This unbounded region represents the solution set.

**Figure 9.8**

The last example in this section shows how a system of linear inequalities can arise in an applied problem.

EXAMPLE 4

An Application of a System of Inequalities

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Set up a system of linear inequalities that describes the minimum daily requirements for calories and vitamins.

SOLUTION

Let

$$\begin{aligned}x &= \text{number of cups of dietary drink X and} \\y &= \text{number of cups of dietary drink Y.}\end{aligned}$$

To meet the minimum daily requirements, the inequalities listed below must be satisfied.

$$\text{For calories: } 60x + 60y \geq 300$$

$$\text{For vitamin A: } 12x + 6y \geq 36$$

$$\text{For vitamin C: } 10x + 30y \geq 90$$

$$x \geq 0$$

$$y \geq 0$$

REMARK

Any point inside the shaded region (or on its boundary) shown in Figure 9.9 meets the minimum daily requirements for calories and vitamins. For instance, 3 cups of dietary drink X and 2 cups of dietary drink Y supply 300 calories, 48 units of vitamin A, and 90 units of vitamin C.



The last two inequalities are included because x and y cannot be negative. The graph of this system of linear inequalities is shown in Figure 9.9.

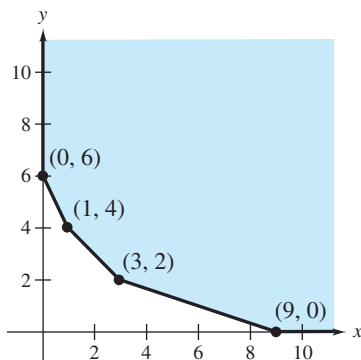


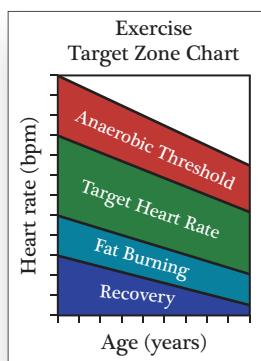
Figure 9.9



LINEAR ALGEBRA APPLIED

A heart rate monitor watch is designed to ensure that a person exercises at a healthy pace. It displays transmissions from a chest monitor that measures the electrical activity of the heart.

Researchers have identified target heart rate ranges, or zones, for achieving specific results from exercise. Fitness facilities and cardiology treadmill rooms often display wall charts like the one shown here, to help determine whether a person's exercise heart rate is in a healthy range. In Exercise 58, you will write a system of inequalities for a person's target heart rates during exercise.



9.1 Exercises

Identifying the Graph of a Linear Inequality In Exercises 1–6, match the linear inequality with its graph. [The graphs are labeled (a)–(f).]

1. $x > 3$

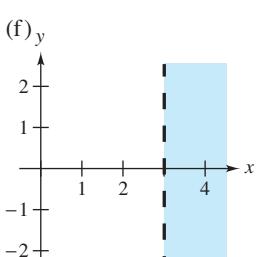
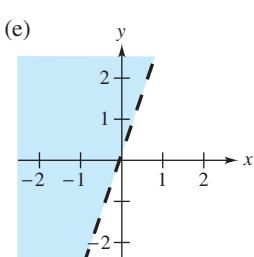
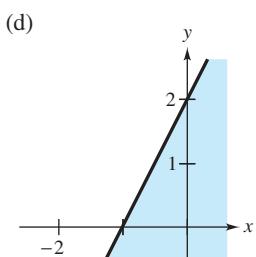
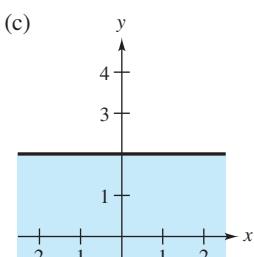
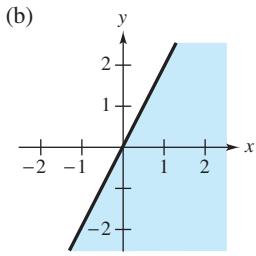
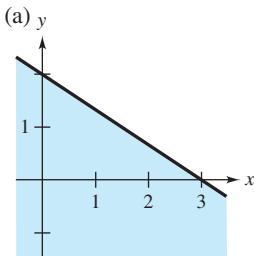
3. $2x + 3y \leq 6$

5. $x \geq \frac{y}{2}$

2. $y \leq 2$

4. $2x - y \geq -2$

6. $y > 3x$



Sketching the Graph of a Linear Inequality In Exercises 7–26, sketch the graph of the linear inequality.

7. $x \geq 2$

9. $y \geq -1$

11. $y < 2 - x$

13. $2y - x \geq 4$

15. $y \leq x$

17. $y \geq 4 - 2x$

19. $3y + 4 \geq x$

21. $4x - 2y \leq 12$

23. $2x + 7y \leq 28$

25. $y + \frac{1}{3}x \geq 6$

8. $x \leq 4$

10. $y \leq 7$

12. $y > 2x - 4$

14. $5x + 3y \geq -15$

16. $3x > y$

18. $y \leq 3 + x$

20. $6 - 2y < x$

22. $y + 3x > 6$

24. $5x - 2y > 10$

26. $y - \frac{1}{2}x < 12$

Solving a System of Inequalities In Exercises 27–30, determine whether each ordered pair is a solution of the system of linear inequalities.

27. $x \geq -4$

$y > -3$

$y \leq -8x - 3$

(a) $(0, 0)$

(b) $(-1, -3)$

(c) $(-4, 0)$

(d) $(-3, 11)$

28. $-2x + 5y \geq 3$

$y < 4$

$-4x - 2y < 7$

(a) $(0, 2)$

(b) $(-6, 4)$

(c) $(-8, -2)$

(d) $(-3, 2)$

29. $x \geq 1$

$y \geq 0$

$y \leq 2x + 1$

(a) $(0, 1)$

(b) $(1, 3)$

(c) $(2, 2)$

(d) $(2, 1)$

30. $x \geq 0$

$y \geq 0$

$y \leq 4x - 2$

(a) $(0, -2)$

(b) $(2, 0)$

(c) $(3, 1)$

(d) $(0, -1)$

Solving a System of Inequalities In Exercises 31–44, sketch the graph of the solution of the system of linear inequalities.

31. $x \geq 0$

$y \geq 0$

$x \leq 2$

$y \leq 4$

32. $x \geq -1$

$y \geq -1$

$x \leq 1$

$y \leq 2$

33. $x + y \leq 1$

$-x + y \leq 1$

$y \geq 0$

34. $3x + 2y < 6$

$x > 0$

$y > 0$

35. $x + y \leq 5$

$x \geq 2$

$y \geq 0$

36. $2x + y \geq 2$

$x \leq 2$

$y \leq 1$

37. $-3x + 2y < 6$

$x + 4y > -2$

$2x + y < 3$

38. $x - 7y > -36$

$5x + 10y > 20$

$6x - 5y > 6$

39. $x \geq 1$

$x - 2y \leq 3$

$3x + 2y \geq 9$

$x + y \leq 6$

40. $x + y < 10$

$2x + y > 10$

$x - y > 2$

41. $2x + y > 2$

$6x + 3y < 2$

42. $-x + 3y \leq 12$

$5x + 2y > 5$

$x - 3y < -3$

43. $-3x + 2y < 6$

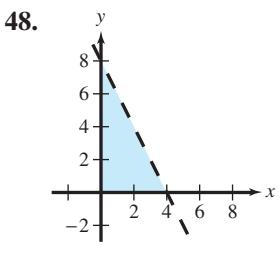
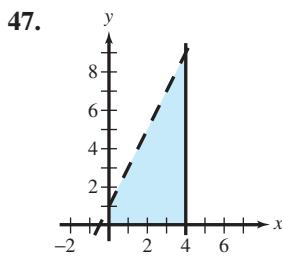
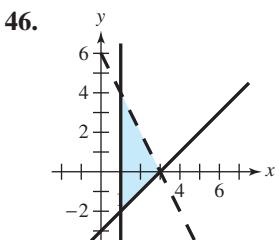
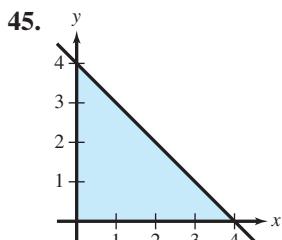
$x - 4y > -2$

$x + 4y < 4$

44. $x - 2y < -6$

$5x - 3y > -9$

Writing a System of Inequalities In Exercises 45–48, derive a set of inequalities that describes the region.



Writing a System of Inequalities In Exercises 49–52, derive a set of inequalities that describes the region.

49. Rectangular region with vertices at $(2, 1)$, $(5, 1)$, $(5, 7)$, and $(2, 7)$.
50. Parallelogram with vertices at $(0, 0)$, $(4, 0)$, $(1, 4)$, and $(5, 4)$.
51. Triangular region with vertices at $(0, 0)$, $(5, 0)$, and $(2, 3)$.
52. Triangular region with vertices at $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

53. **Reasoning** Consider the inequality $8x - 2y < 5$. Without graphing, determine whether the solution points lie in the half-plane above or below the boundary line. Explain your reasoning.



54. **GAPSTONE** Consider the following system of inequalities

$$\begin{aligned} ax + by &\leq c \\ x &\geq d \\ y &> e \end{aligned}$$

where $a > 0$ and $b > 0$. Find values of a , b , c , d , and e such that (a) the origin is a solution of the system and (b) the origin is not a solution of the system.

55. **Investment** A person plans to invest no more than \$20,000 in two different interest-bearing accounts. Each account is to contain at least \$5000. Moreover, one account should have at least twice the amount that is in the other account. Find a system of inequalities describing the various amounts that can be deposited in each account, and sketch the graph of the system.

56. **Laptop Inventory** A store sells two models of laptop computers. Because of the demand, it is necessary to stock at least twice as many units of the Pro Series as units of the Deluxe Series. The costs to the store of the two models are \$800 and \$1200, respectively. The management does not want more than \$20,000 in laptop inventory at any one time, and it wants at least four Pro Series models and two Deluxe Series models in inventory at all times. Devise a system of inequalities describing all possible inventory levels, and sketch the graph of the system.

57. **Furniture Production** A furniture company produces tables and chairs. Each table requires 1 hour in the assembly center and $1\frac{1}{3}$ hours in the finishing center. Each chair requires $1\frac{1}{2}$ hours in the assembly center and $1\frac{1}{2}$ hours in the finishing center. The assembly center is available 12 hours per day, and the finishing center is available 15 hours per day. Let x and y be the numbers of tables and chairs produced per day, respectively. Find a system of inequalities describing all possible production levels, and sketch the graph of the system.

58. **Target Heart Rate** One formula for a person's maximum heart rate is $220 - x$, where x is the person's age in years for $20 \leq x \leq 70$. When a person exercises, it is recommended that the person strive for a heart rate that is at least 50% of the maximum and at most 75% of the maximum. (Source: American Heart Association)

- (a) Write a system of inequalities that describes the region corresponding to these heart rate recommendations.
- (b) Sketch a graph of the region in part (a).
- (c) Find two solutions of the system and interpret their meanings in the context of the problem.

59. **Diet Supplement** A dietitian designs a special diet supplement using two different foods. Each ounce of food X contains 20 units of calcium, 15 units of iron, and 10 units of vitamin B. Each ounce of food Y contains 10 units of calcium, 10 units of iron, and 20 units of vitamin B. The minimum daily requirements for the diet are 300 units of calcium, 150 units of iron, and 200 units of vitamin B. Find a system of inequalities describing the different amounts of food X and food Y that the dietitian can prescribe. Sketch the graph of the system.

60. Rework Exercise 59 using minimum daily requirements of 280 units of calcium, 160 units of iron, and 180 units of vitamin B.

9.2 Linear Programming Involving Two Variables

- Find a maximum or minimum of an objective function subject to a system of constraints.
- Find an optimal solution to a real-world linear programming problem.

SOLVING A LINEAR PROGRAMMING PROBLEM

Many applications in business and economics involve a process called **optimization**, which is used to find the minimum cost, the maximum profit, or the minimum use of resources. In this section, you will study an optimization strategy called **linear programming**.

A two-dimensional linear programming problem consists of a linear **objective function** and a system of linear inequalities called **constraints**. The objective function gives the quantity that is to be maximized (or minimized), and the constraints determine the set of **feasible solutions**.

For example, suppose you are asked to maximize the value of

$$z = ax + by \quad \text{Objective function}$$

subject to a set of constraints that determines the region in Figure 9.10. Because every point in the region satisfies each constraint, it is not clear how to go about finding the point that yields a maximum value of z . Fortunately, it can be shown that when there is an optimal solution, it must occur at one of the vertices of the region. This means that *you can find the maximum value by testing z at each of the vertices*.

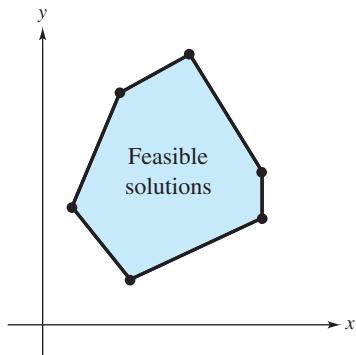


Figure 9.10

THEOREM 9.1 Optimal Solution of a Linear Programming Problem

If a linear programming problem has an optimal solution, then it must occur at a vertex of the set of feasible solutions. If the problem has more than one optimal solution, then at least one of them must occur at a vertex of the set of feasible solutions. In either case, the value of the objective function is unique.

A linear programming problem can include hundreds, and sometimes even thousands, of variables. However, in this section, you will solve linear programming problems that involve only two variables. The graphical method for solving a linear programming problem in two variables is outlined below.

Graphical Method for Solving a Linear Programming Problem

To solve a linear programming problem involving two variables by the graphical method, use the following steps.

1. Sketch the region corresponding to the system of constraints. (The points inside or on the boundary of the region are *feasible solutions*.)
2. Find the vertices of the region.
3. Test the objective function at each of the vertices and select the values of the variables that optimize the objective function. For a bounded region, both a minimum and maximum value will exist. (For an unbounded region, if an optimal solution exists, then it will occur at a vertex.)

EXAMPLE 1**Solving a Linear Programming Problem**

Find the maximum value of

$$z = 3x + 2y \quad \text{Objective function}$$

subject to the constraints listed below.

$$\begin{aligned} & x \geq 0 \\ & y \geq 0 \\ & x + 2y \leq 4 \\ & x - y \leq 1 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Constraints}$$

SOLUTION

The constraints form the region shown in Figure 9.11. At the four vertices of this region, the objective function has the values listed below.

$$\text{At } (0, 0): z = 3(0) + 2(0) = 0$$

$$\text{At } (1, 0): z = 3(1) + 2(0) = 3$$

$$\text{At } (2, 1): z = 3(2) + 2(1) = 8 \quad (\text{Maximum value of } z)$$

$$\text{At } (0, 2): z = 3(0) + 2(2) = 4$$

So, the maximum value of z is 8, and this occurs when $x = 2$ and $y = 1$.

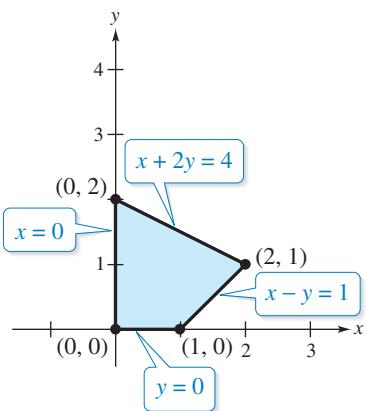


Figure 9.11

In Example 1, try testing some of the interior points in the region. You will see that the corresponding values of z are less than 8. Here are some examples.

$$\text{At } (1, 1): z = 3(1) + 2(1) = 5$$

$$\text{At } \left(1, \frac{1}{2}\right): z = 3(1) + 2\left(\frac{1}{2}\right) = 4$$

$$\text{At } \left(\frac{3}{2}, 1\right): z = 3\left(\frac{3}{2}\right) + 2(1) = \frac{13}{2}$$

To see why the maximum value of the objective function in Example 1 must occur at a vertex, consider writing the objective function in the form

$$y = -\frac{3}{2}x + \frac{z}{2}.$$

This equation represents a family of lines, each of slope $-3/2$. Of these infinitely many lines, you want the one that has the largest z -value, while still intersecting the region determined by the constraints. In other words, of all the lines whose slope is $-3/2$, you want the one that has the largest y -intercept *and* intersects the specified region, as shown in Figure 9.12. Such a line will pass through one (or more) of the vertices of the region.

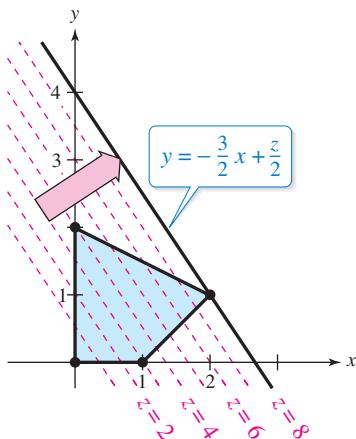


Figure 9.12

The graphical method for solving a linear programming problem will work whether the objective function is to be maximized or minimized. The steps used are precisely the same in either case. Once you have evaluated the objective function at the vertices of the set of feasible solutions, simply choose the largest value as the maximum and the smallest value as the minimum. For instance, the same test used in Example 1 to find the maximum value of z can be used to conclude that the minimum value of z is 0, and that this occurs at the vertex $(0, 0)$.

EXAMPLE 2**Solving a Linear Programming Problem**

Find the maximum value of the objective function

$$z = 4x + 6y \quad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the constraints

$$\left. \begin{array}{l} -x + y \leq 11 \\ x + y \leq 27 \\ 2x + 5y \leq 90 \end{array} \right\} \quad \text{Constraints}$$

SOLUTION

The region bounded by the constraints is shown in Figure 9.13. By testing the objective function at each vertex, you obtain

$$\text{At } (0, 0): z = 4(0) + 6(0) = 0$$

$$\text{At } (0, 11): z = 4(0) + 6(11) = 66$$

$$\text{At } (5, 16): z = 4(5) + 6(16) = 116$$

$$\text{At } (15, 12): z = 4(15) + 6(12) = 132 \quad (\text{Maximum value of } z)$$

$$\text{At } (27, 0): z = 4(27) + 6(0) = 108.$$

So, the maximum value of z is 132, and this occurs when $x = 15$ and $y = 12$. ■

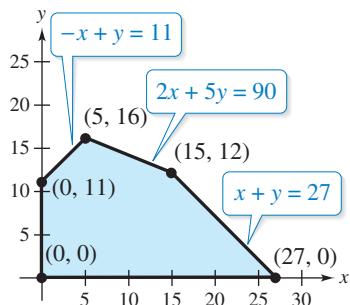


Figure 9.13

EXAMPLE 3**Minimizing an Objective Function**

Find the minimum value of the objective function

$$z = 5x + 7y \quad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the constraints

$$\left. \begin{array}{l} 2x + 3y \geq 6 \\ 3x - y \leq 15 \\ -x + y \leq 4 \\ 2x + 5y \leq 27 \end{array} \right\} \quad \text{Constraints}$$

SOLUTION

The region bounded by the constraints is shown in Figure 9.14. By testing the objective function at each vertex, you obtain

$$\text{At } (0, 2): z = 5(0) + 7(2) = 14 \quad (\text{Minimum value of } z)$$

$$\text{At } (0, 4): z = 5(0) + 7(4) = 28$$

$$\text{At } (1, 5): z = 5(1) + 7(5) = 40$$

$$\text{At } (6, 3): z = 5(6) + 7(3) = 51$$

$$\text{At } (5, 0): z = 5(5) + 7(0) = 25$$

$$\text{At } (3, 0): z = 5(3) + 7(0) = 15.$$

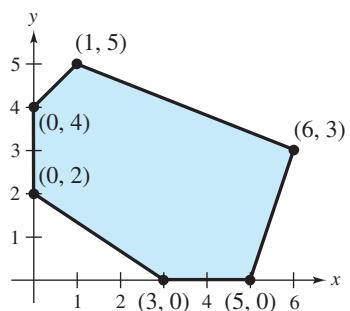


Figure 9.14

So, the minimum value of z is 14, and this occurs when $x = 0$ and $y = 2$. ■

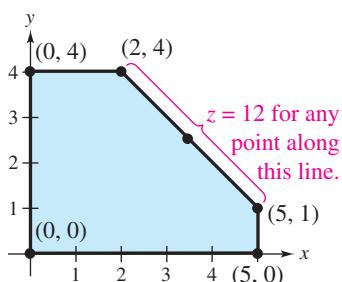


Figure 9.15

When solving a linear programming problem, it is possible that the maximum (or minimum) value occurs at *two* different vertices. For instance, at the vertices of the region shown in Figure 9.15, the objective function

$$z = 2x + 2y \quad \text{Objective function}$$

has the following values.

$$\text{At } (0, 0): z = 2(0) + 2(0) = 0$$

$$\text{At } (0, 4): z = 2(0) + 2(4) = 8$$

$$\text{At } (2, 4): z = 2(2) + 2(4) = 12 \quad (\text{Maximum value of } z)$$

$$\text{At } (5, 1): z = 2(5) + 2(1) = 12 \quad (\text{Maximum value of } z)$$

$$\text{At } (5, 0): z = 2(5) + 2(0) = 10$$

In this case, the objective function has a maximum value (of 12) not only at the vertices $(2, 4)$ and $(5, 1)$, but also at *any point on the line segment connecting these two vertices*.

Some linear programming problems have no optimal solution. This can occur when the region determined by the constraints is *unbounded*. Example 4 illustrates such a problem.

EXAMPLE 4 An Unbounded Region

Find the maximum value of

$$z = 4x + 2y \quad \text{Objective function}$$

where $x \geq 0$ and $y \geq 0$, subject to the constraints

$$\left. \begin{array}{l} x + 2y \geq 4 \\ 3x + y \geq 7 \\ -x + 2y \leq 7 \end{array} \right\} \quad \text{Constraints}$$

SOLUTION

The region determined by the constraints is shown in Figure 9.16.

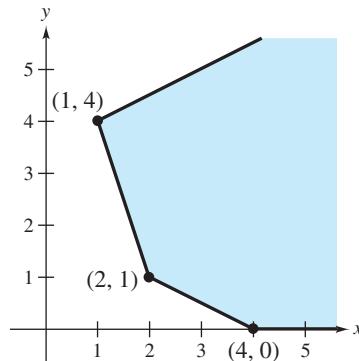


Figure 9.16

REMARK

For this problem, the objective function does have a minimum value, $z = 10$, which occurs at the vertex $(2, 1)$.

For this unbounded region, there is no maximum value of z . To see this, note that the point $(x, 0)$ lies in the region for all values of $x \geq 4$. By choosing large values of x , you can obtain values of

$$\begin{aligned} z &= 4(x) + 2(0) \\ &= 4x \end{aligned}$$

that are as large as you want. So, there is no maximum value of z .



APPLICATION

EXAMPLE 5

An Application: Optimal Cost

Example 4 in Section 9.1 set up a system of linear equations for the following problem. The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Now, suppose that dietary drink X costs \$0.12 per cup and drink Y costs \$0.15 per cup. How many cups of each drink should be consumed each day to minimize the cost and still meet the daily requirements?

SOLUTION

Begin by letting x be the number of cups of dietary drink X and y be the number of cups of dietary drink Y. Moreover, to meet the minimum daily requirements, the inequalities listed below must be satisfied.

$$\left. \begin{array}{l} \text{For calories: } 60x + 60y \geq 300 \\ \text{For vitamin A: } 12x + 6y \geq 36 \\ \text{For vitamin C: } 10x + 30y \geq 90 \\ x \geq 0 \\ y \geq 0 \end{array} \right\} \text{Constraints}$$

The cost C is

$$C = 0.12x + 0.15y. \quad \text{Objective function}$$

The graph of the region corresponding to the constraints is shown in Figure 9.17. To determine the minimum cost, test C at each vertex of the region, as follows.

$$\text{At } (0, 6): C = 0.12(0) + 0.15(6) = 0.90$$

$$\text{At } (1, 4): C = 0.12(1) + 0.15(4) = 0.72$$

$$\text{At } (3, 2): C = 0.12(3) + 0.15(2) = 0.66 \quad \text{(Minimum value of } C\text{)}$$

$$\text{At } (9, 0): C = 0.12(9) + 0.15(0) = 1.08$$

So, the minimum cost is \$0.66 per day, and this occurs when three cups of drink X and two cups of drink Y are consumed each day. ■

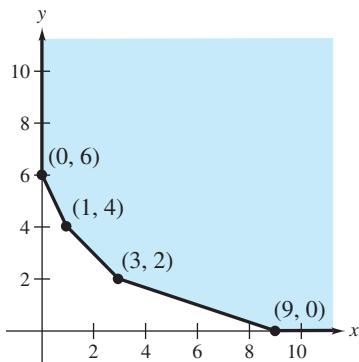


Figure 9.17



LINEAR ALGEBRA APPLIED

Financial institutions that replenish automatic teller machines (ATMs) need to take into account a vast array of variables and constraints to keep the machines stocked appropriately. Demand for cash machines fluctuates with such factors as weather, economic conditions, customer withdrawing habits, day of the week, and even road construction. Further complicating the matter, government regulations require penalty fees for depositing and withdrawing money from the U.S. Federal Reserve in the same week.

To address this complex problem, a company can use high-end optimization software to set up and solve a linear programming problem with several variables and constraints. The company determines an equation for the objective function to minimize total cash in ATMs, while establishing constraints on travel routes, penalty fees, cash limits for trucks, and so on. The optimal solution generated by the software allows the company to build detailed ATM restocking schedules.

9.2 Exercises

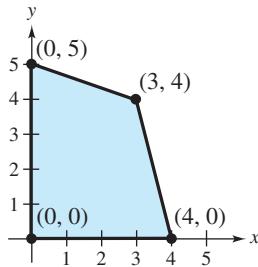
Solving a Linear Programming Problem In Exercises 1 and 2, find the minimum and maximum values of each objective function and where they occur, subject to the given constraints.

1. Objective function:

- (a) $z = 3x + 8y$
 (b) $z = 5x + 0.5y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 3y &\leq 15 \\ 4x + y &\leq 16 \end{aligned}$$

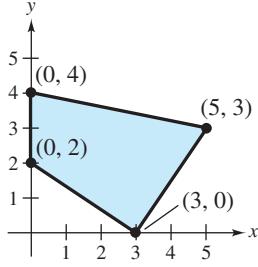


2. Objective function:

- (a) $z = 4x + 3y$
 (b) $z = x + 6y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 3y &\geq 6 \\ 3x - 2y &\leq 9 \\ x + 5y &\leq 20 \end{aligned}$$



Solving a Linear Programming Problem In Exercises 3–8, sketch the region determined by the indicated constraints. Then find the minimum and maximum values of each objective function and where they occur, subject to the constraints.

3. Objective function:

- (a) $z = 10x + 7y$
 (b) $z = 25x + 30y$

Constraints:

$$\begin{aligned} 0 &\leq x \leq 60 \\ 0 &\leq y \leq 45 \\ 5x + 6y &\leq 420 \end{aligned}$$

5. Objective function:

- (a) $z = 4x + 5y$
 (b) $z = 2x + 7y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 4x + 3y &\geq 27 \\ x + y &\geq 8 \\ 3x + 5y &\geq 30 \end{aligned}$$

4. Objective function:

- (a) $z = 50x + 35y$
 (b) $z = 16x + 18y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 8x + 9y &\leq 7200 \\ 8x + 9y &\geq 5400 \end{aligned}$$

6. Objective function:

- (a) $z = 4x + 5y$
 (b) $z = 2x - y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 2y &\leq 10 \\ x + 2y &\leq 6 \end{aligned}$$

7. Objective function:

- (a) $z = 4x + y$
 (b) $z = x + 4y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 40 \\ x + y &\geq 30 \\ 2x + 3y &\geq 72 \end{aligned}$$

8. Objective function:

- (a) $z = x$
 (b) $z = y$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 3y &\leq 60 \\ 2x + y &\leq 28 \\ 4x + y &\leq 48 \end{aligned}$$

Maximizing an Objective Function In Exercises 9–12, maximize the objective function subject to the constraints $3x_1 + x_2 \leq 15$ and $4x_1 + 3x_2 \leq 30$, where $x_1, x_2 \geq 0$.

9. $z = 2x_1 + x_2$

11. $z = x_1 + x_2$

10. $z = 5x_1 + x_2$

12. $z = 3x_1 + x_2$

Recognizing Unusual Characteristics In Exercises 13–18, the linear programming problem has an unusual characteristic. Sketch a graph of the solution region for the problem and describe the unusual characteristic. (In each problem, the objective function is to be maximized.)

13. Objective function:

$$z = 2.5x + y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 3x + 5y &\leq 15 \\ 5x + 2y &\leq 10 \end{aligned}$$

14. Objective function:

$$z = x + y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 1 \\ -x + 2y &\leq 4 \end{aligned}$$

15. Objective function:

$$z = -x + 2y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 10 \\ x + y &\leq 7 \end{aligned}$$

16. Objective function:

$$z = x + y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + y &\leq 0 \\ -3x + y &\geq 3 \end{aligned}$$

17. Objective function:

$$z = 3x + 4y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 1 \\ 2x + y &\geq 4 \end{aligned}$$

18. Objective function:

$$z = x + 2y$$

Constraints:

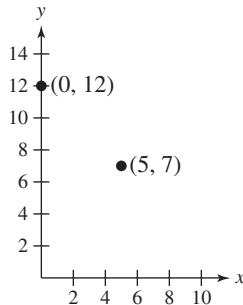
$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ 2x + y &\geq 4 \end{aligned}$$

- 19. Optimal Profit** The costs to a store for two models of Global Positioning System (GPS) receivers are \$80 and \$100. The \$80 model yields a profit of \$25 and the \$100 model yields a profit of \$30. Market tests and available resources indicate the following constraints.

- The merchant estimates that the total monthly demand will not exceed 200 units.
- The merchant does not want to invest more than \$18,000 in GPS receiver inventory.

What is the optimal inventory level for each model? What is the optimal profit?

- 20. CAPSTONE** A company trying to determine an optimal profit finds that the objective function has a maximum value at the vertices shown in the graph.



- Can you conclude that it also has a maximum value at the point (3, 9)? Explain.
- Can you conclude that it also has a maximum value at the point (6, 6)? Explain.
- Find two additional points that maximize the objective function.

- 21. Optimal Cost** A farming cooperative mixes two brands of cattle feed. Brand X costs \$30 per bag, and brand Y costs \$25 per bag. Research and available resources have indicated the following constraints.

- Brand X contains two units of nutritional element A, two units of element B, and two units of element C.
- Brand Y contains one unit of nutritional element A, nine units of element B, and three units of element C.
- The minimum requirements for nutrients A, B, and C are 12 units, 36 units, and 24 units, respectively.

What is the optimal number of bags of each brand that should be mixed? What is the optimal cost?

- 22. Optimal Cost** Rework Exercise 21 given that brand Y now costs \$32 per bag, and it now contains one unit of nutritional element A, twelve units of element B, and four units of element C.

- 23. Optimal Revenue** An accounting firm charges \$2500 for an audit and \$350 for a tax return. The times (in hours) required for staffing and reviewing each are shown in the table.

Component	Audit	Tax Return
Staffing	75	12.5
Reviewing	10	2.5

The firm has 900 hours of staff time and 155 hours of review time available each week. What numbers of audits and tax returns will bring in an optimal revenue?

- 24. Optimal Revenue** The accounting firm in Exercise 23 lowers its charge for an audit to \$2000. What numbers of audits and tax returns will bring in an optimal revenue?

- 25. Optimal Profit** A fruit grower has 150 acres of land for raising crops A and B. The profit is \$185 per acre for crop A and \$245 per acre for crop B. Research and available resources indicate the following constraints.

- It takes 1 day to trim an acre of crop A and 2 days to trim an acre of crop B, and there are 240 days per year available for trimming.
- It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days per year available for picking.

What is the optimal acreage for each fruit? What is the optimal profit?

- 26. Determine**, for each given vertex, t -values such that the objective function has a maximum value at the vertex.

Objective function:

$$z = 3x + ty$$

Constraints:

$$x \geq 0$$

$$y \geq 0$$

$$7x + 14y \leq 84$$

$$5x + 4y \leq 30$$

- (0, 6)
- (2, 5)
- (6, 0)
- (0, 0)

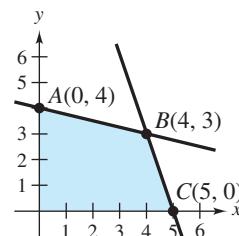
Finding an Objective Function In Exercises 27–30, find an objective function that has a maximum or minimum value at the indicated vertex of the constraint region shown below. (There are many correct answers.)

- 27.** Maximum at vertex A

- 28.** Maximum at vertex B

- 29.** Maximum at vertex C

- 30.** Minimum at vertex C



9.3 The Simplex Method: Maximization

- Write the simplex tableau for a linear programming problem.
- Use pivoting to find an improved solution.
- Use the simplex method to solve a linear programming problem that maximizes an objective function.
- Use the simplex method to optimize an application.

THE SIMPLEX TABLEAU

For linear programming problems involving two variables, the graphical solution method introduced in Section 9.2 is convenient. For problems involving more than two variables or large numbers of constraints, it is better to use methods that are adaptable to technology. One such method is called the **simplex method**, developed by George Dantzig in 1946. It provides a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function.

Suppose you want to find the maximum value of $z = 4x_1 + 6x_2$, where $x_1 \geq 0$ and $x_2 \geq 0$, subject to the following constraints.

$$\begin{aligned} -x_1 + x_2 &\leq 11 \\ x_1 + x_2 &\leq 27 \\ 2x_1 + 5x_2 &\leq 90 \end{aligned}$$

Because the left-hand side of each *inequality* is less than or equal to the right-hand side, there must exist nonnegative numbers s_1, s_2 , and s_3 that can be added to the left side of each equation to produce the following system of linear *equations*.

$$\begin{aligned} -x_1 + x_2 + s_1 &= 11 \\ x_1 + x_2 + s_2 &= 27 \\ 2x_1 + 5x_2 + s_3 &= 90 \end{aligned}$$

The numbers s_1, s_2 , and s_3 are called **slack variables** because they represent the “slack” in each inequality.

REMARK

Note that for a linear programming problem in standard form, the objective function is to be maximized, not minimized. (Minimization problems will be discussed in Sections 9.4 and 9.5.)



Standard Form of a Linear Programming Problem

A linear programming problem is in **standard form** when it seeks to *maximize* the objective function $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. After adding slack variables, the corresponding system of **constraint equations** is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + s_2 &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + s_m &= b_m \end{aligned}$$

where $s_i \geq 0$.

A **basic solution** of a linear programming problem in standard form is a solution $(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_m)$ of the constraint equations in which *at most* m variables are nonzero, and the variables that are nonzero are called **basic variables**. A basic solution for which all variables are nonnegative is called a **basic feasible solution**.

The simplex method is carried out by performing elementary row operations on a matrix called the **simplex tableau**. This tableau consists of the augmented matrix corresponding to the constraint equations together with the coefficients of the objective function written in the form

$$-c_1x_1 - c_2x_2 - \cdots - c_nx_n + (0)s_1 + (0)s_2 + \cdots + (0)s_m + z = 0.$$

In the tableau, it is customary to omit the coefficient of z . For instance, the simplex tableau for the linear programming problem

$$\begin{array}{rcl} z = 4x_1 + 6x_2 & & \text{Objective function} \\ -x_1 + x_2 + s_1 & = 11 \\ x_1 + x_2 + s_2 & = 27 \\ 2x_1 + 5x_2 + s_3 & = 90 \end{array} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Constraints}$$

is as follows.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	
						↑ Current z -value

For this **initial simplex tableau**, the **basic variables** are s_1, s_2 , and s_3 , and the **nonbasic variables** are x_1 and x_2 . Note that the basic variables are labeled to the right of the simplex tableau next to the appropriate rows. This technique is important as you proceed through the simplex method. It helps keep track of the changing basic variables, as shown in Example 1.

Because x_1 and x_2 are the nonbasic variables in this initial tableau, they have an initial value of zero, yielding a current z -value of zero. From the columns that are farthest to the right, you can see that the basic variables have initial values of $s_1 = 11$, $s_2 = 27$, and $s_3 = 90$. So the current solution is

$$x_1 = 0, x_2 = 0, s_1 = 11, s_2 = 27, \text{ and } s_3 = 90.$$

This solution is a basic feasible solution and is often written as

$$(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90).$$

The entry in the lower right corner of the simplex tableau is the current value of z . Note that the bottom-row entries under x_1 and x_2 are the negatives of the coefficients of x_1 and x_2 in the objective function

$$z = 4x_1 + 6x_2.$$

To perform an **optimality check** for a solution represented by a simplex tableau, look at the entries in the bottom row of the tableau. If any of these entries are negative (as above), then the current solution is *not* optimal.

PIVOTING

Once you have set up the initial simplex tableau for a linear programming problem, the simplex method consists of checking for optimality and then, if the current solution is not optimal, improving the current solution. (An improved solution is one that has a larger z -value than the current solution.) To improve the current solution, bring a new basic variable into the solution, the **entering variable**. This implies that one of the current basic variables (the **departing variable**) must leave, otherwise you would have too many variables for a basic solution. Choose the entering and departing variables as follows.

1. The **entering variable** corresponds to the smallest (the most negative) entry in the bottom row of the tableau, excluding the “ b -column.”
2. The **departing variable** corresponds to the smallest nonnegative ratio b_i/a_{ij} in the column determined by the entering variable, when $a_{ij} > 0$.
3. The entry in the simplex tableau in the entering variable’s column and the departing variable’s row is called the **pivot**.

Finally, to form the improved solution, apply Gauss-Jordan elimination to the column that contains the pivot, as illustrated in Example 1. (This process is called **pivotting**.)

EXAMPLE 1

Pivoting to Find an Improved Solution

Use the simplex method to find an improved solution for the linear programming problem represented by the tableau shown below.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	

The objective function for this problem is $z = 4x_1 + 6x_2$.

SOLUTION

Note that the current solution

$$(x_1 = 0, x_2 = 0, s_1 = 11, s_2 = 27, s_3 = 90)$$

corresponds to a z -value of 0. To improve this solution, you determine that x_2 is the entering variable, because -6 is the smallest entry in the bottom row.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	

↑
Entering

To see *why* you choose x_2 as the entering variable, remember that $z = 4x_1 + 6x_2$. So, it appears that a unit change in x_2 produces a change of 6 in z , whereas a unit change in x_1 produces a change of only 4 in z .

REMARK

In the event of a tie when choosing entering and/or departing variables, any of the tied variables may be chosen.

To find the departing variable, locate the b_i 's that have corresponding positive elements in the entering variables column and form the ratios

$$\frac{11}{1} = 11, \quad \frac{27}{1} = 27, \quad \text{and} \quad \frac{90}{5} = 18.$$

Here the smallest nonnegative ratio is 11, so you can choose s_1 as the departing variable.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	(1)	1	0	0	11	$s_1 \leftarrow$ Departing
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	
		↑				Entering

Note that the pivot is the entry in the first row and second column. Now, use Gauss-Jordan elimination to obtain the improved solution shown below.

Before Pivoting		After Pivoting
$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 1 & 1 & 0 & 1 & 0 & 27 \\ 2 & 5 & 0 & 0 & 1 & 90 \\ -4 & -6 & 0 & 0 & 0 & 0 \end{bmatrix}$	→	$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$
		$\begin{array}{l} -R_1 + R_2 \\ -5R_1 + R_3 \\ 6R_1 + R_4 \end{array}$

The new tableau now appears as follows.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	x_2
2	0	-1	1	0	16	s_2
7	0	-5	0	1	35	s_3
-10	0	6	0	0	66	

Note that x_2 has replaced s_1 in the basic variables column and the improved solution

$$(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$$

has a z -value of

$$z = 4x_1 + 6x_2 = 4(0) + 6(11) = 66.$$



In Example 1, the improved solution is not yet optimal because the bottom row still has a negative entry. So, apply another iteration of the simplex method to improve the solution further, as follows. Choose x_1 as the entering variable. Moreover, the smaller of the ratios $16/2 = 8$ and $35/7 = 5$ is 5, so s_3 is the departing variable. Gauss-Jordan elimination produces the following matrices.

$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$		$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$
		$\frac{1}{7}R_3$
	→	$\begin{bmatrix} 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ 0 & 0 & \frac{3}{7} & 1 & -\frac{2}{7} & 6 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{bmatrix}$
		$R_1 + R_3$ $R_2 - 2R_3$ $R_4 + 10R_3$

So, the new simplex tableau is as follows.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	

In this tableau, there is still a negative entry in the bottom row. So, choose s_1 as the entering variable and s_2 as the departing variable, as shown in the next tableau.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$(\frac{3}{7})$	1	$-\frac{2}{7}$	6	$s_2 \leftarrow \text{Departing}$
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	

↑
Entering

One more iteration of the simplex method produces the tableau shown below. (Try checking this.)

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	12	x_2
0	0	1	$\frac{7}{3}$	$-\frac{2}{3}$	14	s_1
1	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	15	x_1
0	0	0	$\frac{8}{3}$	$\frac{2}{3}$	132	← Maximum z -value

In this tableau, there are no negative elements in the bottom row. So, the optimal solution is determined to be

$$(x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)$$

with

$$z = 4x_1 + 6x_2 = 4(15) + 6(12) = 132.$$

Because the linear programming problem in Example 1 involved only two decision variables, you could have used a graphical solution technique, as in Example 2, Section 9.2. Notice in Figure 9.18 that each iteration in the simplex method corresponds to moving from a given vertex to an adjacent vertex with an improved z -value.

$$\begin{array}{cccccc} (0, 0) & \xrightarrow{\hspace{1cm}} & (0, 11) & \xrightarrow{\hspace{1cm}} & (5, 16) & \xrightarrow{\hspace{1cm}} & (15, 12) \\ z = 0 & & z = 66 & & z = 116 & & z = 132 \end{array}$$

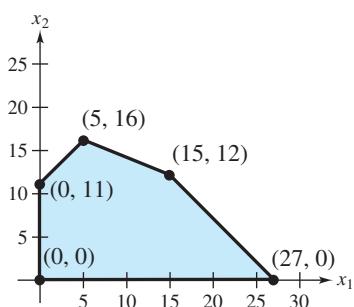


Figure 9.18

THE SIMPLEX METHOD

The steps involved in the simplex method can be summarized as follows.

The Simplex Method (Standard Form)

To solve a linear programming problem in standard form, use the following steps.

1. Convert each inequality in the set of constraints to an equation by adding slack variables.
2. Create the initial simplex tableau.
3. Locate the most negative entry in the bottom row, excluding the “ b -column.” This entry is called the entering variable, and its column is called the **entering column**. (If ties occur, any of the tied entries can be used to determine the entering column.)
4. Form the ratios of the entries in the “ b -column” with their corresponding positive entries in the entering column. (If all entries in the entering column are 0 or negative, then there is no maximum solution.) The **departing row** corresponds to the smallest nonnegative ratio b_i/a_{ij} . (For ties, choose either row.) The entry in the departing row and the entering column is called the **pivot**.
5. Use elementary row operations to change the pivot to 1 and all other entries in the entering column to 0. This process is called **pivoting**.
6. When all entries in the bottom row are zero or positive, this is the final tableau. Otherwise, go back to Step 3.
7. If you obtain a final tableau, then the linear programming problem has a maximum solution. The maximum value of the objective function is given by the entry in the lower right corner of the tableau.

Note that the basic feasible solution of an initial simplex tableau is

$$(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_m) = (0, 0, \dots, 0, b_1, b_2, \dots, b_m).$$

This solution is basic because at most m variables are nonzero (namely, the slack variables). It is feasible because each variable is nonnegative.

In the next two examples, the use of the simplex method to solve a problem involving three decision variables is illustrated.



LINEAR ALGEBRA APPLIED

There are many commercially available optimization software packages to aid operations researchers in allocating resources to maximize profits and minimize costs. Many of these programs use the simplex method as their foundation. As you will see in the next section, the simplex method can also be applied to minimization problems. In addition to optimizing the objective function, the software packages often contain built-in sensitivity reporting to analyze how changes or errors in the data will affect the outcome.

Marcin Balcerzak/Shutterstock.com

EXAMPLE 2**The Simplex Method with Three Decision Variables**

Use the simplex method to find the maximum value of

$$z = 2x_1 - x_2 + 2x_3 \quad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ x_1 + 2x_2 - 2x_3 &\leq 20 \\ x_2 + 2x_3 &\leq 5 \end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

SOLUTION

Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 10, 20, 5)$$

the initial simplex tableau for this problem is as follows. (Try checking these computations, and note the “tie” that occurs when choosing the first entering variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
2	1	0	1	0	0	10	s_1
1	2	-2	0	1	0	20	s_2
0	1	(2)	0	0	1	5	s_3 ← Departing
-2	1	-2	0	0	0	0	
			↑				
			Entering				

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
(2)	1	0	1	0	0	10	s_1 ← Departing
1	3	0	0	1	1	25	s_2
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3
-2	2	0	0	0	1	5	
			↑				
			Entering				

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	5	x_1
0	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	1	20	s_2
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3
0	3	0	1	0	1	15	

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (5, 0, \frac{5}{2}, 0, 20, 0)$$

and the maximum value of z is 15.

Note that $s_2 = 20$. The optimal solution yields a maximum value of $z = 15$ provided that $x_1 = 5$, $x_2 = 0$, and $x_3 = \frac{5}{2}$. Notice that these values satisfy the constraints giving equality in the first and third constraints, yet the second constraint has a slack of 20.



Occasionally, the constraints in a linear programming problem will include an equation. In such cases, add a “slack variable” called an **artificial variable** to form the initial simplex tableau. Technically, this new variable is not a slack variable (because there is no slack to be taken). Once you have determined an optimal solution in such a problem, check to see that any equations in the original constraints are satisfied. Example 3 illustrates such a case.

EXAMPLE 3

The Simplex Method with Three Decision Variables

Use the simplex method to find the maximum value of

$$z = 3x_1 + 2x_2 + x_3 \quad \text{Objective function}$$

subject to the constraints

$$4x_1 + x_2 + x_3 = 30$$

$$2x_1 + 3x_2 + x_3 \leq 60$$

$$x_1 + 2x_2 + 3x_3 \leq 40$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

SOLUTION

Using the basic feasible solution $(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 30, 60, 40)$, the initial simplex tableau for this problem is as follows. (Note that s_1 is an artificial variable, rather than a slack variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
(4)	1	1	1	0	0	30	$s_1 \leftarrow$ Departing
2	3	1	0	1	0	60	s_2
1	2	3	0	0	1	40	s_3
-3	-2	-1	0	0	0	0	
↑							Entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{15}{2}$	x_1
0	$\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	45	$s_2 \leftarrow$ Departing
0	$\frac{7}{4}$	$\frac{11}{4}$	$-\frac{1}{4}$	0	1	$\frac{65}{2}$	s_3
0	$-\frac{5}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	0	0	$\frac{45}{2}$	
↑							Entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	0	$\frac{1}{5}$	$\frac{3}{10}$	$-\frac{1}{10}$	0	3	x_1
0	1	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	0	18	x_2
0	0	$\frac{12}{5}$	$\frac{1}{10}$	$-\frac{7}{10}$	1	1	s_3
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	45	

This implies that the optimal solution is $(x_1, x_2, x_3, s_1, s_2, s_3) = (3, 18, 0, 0, 0, 1)$ and the maximum value of z is 45. (This solution satisfies the equation provided in the constraints, because $4(3) + 1(18) + 1(0) = 30$.)

APPLICATIONS

Example 4 shows how the simplex method can be used to maximize profits in a business application.

EXAMPLE 4

A Business Application: Maximum Profit

A manufacturer produces three types of plastic fixtures. The times required for molding, trimming, and packaging are shown in the table. (Times are given in hours per dozen fixtures, and profits are given in dollars per dozen fixtures.)

<i>Process</i>	<i>Type A</i>	<i>Type B</i>	<i>Type C</i>
<i>Molding</i>	1	2	$\frac{3}{2}$
<i>Trimming</i>	$\frac{2}{3}$	$\frac{2}{3}$	1
<i>Packaging</i>	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
<i>Profit</i>	\$11	\$16	\$15

The maximum amounts of production time that the manufacturer can allocate to each component are as follows.

Molding: 12,000 hours

Trimming: 4600 hours

Packaging: 2400 hours

How many dozen of each type of fixture should be produced to obtain a maximum profit?

SOLUTION

Let x_1 , x_2 , and x_3 represent the numbers of dozens of types A, B, and C fixtures, respectively. The objective function (to be maximized) is

$$\text{Profit} = P = 11x_1 + 16x_2 + 15x_3$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. Moreover, using the information in the table, you can construct the constraints listed below.

$$\begin{aligned} x_1 + 2x_2 + \frac{3}{2}x_3 &\leq 12,000 \\ \frac{2}{3}x_1 + \frac{2}{3}x_2 + x_3 &\leq 4600 \\ \frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 &\leq 2400 \end{aligned}$$

So, the initial simplex tableau with the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 12,000, 4600, 2400)$$

is shown on the next page.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	($\frac{1}{2}$)	$\frac{3}{2}$	1	0	0	12,000	s_1 ← Departing
$\frac{2}{3}$	$\frac{2}{3}$	1	0	1	0	4600	s_2
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	0	0	1	2400	s_3
-11	-16	-15	0	0	0	0	
↑ Entering							

Now, to this initial tableau, apply the simplex method as shown.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{1}{2}$	0	0	6000	x_2
$\frac{1}{3}$	0	$\frac{1}{2}$	$-\frac{1}{3}$	1	0	600	s_2
($\frac{1}{2}$)	0	$\frac{1}{4}$	$-\frac{1}{6}$	0	1	400	s_3 ← Departing
-3	0	-3	8	0	0	96,000	
↑ Entering							

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
0	1	$\frac{3}{8}$	$\frac{3}{4}$	0	$-\frac{3}{2}$	5400	x_2
0	0	($\frac{1}{4}$)	$-\frac{1}{6}$	1	-1	200	s_2 ← Departing
1	0	$\frac{3}{4}$	$-\frac{1}{2}$	0	3	1200	x_1
0	0	$-\frac{3}{4}$	$\frac{13}{2}$	0	9	99,600	
↑ Entering							

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
0	1	0	1	$-\frac{3}{2}$	0	5100	x_2
0	0	1	$-\frac{2}{3}$	4	-4	800	x_3
1	0	0	0	-3	6	600	x_1
0	0	0	6	3	6	100,200	

From this final simplex tableau, the maximum profit is \$100,200, and this is obtained by using the production levels listed below.

Type A: 600 dozen units

Type B: 5100 dozen units

Type C: 800 dozen units



Example 5 shows how the simplex method can be used to maximize the audience in an advertising campaign.

EXAMPLE 5

A Business Application: Media Selection

The advertising alternatives for a company include television, radio, and newspaper advertisements. The costs and estimates of audience coverage are provided in the table.

	<i>Television</i>	<i>Newspaper</i>	<i>Radio</i>
<i>Cost per advertisement</i>	\$2000	\$600	\$300
<i>Audience per advertisement</i>	100,000	40,000	18,000

The local newspaper limits the number of weekly advertisements from a single company to ten. Moreover, in order to balance the advertising among the three types of media, no more than half of the total number of advertisements should occur on the radio, and at least 10% should occur on television. The weekly advertising budget is \$18,200. How many advertisements should be run in each of the three types of media to maximize the total audience?

SOLUTION

To begin, let x_1 , x_2 , and x_3 represent the numbers of advertisements on television, in the newspaper, and on the radio, respectively. The objective function (to be maximized) is

$$z = 100,000x_1 + 40,000x_2 + 18,000x_3 \quad \text{Objective function}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. The constraints for this problem are as follows.

$$\begin{array}{rcl}
 2000x_1 + 600x_2 + 300x_3 & \leq & 18,200 \\
 x_2 & \leq & 10 \\
 x_3 & \leq 0.5(x_1 + x_2 + x_3) \\
 x_1 & \geq 0.1(x_1 + x_2 + x_3)
 \end{array}$$

A more manageable form of this system of constraints is as follows.

$$\left. \begin{array}{rcl} 20x_1 + 6x_2 + 3x_3 & \leq & 182 \\ x_2 & \leq & 10 \\ -x_1 - x_2 + x_3 & \leq & 0 \\ -9x_1 + x_2 + x_3 & \leq & 0 \end{array} \right\} \quad \text{Constraints}$$

So, the initial simplex tableau is as follows.

Now, to this initial tableau, apply the simplex method, as follows.

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	$\frac{3}{10}$	$\frac{3}{20}$	$\frac{1}{20}$	0	0	0	$\frac{91}{10}$	x_1
0	$\frac{1}{10}$	0	0	1	0	0	10	$s_2 \leftarrow$ Departing
0	$-\frac{7}{10}$	$\frac{23}{20}$	$\frac{1}{20}$	0	1	0	$\frac{91}{10}$	s_3
0	$\frac{37}{10}$	$\frac{47}{20}$	$\frac{9}{20}$	0	0	1	$\frac{819}{10}$	s_4
0 - 10,000 - 3000 5000 0 0 0 910,000								
↑ Entering								

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	0	$\frac{3}{20}$	$\frac{1}{20}$	$-\frac{3}{10}$	0	0	$\frac{61}{10}$	x_1
0	1	0	0	1	0	0	10	x_2
0	0	$-\frac{23}{20}$	$\frac{1}{20}$	$\frac{7}{10}$	1	0	$\frac{161}{10}$	$s_3 \leftarrow$ Departing
0	0	$\frac{47}{20}$	$\frac{9}{20}$	$-\frac{37}{10}$	0	1	$\frac{449}{10}$	s_4
0 0 -3000 5000 10,000 0 0 1,010,000								
↑ Entering								

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	0	0	$\frac{1}{23}$	$-\frac{9}{23}$	$-\frac{3}{23}$	0	4	x_1
0	1	0	0	1	0	0	10	x_2
0	0	1	$\frac{1}{23}$	$\frac{14}{23}$	$\frac{20}{23}$	0	14	x_3
0	0	0	$\frac{8}{23}$	$-\frac{118}{23}$	$-\frac{47}{23}$	1	12	s_4
0 0 0 $\frac{118,000}{23}$ $\frac{272,000}{23}$ $\frac{60,000}{23}$ 0 1,052,000								

From this final simplex tableau, you can see that the maximum weekly audience for an advertising budget of \$18,200 is

$$z = 1,052,000 \quad \text{Maximum weekly audience}$$

and this occurs when $x_1 = 4$, $x_2 = 10$, and $x_3 = 14$. The results are summarized as follows.

Media	Number of Advertisements	Cost	Audience
Television	4	\$8000	400,000
Newspaper	10	\$6000	400,000
Radio	14	\$4200	252,000
Total	28	\$18,200	1,052,000



9.3 Exercises

Writing a Simplex Tableau In Exercises 1–4, write the simplex tableau for the linear programming problem. You do not need to solve the problem. (In each case, the objective function is to be maximized.)

1. Objective function:

$$z = x_1 + 2x_2$$

Constraints:

$$2x_1 + x_2 \leq 8$$

$$x_1 + x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

2. Objective function:

$$z = x_1 + 3x_2$$

Constraints:

$$x_1 + x_2 \leq 4$$

$$x_1 - x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

3. Objective function:

$$z = 2x_1 + 3x_2 + 4x_3$$

Constraints:

$$x_1 + 2x_2 \leq 12$$

$$x_1 + x_3 \leq 8$$

$$x_1, x_2, x_3 \geq 0$$

4. Objective function:

$$z = 6x_1 - 9x_2$$

Constraints:

$$2x_1 - 3x_2 \leq 6$$

$$x_1 + x_2 \leq 20$$

$$x_1, x_2 \geq 0$$

Standard Form In Exercises 5–8, explain why the linear programming problem is not in standard form.

5. (Minimize)

Objective function:

$$z = x_1 + x_2$$

Constraints:

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

6. (Maximize)

Objective function:

$$z = x_1 + x_2$$

Constraints:

$$x_1 + 2x_2 \leq 6$$

$$2x_1 - x_2 \leq -1$$

$$x_1, x_2 \geq 0$$

7. (Maximize)

Objective function:

$$z = x_1 + x_2$$

Constraints:

$$x_1 + x_2 + 3x_3 \leq 5$$

$$2x_1 - 2x_3 \geq 1$$

$$x_2 + x_3 \leq 0$$

$$x_1, x_2, x_3 \geq 0$$

8. (Maximize)

Objective function:

$$z = x_1 + x_2$$

Constraints:

$$x_1 + x_2 \geq 4$$

$$2x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Pivoting In Exercises 9 and 10, use one iteration of pivoting to find an improved solution for the linear programming problem represented by the given tableau.

9.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-10	3	1	0	0	32	s_1
2	-2	0	1	0	16	s_2
3	12	0	0	1	28	s_3
-5	-2	0	0	0	0	

10.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
9	15	1	0	0	750	s_1
15	-10	0	1	0	60	s_2
-12	50	0	0	1	3000	s_3
-45	-78	0	0	0	0	

Using the Simplex Method In Exercises 11–24, use the simplex method to solve the linear programming problem. (In each case, the objective function is to be maximized.)

11. Objective function:

$$z = x_1 + 2x_2$$

Constraints:

$$x_1 + 4x_2 \leq 8$$

$$x_1 + x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

12. Objective function:

$$z = x_1 + x_2$$

Constraints:

$$x_1 + 2x_2 \leq 6$$

$$3x_1 + 2x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

13. Objective function:

$$z = 3x_1 + 2x_2$$

Constraints:

$$x_1 + 3x_2 \leq 15$$

$$4x_1 + x_2 \leq 16$$

$$x_1, x_2 \geq 0$$

14. Objective function:

$$z = 4x_1 + 5x_2$$

Constraints:

$$2x_1 + 2x_2 \leq 10$$

$$x_1 + 2x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

15. Objective function:

$$z = x_1 - x_2 + 2x_3$$

Constraints:

$$2x_1 + 2x_2 \leq 8$$

$$x_3 \leq 5$$

$$x_1, x_2, x_3 \geq 0$$

16. Objective function:

$$z = 25x_1 + 35x_2$$

Constraints:

$$18x_1 + 9x_2 \leq 7200$$

$$8x_1 + 9x_2 \leq 3600$$

$$x_1, x_2 \geq 0$$

17. Objective function:

$$z = 4x_1 + 5x_2$$

Constraints:

$$x_1 + x_2 \leq 10$$

$$3x_1 + 7x_2 \leq 42$$

$$x_1, x_2 \geq 0$$

18. Objective function:

$$z = x_1 + 2x_2$$

Constraints:

$$x_1 + 3x_2 \leq 15$$

$$2x_1 - x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

19. Objective function:

$$z = 5x_1 + 2x_2 + 8x_3$$

Constraints:

$$2x_1 - 4x_2 + x_3 \leq 42$$

$$2x_1 + 3x_2 - x_3 \leq 42$$

$$6x_1 - x_2 + 3x_3 \leq 42$$

$$x_1, x_2, x_3 \geq 0$$

20. Objective function:

$$z = 10x_1 + 7x_2$$

Constraints:

$$x_1 \leq 60$$

$$x_2 \leq 45$$

$$5x_1 + 6x_2 \leq 420$$

$$x_1, x_2 \geq 0$$

21. Objective function:

$$z = x_1 - x_2 + x_3$$

Constraints:

$$2x_1 + x_2 - 3x_3 \leq 40$$

$$x_1 + x_3 \leq 25$$

$$2x_2 + 3x_3 \leq 32$$

$$x_1, x_2, x_3 \geq 0$$

22. Objective function:

$$z = 3x_1 + 4x_2 + x_3 + 7x_4$$

Constraints:

$$8x_1 + 3x_2 + 4x_3 + x_4 \leq 7$$

$$2x_1 + 6x_2 + x_3 + 5x_4 \leq 3$$

$$x_1 + 4x_2 + 5x_3 + 2x_4 \leq 8$$

$$x_1, x_2, x_3, x_4 \geq 0$$

23. Objective function:

$$z = x_1 + 2x_2 - x_4$$

Constraints:

$$x_1 + 2x_2 + 3x_3 \leq 24$$

$$3x_2 + 7x_3 + x_4 \leq 42$$

$$x_1, x_2, x_3, x_4 \geq 0$$

24. Objective function:

$$z = x_1 + 2x_2 + x_3 - x_4$$

Constraints:

$$x_1 + x_2 + 3x_3 + 4x_4 \leq 60$$

$$x_2 + 2x_3 + 5x_4 \leq 50$$

$$2x_1 + 3x_2 + 6x_4 \leq 72$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Using Artificial Variables In Exercises 25 and 26, use an artificial variable to solve the linear programming problem, and check your solution.

25. Objective function:

$$z = 10x_1 + 5x_2 + 12x_3$$

Constraints:

$$-2x_1 + x_2 + 2x_3 \leq 80$$

$$x_1 - x_3 = 35$$

$$-x_2 + 3x_3 \leq 60$$

$$x_1, x_2, x_3 \geq 0$$

26. Objective function:

$$z = 9x_1 + x_2 + 2x_3$$

Constraints:

$$2x_1 + 6x_2 = 8$$

$$4x_1 + 2x_3 \leq 21$$

$$-2x_1 - 3x_2 + x_3 \leq 12$$

$$x_1, x_2, x_3 \geq 0$$

27. **Optimal Profit** A grower raises crops X, Y, and Z. The profit is \$60 per acre for crop X, \$20 per acre for crop Y, and \$30 per acre for crop Z. Research and available resources indicate the following constraints.

(a) The grower has 50 acres of land available.

(b) The costs per acre of producing crops X, Y, and Z are \$200, \$80, and \$140, respectively.

(c) The grower's total cost cannot exceed \$10,000.

Use the simplex method to find the optimal number of acres of land for each crop. What is the maximum profit?

- 28. **CAPSTONE** Consider the following initial simplex tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
-1	1	3	1	0	0	15	s_1
1	1	-2	0	1	0	5	s_2
9	3	12	0	0	1	48	s_3
-1	0	-2	0	0	0	0	

(a) List the objective function and constraints corresponding to the tableau.

(b) What solution does the tableau represent? Is it optimal? Explain.

(c) To find an improved solution using an iteration of the simplex method, which entering and departing variables would you select? Explain.

29. **Optimal Profit** A fruit juice company makes two special drinks by blending apple and pineapple juices. The profit per liter is \$0.60 for the first drink and \$0.50 for the second drink. The portions of apple and pineapple juice in each drink are shown in the table.

Ingredient	First drink	Second drink
Apple juice	30%	60%
Pineapple juice	70%	40%

There are 1000 liters of apple juice and 1500 liters of pineapple juice available. Use the simplex method to find the number of liters of each drink that should be produced in order to maximize the profit.

30. **Optimal Profit** Rework Exercise 29 given that the second drink was changed to contain 80% apple juice and 20% pineapple juice.

- 31. Optimal Profit** A manufacturer produces three models of bicycles. The times (in hours) required for assembling, painting, and packaging each model are shown in the table below.

	Model A	Model B	Model C
Assembling	2	2.5	3
Painting	1.5	2	1
Packaging	1	0.75	1.25

The total times available for assembling, painting, and packaging are 4006 hours, 2495 hours, and 1500 hours, respectively. The profits per unit are \$45 for model A, \$50 for model B, and \$55 for model C. What is the optimal production level for each model? What is the optimal profit?

- 32. Optimal Profit** Suppose that in Exercise 31 the total times available for assembling, painting, and packaging are 4000 hours, 2500 hours, and 1500 hours, respectively, and that the profits per unit are \$48 for model A, \$50 for model B, and \$52 for model C. What is the optimal production level for each model? What is the optimal profit?
- 33. Investment** An investor has up to \$250,000 to invest in three types of investments. Type A investments pay 8% annually and have a risk factor of 0. Type B investments pay 10% annually and have a risk factor of 0.06. Type C investments pay 14% annually and have a risk factor of 0.10. To have a well-balanced portfolio, the investor imposes the following conditions. The average risk factor should be no greater than 0.05. Moreover, at least one-fourth of the total portfolio is to be allocated to type A investments and at least one-fourth of the portfolio is to be allocated to type B investments. How much should be allocated to each type of investment to obtain a maximum return?
- 34. Investment** An investor has up to \$450,000 to invest in three types of investments. Type A investments pay 6% annually and have a risk factor of 0. Type B investments pay 10% annually and have a risk factor of 0.06. Type C investments pay 12% annually and have a risk factor of 0.08. To have a well-balanced portfolio, the investor imposes the following conditions. The average risk factor should be no greater than 0.05. Moreover, at least one-half of the total portfolio is to be allocated to type A investments and at least one-fourth of the portfolio is to be allocated to type B investments. How much should be allocated to each type of investment to obtain a maximum return?

Unbounded Solutions In the simplex method, it may happen that in selecting the departing variable all the calculated ratios are negative. This indicates an *unbounded solution*. Demonstrate this in Exercises 35 and 36.

- 35. (Maximize)**
Objective function:
$$z = x_1 + 2x_2$$

Constraints:
$$\begin{aligned}x_1 - 3x_2 &\leq 1 \\ -x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0\end{aligned}$$
- 36. (Maximize)**
Objective function:
$$z = x_1 + 3x_2$$

Constraints:
$$\begin{aligned}-x_1 + x_2 &\leq 20 \\ -2x_1 + x_2 &\leq 50 \\ x_1, x_2 &\geq 0\end{aligned}$$

Other Optimal Solutions If the simplex method terminates and one or more variables *not in the final basis* have bottom-row entries of zero, bringing these variables into the basis will determine other optimal solutions. Demonstrate this in Exercises 37 and 38.

- 37. (Maximize)**
Objective function:
$$z = 2.5x_1 + x_2$$

Constraints:
$$\begin{aligned}3x_1 + 5x_2 &\leq 15 \\ 5x_1 + 2x_2 &\leq 10 \\ x_1, x_2 &\geq 0\end{aligned}$$
- 38. (Maximize)**
Objective function:
$$z = x_1 + \frac{1}{2}x_2$$

Constraints:
$$\begin{aligned}2x_1 + x_2 &\leq 20 \\ x_1 + 3x_2 &\leq 35 \\ x_1, x_2 &\geq 0\end{aligned}$$

 **Using Technology** In Exercises 39 and 40, use a graphing utility or software program to maximize the objective function subject to the constraints

$$\begin{aligned}x_1 + x_2 + 0.83x_3 + 0.5x_4 &\leq 65 \\ 1.2x_1 + x_2 + x_3 + 1.2x_4 &\leq 96 \\ 0.5x_1 + 0.7x_2 + 1.2x_3 + 0.4x_4 &\leq 80\end{aligned}$$

where $x_1, x_2, x_3, x_4 \geq 0$.

- 39.** $z = 2x_1 + 7x_2 + 6x_3 + 4x_4$
40. $z = 1.2x_1 + x_2 + x_3 + x_4$

True or False? In Exercises 41 and 42, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

- 41.** After creating the initial simplex tableau, the entering column is chosen by locating the most negative entry in the bottom row.
- 42.** If all entries in the entering column are 0 or negative, then there is no maximum solution.

9.4 The Simplex Method: Minimization

- Determine the dual of a linear programming problem that minimizes an objective function.
- Use the simplex method to solve a linear programming problem that minimizes an objective function.

THE DUAL OF A MINIMIZATION PROBLEM

In Section 9.3, the simplex method was applied only to linear programming problems in standard form where the objective function was to be maximized. In this section, this procedure will be extended to linear programming problems in which the objective function is to be minimized.

A minimization problem is in **standard form** when the objective function

$$w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. The basic procedure used to solve such a problem is to convert it to a *maximization problem* in standard form, and then apply the simplex method as discussed in Section 9.3.

Consider Example 5 in Section 9.2, where geometric methods were used to solve the minimization problem shown below.

Minimization Problem: Find the minimum value of

$$w = 0.12x_1 + 0.15x_2 \quad \text{Objective function}$$

subject to the constraints

$$\left. \begin{array}{l} 60x_1 + 60x_2 \geq 300 \\ 12x_1 + 6x_2 \geq 36 \\ 10x_1 + 30x_2 \geq 90 \end{array} \right\} \quad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

To solve this problem using the simplex method, it must be converted to a maximization problem. The first step is to form the augmented matrix for this system of inequalities. To this augmented matrix, add a last row that represents the coefficients of the objective function, as follows.

$$\left[\begin{array}{ccc|c} 60 & 60 & 300 \\ 12 & 6 & 36 \\ 10 & 30 & 90 \\ 0.12 & 0.15 & 0 \end{array} \right]$$

Next, form the transpose of this matrix by interchanging its rows and columns.

$$\left[\begin{array}{cccc} 60 & 12 & 10 & 0.12 \\ 60 & 6 & 30 & 0.15 \\ 300 & 36 & 90 & 0 \end{array} \right]$$

Note that the rows of this matrix are the columns of the first matrix, and vice versa. Now, interpret this new matrix as a *maximization problem*, as shown on the next page.

To interpret the transposed matrix as a maximization problem, introduce new variables, y_1 , y_2 , and y_3 . This corresponding maximization problem is called the **dual** of the original minimization problem.

Dual Maximization Problem: Find the maximum value of

$$z = 300y_1 + 36y_2 + 90y_3 \quad \text{Dual objective function}$$

subject to the constraints

$$\begin{aligned} 60y_1 + 12y_2 + 10y_3 &\leq 0.12 \\ 60y_1 + 6y_2 + 30y_3 &\leq 0.15 \end{aligned} \quad \text{Constraints}$$

where $y_1 \geq 0$, $y_2 \geq 0$, and $y_3 \geq 0$.

As it turns out, the solution of the original minimization problem can be found by applying the simplex method to the new dual problem, as follows.

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
(60)	12	10	1	0	0.12	$s_1 \leftarrow$ Departing
60	6	30	0	1	0.15	s_2
-300	-36	-90	0	0	0	
↑						Entering

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{60}$	0	$\frac{1}{500}$	y_1
0	-6	(20)	-1	1	$\frac{3}{100}$	$s_2 \leftarrow$ Departing
0	24	-40	5	0	$\frac{3}{5}$	
↑						Entering

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{4}$	0	$\frac{1}{40}$	$-\frac{1}{120}$	$\frac{7}{4000}$	y_1
0	$-\frac{3}{10}$	1	$-\frac{1}{20}$	$\frac{1}{20}$	$\frac{3}{2000}$	y_3
0	12	0	3	2	$\frac{33}{50}$	
↑	↑					$x_1 \quad x_2$

So, the solution of the dual maximization problem is $z = \frac{33}{50} = 0.66$. This is the same value that was obtained in the minimization problem in Example 5 in Section 9.2. The x -values corresponding to this optimal solution are obtained from the entries in the bottom row corresponding to slack variable columns. In other words, the optimal solution occurs when $x_1 = 3$ and $x_2 = 2$.

The fact that a dual maximization problem has the same solution as its original minimization problem is stated formally in a result called the **von Neumann Duality Principle**, after the American mathematician John von Neumann (1903–1957).

THEOREM 9.2 The von Neumann Duality Principle

The objective value w of a minimization problem in standard form has a minimum value if and only if the objective value z of the dual maximization problem has a maximum value. Moreover, the minimum value of w is equal to the maximum value of z .

SOLVING A MINIMIZATION PROBLEM

The steps used to solve a minimization problem can be summarized, as follows.

Solving a Minimization Problem

A minimization problem is in standard form when the objective function

$$w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

is to be minimized, subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m$$

where $x_i \geq 0$ and $b_i \geq 0$. To solve this problem, use the following steps.

1. Form the **augmented matrix** for the given system of inequalities, and add a bottom row consisting of the coefficients of the objective function.

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \\ c_1 & c_2 & \cdots & c_n & 0 \end{array} \right]$$

2. Form the transpose of this matrix.

$$\left[\begin{array}{ccccc} a_{11} & a_{21} & \cdots & a_{m1} & c_1 \\ a_{12} & a_{22} & \cdots & a_{m2} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} & c_n \\ b_1 & b_2 & \cdots & b_m & 0 \end{array} \right]$$

3. Form the **dual maximization problem** corresponding to this transposed matrix. That is, find the maximum of the objective function given by

$$z = b_1y_1 + b_2y_2 + \cdots + b_my_m$$

subject to the constraints

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \leq c_1$$

$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \leq c_2$$

⋮

$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \leq c_n$$

where $y_1 \geq 0, y_2 \geq 0, \dots, y_m \geq 0$.

4. Apply the **simplex method** to the dual maximization problem. The maximum value of z will be the minimum value of w . Moreover, the values of x_1, x_2, \dots, x_n will occur in the bottom row of the final simplex tableau, in the columns corresponding to the slack variables.

The steps used to solve a minimization problem are illustrated in Examples 1 and 2.

EXAMPLE 1**Solving a Minimization Problem**

Find the minimum value of

$$w = 3x_1 + 2x_2 \quad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} 2x_1 + x_2 &\geq 6 \\ x_1 + x_2 &\geq 4 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

SOLUTION

The augmented matrices corresponding to this problem are as follows.

$$\begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 4 \\ 3 & 2 & 0 \end{bmatrix}$$

Minimization Problem

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 6 & 4 & 0 \end{bmatrix}$$

Dual Maximization Problem

Dual Maximization Problem: Find the maximum value of

$$z = 6y_1 + 4y_2 \quad \text{Dual objective function}$$

subject to the constraints

$$\begin{aligned} 2y_1 + y_2 &\leq 3 \\ y_1 + y_2 &\leq 2 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \text{Constraints}$$

where $y_1 \geq 0$ and $y_2 \geq 0$. Now apply the simplex method to the dual problem, as follows.

REMARK

Both the minimization and maximization linear programming problems in Example 1 could have been solved with a graphical method, as indicated in Figure 9.19. Note that the maximum value of $z = 6y_1 + 4y_2$ in part (a) is the same as the minimum value of $w = 3x_1 + 2x_2$ in part (b).

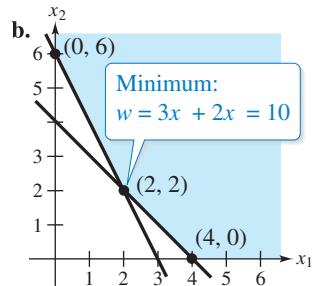
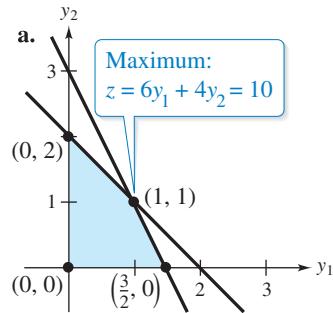


Figure 9.19

y_1	y_2	s_1	s_2	b	Basic Variables
(2)	1	1	0	3	$s_1 \leftarrow$ Departing
1	1	0	1	2	s_2
-6	-4	0	0	0	

↑
Entering

y_1	y_2	s_1	s_2	b	Basic Variables
1	1/2	1/2	0	3/2	y_1
0	(1/2)	-1/2	1	1/2	$s_2 \leftarrow$ Departing
0	-1	3	0	9	

↑
Entering

y_1	y_2	s_1	s_2	b	Basic Variables
1	0	1	-1	1	y_1
0	1	-1	2	1	y_2
0	0	2	2	10	

↑
↑
 x_1 x_2

From this final simplex tableau, the maximum value of z is 10. So, the solution of the original minimization problem is $w = 10$, and this occurs when $x_1 = 2$ and $x_2 = 2$. 

EXAMPLE 2**Solving a Minimization Problem**

Find the minimum value of

$$w = 2x_1 + 10x_2 + 8x_3 \quad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} x_1 + x_2 + x_3 &\geq 6 \\ x_2 + 2x_3 &\geq 8 \\ -x_1 + 2x_2 + 2x_3 &\geq 4 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Constraints}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

SOLUTION

The augmented matrices corresponding to this problem are as follows.

$$\begin{array}{cccc} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ -1 & 2 & 2 & 4 \\ 2 & 10 & 8 & 0 \end{array} \quad \begin{array}{cccc} 1 & 0 & -1 & 2 \\ 1 & 1 & 2 & 10 \\ 1 & 2 & 2 & 8 \\ 6 & 8 & 4 & 0 \end{array}$$

Minimization Problem

Dual Maximization Problem

Dual Maximization Problem: Find the maximum value of

$$z = 6y_1 + 8y_2 + 4y_3 \quad \text{Dual objective function}$$

subject to the constraints

$$\begin{aligned} y_1 - y_3 &\leq 2 \\ y_1 + y_2 + 2y_3 &\leq 10 \\ y_1 + 2y_2 + 2y_3 &\leq 8 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Dual constraints}$$

where $y_1 \geq 0$, $y_2 \geq 0$, and $y_3 \geq 0$. Now apply the simplex method to the dual problem.

y_1	y_2	y_3	s_1	s_2	s_3	b	Basic Variables
1	0	-1	1	0	0	2	s_1
1	1	2	0	1	0	10	s_2
1	(2)	2	0	0	1	8	s_3 ← Departing
-6	-8	-4	0	0	0	0	

↑
Entering

y_1	y_2	y_3	s_1	s_2	s_3	b	Basic Variables
(1)	0	-1	1	0	0	2	s_1 ← Departing
$\frac{1}{2}$	0	1	0	1	$-\frac{1}{2}$	6	s_2
$\frac{1}{2}$	1	1	0	0	$\frac{1}{2}$	4	y_2
-2	0	4	0	0	4	32	

↑
Entering

y_1	y_2	y_3	s_1	s_2	s_3	b	Basic Variables
1	0	-1	1	0	0	2	y_1
0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	5	s_2
0	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	3	y_2
0	0	2	2	0	4	36	
			↑	↑	↑		
			x_1	x_2	x_3		

From this final simplex tableau, the maximum value of z is 36. So, the solution of the original minimization problem is $w = 36$, and this occurs when $x_1 = 2$, $x_2 = 0$, and $x_3 = 4$. ■

EXAMPLE 3

A Business Application: Minimum Cost

A small petroleum company owns two refineries. Refinery 1 costs \$20,000 per day to operate, and refinery 2 costs \$25,000 per day. The table shows the number of barrels of each grade of oil the refineries can produce each day.

Grade	Refinery 1	Refinery 2
High-grade	400	300
Medium-grade	300	400
Low-grade	200	500

The company has orders totaling 25,000 barrels of high-grade oil, 27,000 barrels of medium-grade oil, and 30,000 barrels of low-grade oil. How many days should it run each refinery to minimize its costs and still refine enough oil to meet its orders?

SOLUTION

To begin, let x_1 and x_2 represent the numbers of days the two refineries are operated. Then the total cost is represented by

$$C = 20,000x_1 + 25,000x_2.$$

Objective function

The constraints are

$$\begin{aligned} \text{(High-grade)} \quad & 400x_1 + 300x_2 \geq 25,000 \\ \text{(Medium-grade)} \quad & 300x_1 + 400x_2 \geq 27,000 \\ \text{(Low-grade)} \quad & 200x_1 + 500x_2 \geq 30,000 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Constraints}$$

where $x_1 \geq 0$ and $x_2 \geq 0$. The augmented matrices corresponding to this problem are as follows.

$$\left[\begin{array}{ccc|c} 400 & 300 & 25,000 \\ 300 & 400 & 27,000 \\ 200 & 500 & 30,000 \\ 20,000 & 25,000 & 0 \end{array} \right] \quad \left[\begin{array}{cccc|c} 400 & 300 & 200 & 20,000 \\ 300 & 400 & 500 & 25,000 \\ 25,000 & 27,000 & 30,000 & 0 \end{array} \right]$$

Minimization Problem

Dual Maximization Problem

Now apply the simplex method to the dual problem as follows.

y_1	y_2	y_3	s_1	s_2	b	Basic Variables
400	300	200	1	0	20,000	s_1
300	400	500	0	1	25,000	$s_2 \leftarrow$ Departing
-25,000	-27,000	-30,000	0	0	0	
↑ Entering						
y_1	y_2	y_3	s_1	s_2	b	Basic Variables
280	140	0	1	$-\frac{2}{5}$	10,000	$s_1 \leftarrow$ Departing
$\frac{3}{5}$	$\frac{4}{5}$	1	0	$\frac{1}{500}$	50	y_3
-7000	-3000	0	0	60	1,500,000	
↑ Entering						
y_1	y_2	y_3	s_1	s_2	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{280}$	$-\frac{1}{700}$	$\frac{250}{7}$	y_1
0	$\frac{1}{2}$	1	$-\frac{3}{1400}$	$\frac{1}{350}$	$\frac{200}{7}$	y_3
0	500	0	25	50	1,750,000	
↑ x_1	↑ x_2					

From this final simplex tableau, the solution to the original minimization problem is

$$C = \$1,750,000 \quad \text{Minimum cost}$$

and this occurs when $x_1 = 25$ and $x_2 = 50$. So, the two refineries should be operated for the following numbers of days.

Refinery 1: 25 days

Refinery 2: 50 days

Note that by operating the two refineries for these numbers of days, the company will have produced the amounts of oil shown below.

High-grade oil: $25(400) + 50(300) = 25,000$ barrels

Medium-grade oil: $25(300) + 50(400) = 27,500$ barrels

Low-grade oil: $25(200) + 50(500) = 30,000$ barrels

So, the original production levels have been met (with a surplus of 500 barrels of medium-grade oil). 



LINEAR ALGEBRA APPLIED

A steel manufacturer produces steel coils for a customer in the automotive industry. The customer requires that each type of steel coil be cut to a precise measurement, causing a considerable amount of waste metal.

To minimize steel waste, the company determines an objective function for the waste generated by custom cutting each type of coil. Then, establishing detailed constraints on costs, production time, order specifications, and available steel stocks, the company uses optimization software to develop production plans for each order to minimize the waste. The parameters are then adapted easily for changing conditions or varying customers' needs.

9.4 Exercises

Finding the Dual In Exercises 1–6, determine the dual of the minimization problem.

1. Objective function:

$$w = 3x_1 + 3x_2$$

Constraints:

$$2x_1 + x_2 \geq 4$$

$$x_1 + 2x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

3. Objective function:

$$w = 4x_1 + x_2 + x_3$$

Constraints:

$$3x_1 + 2x_2 + x_3 \geq 23$$

$$x_1 + x_3 \geq 10$$

$$8x_1 + x_2 + 2x_3 \geq 40$$

$$x_1, x_2, x_3 \geq 0$$

5. Objective function:

$$w = 14x_1 + 20x_2 + 24x_3$$

Constraints:

$$x_1 + x_2 + 2x_3 \geq 7$$

$$x_1 + 2x_2 + x_3 \geq 4$$

$$x_1, x_2, x_3 \geq 0$$

2. Objective function:

$$w = 2x_1 + x_2$$

Constraints:

$$5x_1 + x_2 \geq 9$$

$$2x_1 + 2x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

4. Objective function:

$$w = 9x_1 + 6x_2$$

Constraints:

$$x_1 + 2x_2 \geq 5$$

$$2x_1 + 2x_2 \geq 8$$

$$2x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

6. Objective function:

$$w = 9x_1 + 4x_2 + 10x_3$$

Constraints:

$$2x_1 + x_2 + 3x_3 \geq 6$$

$$6x_1 + x_2 + x_3 \geq 9$$

$$x_1, x_2, x_3 \geq 0$$

9. Objective function:

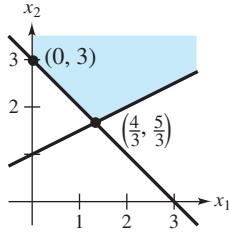
$$w = x_1 + 4x_2$$

Constraints:

$$x_1 + x_2 \geq 3$$

$$-x_1 + 2x_2 \geq 2$$

$$x_1, x_2 \geq 0$$



10. Objective function:

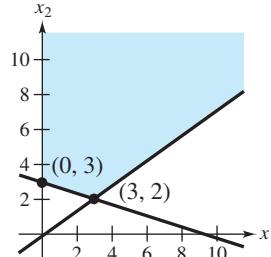
$$w = 2x_1 + 6x_2$$

Constraints:

$$-2x_1 + 3x_2 \geq 0$$

$$x_1 + 3x_2 \geq 9$$

$$x_1, x_2 \geq 0$$



11. Objective function:

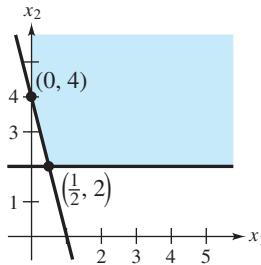
$$w = 6x_1 + 3x_2$$

Constraints:

$$4x_1 + x_2 \geq 4$$

$$x_2 \geq 2$$

$$x_1, x_2 \geq 0$$



12. Objective function:

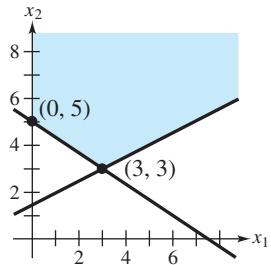
$$w = x_1 + 6x_2$$

Constraints:

$$2x_1 + 3x_2 \geq 15$$

$$-x_1 + 2x_2 \geq 3$$

$$x_1, x_2 \geq 0$$



Solving the Dual by Graphing In Exercises 7–12, (a) solve the minimization problem by the graphical method, (b) formulate the dual problem, and (c) solve the dual problem by the graphical method.

7. Objective function:

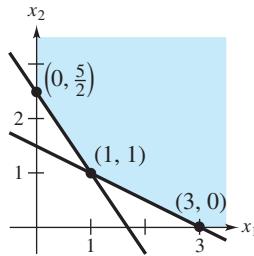
$$w = 2x_1 + 2x_2$$

Constraints:

$$x_1 + 2x_2 \geq 3$$

$$3x_1 + 2x_2 \geq 5$$

$$x_1, x_2 \geq 0$$



8. Objective function:

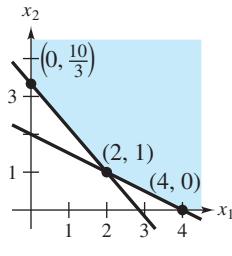
$$w = 14x_1 + 20x_2$$

Constraints:

$$x_1 + 2x_2 \geq 4$$

$$7x_1 + 6x_2 \geq 20$$

$$x_1, x_2 \geq 0$$



Solving the Dual with the Simplex Method In Exercises 13–22, solve the minimization problem by solving the dual maximization problem with the simplex method.

13. Objective function:

$$w = x_2$$

Constraints:

$$x_1 + 5x_2 \geq 10$$

$$-6x_1 + 5x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

15. Objective function:

$$w = 4x_1 + x_2$$

Constraints:

$$x_1 + x_2 \geq 8$$

$$3x_1 + 5x_2 \geq 30$$

$$x_1, x_2 \geq 0$$

14. Objective function:

$$w = 3x_1 + 8x_2$$

Constraints:

$$2x_1 + 7x_2 \geq 9$$

$$x_1 + 2x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

16. Objective function:

$$w = x_1 + x_2$$

Constraints:

$$x_1 + 2x_2 \geq 40$$

$$2x_1 + 3x_2 \geq 72$$

$$x_1, x_2 \geq 0$$

17. Objective function:

$$w = x_1 + 2x_2$$

Constraints:

$$3x_1 + 5x_2 \geq 15$$

$$5x_1 + 2x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

19. Objective function:

$$w = 8x_1 + 4x_2 + 6x_3$$

Constraints:

$$3x_1 + 2x_2 + x_3 \geq 6$$

$$4x_1 + x_2 + 3x_3 \geq 7$$

$$2x_1 + x_2 + 4x_3 \geq 8$$

$$x_1, x_2, x_3 \geq 0$$

21. Objective function:

$$w = 6x_1 + 2x_2 + 3x_3$$

Constraints:

$$3x_1 + 2x_2 + x_3 \geq 28$$

$$6x_1 + x_2 + x_3 \geq 24$$

$$3x_1 + x_2 + 2x_3 \geq 40$$

$$x_1, x_2, x_3 \geq 0$$

23. An electronics manufacturing company has three production plants, each of which produces three different models of a particular MP3 player. The daily capacities (in thousands of units) of the three plants are shown in the table.

	<i>Basic Model</i>	<i>Gold Model</i>	<i>Platinum Model</i>
<i>Plant 1</i>	8	4	8
<i>Plant 2</i>	6	6	3
<i>Plant 3</i>	12	4	8

The total demands are 300,000 units of the Basic model, 172,000 units of the Gold model, and 249,500 units of the Platinum model. The daily operating costs are \$55,000 for plant 1, \$60,000 for plant 2, and \$60,000 for plant 3. How many days should each plant be operated in order to fill the total demand while keeping the operating cost at a minimum?

24. The company in Exercise 23 has lowered the daily operating cost for plant 3 to \$50,000. How many days should each plant be operated in order to fill the total demand while keeping the operating cost at a minimum?
25. **Standard Form** Use a geometric argument to explain why, for a minimization problem in standard form, the left-hand side of each constraint is greater than or equal to the right-hand side.

18. Objective function:

$$w = 3x_1 + 4x_2$$

Constraints:

$$x_1 + 2x_2 \geq 6$$

$$2x_1 + x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

20. Objective function:

$$w = 8x_1 + 16x_2 + 18x_3$$

Constraints:

$$2x_1 + 2x_2 - 2x_3 \geq 4$$

$$-4x_1 + 3x_2 - x_3 \geq 1$$

$$x_1 - x_2 + 3x_3 \geq 8$$

$$x_1, x_2, x_3 \geq 0$$

22. Objective function:

$$w = 42x_1 + 5x_2 + 17x_3$$

Constraints:

$$3x_1 - x_2 + 7x_3 \geq 5$$

$$-3x_1 - x_2 + 3x_3 \geq 8$$

$$6x_1 + x_2 + x_3 \geq 16$$

$$x_1, x_2, x_3 \geq 0$$

- 26. **CAPSTONE** Consider the following related optimization problems.

Problem 1

Objective function:

$$M = 8y_1 + 50y_2 + 10y_3$$

Constraints:

$$2y_1 + y_2 + 2y_3 \leq 5$$

$$y_1 + y_2 \leq 10$$

$$y_3 \geq 6y_1 - 65$$

$$y_1, y_2, y_3 \geq 0$$

Problem 2

Objective function:

$$N = 5x_1 + 10x_2 + 65x_3$$

Constraints:

$$2x_1 + x_2 + 6x_3 \geq 8$$

$$x_1 + x_2 \geq 50$$

$$x_3 \leq 2x_1 - 10$$

$$x_1, x_2, x_3 \geq 0$$

(a) Which is the minimization problem? Explain.

(b) Which is the dual maximization problem? Explain.

(c) Write the corresponding augmented matrices and explain how they are related.

Optimal Cost In Exercises 27–30, two sports drinks are used to supply protein and carbohydrates. Drink A provides 1 unit of protein and 3 units of carbohydrates in each liter. Drink B supplies 2 units of protein and 2 units of carbohydrates in each liter. An athlete requires 3 units of protein and 5 units of carbohydrates. Find the amount of each drink the athlete should consume to minimize the cost and still meet the minimum dietary requirements.

27. Drink A costs \$2 per liter and drink B costs \$3 per liter.

28. Drink A costs \$4 per liter and drink B costs \$2 per liter.

29. Drink A costs \$1 per liter and drink B costs \$3 per liter.

30. Drink A costs \$1 per liter and drink B costs \$2 per liter.

► **Using Technology** In Exercises 31 and 32, use a graphing utility or software program to minimize the objective function subject to the constraints

$$\begin{aligned} 1.5x_1 + x_2 + 2x_4 &\geq 35 \\ 2x_2 + 6x_3 + 4x_4 &\geq 120 \\ x_1 + x_2 + x_3 + x_4 &\geq 50 \\ 0.5x_1 + 2.5x_3 + 1.5x_4 &\geq 75 \end{aligned}$$

where $x_1, x_2, x_3, x_4 \geq 0$.

$$31. w = x_1 + 0.5x_2 + 2.5x_3 + 3x_4$$

$$32. w = 1.5x_1 + x_2 + 0.5x_3 + 2x_4$$

9.5 The Simplex Method: Mixed Constraints

- Find the maximum of an objective function subject to mixed constraints.
- Find the minimum of an objective function subject to mixed constraints.

MIXED CONSTRAINTS AND MAXIMIZATION

In Sections 9.3 and 9.4, you studied linear programming problems in *standard form*. The constraints for the maximization problems all involved \leq inequalities, and the constraints for the minimization problems all involved \geq inequalities.

Linear programming problems for which the constraints involve *both* types of inequalities are called **mixed-constraint** problems. For instance, consider the following linear programming problem.

Mixed-Constraint Problem: Find the maximum value of

$$z = x_1 + x_2 + 2x_3 \quad \text{Objective function}$$

subject to the constraints

$$\left. \begin{array}{l} 2x_1 + x_2 + x_3 \leq 50 \\ 2x_1 + x_2 \geq 36 \\ x_1 + x_3 \geq 10 \end{array} \right\} \quad \text{Constraints}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. Because this is a maximization problem, you would expect each of the inequalities in the set of constraints to involve \leq . The first inequality does involve \leq , so add a slack variable to form the equation

$$2x_1 + x_2 + x_3 + s_1 = 50.$$

For the other two inequalities, a new type of variable, called a **surplus variable**, is introduced, as follows.

$$\begin{aligned} 2x_1 + x_2 - s_2 &= 36 \\ x_1 + x_3 - s_3 &= 10 \end{aligned}$$

Notice that surplus variables are *subtracted from* (not added to) the left side of each equation. They are called *surplus* variables because they represent the amounts by which the left sides of the inequalities exceed the right sides. Surplus variables must be nonnegative.

Now, to solve the problem, form an initial simplex tableau, as follows.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
2	1	1	1	0	0	50	s_1
2	1	0	0	-1	0	36	s_2
1	0	(1)	0	0	-1	10	s_3 ← Departing
-1	-1	-2	0	0	0	0	
							↑ Entering

REMARK

In fact, the values $x_1 = x_2 = x_3 = 0$ do not even satisfy the constraint equations.

Solving mixed-constraint problems can be difficult. One reason for this is that there is no convenient feasible solution to begin the simplex method. Note that the solution represented by the initial tableau above

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 50, -36, -10)$$

is not a feasible solution because the values of the two surplus variables are negative.

In order to eliminate the surplus variables from the current solution on the previous page, “trial and error” is used. That is, in an effort to find a feasible solution, arbitrarily choose new entering variables. For instance, in this tableau, it seems reasonable to select x_3 as the entering variable. After pivoting, the new simplex tableau is as follows.

x_1	x_2	x_3	s_1	s_2	s_3	b	Variables
1	1	0	1	0	1	40	s_1
2	(1)	0	0	-1	0	36	$s_2 \leftarrow$ Departing
1	0	1	0	0	-1	10	x_3
1	-1	0	0	0	-2	20	
		↑					
		Entering					

The current solution $(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 10, 40, -36, 0)$ is still not feasible, so choose x_2 as the entering variable and pivot to obtain the following simplex tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	Variables
-1	0	0	1	1	(1)	4	$s_1 \leftarrow$ Departing
2	1	0	0	-1	0	36	x_2
1	0	1	0	0	-1	10	x_3
3	0	0	0	-1	-2	56	
		↑					
		Entering					

At this point, the feasible solution shown below is finally obtained.

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 36, 10, 4, 0, 0)$$

From here on, you can apply the simplex method as usual. Note that the entering variable here is s_3 because its column has the most negative entry in the bottom row. After pivoting one more time, you obtain the following final simplex tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	Variables
-1	0	0	1	1	1	4	s_3
2	1	0	0	-1	0	36	x_2
0	0	1	1	1	0	14	x_3
1	0	0	2	1	0	64	

Note that this tableau is final because it represents a feasible solution and there are no negative entries in the bottom row. So, the maximum value of the objective function is

$$z = 64 \quad \text{Maximum value}$$

and this occurs when

$$x_1 = 0, x_2 = 36, \text{ and } x_3 = 14.$$

EXAMPLE 1**A Maximization Problem with Mixed Constraints**

Find the maximum value of

$$z = 3x_1 + 2x_2 + 4x_3 \quad \text{Objective function}$$

subject to the constraints

$$\left. \begin{array}{l} 3x_1 + 2x_2 + 5x_3 \leq 18 \\ 4x_1 + 2x_2 + 3x_3 \leq 16 \\ 2x_1 + x_2 + x_3 \geq 4 \end{array} \right\} \quad \text{Constraints}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

SOLUTION

To begin, add a slack variable to each of the first two inequalities and subtract a surplus variable from the third inequality to produce the following system of linear equations.

$$\begin{aligned} 3x_1 + 2x_2 + 5x_3 + s_1 &= 18 \\ 4x_1 + 2x_2 + 3x_3 + s_2 &= 16 \\ 2x_1 + x_2 + x_3 - s_3 &= 4 \end{aligned}$$

Now form the initial simplex tableau, as follows.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
3	2	5	1	0	0	18	s_1
4	2	3	0	1	0	16	s_2
2	1	1	0	0	-1	4	s_3 ← Departing
-3	-2	-4	0	0	0	0	

As it stands, this tableau does not represent a feasible solution (because the value of s_3 is negative). So, s_3 should be the departing variable. There are no real guidelines as to which variable should enter the solution, and in fact, any choice will work. However, some entering variables will require more tedious computations than others. For instance, choosing x_1 as the entering variable produces the following tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
0	$\frac{1}{2}$	$\frac{7}{2}$	1	0	$\frac{3}{2}$	12	s_1
0	0	1	0	1	2	8	s_2
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	2	s_3
0	$-\frac{1}{2}$	$-\frac{5}{2}$	0	0	$-\frac{3}{2}$	6	

Because this tableau contains fractions, try to find another entering variable that results in simpler numbers. For instance, using x_2 as the entering variable on the initial tableau produces the cleaner tableau shown below.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
-1	0	3	1	0	2	10	s_1
0	0	1	0	1	2	8	s_2
2	1	1	0	0	-1	4	x_2
1	0	-2	0	0	-2	8	

Notice that both tableaus represent feasible solutions. Because any choice of entering variables will lead to a feasible solution, use trial and error to find an entering variable that yields “nice” numbers. Once you have reached a feasible solution, follow the standard pivoting procedure to choose entering variables and eventually reach an optimal solution.

In this case, choose the tableau produced by using x_2 as the entering variable. Because this tableau does represent a feasible solution, proceed as usual, choosing the most negative entry in the bottom row to be the entering variable. (In this case, there is a tie, so arbitrarily choose x_3 to be the entering variable.)

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
-1	0	($\hat{3}$)	1	0	2	10	$s_1 \leftarrow$ Departing
0	0	1	0	1	2	8	s_2
2	1	1	0	0	-1	4	x_2
1	0	-2	0	0	-2	8	
		↑					Entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
$-\frac{1}{3}$	0	1	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{10}{3}$	x_3
$\frac{1}{3}$	0	0	$-\frac{1}{3}$	1	$\frac{4}{3}$	$\frac{14}{3}$	$s_2 \leftarrow$ Departing
$\frac{7}{3}$	1	0	$-\frac{1}{3}$	0	$-\frac{5}{3}$	$\frac{2}{3}$	x_2
$\frac{1}{3}$	0	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{44}{3}$	
		↑					Entering

REMARK

Page 474 shows the tableau obtained by choosing x_1 as the entering variable on the initial simplex tableau. Try using the simplex method on that tableau to confirm that it does in fact lead to the same solution found here.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
$-\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	x_3
$\frac{1}{4}$	0	0	$-\frac{1}{4}$	$\frac{3}{4}$	1	$\frac{7}{2}$	s_3
$\frac{11}{4}$	1	0	$-\frac{3}{4}$	$\frac{5}{4}$	0	$\frac{13}{2}$	x_2
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	17	

So, the maximum value of the objective function is $z = 17$, and this occurs when

$$x_1 = 0, x_2 = \frac{13}{2}, \text{ and } x_3 = 1.$$



LINEAR ALGEBRA APPLIED

In Exercise 45, you will use optimization techniques to determine how to best allocate a company’s advertising budget. This type of mixed constraint problem is typical in advertising because market research establishes minimums on the number of each type of ad, while budgets put upper limits on expenses.

To maximize an audience, researchers must first determine the ideal demographic and then estimate the time and place at which those people will most likely see the ad, considering traditional venues as well as digital formats and social media.

MIXED CONSTRAINTS AND MINIMIZATION

In Section 9.4, the solution of minimization problems in *standard form* was discussed. Minimization problems that are not in standard form are more difficult to solve. One technique is to change a mixed-constraint minimization problem to a mixed-constraint maximization problem by multiplying each coefficient in the objective function by -1 . This technique is demonstrated in the next example.

EXAMPLE 2

A Minimization Problem with Mixed Constraints

Find the minimum value of

$$w = 4x_1 + 2x_2 + x_3 \quad \text{Objective function}$$

subject to the constraints

$$\begin{array}{l} 2x_1 + 3x_2 + 4x_3 \leq 14 \\ 3x_1 + x_2 + 5x_3 \geq 4 \\ x_1 + 4x_2 + 3x_3 \geq 6 \end{array} \quad \text{Constraints}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

SOLUTION

First, rewrite the objective function by multiplying each of its coefficients by -1 , as follows.

$$z = -4x_1 - 2x_2 - x_3 \quad \text{Revised objective function}$$

Maximizing this revised objective function is equivalent to minimizing the original objective function. Next, add a slack variable to the first inequality and subtract surplus variables from the second and third inequalities to produce the initial simplex tableau shown below.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
2	3	4	1	0	0	14	s_1
3	(1)	5	0	-1	0	4	$s_2 \leftarrow$ Departing
1	4	3	0	0	-1	6	s_3
4	2	1	0	0	0	0	

↑
Entering

Note that the bottom row contains the negatives of the coefficients of the revised objective function. Another way of looking at this is that for minimization problems (in nonstandard form), the bottom row of the initial simplex tableau consists of the coefficients of the original objective function.

As with maximization problems with mixed constraints, this initial simplex tableau does not represent a feasible solution. By trial and error, choose x_2 as the entering variable and s_2 as the departing variable. After pivoting, you obtain the tableau shown at the top of the next page.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
-7	0	-11	1	3	0	2	s_1
3	1	5	0	-1	0	4	x_2
-11	0	-17	0	4	-1	-10	s_3
-2	0	-9	0	2	0	-8	

To transform the tableau into one that represents a feasible solution, multiply the third row by -1 , as follows.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
-7	0	-11	1	3	0	2	s_1
3	1	5	0	-1	0	4	x_2
11	0	($\frac{1}{17}$)	0	-4	1	10	s_3 ← Departing
-2	0	-9	0	2	0	-8	

↑
 Entering

REMARK

Recall from page 450 that your choices for the entering variable exclude the entry in the “ b -column.” So, although $-\frac{46}{17}$ is the smallest entry in the bottom row here, it does not represent a variable that can enter the solution.



Now that you have obtained a simplex tableau that represents a feasible solution, continue with the standard pivoting operations, as follows.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
$\frac{2}{17}$	0	0	1	$\frac{7}{17}$	$\frac{11}{17}$	$\frac{144}{17}$	s_1
$-\frac{4}{17}$	1	0	0	($\frac{3}{17}$)	$-\frac{5}{17}$	$\frac{18}{17}$	x_2 ← Departing
$\frac{11}{17}$	0	1	0	$-\frac{4}{17}$	$\frac{1}{17}$	$\frac{10}{17}$	x_3
$\frac{65}{17}$	0	0	0	$-\frac{2}{17}$	$\frac{9}{17}$	$-\frac{46}{17}$	

↑
 Entering

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
$\frac{2}{3}$	$-\frac{7}{3}$	0	1	0	$\frac{4}{3}$	6	s_1
$-\frac{4}{3}$	$\frac{17}{3}$	0	0	1	$-\frac{5}{3}$	6	s_2
$\frac{1}{3}$	$\frac{4}{3}$	1	0	0	$-\frac{1}{3}$	2	x_3
$\frac{11}{3}$	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	-2	

Finally, the maximum value of the revised objective function is $z = -2$, and so the minimum value of the original objective function is

$$w = 2$$

(the negative of the entry in the lower-right corner). This occurs when

$$x_1 = 0, x_2 = 0, \text{ and } x_3 = 2.$$



EXAMPLE 3

A Business Application: Minimum Shipment Costs

An automobile company has two factories. One factory has 400 cars (of a certain model) in stock and the other factory has 300 cars (of the same model) in stock. Two customers order this car model. The first customer needs 200 cars, and the second customer needs 300 cars. The costs of shipping cars from the two factories to the customers are shown in the table.

	<i>Customer 1</i>	<i>Customer 2</i>
<i>Factory 1</i>	\$30	\$25
<i>Factory 2</i>	\$36	\$30

How should the company ship the cars in order to minimize the shipping costs?

SOLUTION

To begin, let x_1 and x_2 represent the numbers of cars shipped from factory 1 to the first and second customers, respectively. (See Figure 9.20.)

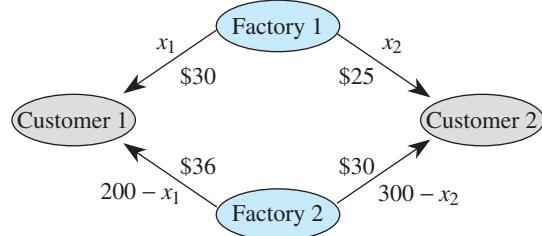


Figure 9.20

The total cost of shipping is

$$C = 30x_1 + 25x_2 + 36(200 - x_1) + 30(300 - x_2) = 16,200 - 6x_1 - 5x_2$$

The constraints for this minimization problem are as follows.

$$x_1 + x_2 \leq 400$$

$x_2) \leq 300$

$$x_1 \leq 200$$

The corresponding maximization problem is to maximize $z = 6x_1 + 5x_2 - 16,200$.

Note that the current z -value is $-16,200$ because the initial solution is

$$(x_1, x_2, s_1, s_2, s_3, s_4) = (0, 0, 400, -200, 200, 300).$$

Now, to this initial tableau, apply the simplex method, as follows.

x_1	x_2	s_1	s_2	s_3	s_4	b	Basic Variables
0	0	1	1	0	0	200	s_1
1	1	0	-1	0	0	200	x_1
0	-1	0	($\frac{1}{2}$)	1	0	0	$s_3 \leftarrow$ Departing
0	1	0	0	0	1	300	s_4
0	1	0	-6	0	0	-15,000	

↑
Entering

x_1	x_2	s_1	s_2	s_3	s_4	b	Basic Variables
0	($\frac{1}{2}$)	1	0	-1	0	200	$s_1 \leftarrow$ Departing
1	0	0	0	1	0	200	x_1
0	-1	0	1	1	0	0	s_2
0	1	0	0	0	1	300	s_4
0	-5	0	0	6	0	-15,000	

↑
Entering

x_1	x_2	s_1	s_2	s_3	s_4	b	Basic Variables
0	1	1	0	-1	0	200	x_2
1	0	0	0	1	0	200	x_1
0	0	1	1	0	0	200	s_2
0	0	-1	0	1	1	100	s_4
0	0	5	0	1	0	-14,000	

From this final simplex tableau, you can see that the minimum shipping cost is \$14,000. Because $x_1 = 200$ and $x_2 = 200$, you can conclude that the numbers of cars that should be shipped from the two factories are as shown in the table.

	<i>Customer 1</i>	<i>Customer 2</i>
<i>Factory 1</i>	200 cars	200 cars
<i>Factory 2</i>	0	100 cars



9.5 Exercises

Slack and Surplus Variables In Exercises 1–6, add the appropriate slack and surplus variables to the system and form the initial simplex tableau.

1. (Maximize)
Objective function:

$$w = 10x_1 + 4x_2$$

Constraints:

$$2x_1 + x_2 \geq 4$$

$$x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

2. (Maximize)
Objective function:

$$w = 3x_1 + x_2 + x_3$$

Constraints:

$$x_1 + 2x_2 + x_3 \leq 10$$

$$x_2 + 5x_3 \geq 6$$

$$4x_1 - x_2 + x_3 \geq 16$$

$$x_1, x_2, x_3 \geq 0$$

3. (Minimize)
Objective function:

$$w = x_1 + x_2$$

Constraints:

$$2x_1 + x_2 \leq 4$$

$$x_1 + 3x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

4. (Minimize)
Objective function:

$$w = 2x_1 + 3x_2$$

Constraints:

$$3x_1 + x_2 \geq 4$$

$$4x_1 + 2x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

5. (Maximize)
Objective function:

$$w = x_1 + x_3$$

Constraints:

$$4x_1 + x_2 \geq 10$$

$$x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + x_2 + 4x_3 \geq 16$$

$$x_1, x_2, x_3 \geq 0$$

6. (Maximize)
Objective function:

$$w = 4x_1 + x_2 + x_3$$

Constraints:

$$2x_1 + x_2 + 4x_3 \leq 60$$

$$x_2 + x_3 \geq 40$$

$$x_1, x_2, x_3 \geq 0$$

Solving a Mixed-Constraint Problem In Exercises 7–12, use the specified entering and departing variables to solve the mixed-constraint problem.

7. (Maximize)
Objective function:

$$w = -x_1 + 2x_2$$

Constraints:

$$x_1 + x_2 \geq 3$$

$$x_1 + x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Entering x_2 , departing s_1

8. (Maximize)
Objective function:

$$w = 2x_1 + x_2$$

Constraints:

$$x_1 + x_2 \geq 4$$

$$x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

Entering x_1 , departing s_1

9. (Minimize)

Objective function:

$$w = x_1 + 2x_2$$

Constraints:

$$2x_1 + 3x_2 \leq 25$$

$$x_1 + 2x_2 \geq 16$$

$$x_1, x_2 \geq 0$$

Entering x_2 , departing s_2

10. (Minimize)

Objective function:

$$w = 3x_1 + 2x_2$$

Constraints:

$$x_1 + x_2 \geq 20$$

$$3x_1 + 4x_2 \leq 70$$

$$x_1, x_2 \geq 0$$

Entering x_1 , departing s_1

11. (Maximize)

Objective function:

$$w = x_1 + x_2$$

Constraints:

$$-4x_1 + 3x_2 + x_3 \leq 40$$

$$-2x_1 + x_2 + x_3 \geq 10$$

$$x_2 + x_3 \leq 20$$

$$x_1, x_2, x_3 \geq 0$$

Entering x_2 , departing s_2

12. (Maximize)

Objective function:

$$w = x_1 + 2x_2 + 2x_3$$

Constraints:

$$x_1 + x_2 \geq 50$$

$$2x_1 + x_2 + x_3 \leq 70$$

$$x_2 + 3x_3 \geq 40$$

$$x_1, x_2, x_3 \geq 0$$

Entering x_2 , departing s_1

Solving a Mixed-Constraint Problem In Exercises 13–18, rework the stated exercise using the given entering and departing variables. Do you obtain the same solution?

13. Exercise 7; entering x_1 , departing s_1

14. Exercise 8; entering x_2 , departing s_1

15. Exercise 9; entering x_1 , departing s_2

16. Exercise 10; entering x_2 , departing s_1

17. Exercise 11; entering x_1 , departing s_2

18. Exercise 12; entering x_1 , departing s_1

Solving a Mixed-Constraint Problem In Exercises 19–26, use the simplex method to solve the problem.

19. (Maximize)

Objective function:

$$w = 2x_1 + 5x_2$$

Constraints:

$$x_1 + 2x_2 \geq 4$$

$$x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

20. (Maximize)

Objective function:

$$w = -x_1 + 3x_2$$

Constraints:

$$2x_1 + x_2 \leq 4$$

$$x_1 + 5x_2 \geq 5$$

$$x_1, x_2 \geq 0$$

21. (Maximize)

Objective function:

$$w = 2x_1 + x_2 + 3x_3$$

Constraints:

$$x_1 + 4x_2 + 2x_3 \leq 85$$

$$x_2 - 5x_3 \geq 20$$

$$3x_1 + 2x_2 + 11x_3 \geq 49$$

$$x_1, x_2, x_3 \geq 0$$

23. (Minimize)

Objective function:

$$w = x_1 + x_2$$

Constraints:

$$x_1 + 2x_2 \geq 25$$

$$2x_1 + 5x_2 \leq 60$$

$$x_1, x_2 \geq 0$$

25. (Minimize)

Objective function:

$$w = -2x_1 + 4x_2 - x_3$$

Constraints:

$$3x_1 - 6x_2 + 4x_3 \leq 30$$

$$2x_1 - 8x_2 + 10x_3 \geq 18$$

$$x_1, x_2, x_3 \geq 0$$

22. (Maximize)

Objective function:

$$w = 3x_1 + 5x_2 + 2x_3$$

Constraints:

$$9x_1 + 4x_2 + x_3 \leq 70$$

$$5x_1 + 2x_2 + x_3 \leq 40$$

$$4x_1 + x_2 \geq 16$$

$$x_1, x_2, x_3 \geq 0$$

24. (Minimize)

Objective function:

$$w = 2x_1 + 3x_2$$

Constraints:

$$3x_1 + 2x_2 \leq 22$$

$$x_1 + x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

26. (Minimize)

Objective function:

$$w = x_1 + x_2 + x_3$$

Constraints:

$$x_1 + 2x_2 + x_3 \geq 30$$

$$6x_2 + x_3 \leq 54$$

$$x_1 + x_2 + 3x_3 \geq 20$$

$$x_1, x_2, x_3 \geq 0$$

Maximizing with Mixed Constraints In Exercises 27–30, maximize the objective function subject to the constraints listed below.

$$x_1 + x_2 \leq 5$$

$$-x_1 + x_2 \leq 3$$

$$x_2 \geq 1$$

$$x_1, x_2 \geq 0$$

$$27. w = 2x_1 + x_2$$

$$28. w = x_1 + 2x_2$$

$$29. w = x_2$$

$$30. w = -x_1 - x_2$$

Maximizing with Mixed Constraints In Exercises 31–34, maximize the objective function subject to the constraints listed below.

$$3x_1 + 2x_2 \geq 6$$

$$x_1 - x_2 \leq 2$$

$$-x_1 + 2x_2 \leq 6$$

$$x_1 \leq 4$$

$$x_1, x_2 \geq 0$$

$$31. w = x_1 + x_2$$

$$32. w = x_1 - 2x_2$$

$$33. w = -4x_1 + x_2$$

$$34. w = 4x_1 - x_2$$

Minimizing Cost In Exercises 35–38, a tire company has two suppliers, S_1 and S_2 . S_1 has 900 tires on hand and S_2 has 800 tires on hand. Customer C_1 needs 500 tires and customer C_2 needs 600 tires. Minimize the cost of filling the orders subject to the data in the table (shipping costs per tire).

	C_1	C_2
S_1	0.60	1.20
S_2	1.00	1.80

	C_1	C_2
S_1	1.20	1.00
S_2	1.00	1.20

	C_1	C_2
S_1	0.80	1.00
S_2	1.00	1.20

	C_1	C_2
S_1	0.80	1.00
S_2	1.00	0.80

39. Minimizing Cost An automobile company has two factories. One factory has 400 cars (of a certain model) in stock and the other factory has 300 cars (of the same model) in stock. Two customers order this car model. The first customer needs 200 cars, and the second customer needs 300 cars. The costs of shipping cars from the two factories to the two customers are shown in the table.

	<i>Customer 1</i>	<i>Customer 2</i>
<i>Factory 1</i>	\$36	\$30
<i>Factory 2</i>	\$30	\$25

- (a) How should the company ship the cars in order to minimize the shipping costs?

- (b) What is the minimum cost?

40. Minimizing Cost Suppose the shipping costs for the two factories in Exercise 39 are as shown in the table below.

	<i>Customer 1</i>	<i>Customer 2</i>
<i>Factory 1</i>	\$25	\$30
<i>Factory 2</i>	\$35	\$30

- (a) How should the company ship the cars in order to minimize the shipping costs?

- (b) What is the minimum cost?

Identifying Feasible Solutions In Exercises 41–43, determine whether the simplex tableau represents a feasible solution. Explain your reasoning.

41.

x_1	x_2	s_1	s_2	b	Basic Variables
5	15	-1	0	60	s_1
2	-2	0	1	8	s_2
25	20	0	0	0	

42.

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
1	2	1	0	0	32	s_1
1	-5	0	1	0	10	s_2
0	2	0	0	-1	-50	s_3
-4	-8	0	0	0	0	

43.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
-10	0	1	-1	0	0	75	s_1
0	0	5	0	-1	0	30	s_2
3	1	12	0	0	1	26	s_3
7	-24	16	0	0	0	0	

44. **CAPSTONE** Consider the following initial simplex tableau.

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
3	-1	8	a	0	0	48	s_1
0	12	4	0	b	0	96	s_2
-2	2	0	0	0	c	5	s_3
-9	-5	-2	0	0	0	0	

- (a) Does this tableau represent a maximization problem or a minimization problem? Explain.
- (b) Find values of a , b , and c such that the solution is feasible.
- (c) Find values of a , b , and c such that the solution is not feasible.
- (d) Let $a = 1$, $b = 1$, and $c = 1$. Which variable should you choose as the departing variable in the first iteration of pivoting? Explain.
- (e) Suppose you perform the simplex method on the tableau above and obtain a final simplex tableau. Does the entry in the lower-right corner represent the optimal value of the original objective function, or its opposite? Explain.

45. **Advertising** A company is determining how to advertise a certain product nationally on television and in a newspaper. Each television ad is expected to be seen by 15 million viewers, and each newspaper ad is expected to be seen by 3 million readers. The company has the following constraints.

- (a) The company has budgeted a maximum of \$600,000 to advertise the product.
- (b) Each minute of television time costs \$60,000 and each one-page newspaper ad costs \$15,000.
- (c) The company's market research department recommends using at least 6 television ads and at least 4 newspaper ads.

How should the advertising budget be allocated to maximize the total audience? What is the maximum audience?

46. Rework Exercise 45 assuming that each one-page newspaper ad costs \$30,000.

Minimizing Cost In Exercises 47 and 48, use the given information. A computer company has two assembly plants, plant A and plant B, and two distribution outlets, outlet I and outlet II. Plant A can assemble 5000 computers in a year and plant B can assemble 4000 computers in a year. Outlet I must have 3000 computers per year and outlet II must have 5000 computers per year. The costs of transportation from each plant to each outlet are shown in the table. Find the shipping schedule that will produce the minimum cost. What is the minimum cost?

	<i>Outlet I</i>	<i>Outlet II</i>
<i>Plant A</i>	\$4	\$5
<i>Plant B</i>	\$5	\$6

	<i>Outlet I</i>	<i>Outlet II</i>
<i>Plant A</i>	\$4	\$5
<i>Plant B</i>	\$6	\$4

True or False? In Exercises 49 and 50, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

- 49. One technique that can be used to change a mixed-constraint minimization problem to a mixed-constraint maximization problem is to multiply each coefficient of the objective function by -1 .
- 50. Surplus variables, like slack variables, are always positive because they represent the amount by which the left side of the inequality is less than the right side.

9 Review Exercises

Graphing a System of Linear Inequalities In Exercises 1–6, sketch a graph of the solution of the system of inequalities.

1. $x + 2y \leq 160$
 $3x + y \leq 180$
 $x \geq 0$
 $y \geq 0$

2. $2x + 3y \leq 24$
 $2x + y \leq 16$
 $x \geq 0$
 $y \geq 0$

3. $3x + 2y \geq 24$
 $x + 2y \geq 12$
 $2 \leq x \leq 15$
 $y \leq 15$

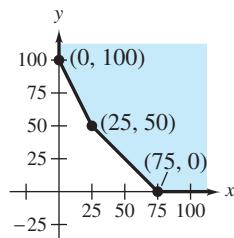
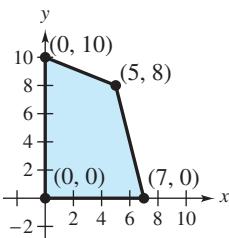
4. $2x + y \geq 16$
 $x + 3y \geq 18$
 $0 \leq x \leq 25$
 $0 \leq y \leq 15$

5. $2x - 3y \geq 0$
 $2x - y \leq 8$
 $y \geq 0$

6. $x - y \leq 10$
 $x \geq 0$
 $y \geq 0$

Using the Graphical Method In Exercises 7–18, find the minimum and/or maximum values of the objective function by the graphical method.

7. Maximize: $z = 3x + 4y$ 8. Minimize: $z = 10x + 7y$

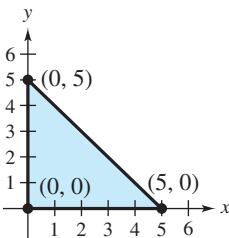


9. Objective function:

$$z = 4x + 3y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 5 \end{aligned}$$

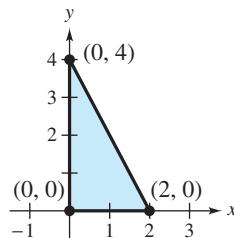


10. Objective function:

$$z = 2x + 8y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x - y &\leq 4 \end{aligned}$$



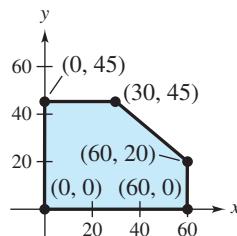
11. Objective function:

$$z = 25x + 30y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 60 \\ y &\leq 45 \end{aligned}$$

$$5x + 6y \leq 420$$

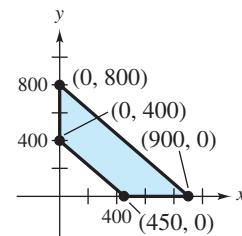


12. Objective function:

$$z = 15x + 20y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 8x + 9y &\leq 7200 \\ 8x + 9y &\geq 3600 \end{aligned}$$

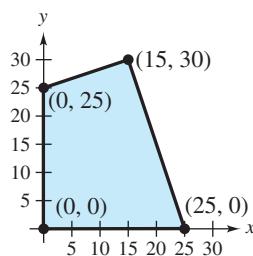


13. Objective function:

$$z = 5x + 0.5y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + 3y &\leq 75 \\ 3x + y &\leq 75 \end{aligned}$$

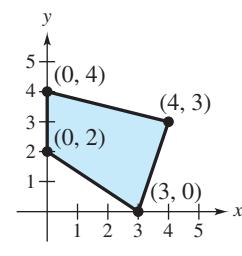


14. Objective function:

$$z = 2x + y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ 2x + 3y &\geq 6 \\ 3x - y &\leq 9 \\ x + 4y &\leq 16 \end{aligned}$$



15. Objective function:

$$z = x + 3y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 5 \\ x + y &\geq 3 \\ x - y &\leq 3 \\ -x + y &\leq 3 \end{aligned}$$

$$3x - y \leq 12$$

16. Objective function:

$$z = 4x - y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 6 \\ x + y &\geq 2 \\ x &\geq y \\ 3x - y &\leq 12 \end{aligned}$$

17. Objective function:

$$z = 3x - y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 3y \\ -x + 2y &\leq 12 \\ 4x + 3y &\leq 40 \\ x + 2y &\leq 15 \end{aligned}$$

18. Objective function:

$$z = x - 2y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 12 \\ 5x + y &\leq 36 \\ 5x - 2y &\geq 4 \\ 2x + 5y &\geq 19 \end{aligned}$$

Using the Simplex Method In Exercises 19–26, use the simplex method to maximize the objective function, subject to the given constraints.

19. Objective function:

$$z = x_1 + 2x_2$$

Constraints:

$$\begin{aligned} 2x_1 + x_2 &\leq 31 \\ x_1 + 4x_2 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

20. Objective function:

$$z = 5x_1 + 4x_2$$

Constraints:

$$\begin{aligned} x_1 - x_2 &\leq 22 \\ x_1 + 2x_2 &\leq 43 \\ x_1, x_2 &\geq 0 \end{aligned}$$

21. Objective function:

$$z = x_1 + 2x_2 + x_3$$

Constraints:

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &\leq 20 \\ x_1 + x_2 - 2x_3 &\leq 23 \\ -2x_1 + x_2 - 2x_3 &\leq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

22. Objective function:

$$z = 4x_1 + 5x_2 + 6x_3$$

Constraints:

$$\begin{aligned} 4x_1 + 2x_2 + x_3 &\leq 30 \\ x_1 + 3x_2 + 2x_3 &\leq 54 \\ x_1 + x_2 + 2x_3 &\leq 24 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

23. Objective function:

$$z = x_1 + x_2$$

Constraints:

$$\begin{aligned} 3x_1 + x_2 &\leq 432 \\ x_1 + 4x_2 &\leq 628 \\ x_1, x_2 &\geq 0 \end{aligned}$$

24. Objective function:

$$z = 6x_1 + 8x_2$$

Constraints:

$$\begin{aligned} 20x_1 + 40x_2 &\leq 200 \\ 30x_1 + 42x_2 &\leq 228 \\ x_1, x_2 &\geq 0 \end{aligned}$$

25. Objective function:

$$z = 3x_1 + 5x_2 + 4x_3$$

Constraints:

$$\begin{aligned} 6x_1 - 2x_2 + 3x_3 &\leq 24 \\ 3x_1 - 3x_2 + 9x_3 &\leq 33 \\ -2x_1 + x_2 - 2x_3 &\leq 25 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

26. Objective function:

$$z = 2x_1 + 5x_2 - x_3$$

Constraints:

$$\begin{aligned} -x_1 + 3x_2 + 2x_3 &\leq 92 \\ -2x_1 + 2x_2 + 12x_3 &\leq 76 \\ 3x_1 + x_2 - 6x_3 &\leq 24 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Finding the Dual In Exercises 27 and 28, determine the dual of the minimization problem.

27. Objective function:

$$w = 7x_1 + 3x_2 + x_3$$

Constraints:

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 30 \\ 3x_1 + 6x_2 + 4x_3 &\geq 75 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

28. Objective function:

$$w = 2x_1 + 3x_2 + 4x_3$$

Constraints:

$$\begin{aligned} x_1 + 5x_2 + 3x_3 &\geq 90 \\ 2x_1 + x_2 + 3x_3 &\geq 60 \\ 3x_1 + 2x_2 + x_3 &\geq 56 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solving a Minimization Problem In Exercises 29–34, solve the minimization problem by solving the dual maximization problem by the simplex method.

29. Objective function:

$$w = 9x_1 + 15x_2$$

Constraints:

$$\begin{aligned} x_1 + 5x_2 &\geq 15 \\ 4x_1 - 10x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

30. Objective function:

$$w = 16x_1 + 18x_2$$

Constraints:

$$\begin{aligned} 2x_1 - 3x_2 &\geq 14 \\ -4x_1 + 9x_2 &\geq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

31. Objective function:

$$w = 24x_1 + 22x_2 + 18x_3$$

Constraints:

$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 &\geq 24 \\ 6x_1 - 2x_3 &\geq 21 \\ -8x_1 - 4x_2 + 8x_3 &\geq 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

32. Objective function:

$$w = 32x_1 + 36x_2 + 4x_3$$

Constraints:

$$\begin{aligned} 4x_1 + 3x_2 - x_3 &\geq 8 \\ -8x_1 + 6x_2 - 6x_3 &\geq 0 \\ -4x_1 + 9x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

33. Objective function:

$$w = 16x_1 + 54x_2 + 48x_3$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\geq 2 \\ 2x_1 + 7x_2 + 4x_3 &\geq 5 \\ x_1 + 3x_2 + 4x_3 &\geq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

34. Objective function:

$$w = 22x_1 + 27x_2 + 18x_3$$

Constraints:

$$\begin{aligned} -2x_1 - 3x_2 + 6x_3 &\geq 0 \\ -2x_1 + 7x_2 + 3x_3 &\geq 4 \\ 2x_1 + x_2 - 3x_3 &\geq 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solving a Mixed-Constraint Problem In Exercises 35–40, use the simplex method to solve the mixed-constraint problem.

35. (Maximize)

Objective function:

$$z = x_1 + 2x_2$$

Constraints:

$$\begin{aligned} -4x_1 + 2x_2 &\leq 26 \\ -3x_1 + x_2 &\geq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

36. (Maximize)

Objective function:

$$z = 2x_1 + 3x_2$$

Constraints:

$$\begin{aligned} -x_1 + x_2 &\geq 40 \\ x_2 &\leq 61 \\ x_1, x_2 &\geq 0 \end{aligned}$$

37. (Maximize)

Objective function:

$$z = 2x_1 + x_2 + x_3$$

Constraints:

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 60 \\ -4x_1 + 2x_2 + x_3 &\geq 52 \\ 2x_1 + x_2 + x_3 &\geq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

38. (Maximize)

Objective function:

$$z = 3x_1 + 2x_2 + x_3$$

Constraints:

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\leq 52 \\ x_1 + x_2 + 2x_3 &\geq 24 \\ 2x_2 + x_3 &\leq 52 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

39. (Minimize)

Objective function:

$$z = 9x_1 + 4x_2 + 10x_3$$

Constraints:

$$\begin{aligned} 32x_1 + 16x_2 + 8x_3 &\leq 344 \\ 20x_1 - 40x_2 + 20x_3 &\geq 200 \\ -45x_1 + 15x_2 + 30x_3 &\leq 525 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

40. (Minimize)

Objective function:

$$z = 4x_1 - 2x_2 - x_3$$

Constraints:

$$\begin{aligned} 2x_1 - x_2 - x_3 &\leq 41 \\ x_1 - 2x_2 - x_3 &\geq 10 \\ -x_1 - 7x_2 + 5x_3 &\leq 11 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Graphing a System of Inequalities In Exercises 41 and 42, determine a system of inequalities that models the description, and sketch a graph of the solution of the system.

41. A Pennsylvania fruit grower has 1500 bushels of apples that are to be divided between markets in Harrisburg and Philadelphia. These two markets need at least 400 bushels and 600 bushels, respectively.

42. A warehouse operator has 24,000 square meters of floor space in which to store two products. Each unit of product I requires 20 square meters of floor space and costs \$12 per day to store. Each unit of product II requires 30 square meters of floor space and costs \$8 per day to store. The total storage cost per day cannot exceed \$12,400.

43. Optimal Revenue A tailor has 12 square feet of cotton, 21 square feet of silk, and 11 square feet of wool. A vest requires 1 square foot of cotton, 2 square feet of silk, and 3 square feet of wool. A purse requires 2 square feet of cotton, 1 square foot of silk, and 1 square foot of wool. If the purse sells for \$80 and the vest sells for \$50, how many purses and vests should be made to maximize the tailor's profit? What is the maximum revenue?

44. Optimal Income A traditional wood carpentry workshop has 400 board-feet of plywood, 487 board-feet of birch, and 795 board-feet of pine. A wooden bar stool requires 1 board-foot of plywood, 2 board-feet of birch, and 1 board-foot of pine. A wooden step stool requires 1 board-foot of plywood, 1 board-foot of birch, and 3 board-feet of pine. A wooden ottoman requires 2 board-feet of plywood, 1 board-foot of birch, and 1 board-foot of pine. If the bar stool sells for \$22, the step stool sells for \$42, and the ottoman sells for \$29, what combination of products would yield the maximum gross income?

Optimal Cost In Exercises 45–48, an athlete uses two dietary supplement drinks that provide the nutritional elements shown in the table.

Drink	Protein	Carbohydrates	Vitamin D
I	4	2	1
II	1	5	1

Find the combination of drinks of minimum cost that will meet the minimum requirements of 4 units of protein, 10 units of carbohydrates, and 3 units of vitamin D.

- 45.** Drink I costs \$5 per liter and drink II costs \$8 per liter.
46. Drink I costs \$7 per liter and drink II costs \$4 per liter.
47. Drink I costs \$1 per liter and drink II costs \$5 per liter.
48. Drink I costs \$8 per liter and drink II costs \$1 per liter.

49. Optimal Cost A company owns three mines that have the daily production levels (in metric tons) shown in the table.

Mine	Grade of Ore		
	High	Medium	Low
A	1	2	3
B	1	2	2
C	2	1	1

The company needs 60 metric tons of high-grade ore, 48 metric tons of medium-grade ore, and 55 metric tons of low-grade ore. How many days should each mine be operated in order to minimize the cost of meeting these requirements when the daily operating costs are \$200 for mine A, \$200 for mine B, and \$100 for mine C, and what would be the minimum total cost?

50. Optimal Cost Rework Exercise 49 using the daily production schedule shown in the table.

Mine	Grade of Ore		
	High	Medium	Low
A	2	1	2
B	1	1	1
C	4	2	1

The company needs 190 metric tons of high-grade ore, 120 metric tons of medium-grade ore, and 200 tons of low-grade ore. The daily operating costs are \$200 for mine A, \$150 for mine B, and \$125 for mine C.

9 Project



Cholesterol in human blood is necessary, but too much can lead to health problems. There are three main types of cholesterol: HDL (high-density lipoproteins), LDL (low-density lipoproteins), and VLDL (very low-density lipoproteins). HDL is considered “good” cholesterol; LDL and VLDL are considered “bad” cholesterol.

A standard fasting cholesterol blood test measures total cholesterol, HDL cholesterol, and triglycerides. These numbers are used to estimate LDL and VLDL, which are difficult to measure directly. It is recommended that your combined LDL/VLDL cholesterol level be less than 130 milligrams per deciliter, your HDL cholesterol level be at least 40 milligrams per deciliter, and your total cholesterol level be no more than 200 milligrams per deciliter. (Source: American Heart Association)

Write a system of linear inequalities for the recommended cholesterol levels. Let x represent the HDL cholesterol level, and let y represent the combined LDL/VLDL cholesterol level.

Graph the system of linear inequalities. Find and label all vertices.

Part I

Determine whether the cholesterol levels listed below are within recommendations. Explain your reasoning.

LDL/VLDL: 120 milligrams per deciliter

HDL: 90 milligrams per deciliter

Total: 210 milligrams per deciliter

Give an example of cholesterol levels in which the LDL/VLDL cholesterol level is too high but the HDL cholesterol level is acceptable.

Another recommendation is that the ratio of total cholesterol to HDL cholesterol be less than 4 (that is, less than 4 to 1). Find a point in the solution region of your system of inequalities that meets this recommendation. Explain why it meets the recommendation.

Part II

Ask a friend or relative what his or her cholesterol levels are (or use your own). See if the levels are within each acceptable range. How far below or above the accepted levels are the results? Compare your results with those of other students in the class.