Riemann-Liouville Fractional Derivatives and the Taylor-Riemann Series

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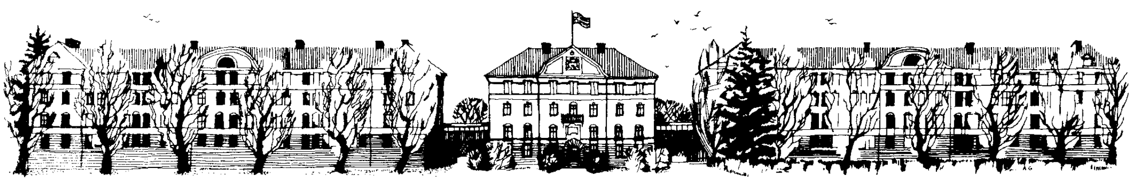

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ABSTRACT. In this paper we give some background theory on the concept of fractional calculus, in particular the Riemann-Liouville operators. We then investigate the Taylor-Riemann series using Osler's theorem and obtain certain double infinite series expansions of some elementary functions. In the process of this we give a proof of the convergence of an alternative form of Heaviside's series. A Semi-Taylor series is introduced as the special case of the Taylor-Riemann series when $\alpha = 1/2$, and some of its relations to special functions are obtained via certain generating functions arising in complex fractional calculus.

1. INTRODUCTION

1.1. **Riemann-Liouville operator.** The concept of non-integral order of integration can be traced back the to the genesis of differential calculus itself: the philosopher and creator of modern calculus G.W.Leibniz made some remarks on the the meaning and possibility of fractional derivative of order 1/2 in the late 17:th century. However a rigorous investigation was first carried out by Liouville in a series of papers from 1832-1837, where he defined the first outcast of an operator of fractional integration. Later investigations and further developments by among others Riemann led to the construction of the integral-based Riemann-Liouville fractional integral operator, which has been a valuable cornerstone in fractional calculus ever since. Prior to Liouville and Riemann, Euler took the first step in the study of fractional integration when he studied the simple case of fractional integrals of monomials of arbitrary real order in the heuristic fashion of the time; it has been said to have lead him to construct the Gamma function for fractional powers of the factorial [2, p. 243]. An early attempt by Liouville was later purified by the Swedish mathematician Holmgren [10], who in 1865 made important contributions to the growing study of fractional calculus. But it was Riemann [4] who reconstructed it to fit Abel's integral equation, and thus made it vastly more useful. Today there exist many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types [1, p. xxxi], but the Riemann-Liouville Operator is still the most frequently used when fractional integration is performed.

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Riemann's modified form of Liouville's fractional integral operator is a direct generalization of Cauchy's formula for an n-fold integral [1, p. 33],

$$\int_a^x dx_1 \int_a^{x_1} dx_2 ... \int_a^{x_{n-1}} f(x_n)dx_n = \frac{1}{(n-1)!}\int_a^x \frac{f(t)}{(x-t)^{1-n}}\,dt, \tag{1}$$

and since $(n-1)! = \Gamma(n)$, Riemann realized that the RHS of (1) might have meaning even when n takes non-integer values. Thus perhaps it was natural to define fractional integration as follows.

Definition 1. *If $f(x) \in C([a,b])$ and $a < x < b$ then*

$$I^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt \tag{2}$$

where $\alpha \in]-\infty,\infty[$, is called the Riemann-Liouville fractional integral of order α. In the same fashion for $\alpha \in]0,1[$ we let

$$D^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha}}\,dt, \tag{3}$$

which is called the Riemann-Liouville fractional derivative of order α.

(It follows from our discussion below Definition 3 that if $0 < \alpha < 1$ then $D^{\alpha}_{a+}f(x)$ exists for all $f \in C^1([a,b])$ and all $x \in]a,b]$.)

These operators are called the Riemann-Liouville fractional integral operators, or simply R-L operators. The special case of the fractional derivative when $\alpha = \frac{1}{2}$ is called the semi-derivative. The connection between the Riemann-Liouville fractional integral and derivative can, as Riemann realized, be traced back to the solvability of Abel's integral equation for any $\alpha \in]0,1[$

$$f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\phi(t)}{(x-t)^{1-\alpha}}\,dt \qquad ,x > 0. \tag{4}$$

Formally equation (4) can be solved by changing x to t and t to s respectively, further by multiplying both sides of the equation by $(x-t)^{-\alpha}$ and integrating we get

$$\int_a^x \frac{dt}{(x-t)^{\alpha}}\int_a^t \frac{\phi(s)ds}{(t-s)^{1-\alpha}} = \Gamma(\alpha)\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}. \tag{5}$$

Interchanging the order of integration in the left hand side by Fubini's theorem we obtain

$$\int_a^x \phi(s)ds \int_s^x \frac{dt}{(x-t)^{\alpha}(t-s)^{1-\alpha}} = \Gamma(\alpha)\int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}. \tag{6}$$

The inner integral is easily evaluated after the change of variable $t = s + \tau(x-s)$ and use of the formulae of the Beta-function (see section 2.1):

$$\int_a^x (x-t)^{-\alpha}(t-s)^{\alpha-1}\,dt = \int_0^1 \tau^{\alpha-1}(1-\tau)^{-\alpha}\,d\tau = B(\alpha,1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha). \tag{7}$$

Therefore we get

$$\int_a^x \phi(s)ds = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^\alpha}. \tag{8}$$

Hence after differentiation we have

$$\phi(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)dt}{(x-t)^\alpha}. \tag{9}$$

Thus if (4) has a solution it is necessarily given by (9) for any $\alpha \in]0,1[$. One observes that (4) is in a sense the α-order integral and the inversion (9) is the α-order derivative.

A very useful fact about the R-L operators is that they satisfy the following important *semi-group property* of fractional integrals.

Theorem 2. *For any $f \in C([a,b])$ the Riemann-Liouville fractional integral satisfies*

$$I_{a+}^\alpha I_{a+}^\beta f(x) = I_{a+}^{\alpha+\beta} f(x) \tag{10}$$

for $\alpha > 0, \beta > 0$.

Proof. The proof is rather direct, we have by definition:

$$I_{a+}^\alpha I_{a+}^\beta f(x) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{dt}{(x-t)^{1-\alpha}} \int_a^t \frac{f(u)}{(t-u)^{1-\beta}}\, du, \tag{11}$$

and since $f(x) \in C([a,b])$ we can by Fubini's theorem interchange order of integration and by setting $t = u + s(x-u)$ we obtain

$$I_{a+}^\alpha I_{a+}^\beta f(x) = \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \frac{f(u)}{(x-u)^{1-\alpha-\beta}}\, du = I_{a+}^{\alpha+\beta} f(x). \tag{12}$$

□

The Riemann-Liouville fractional operators may in many cases be extended to hold for a larger set of α, and a rather technical detail is that we denote $\alpha = [\alpha] + \{\alpha\}$, where $[\alpha]$ denotes the integer part of α, and $\{\alpha\}$ denotes the remainder. This notation is used for convenience, observe the following definition.

Definition 3. *If $\alpha > 0$ is not an integer then we define*

$$D_{a+}^\alpha f = \frac{d^{[\alpha]}}{dx^{[\alpha]}} D_{a+}^{\{\alpha\}} f = \frac{d^{[\alpha]+1}}{dx^{[\alpha]+1}} I_{a+}^{1-\{\alpha\}} f, \tag{13}$$

thus

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \tag{14}$$

for any $f \in C^{[\alpha]+1}([a,b])$ if $n = [\alpha]+1$. If on the other hand $\alpha < 0$ then the notation

$$D_{a+}^\alpha f = I_{a+}^{-\alpha} f, \tag{15}$$

may be used as definition.

Clearly, if $\alpha < 0$ then the fractional derivative $D^{\alpha}_{a+}f(x)$ exists for all $f \in C([a,b])$ and all $x \in [a,b]$. We also remark that for $\alpha > 0$, the fractional derivative $D^{\alpha}_{a+}f(x)$ certainly exists for all $f \in C^{[\alpha]+1}([a,b])$ and all $x \in]a,b]$ (but *not* necessarily for $x = a$). To see this, write $n = [\alpha]+1$ and apply Taylor's formula with remainder:

$$f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k + \frac{1}{(n-1)!}\int_a^t \frac{f^{(n)}(s)}{(t-s)^{n-1}}\,ds, \qquad \forall t \in [a,b].$$

Substituting this into the definition of $D^{\alpha}_{a+}f(x)$ and simplifying the integrals we obtain

$$D^{\alpha}_{a+}f(x) = \frac{d^n}{dx^n}\left(\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k+2-\{\alpha\})}\cdot(x-a)^{k+1-\{\alpha\}} + \frac{1}{\Gamma(n+1-\{\alpha\})}\int_a^x f^{(n)}(s)\cdot(x-s)^{n-\{\alpha\}}\,ds\right). \tag{16}$$

Clearly, this n-fold derivative can be carried out for all $x \in]a,b]$; in particular the integral is unproblematic since $f^{(n)} \in C([a,b])$ and since the exponent $n-\{\alpha\}$ is larger than $n-1$, so that $\frac{d^k}{dx^k}(x-s)^{n-\{\alpha\}}$ is integrable for all $k = 0,1,...,n$. This proves our claim.

For convenience in the later theorems we define the following useful space.

Definition 4. *For $\alpha > 0$ let $I^{\alpha}_{a+}([a,b])$ denote the space of functions which can be represented by an R-L-integral of order α of some $C([a,b])$-function.*

This gives rise to the following manifesting theorem

Theorem 5. *Let $f \in C([a,b])$ and $\alpha > 0$. In order that $f(x) \in I^{\alpha}_{a+}([a,b])$, it is necessary and sufficient that*

$$I^{n-\alpha}_{a+}f \in C^n([a,b]), \tag{17}$$

where $n = [\alpha]+1$, and that

$$\left(\frac{d^k}{dx^k} I^{n-\alpha}_{a+}f(x)\right)_{|x=a} = 0, \qquad k = 0,1,2,...,n-1. \tag{18}$$

Proof. First assume $f(x) \in I^{\alpha}_{a+}([a,b])$; then $f(x) = I^{\alpha}_{a+}g(x)$ for some $g \in C([a,b])$. Hence by the semi-group property (Theorem 2) we have

$$I^{n-\alpha}_{a+}f(x) = I^{n-\alpha}_{a+}I^{\alpha}_{a+}g(x) = I^{n}_{a+}g(x) = \frac{1}{(n-1)!}\int_a^x \frac{g(t)}{(x-t)^{1-n}}\,dt$$
$$= \int_a^x dx_1 \int_a^{x_1} dx_2 ... \int_a^{x_{n-1}} g(x_n)dx_n$$

(cf. (1)). This implies that (17) holds, and by repeated differentiation we also see that (18) holds.

Conversely, assume that $f \in C([a,b])$ satisfies (17) and (18). Then by Taylor's formula applied to the function $I_{a+}^{n-\alpha}f$, we have

$$I_{a+}^{n-\alpha}f(t) = \int_a^t \frac{d^n}{ds^n} I_{a+}^{n-\alpha}f(s) \cdot \frac{(t-s)^{n-1}}{(n-1)!}\, ds \qquad \forall t \in [a,b].$$

Let us write $\varphi(t) = \frac{d^n}{dt^n} I_{a+}^{n-\alpha}f(t)$; then note that $\varphi \in C([a,b])$ by (17). Now, by Definition 1 and the semi-group property (Theorem 2) the above relation implies

$$I_{a+}^{n-\alpha}f(t) = I_{a+}^{n}\varphi(t) = I_{a+}^{n-\alpha}I_{a+}^{\alpha}\varphi(t),$$

and thus

$$I_{a+}^{n-\alpha}\Big(f - I_{a+}^{\alpha}\varphi\Big) \equiv 0.$$

By a general fact about uniqueness of any solution to Abel's integral equation (cf. [1, Lemma 2.5]), and note that we have $n-\alpha > 0$), this implies $f \equiv I_{a+}^{\alpha}\varphi$, and thus $f \in I_{a+}^{\alpha}([a,b])$. □

Theorem 6. *If $\alpha > 0$ then the equality*

$$D_{a+}^{\alpha}I_{a+}^{\alpha}f = f(x) \tag{19}$$

holds for any $f \in C([a,b])$. Now let $f \in C^{[\alpha]+1}([a,b])$; then for the equality

$$I_{a+}^{\alpha}D_{a+}^{\alpha}f(x) = f(x) \tag{20}$$

to hold we need to assume that f satisfies the condition in Theorem 5; otherwise

$$I_{a+}^{\alpha}D_{a+}^{\alpha}f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \frac{d^{n-k-1}}{dx^{n-k-1}} \left(I_{a+}^{n-\alpha}f(x)\right) \tag{21}$$

holds.

Proof. By definition we have

$$D_{a+}^{\alpha}I_{a+}^{\alpha}f = \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{dt}{(x-t)^{\alpha-n+1}} \int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}}. \tag{22}$$

Since the integrals are absolutely convergent we deploy Fubini's theorem and interchange the order of integration and after evaluating the inner integral we obtain

$$D_{a+}^{\alpha}I_{a+}^{\alpha}f = \frac{1}{\Gamma(n)} \frac{d^n}{dx^n} \int_a^x \frac{f(s)}{(x-s)^{n-1}} ds. \tag{23}$$

Then (19) follows from (23) by Cauchy's formula (1). Since f in (20) satisfies the conditions in Theorem 5 and $f \in C^{[\alpha]+1}([a,b])$ it follows immediately by (18) that (19) will hold (because the residue terms of integration will vanish). If on the other hand any function $f \in C^{[\alpha]+1}([a,b])$ does not satisfy the condition (18) given in Theorem 5 the residue terms outside the integral will not disappear like in (19), but as integration is deployed (21) is obtained by induction.

□

Perhaps the second part of Theorem 6 is somewhat surprising, and this gives rise to the following interesting corollary.

Corollary 7. *Let $\alpha > 0$, $n \in \mathbb{Z}^+$ and $f(x) \in C^{[\alpha]+n+1}([a,b])$. Then*

$$f(x) = \sum_{k=-n}^{n-1} \frac{D_{a+}^{\alpha+k} f(x_0)}{\Gamma(\alpha+k+1)} (x-x_0)^{\alpha+k} + R_n(x), \tag{24}$$

for all $a \leq x_0 < x \leq b$, where

$$R_n(x) = I_{a+}^{\alpha+n} D_{a+}^{\alpha+n} f(x) \tag{25}$$

is the remainder.

One obtains (24) by deploying I_{a+}^{α} to $D_{a+}^{\alpha} f$ in (16) and rearrange some. Heuristically when letting n and m tend to infinity, and if f is a sufficiently good function one obtains the Taylor-Riemann expansion which is a fractional generalization of Taylor's theorem, we will return to this in section 2. The concept of studying the R-L operator for $\alpha \geq 1$ leads us to the following useful theorem

An interesting property of the R-L operators is that certain non-differentiable functions such as Weierstrass-function and Riemann- function seem to have fractional derivative of all orders]0,1[, see [8] and [7] for investigations on non-differentiability and its relation to fractional calculus. This adds to the problem that the relation between the ordinary derivative and the fractional derivative is not entirely obvious, but the following theorem might give a picture on some of their covariance.

Theorem 8. *If $f \in C^1([a,b])$, $f(a) \geq 0$ and $\alpha \in]0,1[$, then $D_{a+}^{\alpha} f(x)$ is non-negative if f is increasing on $[a,x]$.*

Proof. Since $f \in C^1([a,b])$ we can deploy (16) if one lets $n = [\alpha]+1 = 1$, then it reduces to two terms and appears like:

$$D_{a+}^{\alpha} f(x) = \frac{f(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_a^x f'(s)(x-s)^{-\alpha}\, ds. \tag{26}$$

Since $\Gamma(1-\alpha) > 0$ for all $\alpha \in]0,1[$ and $x > a$ and then since the fact that $f(a) \geq 0$ was given, we conclude that the first term is non-negative. This leaves us to prove that the integral in the second term is non-negative. Observe that $f'(s) \geq 0$ on $[a,x]$ since f was increasing. Further $(x-s)^{-\alpha} > 0$ for $s \in [a,x]$ which implies that the integral is non-negative. This completes the proof. □

We give below a table of fractional integrals with a chosen to fit (18) (Additional fractional integrals may be found in [1, p. 173-174].)

$f(x)$	$I^{\alpha}_{a+}f(x)$	specifications
$(x-a)^{\beta}$	$\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta}$	$a \in \mathbb{C}$, $\alpha > 0$, $Re(\beta) > -1$
c	$\frac{c}{\Gamma(\alpha+1)}(x-a)^{\alpha}$	$a \in \mathbb{R}$, $\alpha \in \mathbb{R}$
e^{bx}	$b^{-\alpha}e^{bx}$	$a = -\infty$, $\alpha > 0$, $Re(b) > 0$

Here c is an arbitrary constant. Observe the remarkable fact that the fractional derivative of a constant is not constant.

1.2. **Fractional calculus in the complex plane.** As a direct generalization of the Riemann-Liouville fractional integral and derivative for analytic functions we obtain the following definition:

Definition 9. *For any function f analytic on some open simply-connected domain containing the points z and z_0 we define for any $\alpha > 0$:*

$$I^{\alpha}_{z_0}f(z) = \frac{1}{\Gamma(\alpha)}\int_{z_0}^{z}\frac{f(t)dt}{(z-t)^{1-\alpha}}, \tag{27}$$

$$D^{\alpha}_{z_0}f(z) = \frac{d^m}{dz^m}I^{m-\alpha}_{z_0}f(z)\,, m = [\alpha]+1, \tag{28}$$

with integration along the straight line connecting z and z_0. We shall call (27) *the fractional integral of the function $f(z)$ of order α. Like in section 1.1 we shall use the definition $D^{\alpha}_{z_0}f = I^{-\alpha}_{z_0}f$ for $\alpha < 0$.*

We fix the point z and select the principal value of the function $(z-t)^{1-\alpha}$ to interpret the integral (27) uniquely. We choose:

$$0 \leq arg(z-z_0) < 2\pi, \tag{29}$$

in order to coincide with $arg(z-t) = arg(z-z_0)$. Thus we obtain:

$$(z-t)^{1-\alpha} = |z-t|^{1-\alpha}e^{i(1-\alpha)arg(z-z_0)}, \tag{30}$$

thus (27) is uniquely defined. Further the semi-group property of fractional integrals (Theorem 2) holds for (27) for any $\alpha > 0\,, \beta > 0$, the proof is equivalent that of Theorem 2 (cf. [1, p. 417]).

In addition to the complex fractional integral, fractional differentiation by using contour integration in the complex plane has become a valued and frequently used concept in fractional calculus (see [2, p. 250] and [1, p. 414]). We have the familiar Cauchy integral formula

$$D^nF(z) = \frac{n!}{2\pi i}\oint\frac{F(t)}{(t-z)^{n+1}}\,dt \tag{31}$$

for arbitrary positive integral values n. In similar fashion to Riemann-Liouville fractional integral, we observe the consequences as n takes real value. Heuristically $n!$ will be replaced by $\Gamma(-\alpha+1)$, but replacing $(t-z)^{-n-1}$ by $(t-z)^{-\alpha-1}$ is more complicated. Observe that $(t-z)^{-n-1}$ has an isolated singularity at $t = z_0$ while $(t-z)^{-\alpha-1}$ has a branch point at $t = z$. Let the branch line start at $t = z$ and pass through the fixed point z_0, further let the contour of integration be $C(z_0, z^+)$ (see fig.1), and let it be the contour which starts at $t = z_0$, encircles $t = z$ once in the positive sense, and returns to $t = z_0$. Note that the integrand in general assumes *different* values at the start point and the end point of $C(z_0, z^+)$.

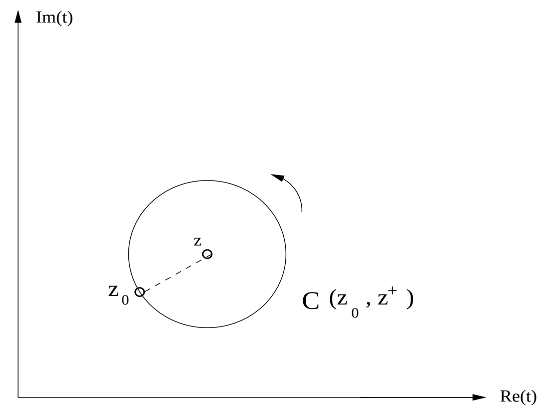

Figure 1.

To make a precise choice, we specify that

$$-\pi \leq \arg(t - z) < \pi \qquad \text{at the } \textit{start point} \text{ of } C(z_0, z^+) \tag{32}$$

(and thus $\pi \leq \arg(t - z) < 3\pi$ at the end point of $C(z_0, z^+)$). We define

Definition 10. *For any function $F(z)$ analytic in some open, simply connected region containing the contour $C(z_0, z^+)$,*

$$D^{\alpha}_{z-z_0} F(z) = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{C(z_0, z^+)} F(t)(t - z)^{-\alpha - 1}\, dt \tag{33}$$

is called the Cauchy-type fractional derivative of order α.

Here follows a theorem connecting the Cauchy-type fractional derivative (33) and the complex generalization of the Riemann-Liouville fractional derivative (28):

Theorem 11. *Let $f(z) = (z - z_0)^{\sigma} g(z)$, where $\sigma > -1$ and $g(z)$ is analytic in some simply connected domain G containing z and z_0, and we choose the principal value* (29) *of $(z - z_0)^{\sigma}$. Then the Cauchy-type fractional integral* (33) *of f equals the complex generalization of Riemann-Liouville fractional derivative:*

$$D^{\alpha}_{z-z_0} f(z) = D^{\alpha}_{z_0} f(z),$$

for all $\alpha \in \mathbb{R}$, except for $\alpha \in \mathbb{Z}$.

Proof. We begin with $\alpha < 0$ (and $\alpha \notin \{-1, -2, ...\}$). Since the integrand of (33) is analytic in the domain G we may deform $C(z_0, z^+)$ into the union of three contours, where C_1 is a straight linesegment from z_0 almost to z, C_2 is a small circle centered at $t = z$, C_3 is C_1 transversed in the opposite direction (see Figure 2). By the Cauchy integral theorem we have

$$D^{\alpha}_{z-z_0}f(z) = \frac{\Gamma(1+\alpha)}{2\pi i}\int_{C(z_0,z^+)}\frac{f(t)dt}{(t-z)^{1+\alpha}} = \frac{\Gamma(1+\alpha)}{2\pi i}\left(\int_{C_1}...\,dt+\int_{C_2}...\,dt+\int_{C_3}...\,dt\right). \tag{34}$$

So by the jump of $(t-z)^{-1-\alpha}$ at the cut we have

$$\frac{\Gamma(1+\alpha)}{2\pi i}\int_{C(z_0,z^+)}\frac{f(t)dt}{(t-z)^{1+\alpha}} = \frac{\Gamma(1+\alpha)}{2\pi i}(1-e^{-2\alpha\pi i})\int_{z_0}^{z}\frac{f(t)dt}{(t-z)^{1+\alpha}} + \frac{\Gamma(1+\alpha)}{2\pi i}\lim_{\epsilon\to 0}\epsilon^{-\alpha}\int_0^{2\pi}f(z+\epsilon e^{i\phi})e^{-i\alpha\phi}\,d\phi. \tag{35}$$

Note that the last term equals 0.

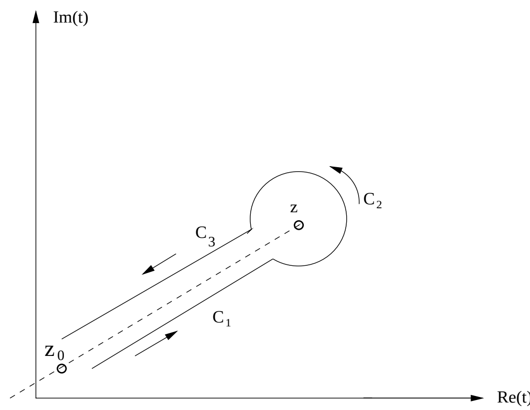

Figure 2.

With condition (29) and (32) we see that $(t-z)^{-1-\alpha} = e^{(1+\alpha)\pi i}(z-t)^{-1-\alpha}$, where $(z-t)^{-1-\alpha}$ coincides with the value chosen in (30) with $1-\alpha$ being replaced by $-1-\alpha$. Since $((e^{i\pi\alpha}-e^{-i\pi\alpha})/2i)\Gamma(\alpha+1) = sin(\pi\alpha)\Gamma(\alpha+1) = -\pi/\Gamma(-\alpha)$, we by (35) obtain

$$\frac{\Gamma(1+\alpha)}{2\pi i}\int_{C(z_0,z^+)}\frac{f(t)dt}{(t-z)^{1+\alpha}} = I^{-\alpha}_{z_0}f(z) \tag{36}$$

we see that (33) equals (27) when $\alpha<0$ except for $\alpha = -1,-2,-3...$.

For the case $\alpha > 0$ we use the complex fractional derivative (28) and end up with (33) after differentiating (36), and by replacing α with $\alpha - \{\alpha\} - 1$ the corresponding number of times under the integral sign. This holds because the contour of integration need only be slightly perturbed so staring point and ending point will be z_0, thus when deploying d^n/dz^n

one only differentiates the integrand with respect to z. This holds for $\alpha \neq 0, 1, 2, ...$, thus this proves our theorem. $\square$

The fractional derivatives are depending on z_0 in a dramatic way, we here give some formulas that arise as z_0 is specifically chosen. Osler [5, p. 69-77] gives some generating functions involving fractional derivatives (except for the incomplete gamma expression, which can be found in [2]):

Name	Derivative represenation
Bessel function	$J_\alpha(z) = (2z)^{-\alpha}\pi^{-\frac{1}{2}}\left(D_0^{-\frac{1}{2}-\alpha}\frac{\cos(\sqrt{w})}{\sqrt{w}}\right)_{\mid w=z^2}$
Struve function	$H_\alpha(z) = (2z)^{-\alpha}\pi^{-\frac{1}{2}}\left(D_0^{-\frac{1}{2}-\alpha}\frac{\sin(\sqrt{w})}{\sqrt{w}}\right)_{\mid w=z^2}$
Incomplete Gamma function	$\gamma(\alpha, z) = \Gamma(\alpha)e^{-z}D_z^{-\alpha}e^z$

Here γ is Euler's constant, and these expressions hold for all $\alpha \in \mathbb{R}, \alpha \neq -1, -2, -3...$. We will return to this in section 3.2.

2. Taylor-Riemann series and related entities

2.1. **The Gamma, incomplete Gamma and Beta functions.** The Gamma function is defined by (for somewhat a comprehensive reading, see [3]):

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\,dt, \qquad \text{for } \operatorname{Re} z > 0. \tag{37}$$

By integration by parts one checks that for all z with $\operatorname{Re}(z) > 0$ we have

$$\Gamma(z+1) = z\Gamma(z). \tag{38}$$

In particular, using $\Gamma(1) = 1$ and (38) we find by induction that $\Gamma(n+1) = n!$ for all non-negative integers n. The relation (38) is used recursively to define $\Gamma(z)$ as a meromorphic function for *all* $z \in \mathbb{C}$. Note that this function is analytic for all z except for simple poles at $z \in \{0, -1, -2, -3, ...\}$.

Theorem 12. *For any $n = 1, 2, 3, ...$ the following holds for the Gamma function*

$$\Gamma(n + \frac{1}{2}) = \frac{(2n-1)!!\sqrt{\pi}}{2^n} \tag{39}$$

,

$$\Gamma(-n + \frac{1}{2}) = \frac{(-1)^n 2^n\sqrt{\pi}}{(2n-1)!!}, \tag{40}$$

where n!! is the double-factorial.

Proof. Clearly, using the relation (38) the theorem follows easily by induction once we have proved that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. To prove this last fact we substitute $t = x^2$ in (37),

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}\,\frac{dt}{\sqrt{t}} = 2\int_0^\infty e^{-x^2}\,dx.$$

This implies that

$$\Gamma(\frac{1}{2})^2 = 4\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy.$$

Hence, using polar coordinates ($x = r\cos\phi$, $y = r\sin\phi$), we obtain

$$\Gamma(\frac{1}{2})^2 = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2} r\,d\phi\,dr = 2\pi\int_0^\infty e^{-r^2} r\,dr = \pi\int_0^\infty e^{-u}\,du = \pi.$$

Hence $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and the proof is complete. $\square$

The Gamma function can for practical reasons when $z \in \mathbb{R}$ and $z > 0$ be divided into two functions on the form:

$$\Gamma(\alpha)Q(\alpha, z) + \gamma(\alpha, z) = \Gamma(\alpha), \tag{41}$$

where $\gamma(\alpha, z)$ is called the incomplete Gamma function and $Q(\alpha, z)$ is called the complementary incomplete Gamma function. We let

$$Q(\alpha, z) = \frac{1}{\Gamma(\alpha)}\int_z^\infty e^{-t}t^{\alpha-1}, \tag{42}$$

it holds for all $\alpha \in \mathbb{R}$, since the poles of the integral will be removable by the $\frac{1}{\Gamma(\alpha)}$ factor, and naturally by (41) we get that if $\alpha > 0$ the incomplete complementary Gamma function as follows [15]:

$$\gamma(\alpha, z) = \int_0^z t^{\alpha-1}e^{-t}\,dt, \tag{43}$$

which holds for all $\alpha > 0$

The Beta function $B(z, w)$, is defined by the Euler integral of the first kind (see [1, p. 17]):

$$B(z, w) = \int_0^1 x^{z-1}(1-x)^{w-1}\,dx,\ Re(z) > 0,\ Re(w) > 0. \tag{44}$$

Its connection to the Gamma function is given by the relation

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{45}$$

2.2. **Bessel and related special functions.** The Bessel function $J_\alpha(z)$ arises as solution to the differential equation [11, p. 357-372].

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + (1 - \frac{\alpha^2}{z^2})y = 0, \tag{46}$$

but a convenient form is given by

$$J_\alpha(z) = \sum_{n=0}^\infty \frac{(-1)^n}{2^{\alpha+2n}n!\Gamma(\alpha+n+1)}(\frac{z}{2})^{\alpha+2n}, \tag{47}$$

which is entire (see [11]). A nice property of the Bessel function is given by [14]

$$J_{-n}(x) = (-1)^n J_n(x). \tag{48}$$

If (46) is non-homogenus and has a RHS given by $\frac{2}{\pi}\frac{2^{n+1}}{(2n-1)!!}$, the solution is given by :

$$H_\alpha(z) = (\frac{1}{2}z)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{1}{2}z)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\alpha+\frac{3}{2})}, \tag{49}$$

which is called the Struve function [11].

2.3. **Total fluctuation and Fourier's theorem.** In the proof of the Taylor-Riemann series we need a strengthend version of Fourier's theorem. Thus we define it here (cf. [11]):

Definition 13. *Let $f(x)$ be a real-valued function, defined when $a \leq x \leq b$. Further let*

$$a \leq x_1 \leq x_2 \leq ... \leq x_n \leq b. \tag{50}$$

Then $|f(a)-f(x_1)|+|f(x_1)-f(x_2)|+...+|f(x_n)-f(b)|$ is called the fluctuation *of $f(x)$ in the range (a,b) for the set of subdivisions $x_1, x_2, ..., x_n$. If the fluctuation has an upper bound independent of n, for all choices of $x_1, x_2, ..., x_n$, then $f(x)$ is said to have a* limited total fluctuation *in the range (a, b).*

One can prove that every function $f(x)$ which has limited total fluctuation in the range (a, b) can be expressed as a sum $f = f_1 + f_2$ of two functions $f_1, f_2 : [a, b] \to R$ such that f_1 is monotonically increasing and f_2 is monotonically decreasing. In particular, for each $x \in (a, b)$, the two one-sided limits $\lim_{h\to 0^+} f(x + h)$ and $\lim_{h\to 0^-} f(x + h)$ both exist.

Theorem 14. *Let f(t) be an L^1 - function for $-\pi \leq t < \pi$, and periodic in the way that $f(t+2\pi) = f(t)$ for all other real values of t. Further let $f \in C([-\pi, \pi])$. Let a_n, b_n be defined by the equations*

$$\pi a_n = \int_{-\pi}^{\pi} f(t) \cos nt \, dt \tag{51}$$

$$\pi b_n = \int_{-\pi}^{\pi} f(t) \sin nt \, dt. \tag{52}$$

Then, if x is an interior point of any interval (a,b) within which f(t) has a limited total fluctuation, the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{53}$$

is convergent and its sum is $\lim_{h\to 0} \frac{1}{2}(f(x + h) + f(x - h))$.

2.4. **Taylor-Riemann series.** In a paper never intended for publication, Riemann scratched down a generalized Taylor-formula based on fractional derivatives. Riemann never gave any proof of this identity and no serious investigation of its validity was to be carried out until Hardy [6] proved convergence in a weak sense (by using Borel-summation). He discussed the series as an asymptotic expansion for the special cases of expanding certain classes of elementary functions involving the exponential function and the binomial function. Osler [5] was the first to give a proof of pointwise convergence of the general Taylor-Riemann series. The series is a form of "fractional" power series of the form $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^{\alpha+n}$. The via fractional calculus generalized Taylor series has drawn some attention among mathematicians over the years, some alternate forms have appeared (see [12]) but Osler's proof seems to be unique in generality. We here give a clarified and simplified proof of Osler's theorem:

Theorem 15. *Let $f(z) = (z-b)^{\sigma}h(z)$, where $\sigma > -1$, $h(z)$ is a function which is analytic in some open set containing the disk $\{|z-a| \leq r\}$ for some $r > 0$, and b belongs to the interior of this disk, i.e. $|b-a| < r$. We then have, for every $\alpha \in \mathbb{R}$ and every z on the circle $|z-a| = |b-a|$, $z \neq b$:*

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{\alpha+n} f(a)}{\Gamma(\alpha+n+1)}(z-a)^{\alpha+n}. \tag{54}$$

Here in the right hand side, for $(z-a)^{\alpha+n}$ we choose the branch which is given by $\arg(b-a) < \arg(z-a) < \arg(b-a) + 2\pi$, where we have fixed $-\pi \leq \arg(b-a) < \pi$.

Proof. For convenience we write $\Theta = \arg(b-a)$, where we require $-\pi \leq \Theta < \pi$. We express z on the circle $|z-a| = |b-a|$ as $z = a + |b-a|e^{i\theta}$, $\theta \in [\Theta, \Theta + 2\pi]$, and expand $\frac{f(z)}{(z-a)^{\alpha}}$ in a Fourier series:

$$\frac{f(z)}{(z-a)^{\alpha}} = \sum_{n=-\infty}^{\infty} a_n e^{i\theta n}. \tag{55}$$

By Fourier's theorem and our conventions about the branches, the coefficients a_n are given by

$$a_n = \frac{1}{2\pi}\int_{\Theta}^{\Theta+2\pi} \frac{f(a+|b-a|e^{i\theta})}{|b-a|^{\alpha}e^{i\alpha\theta}} e^{-in\theta}\, d\theta. \tag{56}$$

Note that the function $\theta \mapsto \frac{f(a+|b-a|e^{i\theta})}{|b-a|^{\alpha}e^{i\alpha\theta}}$ has limited total fluctuation in a neighbourhood of $\theta = \arg(z-a)$, since $z \neq b$; hence by Fourier's Theorem (Theorem 14) the right hand side in (55) is indeed convergent and the equality holds.

On the other hand, we consider the Cauchy-type fractional integral (see (33)) of order $\alpha + n$ of $f(z)$ at the point a:

$$D^{\alpha+n}_{z-b} f(a) = \frac{\Gamma(\alpha+n+1)}{2\pi i} \int_{C(b,a^+)} f(t)(t-a)^{-\alpha-n-1}\, dt. \tag{57}$$

We substitute $t = a + |b-a|e^{i\theta}$ and use θ as parameter of integration, further $dt = |b-a|ie^{i\theta}$ and by the definition of the contour $C(b,a^+)$ the limits of integration are $[\Theta, \Theta + 2\pi]$. Hence, using our conventions regarding branches in (32), we obtain

$$\begin{aligned} D^{\alpha+n}_{z-b} f(a) &= \frac{\Gamma(\alpha+n+1)}{2\pi i} \int_{\Theta}^{\Theta+2\pi} \frac{f(a+|b-a|e^{i\theta})}{|b-a|^{\alpha+n+1} e^{i(n+1)\theta} e^{i\alpha\theta}} |b-a| i e^{i\theta}\, d\theta \\ &= \frac{\Gamma(\alpha+n+1)}{2\pi} \int_{\Theta}^{\Theta+2\pi} \frac{f(a+|b-a|e^{i\theta})}{|b-a|^{\alpha+n} e^{i(\alpha+n)\theta}}\, d\theta. \end{aligned}$$

Comparing this with (56) we see that

$$a_n = \frac{D^{\alpha+n}_{z-b} f(a)}{\Gamma(\alpha+n+1)} \cdot |b-a|^n, \tag{58}$$

so by inserting this in (55) and noticing $(z-a)^{\alpha}|b-a|^n e^{i\theta n} = (z-a)^{\alpha+n}$ we obtain

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{D^{\alpha+n}_{z-b} f(a)}{\Gamma(\alpha+n+1)} (z-a)^{\alpha+n}, \tag{59}$$

on the circle $|z-a| = |b-a|, b \neq z$, which proves the theorem. □

Formally, as special case we see that if we let $\alpha \to 0$ in (54) we obtain the ordinary Taylor series expansion around a point z_0:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n \tag{60}$$

The proof follows directly since $\Gamma(n+1) = n!$ for positive n, and $\frac{1}{\Gamma(n)}$ for negative values of n equals zero, thus the negative-index terms disappear.

As concluded before the fractional calculus generalized Taylor-series, the Taylor-Riemann series, is a "fractional" power series expansion on the form $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^{\alpha+n}$, and it has by many means great similarities with the Fourier and Laurent-series, indeed:

Lemma 16. *Every analytic function expanded in Taylor-Riemann series by Osler's theorem is unique.*

The proof follows directly from the standard uniqueness theorem for the Fourier expansions of L^1-functions.

Here is a table of the two known explicit Taylor-Riemann expansions of elementary functions [6]:

$$e^z = \sum_{n=-\infty}^{\infty} \frac{z^{n+\alpha}}{\Gamma(\alpha+n+1)} \tag{61}$$

$$(z+w)^m = \sum_{n=-\infty}^{\infty} \frac{\Gamma(m+1)}{\Gamma(n+\alpha+1)\Gamma(m-n-\alpha+1)} z^{m-n-\alpha} w^{n+\alpha}\,, \tag{62}$$

Both of these series are divergent series, (61) is called Heaviside's exponential series, it has drawn some attention among mathematicians, including Heaviside, Ingham and Jeffereys (see [6]). Although Riemann wrote down both (61) and (62), the only proof that has appeared was given by Hardy [6]. The equalities in both (61) and (62) seem to only be valid in a special summation-type: weak Borel sense. This exhaustive work was carried out by Hardy (see [9] for more information on Borel-summation). We give a proof of a different form of (61) using Osler's theorem in section 3.

3. Applications of the Taylor-Riemann series

In this section we give a proof of an alternative version of Heaviside's exponential series. Further a Semi-Taylor series is introduced as the special case when $\alpha = 1/2$ in the Taylor-Riemann series. A Semi-Taylor expansion of the exponential function is also obtained and connections to certain special functions are revealed via the generating functions given in table 1.2.

3.1. **Taylor-Riemann expansions.** Firstly we intend to give a proof of an alternate form of Heaviside's series by using D_z^α by means of Osler's theorem.

Theorem 17. *e^{bz} has the following convergent Taylor-Riemann expansion on the ring $|z-a| = a$, $z \neq 0$:*

$$e^{bz} = \sum_{n=-\infty}^{\infty} \frac{e^{ba} b^{\alpha+n}(1-Q(-\alpha-n, ab))}{\Gamma(\alpha+n+1)} (z-a)^{\alpha+n}, \tag{63}$$

for any $b > 0$, $a > 0$ and any $\alpha \in \mathbb{R}$ except for $\alpha \in \mathbb{Z}$.

Proof. For the case $\alpha < 0$ we deploy the definition of the complex fractional derivative (28) to e^{bz} and obtain

$$D_0^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z \frac{e^{bt} dt}{(z-t)^{1+\alpha}}, \tag{64}$$

substitute $t = z - u$, thus $du = -dt$ and after some rearrangements one obtains

$$D_0^\alpha f(z) = \frac{1}{\Gamma(-\alpha)} e^{xb} \Big(\int_0^\infty e^{-ub} u^{-1-\alpha}\, du - \int_z^\infty e^{-ub} u^{-1-\alpha}\, du \Big). \tag{65}$$

By inspection one sees that (65) contains the incomplete complementary gamma-function (42):

$$Q(\alpha, z) = \frac{1}{\Gamma(\alpha)} \int_z^\infty e^{-t} t^{\alpha-1} dt. \tag{66}$$

Inserting it into (65) gives:

$$D_0^\alpha e^{bz} = e^{bz} b^\alpha (1 - Q(-\alpha, bz)), \tag{67}$$

which holds for any $Re(b) > 0$ and any $\alpha \in \mathbb{R}$ ((67) can be found in table 1.2). Further this fractional derivative expression is equivalent to the Cauchy-type fractional derivative (33) used in Osler's theorem for all $\alpha \in \mathbb{R}$ except for $\alpha \in \mathbb{Z}$ by Theorem 11. If we in Osler's theorem let $\sigma = 0$, we obtain the following Taylor-Riemann expansion for e^{bz} on the ring $|z - a| = a$, $Re(z) > 0$, $z \neq a$:

$$e^{bz} = \sum_{n=-\infty}^{\infty} \frac{e^{ba} b^{\alpha+n} (1 - Q(-\alpha - n, ab))}{\Gamma(\alpha + n + 1)} (z - a)^{\alpha+n}, \tag{68}$$

which equals (63). Thus we have shown that the theorem holds for $\alpha < 0$, with some more advanced calculations one can also prove that it holds for $\alpha > 0$, $\alpha \neq 0, 1, 2, 3...$ as well. This completes the proof. □

3.2. **Semi-Taylor series.** In this section we introduce a Semi-Taylor series as a special case of the Taylor-Riemann series, it appears as a "fractional" power expansion where $\alpha = \frac{1}{2}$, further we show some of its expansions and relations to special functions.

Theorem 18. *Any function f which is analytic in some open region containing the disk $|z - a| \leq |a - b|$ has a unique fractional power series expansion for z on the circle $|z - a| = |b - a|$,$z \neq b$. The series appears:*

$$f(z) = \tag{69}$$

$$\frac{D_{z-b}^{-\frac{1}{2}} f(a)}{\sqrt{\pi}} (z - a)^{-\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{2^{n+1} D_{z-b}^{\frac{1}{2}+n} f(a)}{(2n + 1)!! \sqrt{\pi}} (z - a)^{\frac{1}{2}+n}$$
$$+ \sum_{n=2}^{\infty} \frac{(-1)^{n-1} (2n - 3)!! D_{z-b}^{\frac{1}{2}-n} f(a)}{2^{n-1} \sqrt{\pi}} (z - a)^{\frac{1}{2}-n},$$

we shall call (69) *the Semi-Taylor expansion for f on the circle $|z - a| = |b - a|$.*

Proof. Any analytic function $f(z)$ in some open region containing the disk $|z - a| \leq |b - a|$ has a convergent Taylor-Riemann expansion on the circle $|z - a| = |b - a|$, $z \neq b$ by Osler's theorem (Just let $\sigma = 0$ in Theorem 15). Further Lemma 16 proves that the expansion is unique, thus:

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{D_{z-b}^{\alpha+n} f(a)}{\Gamma(\alpha + n + 1)} (z - a)^{\alpha+n}, \tag{70}$$

is pointwise convergence on the ring $|b-a| = |b-a|, z \neq b$. If we let $\alpha = \frac{1}{2}$ then we can by (39) and (40) convert the Gamma-function into two cases where it becomes simple relations containing the double-factorial. This leads us to the following reformulation of (70):

$$f(z) = \frac{D_{z-b}^{-\frac{1}{2}} f(a)}{\sqrt{\pi}}(z-a)^{-\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{2^{n+1} D_{z-b}^{\frac{1}{2}+n} f(a)}{(2n+1)!!\sqrt{\pi}}(z-a)^{\frac{1}{2}+n} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3)!! D_{z-b}^{\frac{1}{2}-n} f(a)}{2^{n-1}\sqrt{\pi}}(z-a)^{\frac{1}{2}-n}, \tag{71}$$

which equals (70). □

If we as an example deploy our series expansion of the exponential series (17), ie expanding e^{bz} in Semi-Taylor, we get the following:

$$e^{bz} = \frac{e^{ba} b^{-\frac{1}{2}}(1-Q(\frac{1}{2}, ba))}{\sqrt{\pi}}(z-a)^{-\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{2^{n+1} e^{ba} b^{\frac{1}{2}+n}(1-Q(-\frac{1}{2}-n, ba))}{(2n+1)!!\sqrt{\pi}}(z-a)^{\frac{1}{2}+n} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-3)!! e^{ba} b^{\frac{1}{2}-n}(1-Q(-\frac{1}{2}+n, ba))}{2^{n-1}\sqrt{\pi}}(z-a)^{\frac{1}{2}-n}, \tag{72}$$

The Semi-Taylor series has direct connections to special functions via the generating functions shown in table 1.2, we give an example below.

In connection with the generating functions given in 1.2 the Semi-Taylor expansion of $\frac{\cos(\sqrt{z})}{\sqrt{(z)}}$ by using Bessel function (47) is perhaps surprisingly simple:

$$\frac{\cos(\sqrt{z})}{\sqrt{z}} = \sum_{m=-\infty}^{\infty} (-1)^m a^{-\frac{m}{2}} J_{|m|}(\sqrt{a})((2|m|-1)!!)^{-\operatorname{sgn}(m)}(z-a)^{m-\frac{1}{2}} \tag{73}$$

(where we agree that $((2|m|-1)!!)^{-\operatorname{sgn}(m)} = 1$ when $m = 0$). This formula is true for all $a > 0$ and all z on the circle $|z-a| = a$, $z \neq 0$. The relation (73) is obtained by using the equality

$$J_{-\alpha-1}(z)(2z)^{-\alpha-1}\sqrt{\pi} = D_z^{\frac{1}{2}+\alpha} \frac{\cos\sqrt{z}}{\sqrt{z}} \tag{74}$$

and after deploying (48) and letting the sign function simplify the sum it will become (73).

The Semi-Taylor series may be used to construct other related expansions by using special functions that can be found in [5, p. 76].

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