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Initial conditions and initialization of linear fractional differential equations

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ABSTRACT

Mastery of the initial conditions of fractional order systems remains an open problem, in spite of a great number of contributions. This paper proposes a solution dedicated to linear fractional differential equations (FDEs), which is based on an equivalence principle between the original system and an exactly equivalent infinite dimensional ordinary differential equation (ODE). This equivalence principle is derived from the fractional integration operator concept and the frequency distributed state space model of this operator. Thanks to this principle, the FDE initial conditions problem is converted into a conventional ODE initialization problem, however with an infinite dimensional state vector. Practical FDE initialization is performed using an observer based technique applied to the equivalent ODE; a numerical example demonstrates the efficiency of this approach.

1. Introduction

It is now well established that fractional differential equations (FDEs) represent an appropriate model for numerous phenomena occurring in physics [2], chemistry [5], biology, etc.; where the dynamics of the considered systems are governed by a diffusion equation [3,19]. These FDEs have also been introduced artificially in feedback systems where the controller incorporates fractional derivative terms (CRONE control [16], fractional PID [18]), in order to improve robustness of the closed-loop. Thus, the analysis of FDEs has become a major theoretical goal in order to master all the new problems aroused by fractional systems. Among all these issues, the FDE initial conditions problem has motivated many researchers. In spite of a great number of papers, FDE initialization remains an open problem, mainly from a practical point of view. Fundamentally, initial conditions are a recurrent issue in fractional derivation, specially in the context of the different definitions (Riemann-Liouville [19], Caputo [4], Grünwald-Letnikov [17,19], etc.). Though a general solution would be necessary, the case of linear fractional systems seems to be more restrictive but with immediate interest. For these systems, the question can be formulated as follows : is it possible to summarize the past behavior of a fractional system by initial conditions (or a function of initial conditions) like in the case of ordinary differential equations (ODEs) [8]?

Historically, Caputo [4] managed to formulate this problem in an adapted form : he modified the definition of the fractional derivative in order to introduce initial conditions composed of the values of the function and of its successive integer order derivatives. Unfortunately, in spite of its apparent obviousness, the Caputo derivation approach is unable to provide a satisfactory solution to the FDE initial conditions problem. Thus, many researchers have proposed different approaches to solve this problem: these contributions can be classed in two categories, direct and indirect. The direct techniques rely mainly on the fractional derivation theory, referring to the different definitions and to the properties of the fractional

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derivatives. In this first category, we can mention the contribution of Ortigueira [15] and those of Lorenzo [9] and Heartley [6], based on the concept of the initialization function. However, it is difficult to use these techniques in practical situations and it is easy to find counter examples [22]. The indirect techniques are based on a state space model of the FDE; they can be considered as indirect approaches because the state model does not refer explicitly to fractional derivation. Owing to diffusive representation [13], Sabatier [22] has defined a continuously distributed state space model which is able to take into account initial conditions. This approach gives satisfactory results in the general case, but the initialization problem remains difficult to solve in practical situations because of the state vector definition. In a previous paper [25], we have proposed an other definition of the state of a fractional system, based on the fractional integration operator concept [24]. Though it would be easy to demonstrate that this definition is equivalent to the one proposed by Sabatier, its main interest is to generalize the conventional concepts of ODEs to FDEs owing to fractional integrators and moreover, this new state vector is better suited to perform practical initialization.

In this paper, we demonstrate that this approach is fundamentally based on an equivalence principle, between a linear FDE and an exactly equivalent infinite dimensional ODE. Moreover, the real challenge is not only to solve correctly the initial conditions problem, but really to give a solution to the initialization problem, i.e. to estimate in practical situations the initial conditions in order to summarize the past behavior of the system. Concretely, we demonstrate that an observer based solution is able to initialize correctly a fractional system.

The paper is organized as follows. Section 2 deals with a necessary recall of definitions related to fractional integration and derivation. Simulation of FDEs is presented in Section 3. The subject treated in Section 4 is the modeling of the fractional integrator while Section 5 deals with the state space representation of FDEs and the resulting equivalence principle. Based on this principle, a solution of the initial conditions problem is presented in Section 6. Finally, a practical observer based technique is proposed in Section 7 and a numerical example illustrates the efficiency of this methodology.

2. Definitions related to fractional systems

2.1. Fractional derivation and integration

Fractional integration is defined by the Riemann-Liouville integral [12,14,19].

The nth order integral (n real positive) of the function $f(t)$ is defined by the relation:

$$I_n(f(t)) = \frac{1}{\Gamma(n)}\int_0^t (t-\tau)^{n-1} f(\tau)\, d\tau \tag{1}$$

where

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \tag{2}$$

is the gamma function.

$I_n(f(t))$ is interpreted as the convolution of the function $f(t)$ with the impulse response:

$$h_n(t) = \frac{t^{n-1}}{\Gamma(n)} \tag{3}$$

of the fractional integration operator whose Laplace transform is

$$I_n(s) = L\{h_n(t)\} = \frac{1}{s^n} \tag{4}$$

Fractional derivation is the dual operation of the fractional integration.

Consider the fractional integration operator $I_n(s)$ whose input/output are respectively $x(t)$ and $y(t)$.

Then

$$y(t) = I_n(x(t)) \tag{5}$$

or

$$Y(s) = \frac{1}{s^n} X(s) \tag{6}$$

Reciprocally, $x(t)$ is the nth order fractional derivative of $y(t)$ defined as

$$x(t) = D_n(y(t)) \tag{7}$$

or

$$X(s) = s^n Y(s) \tag{8}$$

where s^n represents the Laplace transform of the fractional derivation operator.

2.2. Fractional differential equation (FDE)

Consider the general linear FDE:

$$\begin{aligned} &D_{m_N}(y(t)) + a_{N-1} D_{m_{N-1}}(y(t)) + \cdots + a_1 D_{m_1}(y(t)) + a_0 y(t) \\ &\quad = b_M D_{m_M}(u(t)) + \cdots + b_1 D_{m_1}(u(t)) + b_0 u(t) \end{aligned} \tag{9}$$

whose transfer function is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0 + b_1 s^{m_1} + \cdots + b_M s^{m_M}}{a_0 + a_1 s^{m_1} + \cdots + a_{N-1} s^{m_{N-1}} + s^{m_N}} = \frac{B(s)}{A(s)} \tag{10}$$

The fractional derivation orders:

$$m_1 < m_2 < \cdots < m_N \tag{11}$$

are real positive numbers; they are called external or explicit orders. It is necessary to define internal or implicit derivation orders such as

$$\begin{aligned} n_1 &= m_1 \\ &\vdots \\ n_i &= m_i - m_{i-1} \\ &\vdots \\ n_N &= m_N - m_{N-1} \end{aligned} \tag{12}$$

3. Simulation of a FDE and its pseudo state model

3.1. Introduction

W. Thomson (Lord Kelvin) [23] has demonstrated that integration operators are necessary to simulate ordinary differential equations (ODEs). Historically, the first integration operators have been analog ones, first with mechanical hardware and then with electronic devices [8]. Now, we use only numerical algorithms, for example RK4 algorithm, where the integration operator is implicit: consequently, we forget this integration operator concept, however fundamental. Thus, owing to W. Thomson's principle, the simulation of FDEs needs specific integration operators: this is the fractional integration operator concept.

3.2. Simulation of a FDE

Consider the previous FDE (9).
Define

$$X(s)=\frac{1}{A(s)}U(s) \tag{13}$$

and

$$Y(s)=B(s)X(s) \tag{14}$$

which permit to introduce the classical controller canonical state space form [8]:

$$\begin{aligned}
&x_1(t)=x(t)\\
&x_2(t)=D_{n_1}(x_1(t))\\
&\vdots\\
&x_i(t)=D_{n_{i-1}}(x_{i-1}(t))\\
&\vdots\\
&x_N(t)=D_{n_{N-1}}(x_{N-1}(t))\\
&D_{n_N}(x_N(t))=-a_0x_1(t)\cdots-a_{N-1}x_N(t)+u(t)=\varepsilon(t)
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&x_1(t)=I_{n_1}(x_2(t))\\
&\vdots\\
&x_{i-1}(t)=I_{n_{i-1}}(x_i(t))\\
&\vdots\\
&x_{N-1}(t)=I_{n_{N-1}}(x_N(t))\\
&x_N(t)=I_{n_N}(\varepsilon(t))
\end{aligned} \tag{16}$$

This simulation scheme is based on a state space model which requires N fractional integration operators, whose transfer functions are respectively $\{I_{n_N}(s), I_{n_{N-1}}(s), \dots, I_{n_1}(s)\}$ and connected according to the analog simulation scheme of Fig. 1.

Finally, $y(t)$ is obtained using the relation:

$$Y(s)=B(s)X(s) \tag{17}$$

corresponding to

$$y(t)=\sum_{i=0}^{M-1} b_i x_{i+1}(t) \tag{18}$$

3.3. Comment

Because $y(t)$ is the weighted sum of the $x_i(t)$ variables, the fractional derivative orders are constrained to be the same on both sides of Eq. (9). Though it is possible to obtain different fractional derivative orders using modified fractional integrators, we have preferred to restrict this paper to the simple standard case.

3.4. Pseudo state-space model of the FDE

FDE simulation is based on a fractional state-space model which can be expressed as

$$D_{\underline{n}}(\underline{X}(t))=A\underline{X}(t)+\underline{B}u(t) \tag{19}$$

with

$$\underline{X}(t)=\begin{bmatrix} x_1(t)\\ \vdots\\ x_i(t)\\ \vdots\\ x_N(t)\end{bmatrix} \quad \text{and} \quad D_{\underline{n}}(\underline{X}(t))=\begin{bmatrix} D_{n_1}(x_1(t))\\ \vdots\\ D_{n_i}(x_i(t))\\ \vdots\\ D_{n_N}(x_N(t))\end{bmatrix} \tag{20}$$

$$A=\begin{bmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & & \ddots & \vdots\\ 0 & 0 & \dots & \dots & 1\\ -a_0 & -a_1 & \dots & \dots & -a_N \end{bmatrix}; \quad B=\begin{bmatrix}0\\0\\ \vdots\\ 0\\ 1\end{bmatrix}$$

The output is

$$y(t)=\underline{C}^T\underline{X}(t) \tag{21}$$

with

$$\underline{C}^T=[b_0 \ \cdots \ b_M \ 0 \ \cdots \ 0] \tag{22}$$

These relations define the pseudo state space model of the FDE in controller canonical form [8].

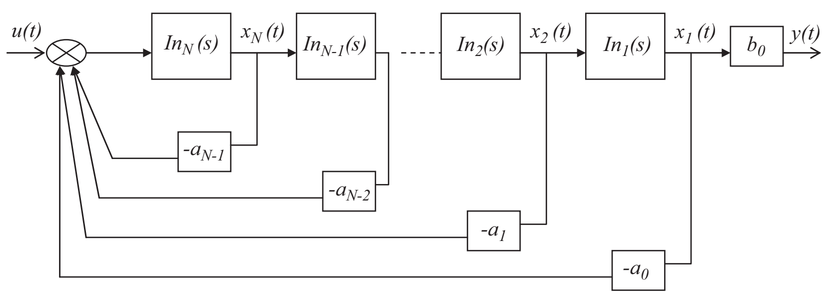

Fig. 1. Simulation of a FDE with fractional integrators.

4. Fractional integration operator

4.1. Introduction

The fractional integration operator $I_n(s)$ is the key element for FDE simulation. However, the realization of $I_n(s)$, either in analog or numerical form, is not a simple problem, as in the integer order case. It is possible to consider the frequency approach [20,24] and the time one [25]. Let us recall the time approach synthesis.

4.2. Principle

Diffusive representation, used by Matignon [7,11] and Montseny [13], provides the theoretical basis for a time approximation of $I_n(s)$.

Consider a linear system such as

$$y(t) = h(t) * v(t),$$

where $h(t)$ is its impulse response. $\mu(\omega)$ is called the diffusive representation (or frequency weighting function) of the impulse response $h(t)$. $h(t)$ and $\mu(\omega)$ verify the pseudo Laplace transform definition [13]

$$h(t) = \int_0^\infty \mu(\omega) e^{-\omega t}\, d\omega \tag{23}$$

A continuous frequency weighted state space model is associated to $\mu(\omega)$, according to

$$\begin{cases} \dfrac{\partial z(t,\omega)}{\partial t} = -\omega z(\omega,t) + v(t) \\ x(t) = \int_0^\infty \mu(\omega) z(\omega,t)\, d\omega \end{cases} \tag{24}$$

For the fractional integration operator:

$$I_n(s) = \frac{1}{s^n} \quad \text{with } 0 < n < 1$$

$$h(t) = \frac{t^{n-1}}{\Gamma(n)}$$

and

$$\mu(\omega) = \frac{\sin(n\pi)}{\pi}\omega^{-n} \tag{25}$$

4.3. Discrete frequency state model

This continuous frequency distributed model is not directly usable. A practical model is obtained by frequency discretization of $\mu(\omega)$, where the function $\mu(\omega)$ is replaced by a multiple step function (with J steps).

For an elementary step, its height is $\mu(\omega_k)$, and its width is $\Delta\omega_k$. Let c_k be the weight of the kth element:

$$c_k = \mu(\omega_k)\Delta\omega_k \tag{26}$$

Thus, the continuous distributed model is converted into a conventional state space model with dimension equal to J.

$$\begin{cases} \dfrac{dz_k(t)}{dt} = -\omega_k z_k(t) + v(t) | k = 1 \ldots J | \\ x(t) = \displaystyle\sum_{k=1}^{J} \mu(\omega_k) z_k(t) \Delta\omega_k = \sum_{k=1}^{J} c_k z_k(t) \end{cases} \tag{27}$$

or equivalently:

$$\dot{\underline{Z}}(t) = A_I \underline{Z}(t) + \underline{B}_I v(t)$$

$$x(t) = \underline{C}_I^T \underline{Z}(t) \tag{28}$$

with

$$\underline{Z}(t) = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_J \end{bmatrix} \quad \text{and} \quad A_I = \begin{bmatrix} -\omega_1 & & 0 \\ & \ddots & \\ 0 & & -\omega_J \end{bmatrix} \tag{29}$$

$$\underline{B}_I^T = [1 \;\; 1 \;\; \cdots \;\; 1]; \quad \underline{C}_I^T = [c_1 \;\; c_2 \;\; \cdots \;\; c_J] \tag{30}$$

With this time approach, we get a modal state model of $I_n(s)$ with the requirements $\omega_1 \to 0$, $\omega_J \to \infty$ and $J \gg 1$.

Remark. Because A is a diagonal matrix, all the modes ω_k are independent. Thus, the output $x(t)$ is the sum of J first order systems:

$$H_k(s) = \frac{c_k}{s + \omega_k} \tag{31}$$

connected to the same input $v(t)$.

5. State space model of FDES

5.1. Introduction

The association of the pseudo-state model of the FDE and of the state model of each fractional integrator leads naturally to the global state model of the FDE, which is an equivalent ODE, with infinite dimension.

5.2. State space model of a FDE

The state model is based on the pseudo state model of the FDE (19), (20), with input $u(t)$, output $y(t)$, and pseudo state variable $\underline{X}(t)$ whose dimension is N, where N is the number of fractional derivatives (or equivalently the number of fractional integrators):

$$D\underline{n}(\underline{X}(t)) = A\underline{X}(t) + \underline{B}u(t)$$

$$y(t) = \underline{C}^T \underline{X}(t)$$

$$\underline{X}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}; \quad \underline{n} = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{bmatrix}; \quad D\underline{n}(\underline{X}(t)) = \begin{bmatrix} D_{n_1}(x_1(t)) \\ D_{n_2}(x_2(t)) \\ \vdots \\ D_{n_N}(x_N(t)) \end{bmatrix} \tag{32}$$

A, $\underline{B}$ and $\underline{C}$ have been expressed in the controller canonical form (20), (22), but it is possible to use other canonical forms [8].

The components $x_i(t)$ of the pseudo state vector are the outputs of the N fractional integrators $I_{n_i}(s)$, their inputs $v_i(t)$ depending on the chosen form for the pseudo state model.

Moreover, in this paper, we assume (only for simplicity) that $0 < n_i < 1\ \forall i$.

According to the definitions of Section 4, there are two possible models for the fractional integrators:

5.2.1. Continuous frequency distributed state

Let $z_i(\omega,t)$ be the continuously distributed state of $I_{n_i}(s)$, verifying the following state model:

$$\begin{cases} \dfrac{\partial}{\partial t} z_i(\omega,t) = -\omega z_i(\omega,t) + v_i(t) \\ x_i(t) = \int_0^\infty \mu_i(\omega) z_i(\omega,t)\, d\omega \end{cases} \tag{33}$$

with

$$\mu_i(\omega) = \frac{\sin(n_i \pi)}{\pi} \omega^{-n_i} \tag{34}$$

$\mu_i(\omega)$ is the frequency weighting function (or frequency distribution) of the distributed state variable $z_i(\omega,t)$ with the fractional order n_i.

5.2.2. Discrete frequency distributed state

Let $\underline{Z}_i(t)$ be the discretely distributed state vector ($\dim \underline{Z}_i(t) = J$); $\underline{Z}_i(t)$ verifies the following state equation:

$$\dot{\underline{Z}}_i(t) = A_i \underline{Z}_i(t) + \underline{B}_i v_i(t)$$

$$x_i(t) = \underline{C}_i^T \underline{Z}_i(t) \tag{35}$$

where A_i, $\underline{B}_i$ and $\underline{C}_i$ (29), (30) correspond to the fractional order n_i.

These two models, (33) and (35), depend on the definition of the inputs $v_i(t)$. In the case of the pseudo state model in controller canonical form (refer to Fig. 1), these inputs $v_i(t)$ verify the following relations:

$$v_i(t) = x_{i+1}(t) \quad (i = 1 \ldots N-1)$$

$$v_i(t) = u(t) - \sum_{i=0}^{N-1} a_i x_{i+1}(t) \quad (i = N) \tag{36}$$

Finally, the output $y(t)$ is expressed owing to (32), i.e.

$$y(t) = \underline{C}^T \underline{X}(t) \tag{37}$$

where the components $x_i(t)$ of $\underline{X}(t)$ correspond to (33) in the continuously distributed case or to (35) in the discretely distributed case.

5.3. Comments

The pseudo state variables $x_i(t)$ are the outputs of the fractional integrators $I_{n_i}(s)$. In the frequency discrete case, $x_i(t)$ is defined as $x_i(t) = \underline{C}_i^T \underline{Z}_i(t)$ (35). This means that $x_i(t)$ is the weighted sum of the components $z_{i,j}(t)$ $(i = 0; \ldots; J)$ of the state vector $\underline{Z}_i(t)$ of the considered integrator. Notice that the $z_{i,j}(t)$ variables are true state variables corresponding to first order systems (31): they are able to memorize an initial condition. On the other hand, because $x_i(t)$ is the weighted sum of these state variables, it is not able to memorize an initial condition and so it is not a true state variable.

The true state vector of the FDE is composed of all the states of the different fractional integrators: thus the state vector of a FDE is infinite dimensional, even when there is only one fractional derivative in the FDE.

Notice that there exists an infinite number of possible linear combinations allowing the reconstruction of $x_i(t)$ from $z_{i,j}(t)$ and particularly at an instant $t = t_0$. Thus the output at $t = t_0$ of the fractional integrator is only an image of its true state which needs rigorously the knowledge of all its internal state variables at $t = t_0$: this is certainly the fundamental reason explaining the failure of the different attempts to solve the initial conditions problem without taking into account the true state of FDEs.

5.4. An equivalence principle

A FDE is fundamentally characterized by fractional derivation. This property is exhibited by the transfer function model (9) (10) or equivalently by the pseudo state space representation (19) (20). Simulation of these models is performed owing to the fractional integration operator concept (refer to 3.1). The main feature of this operator model is to exhibit an infinite dimensional integer order differential system where the frequency modes are continuously distributed (24) (33). The association of the FDE pseudo state model with these integrator distributed models leads to an integer order differential system (33) (36) (37), which is an infinite dimensional ODE, exactly equivalent to the considered FDE.

This a fundamental property which can be expressed as an equivalent principle: owing to the fractional integration operator concept and to the continuously distributed model of this operator, a linear FDE modeled by (9) (10) or (19) (20) is exactly equivalent to an infinite dimensional ODE.

Notice that the original fractional derivatives have been replaced by infinite dimensional integer order differential systems, as a consequence of the use of the integration operators for the simulation of the FDE.

The concept of infinite dimension associated to fractional derivation is classical; we can mention two examples:

- the replacement of the fractional derivatives by Grünwald-Letnikov [17,19] derivation transforms a FDE into an equivalent infinite dimensional ARMA model
- the approximation of fractional derivatives by high order transfer functions transforms a FDE into a large dimensional ODE [1].

However, the resulting models (ARMA or ODE) are only approximate equivalents of the considered FDE.

The distributed state model proposed by Sabatier [22] is also an exact equivalent ODE. Nevertheless, this state model has lost all its references to fractional derivation and to the considered FDE.

The interest of the proposed approach is that the reference to the original FDE remains in the equivalent ODE: basically, this ODE uses the pseudo state space model where only the fractional derivatives have been replaced by the distributed model of the fractional integrators.

Though this equivalence principle was not explicitly formulated in previous papers, it has already been used successfully in applications generalizing classic system theory to FDEs:

- simulation of FDEs with fractional operators [24,20]
- parameter identification of FDEs using an output error technique [21]
- stability analysis of linear and non linear FDEs using Lyapunov's technique [26]
- initial conditions of FDEs [25].

Initial conditions problem is again treated in Sections 6 and 7; the practical initialization problem is solved by an observer based technique applied to the equivalent ODE. This means that observability and controllability FDE properties can be analyzed with the equivalent ODE. Thus, the equivalence principle provides an efficient new methodology for FDEs analysis and applications.

6. The initial conditions problem

6.1. Introduction

The authors who have contributed to solve the initial conditions problem have implicitly used the fractional integrator output values as initial conditions. But we have demonstrated previously that these values are only a linear combination of the internal state variables of each integrator. Thus, it is not possible to reconstruct the true state of the fractional system using only these values, whatever may be the derivative definition used, and particularly the Caputo's one.

The solution of the initial conditions problem has to be based on a true state model of the FDE, i.e. using the internal states of the fractional integrators (or the state model defined by Sabatier [22]).

Fundamentally, the equivalence principle allows the replacement of the FDE initial conditions problem by an equivalent problem with the infinite dimensional ODE, where it is easy to solve a conventional initialization problem. Practically, because each integrator is modelled by an integer order differential system, it is straightforward to express, for each integrator $I_{n_i}(s)$, its output $x_i(t)$ on $t \in [t_0; +\infty[$ thanks to the knowledge of the initial states, $\underline{Z}_i(t_0)$ or $z_i(\omega,t_0)$.

6.2. Continuous frequency distributed state

The solution of system (33) is

$$z_i(\omega,t) = z_i(\omega,t_0)e^{-\omega(t-t_0)} + \int_{t_0}^{t} e^{-\omega(t-\tau)} v_i(\tau)\, d\tau \tag{38}$$

where $z_i(\omega,t_0)$ is the ith initialization function of $I_{n_i}(s)$ at $t=t_0$ and

$$x_i(t) = \int_0^{+\infty} \mu_i(\omega) z_i(\omega,t)\, d\omega \tag{39}$$

6.3. Discrete frequency distributed state

The solution of system (35) is

$$\underline{Z}_i(t) = e^{A_i(t-t_0)}\underline{Z}_i(t_0) + \int_{t_0}^{t} e^{A_i(t-\tau)}\underline{B}_i v_i(\tau)\, d\tau \tag{40}$$

where $\underline{Z}_i(t_0)$ is the ith initialization vector of $I_{n_i}(s)$ at $t=t_0$ and

$$x_i(t) = \underline{C}_i^T \underline{Z}_i(t) \tag{41}$$

Using the controller canonical form, the inputs $v_i(t)$ are expressed by the relations (36). Thus, it is possible to calculate all the integrator outputs ($i=1,\ldots,N$) and as a consequence the FDE output $y(t)$ owing to the relation:

$$y(t) = \underline{C}^T \underline{X}(t) \tag{42}$$

6.4. Conclusion

The FDE output $y(t)$ is obtained thanks to the knowledge of the input $u(\tau)$ (on $\tau \in [t_0; \infty[$) and of all the initial states of the fractional integrators:

$$\{z_1(\omega,t_0)\cdots z_i(\omega,t_0)\cdots z_N(\omega,t_0)\}$$

or

$$\{\underline{Z}_1(t_0)\cdots \underline{Z}_i(t_0)\cdots \underline{Z}_N(t_0)\} \tag{43}$$

Thus, the FDE initial conditions problem has been replaced by that of a conventional ODE, nevertheless with an important difference because each initial state is theoretically infinite dimensional.

It is important to notice that the solution of the problem is no longer connected with the definition of the fractional derivative, but essentially with the definition of a true state model for the fractional system.

Remark. Eqs. (35) and (36) are very easy to use, particularly for FDE simulation, because the corresponding matrices A_i are diagonal, i.e. the internal states are not coupled.

7. The initialization problem

7.1. Introduction

Though the equivalence principle provides a conventional solution to the FDE initial conditions problem, practical FDE initialization remains a difficult problem, owing to the large spectrum of the fractional integrator modes, ranging from $\omega_1 \to 0$ to $\omega_J \to \infty$. As a consequence, it is necessary to solve a wide range of initial values, corresponding to each mode.

Practically, Eqs. (38) and (39) cannot be used directly and the frequency distribution $\mu(\omega)$ has to be discretized. This represents at least two difficulties:

- though theoretically infinite, it is necessary to deal with a limited number of modes : practically, the geometric distribution proposed by Oustaloup [17,24,25] provides an optimal solution with $J+1$ modes

- choice of the lower value ω_1 and of the higher value ω_J with respect to the condition $\omega_J \gg \omega_1$. Moreover, we have to use a supplementary mode $\omega_0 = 0$ in order to avoid a simulation static error [20]; so the total number of modes is equal to $J+1$.

Consider the estimation of $\underline{Z}(t_0)$

$$\underline{Z}^T(t_0) = \{\underline{Z}_1(t_0) \cdots \underline{Z}_i(t_0) \cdots \underline{Z}_N(t_0)\} \tag{44}$$

A direct approach would be to estimate $\underline{Z}(t_0)$ with past input/output values $\{u(t),y(t)\}$ for $t \le t_0$. In order to apply this technique, it is necessary to discretize these values and first to choose a sampling period T_e. Let $T_J = 2\pi/\omega_J$ be the period of the faster mode. In order to take effectively this mode into account, we have to verify the condition $T_e \le T_J$. Because on the other hand the spectrum width $[\omega_1,\omega_J]$ must be equal to several decades (practical experience shows that 4 or 5 decades represent a minimal value) it is necessary to verify the condition $T_e \ll T_1$ (where $T_1 = 2\pi/\omega_1$). This last condition means that the discretized model is very ill conditioned and the calculation of $\underline{Z}(t_0)$ by solving a linear system composed of the sampled values $\{u(kT_e),y(kT_e)\}$ for $k \le t_0/T_e$ is numerically unstable.

Even with a least squares smoothing of these sampled values, the direct approach seems to be inappropriate. Thus, it is necessary to use an indirect approach ; this the reason why we propose an observer based solution.

7.2. Observer based estimation of $\underline{Z}(t_0)$

$\underline{Z}(t_0)$ has to summarize the dynamic behavior of the system output $y(t)$ excited by $u(t)$ for $t \le t_0$.

Assume that $u(t)$ and $y(t)$ are available on a past interval $[t_d,t_0]$ where $t_0 - t_d$ represents a large value. Then an observer, if it exists, is able to reconstruct $\underline{Z}(t)$ on $[t_d,t_0]$ (and thus the particular value $\underline{Z}(t_0)$) with the knowledge of $u(t)$ and $y(t)$ on this interval. The observation of a fractional system is a problem which has not been yet completely solved, either theoretically or practically. So, it is a wise reason to restrict our approach to a one derivative system:

$$\frac{b_0}{a_0+s^n} \quad \text{with } 0<n<1 \tag{45}$$

which uses only one fractional integration. However, state observation of this FDE is not an elementary problem, because a large number of modes are present, with approximately the same influence.

The frequency discretization of $\mu(\omega)$ (34) leads to a state model with the internal state $\underline{Z}(t)$ and the pseudo state variable $x(t)$:

$$\begin{aligned}
&\dot{\underline{Z}}(t) = A\underline{Z}(t) + \underline{B}v(t)\\
&x(t) = \underline{C}^T \underline{Z}(t)\\
&v(t) = u(t) - a_0 x(t)\\
&y(t) = b_0 x(t)
\end{aligned} \tag{46}$$

i.e. the model generating $y(t)$ with the excitation $u(t)$.

Using A, $\underline{B}$, $\underline{C}$, b_0 and a_0, we can formulate a Luenberger observer [10,8] excited by $u(t)$ and $y(t)$ measurements:

$$\begin{aligned}
&\dot{\underline{Z}}_{obs}(t) = A\underline{Z}_{obs}(t) + \underline{B}v_{obs}(t)\\
&x_{obs}(t) = \underline{C}^T \underline{Z}_{obs}(t)\\
&v_{obs}(t) = u(t) + K(y(t) - y_{obs}(t)) - a_0 x_{obs}(t)\\
&y_{obs}(t) = b_0 x_{obs}(t)
\end{aligned} \tag{47}$$

This observer is characterized by its internal state $\underline{Z}_{obs}(t)$ and its pseudo state variable $x_{obs}(t)$. Notice that this observer depends on an initial state $\underline{Z}_{obs}(t_d)$. Because $\underline{Z}(t_d)$ is indeed unknown, the observer is initialized by

$$\underline{Z}_{obs}(t_d) = 0 \tag{48}$$

The convergence speed of the observer depends on the choice of the static gain K.

7.3. Initialization example

Numerical simulations have shown the efficiency of this methodology, i.e.

$$\underline{Z}_{obs}(t_0) \to \underline{Z}(t_0) \tag{49}$$

and the possibility to initialize the system for $t > t_0$ with $\underline{Z}_{obs}(t_0)$.

In order to present a more realistic situation, the dynamic behavior of the system has not been simulated with Eq. (46), but with a theoretical solution. It is well known that the response of a one derivative FDE can be calculated with Mittag-Leffler equations [19].

We have considered a pseudo impulse excitation such as

$$u(t) = E \quad \text{for } 0 < t < t_0 \quad \text{and} \quad u(t) = 0 \quad \text{for } t > t_0 \tag{50}$$

Then, the corresponding Mittag-Leffler response $y(t)$ is calculated with

$$\begin{aligned}
y(t) = {} & \frac{b_0}{a_0} E H(t)\left[1 - \sum_{k=0}^{\infty} \frac{(-a_0)^k t^{nk}}{\Gamma(nk+1)}\right]\\
& - \frac{b_0}{a_0} E H(t-t_0)\left[1 - \sum_{k=0}^{\infty} \frac{(-a_0)^k (t-t_0)^{nk}}{\Gamma(nk+1)}\right]
\end{aligned} \tag{51}$$

where $H(t)$ is the Heaviside function.

The response $y(t)$ to this excitation is represented on Fig. 2, with a_0=1, b_0=1, n=0.5, E=1 and $t_0 = 2$ s.

For $t > t_0$, because $u(t) = 0$, the system exhibits its free response. Thus, we have decided to estimate $\underline{Z}(t_0)$ with the measurements $\{u(t),y(t)\}$ on $[t_d,t_0]$. Concretely, we have chosen $t_d \ne 0$ to avoid steady state for the system at instant t_d, and so to treat a more realistic situation. The numerical values for the simulation are $t_d = 0.1$ s, K=50, $\omega_1 = 0.002$ rd/s, $\omega_J = 500$ rd/s, $T_e = 0.001$ s and $J+1 = 11$.

The responses $y(t)$ and $y_{obs}(t)$ are presented in Fig. 3: after a fast transient, $y_{obs}(t) \to y(t)$.

$\underline{Z}(t)$ does not exist for Eq. (51) because the response $y(t)$ is calculated with a theoretical equation which has no reference with the equivalent ODE; so, it is not possible to appreciate objectively the convergence of the observer. Then, it is the initialization of the ODE model by $\underline{Z}_{obs}(t_0)$ that will be considered as an indirect objective test of this convergence. Notice that the convergence of $y_{obs}(t)$ does not imply necessarily the convergence of $\underline{Z}_{obs}(t)$.

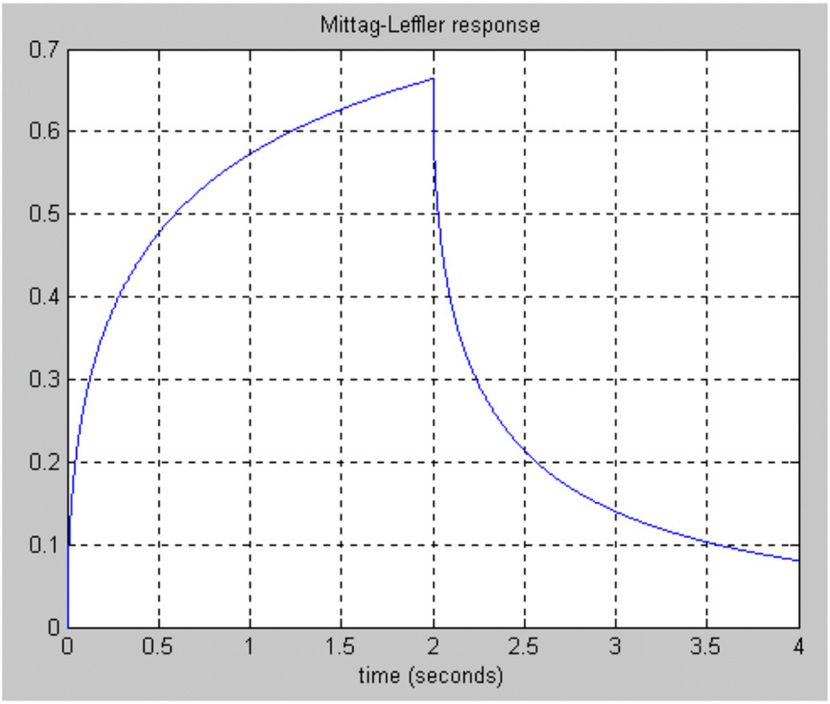

Fig. 2. System response to a pseudo impulse input.

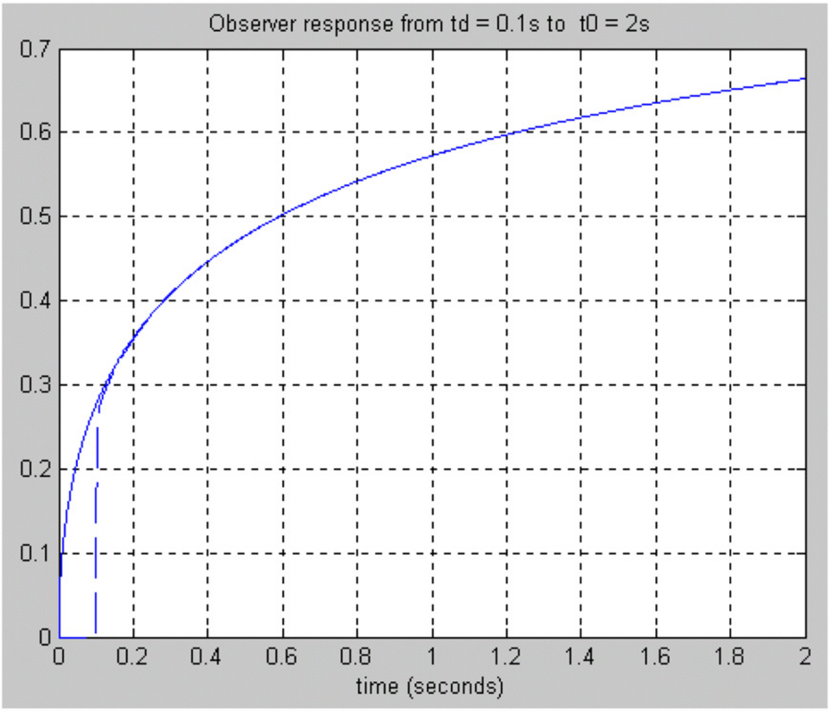

Fig. 3. Response of the observer.

Finally, we have simulated $y_{init}(t)$ for $t > t_0$, i.e.

$$\underline{Z}_{init}(t) = e^{A(t-t_0)}\underline{Z}_{obs}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}\underline{B}v_{init}(\tau)\,d\tau$$

$$x_{init}(t) = \underline{C}^T Z_{init}(t)$$

$$v_{init}(t) = -a_0 x_{init}(t)$$

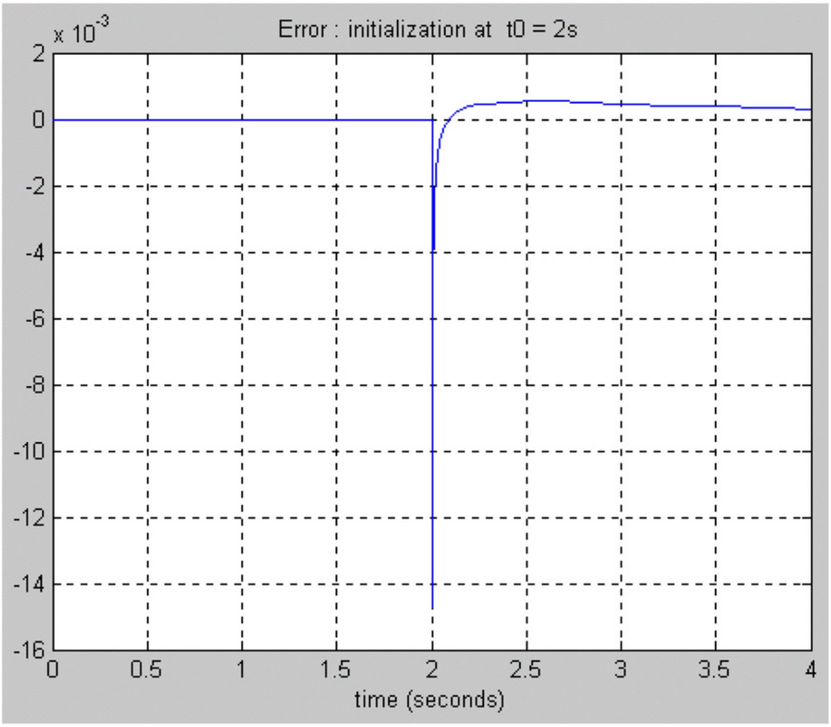

Fig. 4. Initialization error.

$$y_{init}(t) = b_0 x_{init}(t) \tag{52}$$

Because the curves $y(t)$ and $y_{init}(t)$ are very close, we have preferred to represent on Fig. 4:

$$error(t) = y(t) - y_{init}(t) \tag{53}$$

which is a more objective indicator of the initialization efficiency.

Owing to the magnitude of the error, it is obvious that $\underline{Z}_{obs}(t_0)$ has been able to summarize the past behavior of the system. However, we can notice that two residual errors are present:

- a transient error for $t \approx t_0$ caused by the truncation of the faster modes
- a steady state error for $t \gg t_0$ caused by the truncation of the slower modes (long time memory effect).

Complementary tests have confirmed that these two errors can be minimized by increase of ω_J (reciprocally by decrease of T_e) and by decrease of ω_1.

8. Conclusion

In this paper, a solution of the FDE initial conditions problem, based on an equivalence principle, has been proposed. The equivalence principle states that a linear FDE is exactly equivalent to an infinite dimensional ODE. The interest of this equivalence is to transform the original problem into an equivalent one where the classical techniques of ODE system theory can be used. Thus, using the fractional integration operator concept and the frequency distributed state model of the integrators, the FDE initial conditions problem is transformed into conventional ODE initialization, however with an infinite dimensional state vector. Practical initialization is not an elementary problem, because of the high dimensional state vector, and moreover because of the large spectral range of the modes. An observer based technique used to estimate the initial state has been proposed and tested on a realistic numerical example.

Further research works will have to complete these first investigations: applications to more complex systems with several fractional derivatives, and theoretical analysis of state observability of FDEs using the equivalence principle. Generalization of the concepts presented in this paper to non linear fractional order systems will be also a fundamental challenge

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