Relating Rationals To Reals

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Proposition. Regarding numbers of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$. Let, $r \in \mathbb{R}^{>}$. No matter how finitely small r gets; $\exists$ some a, b such that the following holds:

$$0 < a + b\sqrt{2} < r \tag{1}$$

Postponing the proof. We digress for a while, to look at the above from a different perspective. Suppose (1) holds.

Divide by $|b|$, where $b \in \{\mathbb{Z} - 0\}$,

$$0 < \frac{a}{|b|} \pm \sqrt{2} < \frac{r}{|b|} \tag{2a}$$

$$0 < \frac{a}{|b|} < \frac{r}{|b|} \mp \sqrt{2} \tag{2b}$$

$$0 < \frac{a - r}{|b|} < \pm\sqrt{2} \tag{2c}$$

Since **simultaneously**, r gets smaller, while a and b get larger. In (2c), the effect of r becomes more negligble with respect to: a, b, and $\sqrt{2}$. Letting $r \to 0$, and take the absolute value on all sides (left, middle, right) of (2c), to yield:

$$0 < \left|\frac{a}{b}\right| < \sqrt{2} \tag{2d}$$

In essence, we now want to look at positive values: a and b, where their ratio $\frac{a}{b} \to \sqrt{2}$. One way to obtain such values of a and b, is to obtain the Taylor series expansion of $\sqrt{x}$ about the center, x_0. Next, use the derived series expansion to approximate $\sqrt{2}$. For a Taylor series, the error term decreases when x_0 is set closer to the value of x being estimated. Observe that when the

Taylor series is expanded to a finite number of terms. These terms can be expressed as fractions that can be adjusted to a common denominator.

Hence, their sum can be expressed as a fraction. Where a is the numerator, and b the denominator. (3)

Since we want a fractional approximation to $\sqrt{2}$, the differences between x and x_0, are expressed as fractions. Let:

$$(x - x_0) = \frac{1}{d}$$

Since we are using the function $\sqrt{x}$,
to aproximate $\sqrt{2}$. Let $x = 2$,

$$2 - x_0 = \frac{1}{d}$$

Setting to a common denominator,

$$x_0 = 2 - \frac{1}{d} = \frac{2d-1}{d}$$

Substituting: $(x - x_0) = \frac{1}{d}$ and $x_0 = \frac{2d-1}{d}$ into the Taylor series; with k as the number of terms (iterations), using d to establish nearness to $\sqrt{2}$, yields this k'th order Taylor polynomial:

$$T(d,k) = \frac{1}{\sqrt{d}}\left[\sqrt{2d-1} + \sum_{i=1}^{k}\left(-\left(\frac{-1}{2d}\right)^{i}\left(\frac{1}{i!}\right)\prod_{p=1}^{i-1}(2p-1)\right)\right] \tag{4}$$

The values of d, need to satisfy the following critria:

$$\{d, \sqrt{d}, \sqrt{2d-1}\} \subset \mathbb{N} \tag{5}$$

That is, d and it's double minus 1 are perfect squares. Some evaluated results:

$$d \in \left\{\begin{array}{rrr} & \mathbf{1}, & 2\mathbf{5}, \\ 84\mathbf{1}, & 2856\mathbf{1}, & 97022\mathbf{5}, \\ 3295908\mathbf{1}, & 111963852\mathbf{1}, & 3803475062\mathbf{5}, \\ 129206188272\mathbf{1}, & 4389206926188\mathbf{1}, & 149103829302122\mathbf{5}, \\ & \vdots & \end{array}\right\} \tag{6}$$

(also note: OEIS A008844)

While using the Taylor series expansion to approximate $\sqrt{2}$. Two behaviours, are noted. The first occurs when $d = 1$. Here the series gradually converges to $\sqrt{2}$ as k increases.

The second behaviour occurs when using successive values of d above 1. The series converges to a value —but not to $\sqrt{2}$. However, a higher initial value of d sets the convergence to a value that is closer to $\sqrt{2}$. We can observe that (for those, $d > 1$), the nearest approximation to $\sqrt{2}$ occurs up to and including $k = 1$, (origin of k at 0, for the k-th order Taylor polynomial). Higher values of k do not improve accuracy, (as in, Runge's phenomenon). However, successively larger values of d, with a term expansion restricted to $k = 1$, do converge more rapidly to $\sqrt{2}$.

Now we can abridge (4), up to $k = 1$, and to express it as a ratio:

$$T(d) = \frac{\sqrt{2d - 1}}{\sqrt{d}} = \frac{a}{b} \tag{7}$$

Table I is generated by using the first behaviour, (evaluating (4) by iterating to the k-th term, using only $d = 1$).

Table I Output For First Behaviour

a / b	deviation from $\sqrt{2}$
1	−0·414
11 / 8	−0·0392
179 / 128	−0·0157
1439 / 1024	−0·00891
46147 / 32768	−0·00586
369605 / 262144	−0·00427
5917879 / 4194304	−0·00330
47365319 / 33554432	−0·00256
3032383331 / 2147483648	−0·00208
24264959593 / 17179869184	−0·00183

The highlighted blue values of Table II are generated by using the second behaviour, (evaluating (7) by iterating successive values of $d > 1$). What Table II shows in it's entirety, are the successive minima for increasing values of a. That is: the next highest value of a along with it's cooresponding b that yields the next lowest minima ($< |r|$). None of the other preceeding values of a with any b will have a lower value. This observation is made more obvious by the spread sheet output of figure 1. Column indices represent the numerator a, while row indices the denominator b. With what I call the direct approach; the values closest to zero appear as a highlighted diagonal (on the spread sheet). Values obtained from the method of the second behaviour, are differently highlighted. These also appear on the diagonal. Note the diagonal is not quite linear. This introduces offset error if you try to interpolate.

Table II Successive Minima $< |r|$.

a	b	deviation from $\sqrt{2}$	$\pm a \mp b\sqrt{2}$
3	2	$-8{\cdot}58e-02$	$1{\cdot}72e-01$
7	**5**	$-3{\cdot}36e-01$	$-7{\cdot}11e-02$
17	12	$-1{\cdot}31e-01$	$2{\cdot}94e-02$
41	**29**	$-5{\cdot}01e-02$	$-1{\cdot}22e-02$
99	70	$-2{\cdot}06e-02$	$5{\cdot}05e-03$
239	**169**	$-8{\cdot}41e-03$	$-2{\cdot}09e-03$
577	408	$-3{\cdot}48e-03$	$8{\cdot}67e-04$
1393	**985**	$-1{\cdot}44e-03$	$-3{\cdot}59e-04$
3363	2378	$-5{\cdot}95e-04$	$1{\cdot}49e-04$
8119	**5741**	$-2{\cdot}46e-04$	$-6{\cdot}16e-05$

Comparing Table I to Table II, shows that convergence is faster with the second behaviour. Hence, the basis of our algorithm design will be derived from the second behaviour. The problem now becomes to reduce iteration times while finding successively larger values of (6). There are highly optimized efficient methods to determine whether a number is a perfect square. [1]

Iteration pruning to find values of (6), can be established as follows: Notice that, the last digit is **1**, in the first two columns of (6). In column three, it is **5**. Moreover, Since, each value in (6) is a square. The square root of these values, have the following property:

- they are all multiples of $1, 5, 9$

(Perfect squares that end in 1 are either the result of an integer squared that ends in 1 or 9. Similarly, perfect squares that end in 5 are the result of an integer that ends in 5.) So we only need to iterate with every 10'th integer, specifically, only those that end with $1, 5, 9$. (As input, using the three sequences: $1, 11, 21, \cdots$ and $9, 19, 29, \cdots$ and $5, 15, 25, \cdots$). Moreover, these three sequences can be run in parallel; allowing calculation times to be decimated, (reduced to one tenth in duration).

[1] "Using the GNU Compiler Collection (GCC)", 15.5.3 Perfect Square

Iteration search times can be pruned yet again. Instead of searching, the next value can be calculated.

Let me digress, to discuss primitive pythagorean triples that are almost isosceles. These are right angle triangles whose shorter (non-hypotenuse), sides differ by one. [2] More specifically, regard a tree consisting of primitive Pythagorean triples. [3] Described by B. Berggren, (1934). Also note, a matrix multiplication method for generating nodes for this tree is given by F. J. M. Barning (1963).

Exclude the side branches and only use the triples of the central branch, namely:

$$\begin{array}{ccc}
(3, & 4, & 5) \\
(21, & 20, & 29) \\
(119, & 120, & 169) \\
(697, & 696, & 985) \\
(4059, & 4060, & 5741) \\
 & \vdots &
\end{array}$$

Observe that all criteria of (5) can be derived from the central branch. Regarding tuples of the central branch: The rightmost element is $\sqrt{d}$ of criteria (5) . The square of the rightmost element is d of (5) . The sum of the other two elements in the triple yields $\sqrt{2d-1}$ of (5) .

To illustrate: Using the first triple from the central branch. $(3, 4, 5)$.

Being a Pythagorean triangle: $3^2 + 4^2 = 5^2$.

Taking the sum of the first two elements: $3 + 4 = 7$

Going in reverse with the above result:

It's square,	7^2	$= 49$	
Increment by one,	$49+1$	$= 50$	
Divide by two,	$50/2$	$= 25$	
The square root,	$\sqrt{25}$	$= 5$	(RHS of central branch)

To illustrate the deriving calculations, for one iteration; using F. J. M. Barning's method:

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 21 \\ 20 \\ \mathbf{29} \end{bmatrix}$$

This inspired me to deduce a more simpler and direct method. Let, $\delta = \sqrt{d}$, and $n \in \mathbb{Z}^{>}$. Here it is:

[2]Wikipedia, The Free Encyclopedia, "Almost-isosceles Pythagorean triples"

[3]Wikipedia, The Free Encyclopedia, "Tree of primitive Pythagorean triples"

$$\delta_{n+1} = \begin{cases} 6 * \delta_n - \delta_{n-1} & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases} \tag{8}$$

Example: To obtain values for (6) using (8):

$$\begin{aligned} 6 * 1 - 1 &= \mathbf{5} \\ 6 * \mathbf{5} - 1 &= \mathbf{29} \\ 6 * \mathbf{29} - 5 &= \mathbf{169} \\ &\vdots \end{aligned}$$

The values of the RHS above can be squared to obtain (6).

Deriving the polynomial counterpart to (8). The second order linear recurence equation, whose constant coefficients, (8) form this characteristic or auxillary polynomial:

$$u^2 - 6u + 1 = 0 \tag{9}$$

The characteristic roots (using the quadratic formula), are:

$$(\lambda_1, \lambda_2) = 3 \pm 2\sqrt{2} \tag{10}$$

The ansatz substitution of (8), is:

$$\delta(n) = C_1\lambda_1^n + C_2\lambda_2^n, \qquad \text{where, } n \in \mathbb{Z}^{>}. \tag{11}$$

Using the initial conditions from (6):

$$d_0 = 1, \; d_1 = 5$$

let $n = 0$:

$$\begin{aligned} 1 &= C_1\lambda_1^0 + C_2\lambda_2^0 \\ C_2 &= 1 - C_1 \end{aligned}$$

let $n = 1$:

$$\begin{aligned} 5 &= C_1\lambda_1^1 + C_2\lambda_2^1 \\ 5 &= C_1\lambda_1 + (1 - C_1)\lambda_2 \end{aligned}$$

$$C_1 = \frac{5 - \lambda_2}{\lambda_1 - \lambda_2} \qquad C_2 = 1 - \frac{5 - \lambda_2}{\lambda_1 - \lambda_2} \tag{12}$$

Substituting (12) and (10) into (11):

$$\delta(n) = \frac{(1+\sqrt{2})(3+2\sqrt{2})^n + (\sqrt{2}-1)(3-2\sqrt{2})^n}{2\sqrt{2}} \tag{13}$$

The non-highlighted entries alternate with the highlighted entries of Table II. These non-highlighted entries can be generated as follows:

$$b_{i-1} = \quad a_i - b_i \tag{14a}$$
$$a_{i-1} = \quad b_i - b_{i-1} \tag{14b}$$

Where $i \in \{\mathbb{Z}^{>} | (i-1) \equiv 0 \pmod 2\}$. That is: i is a row index of Table II , where i refers to each alternate highlighted row and $i-1$ refers to an un-highlighted row.

Example:

To obtain values for the non-highlighted rows of Table II by using ($10a \cdots b$). We begin with the first highlighted row of Table II, to obtain the un-highlighted row above it:

Referencing entries: The top row contains a_0 and b_0. the next row below it contains a_1 and b_1. Thus:
$a_1 - b_1 = b_0$ becomes, 7 - 5 = 2
$b_1 - b_0 = a_0$ becomes, 5 - 2 = 3

Repeating with the next highlighted row of Table II:
$a_3 - b_3 = b_2$ becomes, 41 -29 = 12
$b_3 - b_2 = a_2$ becomes, 29 - 12 = 17

Returning to the *Proposition* to show (1) holds.

Proof. Using (13), to establish the range of D:

$$\begin{aligned} \lim_{n\to\infty} \delta(n) = & \quad \frac{(1+\sqrt{2})(3+2\sqrt{2})^\infty + (\sqrt{2}-1)(3-2\sqrt{2})^\infty}{2\sqrt{2}} \\ = & \quad \frac{(1+\sqrt{2})(\infty) + (\sqrt{2}-1)(0)}{2\sqrt{2}} = 0 \end{aligned}$$

Where $n \in \mathbb{Z}^{>}$.

We include the above with the first few calculations that made (6) and Table II. We now have,

$$D = \{1, 25, 841, \cdots, = \infty\}; \quad \text{where, } \delta \in D, \text{and } D \subset \mathbb{Z}^{>}$$

Using (7), as a series:

$$P = \left\{ \frac{\sqrt{2\delta - 1}}{\sqrt{\delta}} \right\}_{\delta=1}^{\delta=\infty} \qquad \rho \in P$$

Coorespondingly,

$$\lim_{\delta\to\infty} = \frac{\sqrt{2\delta - 1}}{\sqrt{\delta}} = \lim_{\delta\to\infty} \frac{\sqrt{\delta}\sqrt{2 - \frac{1}{\delta}}}{\sqrt{\delta}} = \sqrt{2}$$

$$P = \left\{ 1, \frac{7}{5}, \frac{41}{29}, \cdots, \sqrt{2} \right\}$$

Therefore, as $\delta \to \infty$, also coorespondingly, $\rho \to \sqrt{2}$.

Substituting (7) into (1). The signs of a and b are arbitrary but the middle term of (1) is always positive. You can take it's absolute value.

$$\begin{aligned} \triangle = & \quad \left|b\sqrt{2} - a\right| \\ \triangle = & \left|\sqrt{\delta}\sqrt{2} - \sqrt{2\delta - 1}\right| \\ \triangle = & \left|\sqrt{2\delta} - \sqrt{2\delta - 1}\right| \end{aligned}$$

$$\triangle = \lim_{\delta\to\infty} \left|\sqrt{2\delta} - \sqrt{2\delta - 1}\right| = \infty - \infty = 0$$

Also,

$$\triangle = \lim_{\delta\to\infty} \left|b\sqrt{2} - a\right| = \left|1\sqrt{2} - \sqrt{2}\right| = 0$$

As $\delta \to 0$, also coorespondingly, $\triangle \to 0$. Hence, for a fixed r, where $r \in \mathbb{R}$, $(r > 0)$. You can choose d (by implication you are also choosing a and b), such that:

$$0 < \triangle < r$$

$\square$

Algorithm. To obtain a and b. Iteratively generate δ values using (8). The successive outputs of (8), are evaluated by (7), to obtain a numerator a, and a denominator b. These form the highlighted rows of (Table II). Then you can use (14a) and (14b) to generate the non-highlighted rows. Refer to Listing 2b, Should you prefer to directly (asyncronously), evaluate a given row use (13), instead of (8). The output of (13) is evaluated by (7). The resulting numerator and denominator can be utilized by (14a) and (14b) to generate the previously non-highlighted row values, should you require them. Refer to Listing 2a.

The problem was also answered: Find successive numbers of the form $|\mp a \pm b\sqrt{2}| \in \mathbb{R}$, where $a, b \in \mathbb{Z}$. Such that each number is the next successive minima.

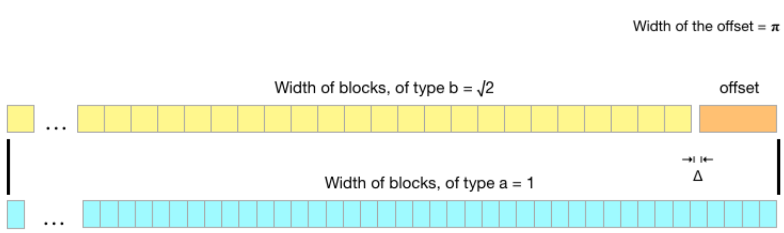
The fractional form output (Table II), was optionally converted to lowest common denominator. Otherwise, excluding this step, the algorithm (Listing 2a), for this runs in $O(c+n)$ time. The direct approach (Listing 1), runs in $O(6^n)$ time. Timing comparisons follow:

a	b	Listing 1	Listing 2a	Listing 2b
7	5	400 µs	214 µs	
41	29	1.37 ms	256 µs	
239	169	6.25 ms	297 µs	
1393	985	35.3 ms	335 µs	
8119	5741	208 ms	379 µs	
47321	33461	1.37 s	417 µs	
275807	195025	7.41 s	458 µs	
1607521	1136689	44.9 s	497 µs	
9369319	6625109	259 s	540 µs	
54608393	38613965	1.51 ks	581 µs	96.8 µs

Showing That. $\exists(a, b)$, such that:

$$\pi < \mp a \pm b\sqrt{2} < \pi + r \qquad \text{where:} \quad (a, b) \in \mathbb{Z}, \quad \pi \in \mathbb{R}$$

Proof. Suppose you have three different types of blocks The first type is one unit long, (of quantity a). The second type is $\sqrt{2}$ units long, (of quantity b). Also, there is one block that is π units in length. We arrange one type of the blocks end to end by their described lengths. Similar arrangement for the other type of blocks. You can add or subtract the type a blocks, while moving the offset block, to minimize $\triangle$, so that both ends of the two rows, (arranged in parallel), align, as shown below:



Δ is the smallest gap distance for a given number of ***b*** blocks plus offset that will align with some amount of ***a*** blocks.

From the above, define a $\triangle$ function as:

$$\triangle(b,\pi) = \begin{cases} \pi + \sqrt{2}b - \lfloor \pi + \sqrt{2}b \rfloor, & \text{if } \pi + \sqrt{2}b - \lfloor \pi + \sqrt{2}b \rfloor \geq \frac{1}{2} \\ 1 - (\pi + \sqrt{2}b - \lfloor \pi + \sqrt{2}b \rfloor), & \text{if } \pi + \sqrt{2}b - \lfloor \pi + \sqrt{2}b \rfloor < \frac{1}{2}. \end{cases}$$

By definition $\triangle(b,\pi)$ has the following range:

$$0 < \triangle(b,\pi) < \frac{1}{2}$$

The graphs of $\triangle(b,0)$ and $\triangle(b,\pi)$ are shown in *Figure* 2. These appear as two sawtooth waveforms offset to each other. Referring to either graph of *Figure* 2; the apex to apex distance is $\frac{\sqrt{2}}{2} = 0.707\cdots \in \mathbb{R}$. This is derived from $\sqrt{2}$, the length of block b. Should the apex to apex distance have been $\in \mathbb{Z}^{>}$. Then, after a given number of cycles, the value of $\triangle(b,\pi)$ would be the same as it would have been at the origin. Succeeding intercepts would repeat. Since, a rational apex to apex distance, would have a multiple that would be an integer. Conversely, since the apex to apex distance is $\in \mathbb{R}$; the sequence of intercept heights never repeats. Since it never repeats, there are an infinite number of different intercept height values in the interval $(0, \frac{1}{2})$.

Collapse the sawtooth waveform, like an accordian, with any non-zero value of π for $\triangle(b,\pi)$. In effect the intercept height values map to a single vertical line segment ranging in height from $(0, \frac{1}{2})$. Since there are an infinite number of values, the segment is continuous in that interval. Thus, choosing a fixed arbitrary value for r, where, $r > 0$, and $r \in \mathbb{R}$, $\exists$ an (a,b) that will generate the hybred number such that: $\pi < \mp a \pm \sqrt{2}b < r + \pi$. $\square$

Listing 1

```
def diagonal_minima(limit):
    #used spread sheet to obtain initial parameter
    s=n(sqrt(2),128)
    m = [[3,2,1000.0]]
    b=m[-1][1]
    for a in range(3,limit):
        r1 = float(a) - s*float(b)
        r2 = float(a) - s*(float(b)+1.0)
        if abs(r1) < abs(r2):
            r=r1
        else:
            r=r2
            b=b+1
        if abs(r) < abs(m[-1][2]):
            m.append([a,b,r])
            b=m[-1][1]
    return m

def generate_diagonal_minima(limit):
    d=diagonal_minima(limit)

    # prints: Table II.
    print '{}'.format(len(d))
    for z in range(1,len(d)):
        print '{:15d}{:15d}{:15.2e}'.format(int(d[z][0]),
            int(d[z][1]), float(100.0*d[z][2]))
    return
```

Listing 2a

```
def  of_criteria (x):
    # input the number of iterations to use (iteration  limit)
    # yields values that meet the criteria  of subset (5)
    last  = [1,5]
    for y in range(x):
        last .append(6*last[-1]-last[-2])

    sq = [z*z for z in last]
    return sq

def num(d):
    # the numerator in equation (8)
    return sqrt(2*d-1)

def dnm(d):
    #the denominator of equation (8)
    return sqrt(d)

fmt = '{0:8}  {1:8}    {2:14.2e}    {3:12.2e}'

def generate_ratios(iterations):
    s=n(sqrt(2),128)
    delta=of_criteria(iterations)

    print  '{0:6} a {0:7} b {0:8} error {0:6} a-sqrt(2)b
        {1:}'.format('  ','\n')
    for x in range(1, iterations):
        a =num(delta[x])
        b =dnm(delta[x])

        b_prev = a - b
        a_prev = b - b_prev

        # err is a/b, approximately equal to sqrt(2)
        err_prev  = s-float(a_prev)/float(b_prev)
        err  = s-float(a)/float(b)

        # where estimate   is  -a + sqrt(2)*b
        # such that  0 < estimate < r
        r_prev  = float(a_prev) - s*float(b_prev)
        r  = float(a) - s*float(b)

        # prints: Table II.
        print fmt.format(int(a_prev), int(b_prev), float(err_prev),
            float(r_prev))
        print fmt.format(int(a), int(b), float(err), float(r))
```

Listing 2b

```
def denominator(n):
    #the denominator of equation (8)
    s = RR(sqrt(2))
    return (((1+s)*((3+2*s)**n)) + ((s-1)*((3-2*s)**n))) / (2*s)

def numerator(d):
    # the numerator in equation (8)
    return sqrt(2*d*d-1)

def pythagorean_minima(x):
    d = ceil(denominator(x))
    n = ceil(numerator(d))
    dprev = n - d
    nprev = d - dprev
    return (nprev, dprev, n, d)

def generate_pythagorean_minima(x):
    pmin = pythagorean_minima(x)
    print '{0:20}{1:20}\n{2:20}{3:20}'.format(int(pmin[0]),
        int(pmin[1]), pmin[2], pmin[3])

def generate_highlighted_minima(x):
    d = ceil(denominator(x))
    n = ceil(numerator(d))
    print '{0:20}{1:20}'.format(int(n), int(d),)
```

Evaluating: $a - (b\sqrt{2})$

	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
2	-0.83	0.17	1.17	2.17	3.17	4.17	5.17	6.17	7.17	8.17	9.17	10.17	11.17	12.17	13.17	14.17	15.17
3	-2.24	-1.24	-0.24	0.76	1.76	2.76	3.76	4.76	5.76	6.76	7.76	8.76	9.76	10.76	11.76	12.76	13.76
4	-3.66	-2.66	-1.66	-0.66	0.34	1.34	2.34	3.34	4.34	5.34	6.34	7.34	8.34	9.34	10.34	11.34	12.34
5	-5.07	-4.07	-3.07	-2.07	-1.07	-0.071	0.93	1.93	2.93	3.93	4.93	5.93	6.93	7.93	8.93	9.93	10.93
6	-6.49	-5.49	-4.49	-3.49	-2.49	-1.49	-0.49	0.51	1.51	2.51	3.51	4.51	5.51	6.51	7.51	8.51	9.51
7	-7.90	-6.90	-5.90	-4.90	-3.90	-2.90	-1.90	-0.90	0.10	1.10	2.10	3.10	4.10	5.10	6.10	7.10	8.10
8	-9.31	-8.31	-7.31	-6.31	-5.31	-4.31	-3.31	-2.31	-1.31	-0.31	0.69	1.69	2.69	3.69	4.69	5.69	6.69
9	-10.73	-9.73	-8.73	-7.73	-6.73	-5.73	-4.73	-3.73	-2.73	-1.73	-0.73	0.27	1.27	2.27	3.27	4.27	5.27
10	-12.14	-11.14	-10.14	-9.14	-8.14	-7.14	-6.14	-5.14	-4.14	-3.14	-2.14	-1.14	-0.14	0.86	1.86	2.86	3.86
11	-13.56	-12.56	-11.56	-10.56	-9.56	-8.56	-7.56	-6.56	-5.56	-4.56	-3.56	-2.56	-1.56	-0.56	0.44	1.44	2.44
12	-14.97	-13.97	-12.97	-11.97	-10.97	-9.97	-8.97	-7.97	-6.97	-5.97	-4.97	-3.97	-2.97	-1.97	-0.97	0.029	1.029
13	-16.38	-15.38	-14.38	-13.38	-12.38	-11.38	-10.38	-9.38	-8.38	-7.38	-6.38	-5.38	-4.38	-3.38	-2.38	-1.38	-0.38
14	-17.80	-16.80	-15.80	-14.80	-13.80	-12.80	-11.80	-10.80	-9.80	-8.80	-7.80	-6.80	-5.80	-4.80	-3.80	-2.80	-1.80

Positive values closest to zero are highlighted as

Negative values closest to zero are highlighted as

Next diagonal minima is highlighted as

Shaded as , column and row indices, correspond to values of a and b, respectively.

Figure: 1 A highlighted diagonal indicates $a - b\sqrt{2}$ minima.

Figure: 2 $\triangle$ function (0 offset —, π offset —).
Intercept points with integer b values are circled.
The $\triangle$ function range is: $0 \cdots 0.5$