Geophys. J. R. astr. Soc. (1973) 31, 447–455.

The Rapid Calculation of Potential Anomalies

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(Received 1972 August 4)

Summary

It is shown how a series of Fourier transforms can be used to calculate the magnetic or gravitational anomaly caused by an uneven, non-uniform layer of material. Modern methods for finding Fourier transforms numerically are very fast and make this approach attractive in situations where large quantities of observations are available.

1. Introduction

The matching of observed potential fields with those produced by crustal models is a traditional method of geophysical data interpretation. The conventional way in which the theoretical fields are found is to break up the model into a set of simpler objects (e.g. prisms or rectangular blocks) whose contributions are calculated separately and summed (see Grant & West 1965; Garland 1965). When the model is complicated and when a large quantity of observations is available, this process can be computationally very time-consuming, since the number of operations increases roughly as the product of the number of output points and the number of points defining the model. In recent times, however, an ingenious factorization method (see special issue of *IEEE*, 1967) has made the computation of Fourier transforms particularly fast: the computation time being proportional to $N \ln N$, where N is the number of input and the number of output points. If the calculation of gravity and magnetic anomalies due to the model could be cast in a form based on Fourier transformation, geophysicists could take full advantage of the remarkable speed of the new algorithm. This fact has been realized by some workers already (Dorman & Lewis 1970; Schouten & McAmy 1972) but until now approximations have been used that ignore the non-linear effects caused by terrain roughness. We give in this note an exact theory for the calculation of potential fields caused by a non-uniform and uneven layer of material; the observation points lie in a plane that is everywhere above the material and, therefore, the proposed technique is most suitable in applications to aeromagnetic or surface oceanographic measurements. We later describe how the results may be found on an uneven surface. The main result of this paper is expressed as an infinite series of Fourier transforms; we discuss the convergence of the series and give a criterion for securing the optimum convergence rate in a given physical situation. A two-dimensional problem is solved, showing extremely fast convergence, which should be typical of oceanographic data.

2. Derivation of the Fourier expansion

For simplicity, we shall consider in detail the calculation of the Bouguer or terrain correction due to the gravitational attraction of a layer of material. We find the Fourier transform of the potential and manipulate the expression until we obtain an expression which is itself a sum of Fourier transforms. The basic result can be elaborated to include the case of many layers and densities varying with position, as well as the analogous magnetic problem.

It is convenient at this point to introduce a slightly unconventional notation. A cartesian axis system is established with $\hat{\mathbf{z}}$ vertically upwards: positions in space are represented by vectors like $\mathbf{r} = (x, y, z)$ and the projection of $\mathbf{r}$ onto the $x-y$ plane is denoted by $\vec{r}$. Thus

$$\vec{r} = \mathbf{r} - \hat{\mathbf{z}}\hat{\mathbf{z}}\cdot\mathbf{r},$$

and the converse of this equation will be written

$$\mathbf{r} = (\vec{r}, z) = (\vec{r}, \hat{\mathbf{z}}\cdot\mathbf{r}).$$

Note that

$$\vec{k}\cdot\mathbf{r} = \vec{k}\cdot\vec{r}.$$

The two-dimensional Fourier transform of a function $f(\vec{r})$ is defined by

$$\mathscr{F}[f(\vec{r})] = \int_X dS\, f(\vec{r}) \exp(i\vec{k}\cdot\vec{r}), \tag{0}$$

where $\vec{k}$ is the wave vector of the transformed function and X is taken to be the whole $x-y$ plane.

Consider the gravitational attraction from a layer of material, whose lower boundary is the plane $z = 0$, and whose upper boundary is defined by the equation $z = h(\vec{r})$. At the outset we shall require that the layer vanishes outside some finite domain, D, i.e. $h(\vec{r}) = 0$ if $|\vec{r}| > R$. The reason for this is that in practical situations we can model only a finite area of terrain and certain problems of convergence are avoided under this assumption. A further assumption is that h is bounded and integrable; both these restrictions are clearly valid for any reasonable model of topography. The gravitational potential at a position $\mathbf{r}_0$ due to the layer is

$$\begin{aligned} U(\mathbf{r}_0) &= G\rho \int_V dV/|\mathbf{r}_0-\mathbf{r}| \\ &= G\rho \int_D dS \int_0^{h(r)} dz/|\mathbf{r}_0-\mathbf{r}|, \end{aligned} \tag{1}$$

where G is Newton's gravitational constant; for the moment ρ, the density, is not a function of position. Suppose that the observation point is confined to the plane $z = z_0$, so that U is now only dependent on $\vec{r}_0$; this plane must lie above all the topography, something aeromagnetic and most oceanographic applications comply with. Take the Fourier transform of (1):

$$\begin{aligned} \mathscr{F}[U(\vec{r}_0)] &= \int_X dS_0\, U(\mathbf{r}_0) \exp(i\vec{k}\cdot\mathbf{r}_0) \\ &= G\rho \int_X dS_0 \int_D dS \exp(i\vec{k}\cdot\mathbf{r}_0) \int_0^{h(r)} dz/|\mathbf{r}_0-\mathbf{r}|. \end{aligned}$$

Interchanging the order of integration we see that

$$\mathscr{F}[U] = G\rho \int_D dS \int_0^{h(r)} dz \int_X dS_0 \exp(i\vec{k}\cdot\mathbf{r}_0)/|\mathbf{r}_0-\mathbf{r}|.$$

The last integral can be carried out analytically by use of polar co-ordinates (see Bracewell 1965); after a little algebra we obtain

$$\mathscr{F}[U] = G\rho \int_D dS \int_0^{h(r)} dz\{2\pi \exp(i\vec{k}\cdot\mathbf{r}-|\vec{k}|(z_0-z))\}/|\vec{k}|.$$

Now the z-integral can be performed explicitly:

$$\mathscr{F}[U] = 2\pi G\rho \int_D dS \exp(i\vec{k}\cdot\vec{r}-|\vec{k}|z_0)\{\exp[|\vec{k}|\,h(\vec{r})]-1\}/|\vec{k}|^2.$$

The integral above is not yet a Fourier transform but, upon expansion of the second exponential function in a Taylor series and rearrangement of summation and integration, we obtain

$$\mathscr{F}[U] = 2\pi G\rho \exp(-|\vec{k}|z_0) \sum_{n=1}^{\infty} \frac{|\vec{k}|^{n-2}}{n!} \mathscr{F}[h^n(\vec{r})], \tag{2}$$

which is a sum of Fourier transforms.

The terrain correction is in fact the vertical attraction of the material, not the potential. To find this we note that, above the masses $(\hat{\mathbf{z}}\cdot\mathbf{r}_0 > \max\{h(\vec{r}_0)\})$, $\nabla^2 U = 0$, so that the potential may be written

$$U(\mathbf{r}_0) = \frac{1}{4\pi^2} \int d^2 k\,\bar{U}(\vec{k}) \exp(-|\vec{k}|\,\hat{\mathbf{z}}\cdot\mathbf{r}_0 - i\vec{k}\cdot\mathbf{r}_0). \tag{3}$$

Thus $\mathscr{F}[U(\vec{r}_0)] = \bar{U}(\vec{k}) \exp(-\,|\vec{k}|\hat{\mathbf{z}}\cdot\mathbf{r}_0)$.

The vertical attraction Δg is by the definition of potential

$$\Delta g = +\partial U/\partial z,$$

and from the above relations it follows that

$$\mathscr{F}[\Delta g] = -|\vec{k}|\,\mathscr{F}[U].$$

With this result we obtain the desired expression

$$\mathscr{F}[\Delta g] = -2\pi G\rho \exp(-|\vec{k}|z_0) \sum_{n=1}^{\infty} \frac{|\vec{k}|^{n-1}}{n!} \mathscr{F}[h^n(\vec{r})]. \tag{4}$$

It is easy to generalize (4) to include the case where the lower boundary of the layer is not flat, but given by $z = g(\vec{r})$, and to allow the density to vary with $\vec{r}$:

$$\mathscr{F}[\Delta g] = -2\pi G \exp(-|\vec{k}|z_0) \sum_{n=1}^{\infty} \frac{|\vec{k}|^{n-1}}{n!} \mathscr{F}[\rho(\vec{r})\{h^n(\vec{r})-g^n(\vec{r})\}], \tag{5}$$

and the extension to many layers is obvious.

The equivalent magnetic problem can be solved by exactly the same procedure. We take a magnetized layer of material with upper and lower boundaries as before. It is commonly assumed in magnetic model calculations that the direction of magnetization is constant, but the intensity may vary: thus

$$\mathbf{M}(\vec{r}) = \hat{\mathbf{M}}_0 M(\vec{r});$$

this restriction is not essential for our technique but simplifies the calculations. Another simplification frequently employed results from the fact that perturbations to the observed field due to the magnetized material are always very small (<10 per cent), and that magnetic measurements at sea are made of the total field $|\mathbf{B}|$. The magnetic anomaly $\Delta|\mathbf{B}|$ can be approximated by

$$\Delta|\mathbf{B}| = \hat{\mathbf{B}}_0 \cdot \Delta\mathbf{B},$$

where $\hat{\mathbf{B}}_0$ is the unit vector in the direction of the unperturbed field and $\Delta\mathbf{B}$ is the perturbing field. With these conditions in force the equivalent magnetic result to (5) is

$$\mathscr{F}[\Delta|\mathbf{B}|] = \tfrac{1}{2}\mu_0 \exp(-|\vec{k}|z_0)\,\hat{\mathbf{B}}_0\cdot(i\vec{k}, |\vec{k}|)\,\hat{\mathbf{M}}_0\cdot(i\vec{k}, |\vec{k}|) \sum_{n=1}^{\infty} \frac{|\vec{k}|^{n-2}}{n!}\,\mathscr{F}[M(\vec{r})\{h^n(\vec{r})-g^n(\vec{r})\}]. \quad (6)$$

When, in addition, a constant thickness of magnetized material is assumed, (6) can be rewritten in a form that is faster computationally:

$$\mathscr{F}[\Delta|\mathbf{B}|] = \tfrac{1}{2}\mu_0 \exp(-|\vec{k}|z_0)\,\hat{\mathbf{B}}_0\cdot(i\vec{k}, |\vec{k}|)\,\hat{\mathbf{M}}_0\cdot(i\vec{k}, |\vec{k}|) (1-\exp(-|\vec{k}|\,h_0)) \sum_{n=0}^{\infty} \frac{|\vec{k}|^{n-2}}{n!}\,\mathscr{F}[M(\vec{r})\,h^n(\vec{r})], \quad (7)$$

where h_0 is the thickness of the layer; note the summation now begins at $n = 0$.

Having obtained the Fourier transforms by one of the above expressions, we can recover the required field by using the inverse transform on the resultant function. It is interesting to note that all the equations hold for a two-dimensional geometry, when a scalar wave number, k, replaces the vector $\vec{k}$.

3. Convergence of the series

Equation (4) has meaning only when the series of Fourier transforms converges and, moreover, rapidity of convergence is vitally important if the expression is to have practical utility. First we need a bound on $\mathscr{F}[h^n(\vec{r})]$ as n becomes large. From the definition of the Fourier transform

$$\begin{aligned} |\mathscr{F}[h^n]| &\leqslant \int_D dS\,|h^n(\vec{r})|\cdot|\exp(i\vec{k}\cdot\vec{r})| \\ &= \int_D dS\,|h^n(\vec{r})| \\ &\leqslant AH^n, \end{aligned}$$

where A is the area of D, the support of h, and $H = \max|h(\vec{r})|$, both quantities being bounded by assumption. Inserting this bound and comparing the series with that for $\exp(|\vec{k}|\,H)$, we find that (4) is uniformly and absolutely convergent in any bounded domain of the k-plane (Whittaker & Watson 1962, p. 581). Practically, an upper bound on $|\vec{k}|$ comes from the non-zero separation of the observation points.

A stronger result, which gives valuable insight into the rate of convergence can be shown as follows. Rearrange (4) thus

$$\mathscr{F}[\Delta g] = -\frac{2\pi G\rho}{k}\sum_{n=1}^{\infty}\frac{k^n \exp(-kz_0)}{n!}\mathscr{F}[h^n] = -\frac{2\pi G\rho}{k}S, \tag{8}$$

where we are writing k for $|\vec{k}|$. Now compare the series for S with

$$S' = \sum_{n=1}^{\infty} A \exp(-kz_0)\frac{(kH)^n}{n!} = \sum_{n=1}^{\infty} AL_n(k);$$

from the bound on $\mathscr{F}[h^n]$ we know every L_n is larger in magnitude than the corresponding term in S. It is easily shown that

$$L_n(k) < (H/z_0)^n,$$

independently of the value of k, when $z_0 > 0$. Therefore, when $H < z_0$ and $z_0 > 0$, the series for S' is uniformly convergent in the whole k-plane, by the Weierstrass M-test (Whittaker & Watson 1962, p. 49), and hence this is true of S also. That $H/z_0 < 1$ and $z_0 > 0$ follows from the condition that the observations plane lies entirely above the material in question, as we have already assumed.

From the computational viewpoint, the useful result is that the series for S converges at least as rapidly as $\sum (H/z_0)^n$, no matter what the value of k. Thus the smaller H/z_0 can be made, the faster the guaranteed rate of convergence. It may not appear at first that we have any control over H/z_0 in a given calculation, but this is in fact not the case. In setting up (4) we chose $z = 0$ to be the bottom of the layer of material; this level is entirely arbitrary* in gravity problems, so that we have complete freedom in our choice of z origin. A displacement of the origin does not affect the validity of (4) but it does alter the numerical values of z_0 and $h(\vec{r})$ and thereby H. The obvious strategy is to position the $z = 0$ plane so as to make H/z_0 as small as possible; with a little thought it can easily be seen that this occurs when $h_{max} = -h_{min} = H$, i.e. when the greatest and smallest values of h are equally distant from $z = 0$. Because this result is based on upper bounds of various terms, faster convergence might occur with a different origin position. Nonetheless, numerical experiments indicate that the choice given here falls very close to the optimum one; this will be illustrated with an example.

An almost identical analysis of convergence can be made on (7), while in the case of (5) and (6) we need only revise the definition of H to be

$$H = \max\{|h(\vec{r})|, \ |g(\vec{r})|\}$$

for the same conclusions to be valid.

4. Numerical example

The numerical implementation of the results in Section 2 is fairly straightforward. It will be obvious to those familiar with 'Fast Fourier Transforms' that the terrain

* If it is important to retain a known thickness of material, the attraction from a uniform slab can always be added to or subtracted from the answer in the shifted frame.

and model functions must be provided on a rectangular* lattice of points in the $x-y$ plane and that the answers appear at those co-ordinates. In a real survey such a regular disposition of observations is quite impossible: interpolation of the measurements onto a grid will always be necessary before the technique can be applied. Furthermore, the use of a discrete transform causes the various Fourier integrals in Section 2 to be approximated by sums; this is a serious defect only when the observation plane approaches the source material more closely than the horizontal spacing between data points. Another artifact of the numerical transform is the introduction of a false periodicity in the data, as if the model repeated itself over and over again. This can give rise to spurious fields at the edges of the model, but they can be reduced by adding a border of dummy points to separate the true model from the neighbouring images.

To illustrate the technique, a simple two-dimensional calculation of the magnetic case was performed. The ocean-bottom topography (shown in Fig. 1) is that found over the Gorda Rise (41° N, 127° W) and was kindly provided by Dr Tanya Atwater. The model consists of a constant thickness layer (500 m thick) magnetized uniformly to an intensity of $1{\cdot}0\,\mathrm{Am}^{-2}$ ($0{\cdot}001\,\mathrm{emu\,cm}^{-3}$) at a dip of $-60°$ and declination 0°, while the regional field was assumed to have dip and declination 60° and 30°: thus the material is reversely magnetized. The profile runs from west to east and the field is calculated at the surface, 2·1 km above the mean level of the bottom, which shows a relief of $\pm 0{\cdot}4$ km. Two short sloping sections have been appended to the ends of the profile to avoid discontinuity anomalies caused by the false periodicity. The magnetic anomaly computed by our method agreed to an accuracy of a few per cent with that given by a standard program (Mudie 1972). The discrepancies between the calculations are due entirely to the different treatment accorded to the ends of the model: the standard program assumes a very long, uniformly magnetized slab is attached to each end, and this naturally gives rise to

* Two-dimensional transforms can also be performed on skewed (i.e. non-orthogonal) axes, so that the basic unit is a parallelogram.

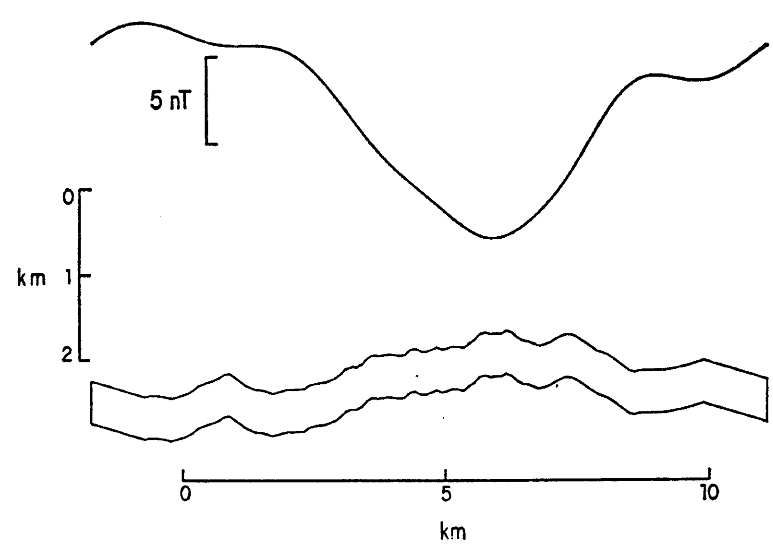

FIG. 1. A uniformly magnetized model and its computed magnetic anomaly at the ocean surface. The scale at the left indicates depth below the surface. The orientation of the profile and the direction of the field and magnetization are given in the text. The tesla is the SI unit of magnetic induction ($1\,\mathrm{n}T = 10^{-9}T = 1\gamma = 10^{-5}G$).

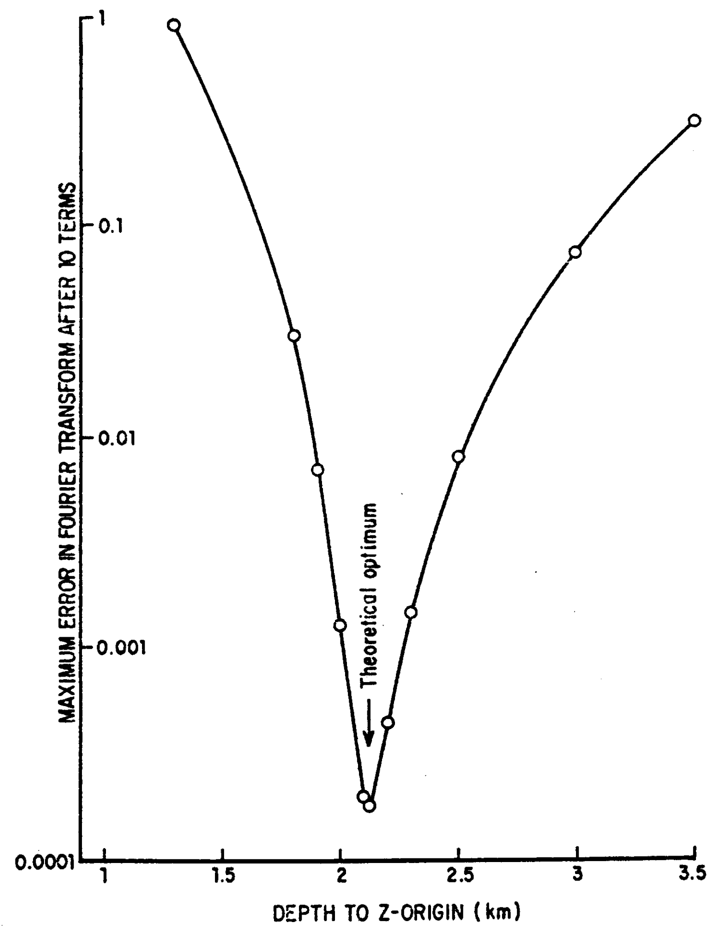

FIG. 2. Rate of convergence as the depth of the $z = 0$ plane is varied for the model shown in Fig. 1. The magnitude of the greatest contribution to the Fourier transform at the tenth term is used as a rough measure of accuracy and hence of convergence rate.

different magnetic anomalies from those of the periodic structure. The calculation time (0·4 s, with 128 data points) was 20 times less on the same machine, a CDC 3600. After four terms in (7) the greatest error in the Fourier transform was 0·6 per cent, when the $z = 0$ level was chosen according to the criterion of Section 3. Such rapid convergence should be typical in other applications with oceanic observations, even though rougher bottom topography is sometimes found. No reversal in the direction of magnetization has been assumed in our model and therefore the field is due solely to the terrain effect (a perfectly horizontal, uniformly magnetized slab exhibits no external magnetic field).

To test the validity of the convergence criterion developed in the previous section, the calculation was repeated with the z origin at different depths. We summed (7) to ten terms and then examined the largest contribution to the sum in the tenth term; roughly speaking, this is an estimate of the error in the sum provided the series is converging fairly rapidly. Fig. 2 shows the values of this error measure as a function of depth to the $z = 0$ plane. The agreement with the theoretical optimum is excellent and, what is more remarkable, we see the extreme sensitivity of the convergence rate to the placement of origin.

5. Extensions

It is worthwhile mentioning some extensions to the method. One restriction of the foregoing analysis has been that the results must all lie on a flat plane above the sources. In certain cases (e.g. land gravity, aeromagnetic surveys or large scale oceanic surveys where the Earth's curvature becomes important) this restriction may be troublesome. Formally, a curved observation surface may be introduced by performing a Taylor series expansion on (3). Suppose now that the results are required on the surface $z = Z(\vec{r}_0)$ so that (3) becomes

$$U(\mathbf{r}_0) = U(\vec{r}_0, Z(\vec{r}_0))$$

$$= \frac{1}{4\pi^2} \int d^2\, k\bar{U}(\vec{k}) \exp\left[-|\vec{k}|\, Z(\vec{r}_0) - i\vec{k}\cdot\vec{r}_0\right].$$

Here we assume that $\bar{U}(\vec{k})$ is the spectrum of the potential field on some level surface and has been calculated by the earlier methods. If a Taylor series for the exponential, $\exp[-|\vec{k}|\, Z(\vec{r}_0)]$ is introduced, an expansion like (4) is obtained with $|\vec{k}|^n$ inside the Fourier transform and Z^n outside. Difficulties with the convergence of the series and with the existence of the transforms occur if the Z surface ever drops below the highest elevation of the terrain: this would seem to present a real problem in the case of land gravity surveys.

Another extension, which may be valuable in magnetic applications, is the inversion of formulas like (7) for the magnetization $M(\vec{r})$. The equation can be rewritten in the form

$$\mathscr{F}[M(\vec{r})] = f(\vec{k}) + T[M(\vec{r})],$$

where T is a functional involving the $|\vec{k}|$ power series (now beginning at $n = 1$). If T can be treated as a small perturbation, this expression can be used iteratively to find M. One technical difficulty here, aside from the convergence of the iteration scheme, is that finding M is tantamount to performing downward continuation on the data, a notoriously unstable process: small errors in the measurements at short wave lengths are magnified by the procedure. The only way to avoid this trouble is to cut off the high frequency components with a somewhat arbitrary filter (see Schouten & McAmy 1972). It is the author's personal prejudice that the use of such inverse methods creates the illusion of uniqueness in the solutions (despite the obvious numerous assumptions built into equations), and that they should therefore be avoided.

6. Conclusions

The methods developed here are fast and practical. It is unlikely that much improvement in calculation time is necessary for the interpretation of single profiles—the existing methods are quite fast enough; however, in recent years detailed, two-dimensional oceanographic surveys have been performed in several areas and for these very large data sets the techniques of this paper should find some utility. Work is under way to complete an interpretation of this type.

Acknowledgments

The author is grateful to Dr John Miles, who suggested the present, more concise proof of the main result.

This work was supported by the National Science Foundation and by the Institute of Geophysics and Planetary Physics.

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