б) $\lim_{n\to\infty} \left(\frac{n^3+n^2+3n+1}{n^3+n^2+2n+1}\right)^n = \lim_{n\to\infty}\left(\frac{n^3+n^2+2n+1+n}{n^3+n^2+2n+1}\right)^n =$

$= \lim_{n\to\infty}\left(1+\frac{n}{n^3+n^2+2n+1}\right)^n = \lim_{n\to\infty}\left(1+\frac{n^2}{\frac{n^3+n^2+2n+1}{n}}\right)^n =$

$= e^{\lim_{n\to\infty}\frac{n^2}{n^3+n^2+2n+1}} = e^0 = 1$ ✓

28.$\overline{X}$

Упражнение 3

$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x, \ x\in\mathbb{R}$

$\lim_{n\to\infty} x_n = x \Rightarrow \lim_{x\to\infty}\left(1+\frac{x_n}{n}\right)^n = e^{\lim_{n\to\infty} x_n} = e$

зад. Да се док, че ред $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}_{n=1}^{\infty}$ е монотонно намаляваща.

$\left\{\left(1+\frac{1}{n}\right)^n\right\}$ – мон. раст

Док. $a_n \geq a_{n+1} \ \forall n$

$\frac{a_n}{a_{n+1}} \geq 1 \longrightarrow \frac{a_{n+1}}{a_n} \leq 1$

$\frac{a_n}{a_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} = \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+1}\cdot\left(1+\frac{1}{n+1}\right)} =$

$= \left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}\right)^{n+1}\cdot\frac{1}{1+\frac{1}{n+1}} = \left(\frac{\frac{n+1}{n}}{\frac{n+2}{n+1}}\right)^{n+1}\cdot\left(\frac{n+1}{n+2}\right) =$

$= \left(\frac{(n+1)}{n}\cdot\frac{(n+1)}{n+2}\right)^{n+1}\cdot\frac{(n+1)}{n+2} = \left(\frac{n^2+2n+1}{n^2+2n}\right)^{n+1}\cdot\frac{(n+1)}{n+2} =$

$$(1+\alpha)^n \geq 1+n\alpha$$

$$= \left(1+\frac{1}{n^2+2n}\right)^{n+1} \cdot \left(\frac{n+1}{n+2}\right) \geq \left(1+(n+1)\cdot\frac{1}{n^2+2n}\right)\frac{(n+1)}{n+2}$$

$$\left(1+\frac{n+1}{n^2+2n}\right)\left(\frac{n+1}{n+2}\right) = \left(\frac{n^2+2n+n+1}{n^2+2n}\right)\frac{(n+1)}{n+2} =$$

$$= \frac{n^3+n^2+2n^2+2n+n^2+n+n+1}{n^3+2n^2+n^3+2n^2+2n^2+4n} =$$

$$= \frac{n^3+4n^2+4n+1}{\cancel{2n^3+6n^2+4n}} = \frac{n^3+4n^2+4n+1}{n^3+4n^2+4n} \geq 1$$

$$\frac{a_n}{a_{n+1}} \geq 1 \iff a_{n+1} \leq a_n, \text{ т.е.}$$

$1+\frac{1}{\dots} > 1$ | редицата е намал

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \cdot \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = e$$

зад. да се док $\frac{1}{n+1} \leq \ln\left(1+\frac{1}{n}\right) \leq \frac{1}{n}$

! $\ln(\dots)$ - обратна ф-я за $e^{(\dots)}$

$\ln(e^x) = x$

$e^{\ln(x)} = x$

$\nearrow e$ (приближаване отдолу)

Т $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ е раст. и $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$\Rightarrow$ И $\left\{\left(1+\frac{1}{n}\right)^{n+1}\right\}$ е нам. и $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = e$

(приближаване отгоре)

От I и II => $(1+\frac{1}{n})^n \le e \le (1+\frac{1}{n})^{n+1}$ | ln

$\ln[(1+\frac{1}{n})^n] \le \ln e \le \ln[(1+\frac{1}{n})^{n+1}]$

$n \ln(1+\frac{1}{n}) \le 1 \le (n+1)\ln(1+\frac{1}{n})$

$n \ln(1+\frac{1}{n}) \le 1$ | : n $\qquad (n+1)\ln(1+\frac{1}{n}) \ge 1$ | : n+1

$\ln(1+\frac{1}{n}) \le \frac{1}{n} \qquad \ln(1+\frac{1}{n}) \ge \frac{1}{n+1}$

Тв. 1 $\{a_n\}$ - ред. с $a_n \ge 0 \ \forall n \in \mathbb{N}$

$a_n \xrightarrow[n\to\infty]{} \ell > 0$, то за $\forall k \in \mathbb{N}$ е изпълнено

$\sqrt[k]{a_n} \xrightarrow[n\to\infty]{} \sqrt[k]{\ell}$

1.19 ръководство

3 а) $\lim\limits_{n\to\infty} \sqrt{9+\frac{1}{n}} = 3$

$\lim\limits_{n\to\infty}(9+\frac{1}{n}) = \lim\limits_{n\to\infty} 9 + \lim\limits_{n\to\infty}\frac{1}{n} = 9$

б) $\lim\limits_{n\to\infty}(8-\frac{1}{n^2})^{-\frac{1}{3}}$

$\lim\limits_{n\to\infty}\sqrt[3]{(8-\frac{1}{n^2})^{-1}} = \lim\limits_{n\to\infty}\sqrt[3]{\frac{1}{(8-\frac{1}{n^2})}}$ Разгл. пр. без корен

$\lim\limits_{n\to\infty} \frac{1}{\frac{8n^2-1}{n^2}} = \lim\limits_{n\to\infty}\frac{n^2}{8n^2-1} = \frac{1}{8}$

Получ. $\lim\limits_{n\to\infty}\sqrt[3]{\frac{1}{(8-\frac{1}{n^2})}} = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$

в) $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n^2+n}}{n+2} = \lim\limits_{n\to\infty} \sqrt{\frac{n^2+n}{(n+2)^3}} = \lim\limits_{n\to\infty} \frac{n^2+n}{n^3+6n^2+12n+8} = 0$

По т6. $\Rightarrow \lim\limits_{n\to\infty} \sqrt[3]{\frac{n^2+n}{(n+2)^3}} = \sqrt[3]{0} = 0$

4) $\lim\limits_{n\to\infty} \underbrace{\sqrt{n^2+1}}_{a} - \underbrace{(n+1)}_{b} -$

$a - b = \frac{a^2 - b^2}{a+b}$

$\Rightarrow \lim\limits_{n\to\infty} \frac{(n^2+1) - (n+1)^2}{\sqrt{n^2+1} + (n+1)} =$

$= \lim\limits_{n\to\infty} \frac{\cancel{n^2}+1 - (\cancel{n^2}+2n+\cancel{1})}{\sqrt{n^2+1} + n+1} = \lim\limits_{n\to\infty} \frac{-2n}{\sqrt{n^2+1}+n+1} =$

$= \lim\limits_{n\to\infty} \frac{-2\cancel{n}}{\cancel{n}\left(\sqrt{1+\frac{1}{n^2}} + 1 + \frac{1}{n}\right)} = \lim\limits_{n\to\infty} \frac{-2}{\sqrt{1+\underbrace{\frac{1}{n^2}}_{0}} + 1 + \underbrace{\frac{1}{n}}_{0}} =$

$= -\frac{2}{2} = -1$

Зад. 5 $\lim\limits_{n\to\infty} \left(\frac{1+2+\ldots+n}{n+2} - \frac{n}{3} \right)$. Виждаме, че

$\sum\limits_{i=1}^{n} i = 1+2+\ldots+n = \frac{n(n+1)}{2} \Rightarrow \lim\limits_{n\to\infty} \frac{n(n+1)}{2(n+2)} - \frac{n}{2} =$

$= \lim\limits_{n\to\infty} \frac{\cancel{n^2}+n-\cancel{n^2}-2n}{2(n+2)} = \lim\limits_{n\to\infty} \frac{-n}{2n+4} = -\frac{1}{2}$

⑥ $\lim\limits_{n\to\infty} n^3\left(\sqrt{n^2+\sqrt{n^4+1}}-n\sqrt{2}\right)=$

$=\lim\limits_{n\to\infty} n^3\dfrac{\left(n^2+\sqrt{n^4+1}-2n^2\right)}{\sqrt{n^2+\sqrt{n^4+1}}+n\sqrt{2}}=$

$=\lim\limits_{n\to\infty} n^3\dfrac{(\overbrace{\sqrt{n^4+1}}^{a}-\overbrace{n^2}^{b})}{\sqrt{n^2+\sqrt{n^4+1}}+n\sqrt{2}}=$

$=\lim\limits_{n\to\infty} n^3\dfrac{\left(\cancel{n^4}+1-\cancel{n^4}\right)}{\left(\sqrt{n^2+\sqrt{n^4+1}}+n\sqrt{2}\right)\left(\left(\sqrt{n^4+1}\right)+n^2\right)}=$

$=\lim\limits_{n\to\infty}\dfrac{\cancel{n^3}}{\cancel{n^2}\left(\sqrt{1+\frac{1}{n^4}}+1\right)\cdot\cancel{n}\left(1+\sqrt{1+\frac{1}{n^4}}+\sqrt{2}\right)}=$

$=\lim\limits_{n\to\infty}\dfrac{1}{\left(\sqrt{1+\frac{1}{n^4}}+1\right)\left(\sqrt{1+\sqrt{1+\frac{1}{n^4}}}+\sqrt{2}\right)}=$

$=\dfrac{1}{2\cdot2\sqrt{2}}=\dfrac{\sqrt{2}}{8}$

⑦ $\lim\limits_{n\to\infty}\dfrac{2^{n+2}+3^{n+2}}{2^n+3^n}=$ $\qquad q^n\xrightarrow[n\to\infty]{}0,\ |q|<1$

$=\lim\limits_{n\to\infty}\dfrac{2^n\cdot4+9\cdot3^n}{2^n+3^n}$ $\quad |:3^n$

$\lim\limits_{n\to\infty}\dfrac{\frac{4\cdot2^n+9\cdot3^n}{3^n}}{\frac{2^n+3^n}{3^n}}=\lim\limits_{n\to\infty}\dfrac{4\left(\frac{2}{3}\right)^n+9}{\left(\frac{2}{3}\right)^n+1}=$

$=\dfrac{\lim\limits_{n\to\infty}4\cdot\left(\frac{2}{3}\right)^n+\lim\limits_{n\to\infty}9}{\lim\left(\frac{2}{3}\right)^n+\lim\limits_{n\to\infty}1}=\dfrac{0+9}{0+1}=9$

8) $\lim\limits_{n\to\infty} 4^{\frac{n+2}{n+1}} = \lim\limits_{n\to\infty} 4^{\left(1+\frac{1}{n+1}\right)} = \lim\limits_{n\to\infty} 4 \cdot \underbrace{\lim\limits_{n\to\infty} \sqrt[n+1]{4}}_{1} =$

$= 4$

9) $\lim\limits_{n\to\infty} \frac{\sqrt[n]{8}-1}{\sqrt[n]{2}-1} = \lim\limits_{n\to\infty} \frac{\cancel{(\sqrt[n]{2}-1)}(\sqrt[n]{4}+\sqrt[n]{2}+1)}{\cancel{\sqrt[n]{2}-1}} =$

$= \lim\limits_{n\to\infty} \sqrt[n]{4}+\sqrt[n]{2}+1 = 1+1+1=3$

ДР! $\lim\limits_{n\to\infty} \frac{3^n+(-2)^n}{3^{n+1}+(-2)^{n+1}} = \lim\limits_{n\to\infty} \frac{3^n+(-2)^n}{3\cdot 3^n+(-2)\cdot(-2)^n} \quad \Big| : 3^n$

$\lim\limits_{n\to\infty} \frac{1+\left(-\frac{2}{3}\right)^n}{3+(-2)\left(-\frac{2}{3}\right)^n} \quad \ldots$

Рекурентни редици

Пример: $n!$ $\begin{cases} 0! = 1 \\ (n+1)! = (n+1)\cdot n! \end{cases}$

Дадена е редица $\{a_n\}_{n=1}^{\infty}$, за която знаем първите k члена $a_1, a_2, \ldots, a_k$, а другите се получават като $a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k})$

Зад. Да се провери за сходимост и да се намери гр. на рекур. зададената редица

$\begin{cases} a_1 = \sqrt{2} \\ a_{n+1} = \sqrt{2-a_n} \end{cases}$

Допускаме, че редицата е сходяща

$$a_n \xrightarrow[n\to\infty]{} \ell$$

$$a_{n+1} \xrightarrow[n\to\infty]{} \ell \Rightarrow \sqrt{2a_n} \xrightarrow[n\to\infty]{} \sqrt{2\ell}$$

$a_{n+1} = \sqrt{2a_n}$

граничен преход $\ell = \sqrt{2\ell} \quad \uparrow^2$

$\ell^2 = 2\ell$

$\ell = 0$ или $\ell = 2$

! член е >0
$\Rightarrow \ell = 0$ не може

За док. сходимост (мон + огр.)

$$\frac{a_{n+1}}{a_n} \gtrless 1, \quad a_{n+1} - a_n = \overbrace{\sqrt{2a_n}}^{a} - \overbrace{a_n}^{b} =$$

$$= \frac{2a_n - a_n^2}{\sqrt{2a_n} + a_n} = \frac{a_n(2 - a_n)}{\underbrace{\sqrt{2a_n} + a_n}_{>0}}. \text{ зависи}$$

само от множителя $(2 - a_n)$

С индукция ще док., че $\forall n \in \mathbb{N}$
$a_n \in (0, 2)$
$0 < a_n < 2$

1) База - $n = 1$, $0 < \sqrt{2} < 2$ - очевидно

2) Допускаме, че за някое $k \in \mathbb{N}$ е вярно
$0 < a_k < 2$

3) Разгл. за $k+1$
$a_{k+1} = \sqrt{2a_k}$. От инд. предпол. $0 < a_k < 2$
$0 < 2a_k < 4$
$0 < \sqrt{2a_k} < 2$

Доказахме, че за $\forall n \in \mathbb{N}$, $0 < a_n < 2 \Rightarrow a_{n+1} - a_n > 0 \quad \forall n \in \mathbb{N}$
редицата е растяща и от док. огр. от 2

Следователно границата ӑ е $\ell = 2$ $\square$