Homework 01

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Contents

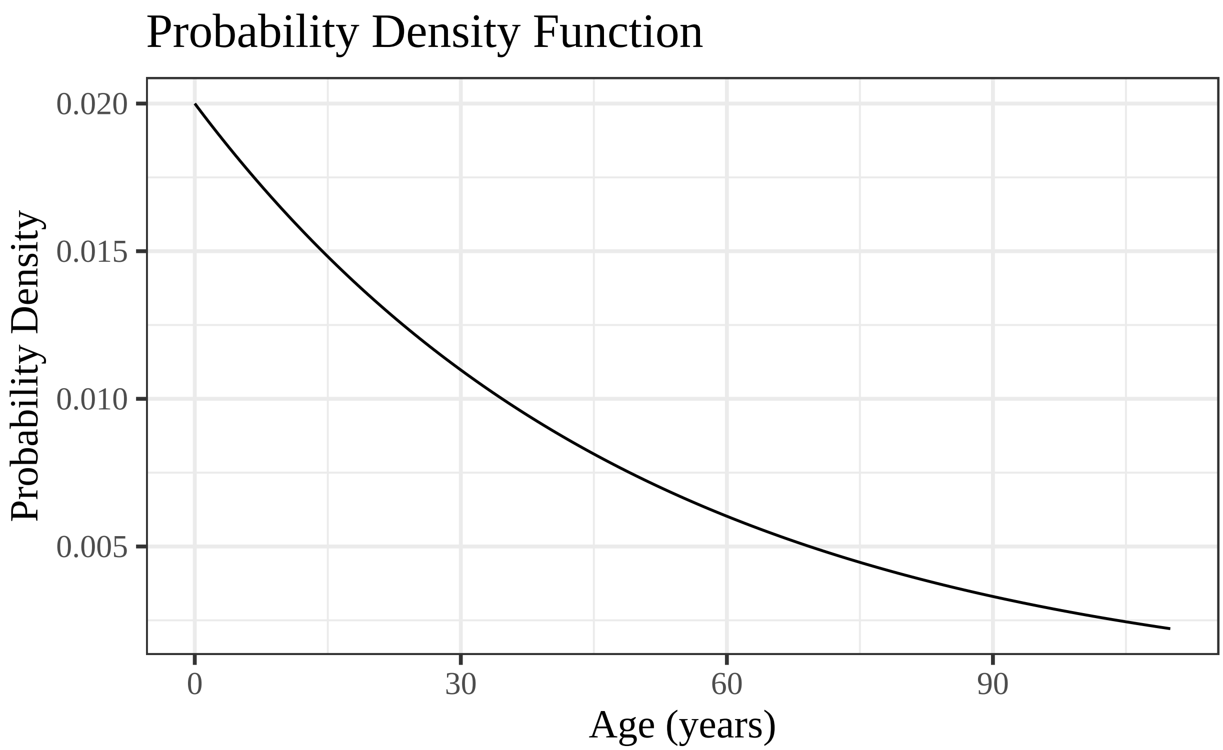
1 *Q1*

If the instantaneous mortality rate is constant (0.02) — independent of age, the probability distribution follows an exponential distribution. Let x equals to age at death; hence the relevant probability density function $f(x)$, cumulative distribution function $F(x)$, survival function $S(x)$:

$$
\begin{aligned}
f(x) &= 0.02e^{-0.02x} \\
F(x) &= 1 - e^{0.02x} \\
S(x) &= e^{-0.02x}
\end{aligned}
$$

1.1 (a)

Plotting the probability density function:



1.2 (b)

Probability that a member of this population is still alive at age 70: $S(70) = 0.246597$

1.3 (c)

Probability that a member of this population dies before age 6: $F(6) = 0.1130796$

1.4 (d)

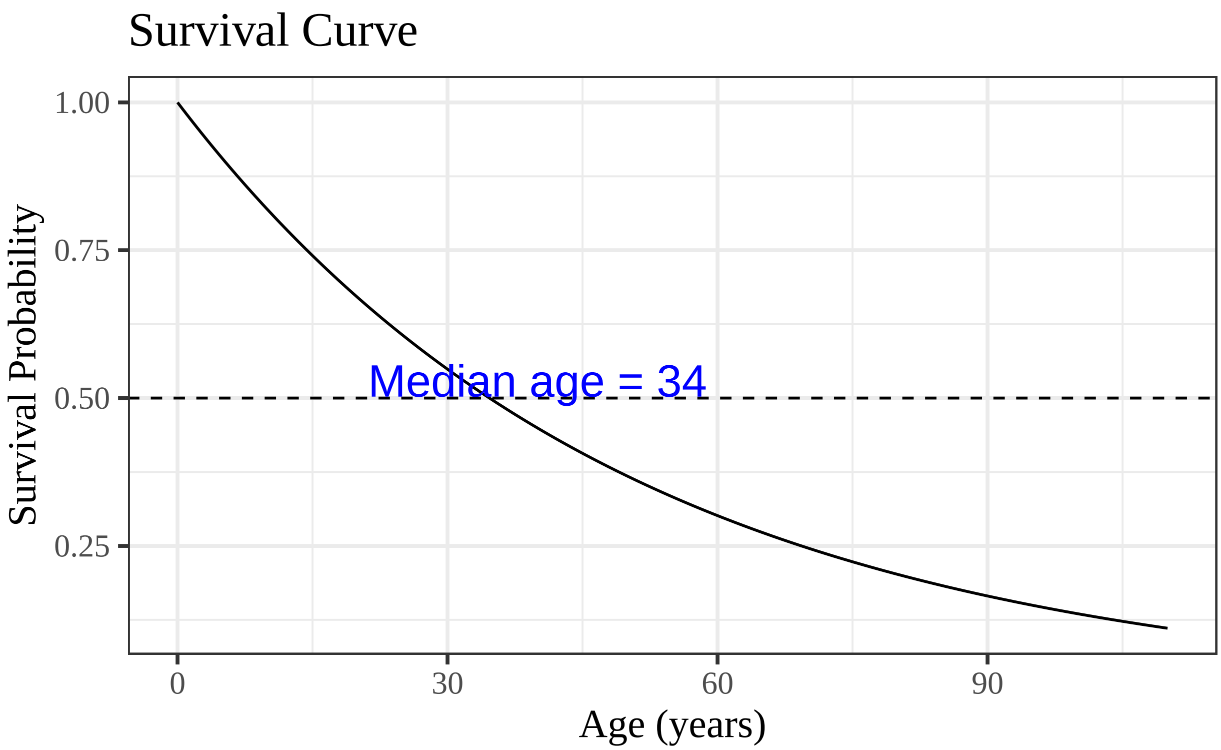
Life expectancy at birth for a member of this population: $e_0 = 50$

1.5 (e)

Life expectancy at age 50 for a member of this population: $e_{50} = 50$

1.6 (f)

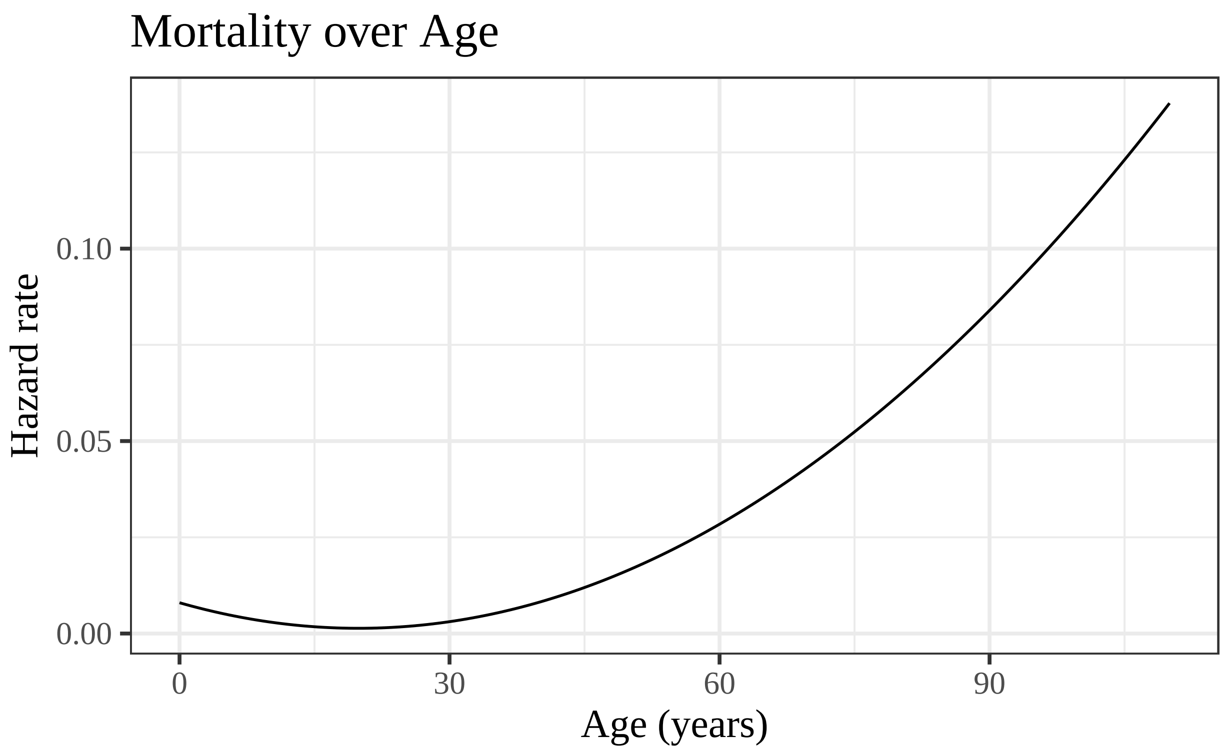
Median age at death for this population: $S(50) = 0.506617$



2 *Q2*

2.1 (a)

For ages 0 to 110, this mortality rate plot looks like:

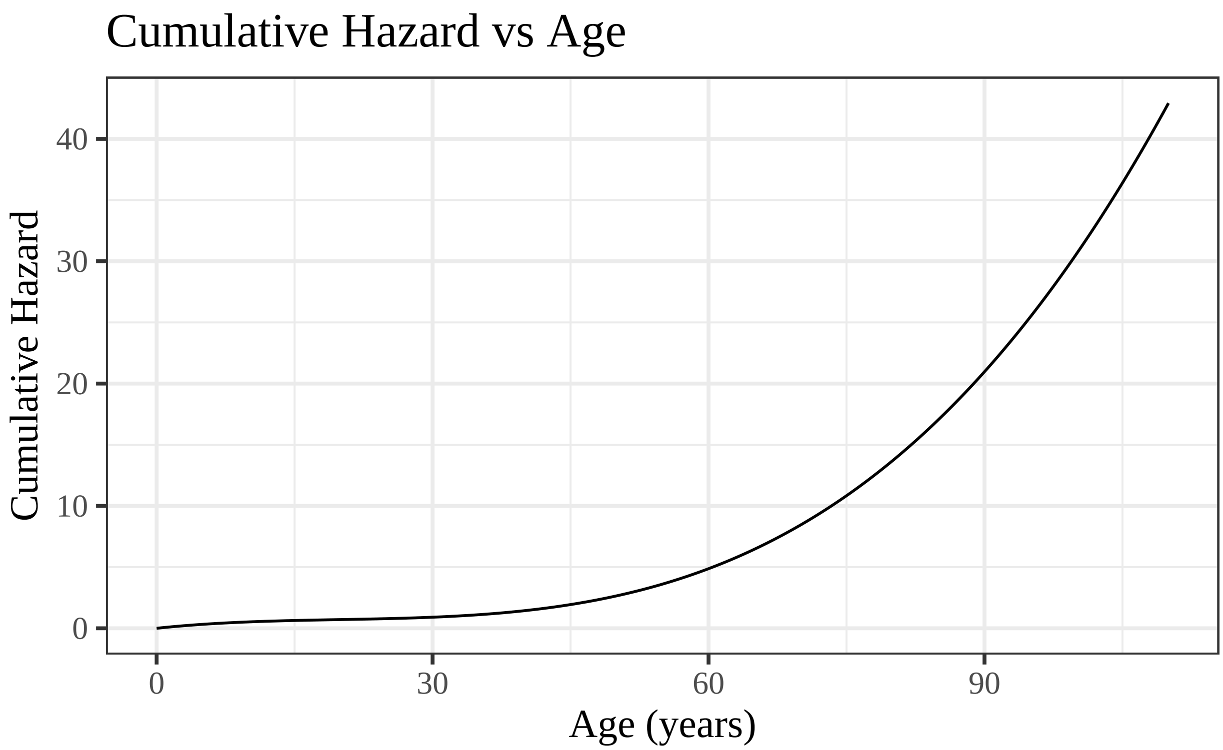


2.2 (b)

For a given instantaneous mortality function $\mu(x) = (0.0168x^2 - 0.668x + 8)/1000$, defined as the total area under the curve of $\mu(x)$ bounded on the interval $[0, x]$, or put another way:

$$
\begin{aligned}
\Lambda(x) &= \int_0^x \mu(u)du \\
&= \int_0^x \left[(0.0168x^2 - 0.668x + 8)/1000\right] dx \\
&= \int_0^x \frac{21x^2 - 835x + 10000}{1250000} dx \\
&= \frac{21}{125000}\int_0^x x^2 dx - \frac{167}{250000}\int_0^x x dx + \frac{1}{125}\int_0^x 1 dx \\
&= \frac{x \cdot (14x^2 - 835x + 20000)}{250000} + C
\end{aligned}
$$

For ages 0 to 110, this cumulative hazard functions looks like:

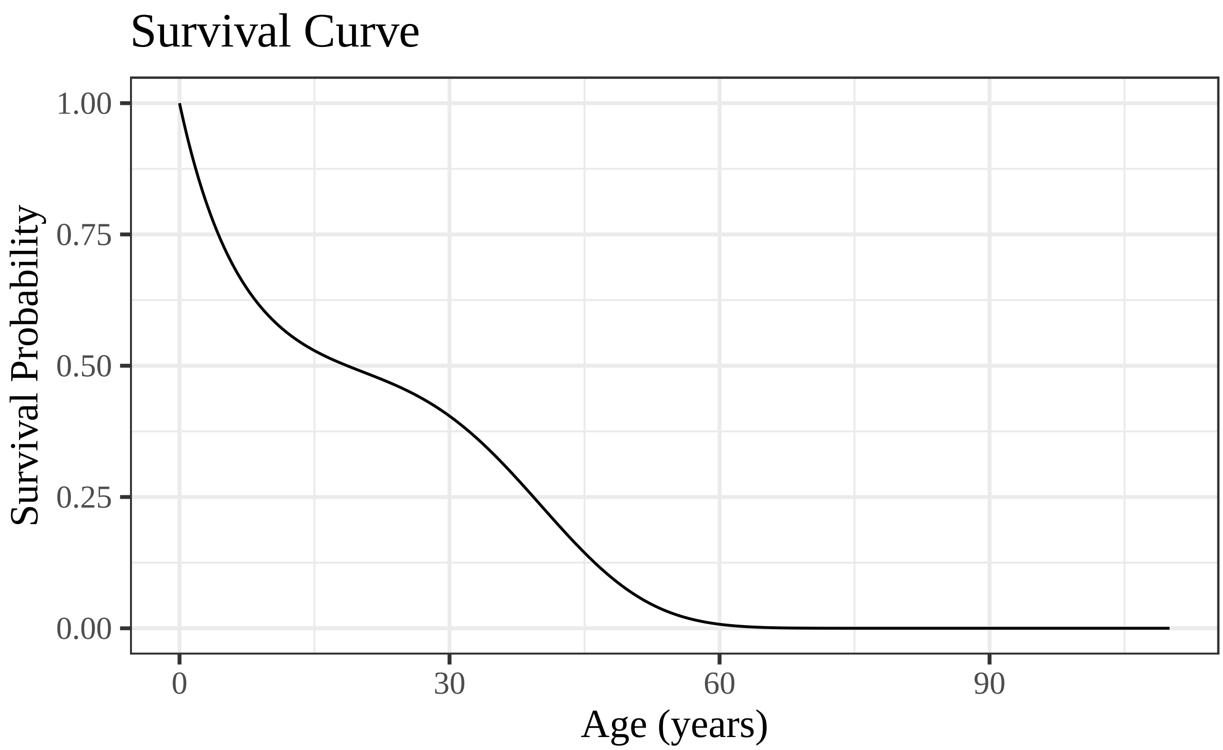


The survival function, $S(x)$, is defined as the exponentiated negative cumulative hazard function, $e^{-\Lambda(x)}$. Using our calculated cumulative hazard function, the survival function is then:

$$S(x) = \exp\left[\frac{-x \cdot (14x^2 - 835x + 20000)}{250000}\right]$$

2.3 (c)

For ages 0 to 110, the survival function then looks like:

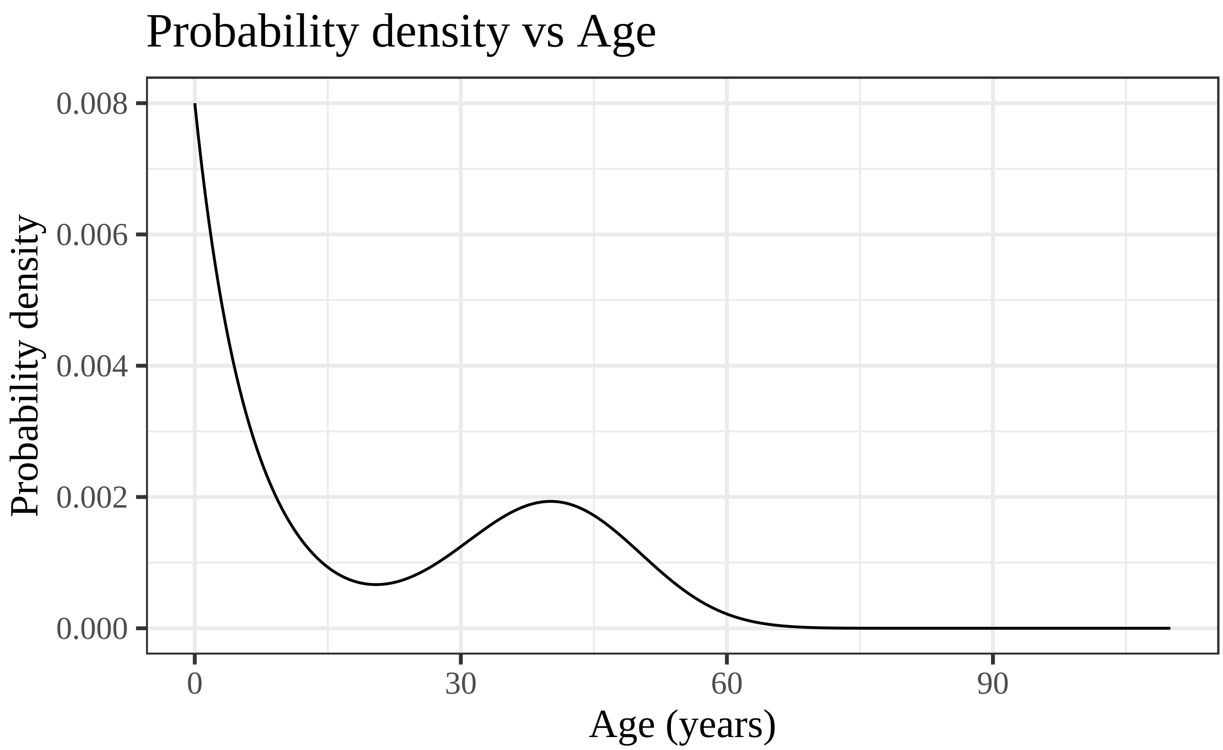


2.4 (d)

The probability density function of X, $f(x)$, is the negative derivative of the survival function with respect to x, $f(x) = -\frac{dS(x)}{dx}$. Using our calculated survival function, the probability density function of X is then:

$$f(x) = \frac{-d}{dx} = \mu(x)S(x)$$

For ages 0 to 110, the probability density function looks like:



2.5 (e)

Life expectancy at age x, e_x, is defined as:

$$e_x = \frac{\int_x^\infty S(u)du}{S(x)}$$

which simplifies to $\int_0^\infty S(u)du$ for life expectancy at birth, e_0. Using numerical integration, the life expectancy at birth for our cohort is calculated to be **22.399**.

2.6 (f)

The life expectancy at age 10 (e_{10}) for a member of this cohort is numerically calculated to be **25.156**.

2.7 (g)

The probability that a person aged x dies within the next n years is defined as:

$${}_nq_x = \frac{S(x) - S(x+n)}{S(x)}$$

The ${}_{45}q_{15}$ value for this cohort is then **0.986**.

3 *Q3*

3.1 (a) & (b)

$${}_nd_x = l_x - l_{x+n}$$

Table 1: UN Prospects for Peru

age	2015	2020	ndx
70-74	275.30	352.75	31.54
75-79	202.70	243.75	41.53
80-84	125.20	161.17	54.06
85-89	52.70	71.14	30.86
90-94	12.83	21.84	9.19
95-99	2.18	3.65	1.79

4 *Q4*

4.1 (a)

$${}_nM_x = \frac{{}_nd_x}{{}_nL_x}$$

Table 2: UN Life Table for Peru

age	2015	2020	ndx	nLx	nMx
70-74	275.30	352.75	31.54	1570.11	0.02
75-79	202.70	243.75	41.53	1116.12	0.04
80-84	125.20	161.17	54.06	715.92	0.08
85-89	52.70	71.14	30.86	309.60	0.10
90-94	12.83	21.84	9.19	86.68	0.11
95-99	2.18	3.65	1.79	14.57	0.12

4.2 (b)

Solution 1:

$${}_nq_x \approx 1 - e^{-{}_nM_x}$$

Solution 2:

$${}_nq_x \approx \frac{n_nM_x}{1 + n_nM_x/2}$$

Table 3: First Method

age	2015	2020	${}_nd_x$	${}_nL_x$	${}_nM_x$	${}_nq_x$
70-74	275.30	352.75	31.54	1570.11	0.02	0.10
75-79	202.70	243.75	41.53	1116.12	0.04	0.17
80-84	125.20	161.17	54.06	715.92	0.08	0.31
85-89	52.70	71.14	30.86	309.60	0.10	0.39
90-94	12.83	21.84	9.19	86.68	0.11	0.41
95-99	2.18	3.65	1.79	14.57	0.12	0.46

Table 4: Second Method

age	2015	2020	${}_nd_x$	${}_nL_x$	${}_nM_x$	${}_nq_x$
70-74	275.30	352.75	31.54	1570.11	0.02	0.10
75-79	202.70	243.75	41.53	1116.12	0.04	0.17
80-84	125.20	161.17	54.06	715.92	0.08	0.32
85-89	52.70	71.14	30.86	309.60	0.10	0.40
90-94	12.83	21.84	9.19	86.68	0.11	0.42
95-99	2.18	3.65	1.79	14.57	0.12	0.47

4.3 (c)

$$ {}_nm_x = \frac{{}_nd_x}{{}_nL_x} = \frac{{}_nd_x}{nl_x - 1/2 * n {}_nd_x} $$

Table 5: Life Table ${}_nm_x$

age	2015	2020	${}_nd_x$	${}_nL'_x$	${}_nm_x$
70-74	275.30	352.75	31.54	1297.62	0.02
75-79	202.70	243.75	41.53	909.68	0.05
80-84	125.20	161.17	54.06	490.83	0.11
85-89	52.70	71.14	30.86	186.36	0.17
90-94	12.83	21.84	9.19	41.19	0.22
95-99	2.18	3.65	1.79	6.44	0.28

4.4 (d)

Table 6: Comparison

age	${}_nq_x^1$	${}_nq_x^2$
70-74	0.096	0.096
75-79	0.170	0.170
80-84	0.314	0.318
85-89	0.393	0.399
90-94	0.411	0.419
95-99	0.458	0.469

As seen from the table above, the estimates from two methods are quite similiar. The difference occurs in later ages because ${}_nM_x$ becomes larger. I think second method is better because it doesn't assume that mortality rate is constant within the age interval. Instead, it is modeled as a linear function of age. As we move from 5-year age groups to single age groups, this became more important because even two-year age difference matters in later ages.

4.5 (e)

Table 7: Comparison: ${}_nM_x$ and ${}_nm_x$

age	2015	${}_nd_x$	${}_nL_x$	${}_nL'_x$	${}_nM_x$	${}_nm_x$
70-74	275.296	31.544	1570.110	1297.620	0.020	0.024
75-79	202.698	41.525	1116.125	909.678	0.037	0.046
80-84	125.197	54.061	715.925	490.832	0.076	0.110
85-89	52.703	30.861	309.597	186.362	0.100	0.166
90-94	12.832	9.186	86.685	41.195	0.106	0.223
95-99	2.181	1.786	14.567	6.440	0.123	0.277

${}_nM_x$ uses person-years (${}_nL_x$) assuming mortality rate is constant within 5-year age groups. "${}_nm_x$" uses person-years (${}_nL'_x$) assuming mortality rate is a linear function of age. As a result, former estimates more person-years because it doesn't take into account the deaths occur within the early years of the age group which happens more in older ages. As expected, ${}_nM_x$ underestimates mortality rate in older ages comparing to ${}_nm_x$

5 *Q5*

5.1 (a)

Table 8: Life Table - First Method

age	${}_nq_x$	${}_np_x$	${}_nd_x$	l_x	L_x	T_x	e_x
70-74	0.10	0.90	10000.00	955.71	47610.72	163332.68	16.33
75-79	0.17	0.83	9044.29	1535.23	41383.38	115721.96	12.80
80-84	0.31	0.69	7509.06	2361.36	31641.90	74338.59	9.90
85-89	0.39	0.61	5147.70	2020.48	20687.30	42696.68	8.29
90-94	0.41	0.59	3127.22	1286.25	12420.49	22009.38	7.04
95-99	0.46	0.54	1840.97	843.68	7095.66	9588.89	5.21
100+	1.00	0.00	997.29	997.29	2493.23	2493.23	2.50

Table 9: Life Table - Second Method

age	${}_nq_x$	${}_np_x$	${}_nd_x$	l_x	L_x	T_x	e_x
70-74	0.10	0.90	10000.00	956.48	47608.81	162423.19	16.24
75-79	0.17	0.83	9043.52	1539.15	41369.76	114814.38	12.70
80-84	0.32	0.68	7504.38	2383.42	31563.35	73444.63	9.79
85-89	0.40	0.60	5120.96	2043.16	20496.93	41881.27	8.18
90-94	0.42	0.58	3077.81	1289.23	12165.97	21384.34	6.95
95-99	0.47	0.53	1788.58	839.20	6844.91	9218.37	5.15
100+	1.00	0.00	949.38	949.38	2373.46	2373.46	2.50

5.2 (b)

Table 10: Life Expectancy Comparison

age	e_x^1	e_x^2
70-74	16.33	16.24
75-79	12.80	12.70
80-84	9.90	9.79
85-89	8.29	8.18
90-94	7.04	6.95
95-99	5.21	5.15
100+	2.50	2.50

The first approximation method generated slightly higher life expectancies for all ages, except 100+. This happens because the first method estimates lower probability of death (${}_nq_x$).

6 Code Appendix

```
# Prep work ---------------------------------------------------------------

# Load libraries
library(tidyverse, quietly = TRUE)
library(wpp2019)

# Make data
age_range <- c(0, 110)
age_data <- tibble(age = seq(age_range[1], age_range[2], .1))

# Question 1 ---------------------------------------------------------------
pdf = function(x) 0.02*exp(-0.02*x)
cdf = function(x) 1 - exp(-0.02*x)
survf = function(x) exp(-0.02*x)

pdf_plot =
  age_data %>%
  mutate(PDF = pdf(age)) %>%
  ggplot(aes(x=age, y=PDF)) +
  geom_line() +
  theme_bw(base_size = plot_size) +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Probability Density Function",
    x = "Age (years)",
    y = "Probability Density"
    )

median_plot =
  age_data %>%
  mutate(surv = survf(age)) %>%
  ggplot(aes(x=age, y=surv)) +
  geom_line() +
  geom_hline(yintercept = 0.5, linetype="dashed") +
  annotate("text", x=40, y=0.53, label="Median age = 34", color = "blue", size=6) +
  theme_bw(base_size = plot_size) +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Survival Curve",
    x = "Age (years)",
    y = "Survival Probability"
    )
pdf_plot
median_plot
# Question 2 ---------------------------------------------------------------
hazard_fun = function(x) (0.0168*x^2 - 0.668*x + 8)/1000

hazard_plot =
  age_data %>%
  mutate(mortality = hazard_fun(age)) %>%
  ggplot(aes(x=age, y=mortality)) +
```

```r
  geom_line() +
  theme_bw(base_size = plot_size) +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Mortality over Age",
    x = "Age (years)",
    y = "Hazard rate"
    )

cum_hazard_fun <- function(x) x*(14*x^2 - 835*x + 20000)/250000

chf_plot <-
  ggplot(age_data, aes(x = age, y = cum_hazard_fun(age))) +
  geom_line() +
  theme_bw(base_size = plot_size) +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Cumulative Hazard vs Age",
    x = "Age (years)",
    y = "Cumulative Hazard"
  )

survival_fun <- function(x) exp(-1 * cum_hazard_fun(x))

surv_plot =
  age_data %>%
  mutate(surv = survival_fun(age)) %>%
  ggplot(aes(x=age, y=surv)) +
  geom_line() +
  theme_bw(base_size = plot_size) +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Survival Curve",
    x = "Age (years)",
    y = "Survival Probability"
    )

pdf_fun = function(x) survival_fun(x) * hazard_fun(x)

pdfun_plot <-
  ggplot(age_data, aes(x = age, y = pdf_fun(age))) +
  geom_line() +
  theme_bw(base_size = plot_size) +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Probability density vs Age",
    x = "Age (years)",
    y = "Probability density"
  )


e0 <- integrate(survival_fun, lower = 0, upper = Inf)
e0_val <- round(e0$value, 3)
```

```r
e10 <- integrate(survival_fun, lower = 10, upper = Inf)
e10_val <- round(e10$value / survival_fun(10), digits = 3)

nqx <- function(x, n) (survival_fun(x) - survival_fun(x + n)) / survival_fun(x)
q45_15 <- round(nqx(15, 45), 3)
hazard_plot
chf_plot
surv_plot
pdfun_plot

# Question 3 ---------------------------------------------------------------
## download estimates of males
data("popM")

peru =
  popM %>%
  filter(name == "Peru") %>%
  filter(age == c("70-74", "75-79", "80-84", "85-89", "90-94", "95-99", "100+")) %>%
  select(age, "2015", "2020") %>%
  mutate(ndx = `2015` - lead(`2020`)) %>%
  filter(age != "100+")




peru %>%
  knitr::kable(digits = 2, booktabs = TRUE, escape = FALSE,
               caption = "UN Prospects for Peru")

# Question 4 ---------------------------------------------------------------
peru_LT =
  peru %>%
  mutate(nLx = (5*(`2015`+`2020`)/2)) %>%
  mutate(nMx = ndx/nLx)

peru_LT %>%
 knitr::kable(digits = 2, booktabs = TRUE, escape = FALSE,
              caption = "UN Life Table for Peru")

## first method
peru_LT %>%
 mutate(nqx = 1-exp(-5*nMx)) %>%
 knitr::kable(digits = 2, booktabs = TRUE, escape = FALSE,
              caption = "First Method",
             col.names=c("age", "2015", "2020", "$_nd_x$", "$_nL_x$", "$_nM_x$", "$_nq_x$"))


## second method
peru_LT %>%
  mutate(nqx = (5*nMx)/(1+5*nMx/2)) %>%
  knitr::kable(digits = 2, booktabs = TRUE, escape = FALSE,
               caption = "Second Method",
               col.names=c("age", "2015", "2020", "$_nd_x$", "$_nL_x$", "$_nM_x$", "$_nq_x$"))
```

```r
peru_LT %>%
  mutate(nLx = 5*`2015`-(1/2)*5*ndx) %>%
  mutate(nmx = ndx/nLx) %>%
  select(-nMx) %>%
  knitr::kable(digits = 2, booktabs = TRUE, escape = FALSE,
               caption = "Life Table $_nm_x$",
               col.names = c("age", "2015", "2020", "$_nd_x$", "$_nL_x'$", "$_nm_x$"))

## comparison
peru_LT %>%
  mutate(nqx_1 = 1-exp(-5*nMx)) %>%
  mutate(nqx_2 = (5*nMx)/(1+5*nMx/2)) %>%
  select(age, nqx_1, nqx_2) %>%
  knitr::kable(digits = 3, booktabs = TRUE, escape = FALSE,
               caption = "Comparison", col.names = c("age", "$_nq_x^1$", "$_nq_x^2$"))
peru_LT %>%
  mutate(nLx2 = 5*`2015`-(1/2)*5*ndx) %>%
  mutate(nmx = ndx/nLx2) %>%
  select(age, `2015`, ndx, nLx, nLx2, nMx, nmx) %>%
  knitr::kable(digits = 3, booktabs = TRUE, escape = FALSE,
               caption = "Comparison: $_nM_x$ and $_nm_x$",
               col.names = c("age", "2015", "$_nd_x$", "$_nL_x$", "$_nL_x'$" , "$_nM_x$", "$_nm_x$"))

# Question 5 --------------------------------------------------------------
## Life Table using first method
LT_1 = peru_LT %>%
  mutate(nqx = 1-exp(-5*nMx)) %>%
  select(age, nqx) %>%
  mutate(n = 5, npx = 1-nqx) %>%
  bind_rows(a=list(age="100+", n=5, nqx=1, npx=0))

npx <- LT_1$npx

age <- seq(70,100,5)
lx <- 10000

for (a in age[-length(age)]) {
  l <- lx[which(age == a)] * npx[which(age == a)]
  lx <- c(lx, l)
}

LT_1 <- LT_1 %>%
  mutate(
    x = age,
    lx = lx,
    ndx = lx*nqx,
    Lx = n*lx - n*ndx/2,
    Tx = rev(Lx) %>% coalesce(0) %>% cumsum() %>% rev(),
    ex = Tx / lx
)

## Life Table using second method
```

```
LT_2 = peru_LT %>%
  mutate(nqx = (5*nMx)/(1+5*nMx/2)) %>%
  select(age, nqx) %>%
  mutate(n = 5, npx = 1-nqx) %>%
  bind_rows(a=list(age="100+", n=5, nqx=1, npx=0))

npx <- LT_2$npx

age <- seq(70,100,5)
lx <- 10000

for (a in age[-length(age)]) {
  l <- lx[which(age == a)] * npx[which(age == a)]
  lx <- c(lx, l)
}

LT_2 <- LT_2 %>%
  mutate(
    x = age,
    lx = lx,
    ndx = lx*nqx,
    Lx = n*lx - n*ndx/2,
    Tx = rev(Lx) %>% coalesce(0) %>% cumsum() %>% rev(),
    ex = Tx / lx
)
LT_1 %>%
  select(-n, -x) %>%
  knitr::kable(digits = 2, booktabs = TRUE, escape = FALSE,
               caption = "Life Table - First Method",
               col.names=c("age", "$_nq_x$", "$_np_x$", "$_nd_x$", "$l_x$", "$L_x$", "$T_x$", "$e_x$"))


LT_2 %>%
  select(-n, -x) %>%
  knitr::kable(digits = 2, booktabs = TRUE, escape = FALSE,
               caption = "Life Table - Second Method",
               col.names=c("age", "$_nq_x$", "$_np_x$", "$_nd_x$", "$l_x$", "$L_x$", "$T_x$", "$e_x$"))

LT_1 %>%
  select(age, ex_1 = ex) %>%
  cbind(ex_2 = LT_2$ex) %>%
  knitr::kable(digits = 2, booktabs = TRUE, escape = FALSE,
               caption = "Life Expectancy Comparison", col.names = c("age", "$e_x^1$", "$e_x^2$" ))
```