Homework 01

Spencer Pease

4/6/2020

Questions

Q1

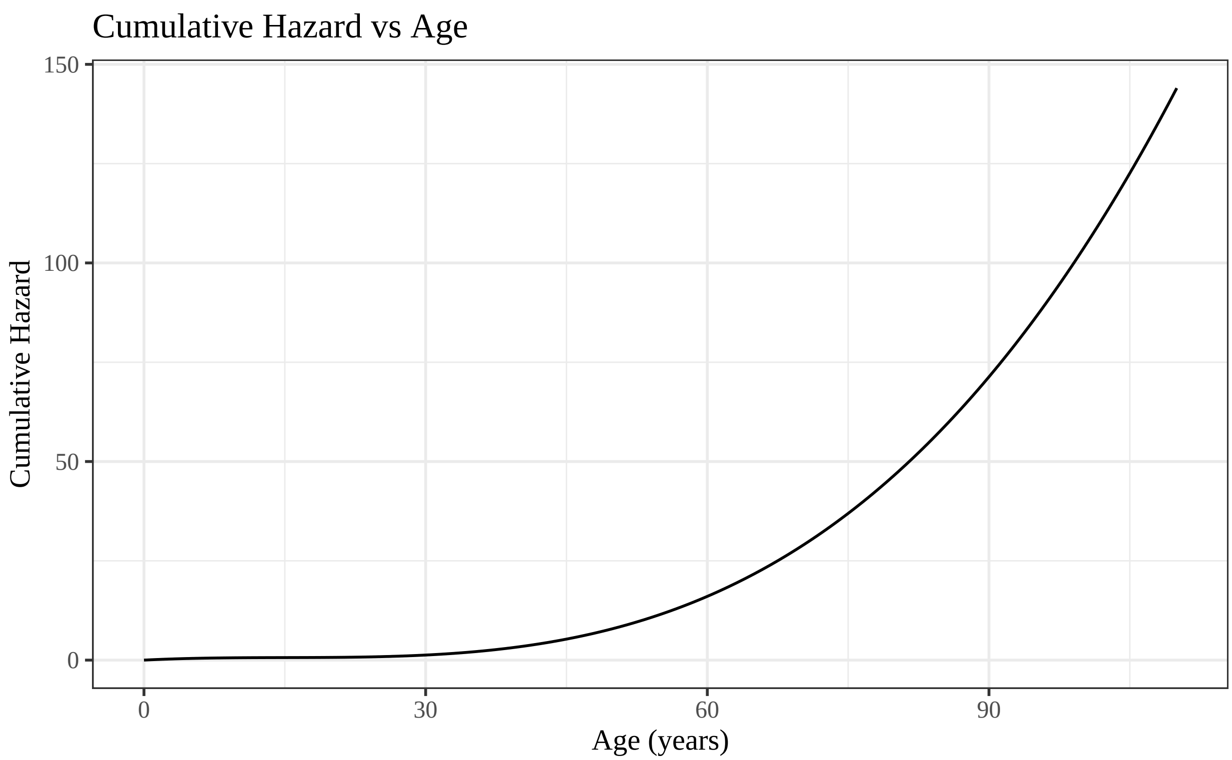
For a given force of mortality function, $\mu(x)$, the cumulative hazard is defined as the total area under the curve of $\mu(x)$ bounded on the interval $[0, x]$, or put another way:

$$\Lambda(x) = \int_0^x \mu(u)du$$

For the force of mortality function $\mu(x) = 0.005 + 0.0005(x - 15)^2$, the cumulative hazard function is then:

$$\begin{aligned}
\Lambda(x) &= \int_0^x \mu(u)du \\
&= \int_0^x \left[0.005 + 0.0005(u - 15)^2\right] du \\
&= 0.005x + 0.0005 \int_0^x \left(u^2 - 30u + 225\right) du \\
&= 0.005x + 0.0005 \left(\frac{1}{3}u^3 - 15u^2 + 225u\right)\Big|_0^x \\
&= 0.005x + 0.0005 \left(\frac{1}{3}x^3 - 15x^2 + 225x\right) \\
&= \frac{1}{2000} \left(\frac{1}{3}x^3 - 15x^2 + 235x\right)
\end{aligned}$$

For ages 0 to 110, this cumulative hazard functions looks like:

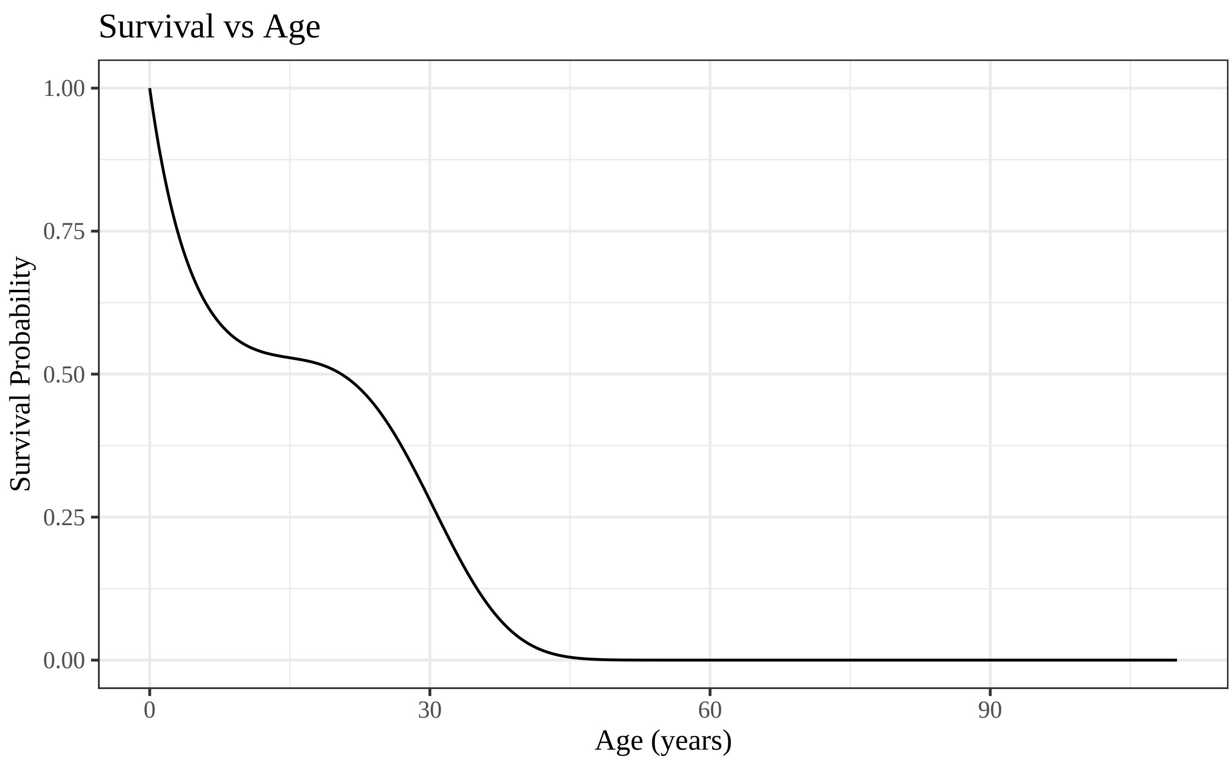


Q2

The survival function, $S(x)$, is defined as the exponentiated negative cumulative hazard function, $e^{-\Lambda(x)}$. Using our calculated cumulative hazard function, the survival function is then:

$$S(x) = \exp\left[\frac{-1}{2000}\left(\frac{1}{3}x^3 - 15x^2 + 235x\right)\right]$$

For ages 0 to 110, the survival function then looks like:

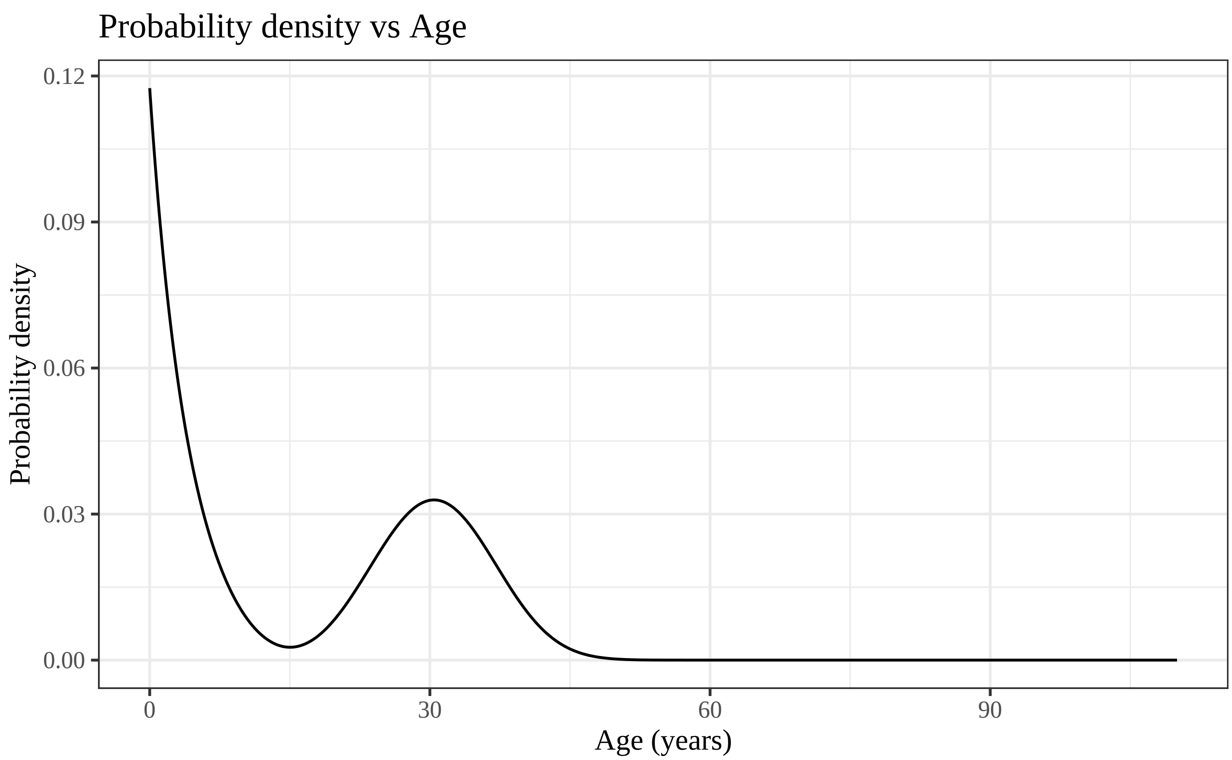


Q3

The probability density function of X, $f(x)$, is the negative derivative of the survival function with respect to x, $f(x) = -\frac{dS(x)}{dx}$. Using our calculated survival function, the probability density function of X is then:

$$\begin{aligned} f(x) &= \frac{-d}{dx} \exp\left[\frac{-1}{2000}\left(\frac{1}{3}x^3 - 15x^2 + 235x\right)\right] \\ &= \frac{1}{2000}\frac{d}{dx}\left(\frac{1}{3}x^3 - 15x^2 + 235x\right) S(x) \\ &= \frac{1}{2000}\left(x^2 - 30x + 235\right) S(x) \\ &= \mu(x)S(x) \end{aligned}$$

For ages 0 to 110, the probability density function looks like:



Q4

Assuming an initial cohort size of $100,000$, we can actuate a single year life table using our survival function $S(x)$, and by treating l_x as linear over small intervals.

Table 1: Cohort life table for single year ages 0 to 50

age	l_x	${}_1d_x$	${}_1q_x$	${}_1m_x$
0	100000.0	10431.5	0.104	0.110
1	89568.5	8212.3	0.092	0.096
2	81356.2	6492.4	0.080	0.083
3	74863.8	5142.6	0.069	0.071
4	69721.1	4071.2	0.058	0.060
5	65650.0	3212.2	0.049	0.050
6	62437.8	2518.2	0.040	0.041
7	59919.6	1954.7	0.033	0.033
8	57964.9	1497.1	0.026	0.026
9	56467.8	1127.4	0.020	0.020
10	55340.4	833.0	0.015	0.015
11	54507.4	605.3	0.011	0.011
12	53902.1	438.4	0.008	0.008
13	53463.7	328.7	0.006	0.006
14	53135.1	273.8	0.005	0.005
15	52861.2	272.4	0.005	0.005
16	52588.8	323.3	0.006	0.006
17	52265.5	425.1	0.008	0.008
18	51840.4	575.7	0.011	0.011

age	l_x	${}_1d_x$	${}_1q_x$	${}_1m_x$
19	51264.8	771.6	0.015	0.015
20	50493.1	1008.1	0.020	0.020
21	49485.0	1278.1	0.026	0.026
22	48207.0	1572.6	0.033	0.033
23	46634.3	1880.8	0.040	0.041
24	44753.5	2189.7	0.049	0.050
25	42563.8	2485.4	0.058	0.060
26	40078.4	2753.1	0.069	0.071
27	37325.3	2978.6	0.080	0.083
28	34346.6	3149.2	0.092	0.096
29	31197.5	3254.4	0.104	0.110
30	27943.1	3287.5	0.118	0.125
31	24655.6	3246.0	0.132	0.141
32	21409.5	3132.1	0.146	0.158
33	18277.5	2952.2	0.162	0.176
34	15325.3	2717.2	0.177	0.195
35	12608.1	2440.8	0.194	0.214
36	10167.2	2138.7	0.210	0.235
37	8028.5	1826.8	0.228	0.257
38	6201.8	1520.0	0.245	0.279
39	4681.7	1231.3	0.263	0.303
40	3450.4	970.3	0.281	0.327
41	2480.2	743.2	0.300	0.352
42	1737.0	552.9	0.318	0.379
43	1184.1	399.2	0.337	0.405
44	784.9	279.5	0.356	0.433
45	505.4	189.6	0.375	0.462
46	315.8	124.5	0.394	0.491
47	191.3	79.1	0.413	0.521
48	112.3	48.5	0.432	0.552
49	63.7	28.8	0.451	0.583
50	35.0	-	-	-

Q5

Life expectancy at age x, e_x, is defined as:

$$e_x = \frac{\int_x^\infty S(u)du}{S(x)}$$

which simplifies to $\int_0^\infty S(u)du$ for life expectancy at birth, e_0. Using numerical integration, the life expectancy at birth for our cohort is calculated to be **17.834**.

Q6

The life expectancy at age 10 (e_{10}) for a member of this cohort is numerically calculated to be **19.673**.

Q7

The probability that a person aged x dies within the next n years is defined as:

$$
{}_nq_x = \frac{S(x) - S(x+n)}{S(x)}
$$

The ${}_{15}q_{15}$ value for this cohort is then **0.471**.

Appendix

```r
# Prep work ---------------------------------------------------------------

# Load libraries
library(dplyr)
library(ggplot2)

# Make data
age_range <- c(0, 110)
age_data <- tibble(age = seq(age_range[1], age_range[2], .1))


# Question 1 --------------------------------------------------------------

hazard_fun <- function(x) 0.005 + 0.0005 * (x - 15)^2
cum_hazard_fun <- function(x) .0005 * ((x^3 / 3) - (15 * x^2) + (235 * x))

chf_plot <- ggplot(age_data, aes(x = age, y = cum_hazard_fun(age))) +
  geom_line() +
  theme_bw() +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Cumulative Hazard vs Age",
    x = "Age (years)",
    y = "Cumulative Hazard"
  )

chf_plot


# Question 2 --------------------------------------------------------------

survival_fun <- function(x) exp(-1 * cum_hazard_fun(x))

survf_plot <- ggplot(age_data, aes(x = age, y = survival_fun(age))) +
  geom_line() +
  theme_bw() +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Survival vs Age",
    x = "Age (years)",
    y = "Survival Probability"
  )

survf_plot
```

```r
# Question 3 -------------------------------------------------------------

pdfun_plot <-
  ggplot(age_data, aes(x = age, y = survival_fun(age) * hazard_fun(age))) +
  geom_line() +
  theme_bw() +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Probability density vs Age",
    x = "Age (years)",
    y = "Probability density"
  )

pdfun_plot


# Question 4 -------------------------------------------------------------

cohort_size <- 100000

lt_data <- tibble(
  age = 0:50,
  lx = survival_fun(age) * cohort_size,
  dx = lx - lead(lx),
  qx = dx / lx,
  mx =  dx / (lx - .5 * dx)
)

lt_names <- c("age", "$l_x$", "$_{1}d_x$", "$_{1}q_x$", "$_{1}m_x$")

knitr::kable(lt_data,
  booktabs = TRUE,
  escape = FALSE,
  digits = c(0, 1, 1, 3, 3),
  col.names = lt_names,
  caption = "Cohort life table for single year ages 0 to 50"
)


# Question 5 -------------------------------------------------------------

e0 <- integrate(survival_fun, lower = 0, upper = Inf)
e0_val <- round(e0$value, 3)


# Question 6 -------------------------------------------------------------

e10 <- integrate(survival_fun, lower = 10, upper = Inf)
e10_val <- round(e10$value / survival_fun(10), digits = 3)


# Question 7 -------------------------------------------------------------
```

```r
nqx <- function(x, n) (survival_fun(x) - survival_fun(x + n)) / survival_fun(x)
q15_15 <- round(nqx(15, 15), 3)
```