Homework 02

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Questions

Q1

Table 1: Q1 one-sex closed population

Age	Population (N_x)	Fertility Rate ($\tilde{F}_x$)	Survival Prob. (s_x)
1	18000	0.0	0.65
2	17000	0.9	0.75
3+	14000	0.2	0.15

Q1.a

The crude birth rate (CBR) is defined as the number of births over the person-years lived in the period $[T_1, T_2]$. Since our period is a single year, we can calculate CBR as:

$$CBR = \sum \frac{N_x \tilde{F}_x}{N_x s_x}$$

where we sum over all age groups. The crude birth rate for this population in the next time period is then **0.682**.

Q1.b

The total fertility rate in the population in the period $[T_1, T_2]$ is defined as the sum of the age-specific fertility rates across all age groups, multiplied by the length of the age interval, n. With $T_2 - T_1 = n = 1$, the total fertility rate represents the single-year cohort total fertility rate:

$$\text{TFR}[T_1, T_1 + 1] = \sum {}_1F_x[T_1, T_1 + 1]$$

We can convert between $\tilde{F}_x$ and ${}_1F_x$ using the equation

$$\tilde{F}_x = {}_1F_x \times \frac{1}{1 + SRB} \times \frac{1}{2}\left(1 + s_{x-1}\frac{N_{x-1,t}}{N_{x,t}}\right) \times \left(1 - \frac{q_0}{2}\right)$$

where we assume $SRB = 1.05$ and take $q_0 = 1 - s_0$. After converting to age-specific fertility rates, we calculate a total fertility rate of $0 + 2.649 + 0.52 = \mathbf{3.17}$ for this population.

Q1.c

The Leslie matrix, L, for this population is defined as:

$$L = \begin{bmatrix} \tilde{F}_{A-3} & \tilde{F}_{A-2} & \tilde{F}_{A-1} \\ s_{A-3} & 0 & 0 \\ 0 & s_{A-2} & s_{A-1} \end{bmatrix} = \begin{bmatrix} 0 & 0.9 & 0.2 \\ 0.65 & 0 & 0 \\ 0 & 0.75 & 0.15 \end{bmatrix}$$

Where $(A-1)+$ is the highest age group that can be reached in this population, 3+, s_x denotes the probability of survival to the next age group for age group x, and $\tilde{F}_x$ is the expected number of female births to a woman age x, who survives to the next time interval.

Q1.d

We can project this population forward using the *cohort-component method of population projection*, which states that the age-specific populations one time period ahead (N_{t+1}) can be calculated from the matrix multiplication of the age-specific population in the current period (N_t) and the Leslie matrix (L) of the population. The population by age one period forward from our given initial population is then:

$$\begin{aligned} N_{t+1} &= LN_t \\ &= \begin{bmatrix} 0 & 0.9 & 0.2 \\ 0.65 & 0 & 0 \\ 0 & 0.75 & 0.15 \end{bmatrix} \begin{bmatrix} 18000 \\ 17000 \\ 14000 \end{bmatrix} \\ &= \begin{bmatrix} 18100 \\ 11700 \\ 14850 \end{bmatrix} \end{aligned}$$

Q1.e

This method can be extended to projecting age-specific population k periods ahead by raising the Leslie matrix to the k^{th} power (L^k). Our given population, projected 10 periods into the future is then:

$$\begin{aligned} N_{t+10} &= L^{10}N_t \\ &= \begin{bmatrix} 0 & 0.9 & 0.2 \\ 0.65 & 0 & 0 \\ 0 & 0.75 & 0.15 \end{bmatrix}^{10} \begin{bmatrix} 18000 \\ 17000 \\ 14000 \end{bmatrix} \\ &= \begin{bmatrix} 0.102 & 0.104 & 0.022 \\ 0.056 & 0.102 & 0.022 \\ 0.084 & 0.084 & 0.018 \end{bmatrix} \begin{bmatrix} 18000 \\ 17000 \\ 14000 \end{bmatrix} \\ &= \begin{bmatrix} 3903.625 \\ 3051.594 \\ 3189.032 \end{bmatrix} \end{aligned}$$

Q1.f

The crude birth rate for this population 10 time periods ahead is **0.638**.

The total fertility rate 10 time periods ahead, after converting $\tilde{F}_x$ to ${}_1F_x$, is **3.021** for this population.

Q1.g

From the theorem that N_t converges to $\lambda^t u$ as t approaches infinity, $log(\lambda)$ is the *instantaneous rate of increase of the population*. Here, λ is defined as the dominant right eigenvalue of the Leslie matrix, or for the equation:

$$Lv = \lambda v$$

it is the eigenvalue λ with the largest magnitude. For our calculated Leslie matrix, the instantaneous rate of increase is **-0.161.**

Q1.h

Again, from the the formula $\lambda^t u$, u is the *stable age distribution*, and is defined as the dominant right eigenvector of the Leslie matrix, which is the column vector v from the eigendecomposition of L corresponding to the eigenvalue λ with the largest magnitude.

For our calculated Leslie matrix, the stable age distribution is $\begin{bmatrix} 0.666 \\ 0.509 \\ 0.545 \end{bmatrix}$.

Q1.i

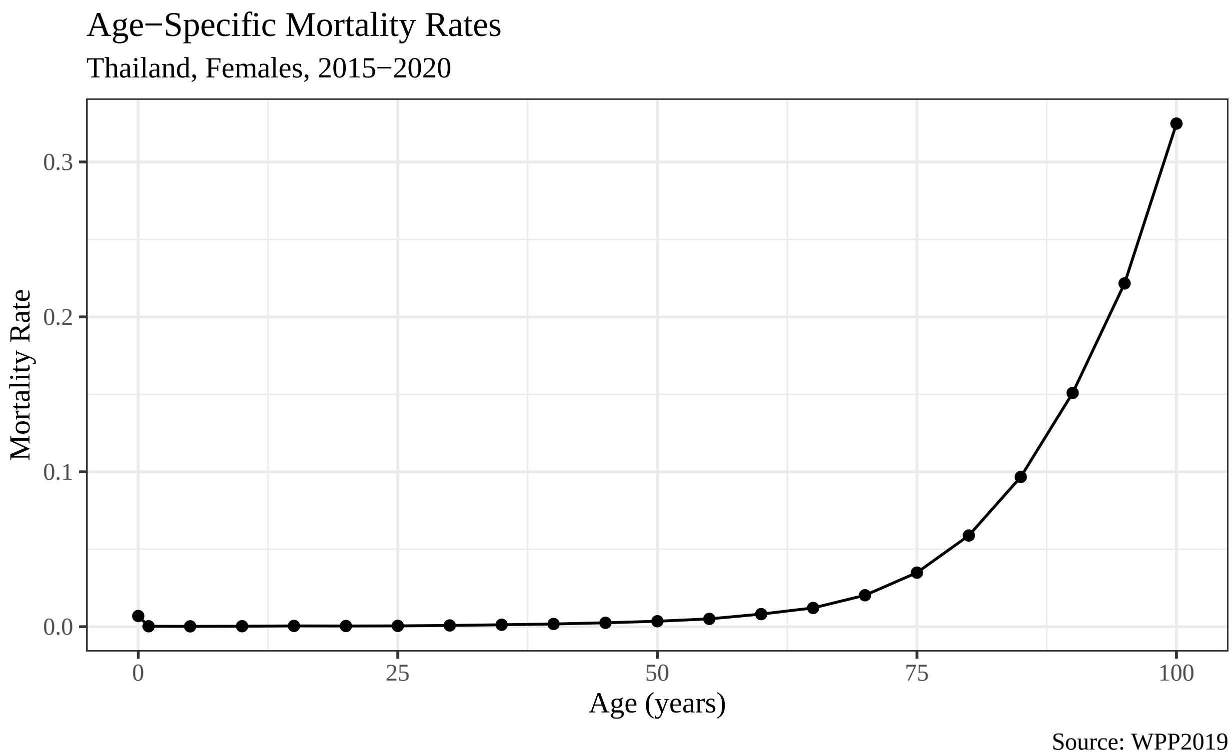
The reproductive value vector (v) is a vector of expected the number of future offspring of an individual for each age group. A theorem states that v is the dominant left eigenvector of the Leslie matrix for the population. The left dominant eigenvector of a matrix A is equivalent to the right dominant eigenvector of the transpose of matrix, $A^\top$. So, in the formula:

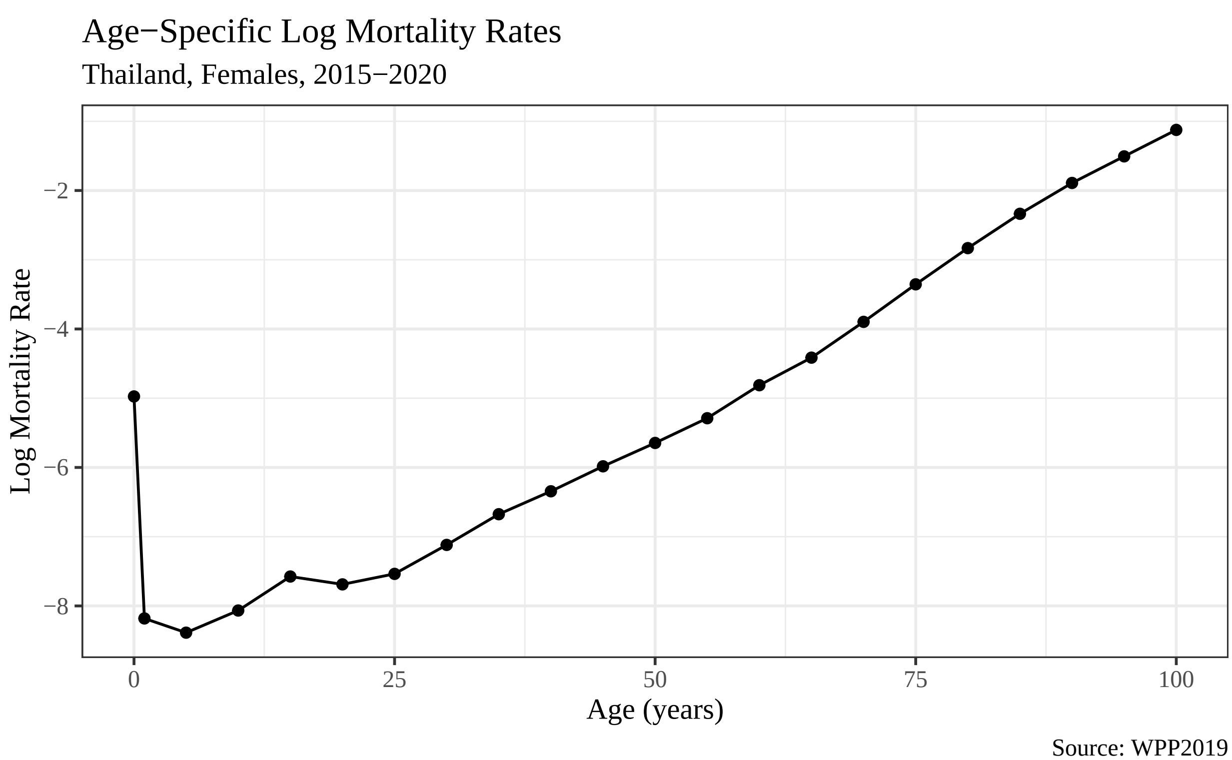
$$L^\top u = \kappa u$$

the dominant eigenvector u represents the reproductive values. For our Leslie matrix, the reproductive value matrix is then $\begin{bmatrix} -0.598 \\ -0.783 \\ -0.171 \end{bmatrix}$.

Q2

Q2.a





One unusual feature of these mortality rates is the the bump in log-mortality around the 15-20 year group.

This is a common feature in male populations (called the accident hump), but is not as common in female populations. Another feature to note is the incredibly high infant mortality rate, which is at the same level as older adults.

Q2.b

To derive a life table from the given ${}_nM_x$ values, we use the following formulas:

$$
\begin{aligned}
{}_nq_x &= 1 - e^{-n \cdot {}_nM_x} \\
{}_ns_x &= 1 - {}_nq_x \\
l_{x+n} &= (1 - {}_nq_x) \cdot l_x \\
{}_nd_x &= l_x - l_{x+n} \\
{}_nL_x &= l_{x+n} + a_x \cdot {}_nd_x \\
T_x &= \sum_{age=x}^{\infty} {}_nL_x \\
e_x &= T_x \div L_x
\end{aligned}
$$

Where we are using the approximation ${}_nq_x = 1 - e^{-n \cdot {}_nM_x}$ to to get ${}_nq_x$ from ${}_nM_x$, and a_x is calculated as $\frac{n}{2}$ except for the 0-1 age group, where it is $0.07 + 1.7{}_1M_0$.

Table 2: Life table for 2015-2020 Thailand female population

Age	n	${}_5m_x$	${}_1q_x$	${}_1s_x$	l_x	${}_5d_x$	a_x	${}_5L_x$	${}_5T_x$	e_x
0	1	0.007	0.007	0.993	100000	689	0.08	496612	8562458	85.62
1-4	4	0.000	0.001	0.999	99311	111	2.00	496222	8065846	81.22
5-9	5	0.000	0.001	0.999	99200	113	2.50	495717	7569624	76.31
10-14	5	0.000	0.002	0.998	99087	155	2.50	495046	7073907	71.39
15-19	5	0.001	0.003	0.997	98931	253	2.50	494024	6578862	66.50
20-24	5	0.000	0.002	0.998	98678	226	2.50	492826	6084838	61.66
25-29	5	0.001	0.003	0.997	98452	262	2.50	491606	5592012	56.80
30-34	5	0.001	0.004	0.996	98190	397	2.50	489959	5100406	51.94
35-39	5	0.001	0.006	0.994	97793	615	2.50	487429	4610446	47.14
40-44	5	0.002	0.009	0.991	97178	850	2.50	483766	4123017	42.43
45-49	5	0.003	0.013	0.987	96328	1206	2.50	478626	3639251	37.78
50-54	5	0.004	0.018	0.982	95122	1665	2.50	471449	3160625	33.23
55-59	5	0.005	0.025	0.975	93458	2329	2.50	461464	2689175	28.77
60-64	5	0.008	0.040	0.960	91128	3629	2.50	446568	2227711	24.45
65-69	5	0.012	0.059	0.941	87499	5140	2.50	424646	1781143	20.36
70-74	5	0.020	0.097	0.903	82359	7950	2.50	391921	1356497	16.47
75-79	5	0.035	0.160	0.840	74409	11918	2.50	342251	964576	12.96
80-84	5	0.059	0.255	0.745	62491	15938	2.50	272612	622325	9.96
85-89	5	0.097	0.383	0.617	46553	17844	2.50	188157	349713	7.51
90-94	5	0.151	0.530	0.470	28709	15210	2.50	105522	161556	5.63
95-99	5	0.222	0.670	0.330	13499	9042	2.50	44892	56035	4.15
100+	5	0.325	1.000	0.000	4457	4457	2.50	11143	11143	2.50

Q2.c

The life expectancy at birth is **85.625**, and the life expectancy at age 10 is **71.391**

Q3

Q3.a

To calculate ${}_5\tilde{F}_x$, we first use the provided proportional age-specific fertility rate and total fertility rate for Thailand in 2015 to get age-specific fertility rate with the formula:

$$\frac{TFR[T_1, T_2] \times {}_nPASFR_x}{n} = {}_nF_x[T_1, T_2]$$

where $n = 5$ and $[T_1, T_2] = [2015, 2020]$. Then we use the provided population and previously calculated mortality rates ${}_5s_x$ and ${}_5q_0$ to calculate ${}_5\tilde{F}_x$ using the formula:

$${}_n\tilde{F}_x = {}_nF_x \times \frac{1}{1 + SRB} \times \frac{1}{2}\left(1 + {}_ns_x \frac{{}_5N_{x-1,t}}{{}_5N_{x,t}}\right) \times \left(1 - \frac{{}_nq_0}{2}\right)$$

Note that ${}_5q_0$ was calculated as $1 - {}_5s_0$, which was in turn calculated from ${}_1s_0 \times {}_4s_1$.

With these calculations, our resulting ${}_5\tilde{F}_x$ is:

Table 3: Expected number of live female births per woman per five-year period in Thailand, 2015-2020

Age	${}_5\tilde{F}_x$	${}_5F_x$	${}_5N_x$	${}_5s_x$
15-19	0.000	4.491	2356	0.997
20-24	3.633	7.587	2392	0.998
25-29	4.014	8.037	2234	0.997
30-34	3.138	6.748	2408	0.996
35-39	1.353	2.989	2745	0.994
40-44	0.342	0.737	2954	0.991
45-49	0.049	0.103	2983	0.987
0-4	0.000	0.000	1867	0.992
5-9	0.000	0.000	1998	0.999
10-14	0.000	0.000	2135	0.998
50-54	0.000	0.000	2733	0.982
55-59	0.000	0.000	2405	0.975
60-64	0.000	0.000	1853	0.960
65-69	0.000	0.000	1373	0.941
70-74	0.000	0.000	1031	0.903
75-79	0.000	0.000	797	0.840
80-84	0.000	0.000	512	0.745
85-89	0.000	0.000	265	0.617
90-94	0.000	0.000	87	0.470
95-99	0.000	0.000	22	0.330
100+	0.000	0.000	4	0.000

Q3.b

Using our calculated ${}_5\tilde{F}_x$ and ${}_5s_x$, we can build a Leslie matrix for this population:

Table 4: Leslie matrix for Thailand females, 2015-2020, (sparse format)

Row	Column	Value
1	2	3.633
1	3	4.014
1	4	3.138
1	5	1.353
1	6	0.342
1	7	0.049
2	1	0.997
3	2	0.998
4	3	0.997
5	4	0.996
6	5	0.994
7	6	0.991
8	7	0.987
9	8	0.992
10	9	0.999
11	10	0.998
12	11	0.982
13	12	0.975
14	13	0.960
15	14	0.941
16	15	0.903
17	16	0.840
18	17	0.745
19	18	0.617
20	19	0.470
21	20	0.330

Q3.c

Using the given female population in Thailand, 2015-2020 (*table 3*), we can calculate the population one 5-year period ahead to 2020, assuming fertility and mortality are constant over time, and that there is no migration.

$$N_{t+1} = LN_t = \begin{bmatrix} 30085 \\ 2350 \\ 2386 \\ 2229 \\ 2398 \\ 2728 \\ 2928 \\ 2946 \\ 1852 \\ 1996 \\ 2131 \\ 2685 \\ 2345 \\ 1779 \\ 1293 \\ 932 \\ 670 \\ 381 \\ 164 \\ 41 \\ 7 \end{bmatrix}$$

Q3.d

Under the same assumptions as the previous question, we can instead project the population 15 years (3 time periods) into the future.

$$N_{t+3} = L^3 N_t = \begin{bmatrix} 129846 \\ 29355 \\ 29939 \\ 2338 \\ 2370 \\ 2206 \\ 2362 \\ 2670 \\ 2868 \\ 2919 \\ 1847 \\ 1958 \\ 2042 \\ 2514 \\ 2119 \\ 1513 \\ 981 \\ 583 \\ 308 \\ 111 \\ 25 \end{bmatrix}$$

Q3.e

Under the assumption that the age-specific migration rates follow the same distribution as the population distribution, we can calculate age-specific migration rates using the provided net migration rate (after converting the rate from *migrations per 1,000 person-years* to *migrant per person-year*).

Table 5: Age-specific migration rate in Thailand, 205-2020

Age	${}_5G_x$
0-4	5.16e-03
5-9	5.53e-03
10-14	5.90e-03
15-19	6.52e-03
20-24	6.61e-03
25-29	6.18e-03
30-34	6.66e-03
35-39	7.59e-03
40-44	8.17e-03
45-49	8.25e-03
50-54	7.56e-03
55-59	6.65e-03
60-64	5.13e-03
65-69	3.80e-03
70-74	2.85e-03
75-79	2.21e-03
80-84	1.42e-03
85-89	7.34e-04
90-94	2.41e-04
95-99	5.99e-05
100+	9.91e-06

Q3.f

To incorporate migration into the population projection to 2030, we use the assumption that all migration happens half-way through a time interval:

$$\begin{aligned} N_{t+1} &= LN_t + L^{\frac{1}{2}} G_t \\ &\approx LN_t + \frac{1}{2}(I + L)G^t \end{aligned}$$

and

$$\begin{aligned} N_{t+3} &= L^3 N_t + L^{\frac{3}{2}} G_t \\ &\approx L^3 N_t + \frac{1}{2}(I + L^3)G^t \end{aligned}$$

where N and G refer to female population and migrants, and I is the identity matrix. Using this equation, we calculate the with-migration female population in 2020 and 2030:

Table 6: Thailand female with-migration population projections

Age	2020 Pop.	2030 Pop.
0-4	30278	130305
5-9	2465	29537
10-14	2502	30132
15-19	2345	2454
20-24	2516	2486
25-29	2846	2322
30-34	3045	2480
35-39	3064	2789
40-44	1973	2985
45-49	2118	3037
50-54	2254	1968
55-59	2805	2080
60-64	2463	2164
65-69	1894	2634
70-74	1406	2236
75-79	1044	1628
80-84	781	1093
85-89	492	694
90-94	274	418
95-99	151	221
100+	118	136

Q3.g

We can compare our population projections for 2020 and 2030 with migration to the same projections without migration:

Table 7: Comparison of Thailand female population projections for 2020 and 2030, with and without migration

Age	Pop. 2020	+Mig. 2020	Pop. 2030	+Mig. 2030
0-4	30085	30278	129846	130305
5-9	2350	2465	29355	29537
10-14	2386	2502	29939	30132
15-19	2229	2345	2338	2454
20-24	2398	2516	2370	2486
25-29	2728	2846	2206	2322
30-34	2928	3045	2362	2480
35-39	2946	3064	2670	2789
40-44	1852	1973	2868	2985
45-49	1996	2118	2919	3037
50-54	2131	2254	1847	1968
55-59	2685	2805	1958	2080
60-64	2345	2463	2042	2164
65-69	1779	1894	2514	2634
70-74	1293	1406	2119	2236
75-79	932	1044	1513	1628
80-84	670	781	981	1093
85-89	381	492	583	694

Age	Pop. 2020	+Mig. 2020	Pop. 2030	+Mig. 2030
90-94	164	274	308	418
95-99	41	151	111	221
100+	7	118	25	136

Here we see that the with-migration projections are always larger than the without-migration projections, since Thailand has a net positive migration rate in all age groups.

Appendix

```r
# Prep work ---------------------------------------------------------------

# Load libraries
library(dplyr)
library(tidyr)
library(magrittr)
library(ggplot2)
library(wpp2019)

# Helper functions

write_matex <- function(x, digits = 3) {
  # From: https://stackoverflow.com/a/54088015/8866058
  x <- round(x, digits = digits)
  mat_string <- apply(x, 1, function(y) paste(y, collapse = "&"))
  paste("\\begin{bmatrix}", paste0(mat_string, collapse = "\\\\"), "\\end{bmatrix}")
}

"%^%" <- function(A, n) {
  if (n == 1) {
    A
  } else {
    A %*% (A %^% (n - 1))
  }
}

make_leslie_matrix <- function(f, s) {

  if (length(f) != length(s)) {
    stop("f and s must be the same length")
  }

  n_size <- length(f)
  l_mat <- matrix(0, nrow = n_size, ncol = n_size)

  l_mat[1, ] <- f
  diag(l_mat[-1, ]) <- s[1:(n_size - 1)]
  l_mat[n_size, n_size] <- s[n_size]

  l_mat
```

```
}


# Question 1 -------------------------------------------------------------

pop_table <- tibble(
  age = c("1", "2", "3+"),
  pop = c(18, 17, 14) * 1000,
  fr = c(0, .9, .2),
  surv = c(.65, .75, .15)
)

knitr::kable(
  pop_table,
  booktabs = TRUE,
  caption = "Q1 one-sex closed population",
  col.names = c(
    "Age",
    "Population ($N_x$)",
    "Fertility Rate ($\\tilde{F}_x$)",
    "Survival Prob. ($s_x$)"
  ),
  eval = FALSE
)


# Question 1a ------------------------------------------------------------

CBR <- pop_table %>%
  mutate(
    births = pop * fr,
    person_years = pop * surv
  ) %>%
  summarise(cbr = sum(births) / sum(person_years))


# Question 2b ------------------------------------------------------------

f_tilde_2_asfr <- function(F_tilde, srb, Sxm1, Nxm1, Nx, q0) {
  F_tilde * (1 + srb) * 2/(1 + Sxm1 * (Nxm1/Nx)) / (1 - q0/2)
}

pop_asfr_1 <- 0
pop_asfr_2 <- f_tilde_2_asfr(.9, 1.05, .65, 18000, 17000, 1-.65)
pop_asfr_3 <- f_tilde_2_asfr(.2, 1.05, .75, 17000, 14000, 1-.65)

pop_asfr <- c(pop_asfr_1, pop_asfr_2, pop_asfr_3)

TFR <- sum(pop_asfr)
tfr_eqn <- paste0(round(pop_asfr, 3), collapse = " + ")


# Question 1c ------------------------------------------------------------
```

```r
pop_leslie <- make_leslie_matrix(pop_table$fr, pop_table$surv)


# Question 1d -------------------------------------------------------------

pop_t0 <- matrix(pop_table$pop)
pop_t1 <- pop_leslie %*% pop_t0


# Question 1e -------------------------------------------------------------

pop_t10 <- (pop_leslie %^% 10) %*% pop_t0


# Question 1f -------------------------------------------------------------

CBR_t10 <- pop_table %>%
  mutate(
    pop = as.vector(pop_t10),
    births = pop * fr,
    person_years = pop * surv
  ) %>%
  summarise(cbr = sum(births) / sum(person_years))

pop_asfr_1_t10 <- 0
pop_asfr_2_t10 <- f_tilde_2_asfr(.9, 1.05, .65, pop_t10[1], pop_t10[2], 1-.65)
pop_asfr_3_t10 <- f_tilde_2_asfr(.2, 1.05, .75, pop_t10[2], pop_t10[3], 1-.65)

pop_asfr_t10 <- c(pop_asfr_1_t10, pop_asfr_2_t10, pop_asfr_3_t10)

TFR_t10 <- sum(pop_asfr_t10)


# Question 1g -------------------------------------------------------------

pop_right_eigen <- eigen(pop_leslie)
dominant_right_index <- which.max(abs(pop_right_eigen$values))

pop_iroi <- log(pop_right_eigen$values[dominant_right_index])


# Question 1h -------------------------------------------------------------

pop_sad <- matrix(pop_right_eigen$vectors[, dominant_right_index])


# Question 1i -------------------------------------------------------------

pop_left_eigen <- eigen(t(pop_leslie))
dominant_left_index <- which.max(abs(pop_left_eigen$values))

pop_repv <- matrix(pop_left_eigen$vectors[, dominant_left_index])
```

```r
# Question 2 -------------------------------------------------------------

data(mxF)

thailand_mx <- mxF %>%
  filter(name == "Thailand") %>%
  select(age, mx = `2015-2020`)


# Question 2a ------------------------------------------------------------

plot_mx <- function(data, x, y) {
  ggplot(data, aes(x = {{ x }}, y = {{ y }})) +
    geom_point() +
    geom_line() +
    theme_bw() +
    theme(text = element_text(family = "serif")) +
    labs(
      title = "Age-Specific Mortality Rates",
      subtitle = "Thailand, Females, 2015-2020",
      caption = "Source: WPP2019",
      x = "Age (years)",
      y = "Mortality Rate"
    )
}

thailand_mx_plot <- plot_mx(thailand_mx, age, mx)
thailand_log_mx_plot <- plot_mx(thailand_mx, age, log(mx)) +
  labs(title = "Age-Specific Log Mortality Rates", y = "Log Mortality Rate")

plot(thailand_mx_plot)
plot(thailand_log_mx_plot)


# Question 2b ------------------------------------------------------------

thailand_l0 <- 100000
lx = numeric(nrow(thailand_mx))
lx[1] <- thailand_l0
n <- c(1, 4, rep(5, 20))

for (i in 2:length(lx)) {
  lx[i] <- (1 - (1 - exp(-1 * n[i - 1] * thailand_mx$mx[i - 1]))) * lx[i - 1]
}

thailand_lt <- thailand_mx %>%
  mutate(
    age_str =
      paste0(age, "-", age + 4) %>%
      inset(c(1, 2, length(age)), c("0", "1-4", "100+")),
    n = n,
    qx = 1 - exp(-1 * n * mx),
    sx = 1 - qx,
```

```
    lx = lx,
    dx = lx - if_else(!is.na(lead(lx)), lead(lx), 0),
    ax = n / 2
  ) %>%
  select(age, age_str, n, everything())

thailand_lt[["ax"]][1] <- 0.07 + 1.7 * thailand_lt[["mx"]][1]
thailand_lt[["qx"]][nrow(thailand_lt)] <- 1
thailand_lt[["sx"]][nrow(thailand_lt)] <- 0

thailand_lt <- thailand_lt %>%
  mutate(
    Lx = 5 * if_else(!is.na(lead(lx)), lead(lx), 0) + ax * dx,
    Tx = rev(cumsum(rev(Lx))),
    ex = Tx / lx
  )

thailand_lt_names <- c(
  "Age",
  "$n$",
  "$_{5}m_x$",
  "$_{1}q_x$",
  "$_{1}s_x$",
  "$l_x$",
  "$_{5}d_x$",
  "$a_x$",
  "${}_{5}L_x$",
  "${}_{5}T_x$",
  "$e_x$"
)

knitr::kable(
  select(thailand_lt, -age),
  booktabs = TRUE,
  col.names = thailand_lt_names,
  eval = FALSE,
  digits = c(0, 1, 3, 3, 3, 0, 0, 2, 0, 0, 2),
  caption = "Life table for 2015-2020 Thailand female population"
)

# Question 3 -------------------------------------------------------------

data(tfr)
data(percentASFR)
data(sexRatio)
data(popF)
data(migration)


# Question 3a ------------------------------------------------------------

# Collapse 0 and 1-4 age group sx and qx to 0-4 sx and qx
thailand_mort_0to5 <- thailand_lt %>%
```

```r
  select(age, sx) %>%
  filter(age < 5) %>%
  summarise(age = "0-4", sx = prod(sx), qx = 1 - prod(sx))

thailand_q0 <- thailand_mort_0to5$qx[1]

# Create standard 5-year age goup sx and qx mortality
thailand_mort_std <- thailand_lt %>%
  filter(age >= 5) %>%
  select(age = age_str, sx, qx) %>%
  bind_rows(thailand_mort_0to5, .)

asfr_2_f_tilde <- function(asfr, srb, Sxm1, Nxm1, Nx, q0) {
  asfr * (1 / (1 + srb)) * .5 * (1 + Sxm1 * (Nxm1 / Nx)) * (1 - q0 / 2)
}

thailand_pop <- popF %>% filter(name == "Thailand") %>% select(age, pop = `2015`)
thailand_srb <- sexRatio %>% filter(name == "Thailand") %>% pull(`2015-2020`)
thailand_tfr <- tfr %>% filter(name == "Thailand") %>% pull(`2015-2020`)

thailand_fertilty <- percentASFR %>%
  filter(name == "Thailand") %>%
  select(age, pasfr = `2015-2020`) %>%
  mutate(asfr = pasfr * thailand_tfr / 5) %>%
  right_join(thailand_pop, by = "age") %>%
  replace_na(list(pasfr = 0, asfr = 0)) %>%
  left_join(thailand_mort_std, by = "age") %>%
  mutate(
    f_tilde =
      asfr_2_f_tilde(asfr, thailand_srb, lag(sx), lag(pop), pop, thailand_q0)
  ) %>%
  replace_na(list(f_tilde = 0))

knitr::kable(
  select(thailand_fertilty, age, f_tilde, asfr, pop, sx),
  booktabs = TRUE,
  col.names =
    c("Age", "${}_{5}\\tilde{F}_x$", "${}_{5}F_x$", "${}_{5}N_x$", "${}_{5}s_x$"),
  eval = FALSE,
  digits = c(0, 3, 3, 0, 3),
  caption = paste(
    "Expected number of live female births per woman per five-year period",
    "in Thailand, 2015-2020"
  )
)

# Question 3b ------------------------------------------------------------

thailand_leslie <- with(thailand_fertilty, make_leslie_matrix(f_tilde, sx))

thailand_leslie_idx <- which(thailand_leslie != 0, arr.ind = TRUE)

thailand_leslie_tbl <-
```

```r
  thailand_leslie_idx %>%
  as_tibble() %>%
  mutate(value = thailand_leslie[thailand_leslie_idx]) %>%
  arrange(row, col)

knitr::kable(
  thailand_leslie_tbl,
  booktabs = TRUE,
  col.names = c("Row", "Column", "Value"),
  digits = c(0, 0, 3),
  caption = "Leslie matrix for Thailand females, 2015-2020, (sparse format)"
)

# Question 3c -------------------------------------------------------------

thailand_pop_t1 <- thailand_leslie %*% matrix(thailand_fertilty$pop)


# Question 3d -------------------------------------------------------------

thailand_pop_t3 <- (thailand_leslie %^% 3) %*% matrix(thailand_fertilty$pop)


# Question 3e -------------------------------------------------------------

# Net migration is per 1000 person-years
thailand_net_mig <- migration %>%
  filter(name == "Thailand") %>%
  pull(`2015-2020`)

thailand_mig <- thailand_pop %>%
  mutate(
    mig_rate = (thailand_net_mig / 1000) * (pop / sum(pop)),
    mig_rate_fmt = formatC(mig_rate, digits = 2, format = "e"),
    mig_num = pop * mig_rate
  )

knitr::kable(
  select(thailand_mig, age, mig_rate_fmt),
  booktabs = TRUE,
  col.names = c("Age", "${}_{5}G_x$"),
  eval = FALSE,
  caption = "Age-specific migration rate in Thailand, 205-2020"
)

# Question 3f -------------------------------------------------------------

thailand_pop_mig_t1 <- `+`(
  (thailand_leslie %*% matrix(thailand_fertilty$pop)),
  (.5 * (1 + thailand_leslie)) %*% matrix(thailand_mig$mig_num)
)

thailand_pop_mig_t3 <- `+`(
```

```r
  ((thailand_leslie %^% 3) %*% matrix(thailand_fertilty$pop)),
  (.5 * (1 + (thailand_leslie %^% 3))) %*% matrix(thailand_mig$mig_num)
)

thailand_pop_mig_tbl <- tibble(
  age = thailand_pop$age,
  mig_pop_2020 = thailand_pop_mig_t1,
  mig_pop_2030 = thailand_pop_mig_t3
)


knitr::kable(
  mutate_if(thailand_pop_mig_tbl, is.numeric, round),
  booktabs = TRUE,
  col.names = c("Age", "2020 Pop.", "2030 Pop."),
  caption = "Thailand female with-migration population projections"
)

# Question 3g ------------------------------------------------------------

thailand_pop_all_tbl <- thailand_pop_mig_tbl %>%
  mutate(
    pop_2020 = thailand_pop_t1,
    pop_2030 = thailand_pop_t3
  ) %>%
  select(age, pop_2020, mig_pop_2020, pop_2030, mig_pop_2030)

knitr::kable(
  mutate_if(thailand_pop_all_tbl, is.numeric, round),
  booktabs = TRUE,
  col.names = c("Age", "Pop. 2020", "+Mig. 2020", "Pop. 2030", "+Mig. 2030"),
  caption = paste(
    "Comparison of Thailand female population projections for 2020 and 2030,",
    "with and without migration"
  )
)
```