Supplemental data for "Fully Bayesian Benchmarking of Small Area Estimation Models"

1. Derivation of Posterior Distribution in the Analytical Example

Derivation of the posterior distribution for the unbenchmarked model is straightforward. The posterior distribution for the model with exact benchmarking is obtained as follows. Under the unbenchmarked model, the joint posterior distribution of γ_i, γ_j ($1 \leq i, j \leq n$ and $i \neq j$) and ψ is

$$\begin{pmatrix} \gamma_i \\ \gamma_j \\ \psi \end{pmatrix} \Bigg| \boldsymbol{y} \sim \mathrm{N}\left[\begin{pmatrix} \frac{\sigma^2}{\sigma^2+\tau^2}\mu_0 + \frac{\tau^2}{\sigma^2+\tau^2}y_i \\ \frac{\sigma^2}{\sigma^2+\tau^2}\mu_0 + \frac{\tau^2}{\sigma^2+\tau^2}y_j \\ \frac{\sigma^2}{\sigma^2+\tau^2}\mu_0 + \frac{\tau^2}{\sigma^2+\tau^2}m \end{pmatrix}, \frac{\sigma^2\tau^2}{\sigma^2+\tau^2} \begin{pmatrix} 1 & 0 & w_i \\ 0 & 1 & w_j \\ w_i & w_j & \boldsymbol{w}^\top \boldsymbol{w} \end{pmatrix} \right].$$

Under exact benchmarking, $\psi = m$. Using the property that the conditional distribution of a subvector of a multivariate normally distributed random vector is multivariate normal, it is easy to show that

$$\begin{pmatrix} \gamma_i \\ \gamma_j \end{pmatrix} \Bigg| \boldsymbol{y}, \{\psi = m\} \sim \mathrm{N}\left[\begin{pmatrix} \frac{\sigma^2}{\sigma^2+\tau^2}\left[1 - \frac{w_i}{\boldsymbol{w}^\top \boldsymbol{w}}\right]\mu_0 + \frac{\sigma^2}{\sigma^2+\tau^2}\frac{w_i}{\boldsymbol{w}^\top \boldsymbol{w}}m + \frac{\tau^2}{\sigma^2+\tau^2}y_i \\ \frac{\sigma^2}{\sigma^2+\tau^2}\left[1 - \frac{w_j}{\boldsymbol{w}^\top \boldsymbol{w}}\right]\mu_0 + \frac{\sigma^2}{\sigma^2+\tau^2}\frac{w_j}{\boldsymbol{w}^\top \boldsymbol{w}}m + \frac{\tau^2}{\sigma^2+\tau^2}y_j \end{pmatrix}, \right.$$

$$\left. \frac{\sigma^2\tau^2}{\sigma^2+\tau^2} \begin{pmatrix} 1 - \frac{w_i^2}{\boldsymbol{w}^\top \boldsymbol{w}} & -\frac{w_i w_j}{\boldsymbol{w}^\top \boldsymbol{w}} \\ -\frac{w_i w_j}{\boldsymbol{w}^\top \boldsymbol{w}} & 1 - \frac{w_j^2}{\boldsymbol{w}^\top \boldsymbol{w}} \end{pmatrix} \right].$$

The exchangeability of the γ_i gives Equations (17) and (18).

The posterior distribution with inexact benchmarking can be obtained by multiplying the unbenchmarked posterior distribution by Equation (14)

$$f(\boldsymbol{\gamma}|\boldsymbol{y}) \propto \exp\left\{ -\frac{1}{2}\frac{\sigma^2+\tau^2}{\sigma^2\tau^2}\left(\boldsymbol{\gamma} - \frac{\sigma^2}{\sigma^2+\tau^2}\mathbf{1}_n\mu_0 - \frac{\tau^2}{\sigma^2+\tau^2}\boldsymbol{y}\right)^\top \left(\boldsymbol{\gamma} - \frac{\sigma^2}{\sigma^2+\tau^2}\mathbf{1}_n\mu_0 - \frac{\tau^2}{\sigma^2+\tau^2}\boldsymbol{y}\right) \right.$$

$$\left. -\frac{(m - \boldsymbol{\gamma}^\top \mathbf{1}_n)^2}{2\lambda\sigma^2 \boldsymbol{w}^\top \boldsymbol{w}} \right\}.$$

By completing the squares, it can be shown that

$$f(\boldsymbol{\gamma}|\boldsymbol{y}) \propto \exp\left\{-\frac{1}{2}(\boldsymbol{\gamma}-\boldsymbol{\mu}^{\mathrm{IB}})^{\top}(\Sigma^{\mathrm{IB}})^{-1}(\boldsymbol{\gamma}-\boldsymbol{\mu}^{\mathrm{IB}})\right\},$$

with

$$\Sigma^{\mathrm{IB}} = \left(\frac{\sigma^2+\tau^2}{\sigma^2\tau^2}\boldsymbol{I}_n + \frac{1}{\lambda\sigma^2\boldsymbol{w}^{\top}\boldsymbol{w}}\boldsymbol{w}\boldsymbol{w}^{\top}\right)^{-1},$$

$$\boldsymbol{\mu}^{\mathrm{IB}} = \Sigma^{\mathrm{IB}}\left[\frac{\sigma^2+\tau^2}{\sigma^2\tau^2}\left(\frac{\sigma^2}{\sigma^2+\tau^2}\mathbf{1}_n\mu_0 + \frac{\tau^2}{\sigma^2+\tau^2}\boldsymbol{y}\right) + \frac{1}{\lambda\sigma^2\boldsymbol{w}^{\top}\boldsymbol{w}}\boldsymbol{w}m\right].$$

Using the properties that $(\boldsymbol{w}\boldsymbol{w}^{\top})(\boldsymbol{w}\boldsymbol{w}^{\top}) = (\boldsymbol{w}^{\top}\boldsymbol{w})(\boldsymbol{w}\boldsymbol{w}^{\top})$, $\boldsymbol{w}\boldsymbol{w}^{\top}\mathbf{1}_n = \boldsymbol{w}$ (since $\boldsymbol{w}^{\top}\mathbf{1}_n = 1$), and $\boldsymbol{w}\boldsymbol{w}^{\top}\boldsymbol{y} = \boldsymbol{w}m$ (since $\boldsymbol{w}^{\top}\boldsymbol{y} = m$), we have

$$\begin{aligned}
\Sigma^{\mathrm{IB}} &= \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}\left[\boldsymbol{I}_n + \frac{\tau^2}{\lambda(\sigma^2+\tau^2)\boldsymbol{w}^{\top}\boldsymbol{w}}\boldsymbol{w}\boldsymbol{w}^{\top}\right]^{-1} \\
&= \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}\left[\boldsymbol{I}_n - \frac{\tau^2}{(\lambda\sigma^2+(\lambda+1)\tau^2)\boldsymbol{w}^{\top}\boldsymbol{w}}\boldsymbol{w}\boldsymbol{w}^{\top}\right], \\
\boldsymbol{\mu}^{\mathrm{IB}} &= \frac{\tau^2}{\sigma^2+\tau^2}\left[\boldsymbol{I}_n - \frac{\tau^2}{(\lambda\sigma^2+(\lambda+1)\tau^2)\boldsymbol{w}^{\top}\boldsymbol{w}}\boldsymbol{w}\boldsymbol{w}^{\top}\right]\left(\frac{\sigma^2}{\tau^2}\mathbf{1}_n\mu_0 + \frac{1}{\lambda\boldsymbol{w}^{\top}\boldsymbol{w}}\boldsymbol{w}m + \boldsymbol{y}\right) \\
&= \frac{\sigma^2}{\sigma^2+\tau^2}\left[\mathbf{1}_n - \frac{\tau^2}{(\lambda\sigma^2+(\lambda+1)\tau^2)\boldsymbol{w}^{\top}\boldsymbol{w}}\boldsymbol{w}\right]\mu_0 \\
&\quad + \frac{\sigma^2\tau^2}{\sigma^2+\tau^2}\frac{1}{(\lambda\sigma^2+(\lambda+1)\tau^2)\boldsymbol{w}^{\top}\boldsymbol{w}}\boldsymbol{w}m \\
&\quad + \frac{\tau^2}{\sigma^2+\tau^2}\boldsymbol{y} \\
&= \left[1 - \frac{\tau^2}{\lambda\sigma^2+(\lambda+1)\tau^2}\right]\boldsymbol{\mu}^{\mathrm{NB}} + \frac{\tau^2}{\lambda\sigma^2+(\lambda+1)\tau^2}\boldsymbol{\mu}^{\mathrm{EB}}.
\end{aligned}$$

2. Proof that the MCMC Samplers are Valid for Linear Aggregation Functions

2.1. Proof that the MCMC Sampler Under Exact Benchmarking is Valid

Under exact benchmarking, (4) is a singular distribution concentrated on the region $\{(\boldsymbol{\gamma}, \boldsymbol{\phi}) : \boldsymbol{B}^{\top}\boldsymbol{\gamma} = m\}$. But a nonsingular distribution can be derived by reparameterization. Let $\boldsymbol{A}$ denote an $n \times (n-d)$ matrix whose column vectors form a basis of the null space $\{(\boldsymbol{\gamma} : \boldsymbol{B}^{\top}\boldsymbol{\gamma} = \mathbf{0}_d\}$, where $\mathbf{0}_d$ is a vector of d zeros. Let $\boldsymbol{\gamma}_0$ denote an $n \times 1$ vector that satisfies $\boldsymbol{B}^{\top}\gamma_0 = \boldsymbol{m}$. Each element in the set $\{\boldsymbol{\gamma} : \boldsymbol{B}^{\top}\boldsymbol{\gamma} = \boldsymbol{m}\}$ can be reparameterized as $\boldsymbol{\gamma} = \boldsymbol{A}\boldsymbol{\kappa} + \boldsymbol{\gamma}_0$ where $\boldsymbol{\kappa}$ is an $(n-d) \times 1$ vector. The distribution Equation (4) implies that, for any $\boldsymbol{A}$ and $\boldsymbol{\gamma}_0$,

$$p(\kappa, \boldsymbol{\phi} \,|\, \boldsymbol{y}, \boldsymbol{m}, \boldsymbol{A}, \boldsymbol{\gamma}_0) \propto p\big(\boldsymbol{\phi}\big)p(\boldsymbol{A}\boldsymbol{\kappa} + \boldsymbol{\gamma}_0 \,|\, \boldsymbol{\phi})p(\boldsymbol{y} \,|\, \boldsymbol{A}\boldsymbol{\kappa} + \boldsymbol{\gamma}_0). \tag{S-1}$$

which is a nonsingular distribution.

We only need to show that the Metropolis-Hastings step in E1 is valid in the sense that it leaves the posterior distribution of $(\boldsymbol{\kappa}, \boldsymbol{\phi})$ in Equation (S−1) invariant for any $\boldsymbol{A}$ and $\boldsymbol{\gamma}_0$. The other steps are clearly valid.

Suppose that i_1 and i_2 have been randomly selected and they both belong to δ_j. Let $\mathcal{I} = \{i_{\text{sub},1}, \cdots, i_{\text{sub},d}\}$ denote a subset of $\{1, \cdots, n\}$, including one area from each $\delta_{j'}$ for $j' \in \{1, \cdots, d\}$, with $i_{\text{sub},j} = i_2$. Let $\mathcal{I}_{\text{res}} = \{i_{\text{res},1}, \cdots, i_{\text{res},n-d}\}$ denote the rest of areas which are not in $\mathcal{I}$. Let $\boldsymbol{\gamma} - \mathcal{I}$ denote the $(n-d) \times 1$ vector of γ_i for areas $i \in \mathcal{I}_{\text{res}}$, which does not include γ_{i_2} but includes γ_{i_1}. Let $\tilde{\boldsymbol{A}}$ denote an $n \times (n-d)$ matrix that satisfies: (a) in each row $i_{\text{res},q} (q = 1, \cdots, n-d)$, the qth element is one, and the other elements are zero; (b) in each row $i_{\text{sub},j'} (j' = 1, \cdots, d)$, the qth element is $-b_{i_{\text{res},q},j'} / b_{i_{\text{sub},j'},j'}$ if $i_{\text{res},q} \in \delta_{j'}$, and is zero otherwise. It is easy to show that $\tilde{\boldsymbol{A}}$ is full rank and $\boldsymbol{B}^\top \tilde{\boldsymbol{A}} = \boldsymbol{0}_{d \times (n-d)}$, and therefore the column vectors of $\tilde{\boldsymbol{A}}$ form a basis of the null space $\{\boldsymbol{\gamma} : \boldsymbol{B}^\top \boldsymbol{\gamma} = \boldsymbol{0}_d\}$. Let $\tilde{\boldsymbol{\gamma}}_0$ denote an $n \times 1$ vector with elements equal to $m_{j'} / b_{i_{\text{sub},j'},j'}$ in positions $i_{\text{sub},j'}$ for $j' \in \{1, \cdots, d\}$, and zero elsewhere. It is easy to show that $\boldsymbol{B}^\top \tilde{\boldsymbol{\gamma}}_0 = \boldsymbol{m}$.

For example, suppose that $n = 6$, $d = 2$, $\delta_1 = \{1, 2, 3\}$ and $\delta_2 = \{4, 5, 6\}$. Then

$$\boldsymbol{B}^\top = \begin{pmatrix} b_{11} & b_{21} & b_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{42} & b_{52} & b_{62} \end{pmatrix},$$

$$\tilde{\boldsymbol{A}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\dfrac{b_{11}}{b_{31}} & -\dfrac{b_{21}}{b_{31}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{b_{42}}{b_{62}} & -\dfrac{b_{52}}{b_{62}} \end{pmatrix},$$

$$\tilde{\boldsymbol{\gamma}}_0 = \begin{pmatrix} 0 & 0 & \dfrac{m_1}{b_{31}} & 0 & 0 & \dfrac{m_2}{b_{62}} \end{pmatrix}^\top.$$

It is easy to verify that $\tilde{\boldsymbol{A}}$ is full rank, $B^\top \tilde{\boldsymbol{A}} = \boldsymbol{0}_{2 \times 4}$, and $\boldsymbol{B}^\top \tilde{\boldsymbol{\gamma}}_0 = (m_1, m_2)^\top$.

Apparently, any $\boldsymbol{\gamma}$ satisfying $\boldsymbol{B}^\top \boldsymbol{\gamma} = \boldsymbol{m}$ can be written as $\boldsymbol{\gamma} = \tilde{\boldsymbol{A}}_{\boldsymbol{\gamma} - \mathcal{I}} + \tilde{\boldsymbol{\gamma}}_0$. If $\boldsymbol{\gamma}$ comes from Equation (4), then Equation (S−1) implies that

$$p(\boldsymbol{\gamma}_{-\mathcal{I}}, \boldsymbol{\phi} \mid \boldsymbol{y}) \propto p(\boldsymbol{\phi}) p(\tilde{\boldsymbol{A}} \boldsymbol{\gamma}_{-\mathcal{I}} + \tilde{\boldsymbol{\gamma}}_0 \mid \boldsymbol{\phi}) p(\boldsymbol{y} \mid \tilde{\boldsymbol{A}} \boldsymbol{\gamma}_{-\mathcal{I}} + \tilde{\boldsymbol{\gamma}}_0). \tag{S−2}$$

Conditional on i_1 and i_2 being selected, the Metropolis-Hastings step in E1 can be treated as an update of the single element γ_{i_1} in $\boldsymbol{\gamma}_{-\mathcal{I}}$, which can be done in two ways: (a) drawing $\gamma_{i_1}^*$ from $J(\gamma_{i_1}^* \mid \gamma_{i_1}^{(t-1)})$; (b) drawing proposal for

$$\gamma_{i_2} = \frac{m_j - \sum_{i \in \delta_j; i \notin \{i_1, i_2\}} b_{ij} \gamma_i^{(t-1)} - b_{i_1 j} \gamma_{i_1}}{b_{i_2 j}},$$

a linear function of γ_{i_1}, from $J(\gamma_{i_2}^* \mid \gamma_{i_2}^{(t-1)})$, and then calculating $\gamma_{i_1}^* = \gamma_{i_1}^{(t-1)} + b_{i_2 j} / b_{i_1 j} \left(\gamma_{i_2}^{(t-1)} - \gamma_{i_2}^*\right)$. The proposal distribution for γ_{i_1} is hence

$$\frac{1}{2}J(\gamma_{i_1}^* \mid \gamma_{i_1}^{(t-1)}) + \frac{1}{2}\left|\frac{b_{i_1 j}}{b_{i_2 j}}\right| J(\gamma_{i_1}^* \mid \gamma_{i_2}^{(t-1)}).$$

It is easy to see that this proposal distribution coupled with the acceptance ratio in (19) is valid for the posterior distribution in Equation (S−2).

Now consider any other reparameterization of $\boldsymbol{\gamma}$ in the form of $\boldsymbol{\gamma} = \boldsymbol{A}\boldsymbol{\kappa} + \boldsymbol{\gamma}_0$, where $\boldsymbol{A}$ is an $n \times (n-d)$ matrix whose column vectors form a basis of the null space $\{\boldsymbol{\gamma} : \boldsymbol{B}^\top \boldsymbol{\gamma} = \mathbf{0}_d\}$, and $\boldsymbol{\gamma}_0$ is an $n \times 1$ vector that satisfies $\boldsymbol{B}^\top \boldsymbol{\gamma}_0 = \boldsymbol{m}$. We have

$$\boldsymbol{\gamma}^\dagger \equiv \boldsymbol{A}\boldsymbol{\kappa} = \tilde{\boldsymbol{A}}_{\boldsymbol{\gamma}-\mathcal{I}} + \tilde{\boldsymbol{\gamma}}_0 - \boldsymbol{\gamma}_0. \tag{S−3}$$

It is easy to see that $\boldsymbol{B}^\top(\tilde{\boldsymbol{A}}_{\boldsymbol{\gamma}-\mathcal{I}} + \tilde{\boldsymbol{\gamma}}_0 - \boldsymbol{\gamma}_0) = \mathbf{0}_d$, and hence $\boldsymbol{\gamma}^\dagger$ belongs to $\{\boldsymbol{\gamma} : \boldsymbol{B}^\top \boldsymbol{\gamma} = \mathbf{0}_d\}$. Since the column vectors of $\boldsymbol{A}$ form a basis of this null space, there is a unique vector $\boldsymbol{\kappa}$ that satisfies $\boldsymbol{\gamma}^\dagger = \boldsymbol{A}\boldsymbol{\kappa}$. Hence $\boldsymbol{\kappa}$ is a one-to-one linear tranformation of $\boldsymbol{\gamma}_{-\mathcal{I}}$. Since the Metropolis-Hastings step in E1 leaves the posterior distribution of $(\boldsymbol{\gamma}_{-\mathcal{I}}\ \boldsymbol{\phi})$ invariant, it also leaves the posterior distribution of $(\boldsymbol{\kappa},\ \boldsymbol{\phi})$ in Equation (S−1) invariant.

2.2. *Proof that the Gibbs Sampler Under Inexact Benchmarking is Valid*

We only need to show that step I1 leaves the posterior distribution in Equation (4) invariant.

Suppose that i_1 and i_2 have been randomly selected and they both belong to δ_j. Consider the reparameterization of $\boldsymbol{\gamma}$ in the form of $\boldsymbol{\gamma} = \tilde{\boldsymbol{A}}_{\boldsymbol{\gamma}-\mathcal{I}} + \tilde{\boldsymbol{\gamma}}_{0,\boldsymbol{\psi}}$, where $\tilde{\boldsymbol{\gamma}}_{0,\boldsymbol{\psi}}$ denotes an $n \times 1$ vector with elements equal to $\psi_{j'}/b_{i_{\text{sub},j'},j'}$ in positions $i_{\text{sub},j'}$ for $j' \in \{1, \cdots, d\}$, and zero elsewhere. Then $(\boldsymbol{\gamma}_{-\mathcal{I}}, \boldsymbol{\psi})$ is a one-to-one linear transformation of $\boldsymbol{\gamma}$. According to Equation (4), the conditional distribution of $(\boldsymbol{\gamma}_{-\mathcal{I}}, \phi)$ given $\boldsymbol{\psi}$ is in the form:

$$p(\boldsymbol{\gamma}_{-\mathcal{I}}, \boldsymbol{\phi} \mid \boldsymbol{y}, \boldsymbol{m}, \boldsymbol{\psi}) \propto p\big(\boldsymbol{\phi}\big) p(\tilde{\boldsymbol{A}}_{\boldsymbol{\gamma}-\mathcal{I}} + \tilde{\boldsymbol{\gamma}}_{0,\boldsymbol{\psi}} \mid \boldsymbol{\phi}) p(\boldsymbol{y} \mid \tilde{\boldsymbol{A}}_{\boldsymbol{\gamma}-\mathcal{I}} + \tilde{\boldsymbol{\gamma}}_{0,\boldsymbol{\psi}}).$$

The proof in Subsection 2.1 indicates that step I1 leaves this conditional distribution invariant. Hence, step I1 also leaves the joint distribution of $(\boldsymbol{\gamma}_{-\mathcal{I}}, \boldsymbol{\psi}, \boldsymbol{\phi})$ invariant, and equivalently leaves the joint distribution of $(\boldsymbol{\gamma},\ \boldsymbol{\phi})$ in Equation (4) invariant.

3. Application of the MCMC Samplers to a Family of Area-level Models

The parts of the algorithms described in Subsection 2.3 that directly involve benchmarking and that depend on the specific details of the model are step E1, step I1 (which is similar to step E1), and step I2.

We begin with step E1. Let $\eta_i = g(\gamma_i)$ for $i = 1, \cdots, n$. We obtain a proposal for γ_{i_1} by drawing $\eta_{i_1}^*$ from $\mathrm{N}(\eta_{i_1}^{(t-1)}, \rho_1^2)$, with $\rho_1^2 > 0$, and set $\gamma_{i_1}^* = g^{-1}(\eta_{i_1}^*)$. The proposal density for $\gamma_{i_1}^*$ is then

$$J(\gamma_{i_1}^* \mid \gamma_{i_1}^{(t-1)}) = g'(\gamma_{i_1}^*)\mathrm{N}\Big(g(\gamma_{i_1}^*) \mid g(\gamma_{i_1}^{(t-1)}), \rho_1^2\Big), \tag{S−4}$$

where $g'(\cdot)$ is the first derivative of $g(\cdot)$. We then set

$$\gamma_{i_2}^* = \gamma_{i_2}^{(t-1)} + \frac{b_{i_1 j}}{b_{i_2 j}}\left(\gamma_{i_1}^{(t-1)} - \gamma_{i_1}^*\right). \tag{S−5}$$

The Metropolis-Hastings ratio in Equation (19) becomes

$$
\begin{aligned}
r = & \left[\frac{p(y_{i_1} \mid \gamma_{i_1}^*, w_{i_1}, \sigma^2)}{p(y_{i_1} \mid \gamma_{i_1}^{(t-1)}, w_{i_1}, \sigma^2)} \frac{g'(\gamma_{i_1}^*)\mathrm{N}\left(g(\gamma_{i_1}^*) \mid \boldsymbol{x}_{i_1}\boldsymbol{\beta}^{(t-1)}, \boldsymbol{\tau}^{2(t-1)}\right)}{g'(\gamma_{i_1}^{(t-1)})\mathrm{N}\left(g(\gamma_{i_1}^{(t-1)}) \mid \boldsymbol{x}_{i_1}\boldsymbol{\beta}^{(t-1)}, \boldsymbol{\tau}^{2(t-1)}\right)}\right] \\
& \times \left[\frac{p(y_{i_2} \mid \gamma_{i_2}^*, w_{i_2}, \sigma^2)}{p(y_{i_2} \mid \gamma_{i_2}^{(t-1)}, w_{i_2}, \sigma^2)} \frac{g'(\gamma_{i_2}^*)\mathrm{N}\left(g(\gamma_{i_2}^*) \mid \boldsymbol{x}_{i_2}\boldsymbol{\beta}^{(t-1)}, \boldsymbol{\tau}^{2(t-1)}\right)}{g'(\gamma_{i_2}^{(t-1)})\mathrm{N}\left(g(\gamma_{i_2}^{(t-1)}) \mid \boldsymbol{x}_{i_2}\boldsymbol{\beta}^{(t-1)}, \boldsymbol{\tau}^{2(t-1)}\right)}\right] \\
& \times \frac{g'(\gamma_{i_1}^{(t-1)})\mathrm{N}\left(g(\gamma_{i_1}^{(t-1)}) \mid g(\gamma_{i_1}^*), \rho_1^2\right) + \mid b_{i_1 j}/b_{i_2 j} \mid g'(\gamma_{i_2}^{(t-1)})\mathrm{N}\left(g(\gamma_{i_2}^{(t-1)}) \mid g(\gamma_{i_2}^*), \rho_1^2\right)}{g'(\gamma_{i_1}^*)\mathrm{N}\left(g(\gamma_{i_1}^*) \mid g(\gamma_{i_1}^{(t-1)}), \rho_1^2\right) + \mid b_{i_1 j}/b_{i_2 j} \mid g'(\gamma_{i_2}^*)\mathrm{N}\left(g(\gamma_{i_2}^*) \mid g(\gamma_{i_2}^{(t-1)}), \rho_1^2\right)}.
\end{aligned}
\tag{S-6}
$$

Step I2 is carried out as follows. For $j = 1, \cdots, d$, let $\boldsymbol{\gamma}_j$ denote the $n_j \times 1$ vector of γ_i for $i \in \delta_j$. We iteratively sample $\boldsymbol{\gamma}_j$ $(j = 1, \cdots, d)$ using a Metropolis-Hastings step. Let n_j denote the number of elements in δ_j, let $\boldsymbol{b}_j$ denote the $n_j \times 1$ vector of b_{ij} for $i \in \delta_j$, and let $\boldsymbol{\eta}_j = g(\boldsymbol{\gamma}_j)$, where g is applied element by element. Let $\boldsymbol{\gamma}_j^{(t-\frac{1}{2})}$ and $\boldsymbol{\eta}_j^{(t-\frac{1}{2})}$ denote the values of $\boldsymbol{\gamma}_j$ and $\boldsymbol{\eta}_j$ after step I1. We set

$$
\boldsymbol{\eta}_j^* \sim \mathrm{N}(\boldsymbol{\eta}_j^{\left(t-\frac{1}{2}\right)}, \rho_2^2 I_{n_j}),
\tag{S-7}
$$

where $\rho_2^2 > 0$. We then set $\boldsymbol{\gamma}_j^* = g^{-1}(\boldsymbol{\eta}_j^*)$, where g^{-1} is applied element by element. For instance, with the model for the benchmarks given by Equation (6), the acceptance ratio is

$$
\begin{aligned}
r = & \prod_{i \in \delta_j} \left[\frac{p\left(y_i \mid \gamma_i^*, w_i, \sigma^2\right)}{p\left(y_i \mid \gamma_i^{\left(t-\frac{1}{2}\right)}, w_i, \sigma^2\right)} \frac{\mathrm{N}\left(g(\gamma_i^*) \mid \boldsymbol{x}_i\boldsymbol{\beta}^{(t-1)}, \tau^2\right)}{\mathrm{N}\left(g\left(\gamma_i^{\left(t-\frac{1}{2}\right)}\right) \mid \boldsymbol{x}_i\boldsymbol{\beta}^{(t-1)}, \tau^2\right)}\right] \\
& \times \exp\left(\frac{\left(m_j - \sum_{i \in \delta_j} b_{ij}\gamma_i^{\left(t-\frac{1}{2}\right)}\right)^2 - \left(m_j - \sum_{i \in \delta_j} b_{ij}\gamma_i^*\right)^2}{2\lambda s_j^2}\right).
\end{aligned}
\tag{S-8}
$$

4. Number of Replicates Used in Simulation Study

In a simulation study, it is important to carry out enough replicates for Monte Carlo error to be reduced to low levels. The number of replicates required to achieve low levels of Monte Carlo error depends on the quantities being simulated, and varies from simulation study to simulation study. In our case, the number of replicates required is relatively low. Figures 1 and 2 below, which are versions of Figure 1 in the main article, compare overall results for 50 replicates and 200 replicates. Any differences between the two graphs are minor, and do not affect the substantive conclusions.

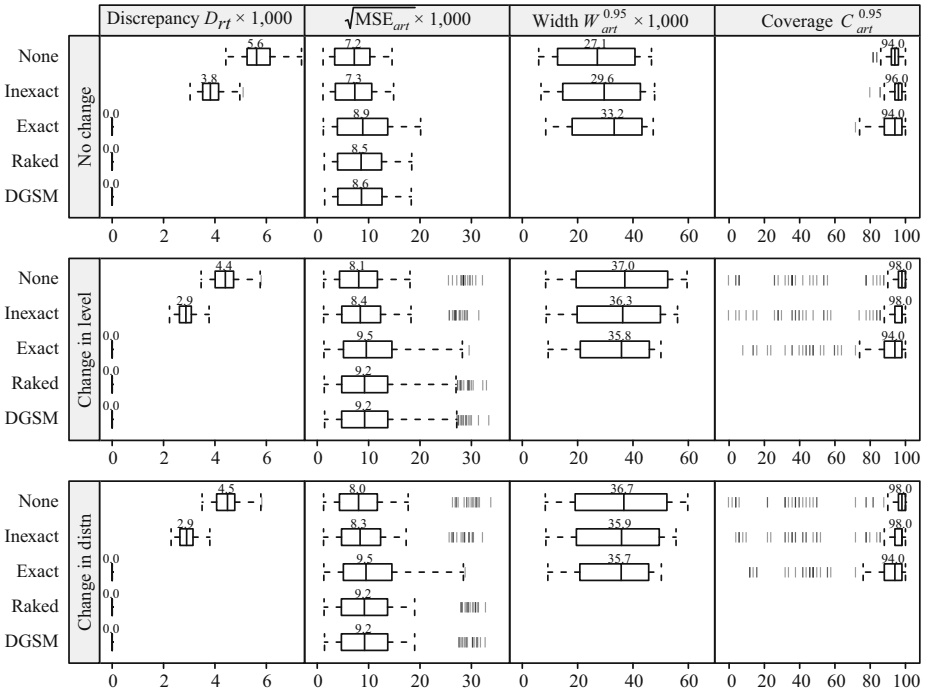

Fig. 1. Performance of models of fertility rates, by type of benchmarking and dataset (50 replicates).

5. Extension to Unit-Level Models

In some applications, unit-level covariates are available and can be used to improve the precision of the estimates of area-level quantities. Suppose that a sample of size l_i is taken from the L_i units in area i, and that for each sampled unit $k = 1,..., l_i$ in area i, there is a response y_{ik} and a vector of covariates $\boldsymbol{z}_{ik} = (z_{ik1}, \ldots, z_{ikp})^\top$, not including the constant term. A random intercept model, for intance, can be fitted that combines unit-level and area-level covariates,

$$y_{ik} \sim G\big(\theta_{ik}, \xi\big) \tag{S–9}$$

$$g(\theta_{ik}) = \boldsymbol{z}_{ik}^\top \boldsymbol{\alpha} + \boldsymbol{x}_i^\top \boldsymbol{\beta} + \zeta_i \tag{S–10}$$

$$\zeta_i \sim \mathrm{N}\big(0, \sigma^2\big), \tag{S–11}$$

where G typically is the normal or Bernoulli distribution, ξ is a standard deviation (used only with the normal distribution), g is the identity or logit link function, and ζ_i is an area-specific random effect, with $\boldsymbol{\zeta} = (\zeta_1, \cdots, \zeta_n)$.

We assume that the aim of the modelling is to estimate area-level quantities such as $\gamma_i = \sum_{k=1}^{L_i} \theta_{ik}/L_i$. For simplicity, we assume that observations on $\boldsymbol{z}_{ik}$ are available for all non-sampled units. Parameter γ_i can then be written as $\sum_{k=1}^{L_i} g^{-1}(\boldsymbol{z}_{ik}^\top \boldsymbol{\alpha} + \boldsymbol{x}_i^\top \boldsymbol{\beta} + \zeta_i)/L_i$.

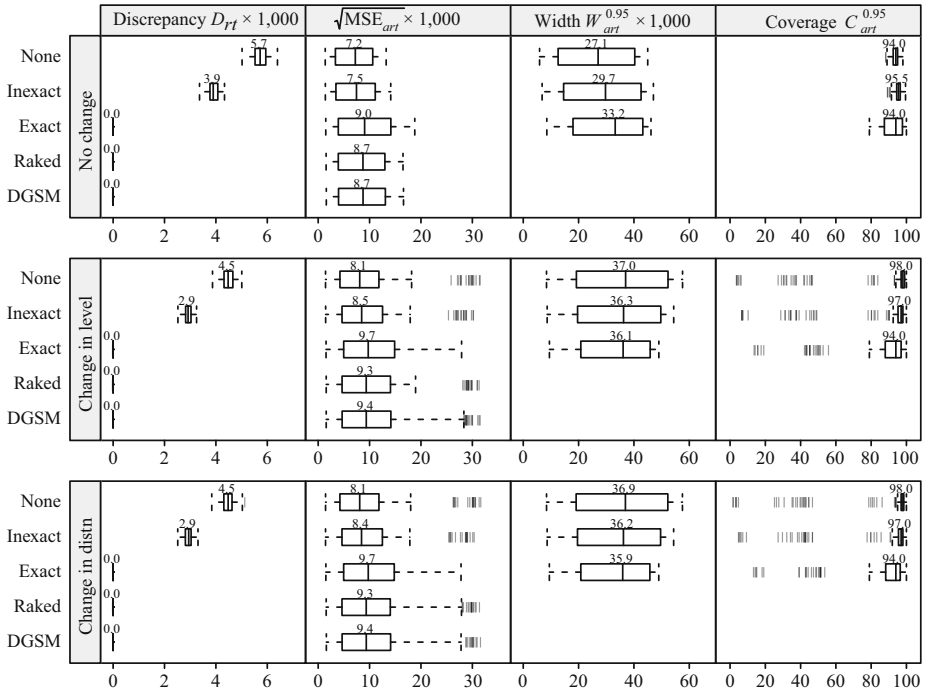

Fig. 2. Performance of models of fertility rates, by type of benchmarking and dataset (200 replicates).

Parameter ψ_j has the form

$$\psi_j = f_j\left(\sum_{k=1}^{L_1}\frac{g^{-1}\left(z_{1k}^\top\boldsymbol{\alpha}+x_1^\top\boldsymbol{\beta}+\zeta_1\right)}{L_1},\cdots,\sum_{k=1}^{L_n}\frac{g^{-1}\left(z_{nk}^\top\boldsymbol{\alpha}+x_n^\top\boldsymbol{\beta}+\zeta_n\right)}{L_n}\right). \quad \text{(S - 12)}$$

In area-level models, conditional on the hyperparameters $\boldsymbol{\phi}$, the γ_i are independent. In contrast, in unit-level models, even after conditioning on $\boldsymbol{\phi}$, the γ_i are not independent, because they depend on the same set of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\zeta}$. In area-level models, the γ_i can be updated on their own or in pairs, while in unit-level models, if $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ is updated, then all γ_i must be updated as well. Exact benchmarking requires that the updated γ_i satisfy $\psi_j = m_j$ for all j. Satisfying these constraints while updating all γ_i would be extremely difficult. Exact benchmarking is therefore likely to be impractical for unit-level models. In contrast, because inexact benchmarking does not require strict constraints, it can be implemented through a Gibbs sampler in which vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\zeta}$ are each updated using a Metropolis-Hastings step. Exact benchmarking can be approximated by inexact benchmarking with a suitable value for the tuning parameter λ.

Once the model of Equation (S–9) to (S–12) has been fitted, it is possible to draw values of y_{ik} for non-sampled individuals. Combining these values with values for sampled individuals can provide finite-population area-level poverty estimates comparable to those of the World Bank method (Elbers et al. 2003). However, because the model is fitted using

fully Bayesian methods, which, in contrast to standard implementations of the World Bank methods, treat all hyper-parameters as uncertain, the associated measures of uncertainty are likely to be more satisfactory.

6. Extension to Multiple Sets of Benchmarks

In the formulation in the main paper, we have assumed that there is only one set of benchmarks corresponding to mutually exclusive sets of small areas. That is, we require each area i belong to at most one δ_j. In some applications, however, small area estimates must agree with multiple sets of benchmarks. For instance, there might be one set of benchmarks estimating labor force participation rates by sex, and a second set of benchmarks estimating labor force participation rates by age.

Suppose there are K sets of benchmarks. Let $\boldsymbol{m}^{(k)} = (m_1^{(k)}, \cdots, m_{d_k}^{(k)})^\top$ denote the kth set of benchmarks with dimension d_k. Let $\boldsymbol{\psi}^{(k)} = (\psi_1^{(k)}, \cdots, \psi_{d_k}^{(k)})^\top$ denote the set of underlying parameters that the kth set of benchmarks estimate. We combine $\boldsymbol{m}^{(1)}, \cdots, \boldsymbol{m}^{(K)}$ into the vector $\boldsymbol{m}$, with $d = d_1 + \cdots + d_k$. Correspondingly, we combine $\boldsymbol{\psi}^{(1)}, \cdots, \boldsymbol{\psi}^{(K)}$ into the vector $\boldsymbol{\psi}$. We again let δ_j denote the set of areas i such that γ_i contributes to ψ_j $(j = 1, \cdots, d)$. An area i may belong to more than one δ_j.

Under exact benchmarking, the distribution $p^{[\boldsymbol{m} \mid \boldsymbol{\psi}]}(\boldsymbol{m} \mid \boldsymbol{\psi})$ can still be given by (5). Under inexact benchmarking, we can specify

$$p^{[\boldsymbol{m} \mid \boldsymbol{\psi}]}(\boldsymbol{m} \mid \boldsymbol{\psi}) = \prod_{k=1}^{K} p^{[\boldsymbol{m}^{(k)} \mid \boldsymbol{\psi}^{(k)}]}\left(\boldsymbol{m}^{(k)} \mid \boldsymbol{\psi}^{(k)}\right), \tag{S-13}$$

where $p^{[\boldsymbol{m}^{(k)} \mid \boldsymbol{\psi}^{(k)}]}\left(\boldsymbol{m}^{(k)} \mid \boldsymbol{\psi}^{(k)}\right)$ may take the form in Equation (8) or Equation (10), possibly with different λ_k for each set of benchmarks.

We next discuss the Markov chain Monte Carlo strategy for sampling from the benchmarked posterior distribution. We first focus on the case where the components of the benchmarking function are linear. We can no longer use our original method for generating proposals in step E1 or I1, in which we update at most two elements of $\boldsymbol{\gamma}$ within the same δ_j, γ_{i_1} and γ_{i_2} with $i_1 \in \delta_j$, $i_2 \in \delta_j$ and $i_1 \neq i_2$. Although such an update can still maintain the constraint $\sum_{i=1}^{n} b_{ij}\gamma_i = m_j$, it will break the constraints for other $m_{j'}$'s with $i_1 \in \delta_{j'}$ or $i_2 \in \delta_{j'}$.

Let $\boldsymbol{A}$ denote a $n \times (n - d)$ matrix whose columns span the null space of $\boldsymbol{B}^\top$ (the null space consists of all $n \times 1$ vectors $\boldsymbol{v}$ such that $\boldsymbol{B}^\top \boldsymbol{v} = 0$). A proposal satisfying $\boldsymbol{B}^\top \boldsymbol{\gamma}^* = \boldsymbol{m}$ can be generated by setting $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}^{(t-1)} + \boldsymbol{A}\boldsymbol{u}$, where $\boldsymbol{u} \sim \mathrm{N}(\boldsymbol{0}_{n-d}, \lambda_3^2 \boldsymbol{I}_{n-d})$. To ensure that $\boldsymbol{\gamma}^*$ lies within its support, vector $\boldsymbol{u}$ must be drawn from a truncated multivariate normal distribution with linear inequality constraints. Efficient algorithms for generating such draws do exist (Geweke 1996; Rodriguez-Yam et al. 2004; Li and Ghosh 2013), but computational considerations may limit the number of benchmarks that can be used in practice.

When there is computational difficulty in implementing step E1 or I1 with linear benchmarking function, or when the components of the benchmarking function are nonlinear, we can use steps I2 and I3 for inexact benchmarking, and approximate exact benchmarking by using inexact benchmarking with λ_k's close to zero.

7. References

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