Homework 02

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Questions

Q1

Table 1: Q1 one-sex closed population

Age	Population (N_x)	Fertility Rate ($\tilde{F}_x$)	Survival Prob. (s_x)
1	18000	0.0	0.65
2	17000	0.9	0.75
3+	14000	0.2	0.15

Q1.a

The crude birth rate (CBR) is defined as the number of births over the person-years lived in the period $[T_1, T_2]$. Since our period is a single year, we can calculate CBR as:

$$CBR = \sum \frac{N_x \tilde{F}_x}{N_x s_x}$$

where we sum over all age groups. The crude birth rate for this population in the next time period is then **0.682**.

Q1.b

The total fertility rate in the population in the period $[T_1, T_2]$ is defined as the sum of the age-specific fertility rates across all age groups, multiplied by the length of the age interval, n. With $T_2 - T_1 = n = 1$, the total fertility rate represents the single-year cohort total fertility rate:

$$\text{TFR}[T_1, T_1 + 1] = \sum {}_1F_x[T_1, T_1 + 1]$$

We can convert between $\tilde{F}_x$ and ${}_1F_x$ using the equation

$$\tilde{F}_x = {}_1F_x \times \frac{1}{1 + SRB} \times \frac{1}{2}\left(1 + s_{x-1}\frac{N_{x-1,t}}{N_{x,t}}\right) \times \left(1 - \frac{q_0}{2}\right)$$

where we assume $SRB = 1.05$ and take $q_0 = 1 - s_0$. After converting to age-specific fertility rates, we calculate a total fertility rate of $0 + 2.649 + 0.52 = \mathbf{3.17}$ for this population.

Q1.c

The Leslie matrix, L, for this population is defined as:

$$L = \begin{bmatrix} \tilde{F}_{A-3} & \tilde{F}_{A-2} & \tilde{F}_{A-1} \\ s_{A-3} & 0 & 0 \\ 0 & s_{A-2} & s_{A-1} \end{bmatrix} = \begin{bmatrix} 0 & 0.9 & 0.2 \\ 0.65 & 0 & 0 \\ 0 & 0.75 & 0.15 \end{bmatrix}$$

Where $(A-1)+$ is the highest age group that can be reached in this population, 3+, s_x denotes the probability of survival to the next age group for age group x, and $\tilde{F}_x$ is the expected number of female births to a woman age x, who survives to the next time interval.

Q1.d

We can project this population forward using the *cohort-component method of population projection*, which states that the age-specific populations one time period ahead (N_{t+1}) can be calculated from the matrix multiplication of the age-specific population in the current period (N_t) and the Leslie matrix (L) of the population. The population by age one period forward from our given initial population is then:

$$\begin{aligned} N_{t+1} &= LN_t \\ &= \begin{bmatrix} 0 & 0.9 & 0.2 \\ 0.65 & 0 & 0 \\ 0 & 0.75 & 0.15 \end{bmatrix} \begin{bmatrix} 18000 \\ 17000 \\ 14000 \end{bmatrix} \\ &= \begin{bmatrix} 18100 \\ 11700 \\ 14850 \end{bmatrix} \end{aligned}$$

Q1.e

This method can be extended to projecting age-specific population k periods ahead by raising the Leslie matrix to the k^{th} power (L^k). Our given population, projected 2 periods into the future is then:

$$\begin{aligned} N_{t+2} &= L^{10} N_t \\ &= \begin{bmatrix} 0 & 0.9 & 0.2 \\ 0.65 & 0 & 0 \\ 0 & 0.75 & 0.15 \end{bmatrix}^2 \begin{bmatrix} 18000 \\ 17000 \\ 14000 \end{bmatrix} \\ &= \begin{bmatrix} 0.585 & 0.15 & 0.03 \\ 0 & 0.585 & 0.13 \\ 0.488 & 0.112 & 0.022 \end{bmatrix} \begin{bmatrix} 18000 \\ 17000 \\ 14000 \end{bmatrix} \\ &= \begin{bmatrix} 13500 \\ 11765 \\ 11002.5 \end{bmatrix} \end{aligned}$$

Q1.f

The crude birth rate for this population from time period 1 to time period 2 is **0.607**.

The total fertility rate between time periods 1&2 is **2.907** for this population.

Q1.g

From the theorem that N_t converges to $\lambda^t u$ as t approaches infinity, $log(\lambda)$ is the *instantaneous rate of increase of the population*. Here, λ is defined as the dominant right eigenvalue of the Leslie matrix, or for the equation:

$$Lv = \lambda v$$

it is the eigenvalue λ with the largest magnitude. For our calculated Leslie matrix, the instantaneous rate of increase is **-0.161.**

Q1.h

Again, from the the formula $\lambda^t u$, u is the *stable age distribution*, and is defined as the dominant right eigenvector of the Leslie matrix, which is the column vector v from the eigendecomposition of L corresponding to the eigenvalue λ with the largest magnitude.

For our calculated Leslie matrix, the stable age distribution is $\begin{bmatrix} 0.666 \\ 0.509 \\ 0.545 \end{bmatrix}$.

Q1.i

The reproductive value vector (v) is a vector of expected the number of future offspring of an individual for each age group. A theorem states that v is the dominant left eigenvector of the Leslie matrix for the population. The left dominant eigenvector of a matrix A is equivalent to the right dominant eigenvector of the transpose of matrix, $A^\top$. So, in the formula:

$$L^\top u = \kappa u$$

the dominant eigenvector u represents the reproductive values. For our Leslie matrix, the reproductive value matrix is then $\begin{bmatrix} -0.598 \\ -0.783 \\ -0.171 \end{bmatrix}$.

Appendix

```
# Prep work -----------------------------------------------------------------

# Load libraries
library(tidyverse, quietly = TRUE)
library(wpp2019)

# Helper functions

write_matex <- function(x, digits = 3) {
  # From: https://stackoverflow.com/a/54088015/8866058
  x <- round(x, digits = digits)
  mat_string <- apply(x, 1, function(y) paste(y, collapse = "&"))
  paste("\\begin{bmatrix}", paste0(mat_string, collapse = "\\\\"), "\\end{bmatrix}")
}

"%^%" <- function(A, n) {
  if (n == 1) {
    A
  } else {
    A %*% (A %^% (n - 1))
  }
}

make_leslie_matrix <- function(f, s) {

  if (length(f) != length(s)) {
    stop("f and s must be the same length")
  }

  n_size <- length(f)
  l_mat <- matrix(0, nrow = n_size, ncol = n_size)

  l_mat[1, ] <- f
  diag(l_mat[-1, ]) <- s[1:(n_size - 1)]
  l_mat[n_size, n_size] <- s[n_size]

  l_mat
}


# Question 1 ----------------------------------------------------------------

pop_table <- tibble(
  age = c("1", "2", "3+"),
  pop = c(18, 17, 14) * 1000,
  fr = c(0, .9, .2),
  surv = c(.65, .75, .15)
)

knitr::kable(
  pop_table,
```

```r
  booktabs = TRUE,
  caption = "Q1 one-sex closed population",
  col.names = c(
    "Age",
    "Population ($N_x$)",
    "Fertility Rate ($\\tilde{F}_x$)",
    "Survival Prob. ($s_x$)"
  ),
  eval = FALSE
)


# Question 1a ------------------------------------------------------------

CBR <- pop_table %>%
  mutate(
    births = pop * fr,
    person_years = pop * surv
  ) %>%
  summarise(cbr = sum(births) / sum(person_years))


# Question 1b ------------------------------------------------------------

f_tilde_2_asfr <- function(F_tilde, srb, Sxm1, Nxm1, Nx, q0) {
  F_tilde * (1 + srb) * 2/(1 + Sxm1 * (Nxm1/Nx)) / (1 - q0/2)
}

pop_asfr_1 <- 0
pop_asfr_2 <- f_tilde_2_asfr(.9, 1.05, .65, 18000, 17000, 1-.65)
pop_asfr_3 <- f_tilde_2_asfr(.2, 1.05, .75, 17000, 14000, 1-.65)

pop_asfr <- c(pop_asfr_1, pop_asfr_2, pop_asfr_3)

TFR <- sum(pop_asfr)
tfr_eqn <- paste0(round(pop_asfr, 3), collapse = " + ")


# Question 1c ------------------------------------------------------------

pop_leslie <- make_leslie_matrix(pop_table$fr, pop_table$surv)


# Question 1d ------------------------------------------------------------

pop_t0 <- matrix(pop_table$pop)
pop_t1 <- pop_leslie %*% pop_t0


# Question 1e ------------------------------------------------------------
pop_t2 <- (pop_leslie %^% 2) %*% pop_t0
```

```r
# Question 1f ------------------------------------------------------------

pop_t2 <- (pop_leslie %^% 2) %*% pop_t1


CBR_t2 <- pop_table %>%
  mutate(
    pop = as.vector(pop_t2),
    births = pop * fr,
    person_years = pop * surv
  ) %>%
  summarise(cbr = sum(births) / sum(person_years))

pop_asfr_1_t2 <- 0
pop_asfr_2_t2 <- f_tilde_2_asfr(.9, 1.05, .65, pop_t2[1], pop_t2[2], 1-.65)
pop_asfr_3_t2 <- f_tilde_2_asfr(.2, 1.05, .75, pop_t2[2], pop_t2[3], 1-.65)

pop_asfr_t2 <- c(pop_asfr_1_t2, pop_asfr_2_t2, pop_asfr_3_t2)

TFR_t2 <- sum(pop_asfr_t2)


# Question 1g ------------------------------------------------------------

pop_right_eigen <- eigen(pop_leslie)
dominant_right_index <- which.max(abs(pop_right_eigen$values))

pop_iroi <- log(pop_right_eigen$values[dominant_right_index])


# Question 1h ------------------------------------------------------------

pop_sad <- matrix(pop_right_eigen$vectors[, dominant_right_index])


# Question 1i ------------------------------------------------------------

pop_left_eigen <- eigen(t(pop_leslie))
dominant_left_index <- which.max(abs(pop_left_eigen$values))

pop_repv <- matrix(pop_left_eigen$vectors[, dominant_left_index])
```