Homework 01

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Questions

Q1

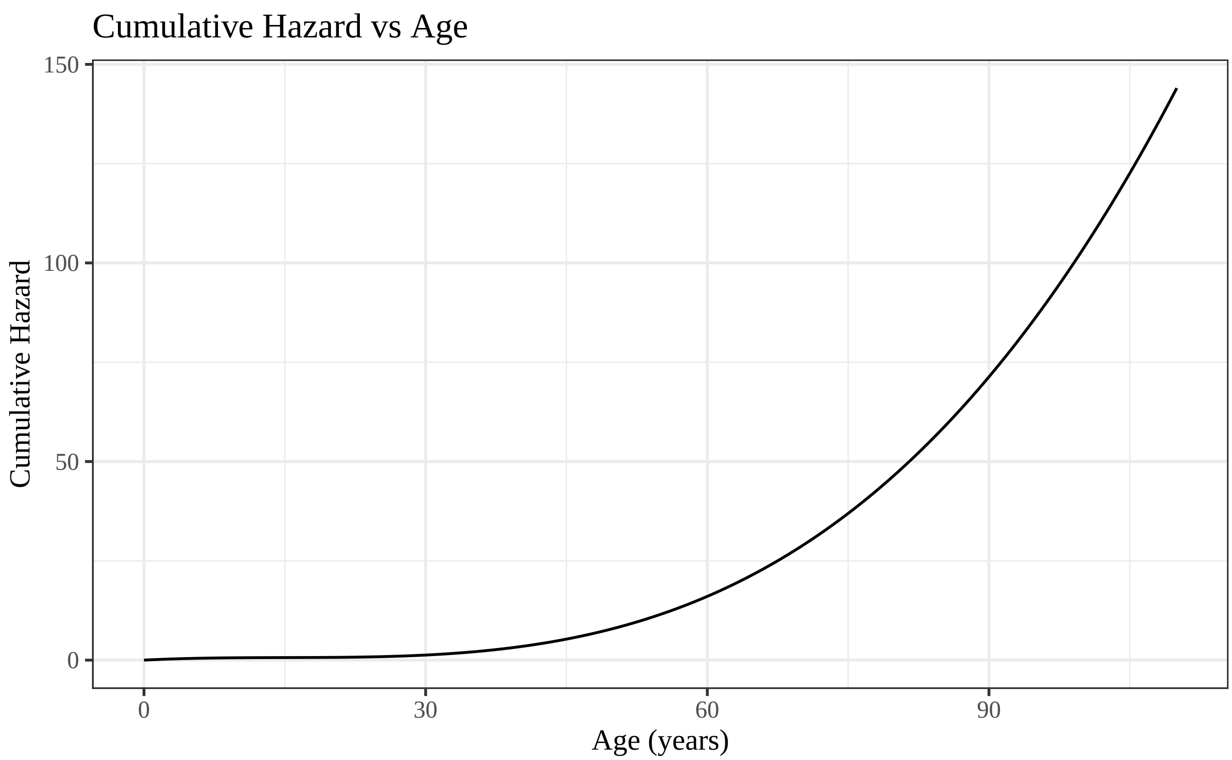
For a given force of mortality function, $\mu(x)$, the cumulative hazard is defined as the total area under the curve of $\mu(x)$ bounded on the interval $[0, x]$, or put another way:

$$\Lambda(x) = \int_0^x \mu(u) du$$

For the force of mortality function $\mu(x) = 0.005 + 0.0005(x - 15)^2$, the cumulative hazard function is then:

$$\begin{aligned}
\Lambda(x) &= \int_0^x \mu(u) du \\
&= \int_0^x \left[0.005 + 0.0005(u - 15)^2\right] du \\
&= 0.005x + 0.0005 \int_0^x \left(u^2 - 30u + 225\right) du \\
&= 0.005x + 0.0005 \left(\frac{1}{3}u^3 - 15u^2 + 225u\right)\Big|_0^x \\
&= 0.005x + 0.0005 \left(\frac{1}{3}x^3 - 15x^2 + 225x\right) \\
&= \frac{1}{2000} \left(\frac{1}{3}x^3 - 15x^2 + 235x\right)
\end{aligned}$$

For ages 0 to 110, this cumulative hazard functions looks like:

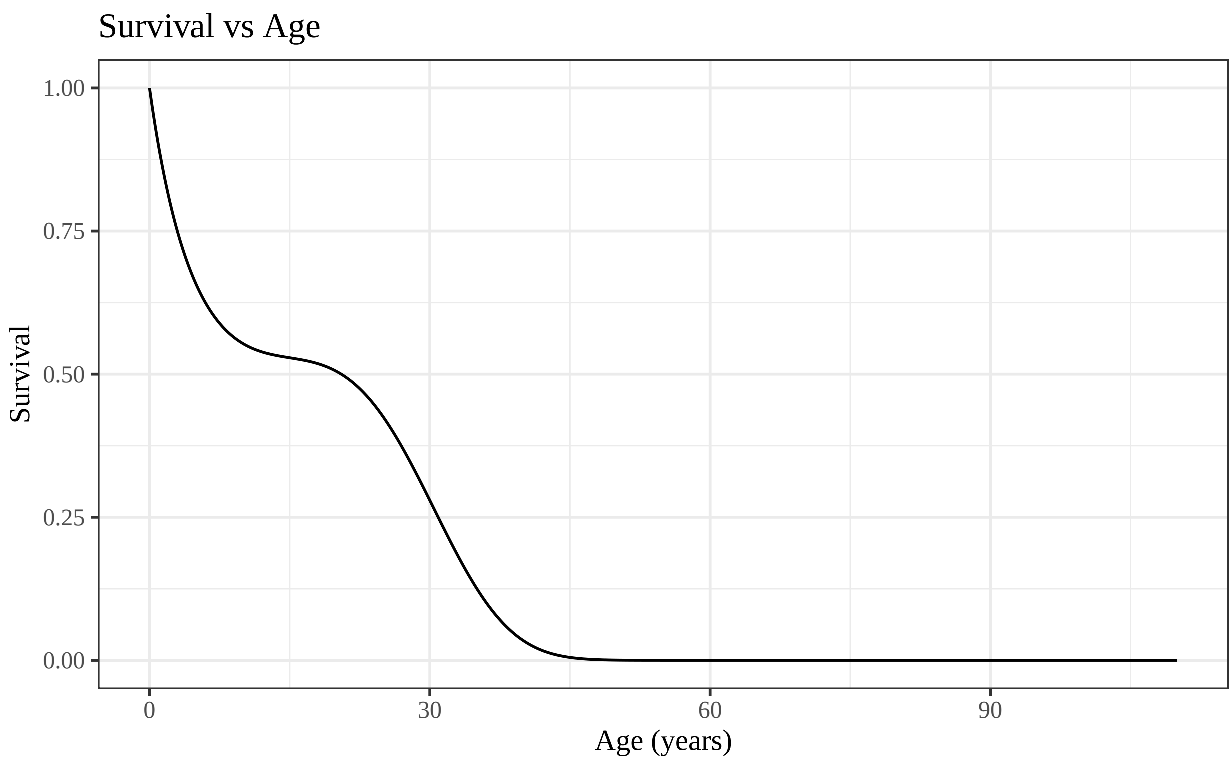


Q2

The survival function, $S(x)$, is defined as the exponentiated negative cumulative hazard function, $e^{-\Lambda(x)}$. Using our calculated cumulative hazard function, the survival function is then:

$$S(x) = \exp\left[\frac{-1}{2000}\left(\frac{1}{3}x^3 - 15x^2 + 235x\right)\right]$$

For ages 0 to 110, the survival function then looks like:

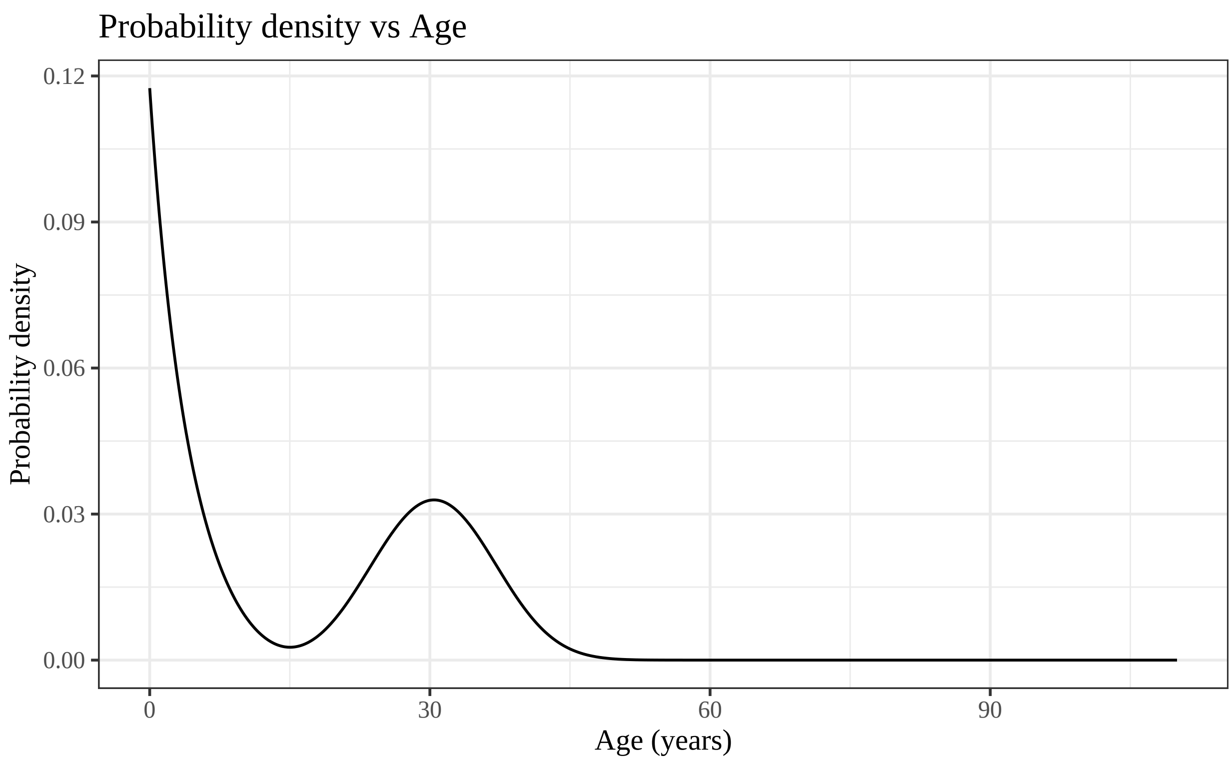


Q3

The probability density function of X, $f(x)$, is the negative derivative of the survival function with respect to x, $f(x) = -\frac{dS(x)}{dx}$. Using our calculated survival function, the probability density function of X is then:

$$\begin{aligned} f(x) &= \frac{-d}{dx}\exp\left[\frac{-1}{2000}\left(\frac{1}{3}x^3 - 15x^2 + 235x\right)\right] \\ &= \frac{1}{2000}\frac{d}{dx}\left(\frac{1}{3}x^3 - 15x^2 + 235x\right)S(x) \\ &= \frac{1}{2000}\left(x^2 - 30x + 235\right)S(x) \\ &= \mu(x)S(x) \end{aligned}$$

For ages 0 to 110, the probability density function looks like:



Q4

Table 1: Cohort life table for single year ages 0 to 50

age	l_x	${}_1d_x$	${}_1q_x$	${}_1m_x$
0	100000.000	10431.516	0.104	0.110
1	89568.484	8212.300	0.092	0.096
2	81356.184	6492.405	0.080	0.083
3	74863.779	5142.643	0.069	0.071
4	69721.136	4071.185	0.058	0.060
5	65649.951	3212.193	0.049	0.050
6	62437.758	2518.166	0.040	0.041
7	59919.591	1954.738	0.033	0.033
8	57964.853	1497.075	0.026	0.026
9	56467.779	1127.361	0.020	0.020
10	55340.418	832.997	0.015	0.015
11	54507.421	605.280	0.011	0.011
12	53902.140	438.408	0.008	0.008
13	53463.732	328.679	0.006	0.006
14	53135.054	273.823	0.005	0.005
15	52861.230	272.412	0.005	0.005
16	52588.818	323.300	0.006	0.006
17	52265.519	425.097	0.008	0.008
18	51840.422	575.665	0.011	0.011
19	51264.757	771.649	0.015	0.015
20	50493.108	1008.079	0.020	0.020
21	49485.029	1278.064	0.026	0.026

age	l_x	${}_1d_x$	${}_1q_x$	${}_1m_x$
22	48206.965	1572.641	0.033	0.033
23	46634.325	1880.801	0.040	0.041
24	44753.524	2189.750	0.049	0.050
25	42563.774	2485.402	0.058	0.060
26	40078.373	2753.117	0.069	0.071
27	37325.255	2978.639	0.080	0.083
28	34346.617	3149.151	0.092	0.096
29	31197.465	3254.369	0.104	0.110
30	27943.097	3287.510	0.118	0.125
31	24655.587	3246.042	0.132	0.141
32	21409.545	3132.056	0.146	0.158
33	18277.489	2952.212	0.162	0.176
34	15325.276	2717.209	0.177	0.195
35	12608.067	2440.833	0.194	0.214
36	10167.234	2138.687	0.210	0.235
37	8028.547	1826.764	0.228	0.257
38	6201.783	1520.041	0.245	0.279
39	4681.742	1231.296	0.263	0.303
40	3450.446	970.252	0.281	0.327
41	2480.194	743.176	0.300	0.352
42	1737.019	552.894	0.318	0.379
43	1184.124	399.196	0.337	0.405
44	784.928	279.490	0.356	0.433
45	505.438	189.591	0.375	0.462
46	315.847	124.500	0.394	0.491
47	191.348	79.076	0.413	0.521
48	112.272	48.536	0.432	0.552
49	63.737	28.763	0.451	0.583
50	34.974	-	-	-

Q5

Life expectancy at age x, e_x, is defined as:

$$e_x = \frac{\int_x^\infty S(u)du}{S(x)}$$

which simplifies to $\int_0^\infty S(u)du$ for life expectancy at birth, e_0. Using numerical integration, the life expectancy at birth for our cohort is calculated to be **17.834**.

Q6

The life expectancy at age 10 (e_{10}) for a member of this cohort is numerically calculated to be **19.673**.

Q7

The probability that a person aged x dies within the next n years is defined as:

$$_nq_x = \frac{S(x) - S(x+n)}{S(x)}$$

The $_{15}q_{15}$ value for this cohort is then **0.471**.

Appendix

```
# Prep work -------------------------------------------------------------

# Load libraries
library(dplyr)
library(ggplot2)

# Make data
age_range <- c(0, 110)
age_data <- tibble(age = seq(age_range[1], age_range[2], .1))


# Question 1 ------------------------------------------------------------

hazard_fun <- function(x) 0.005 + 0.0005 * (x - 15)^2
cum_hazard_fun <- function(x) .0005 * ((x^3 / 3) - (15 * x^2) + (235 * x))

chf_plot <- ggplot(age_data, aes(x = age, y = cum_hazard_fun(age))) +
  geom_line() +
  theme_bw() +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Cumulative Hazard vs Age",
    x = "Age (years)",
    y = "Cumulative Hazard"
  )

chf_plot


# Question 2 ------------------------------------------------------------

survival_fun <- function(x) exp(-1 * cum_hazard_fun(x))

survf_plot <- ggplot(age_data, aes(x = age, y = survival_fun(age))) +
  geom_line() +
  theme_bw() +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Survival vs Age",
    x = "Age (years)",
    y = "Survival"
  )

survf_plot
```

```r
# Question 3 ------------------------------------------------------------

pdfun_plot <-
  ggplot(age_data, aes(x = age, y = survival_fun(age) * hazard_fun(age))) +
  geom_line() +
  theme_bw() +
  theme(text = element_text(family = "serif")) +
  labs(
    title = "Probability density vs Age",
    x = "Age (years)",
    y = "Probability density"
  )

pdfun_plot


# Question 4 ------------------------------------------------------------

cohort_size <- 100000

lt_data <- tibble(
  age = 0:50,
  lx = survival_fun(age) * cohort_size,
  dx = lx - lead(lx),
  qx = dx / lx,
  mx =  dx / (lx - .5 * dx)
)

lt_names <- c("age", "$l_x$", "$_{1}d_x$", "$_{1}q_x$", "$_{1}m_x$")

knitr::kable(lt_data,
  booktabs = TRUE,
  escape = FALSE,
  digits = 3,
  col.names = lt_names,
  caption = "Cohort life table for single year ages 0 to 50"
)


# Question 5 ------------------------------------------------------------

e0 <- integrate(survival_fun, lower = 0, upper = Inf)
e0_val <- round(e0$value, 3)


# Question 6 ------------------------------------------------------------

e10 <- integrate(survival_fun, lower = 10, upper = Inf)
e10_val <- round(e10$value / survival_fun(10), digits = 3)


# Question 7 ------------------------------------------------------------
```

```r
nqx <- function(x, n) (survival_fun(x) - survival_fun(x + n)) / survival_fun(x)
q15_15 <- round(nqx(15, 15), 3)
```