# ON MINIMIZERS OF INTERACTION FUNCTIONALS WITH COMPETING ATTRACTIVE AND REPULSIVE POTENTIALS

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Abstract. We consider a class of interaction functionals consisting of power-law potentials of attractive and repulsive parts. In the first part of this article, we use the concentration compactness principle to establish existence of global minimizers in certain classes. When the power associated with the repulsion is negative, we prove existence in the class of radially symmetric uniformly bounded densities. We motivate why these assumptions are natural but also show how to relax the condition on radial symmetry. When the power associated with the repulsion is positive, simulations show steady states that concentrate and are not necessarily radially symmetric. We thus prove existence in the class of probability measures. In the second part, we address local minimizers by deriving two necessary conditions associated with criticality. In certain cases, this allows us to characterize the ground state. Finally, we consider these functionals minimized over binary radial densities, deriving conditions for vanishing first variation and positive second variation.

## 1. Introduction

We consider the minimization of energies of the form

$$E[\rho] := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho(x)\rho(y)\,dxdy, \tag{1.1}$$

where

$$K(x) := \frac{1}{q}|x|^q - \frac{1}{p}|x|^p, \qquad \text{for } -N < p < q. \tag{1.2}$$

These functionals are directly connected to a class of self-assembly/aggregation models which recently have received much attention (see for example, [9, 11, 12, 18, 28, 30, 32, 33, 34, 41, 48, 51, 52]). The aggregation models consist of the following active transport equation in $\mathbb{R}^N$ for the population density $\rho$:

$$\rho_t + \nabla\cdot(\rho\vec{v}) = 0, \quad \vec{v} = -\nabla K * \rho, \tag{1.3}$$

where $K$ represents the interaction potential and $*$ denotes spatial convolution. This partial differential equation is the gradient flow of the energy (1.1) with respect to the Wasserstein metric [1]. Indeed, the evolution equation (1.3) can be written in the form

$$\partial_t\rho = \nabla\cdot\Big(\rho\nabla\frac{\delta E[\rho]}{\delta\rho}\Big),$$

which is the standard form for the Wasserstein gradient flow [1] of the energy (1.1).

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Model (1.3) has a wide range of applications, including biological swarms [48, 51], granular media [9, 52], self-assembly of nanoparticles [32, 33] and molecular dynamics simulations of matter [31]. The study of solutions to (1.3) (well-posedness, finite or infinite time blow-up, long-time behavior) has been a very active area of research during the past decade [11, 12, 18, 28, 41]. It is important to note that the analysis and behavior of solutions to (1.3) depend essentially on the properties of the potential $K$. In the context of biological swarms, $K$ incorporates social interactions (attraction and repulsion) between group individuals. Potentials which are attractive in nature typically lead to blow-up [11, 34], while attractive-repulsive potentials may generate finite-size, confined aggregations [30, 41].

A significant number of recent works exploited the gradient flow structure in the particle (individual-based) model associated to (1.3). Specifically, the PDE model (1.3) can be regarded as the continuum limit of the following particle model describing the pairwise interaction of $M$ particles in $\mathbb{R}^N$ [17]:

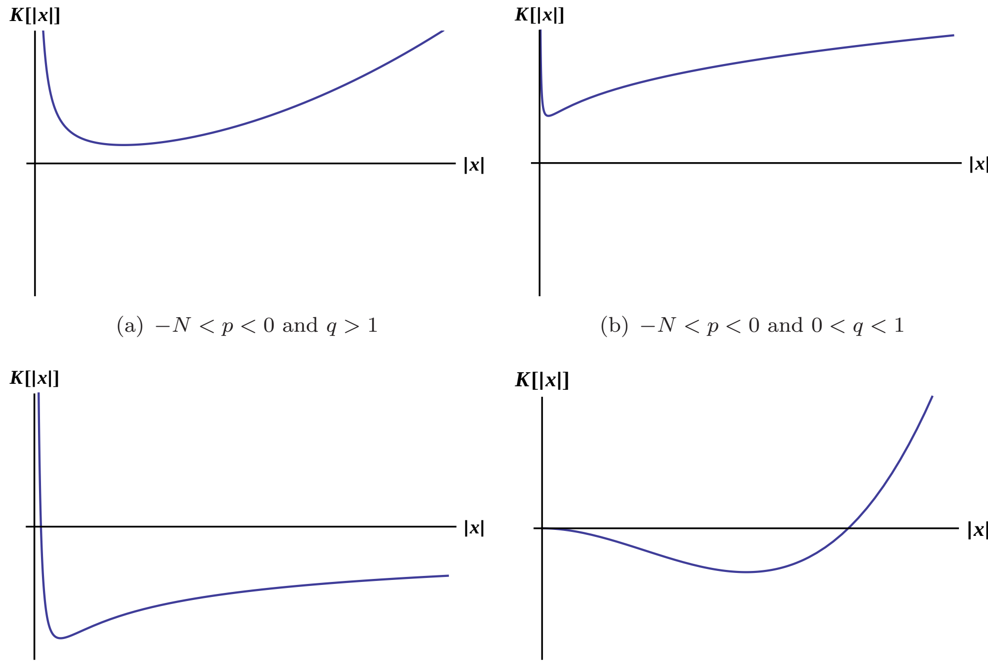
$$\frac{dX_i}{dt} = -\frac{1}{M}\sum_{\substack{j=1\\ j\neq i}}^{M} \nabla K(X_i - X_j), \qquad i = 1\ldots M, \tag{1.4}$$

where $X_i(t)$ represents the spatial location of the $i$-th individual at time $t$. The particle model (1.4) is the gradient flow of the interaction energy which is the discrete version of (1.1) [38]. It has been shown that simple choices of interaction potentials in (1.4) can lead to very diverse and complex equilibrium solutions (e.g., disks, rings and annular regions in 2D, balls, spheres and soccer balls in 3D) [38, 19, 30].

By staying entirely at the continuum level (that is, working with (1.3) without resorting to the particle system (1.4)) it is more difficult to identify equilibria. There are only a few works in this direction. In [29, 30] the authors study equilibrium solutions to (1.3) which are supported in a ball, while in [5] the focus is equilibria that are uniformly distributed on spherical shells. Such equilibria, along with those revealed by simulations of the discrete model, constitute the main motivation for this work. In this article, we directly study the problem from the variational point of view, i.e., minimizers of the nonlocal energy (1.1). In particular, we show the existence of a global minimizer of (1.1) over the classes of radially symmetric and uniformly bounded functions (for $p < 0$) and probability measures (for $p > 0$). We address the assumptions of radial symmetry and uniform boundedness in the summary below, in particular, we point out in which cases these assumptions are natural and in the case of radial symmetry, how one can in fact remove this assumption.

By inspecting the equation for $\vec{v}$ in (1.3) one notes that the nature of a symmetric potential $K(x) = K(|x|)$ is dictated by the sign of its derivative ($K' > 0$ corresponds to attraction and $K' < 0$ to repulsion). Hence, for $K$ given by (1.2), the exponent $q$ refers to attraction and $p$ to repulsion ($p$ and $q$ can be of any sign). The condition $q > p$ is needed to ensure that the potential is repulsive at short ranges and attractive in the far field — see Figures 1(a)–1(c) for a generic illustration of $K$ with $p < 0$ and Figure 1(d) for an example of $K$ with $p > 0$. Note that in the regime $p < 0$ when $q \geqslant 1$, the potential $K$ is positive, convex and $K \to \infty$ as $|x| \to \infty$ whereas when $0 < q < 1$, $K$ is still positive and grows indefinitely with $|x|$; however, it is not convex. Finally for $-N < q < 0$, $K$ becomes negative, approaches 0 as $|x| \to \infty$ and is not convex.

Potentials in power-law form have been frequently considered in the recent literature on the aggregation model (1.3) [5, 19, 29, 30, 38]. As shown in these works, the delicate balance




(a) $-N < p < 0$ and $q > 1$

(b) $-N < p < 0$ and $0 < q < 1$

(c) $-N < p < q < 0$

(d) $q > p > 0$

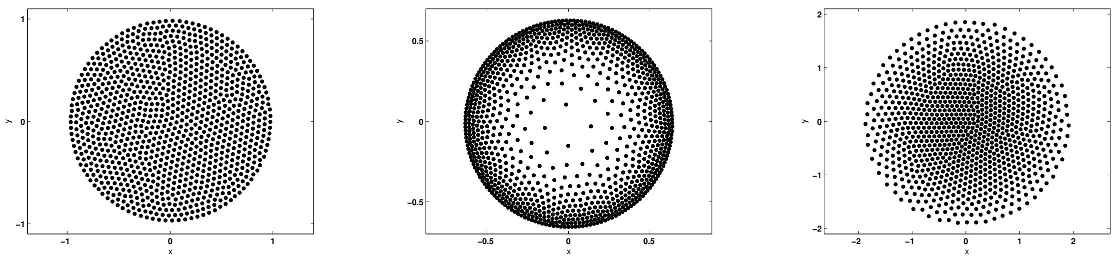
FIGURE 1. Generic examples of $K$ for various values of $p$ and $q$.

between attraction and repulsion often leads to complex equilibrium configurations, supported on sets of various dimensions. Indeed, a simple particle model simulation in two dimensions shows accumulation of the density in different states depending on the powers of the interaction potential $K$ (see Figures 2 and 3). The dimensionality of local minimizers of (1.1) with $K$ given by (1.2) was recently investigated in [4]. The repulsion exponent $p$ in [4] is restricted to be above the Newtonian singularity, i.e., $p > 2 - N$. In our results we only need integrability at the origin, so $p$ can take values in the larger range $p > -N$.

1.1. **Summary of Our Results.** We summarize our results, placing them in the context of other recent work. To this end, we consider two separate cases: $p < 0$ and $p > 0$.

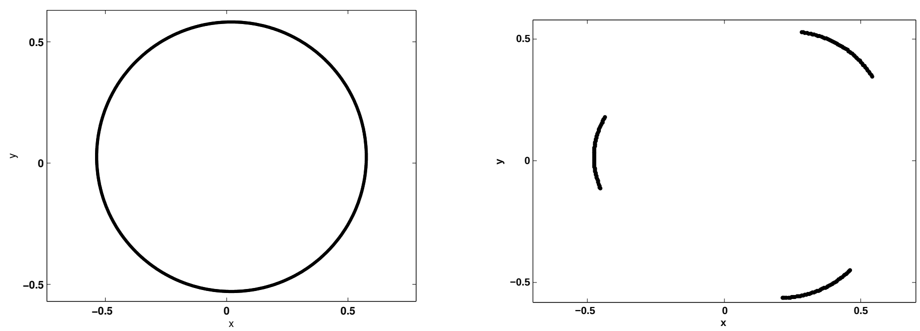
**Negative power repulsion** $p < 0$. Figures 2 (a) - (c) show examples of particle simulations for Newtonian potential repulsion[1] and various values of $q$. In the case of negative $p$, one expects densities to not accumulate on lower-dimensional sets. For example, in [30] the time-dependent density $\rho(\cdot, t)$ is shown to be uniformly bounded in $L^\infty$ for all $t > 0$ provided the initial condition $\rho_0$ is in $L^\infty$. Also in [4], the authors prove that for any $N$ and $-N < p < 0$, minimizers do not accumulate on a set of dimension less than $2 - p$ and point out that they never observed minimizers with support of non-integer Hausdorff dimension. This means that when $N = 3$ the minimizers are indeed functions; however, for $N > 3$ the result is weaker and only gives a lower bound $2 - p < 3$. Nonetheless, for $p < 0$, we expect that minimizers exist in the space of density *functions* (and not measures). As a matter of fact, we immediately see

[1]In $\mathbb{R}^N$, Newtonian potential is given by the repulsive part of (1.2) with $p = 2 - N$. For 2-D particle simulations we use $-\log|x|$ as the Newtonian repulsion. See Remark 2.8 for details.



(a) Newtonian repulsion, quadratic attraction (b) Newtonian repulsion, large positive attraction (c) Below Newtonian repulsion, small negative attraction

FIGURE 2. 2D Particle model simulations in the regime $p < 0$. In 2D, the Newtonian repulsion is given by $-\log|x|$ and below Newtonian repulsion corresponds to $-2 < p < 0$ (see also Remark 2.8).



(a) $q = 10$, $p = 1$ (b) $q = 7$, $p = 1.5$

FIGURE 3. 2D Particle model simulations in the regime $q > p > 0$.

that a Dirac delta integrated against an interaction potential $1/|x|^a$ with $0 < a < N$ cannot be a minimizer of the energy (2.1). However, with only an $L^1$-bound on the density, accumulation along a set of Hausdorff dimension less than $N$ is a possibility – in fact as we discuss below, such possibilities are indeed generic when $p > 0$. To this end, for $p < 0$ we take our admissible class of densities $\rho$ as the space $\mathcal{A}$ of non-negative, uniformly bounded $L^1$ functions with fixed mass $m$ (see equation (2.2) for the precise definition of $\mathcal{A}$). Note that the uniform boundedness condition is necessary to prevent concentrations as the energy does not bound any $L^s$ norm for $s > 1$. We repeat that while the uniform boundedness condition on the density function $\rho$ is a strong assumption, it is supported by results in [29, 30] with $p = 2 - N$, as well as other works that consider power kernels with negative repulsion exponent [4, 5, 19]. The uniform boundedness of the density when minimizing similar energies is also assumed, for example, in [2].

For $p < 0$, even though there might be some minor symmetry defects depending on the choice of number of particles, the simulations with random initial data suggest that the steady states are radially symmetric. Also, considering the isotropic nature of the interaction potentials it seems reasonable to conjecture that the minimizers of the energy (1.1) are radially symmetric. We thus state and prove our existence theorem, Theorem 2.5, in the class of radially symmetric, uniformly bounded densities. In Remark 2.9, we show how this assumption can be relaxed: In the case where $q > 0$, this is straightforward and we provide the details. We also conjecture that minimizers over the wider class of densities are radially symmetric but also note that, unlike in the works [6, 22], here re-arrangement arguments do not necessarily carry over. As we have mentioned, the assumption on uniform boundedness is natural in this case but in principle should not be required (see Remark 2.10). However, it is a technical requirement of our relatively simple proof following the direct method in the calculus of variations. The key idea of this proof is to apply Lions' concentration compactness lemma (Lemma 2.2) to a minimizing sequence and to extract a subsequence which is tight in the sense of measures, and use the uniform boundedness to infer weak convergence in $L^s$ for any $1 < s < \infty$. Weak convergence of the minimizing (sub)sequence is sufficient for lower semicontinuity of the energy functionals (Lemma 2.4).

In Section 3, we give two necessary conditions for critical points of the energy. Using these conditions, we address the case $p = 2 - N$ (Newtonian potential for $N > 2$) and prove, for the special case $q = 2$, that the rescaled characteristic function of a ball is the global minimizer. Given that we are working with uniformly bounded functions, we also consider the problem of minimizing $E$ over binary densities $\rho \in \{0, 1\}$. In particular, we derive expressions for the vanishing first and positive second variation conditions. In Theorem 4.2, we prove that, for the special case $p = 2 - N, q = 2$, the characteristic function of a ball is again the unique global minimizer.

For Newtonian repulsion with stronger attraction $q > 2$, simulations suggest that the density concentrates on an annular shell (cf. [29, 30], see also Figure 2 (b)). This might indicate that annular spherical shells are global minimizers of $E$ over $\rho \in \mathcal{A}$. However, we show that, for the binary variational problem, there do not exist such annular critical points. This also suggests that, while particle simulations do indeed show a density accumulation to an annulus, the minimizing density of (1.1) is still supported in the entire ball as otherwise we would expect an annular critical point of uniform density.

**Positive power repulsion** $p > 0$. In Section 2.2, we address the case of positive $p$ and $q$. The character of the interaction potential $K$ is very different when $p > 0$. In this regime $K$ does not have a singularity at zero and it allows concentration of densities on sets of dimension less than $N$. Note that Figure 3(a) shows an associated particle simulation for positive $p$ and $q$ with accumulation along a circle. The results of [4] support this observation via rigorous bounds on the Hausdorff dimension of the support of minimizers. Moreover, for $p > 0$, simulations shows that minimizers need not be radially symmetric (see Figure 3(b)). Therefore we will define the energy $E$ over probability measures (cf. [4, 5]), that is, in the regime $q > p > 0$, we consider the problem:

$$\text{minimize} \quad E(\mu) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left( \frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p \right) d\mu(x) d\mu(y)$$

over probability measures $\mu \in \mathcal{P}(\mathbb{R}^N)$. Following similar steps as in Theorem 2.5, we prove the existence of a global minimizer.

Related to our positive power repulsion case, in [23] the authors consider a class of aggregation equations with interaction potentials which satisfy certain growth and convexity conditions. Using an approach based on the theory of gradient flows they establish existence and uniqueness of global-in-time weak measure solutions. Later in [24], again working with measure solutions, they find sufficient conditions on the interaction potential which guarantee the confinement of localized solutions for all times. In both works their use of a measure theoretic setting as the class of admissible functions enables them a unified analysis of both particle and continuum models.

We conclude the introduction with two remarks concerning related problems.

**Remark 1.1** (Repulsion via nonlinear diffusion)**.** A related, well-studied model is to consider a kernel which is purely attractive and bring in the repulsive component by adding nonlinear diffusion to the dynamic PDE, i.e.

$$\rho_t + \nabla \cdot (\rho \vec{v}) = \Delta \rho^m, \qquad \vec{v} = -\nabla K * \rho,$$

where $K$ is purely *attractive* (cf. [7, 8, 14, 16, 20, 21, 22]). Here the associated energy functional is given by

$$E_{\text{nld}}[\rho] := \frac{1}{2} \int \int K(x-y)\rho(x)\rho(y) dx dy + \frac{1}{m-1} \int \rho^m(x)\, dx.$$

Using Lions' concentration compactness lemma, a proof of global existence for this functional was given in [6]. A detailed study of steady state solutions is given in [22]. Note that these results do not need a uniform $L^\infty$-bound on the admissible densities as the energy $E_{\text{nld}}$ controls some $L^s$-norm of the density function $\rho$. Also, since the interaction term in the energy $E_{\text{nld}}$ is purely attractive, using symmetric decreasing rearrangement type arguments one sees that the minimizers have to be radially symmetric.

Similar energies also appear in astrophysics and quantum mechanics and have been extensively studied [2, 3, 46]. In fact, in the seminal paper of Lions [46] wherein he introduces the concentration compactness lemma, a direct application is the existence of minimizers to a class of these $L^1$ minimization problems. Many of the arguments in our present article follow his application.

**Remark 1.2** (Thomas-Fermi-Dirac-von Weizsäcker Functionals and the Nonlocal Isoperimetric Problem)**.** Many other functions with interacting attractive and repulsive components have been studied, for example the Thomas-Fermi-Dirac-von Weizsäcker functional in mathematical physics [39, 40, 43, 47]. Recently, a binary *nonlocal isoperimetric functional* appeared in connection with the modeling of self-assembly of diblock copolymers [25, 26, 27]. Existence and non-existence results have been presented in [35, 36, 37, 47]. In dimension $N = 3$, the functional has a Newtonian repulsive component as in 1.1 with $p = -1$. However, the attractive component does not come from an interaction term but rather by adding a higher-order regularization. Precisely, for $m > 0$, the nonlocal isoperimetric problem is to minimize

$$E_{\text{nlip}}[\rho] := \int_{\mathbb{R}^3} |\nabla \rho| \; + \; \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{4\pi |x-y|}\, dx\, dy$$

over

$$\rho \in BV(\mathbb{R}^3, \{0,1\}) \;\text{ with }\; \int_{\mathbb{R}^3} \rho \;=\; m.$$

Since admissible densities $\rho$ are characteristic functions, the first term in the energy is simply the perimeter of the support. Not only is there a competition between the two terms in $E_{\text{nlip}}$ but they are in *direct competition* in the following sense: balls are *best* (least energy) for the first (attractive) term and *worst* (greatest energy) for the second (repulsive) term. The latter point has an interesting history. Poincaré [49] considered the problem of determining the equilibrium shapes of a fluid body of mass $m$. In simplified form, this amounts to minimizing the total potential energy of the region of fluid $E \subset \mathbb{R}^3$

$$\int_E \int_E -\frac{1}{C\,|x-y|}\,dx\,dy,$$

where $-(C|x-y|)^{-1}$ $(C>0)$ is the potential resulting from the gravitational attraction between two points $x$ and $y$ in the fluid. Poincaré showed that, under some smoothness assumptions, a body has the lowest energy if and only if it is a ball. It was not until almost a century later that the essential details were sorted via the rearrangement ideas of Steiner for the isoperimetric inequality. These ideas are captured in the *Riesz Rearrangement Inequality* and its development (cf. [42, 45]).

In [25, 26], it was conjectured that there exists a critical mass $m_0$ below which, a unique minimizer of $E_{\text{nlip}}$ exists and is the characteristic function of a ball, and above which, the minimizer fails to exist because of "mass" escaping to infinity. Note this is in stark contrast to minimizers of (1.1). The non-existence for sufficiently large $m$ has recently been proved in [47]. Existence of a radially symmetric minimizer (i.e. a ball) for $m$ sufficiently small has recently been proved in [35, 37]. Whether or not balls are the *only* minimizers remains open.

## 2. Existence of Global Minimizers

2.1. **Negative-Power Repulsion ($p<0$).** For $p<0$, $q>p$, $m>0$ and $M>0$, we consider the following variational problem:

$$\text{minimize}\quad E(\rho)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{1}{q}|x-y|^q-\frac{1}{p}|x-y|^p\right)\rho(x)\rho(y)\,dxdy \tag{2.1}$$

over

$$\mathcal{A}:=\Big\{\rho\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N):\ \rho=\rho(|x|)\geqslant 0,\ \|\rho\|_{L^\infty}\leqslant M,\ \text{and}\ \int_{\mathbb{R}^N}\rho(x)\,dx=m\Big\}. \tag{2.2}$$

**Remark 2.1.** We note that the case $q=2$ is rather special. Indeed, by the radial symmetry assumption the center of mass is conserved, that is,

$$\int_{\mathbb{R}^N} x\rho(x)\,dx=0.$$

A simple calculation leads to

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|x-y|^2\rho(x)\rho(y)\,dxdy&=\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(|x|^2-2x\cdot y+|y|^2\right)\rho(x)\rho(y)\,dxdy\\&=m\int_{\mathbb{R}^N}|x|^2\rho(x)\,dx.\end{aligned}$$

With the attractive term being simplified, the energy (2.1) can be written as

$$E(\rho)=m\int_{\mathbb{R}^N}|x|^2\rho(x)\,dx-\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|x-y|^p\rho(x)\rho(y)\,dxdy. \tag{2.3}$$

We now prove the existence of a minimizer for (2.1) using the direct method of the calculus of variations. The key tool we use in establishing the existence of minimizers is the concentration-compactness lemma by Lions [46, Lemma I.1].

**Lemma 2.2** (Concentration-compactness lemma [46])**.** *Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$\rho_n \geqslant 0 \text{ in } \mathbb{R}^N, \quad \int_{\mathbb{R}^N} \rho_n(x)\,dx = m,$$

*for some fixed $m > 0$. Then there exists a subsequence $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ satisfying one of the three following possibilities:*

(i) *(tightness up to translation) there exists $y_k \in \mathbb{R}^N$ such that $\rho_{n_k}(\cdot + y_k)$ is tight, that is, for all $\epsilon > 0$ there exists $R > 0$ such that*

$$\int_{B(y_k,R)} \rho_{n_k}(x)\,dx \geqslant m - \epsilon;$$

(ii) *(vanishing)* $\displaystyle\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \rho_{n_k}(x)\,dx = 0$, *for all $R > 0$;*

(iii) *(dichotomy) there exists $\alpha \in (0, m)$ such that for all $\epsilon > 0$, there exist $k_0 \geqslant 1$ and $\rho_{1,k}$, $\rho_{2,k} \in L^1_+(\mathbb{R}^N)$ satisfying for $k \geqslant k_0$*

$$\|\rho_{n_k} - (\rho_{1,k} + \rho_{2,k})\|_{L^1(\mathbb{R}^N)} \leqslant \epsilon,$$
$$\Big| \|\rho_{1,k}\|_{L^1(\mathbb{R}^N)} - \alpha \Big| \leqslant \epsilon, \quad \Big| \|\rho_{2,k}\|_{L^1(\mathbb{R}^N)} - (m-\alpha) \Big| \leqslant \epsilon,$$

*and*

$$\mathrm{dist}(\mathrm{supp}(\rho_{1,k}), \mathrm{supp}(\rho_{2,k})) \to \infty \text{ as } k \to \infty.$$

In certain cases, we will use the following special form of the Hardy–Littlewood–Sobolev inequality to bound the energy from below.

**Proposition 2.3** (cf. Theorem 3.1 in [44])**.** *For any $-N < p < 0$ and $f \in L^{2N/(2N+p)}(\mathbb{R}^N)$ we have*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p f(x) f(y)\,dxdy \leqslant C(p)\|f\|^2_{L^{2N/(2N+p)}(\mathbb{R}^N)}$$

*where the sharp constant $C(p)$ is given by*

$$C(p) = \pi^{-p/2}\frac{\Gamma(N/2 + p/2)}{\Gamma(N + p/2)}\left(\frac{\Gamma(N/2)}{\Gamma(N)}\right)^{-1-p/N}$$

*with $\Gamma$ denoting the Gamma function.*

Finally, we state and prove a lemma which we will use in establishing the lower semicontinuity of the energy $E$. A similar argument appears in the proof of Theorem II.1 in [46].

**Lemma 2.4.** *Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ and $f \in \mathcal{A}$ be given such that $f_n \rightharpoonup f$ weakly in $L^s(\mathbb{R}^N)$ for some $1 < s < \infty$. Then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f_n(x) f_n(y)}{|x-y|^a}\,dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x) f(y)}{|x-y|^a}\,dxdy$$

*where $0 < a < N$.*

*Proof.* First note that

$$\begin{aligned}\left|\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f_n(x)f_n(y)}{|x-y|^a}\,dxdy\right)^{1/2}-\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)f(y)}{|x-y|^a}\,dxdy\right)^{1/2}\right|&\\ \leqslant\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\,dxdy\right|^{1/2}&.\end{aligned}\tag{2.4}$$

On the other hand, for $R>0$ we have that

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\,dxdy\right|\\ &\qquad\leqslant\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{|x|<1/R\}}(|x-y|)\,dxdy\right|\\ &\qquad\quad+\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{|x|>R\}}(|x-y|)\,dxdy\right|\\ &\qquad\quad+\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{1/R<|x|<R\}}(|x-y|)\,dxdy\right|,\end{aligned}$$

where $\chi_A$ denotes the characteristic function of the set $A$. Since $f_n$ and $f$ are in $\mathcal{A}$, they are uniformly bounded and $\|f_n\|_{L^1(\mathbb{R}^N)}=\|f\|_{L^1(\mathbb{R}^N)}=m$. Hence, the above inequality yields

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\,dxdy\right|&\leqslant C_1\frac{1}{R^{N-a}}+C_2\frac{1}{R^a}\\ +\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{1/R<|x|<R\}}(|x-y|)\,dxdy\right|&,\end{aligned}\tag{2.5}$$

for some constants $C_1$, $C_2>0$ depending only on $a$, $M$ and $N$.

For simplicity of presentation, define

$$g(x-y):=\frac{1}{|x-y|^a}\chi_{\{1/R<|x|<R\}}(|x-y|)$$

and note that $g(x-\cdot)\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Also, define

$$F_n(x):=\int_{\mathbb{R}^N}f_n(y)g(x-y)\,dy\quad\text{ and }\quad F(x):=\int_{\mathbb{R}^N}f(y)g(x-y)\,dy.$$

Since $f_n\rightharpoonup f$ weakly in $L^s(\mathbb{R}^N)$, for all $x\in\mathbb{R}^N$ we have that

$$F_n(x)\to F(x),\qquad\text{as } n\to\infty.$$

Moreover, integrating $F_n$ over $\mathbb{R}^N$ and using $\|f_n\|_{L^1(\mathbb{R}^N)}=\|f\|_{L^1(\mathbb{R}^N)}=m$, we find

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}f_n(y)g(x-y)\,dydx&=\left(\int_{\mathbb{R}^N}g(s)\,ds\right)\left(\int_{\mathbb{R}^N}f_n(y)\,dy\right)\\ &=\left(\int_{\mathbb{R}^N}g(s)\,ds\right)\left(\int_{\mathbb{R}^N}f(y)\,dy\right)\\ &=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}f(y)g(x-y)\,dydx,\end{aligned}$$

which implies that $\|F_n\|_{L^1(\mathbb{R}^N)} \to \|F\|_{L^1(\mathbb{R}^N)}$. Since $|F_n - F| \leqslant |F_n| + |F|$, the function $|F_n| + |F| - |F_n - F|$ is positive. So, applying Fatou's theorem we get that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |F_n| + |F| - |F_n - F| \, dx \geqslant \int_{\mathbb{R}^N} \liminf_{n\to\infty} |F_n| + |F| - |F_n - F| \, dx;$$

hence,

$$2\|F\|_{L^1(\mathbb{R}^N)} - \limsup_{n\to\infty} \int_{\mathbb{R}^N} |F_n - F| \, dx \geqslant 2\|F\|_{L^1(\mathbb{R}^N)}.$$

Thus $\limsup_{n\to 0} \int_{\mathbb{R}^N} |F_n - F| \, dx = 0$, that is, $F_n \to F$ strongly in $L^1(\mathbb{R}^N)$.

Consequently, $F_n \to F$ in $L^s(\mathbb{R}^N)$. Now, since $f_n \rightharpoonup f$ weakly in $L^s(\mathbb{R}^N)$, we have that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (f_n(x) - f(x))(f_n(y) - f(y)) g(x-y) \, dxdy \to 0$$

as $n \to \infty$. Letting $R \to \infty$ in (2.5) yields, by (2.4), the desired result. □ □

We now prove the existence theorem for $p < 0$.

**Theorem 2.5.** *There exists a minimum of* (2.1) *when* $-N < p < q < 0$ *or* $-N < p < 0 < q$.

*Proof.* Let $\{\rho_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ be a minimizing sequence of the problem (2.1), that is, let $\{\rho_n\} \subset \mathcal{A}$ be a sequence such that

$$\lim_{n\to\infty} E(\rho_n) = \inf\{E(\rho) \colon \rho \in \mathcal{A}\}.$$

To prove the theorem we consider the two regimes of $q$ separately, as the character of the interaction potential $K$ is quite different in the two cases (see Figures 1(a), 1(b) and 1(c)).

**Case 1:** $-N < p < 0 < q$. In this regime both terms of the energy are positive. Hence, $E(\rho) \geqslant 0$ for all $\rho \in \mathcal{A}$, so the above infimum exists and is nonnegative. As $\{\rho_n\}_{n\in\mathbb{N}}$ is a minimizing sequence, for sufficiently large $n$ the energy $E(\rho_n)$ is uniformly bounded.

By the concentration-compactness lemma (Lemma 2.2) the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ has a subsequence which satisfies one of the three possibilities: "tightness up to translation", "vanishing" or "dichotomy". We will show that "tightness up to translation" is the only possibility. To this end, suppose "vanishing" occurs. Let $0 < \epsilon < m$ and $R > 0$. Then for $k$ large enough

$$\int_{B(0,R)} \rho_{n_k}(x) \, dx \leqslant \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \rho_{n_k}(x) \, dx < \epsilon.$$

Since $\rho_{n_k} \in \mathcal{A}$ this implies that

$$\int_{\mathbb{R}^N\setminus B(0,R)} \rho_{n_k}(x) \, dx \geqslant m - \epsilon > 0. \tag{2.6}$$

Now we are going to use the fact that the attractive term grows indefinitely at infinity. Since $\rho_{n_k}$ are positive we have that

$$\begin{aligned} &\frac{1}{q} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |x-y|^q \rho_{n_k}(x) \rho_{n_k}(y) \, dxdy \\ &\qquad\qquad \geqslant \frac{1}{q} \int_{\mathbb{R}^N\setminus B(0,R)} \int_{\mathbb{R}^N\setminus B(0,R)} |x-y|^q \rho_{n_k}(x) \rho_{n_k}(y) \, dxdy. \end{aligned} \tag{2.7}$$

Due to radial symmetry, integration in the right-hand-side of (2.7) can be done on rays passing through the origin. Since one can always take $x$ and $y$ lying on opposite rays, so that $|x-y|>2R$, we have by using (2.6),

$$\int_{\mathbb{R}^N\setminus B(0,R)}\int_{\mathbb{R}^N\setminus B(0,R)} |x-y|^q\rho_{n_k}(x)\rho_{n_k}(y)\,dxdy \geqslant (2R)^q(m-\epsilon)^2. \tag{2.8}$$

Combining (2.7) and (2.8), we find

$$E(\rho_{n_k}) \geqslant \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q\rho_{n_k}(x)\rho_{n_k}(y)\,dxdy \geqslant C\,R^q.$$

As $q>0$, this inequality yields that $E(\rho_{n_k})\to\infty$ as $R\to\infty$, contradicting the fact that $\rho_{n_k}$ is a minimizing sequence. Therefore "vanishing" does not occur.

Next, suppose "dichotomy" occurs. Using the notation of Lemma 2.2(iii), let

$$d_k := \operatorname{dist}(\operatorname{supp}(\rho_{1,k}),\operatorname{supp}(\rho_{2,k}))$$

denote the distance between the supports of $\rho_{1,k}$ and $\rho_{2,k}$. We can further assume that the supports of $\rho_{1,k}$ and $\rho_{2,k}$ are disjoint. Inspecting again the attraction term we get that for some constant $C>0$,

$$\begin{aligned}
&\frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q\rho_{n_k}(x)\rho_{n_k}(y)\,dxdy\\
&\qquad\geqslant \frac{C}{q}\int_{\operatorname{supp}(\rho_{1,k})}\int_{\operatorname{supp}(\rho_{2,k})} |x-y|^q\rho_{1,k}(x)\rho_{2,k}(y)\,dxdy\\
&\qquad\geqslant \frac{C}{q}\,d_k^q\,\|\rho_{1,k}\|_{L^1(\mathbb{R}^N)}\|\rho_{2,k}\|_{L^1(\mathbb{R}^N)}.
\end{aligned}$$

Since $d_k\to\infty$ as $k\to\infty$ it gives that $E(\rho_{n_k})\to\infty$, contradicting again the fact that $\rho_{n_k}$ is a minimizing sequence. Thus "dichotomy" does not occur.

Therefore "tightness up to translation" is the only possibility; hence, there exists a sequence $\{y_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$\text{for all } \epsilon>0 \text{ there exists } R>0 \text{ satisfying } m \geqslant \int_{B(y_k,R)} \rho_{n_k}(x)\,dx \geqslant m-\epsilon. \tag{2.9}$$

Now, let $\overline{\rho}_{n_k}(x)=\rho_{n_k}(x+y_k)$ and note that $E(\rho_{n_k})=E(\overline{\rho}_{n_k})$ by translation invariance of the energy $E$. Thus, $\{\overline{\rho}_{n_k}\}_{k\in\mathbb{N}}$ is also a minimizing sequence. Since $\{\rho_{n_k}\}_{k\in\mathbb{N}}\subset\mathcal{A}$, all members of the sequence are uniformly bounded in $L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and passing to a subsequence if necessary, we may assume that

$$\overline{\rho}_{n_k} \rightharpoonup \rho_0 \text{ weakly in } L^s(\mathbb{R}^N)$$

for some $1<s<\infty$ and some $\rho_0\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ [2]. Moreover, by (2.9),

$$\int_{\mathbb{R}^N}\rho_0(x)\,dx = m,$$

[2]In fact, the sequence $\overline{\rho}_{n_k}$ converges weakly to $\rho_0$ in $L^s(\mathbb{R}^N)$ for every $1<s<\infty$ because of the uniform bound on the sequence. The weak convergence holds for $s=1$, as well, by (2.9) and since the translation sequence $\{y_k\}_{k\in\mathbb{N}}$ can be taken to be zero by the translation invariance of the energy.

or in other words, when passing to the limit as $k \to \infty$ the sequence $\overline{\rho}_{n_k}$ does not "leak-out" at infinity. To show that $\rho_0 \geqslant 0$ a.e. let

$$S := \{x \in \mathbb{R}^N \colon \rho_0(x) < 0\}.$$

Then the characteristic function of $S$, $\chi_S$, is an admissible test function for the weak convergence of $\overline{\rho}_{n_k}$, so we get that

$$\int_S \overline{\rho}_{n_k}(x)\,dx \to \int_S \rho_0(x)\,dx < 0.$$

However, since $\overline{\rho}_{n_k} \in \mathcal{A}$, we see that

$$\liminf_{k\to\infty} \int_S \overline{\rho}_{n_k}(x)\,dx \geqslant 0;$$

hence, $S$ has measure zero. Similarly we can show that $\|\rho_0\|_{L^\infty(\mathbb{R}^N)} \leqslant M$. Thus $\rho_0 \in \mathcal{A}$.

Next we need to show that the energy is weakly lower semicontinuous. Here, with an abuse of notation, we will drop the bar on $\overline{\rho}_n$, and simply denote them by $\rho_n$.

By Lemma 2.4, the repulsive part is weakly lower semicontinuous and we have that

$$-\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho_n(x)\rho_n(y)\,dxdy \to -\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho_0(x)\rho_0(y)\,dxdy \tag{2.10}$$

as $n \to \infty$.

On the other hand, for the attractive part, define

$$G_n(x) = \int_{B(0,R)} |x-y|^q \rho_n(y)\,dy \quad \text{and} \quad G_0(x) = \int_{B(0,R)} |x-y|^q \rho_0(y)\,dy,$$

for any fixed $R > 0$. Note that since $\|\rho_0\|_{L^\infty(\mathbb{R}^N)} \leqslant M$ and $q > 0$, we see that $G_0 \in L^\infty(B(0,R))$, in particular, $G_0 \in L^{s/(s-1)}(B(0,R))$. Therefore, by the weak convergence of $\rho_n$ in $L^s(B(0,R))$,

$$\int_{B(0,R)} G_0(x)[\rho_n(x) - \rho_0(x)]\,dx \to 0 \tag{2.11}$$

as $n \to \infty$. Also, since $\rho_n$ are uniformly bounded, taking $\int_{B(0,R)} |\cdot - y|^q\,dy \in L^{s/(s-1)}(B(0,R))$ as a test function, we see that

$$\int_{B(0,R)} \rho_n(x)[G_n(x) - G_0(x)]\,dx \to 0 \tag{2.12}$$

as $n \to \infty$, by the weak convergence of $\rho_n$ in $L^s(B(0,R))$.

Thus, using (2.11) and (2.12), we have that

$$\begin{aligned}\int_{B(0,R)} G_n(x)\rho_n(x)\,dx &= \int_{B(0,R)} G_0(x)[\rho_n(x) - \rho_0(x)]\,dx \\ &\quad + \int_{B(0,R)} \rho_n(x)[G_n(x) - G_0(x)]\,dx + \int_{B(0,R)} G_0(x)\rho_0(x)\,dx\end{aligned}$$

converges to

$$\int_{B(0,R)} G_0(x)\rho_0(x)\,dx$$

as $n \to \infty$. Hence,

$$\liminf_{n\to\infty} \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy = \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_0(x)\rho_0(y)\,dxdy.$$

Now, by (2.9), for given $\epsilon > 0$, we can choose $R > 0$ such that

$$\int_{B(0,R)} \rho_0(x)\,dx \geqslant m - \epsilon.$$

Then, for such $R$, since $E(\rho_0) < \infty$, we can control the excess of the attractive part on $\mathbb{R}^N \setminus B(0,R)$ and we get that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_0(x)\rho_0(y)\,dxdy &\leqslant \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_0(x)\rho_0(y)\,dxdy + C\epsilon \\ &\leqslant \liminf_{n\to\infty} \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy + C\epsilon \qquad (2.13) \\ &\leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy + C\epsilon.\end{aligned}$$

Letting $\epsilon \to 0$ and combining with (2.10) yields

$$\inf\{E(\rho) : \rho \in \mathcal{A}\} \leqslant E(\rho_0) \leqslant \liminf_{n\to\infty} E(\rho_n) = \inf\{E(\rho) : \rho \in \mathcal{A}\},$$

that is, $\rho_0$ is a solution to the minimization problem (2.1) in the regime $-N < p < 0 < q$.

**Case 2:** $-N < p < q < 0$. Note that in this case, the character of the interaction potential is quite different than in the previous regime. Now the attractive term is strictly negative whereas the repulsive part of the energy $E$ is still strictly positive. However, using Proposition 2.3 we see that the attractive term is bounded below, and we conclude that in this regime

$$\inf\{E(\rho)\colon \rho \in \mathcal{A}\} > -\infty.$$

Looking at the scaling

$$\rho_\lambda(x) = \frac{1}{\lambda^N}\rho\left(\frac{x}{\lambda}\right)$$

we see that $\rho_\lambda \in \mathcal{A}$ for $\lambda \geqslant 1$, and the energy of $\rho_\lambda$ is given by

$$E(\rho_\lambda) = \frac{\lambda^q}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho(x)\rho(y)\,dxdy - \frac{\lambda^p}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho(x)\rho(y)\,dxdy,$$

for any given function $\rho \in \mathcal{A}$. Note that, in particular, we can choose $\rho$ to be the characteristic function of a ball. Since $-N < p < q < 0$, for $\lambda$ large enough we get that $E(\rho_\lambda) < 0$, and hence,

$$I_m := \inf\{E(\rho)\colon \rho \in \mathcal{A}\} < 0.$$

Again, we will make use of the concentration compactness lemma, Lemma 2.9, and show that for a minimizing sequence $\rho_n$ the possibilities of “vanishing” and “dichotomy” do not occur.

Suppose “vanishing” occurs. Since $I_m < 0$ in this regime and since the repulsive part is strictly positive, looking at the attractive part we have that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy > 0. \qquad (2.14)$$

Let $R > 1$ and $q = -a$ for $0 < a < N$. Then, as in the proof of Lemma 2.4,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_n(x)\rho_n(y)}{|x-y|^a}\chi_{\{|x|<1/R\}}(|x-y|)\,dxdy \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_n(x)\rho_n(y)}{|x-y|^a}\chi_{\{1/R<|x|<R\}}(|x-y|)\,dxdy \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_n(x)\rho_n(y)}{|x-y|^a}\chi_{\{|x|>R\}}(|x-y|)\,dxdy \\ &\leqslant \frac{CmM}{R^{N-a}} + \frac{m^2}{R^a} + R^a \int_{\mathbb{R}^N}\rho_n(x)\int_{|x-y|<R}\rho_n(y)\,dydx \\ &\leqslant \frac{CmM}{R^{N-a}} + \frac{m^2}{R^a} + R^a\, m \sup_{x\in\mathbb{R}^N}\left(\int_{|x-y|<R}\rho_n(y)\,dy\right)\end{aligned}$$

where $C$ is positive constant depending only on $a$ and $N$, $M$ is the uniform bound on $\rho_n$ and $m$ is the mass of $\rho_n$, as before.

Since $\rho_n$ vanishes by Lemma 2.2 (ii), we get that as $n \to \infty$ the last term in the above inequality is zero; hence,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy \leqslant \frac{CmM}{R^{N-a}} + \frac{m^2}{R^a}.$$

Letting $R \to \infty$, since $0 < a < N$, this yields that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy \leqslant 0,$$

contradicting (2.14). Thus "vanishing" does not occur.

To show that "dichotomy" does not occur, first we need to prove a subadditivity condition similar to the one in [46]. As in [6, Lemma 1], we can prove a weaker subadditivity condition which states that

$$\text{for } m_1 > m_2 \text{ we have } I_{m_1} < I_{m_2}, \tag{2.15}$$

where, as above, $I_{m_i}$ denotes the infimum of $E$ over $\mathcal{A}_{m_i}$ with mass constraint $\int_{\mathbb{R}^N}\rho(x)\,dx = m_i$. Here we choose to display the dependence of the admissible class $\mathcal{A}$ on the mass by using the notation $\mathcal{A}_{m_i}$ to avoid confusion.

Suppose $m_1 > m_2$ and let $\psi \in \mathcal{A}_{m_2}$ be an arbitrary function. For

$$c := \frac{m_2}{m_1} < 1$$

define $\rho \in \mathcal{A}_{m_1}$ such that

$$\psi = c\,\rho.$$

Then we have that

$$E[\psi] = c^2\,E[\rho].$$

Note that since $I_{m_1} < 0$ in this regime and since $c^2 < 1$ we have that

$$c^2\, I_{m_1} > I_{m_1}.$$

Thus

$$E[\psi] = c^2\,E[\rho] \geqslant c^2\, I_{m_1} > I_{m_1},$$

and taking the infimum of both sides over $\mathcal{A}_{m_2}$ implies that

$$I_{m_2} > I_{m_1}.$$

Now, suppose "dichotomy" occurs, that is, there exists $\alpha \in (0, m)$ such that for all $\epsilon > 0$, there exist $k_0 \geqslant 1$ and $\rho_{1,k},\ \rho_{2,k} \in L^1_+(\mathbb{R}^N)$ satisfying for $k \geqslant k_0$

$$\|\rho_{n_k} - (\rho_{1,k} + \rho_{2,k})\|_{L^1(\mathbb{R}^N)} \leqslant \epsilon,$$
$$\Big| \|\rho_{1,k}\|_{L^1(\mathbb{R}^N)} - \alpha \Big| \leqslant \epsilon, \quad \Big| \|\rho_{2,k}\|_{L^1(\mathbb{R}^N)} - (m - \alpha) \Big| \leqslant \epsilon,$$

and

$$\mathrm{dist}(\mathrm{supp}(\rho_{1,k}), \mathrm{supp}(\rho_{2,k})) \to \infty \text{ as } k \to \infty.$$

Furthermore, after defining $v_k := \rho_{n_k} - (\rho_{1,k} + \rho_{2,k})$ we can assume that

$$0 \leqslant \rho_{1,k}, \rho_{2,k}, v_k \leqslant \rho_{n_k} \text{ and } \rho_{1,k}\rho_{2,k} = \rho_{1,k}v_k = \rho_{2,k}v_k = 0 \text{ a.e.} \tag{2.16}$$

We have that, for any $0 < a < N$,

$$\begin{aligned} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{n_k}(x)\rho_{n_k}(y)}{|x-y|^a}\,dxdy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{1,k}(x)\rho_{1,k}(y)}{|x-y|^a}\,dxdy \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{2,k}(x)\rho_{2,k}(y)}{|x-y|^a}\,dxdy + 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{1,k}(x)\rho_{2,k}(y)}{|x-y|^a}\,dxdy \\ &+ 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{n_k}(x)v_k(y)}{|x-y|^a}\,dxdy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v_k(x)v_k(y)}{|x-y|^a}\,dxdy. \end{aligned} \tag{2.17}$$

The last two terms above vanish as $k \to \infty$ using the integrability of the kernel around zero, the uniform bound on $\rho_{n_k}$ and the fact that $\|v_k\|_{L^1(\mathbb{R}^N)} \to 0$. Since $\mathrm{dist}(\mathrm{supp}(\rho_{1,k}), \mathrm{supp}(\rho_{2,k})) \to \infty$ as $k \to \infty$, and $\lim_{|x|\to\infty} K(|x|) = 0$ in this regime, the third term on the right hand side of (2.17) goes to zero as $k \to \infty$.

Again, since $\mathrm{dist}(\mathrm{supp}(\rho_{1,k}), \mathrm{supp}(\rho_{2,k})) \to \infty$ as $k \to \infty$, we see that one of the components of $\rho_n$ is localized and the other component (say $\rho_{2,k}$, without loss of generality) spreads to infinity, i.e., $\mathrm{dist}(0, \mathrm{supp}(\rho_{2,k})) \to \infty$ as $k \to \infty$. Also, as the supports of $\rho_{1,k}$, $\rho_{2,k}$ and $v_k$ are disjoint as noted in (2.16) and the functions $\rho_{n_k}$ are radially symmetric, so are the functions $\rho_{2,k}$. Using the radial symmetry of $\rho_{2,k}$ and considering the energy $E[\rho_{2,k}]$ on rays in opposite directions (see the argument leading to (2.8)), and recalling that the kernel $K(|x|)$ approaches zero as $|x| \to \infty$, we get that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho_{2,k}(x)\rho_{2,k}(y)\,dxdy = 0,$$

as well.

These observations combined with (2.17) imply that

$$\begin{aligned} I_m &= \lim_{k\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho_{n_k}(x)\rho_{n_k}(y)\,dxdy \\ &\geqslant \liminf_{k\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho_{1,k}(x)\rho_{1,k}(y)\,dxdy \\ &= I_\alpha, \end{aligned} \tag{2.18}$$

which contradicts the weak subadditivity condition (2.15). Thus "dichotomy" does not occur.

As in the first case, "tightness up to translation" is the only possibility. Therefore the weak limit $\rho_0$ of the translated sequence $\overline{\rho}_n$ satisfies the mass constraint and hence, is a member of $\mathcal{A}$.

The weak lower semicontinuity in this regime follows directly from Lemma 2.4 as both attractive and repulsive terms of the energy are of the form considered in the lemma and by (2.9) the assumptions of the lemma are satisfied. We conclude that the minimization problem (2.1) has a solution when $-N < p < q < 0$. □ □

**Remark 2.6.** The concentration-compactness principle suffices to establish a weaker form compactness so that we can pass to a weak limit in the sequence $\rho_n$. However, the sequence does not necessarily convergence strongly to $\rho$ in any $L^s(\mathbb{R}^N)$. Indeed, strong convergence can fail due to mass leaking out at infinity and/or because of oscillations. By the tightness of the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ the former does not happen; but, we cannot rule out the oscillations of $\rho_n$.

On the other hand, we note that for functionals which contain a term that is convex in $\rho$ (cf. Remark 1.1), one can further show that the convergence of $\{\rho_n\}_{n\in\mathbb{N}}$ is strong (cf. [6, 46]).

**Remark 2.7.** Note that the proof of existence of minimizers also applies for potentials of the form

$$K(x) = f(|x|) - \frac{1}{p}|x|^p,$$

where $f(|x|) \to \infty$ as $|x| \to \infty$ and $-N < p < 0$.

**Remark 2.8.** When $N = 2$ the Newtonian potential is given by $-\frac{1}{2\pi}\log|x|$. Either considering the logarithmic term as the repulsion in

$$K(x) = \frac{1}{q}|x|^q - \log|x|, \qquad q > 0, \quad x \in \mathbb{R}^2$$

or as the attraction in

$$K(x) = \log|x| - \frac{1}{p}|x|^p, \qquad -2 < p < 0, \quad x \in \mathbb{R}^2$$

the proof of Theorem 2.5 applies since the properties of the interaction potential (singularity at the origin and blow-up at infinity) remain the same as in higher dimensions.

**Remark 2.9 (Radial Symmetry).** We have already noted in the introduction that in the regime $p < 0$, radial symmetry is observed in numerical simulations, and hence one does expect the minimizer to be radially symmetric. For this reason and for simplicity of arguments, we have presented the proof of global existence in the class of radially symmetric densities. Ideally, one would prove existence in a general class of densities and then prove that the minimizer is indeed radial. Proving that the minimizer is radial, though, is a complicated task and an open problem. Unlike in the case of purely attractive kernels (cf. Remark 1.1), symmetrization via Riesz rearrangement techniques [45] do not immediately apply here because of the repulsive-attractive combination in the kernel.

Nevertheless, here we note that in the regime $-N < p < 0 < q$, one can accomplish the first part, that is, one can relax the radial symmetry and obtain existence in the general class.

In proof of Theorem 2.5 Case 1, we use the radial symmetry of admissible densities only to dismiss the possibility of "vanishing". However, since $q > 0$, the attraction term is very strong, and instead of (2.7) and (2.8), "vanishing" can be eliminated by using the inequality

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho(x)\rho(y)\,dxdy &\geqslant \int_{\mathbb{R}^N}\int_{|x-y|>R} |x-y|^q \rho(x)\rho(y)\,dxdy \\ &\geqslant R^q \int_{\mathbb{R}^N} \rho(x)\left(\int_{|x-y|>R} \rho(y)\,dy\right)dx \\ &\geqslant R^q\, m(m-\epsilon).\end{aligned}$$

along a subsequence of a minimizing sequence instead.

In the proof of Theorem 2.5 Case 2, i.e., when $-N < p < q < 0$, we use the radial symmetry of admissible densities to rule out "dichotomy" and we believe that radial symmetry assumption can be relaxed in this regime, as well; however, the above argument does not apply directly in this case as the kernel $K$ does not grow indefinitely but approach zero in this regime (see Figure 1(c)). In fact, in Section 2.2 we use a similar trick in the existence proof and do not assume radial symmetry on the admissible class.

**Remark 2.10** (**Uniform Boundedness**)**.** The uniform $L^\infty$ bound is used for admissible densities $\rho$ in the proof of Theorem 2.5 to control the energy near the singularity of $K$, and provide weak compactness in some function space. We have motivated, for example from the point of view of gradient flow dynamics, why this assumption is natural, or more precisely, acceptable. However, we do not believe it is essential but rather convenient for our proof. In fact, we believe that the assumption can be relaxed by just taking densities in some $L^s$ (not necessarily uniformly bounded), using tightness of a minimizing sequence to imply convergence of measures, and then showing that, due to the negative-power repulsion, finite energy rules out concentrations (i.e. the densities remain *functions*). It would be interesting in the future to document these details.

2.2. **Positive-Power Repulsion** (**$p > 0$**)**.** As we have already mentioned, the character of the interaction potential $K$ is very different when $p$ is positive. In this regime $K$ does not have a singularity at zero and hence concentrations of densities on sets of dimension less than $N$ is possible. In this regime, particle simulations also reveal non-radially symmetric steady states (see Figure 3). We thus consider the class of admissible densities to be probability measures, $\mathcal{P}(\mathbb{R}^N)$, endowed with the weak-* topology. Note that when $q > p > 0$ we do not assume radial symmetry on the admissible class. The variational problem is to

$$\text{minimize} \quad E(\mu) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right) d\mu(x)d\mu(y), \tag{2.19}$$

over probability measures $\mu \in \mathcal{P}(\mathbb{R}^N)$.

The main tool in establishing the existence of minimizers for (2.19) will be, again, the concentration-compactness principle. In particular, we will cite [50, Section 4.3] for the following version of Lemma 2.2.

**Lemma 2.11** (Concentration-compactness lemma for measures)**.** *Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}^N$. Then there exists a subsequence $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ satisfying one of the three following possibilities:*

(i) *(tightness up to translation) there exists $y_k \in \mathbb{R}^N$ such that for all $\epsilon > 0$ there exists $R > 0$ with the property that*
$$\int_{B(y_k,R)} d\mu_{n_k}(x) \geqslant 1-\epsilon \qquad \text{for all } k.$$

(ii) *(vanishing)* $\displaystyle\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} d\mu_{n_k}(x) = 0$*, for all $R > 0$;*

(iii) *(dichotomy) there exists $\alpha \in (0,1)$ such that for all $\epsilon > 0$, there exist a number $R > 0$ and a sequence $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$ with the following property:*

*Given $R' > R$ there are non-negative measures $\mu_k^1$ and $\mu_k^2$ such that*
$$0 \leqslant \mu_k^1 + \mu_k^2 \leqslant \mu_{n_k}\,,$$
$$\operatorname{supp}(\mu_k^1) \subset B(y_k, R),\ \operatorname{supp}(\mu_k^2) \subset \mathbb{R}^N \setminus B(y_k, R')\,,$$
$$\limsup_{k\to\infty} \left( \left| \alpha - \int_{\mathbb{R}^N} d\mu_k^1(x) \right| + \left| (1-\alpha) - \int_{\mathbb{R}^N} d\mu_k^2(x) \right| \right) \leqslant \epsilon.$$

**Theorem 2.12.** *There exists a minimum in $\mathcal{P}(\mathbb{R}^N)$ of the problem* (2.19) *when $q > p > 0$.*

*Proof.* For any $\mu \in \mathcal{P}(\mathbb{R}^N)$ we have that $\int_{\mathbb{R}^N} d\mu(x) = 1$. Also when $q > p > 0$ the interaction potential satisfies $K(|x|) \geqslant 1/q - 1/p$. Thus
$$\inf\{E(\mu) \colon \mu \in \mathcal{P}(\mathbb{R}^N)\} > -\infty.$$
Since $K(|x|) \leqslant 0$ when $0 \leqslant |x| \leqslant (q/p)^{1/(q-p)}$ we see that the above infimum is negative.

Now let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{R}^N)$ be a minimizing sequence of the problem (2.19). Then by the concentration compactness lemma for measures there is a subsequence of $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ which satisfies one of the three possibilities in Lemma 2.11.

Suppose "vanishing" occurs, i.e., for $0 < \epsilon < 1$ and $R > 0$ and for $k$ sufficiently large enough we have that
$$\int_{B(y,R)} d\mu_{n_k}(x) < \epsilon$$
for any $y \in \mathbb{R}^N$. This implies that
$$\int_{\mathbb{R}^N\setminus B(0,R)} d\mu_{n_k}(x) \geqslant 1-\epsilon > 0.$$

Note that since $K$ is a polynomial of $|x|$ there exists a constant $C_{p,q} > 0$ depending on $p$ and $q$ only such that
$$K(x) \geqslant |x|^{q-p} - C_{p,q} \tag{2.20}$$
with $q - p > 0$.

Now looking at the energy and using the indefinite growth of the interaction potential $K$ as $|x| \to \infty$ (see Figure 1(d)) we see that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\, d\mu_{n_k}(x) d\mu_{n_k}(y) &\geqslant \int_{\mathbb{R}^N}\int_{|x-y|>R} \left(|x-y|^{q-p} - C_{p,q}\right)\, d\mu_{n_k}(x) d\mu_{n_k}(y) \\ &\geqslant R^{q-p} \int_{\mathbb{R}^N} \left(\int_{|x-y|>R} d\mu_{n_k}(y)\right) d\mu_{n_k}(x) - C_{p,q} \\ &\geqslant R^{q-p}\,(1-\epsilon) - C_{p,q};\end{aligned}$$

hence, by letting $R \to \infty$, the energy $E[\mu_{n_k}] \to \infty$ as $k \to \infty$. This contradicts the fact that $\{\mu_{n_k}\}_{k\in\mathbb{N}}$ is a minimizing sequence.

Similarly, if we assume that "dichotomy" occurs, looking at the energy and using (2.20) we get that

$$\begin{aligned}\liminf_{k\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\, d\mu_{n_k}(x) d\mu_{n_k}(y) &\geqslant \liminf_{k\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\, d\mu_k^1(x) d\mu_k^2(y) \\ &\geqslant (R'-R)^{q-p}\, \alpha(1-\alpha) - C_{p,q}.\end{aligned}$$

Again, since $q > p > 0$ and $K(|x|) \nearrow \infty$ as $|x| \to \infty$, by letting $R' \to \infty$ we get that

$$\liminf_{k\to\infty} E[\mu_{n_k}] = \infty,$$

a contradiction.

Therefore "tightness up to translation" is the only possibility. As in the case of Theorem 2.5 for $q > 0$, the centers $y_k$ associated with the translation can be taken to be zero by the translation invariance of the energy. Hence we may assume the sequence of probability measures, $\{\mu_n\}_{n\in\mathbb{N}}$ is tight. Then, by the Prokhorov's theorem (cf. [15, Theorem 4.1]) there exists a further subsequence of $\{\mu_n\}_{n\in\mathbb{N}}$ which we still index by $n$, and a measure $\mu_0 \in \mathcal{P}(\mathbb{R}^N)$ such that

$$\mu_n \overset{\text{weak}-*}{\rightharpoonup} \mu_0$$

in $\mathcal{P}(\mathbb{R}^N)$ as $n \to \infty$.

To show weak lower semi-continuity of $E(\mu)$ we will proceed as in the proof of Theorem 2.5, paying attention to the fact that in this regime $K$ becomes negative.

Since the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is tight, for any given $\epsilon > 0$ there exists $r > 0$ such that

$$\int_{B(0,r)} d\mu_0(x) \geqslant 1 - \epsilon.$$

Choose $R := \max\{(q/p)^{1/(q-p)} + 1, r\}$, and define

$$\tilde{G}_n(x) := \int_{B(0,R)} K(x,y)\, d\mu_n(y) \quad \text{and} \quad \tilde{G}_0(x) := \int_{B(0,R)} K(x,y)\, d\mu_0(y).$$

As $K(x,y)$ is continuous in $x$ on $B(0,R)$, the sequence of functions $\tilde{G}_n$ converges uniformly to $\tilde{G}$ on $C(\overline{B(0,R)})$ by the Arzela–Ascoli theorem, using the compactness of the closed ball and the equicontinuity of $\tilde{G}_n$. Then, by the uniform convergence of $\tilde{G}_n$ and the weak-$*$ convergence of $\mu_n$ we get that

$$\liminf_{n\to\infty} \int_{B(0,R)}\int_{B(0,R)} K(x,y)\, d\mu_n(x) d\mu_n(y) = \int_{B(0,R)}\int_{B(0,R)} K(x,y)\, d\mu_0(x) d\mu_0(y).$$

Since $E(\mu_0) < \infty$, again, the energy on $\mathbb{R}^N \setminus B(0,R)$ is controlled and the above equality, as in (2.13), yields

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)\,d\mu_0(x)d\mu_0(y) &\leqslant \int_{B(0,R)}\int_{B(0,R)} K(x,y)\,d\mu_0(x)d\mu_0(y) + C\epsilon \\ &\leqslant \liminf_{n\to\infty} \int_{B(0,R)}\int_{B(0,R)} K(x,y)\,d\mu_n(x)d\mu_n(y) + C\epsilon \\ &\leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)\,d\mu_n(x)d\mu_n(y) + C\epsilon.\end{aligned}$$

Sending $\epsilon$ to 0 gives the weak lower semi-continuity of $E$; hence, $\mu_0 \in \mathcal{P}(\mathbb{R}^N)$ is a solution of the minimization problem (2.19). □ □

## 3. Characterization of Critical Points of $E$ for $p < 0$

We begin by noting that equipartition (up to constants) of energy is a necessary condition for criticality. Indeed, the next proposition can be viewed as a weak formulation of the Euler-Lagrange equation.

**Proposition 3.1** (Equipartition of Energy)**.** *If $\rho_0$ is a critical point of $E$, then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_0(x)\rho_0(y)\,dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho_0(x)\rho_0(y)\,dxdy \tag{3.1}$$

*Proof.* Let $\rho_0 \in \mathcal{A}$ be a critical point of (2.1) and consider the rescaled function $\rho_\lambda$ given by

$$\rho_\lambda(x) := \frac{1}{\lambda^N}\rho_0\left(\frac{x}{\lambda}\right)$$

as before. Then a simple change of variables reveals that

$$E(\rho_\lambda) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{\lambda^q}{q}|x-y|^q - \frac{\lambda^p}{p}|x-y|^p\right)\rho_0(x)\rho_0(y)\,dxdy.$$

Since $\rho_0$ is a critical point, we have $\frac{d}{d\lambda}\Big|_{\lambda=1} E(\rho_\lambda) = 0$ which in turn yields that the equation (3.1) needs to be satisfied by $\rho_0$ as a necessary condition of criticality. □ □

Next we derive the Euler–Lagrange equations for the energy $E$. The same equations were formally obtained in [10] and derived in the context of minimization with respect to the Wasserstein distance in [4]. Here we take a more direct and elementary approach.

**Proposition 3.2.** *Let $\rho_0 \in \mathcal{A}$ be a minimizer of the energy $E$. Then we have*

$$\begin{aligned} \Lambda(x) &\geqslant \eta \quad \text{a.e. on the set} \quad \{x : \rho_0(x) = 0\} \\ \Lambda(x) &= \eta \quad \text{a.e. on the set} \quad \{x : \rho_0(x) > 0\} \end{aligned} \tag{3.2}$$

*where*

$$\Lambda(x) := 2\int_{\mathbb{R}^N} \left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right)\rho_0(y)\,dy, \tag{3.3}$$

*and $\eta$ is a constant.*

*Proof.* Proceeding as in [45], let $\rho_0$ be a minimizer of $E$ and let $\zeta \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be an arbitrary function. For $0 \leqslant \epsilon \leqslant 1$, consider

$$\rho_\epsilon(x) := \rho_0(x) + \epsilon \left( \zeta(x) - \frac{\int_{\mathbb{R}^N} \zeta(x)\,dx}{m} \rho_0(x) \right). \tag{3.4}$$

Clearly $\int_{\mathbb{R}^N} \rho_\epsilon(x)\,dx = m$. Also, we have $\rho_\epsilon \geqslant 0$ provided

$$\zeta(x) \geqslant -\frac{\rho_0(x)}{2} \quad \text{and} \quad \int_{\mathbb{R}^N} \zeta(x)\,dx \leqslant \frac{m}{2}. \tag{3.5}$$

By minimality of $\rho_0$ we have

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0^+} E(\rho_\epsilon) \geqslant 0. \tag{3.6}$$

Plugging $\rho_\epsilon$ into $E$, taking the derivative with respect to $\epsilon$ and evaluating at $\epsilon = 0$ gives

$$\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0^+} E(\rho_\epsilon) &= 2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left( \frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p \right) \rho_0(y)\zeta(x)\,dxdy \\
&\quad - 2 \left( \frac{\int_{\mathbb{R}^N} \zeta(x)\,dx}{m} \right) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left( \frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p \right) \rho_0(x)\rho_0(y)\,dxdy \\
&= \int_{\mathbb{R}^N} \Lambda(x)\zeta(x)\,dx - \eta \int_{\mathbb{R}^N} \zeta(x)\,dx \\
&= \int_{\mathbb{R}^N} (\Lambda(x) - \eta)\zeta(x)\,dx,
\end{aligned}$$

where $\Lambda$ was defined in (3.3), and

$$\eta := \frac{\int_{\mathbb{R}^N} \Lambda(x)\rho_0(x)\,dx}{m}. \tag{3.7}$$

Hence, we get that

$$\int_{\mathbb{R}^N} (\Lambda(x) - \eta)\zeta(x)\,dx \geqslant 0. \tag{3.8}$$

The inequality (3.8) above holds, in particular, for all nonnegative functions $\zeta \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} \zeta(x)\,dx \leqslant \frac{m}{2}.$$

This, in turn, implies that

$$\Lambda(x) - \eta \geqslant 0 \quad \text{a.e.}$$

Moreover, note that $\eta$ is the average of $\Lambda$ with respect to the measure $\rho_0(x)dx$. Then, the condition $\Lambda(x) \geqslant \eta$ a.e. implies that $\Lambda(x) = \eta$ for a.e. $x$ where $\rho_0(x) > 0$. Thus, we establish (3.2). □ □

Note that in Prop. 3.2, we actually do not need $\rho_0$ to be a minimizer but simply a critical point (vanishing first variation) in the sense of (3.6) holding for all $\zeta$ satisfying (3.4) and (3.5).

Next we consider the special case $p = 2 - N$, for which the repulsion component of (2.1) reduces to the Coulomb energy (see Chapter 9 in [45]). In three dimensions, $p = -1$ corresponds to the well-known electrostatic potential energy or the Newton's gravitational potential. Case $p = 2 - N$ was investigated in the context of the evolution equation (1.3) in [30, 29]. There, the authors focused on the existence of symmetric, bounded and compactly-supported steady states and they showed that for any attraction component $q > 2 - N$, a unique such steady state exists. Moreover, numerical experiments suggest that these equilibrium solutions are global attractors for solutions of (1.3).

In particular, for $q = 2$, the steady state considered in [30] consists in a uniform density in a ball. It was shown in [13] that such uniform states (called patch solutions by the authors) are global attractors for the dynamics of (1.3). We study these steady states here from a variational point of view, and show that they are global minimizers of (2.1). Below, $\omega_N = \frac{\pi^{N/2}}{\Gamma(N/2+1)}$ denotes the volume of the unit ball in $\mathbb{R}^N$.

**Theorem 3.3.** *For any $m > 0$ and $M \geqslant \frac{m}{\omega_N}$, the function $\rho(x) = \dfrac{m}{\omega_N}\chi_{B(0,1)}(x)$ is the global minimizer of the problem* (2.1) *when $q = 2$, $p = 2 - N$.*

*Proof.* First, note that since $M \geqslant \frac{m}{\omega_N}$, the function $\rho$ is in the admissible class $\mathcal{A}$. Next, we check that $\rho$ is a critical point of the functional $E$, i.e., that $\rho$ satisfies (3.2). As noted in Remark 2.1, the attractive term of the energy simplifies when $q = 2$. On the other hand, when $p = 2-N$ the repulsive part is the Newtonian potential (i.e., $-\Delta_y(|x-y|^{2-N}) = N(N-2)\omega_N\delta_x$), and

$$\Phi(x) := \int_{B(0,1)} \frac{1}{N(N-2)\omega_N|x-y|^{N-2}}\,dy$$

solves the Poisson problem

$$-\Delta\Phi(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Since the right-hand side of the Poisson problem is radial, so is $\Phi(x)$. We use the expression of the Laplacian on $\mathbb{R}^N$ in hyper-spherical coordinates,

$$-\Delta\Phi(x) = -\frac{1}{r^{N-1}}\frac{d}{dr}\left(r^{N-1}\frac{d\Phi(r)}{dr}\right),$$

with $r = |x|$, and integrate once to get

$$\frac{d\Phi(r)}{dr} = \begin{cases} -\frac{r}{N} & \text{if } r \leqslant 1, \\ -\frac{1}{Nr^{N-1}} & \text{if } r > 1. \end{cases}$$

Integrating one more time and using the fact that $\Phi \in C^1$ by elliptic regularity, we get that

$$\Phi(r) = \begin{cases} -\frac{r^2}{2N} + \frac{1}{2(N-2)} & \text{if } r \leqslant 1, \\ \frac{1}{N(N-2)r^{N-2}} & \text{if } r > 1. \end{cases}$$

Then we calculate the function $\Lambda(x)$ given by (3.3) to find

$$\begin{aligned} \Lambda(x) &= 2\int_{\mathbb{R}^N}\left(\frac{1}{2}|x-y|^2 - \frac{1}{2-N}|x-y|^{2-N}\right)\frac{m}{\omega_N}\chi_{B(0,1)}(y)\,dy \\ &= \frac{m}{\omega_N}\left(\int_{B(0,1)} |x|^2 + |y|^2\,dy\right) + \frac{2m}{\omega_N(N-2)}\int_{B(0,1)}\frac{1}{|x-y|^{N-2}}\,dy \\ &= \begin{cases} \frac{2mN^2}{N^2-4} & \text{if } |x| \leqslant 1, \\ m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}} + \frac{mN}{N+2}, & \text{if } |x| > 1. \end{cases} \end{aligned}$$

Clearly, $\mathrm{supp}(\rho) = \{x \in \mathbb{R}^N : |x| \leqslant 1\}$, and when $|x| \leqslant 1$, we have $\Lambda \equiv \eta$ by (3.7). For $|x| > 1$, note that $\Lambda(x)$ is an increasing function of $|x|$ and equals $\eta$ when $|x| = 1$; hence, $\rho$ satisfies (3.2), and is a critical point of $E$.

Note that we can write the repulsive part in (2.3) using the $H^{-1}$-norm [3], and write the energy as

$$E(\rho) = m \int_{\mathbb{R}^N} |x|^2 \rho(x)\, dx + N(N-2)\omega_N \|\rho\|^2_{H^{-1}(\mathbb{R}^N)}.$$

Here, both terms in the energy are strictly convex [4]. Since the energy is strictly convex in every direction and $\rho(x) = \frac{m}{\omega_N}\chi_{B(0,1)}(x)$ is a critical point, it is the global minimizer of the problem (2.1). □ □

**Remark 3.4.** When $p = 2-N$ the repulsive term is always strictly convex as it can be written as the square of the $H^{-1}$-norm of $\rho$; however, for $q > 2-N$, $q \neq 2$, it is difficult to check the convexity of the attractive term due to cross-integral terms in the energy.

**Remark 3.5.** The scaling of the uniform distribution $(m/\omega_N)\chi_{B(0,1)}$ can be determined by looking at the weak criticality condition (3.1). Indeed, when $q = 2$ and $p = 2-N$, an explicit calculation shows that for any given $m > 0$ the function

$$\rho_R(x) := \frac{m}{\omega_N R^N} \chi_{B(0,R)}(x)$$

satisfies the weak condition (3.1) if and only if $R = 1$.

## 4. Binary Density Version

We consider the energy $E$ where the density function $\rho$ takes on only values $\{0, 1\}$. Then $\rho$ is the characteristic function $\chi_A$ of the set $A := \{x \in \mathbb{R}^N : \rho(x) = 1\}$ and the problem (2.1) can be re-written using a set functional notation as

$$\text{minimize} \quad \mathcal{E}(A) = \int_A \int_A K(x-y)\, dxdy \tag{4.1}$$

over radial sets $A$ of finite measure subject to the constraint

$$|A| = m.$$

Following the calculations in [27] we can find the first and second variations of $\mathcal{E}$. The idea there is to define an admissible perturbation of $A$, a family of sets $\{A_\epsilon\}_{\epsilon \in (-\tau,\tau)}$ for some $\tau > 0$, which satisfies the following three conditions: (i) $\chi_{A_\epsilon} \to \chi_A$ in $L^1(\mathbb{R}^N)$ as $\epsilon \to 0$, (ii) $\partial A_\epsilon$ is of class $C^2$, and (iii) $d/d\epsilon|_{\epsilon=0}|A_\epsilon| = 0$. For the calculation of the second variation the family needs to be modified to further satisfy the condition that it preserves mass up to second order, that is, the modified family $\{\tilde{A}_\epsilon\}$ needs to satisfy the extra condition: (iv) $d^2/d\epsilon^2|_{\epsilon=0}|\tilde{A}_\epsilon| = 0$.

[3]For a function $\rho \in L^2(\mathbb{R}^N)$ the $H^{-1}$-norm is defined as

$$\|\rho\|^2_{H^{-1}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\rho(x)\rho(y)}{N(N-2)\omega_N |x-y|^{N-2}}\, dxdy.$$

[4]A functional $E : \mathcal{A} \to \mathbb{R}$ is strictly convex if for all $f$ and $g$ in $\mathcal{A}$, $f \neq g$, and $t \in (0,1)$ we have $E[tf + (1-t)g] < tE[f] + (1-t)E[g]$.

For such an admissible family, $\chi_A$ is called a critical point of $\mathcal{E}$ if

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{E}(A_\epsilon) = 0,$$

and a critical point is stable if

$$\left.\frac{d^2}{d\epsilon^2}\right|_{\epsilon=0} \mathcal{E}(\tilde{A}_\epsilon) \geqslant 0.$$

Now we can state the result concerning the stable critical points of $\mathcal{E}(A)$.

**Proposition 4.1.** *Let $\rho$ be a critical point of $\mathcal{E}$ where $A := \{x \in \mathbb{R}^N : \rho(x) = 1\}$. Then*

$$\Lambda(x) = \lambda \quad \text{for all } x \in \partial A, \tag{4.2}$$

*where*

$$\Lambda(x) = \int_A K(x-y)\, dy$$

*and the Lagrange multiplier $\lambda$ is a constant.*

*Moreover, if $\rho$ is a stable critical point, then for any smooth function $\xi$ on $\partial A$ satisfying the condition*

$$\int_{\partial A} \xi(x)\, d\mathcal{H}^{N-1}_x = 0,$$

*we have that*

$$\int_{\partial A}\int_{\partial A} K(x-y)\xi(x)\xi(y)\, d\mathcal{H}^{N-1}_x d\mathcal{H}^{N-1}_y + \int_{\partial A} (\nabla\Lambda(x)\cdot\nu(x))\xi^2(x)\, d\mathcal{H}^{N-1}_x \geqslant 0, \tag{4.3}$$

*where $\mathcal{H}^{N-1}$ denotes the $N-1$-dimensional Hausdorff measure, and $\nu$ denotes the unit normal on $\partial A$ pointing out of $A$.*

The proof follows directly from the proof of Theorems 2.3 and 2.6 in [27]. In [27], the nonlocal isoperimetric functional (cf. Remark 1.2) consists of a perimeter term plus a repulsive interaction involving the Green's function for the Laplace operator on the flat $N$-torus. However, their calculation performed on each term separately applies to any radially symmetric kernel $K(|x|)$.

In Theorem 3.3 we showed that the global minimizer of the problem (2.1) for $q = 2$ and $p = 2 - N$ is a uniform density in a ball. It is clear that when we restrict the density functions to binary functions the result of Theorem 3.3 fails for any $m \neq \omega_N$. However, we can still prove explicitly for the binary density version that the ball is the global minimizer when $q = 2$ and $p = 2 - N$.

**Theorem 4.2.** *For any $m > 0$ let $R := \left(\frac{m}{\omega_N}\right)^{1/N}$. Then the ball $B = B(0,R) \subset \mathbb{R}^N$, or rather its characteristic function $\chi_B$, is the global minimizer of the problem (4.1) when $q = 2$ and $p = 2 - N$.*

*Proof.* Clearly $\chi_B$ satisfies the criticality condition (4.2). Now let $A \subset \mathbb{R}^N$ be any radially symmetric set of finite measure such that $|A| = m$. Using a similar idea to the one in [35], consider

$$
\begin{aligned}
\mathcal{E}(\chi_A) - \mathcal{E}(\chi_B) &= m \int_{\mathbb{R}^N} |x|^2 (\chi_A(x) - \chi_B(x))\, dx \\
&\quad + \frac{1}{N-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_A(x)\chi_A(y)}{|x-y|^{N-2}}\, dxdy - \frac{1}{N-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_B(x)\chi_B(y)}{|x-y|^{N-2}}\, dxdy \\
&= m \int_{\mathbb{R}^N} |x|^2 (\chi_A(x) - \chi_B(x))\, dx + \frac{2}{N-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_B(y)(\chi_A(x) - \chi_B(x))}{|x-y|^{N-2}}\, dxdy \\
&\quad + \frac{1}{N-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(\chi_A(x) - \chi_B(x))(\chi_A(y) - \chi_B(y))}{|x-y|^{N-2}}\, dxdy \\
&\geqslant m \int_{\mathbb{R}^N} |x|^2 (\chi_A(x) - \chi_B(x))\, dx + \frac{2}{N-2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_B(y)(\chi_A(x) - \chi_B(x))}{|x-y|^{N-2}}\, dxdy \\
&= m \int_{\mathbb{R}^N} |x|^2 (\chi_A(x) - \chi_B(x))\, dx + 2 \int_B \Theta(y)\, dy,
\end{aligned}
$$

where $\Theta$ is given by

$$
\Theta(y) = \frac{1}{N-2} \int_{\mathbb{R}^N} \frac{\chi_A(x) - \chi_B(x)}{|x-y|^{N-2}}\, dx.
$$

Note that for any point $x \in B = B(0,R)$ the function $\chi_A(x) - \chi_B(x)$ is either 0 or $-1$. Hence,

$$
-\Delta\Theta = N\omega_N(\chi_A(x) - \chi_B(x)) \leqslant 0 \text{ on } B,
$$

which means, $\Theta$ is subharmonic on $B$. Then, by the mean value property, we have that

$$
\int_B \Theta(y)\, dy \geqslant m\Theta(0) = \frac{m}{N-2} \int_{\mathbb{R}^N} \frac{\chi_A(x) - \chi_B(x)}{|x|^{N-2}}\, dx
$$

Thus, from the calculation above, we get that

$$
\begin{aligned}
\mathcal{E}(\chi_A) - \mathcal{E}(\chi_B) &\geqslant m \int_{\mathbb{R}^N} |x|^2 (\chi_A(x) - \chi_B(x))\, dx + \frac{2m}{N-2} \int_{\mathbb{R}^N} \frac{\chi_A(x) - \chi_B(x)}{|x|^{N-2}}\, dx \\
&= \int_{\mathbb{R}^N} \left( m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}} \right) (\chi_A(x) - \chi_B(x))\, dx \\
&= \int_A \left( m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}} \right) dx - \int_B \left( m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}} \right) dx, \\
&\geqslant 0
\end{aligned}
$$

where the last inequality follows from the fact that $A$ is a radially symmetric set. Indeed, writing the integral

$$
\int_A \left( m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}} \right) dx = \int_{\mathbb{R}^N} \left( m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}} \right) \chi_A(x)\, dx
$$

in spherical coordinates yields

$$
mN\omega_N \int_0^\infty \left( r^{N+1} + \frac{2}{N-2} r \right) \chi_A(r)\, dr.
$$

Clearly, $r^{N+1} + \frac{2}{N-2} r$ is an increasing function; hence,

$$
mN\omega_N \int_0^R \left( r^{N+1} + \frac{2}{N-2} r \right) dr \leqslant mN\omega_N \int_0^\infty \left( r^{N+1} + \frac{2}{N-2} r \right) \chi_A(r)\, dr
$$

proving that the ball of radius $R = \left(\frac{m}{\omega_N}\right)^{1/N}$ is the global minimizer of the binary density problem. □ □

According to the computations in [30], we see that when one considers the Newtonian potential as the repulsive interaction, that is when $p = 2 - N$, increasing the power of the attractive interaction, namely $q$, results in accumulation of the density $\rho$ around the boundary of the support of $\rho$. Moreover, the resulting density profiles indicate the possibility of annulus shaped attractors. However, we will show that when we consider the binary minimization problem (4.1) in $\mathbb{R}^3$, spherical annuli are not even critical points for $p = -1$ and $q > 2$. Our arguments here rely on explicit calculations; hence, we chose to work in 3 dimensions. We believe that the calculations can be extended to $N$ dimensions.

Without loss of generality take $m = 4\pi/3$, the volume of the unit ball in $\mathbb{R}^3$, and consider the spherical annulus

$$A := \{x \in \mathbb{R}^3 \colon R < |x| < (1+R^3)^{1/3}\}$$

for some $R > 0$. Recalling (4.2) let us define

$$\Lambda_{\text{attr}}(x) := \int_A \frac{|x-y|^q}{q}\,dy \qquad \text{and} \qquad \Lambda_{\text{rep}}(x) := \int_A \frac{1}{|x-y|^{N-2}}\,dy$$

for some $q > 2$. Let $x_1$ and $x_2 \in \partial A$ be two collinear points with the origin given in spherical coordinates by

$$x_1 = (R, 0, 0) \qquad \text{and} \qquad x_2 = ((1+R^3)^{1/3}, 0, 0),$$

that is, $x_1$ and $x_2$ are two points on the same ray from the origin such that $x_1$ lies on the inner boundary component of $A$ whereas $x_2$ lies on the outer boundary component.

Assume, for a contradiction, that there exists a number $R > 0$, depending on $q$, such that $A$ is a critical point of the minimization problem (4.1). Then the Euler–Lagrange equation (4.2) is satisfied at $x_1$ and $x_2$ simultaneously, and this implies that

$$(\Lambda_{\text{attr}}(x_2) - \Lambda_{\text{attr}}(x_1)) + (\Lambda_{\text{rep}}(x_2) - \Lambda_{\text{rep}}(x_1)) = 0. \tag{4.4}$$

Since $\Lambda_{\text{rep}}$ solves the Poisson equation

$$-\Delta\Lambda_{\text{rep}}(x) = 4\pi\chi_A(x) \quad \text{in} \quad \mathbb{R}^3$$

and is a radial function we can find the solution explicitly as

$$\frac{1}{4\pi}\Lambda_{\text{rep}}(|x|) = \begin{cases} \frac{(1+R^3)^{2/3}-R^2}{2} & \text{if} \quad 0 \leqslant |x| < R, \\ -\frac{|x|^2}{6} - \frac{R^3}{3|x|} + \frac{(1+R^3)^{2/3}}{2} & \text{if} \quad R \leqslant |x| < (1+R^3)^{1/3}, \\ \frac{1}{3|x|} & \text{if} \quad |x| \geqslant (1+R^3)^{1/3}. \end{cases}$$

Thus the difference of the repulsive terms on two boundary components is given by

$$\frac{1}{4\pi}(\Lambda_{\text{rep}}(x_2) - \Lambda_{\text{rep}}(x_1)) = \frac{1}{3(1+R^3)^{1/3}} - \frac{(1+R^3)^{2/3}-R^2}{2}; \tag{4.5}$$

hence,

$$\lim_{R\to 0} \Lambda_{\text{rep}}(x_2) - \Lambda_{\text{rep}}(x_1) = -\frac{4\pi}{6} < 0.$$

Now referring back to (4.5) and taking the derivative of the right-hand side with respect to $R$ we get that

$$\Lambda_{\text{rep}}(x_2) - \Lambda_{\text{rep}}(x_1)$$

is an increasing function of $R$ for all $R > 0$. Moreover, $\Lambda_{\mathrm{rep}}(x_2) - \Lambda_{\mathrm{rep}}(x_1) < 0$ for all $R$ and approaches zero as $R$ increases.

Now, returning to the difference of attractive terms in (4.4) we see that by the choice of $x_1$ and $x_2$, using spherical coordinates, it can be expressed as

$$\Lambda_{\mathrm{attr}}(x_2) - \Lambda_{\mathrm{attr}}(x_1) = 2\pi \int_0^{\pi} \int_R^{(1+R^3)^{1/3}} \left( \frac{((1+R^3)^{2/3} + r^2 - 2r(1+R^3)^{1/3}\cos\phi)^{q/2}}{q} - \frac{(R^2 + r^2 - 2rR\cos\phi)^{q/2}}{q} \right) r^2 \sin\phi \, d\phi dr,$$

which is a function of the single variable $R$ any fixed $q$.

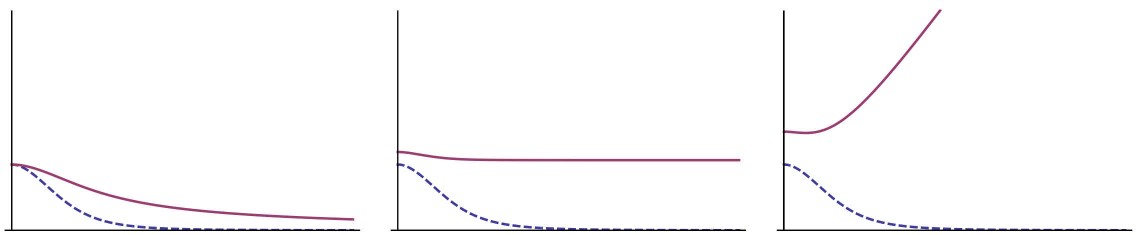


Figure 4. The graphs of $\Lambda_{\mathrm{attr}}(x_2) - \Lambda_{\mathrm{attr}}(x_1)$ (solid) and $-[\Lambda_{\mathrm{rep}}(x_2) - \Lambda_{\mathrm{rep}}(x_1)]$ (dashed) as a function of $R$ with $q = 2, 3, 4$, respectively.

First, note that, the above difference is an increasing function of $q$ for $q > 2$. Furthermore, calculating it numerically for increasing values of $q > 2$ we see that $\Lambda_{\mathrm{attr}}(x_2) - \Lambda_{\mathrm{attr}}(x_1)$ does not intersect with $-[\Lambda_{\mathrm{rep}}(x_2) - \Lambda_{\mathrm{rep}}(x_1)]$ for any value of $R > 0$. However, this contradicts the equation (4.4); hence, there does not exist a positive number $R > 0$ such that the spherical annulus

$$A := \{x \in \mathbb{R}^3 \colon R < |x| < (1+R^3)^{1/3}\}$$

is a critical point of the problem (4.1) in $\mathbb{R}^3$.

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## References


[1] Ambrosio, L., Gigli, N., Savaré, G.: Gradient flows in metric spaces and in the space of probability measures, second edn. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel (2008)

[2] Auchmuty, J.: Existence of axisymmetric equilibrium figures. Arch. Rational Mech. Anal. **65**(3), 249–261 (1977)

[3] Auchmuty, J., Beals, R.: Variational solutions of some nonlinear free boundary problems. Arch. Rational Mech. Anal. **43**, 255–271 (1971)

[4] Balagué, D., Carrillo, J., Laurent, T., Raoul, G.: Dimensionality of local minimizers of the interaction energy. Arch. Rat. Mech. Anal. **209**, 1055–1088 (2013)

[5] Balagué, D., Carrillo, J., Laurent, T., Raoul, G.: Nonlocal interactions by repulsive-attractive potentials: radial ins/stability. Physica D **260**, 5–25 (2013)

[6] Bedrossian, J.: Global minimizers for free energies of subcritical aggregation equations with degenerate diffusion. Appl. Math. Lett. **24**(11), 1927–1932 (2011)

[7] Bedrossian, J.: Intermediate asymptotics for critical and supercritical aggregation equations and Patlak-Keller-Segel models. Commun. Math. Sci. **9**(4), 1143–1161 (2011)
[8] Bedrossian, J., Rodríguez, N., Bertozzi, A.: Local and global well-posedness for aggregation equations and Patlak-Keller-Segel models with degenerate diffusion. Nonlinearity **24**, 1683–1714 (2011)
[9] Benedetto, D., Caglioti, E., Pulvirenti, M.: A kinetic equation for granular media. RAIRO Modél. Math. Anal. Numér. **31**(5), 615–641 (1997)
[10] Bernoff, A., Topaz, C.: A primer of swarm equilibria. SIAM J. Appl. Dyn. Syst. **10**(1), 212–250 (2011)
[11] Bertozzi, A., Carrillo, J., Laurent, T.: Blow-up in multidimensional aggregation equations with mildly singular interaction kernels. Nonlinearity **22**(3), 683–710 (2009)
[12] Bertozzi, A., Laurent, T.: Finite-time blow-up of solutions of an aggregation equation in $\vec{R}^n$. Comm. Math. Phys. **274**(3), 717–735 (2007)
[13] Bertozzi, A., Laurent, T., Flavien, L.: Aggregation and Spreading via the Newtonian Potential: The Dynamics of Patch Solutions. Math. Models Methods Appl. Sci. **22**(Supp. 1), 1140,005 (2012)
[14] Bertozzi, A., Slepčev, D.: Existence and uniqueness of solutions to an aggregation equation with degenerate diffusion. Commun. Pure Appl. Anal. **9**(6), 1617–1637 (2010)
[15] Billingsley, P.: Weak convergence of measures: Applications in probability. Society for Industrial and Applied Mathematics, Philadelphia, Pa. (1971). Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 5
[16] Blanchet, A., Carrillo, J., Laurençot, P.: Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions. Calc. Var. Partial Differential Equations **35**(2), 133–168 (2009)
[17] Bodnar, M., Velasquez, J.: Derivation of macroscopic equations for individual cell-based models: a formal approach. Math. Meth. Appl. Sci. (M3AS) **28**, 1757–1779 (2005)
[18] Bodnar, M., Velazquez, J.: An integro-differential equation arising as a limit of individual cell-based models. J. Differential Equations **222**(2), 341–380 (2006)
[19] von Brecht, J., Uminsky, D., Kolokolnikov, T., Bertozzi, A.: Predicting pattern formation in particle interactions. Math. Models Methods Appl. Sci. **22**(Supp. 1), 1140,002 (2012)
[20] Burger, M., Capasso, V., Morale, D.: On an aggregation model with long and short range interactions. Nonlinear Anal. Real World Appl. **8-3**, 939–958 (2007)
[21] Burger, M., DiFrancesco, M.: Large time behavior of nonlocal aggregation models with nonlinear diffusion. Netw. Heterog. Media **3-4**, 749–785 (2008)
[22] Burger, M., DiFrancesco, M., Franek, M.: Stationary states of quadratic diffusion equations with long-range attraction. Comm. Math. Sci. **3**, 709–738 (2013)
[23] Carrillo, J.A., DiFrancesco, M., Figalli, A., Laurent, T., Slepčev, D.: Global-in-time weak measure solutions and finite-time aggregation for nonlocal interaction equations. Duke Math. J. **156**(2), 229–271 (2011)
[24] Carrillo, J.A., DiFrancesco, M., Figalli, A., Laurent, T., Slepčev, D.: Confinement in nonlocal interaction equations. Nonlinear Anal. **75**(2), 550–558 (2012)
[25] Choksi, R., Peletier, M.: Small volume fraction limit of the diblock copolymer problem: I. sharp interface functional. SIAM J. Math. Anal. **42-3**, 1334–1370 (2010)
[26] Choksi, R., Peletier, M.: Small volume fraction limit of the diblock copolymer problem: Ii. diffuse interface functional. SIAM J. Math. Anal. **43-2**, 739–763 (2011)
[27] Choksi, R., Sternberg, P.: On the first and second variations of a nonlocal isoperimetric problem. J. Reine Angew. Math. **611**, 75–108 (2007)
[28] Fellner, K., Raoul, G.: Stable stationary states of non-local interaction equations. Math. Models Methods Appl. Sci. **20**(12), 2267–2291 (2010)
[29] Fetecau, R.C., Huang, Y.: Equilibria of biological aggregations with nonlocal repulsiveattractive interactions. Physica D **260**, 49–64 (2013)
[30] Fetecau, R.C., Huang, Y., Kolokolnikov, T.: Swarm dynamics and equilibria for a nonlocal aggregation model. Nonlinearity **24**(10), 2681–2716 (2011)
[31] Haile, J.: Molecular Dynamics Simulation: Elementary Methods. John Wiley and Sons, Inc., New York (1992)
[32] Holm, D., Putkaradze, V.: Aggregation of finite-size particles with variable mobility. Phys Rev Lett. **95:226106** (2005)
[33] Holm, D., Putkaradze, V.: Formation of clumps and patches in selfaggregation of finite-size particles. Physica D. **220**(2), 183–196 (2006)

[34] Huang, Y., Bertozzi, A.: Self-similar blowup solutions to an aggregation equation in $\mathbb{R}^n$. SIAM J. Appl. Math. **70**(7), 2582–2603 (2010)
[35] Julin, V.: Isoperimetric problem with a coulombic repulsive term. Preprint (2012)
[36] Knuepfer, H., Muratov, C.: On an isoperimetric problem with a competing non-local term. i. the planar case. Comm. Pure Appl. Math. **66-7**, 1129–1162 (2013)
[37] Knuepfer, H., Muratov, C.: On an isoperimetric problem with a competing non-local term. ii. the general case. Comm. Pure Appl. Math. **in press**
[38] Kolokolnikov, T., Sun, H., Uminsky, D., Bertozzi, A.: A theory of complex patterns arising from 2D particle interactions. Phys. Rev. E, Rapid Communications **84**, 015,203(R) (2011)
[39] Le Bris, C.: Some results on the Thomas-Fermi-Dirac-von Weizsäcker model. Diff. Int. Equ. **6**, 337–353 (1993)
[40] Le Bris, C., Lions, P.L.: From atoms to crystals: a mathematical journey. Bull. Amer. Math. Soc. **42** (2005)
[41] Leverentz, A., Topaz, C., Bernoff, A.: Asymptotic dynamics of attractive-repulsive swarms. SIAM J. Appl. Dyn. Syst. **8**(3), 880–908 (2009)
[42] Lieb, E.: Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. Studies in Appl. Math. **57**, 93–105 (1977)
[43] Lieb, E.: Thomas-Fermi and related theories of atoms and molecules. Rev. Mod. Phys. **52**, 603641 (1981)
[44] Lieb, E.: Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann. of Math. (2) **118**(2), 349–374 (1983)
[45] Lieb, E., Loss, M.: Analysis, *Graduate Studies in Mathematics*, vol. 14, second edn. American Mathematical Society, Providence, RI (2001)
[46] Lions, P.L.: The concentration-compactness principle in the calculus of variations. The locally compact case. I. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**(2), 109–145 (1984)
[47] Lu, J., Otto, F.: Nonexistence of a minimizer for the Thomas-Fermi-Dirac-von Weizsäcker model. Comm. Pure Appl. Math. **in press**
[48] Mogilner, A., Edelstein-Keshet, L.: A non-local model for a swarm. J. Math. Biol. **38**, 534–570 (1999)
[49] Poincaré, H.: Sur une théorème de M. Liapunoff rélatif a l'équilibre d'une masse fluide. Comptes Rendus de L'Academie des Sciences **104**, 622–625 (1887)
[50] Struwe, M.: Variational methods, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, vol. 34, third edn. Springer-Verlag, Berlin (2000). Applications to nonlinear partial differential equations and Hamiltonian systems
[51] Topaz, C., Bertozzi, A., Lewis, M.: A nonlocal continuum model for biological aggregation. Bull. Math. Bio. **68**, 1601–1623 (2006)
[52] Toscani, G.: One-dimensional kinetic models of granular flows. M2AN Math. Model. Numer. Anal. **34**(6), 1277–1291 (2000)

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