ON MINIMIZERS OF INTERACTION FUNCTIONALS WITH COMPETING ATTRACTIVE AND REPULSIVE POTENTIALS*

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Abstract. We consider a class of interaction functionals consisting of power-law potentials of attractive and repulsive parts. Using the concentration compactness principle, we establish existence of minimizers in appropriate classes. In certain cases, we characterize the ground state. We also address local minimizers based upon first and second variation arguments. Finally, we address these functionals minimized over binary densities.

Key words. interaction of attractive and repulsive potentials, aggregation, local and global minimizers.

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1. Introduction.

We consider the minimization of energies of the form

$$E[\rho] := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho(x)\rho(y)\,dxdy, \tag{1.1}$$

where

$$K(x) := \frac{1}{q}|x|^q - \frac{1}{p}|x|^p, \qquad \text{for } -N < p < q. \tag{1.2}$$

These functionals are directly connected to a class of aggregation models which recently have received much attention (see for example, [47, 50, 18, 12, 11, 27, 40, 29, 40, 11, 33, 9, 51, 31, 32]). The aggregation models consist of the following active transport equation in $\mathbb{R}^N$ for the population density ρ:

$$\rho_t + \nabla\cdot(\rho\mathbf{v}) = 0, \quad \mathbf{v} = -\nabla K * \rho, \tag{1.3}$$

where K represents the interaction potential and $*$ denotes spatial convolution. This partial differential equation is the gradient flow of the energy (1.1) with respect to the Wasserstein metric [1]. Indeed, the evolution equation (1.3) can be written in the form

$$\partial_t\rho = \nabla\cdot\Big(\rho\nabla\frac{\delta E[\rho]}{\delta\rho}\Big),$$

which is the standard form for the Wasserstein gradient flow [1] of the energy (1.1).

Model (1.3) has a wide range of applications, including biological swarms [47, 50], granular media [9, 51], self-assembly of nanoparticles [31, 32] and molecular dynamics simulations of matter [30]. The study of solutions to (1.3) (well-posedness, finite or infinite time blow-up, long-time behavior) has been a very active area of research during the past decade [18, 12, 11, 27, 40]. It is important to note that the analysis

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and behavior of solutions to (1.3) depend essentially on the properties of the potential K. In the context of biological swarms, K incorporates social interactions (attraction and repulsion) between group individuals. Potentials which are attractive in nature typically lead to blow-up [11, 33], while attractive-repulsive potentials may generate finite-size, confined aggregations [29, 40].

A significant number of recent works exploited the gradient flow structure in the particle (individual-based) model associated to (1.3). Specifically, the PDE model (1.3) can be regarded as the continuum limit of the following particle model describing the pairwise interaction of M particles in $\mathbb{R}^N$ [17]:

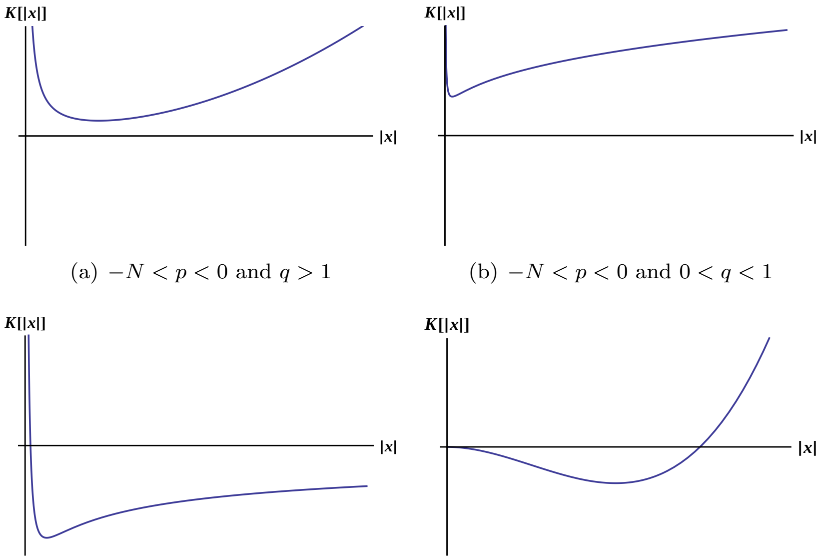
$$\frac{dX_i}{dt} = -\frac{1}{M} \sum_{\substack{i,j=1 \\ j \neq i}}^{M} \nabla_i K(X_i - X_j), \qquad i = 1 \dots M, \tag{1.4}$$

where $X_i(t)$ represents the spatial location of the i-th individual at time t. The particle model (1.4) is the gradient flow of the interaction energy which is the discrete version of (1.1) [37]. It has been shown that simple choices of interaction potentials in (1.4) can lead to very diverse and complex equilibrium solutions (e.g., disks, rings and annular regions in 2D, balls, spheres and soccer balls in 3D) [37, 52, 29].

By staying entirely at the continuum level (that is, working with (1.3) without resorting to the particle system (1.4)) it is more difficult to identify equilibria. There are only a few works in this direction. In [29, 28] the authors study equilibrium solutions to (1.3) which are supported in a ball, while in [5] the focus is equilibria that are uniformly distributed on spherical shells. Such equilibria, along with those revealed by simulations of the discrete model, constitute the main motivation for this work. In this article, we directly study the problem from the variational point of view, i.e. minimizers of (1.1). In particular, we show the existence of a global minimizer of (1.1) by using the direct method in the calculus of variations.

By inspecting the equation for $\mathbf{v}$ in (1.3) one notes that the nature of a symmetric potential $K(x) = K(|x|)$ is dictated by the sign of its derivative ($K' > 0$ corresponds to attraction and $K' < 0$ to repulsion). Hence, for K given by (1.2), the exponent q refers to attraction and p to repulsion (p and q can be of any sign). Condition $q > p$ is needed to ensure that the potential is repulsive at short ranges and attractive in the far field — see Figure 1.1 for a generic illustration of K with $p = -1$ (Newtonian repulsion in three dimensions). Note that when $q > 1$, the potential K is positive, strictly convex and $K \to \infty$ as $|x| \to \infty$ whereas when $q = 1$ it is merely convex. When $0 < q < 1$, K is still positive and grows indefinitely with $|x|$; however, it is not convex. Finally for $-N < q < 0$, K becomes negative, approaches 0 as $|x| \to \infty$ and is not convex.

Potentials in power-law form have been frequently considered in the recent literature on the aggregation model (1.3) [37, 52, 29, 28, 5]. As shown in these works, the delicate balance between attraction and repulsion often leads to complex equilibrium configurations, supported on sets of various dimensions. The dimensionality of local minimizers of (1.1) with K given by (1.2) was recently investigated in [4]. The repulsion exponent p in [4] is restricted to be above the Newtonian singularity, i.e., $p > 2 - N$. In our results we only need integrability at the origin, so p can take values in the larger range $p > -N$. Indeed, a simple particle model simulation in two dimensions shows accumulation of the density in different states depending on the powers of the interaction potential K (see Figure 1.2).



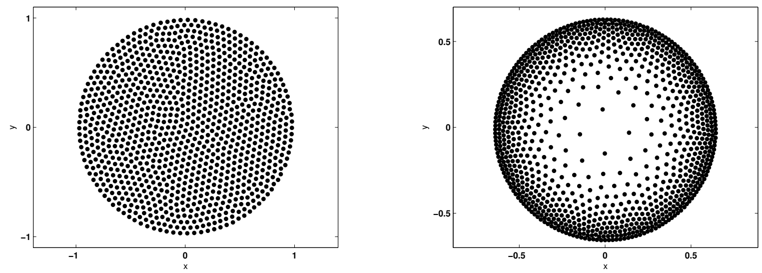
(a) $-N < p < 0$ and $q > 1$ (b) $-N < p < 0$ and $0 < q < 1$ (c) $-N < p < q < 0$ (d) $q > p > 0$

FIG. 1.1. *Examples of K with varying p and q.*

We summarize our results, placing them in the context of other recent work. To this end, we consider two separate cases: $p < 0$ and $p > 0$.

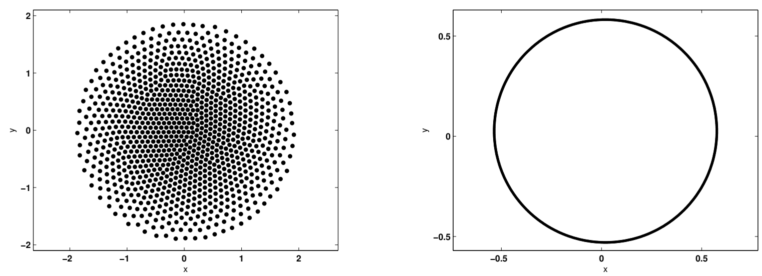
Negative power repulsion $p < 0$. Figure 1.2 (a) - (c) show examples of particle simulations for $p = -1$ and various values of q. In the case of negative p, one expects densities to not accumulate on lower-dimensional sets. For example, in [29] the time-dependent density $\rho(\cdot, t)$ is shown to be uniformly bounded in L^∞ for all $t > 0$ provided the initial condition ρ_0 is in L^∞. Also in [4], the authors prove that for $N = 3$ and $p < 0$, minimizers do not accumulate on a set of dimension less than 3. Thus for $p < 0$, we expect that minimizers exist in the space of density *functions* (and not measures). As a matter of fact, we immediately see that a Dirac delta integrated against an interaction potential $1/|x|^a$ with $0 < a < N$ cannot be a minimizer of the energy (2.1). However, with only an L^1-bound on the density, accumulation along a set of Hausdorff dimension less than N is a possibility – in fact as we discuss below, such possibilities are indeed generic when $p > 0$. To this end, for $p < 0$ our admissible class of densities ρ is the space $\mathcal{A}$ of non-negative, uniformly bounded L^1 functions with fixed mass m (see equation (2.2) for the precise definition of $\mathcal{A}$). Note that the uniform boundedness condition is necessary to prevent concentrations as the energy does not bound any L^s norm for $s > 1$. We repeat that while the uniform boundedness condition on the density function ρ is a strong assumption, it is supported by results in [28, 29] with $p = 2 - N$, as well as other works that consider power kernels with negative repulsion exponent [52, 5, 4]. The uniform boundedness of the density when minimizing similar energies is also assumed, for example, in [2].

In Section 2, we use the direct method in the calculus of variations to show global existence of a minimizer of (1.1) within the class $\mathcal{A}$ (cf. Theorem 2.5). Here the key tool is Lions' concentration compactness lemma (Lemma 2.2) to extract from a minimizing sequence a subsequence which is tight in the sense of measures. We then use the uniform boundedness to infer weak convergence in L^s for any $1 < s <$



(a) Newtonian repulsion, quadratic attraction

(b) Newtonian repulsion, large positive attraction



(c) Below Newtonian repulsion, small negative attraction

(d) Positive repulsion, positive attraction

FIG. 1.2. *2D Particle model simulations with varying p and q. In 2D, the Newtonian repulsion is given by $-\log |x|$ and below Newtonian repulsion corresponds to $-2 < p < 0$ (see also Remark 2.8).*

∞. Weak convergence is sufficient for lower semicontinuity of the energy functionals (Lemma 2.4).

In Section 3, we give necessary conditions for critical points of the energy. Using these conditions, we address the case $p = 2 - N$ (Newtonian potential for $N > 2$) and prove, for the special case $q = 2$, that the rescaled characteristic function of a ball is the global minimizer. Given that we are working with uniformly bounded functions, we also consider the problem of minimizing E over binary densities $\rho \in \{0, 1\}$. In particular, we derive expressions for the vanishing first and positive second variation conditions. In Theorem 4.2, we prove that, for the special case $p = 2 - N, q = 2$, the characteristic function of a ball is again the unique global minimizer.

For Newtonian repulsion with stronger attraction $q > 2$, simulations suggest that the density concentrates on an annular shell (cf. [29, 28], see also Figure 1.2 (b)). This might indicate that annular spherical shells are global minimizers of E over $\rho \in \mathcal{A}$. However, we show that, for the binary variational problem, there do not exist such annular critical points. This suggests that, while particle simulations do indeed show a density accumulation to an annulus, the minimizing density is still supported in the entire ball.

Positive power repulsion $p > 0$. In Section 5, we address the case of positive p and q. The character of the interaction potential K is very different when $p > 0$. In this regime K does not have a singularity at zero and it allows concentration of densities on sets of dimension less than N. Note that Figure 1.2 (d) shows an associated particle simulation for positive p and q with accumulation along a circle. Moreover, the results of [4] support this observation via rigorous bounds on the Hausdorff dimension of the support of minimizers. Therefore we will define the energy E over probability measures (cf. [5, 4]). The minimization problem that we consider in the regime $q > p > 0$ is given by:

$$\text{minimize} \quad E(\mu) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right) d\mu(x)d\mu(y)$$

over radially symmetric probability measures $\mu \in \mathcal{P}^r(\mathbb{R}^N)$ (see Section 5). Following the same steps as in Theorem 2.5, we prove the existence of a global minimizer.

Related to our positive power repulsion case, in [22] the authors consider a class of aggregation equations with interaction potentials which satisfy certain growth and convexity conditions. Using an approach based on the theory of gradient flows they establish existence and uniqueness of global-in-time weak measure solutions. Later in [23], again working with measure solutions, they find sufficient conditions on the interaction potential which guarantee the confinement of localized solutions for all times. In both works their use of a measure theoretic setting as the class of admissible functions enables them a unified analysis of both particle and continuum models.

Remark 1.1 (Radial Symmetry). In both cases, $p < 0$ and $p > 0$, to prove the existence of minimizers of the energy (1.1) we assume that the members of the admissible class are radially symmetric. We believe that due to the radially symmetric structure of the interaction potential this assumption is natural. Indeed, even though there might be some minor symmetry defects depending on the choice of number of particles, the simulations with random initial data suggest that the steady states are radially symmetric. Also, considering the isotropic nature of the interaction potentials it seems reasonable to conjecture that the minimizers of the energy (1.1) are radially symmetric. It would be interesting to prove this claim, at least in certain p and q regimes, by constraining the first moment, i.e., assuming the conservation of the center of mass.

We conclude the introduction with a two remarks concerning related problems.

Remark 1.2 (Repulsion via nonlinear diffusion). A related, well-studied model (cf. [21, 19, 20, 7, 8, 14, 16]) is to consider a kernel which is purely attractive and bring in the repulsive component by adding nonlinear diffusion to the dynamic PDE, i.e.

$$\rho_t + \nabla\cdot(\rho\mathbf{v}) = \Delta\rho^m, \qquad \mathbf{v} = -\nabla K * \rho,$$

where K is purely *attractive*. Here the associated energy functional is given by

$$E_{\mathrm{nld}}[\rho] := \frac{1}{2}\int\int K(x-y)\rho(x)\rho(y)dxdy + \frac{1}{m-1}\int \rho^m.$$

Using Lions' concentration compactness lemma, a proof of global existence for this functional was given in [6]. A detailed study of steady state solutions is given in [21].

Similar energies also appear in Astrophysics and Quantum Mechanics and have extensively studied [2, 3, 45]. In fact, in the seminal paper of Lions [45] wherein he introduces the concentration compactness lemma, a direct application is the existence of minimizers to a class of these L^1 minimization problems. Many of the arguments in our present article follow his application.

Remark 1.3 (Thomas-Fermi-Dirac-von Weizsäcker Functionals and the Nonlocal Isoperimetric Problem). Many other functions with interacting attractive and repulsive components have been studied, for example the Thomas-Fermi-Dirac-von Weizsäcker functional in mathematical physics [42, 38, 39, 46]. Recently, a binary *nonlocal isoperimetric functional* appeared in connection with the modeling of self-assembly of diblock copolymers [26, 24, 25]. Existence and non-existence results have been presented in [36, 46, 35, 34]. In dimension $N = 3$, the functional has a Newtonian repulsive component as in 1.1 with $p = -1$. However, the attractive component does not come from an interaction term but rather by adding a higher-order regularization. Precisely, for $m > 0$, the nonlocal isoperimetric problem is to minimize

$$E_{\mathrm{nlip}}[\rho] := \int_{\mathbb{R}^3} |\nabla \rho| \; + \; \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{4\pi|x-y|}\, dx\, dy$$

over

$$\rho \in BV(\mathbb{R}^3, \{0,1\}) \;\text{ with }\; \int_{\mathbb{R}^3} \rho \,=\, m.$$

Since admissible densities ρ are characteristic functions, the first term in the energy is simply the perimeter of the support. Not only is there a competition between the two terms in E_{nlip} but they are in *direct competition* in the following sense: balls are *best* (least energy) for the first (attractive) term and *worst* (greatest energy) for the second (repulsive) term. The latter point has an interesting history. Poincaré [48] considered the problem of determining the equilibrium shapes of a fluid body of mass m. In simplified form, this amounts to minimizing the total potential energy of the region of fluid $E \subset \mathbb{R}^3$

$$\int_E \int_E -\frac{1}{C\,|x-y|}\, dx\, dy,$$

where $-(C|x-y|)^{-1}$ $(C > 0)$ is the potential resulting from the gravitational attraction between two points x and y in the fluid. Poincaré showed that, under some smoothness assumptions, a body has the lowest energy if and only if it is a ball. It was not until almost a century later that the essential details were sorted via the rearrangement ideas of Steiner for the isoperimetric inequality. These ideas are captured in the *Riesz Rearrangement Inequality* and its development (cf. [44, 41]).

In [24, 25], it was conjectured that there exists a critical mass m_0 below which, a unique minimizer of E_{nlip} exists and is the characteristic function of a ball, and above which, the minimizer fails to exist because of “mass” escaping to infinity. Note this is in stark contrast to minimizers of (1.1). The non-existence for sufficiently large m has recently been proved in [46]. Existence of a radially symmetric minimizer (i.e. a ball) for m sufficiently small has recently been proved in [35, 34]. Whether or not balls are the *only* minimizers remains open.

2. Existence of Minimizers for Negative-Power Repulsion. For $p < 0$, $q > p$, $m > 0$ and $M > 0$, we consider the following variational problem:

$$\text{minimize} \quad E(\rho) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right) \rho(x)\rho(y)\, dxdy \tag{2.1}$$

over

$$\mathcal{A} := \big\{\rho \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \colon \\ \rho = \rho(|x|) \geqslant 0,\ \|\rho\|_{L^\infty} \leqslant M,\ \text{and} \int_{\mathbb{R}^N} \rho(x)\,dx = m\Big\}. \tag{2.2}$$

REMARK 2.1. We note that the case $q = 2$ is rather special. Indeed, by the radial symmetry assumption the center of mass is conserved, that is,

$$\int_{\mathbb{R}^N} x\rho(x)\,dx = 0.$$

A simple calculation leads to

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^2\rho(x)\rho(y)\,dxdy &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(|x|^2 - 2x\cdot y + |y|^2\right)\rho(x)\rho(y)\,dxdy \\ &= m\int_{\mathbb{R}^N} |x|^2\rho(x)\,dx.\end{aligned}$$

With the attractive term being simplified, the energy (2.1) can be written as

$$E(\rho) = m\int_{\mathbb{R}^N} |x|^2\rho(x)\,dx - \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p\rho(x)\rho(y)\,dxdy. \tag{2.3}$$

We now prove the existence of a minimizer for (2.1) using the direct method of the calculus of variations. The key tool we use in establishing the existence of minimizers is the concentration-compactness lemma by Lions [45, Lemma I.1] which we state below.

LEMMA 2.2 (Concentration-compactness lemma [45]). *Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$\rho_n \geqslant 0 \ \text{in } \mathbb{R}^N, \quad \int_{\mathbb{R}^N} \rho_n(x)\,dx = m,$$

for some fixed $m > 0$. Then there exists a subsequence $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ satisfying one of the three following possibilities:

(i) (tightness up to translation) there exists $y_k \in \mathbb{R}^N$ such that $\rho_{n_k}(\cdot + y_k)$ is tight, that is, for all $\epsilon > 0$ there exists $R > 0$ such that

$$\int_{B(y_k,R)} \rho_{n_k}(x)\,dx \geqslant m - \epsilon;$$

(ii) (vanishing) $\displaystyle\lim_{k\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,R)} \rho_{n_k}(x)\,dx = 0$, for all $R > 0$;

(iii) (dichotomy) there exists $\alpha \in (0, m)$ such that for all $\epsilon > 0$, there exist $k_0 \geqslant 1$ and $\rho_{1,k},\ \rho_{2,k} \in L^1_+(\mathbb{R}^N)$ satisfying for $k \geqslant k_0$

$$\begin{gathered}\|\rho_{n_k} - (\rho_{1,k} + \rho_{2,k})\|_{L^1(\mathbb{R}^N)} \leqslant \epsilon, \\ \big|\|\rho_{1,k}\|_{L^1(\mathbb{R}^N)} - \alpha\big| \leqslant \epsilon, \quad \big|\|\rho_{2,k}\|_{L^1(\mathbb{R}^N)} - (m-\alpha)\big| \leqslant \epsilon,\end{gathered}$$

and

$$\operatorname{dist}(\operatorname{supp}(\rho_{1,k}), \operatorname{supp}(\rho_{2,k})) \to \infty \ \text{as } k \to \infty.$$

In certain cases, we will use the following special form of the Hardy–Littlewood–Sobolev inequality to bound the energy from below.

PROPOSITION 2.3 (cf. Theorem 3.1 in [43]). *For any* $-N < p < 0$ *and* $f \in L^{6/(6+p)}(\mathbb{R}^N)$ *we have*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p f(x)f(y)\,dxdy \leqslant C(p)\|f\|^2_{L^{6/(6+p)}(\mathbb{R}^N)}$$

where the sharp constant $C(p)$ *is given by*

$$C(p) = \pi^{-p/2}\frac{\Gamma(N/2+p/2)}{\Gamma(N+p/2)}\left(\frac{\Gamma(N/2)}{\Gamma(N)}\right)^{-1-p/N}$$

with Γ *denoting the Gamma function.*

Finally, we state and prove a lemma which we will use in establishing the lower semicontinuity of the energy E. A similar argument appears in the proof of Theorem II.1 in [45].

LEMMA 2.4. *Let* $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ *and* $f \in \mathcal{A}$ *be given such that* $f_n \rightharpoonup f$ *weakly in* $L^s(\mathbb{R}^N)$ *for some* $1 < s < \infty$. *Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f_n(x)f_n(y)}{|x-y|^a}\,dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)f(y)}{|x-y|^a}\,dxdy$$

where $0 < a < N$.

Proof. First note that

$$\left|\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f_n(x)f_n(y)}{|x-y|^a}\,dxdy\right)^{1/2} - \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)f(y)}{|x-y|^a}\,dxdy\right)^{1/2}\right| \leqslant \left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\,dxdy\right|^{1/2}. \tag{2.4}$$

On the other hand, for $R > 0$ we have that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\,dxdy\right| \\
&\quad\leqslant \left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{|x|<1/R\}}(|x-y|)\,dxdy\right| \\
&\qquad+\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{|x|>R\}}(|x-y|)\,dxdy\right| \\
&\qquad+\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{1/R<|x|<R\}}(|x-y|)\,dxdy\right|,
\end{aligned}$$

where χ_A denotes the characteristic function of the set A. Since f_n and f are in $\mathcal{A}$, they are uniformly bounded and $\|f_n\|_{L^1(\mathbb{R}^N)} = \|f\|_{L^1(\mathbb{R}^N)} = m$. Hence, the above inequality yields

$$\begin{aligned}
\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\,dxdy\right| &\leqslant C_1\frac{1}{R^{N-a}} + C_2\frac{1}{R^a} \\
&\quad+\left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(f_n(x)-f(x))(f_n(y)-f(y))}{|x-y|^a}\chi_{\{1/R<|x|<R\}}(|x-y|)\,dxdy\right|,
\end{aligned} \tag{2.5}$$

for some constants C_1, $C_2 > 0$ depending only on a and N.

For simplicity of presentation, define

$$g(x-y) := \frac{1}{|x-y|^a}\chi_{\{1/R<|x|<R\}}(|x-y|)$$

and note that $g(x-\cdot) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Also, define

$$F_n(x) := \int_{\mathbb{R}^N} f_n(y)g(x-y)\,dy \quad \text{and} \quad F(x) := \int_{\mathbb{R}^N} f(y)g(x-y)\,dy.$$

Since $f_n \rightharpoonup f$ weakly in $L^s(\mathbb{R}^N)$, for all $x \in \mathbb{R}^N$ we have that

$$F_n(x) \to F(x), \qquad \text{as } n \to \infty.$$

Moreover, integrating F_n over $\mathbb{R}^N$ and using $\|f_n\|_{L^1(\mathbb{R}^N)} = \|f\|_{L^1(\mathbb{R}^N)} = m$, we find

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_n(y)g(x-y)\,dydx &= \left(\int_{\mathbb{R}^N} g(s)\,ds\right)\left(\int_{\mathbb{R}^N} f_n(y)\,dy\right) \\
&= \left(\int_{\mathbb{R}^N} g(s)\,ds\right)\left(\int_{\mathbb{R}^N} f(y)\,dy\right) \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(y)g(x-y)\,dydx,
\end{aligned}$$

which implies that $\|F_n\|_{L^1(\mathbb{R}^N)} \to \|F\|_{L^1(\mathbb{R}^N)}$. Since $|F_n - F| \leqslant |F_n| + |F|$, the function $|F_n| + |F| - |F_n - F|$ is positive. So, applying Fatou's theorem we get that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |F_n| + |F| - |F_n - F|\,dx \geqslant \int_{\mathbb{R}^N} \liminf_{n\to\infty} |F_n| + |F| - |F_n - F|\,dx;$$

hence,

$$2\|F\|_{L^1(\mathbb{R}^N)} - \limsup_{n\to\infty} \int_{\mathbb{R}^N} |F_n - F|\,dx \geqslant 2\|F\|_{L^1(\mathbb{R}^N)}.$$

Thus $\limsup_{n\to 0} \int_{\mathbb{R}^N} |F_n - F|\,dx = 0$, that is, $F_n \to F$ strongly in $L^1(\mathbb{R}^N)$.

Consequently, $F_n \to F$ in $L^s(\mathbb{R}^N)$. Now, since $f_n \rightharpoonup f$ weakly in $L^s(\mathbb{R}^N)$, we have that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (f_n(x) - f(x))(f_n(y) - f(y))g(x-y)\,dxdy \to 0$$

as $n \to \infty$. Letting $R \to \infty$ in (2.5) yields, by (2.4), the desired result. □

We now prove the existence theorem for $p < 0$.

Theorem 2.5. *There exists a minimum of* (2.1) *when* $-N < p < q < 0$ *or* $-N < p < 0 < q$.

Proof. Let $\{\rho_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ be a minimizing sequence of the problem (2.1), that is, let $\{\rho_n\} \subset \mathcal{A}$ be a sequence such that

$$\lim_{n\to\infty} E(\rho_n) = \inf\{E(\rho)\colon \rho \in \mathcal{A}\}.$$

To prove the theorem we consider the two regimes of q separately, as the character of the interaction potential K is quite different in the two cases (see Figures 1.1(a), 1.1(b) and 1.1(c)).

Case 1: $-N < p < 0 < q$. In this regime both terms of the energy are positive. Hence, $E(\rho) \geqslant 0$ for all $\rho \in \mathcal{A}$, so the above infimum exists and is nonnegative. As $\{\rho_n\}_{n\in\mathbb{N}}$ is a minimizing sequence, for sufficiently large n the energy $E(\rho_n)$ is uniformly bounded.

By the concentration-compactness lemma (Lemma 2.2) the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ has a subsequence which satisfies one of the three possibilities: "tightness up to translation", "vanishing" or "dichotomy". We will show that "tightness up to translation" is the only possibility. To this end, suppose "vanishing" occurs. Let $0 < \epsilon < m$ and $R > 0$. Then for k large enough

$$\int_{B(0,R)} \rho_{n_k}(x)\,dx \leqslant \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \rho_{n_k}(x)\,dx < \epsilon.$$

Since $\rho_{n_k} \in \mathcal{A}$ this implies that

$$\int_{\mathbb{R}^N\setminus B(0,R)} \rho_{n_k}(x)\,dx \geqslant m - \epsilon > 0. \tag{2.6}$$

Now we are going to use the fact that the attractive term grows indefinitely at infinity. Since ρ_{n_k} are positive we have that

$$\begin{aligned}\frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} &|x-y|^q \rho_{n_k}(x)\rho_{n_k}(y)\,dxdy \\ &\geqslant \frac{1}{q}\int_{\mathbb{R}^N\setminus B(0,R)}\int_{\mathbb{R}^N\setminus B(0,R)} |x-y|^q \rho_{n_k}(x)\rho_{n_k}(y)\,dxdy.\end{aligned} \tag{2.7}$$

Due to radial symmetry, integration in the right-hand-side of (2.7) can be done on rays passing through the origin. Since one can always take x and y lying on opposite rays, so that $|x-y| > 2R$, we have by using (2.6),

$$\int_{\mathbb{R}^N\setminus B(0,R)}\int_{\mathbb{R}^N\setminus B(0,R)} |x-y|^q \rho_{n_k}(x)\rho_{n_k}(y)\,dxdy \geqslant (2R)^q(m-\epsilon)^2. \tag{2.8}$$

Combining (2.7) and (2.8), we find

$$E(\rho_{n_k}) \geqslant \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_{n_k}(x)\rho_{n_k}(y)\,dxdy \geqslant C\,R^q.$$

As $q > 0$, this inequality yields that $E(\rho_{n_k}) \to \infty$ as $R \to \infty$, contradicting the fact that ρ_{n_k} is a minimizing sequence. Therefore "vanishing" does not occur.

Next, suppose "dichotomy" occurs. Using the notation of Lemma 2.2(ii), let

$$d_k := \operatorname{dist}(\operatorname{supp}(\rho_{1,k}), \operatorname{supp}(\rho_{2,k}))$$

denote the distance between the supports of $\rho_{1,k}$ and $\rho_{2,k}$. We can further assume that the supports of $\rho_{1,k}$ and $\rho_{2,k}$ are disjoint. Inspecting again the attraction term we get that for some constant $C > 0$,

$$\begin{aligned}\frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_{n_k}(x)\rho_{n_k}(y)\,dxdy \\ &\geqslant \frac{C}{q}\int_{\operatorname{supp}(\rho_{1,k})}\int_{\operatorname{supp}(\rho_{2,k})} |x-y|^q \rho_{1,k}(x)\rho_{2,k}(y)\,dxdy \\ &\geqslant \frac{C}{q}\, d_k^q\, \|\rho_{1,k}\|_{L^1(\mathbb{R}^N)}\|\rho_{2,k}\|_{L^1(\mathbb{R}^N)}.\end{aligned}$$

Since $d_k \to \infty$ as $k \to \infty$ it gives that $E(\rho_{n_k}) \to \infty$, contradicting again the fact that ρ_{n_k} is a minimizing sequence. Thus "dichotomy" does not occur.

Therefore "tightness up to translation" is the only possibility; hence, there exists a sequence $\{y_k\}_{k\in\mathbb{N}}$ in $\mathbb{R}^N$ such that

$$\text{for all } \epsilon > 0 \text{ there exists } R > 0 \text{ satisfying } m \geqslant \int_{B(y_k,R)} \rho_{n_k}(x)\,dx \geqslant m - \epsilon. \tag{2.9}$$

Note that if $\{y_k\}_{k\in\mathbb{N}}$ were unbounded, then, since $|y_k| \to \infty$ as $k \to \infty$, we would get

$$\lim_{k\to\infty}\int_{B(0,R)} \rho_{n_k}(x)\,dx = 0$$

for all $R > 0$; hence, a contradiction as in the "vanishing" case. Therefore $\{y_k\}_{k\in\mathbb{N}}$ is bounded.

Now, let $\overline{\rho}_{n_k}(x) = \rho_{n_k}(x+y_k)$ and note that $E(\rho_{n_k}) = E(\overline{\rho}_{n_k})$. Since $\{\rho_{n_k}\}_{k\in\mathbb{N}} \subset \mathcal{A}$, all members of the sequence are uniformly bounded in $L^1(\mathbb{R}^N) \cup L^\infty(\mathbb{R}^N)$ and passing to a subsequence if necessary, we may assume that

$$\overline{\rho}_{n_k} \rightharpoonup \rho_0 \text{ weakly in } L^s(\mathbb{R}^N)$$

for some $1 < s < \infty$ and some $\rho_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ [1]. Moreover, by (2.9),

$$\int_{\mathbb{R}^N} \rho_0(x)\,dx = m,$$

or in other words, when passing to the limit as $k \to \infty$ the sequence $\overline{\rho}_{n_k}$ does not "leak-out" at infinity. To show that $\rho_0 \geqslant 0$ a.e. let

$$S := \{x \in \mathbb{R}^N : \rho_0(x) < 0\}.$$

Then the characteristic function of S, χ_S, is an admissible test function for the weak convergence of $\overline{\rho}_{n_k}$, so we get that

$$\int_S \overline{\rho}_{n_k}(x)\,dx \to \int_S \rho_0(x)\,dx < 0.$$

However, since $\overline{\rho}_{n_k} \in \mathcal{A}$, we see that

$$\liminf_{k\to\infty}\int_S \overline{\rho}_{n_k}(x)\,dx \geqslant 0;$$

[1] In fact, the sequence $\overline{\rho}_{n_k}$ converges weakly to ρ_0 in $L^s(\mathbb{R}^N)$ for every $1 < s < \infty$ because of the uniform bound on the sequence. The weak convergence holds for $s = 1$, as well, by (2.9) and since the translation sequence $\{y_k\}_{k\in\mathbb{N}}$ is bounded.

hence, S has measure zero. Similarly we can show that $\|\rho_0\|_{L^\infty(\mathbb{R}^N)} \leqslant M$. Thus $\rho_0 \in \mathcal{A}$.

Next we need to show that the energy is weakly lower semicontinuous. Here, with an abuse of notation, we will drop the bar on $\overline{\rho}_n$, and simply denote them by ρ_n.

By Lemma 2.4, the repulsive part is weakly lower semicontinuous and we have that

$$-\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho_n(x)\rho_n(y)\,dxdy \to -\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho_0(x)\rho_0(y)\,dxdy \tag{2.10}$$

as $n \to \infty$.

On the other hand, for the attractive part, define

$$G_n(x) = \int_{B(0,R)} |x-y|^q \rho_n(y)\,dy \quad \text{and} \quad G_0(x) = \int_{B(0,R)} |x-y|^q \rho_0(y)\,dy,$$

for any fixed $R > 0$. Note that since $\|\rho_0\|_{L^\infty(\mathbb{R}^N)} \leqslant M$ and $q > 0$, we see that $G_0 \in L^\infty(B(0,R))$, in particular, $G_0 \in L^{s/(s-1)}(B(0,R))$. Therefore, by the weak convergence of ρ_n in $L^s(B(0,R))$,

$$\int_{B(0,R)} G_0(x)[\rho_n(x) - \rho_0(x)]\,dx \to 0 \tag{2.11}$$

as $n \to \infty$. Also, since ρ_n are uniformly bounded, taking $\int_{B(0,R)} |\cdot - y|^q\,dy \in L^{s/(s-1)}(B(0,R))$ as a test function, we see that

$$\int_{B(0,R)} \rho_n(x)[G_n(x) - G_0(x)]\,dx \to 0 \tag{2.12}$$

as $n \to \infty$, by the weak convergence of ρ_n in $L^s(B(0,R))$.

Thus, using (2.11) and (2.12), we have that

$$\begin{aligned}\int_{B(0,R)} G_n(x)\rho_n(x)\,dx = &\int_{B(0,R)} G_0(x)[\rho_n(x)-\rho_0(x)]\,dx \\ &+ \int_{B(0,R)} \rho_n(x)[G_n(x)-G_0(x)]\,dx + \int_{B(0,R)} G_0(x)\rho_0(x)\,dx\end{aligned}$$

converges to

$$\int_{B(0,R)} G_0(x)\rho_0(x)\,dx$$

as $n \to \infty$. Hence,

$$\liminf_{n\to\infty} \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy = \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_0(x)\rho_0(y)\,dxdy.$$

Now, by (2.9), for given $\epsilon > 0$, we can choose $R > 0$ such that

$$\int_{B(0,R)} \rho_0(x)\,dx \geqslant m - \epsilon.$$

Then, for such R, since $E(\rho_0) < \infty$, we can control the excess of the attractive part on $\mathbb{R}^N \setminus B(0,R)$ and we get that

(2.13)
$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_0(x)\rho_0(y)\,dxdy &\leqslant \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_0(x)\rho_0(y)\,dxdy + C\epsilon \\ &\leqslant \liminf_{n\to\infty} \int_{B(0,R)}\int_{B(0,R)} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy + C\epsilon \\ &\leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy + C\epsilon.\end{aligned}$$

Letting $\epsilon \to 0$ and combining with (2.10) yields

$$\inf\{E(\rho) : \rho \in \mathcal{A}\} \leqslant E(\rho_0) \leqslant \liminf_{n\to\infty} E(\rho_n) = \inf\{E(\rho) : \rho \in \mathcal{A}\},$$

that is, ρ_0 is a solution to the minimization problem (2.1) in the regime $-N < p < 0 < q$.

Case 2: $-N < p < q < 0$. Note that in this case, the character of the interaction potential is quite different than in the previous regime. Now the attractive term is strictly negative whereas the repulsive part of the energy E is still strictly positive. However, using Proposition 2.3 we see that the attractive term is bounded below, and we conclude that in this regime

$$\inf\{E(\rho)\colon \rho \in \mathcal{A}\} > -\infty.$$

Looking at the scaling

$$\rho_\lambda(x) = \frac{1}{\lambda^N}\rho\left(\frac{x}{\lambda}\right)$$

we see that $\rho_\lambda \in \mathcal{A}$ for $\lambda \geqslant 1$, and the energy of ρ_λ is given by

$$E(\rho_\lambda) = \frac{\lambda^q}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho(x)\rho(y)\,dxdy - \frac{\lambda^p}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho(x)\rho(y)\,dxdy,$$

for any given function $\rho \in \mathcal{A}$. Note that, in particular, we can choose ρ to be the characteristic function of a ball. Since $-N < p < q < 0$, for λ large enough we get that $E(\rho_\lambda) < 0$, and hence,

$$I_m := \inf\{E(\rho)\colon \rho \in \mathcal{A}\} < 0.$$

Again, we will make use of the concentration compactness lemma, Lemma 2.9, and show that for a minimizing sequence ρ_n the possibilities of "vanishing" and "dichotomy" do not occur.

Suppose "vanishing" occurs. Since $I_m < 0$ in this regime and since the repulsive part is strictly positive, looking at the attractive part we have that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy > 0. \tag{2.14}$$

Let $R > 1$ and $q = -a$ for $0 < a < N$. Then, as in the proof of Lemma 2.4,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_n(x)\rho_n(y)}{|x-y|^a}\chi_{\{|x|<1/R\}}(|x-y|)\,dxdy \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_n(x)\rho_n(y)}{|x-y|^a}\chi_{\{1/R<|x|<R\}}(|x-y|)\,dxdy \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_n(x)\rho_n(y)}{|x-y|^a}\chi_{\{|x|>R\}}(|x-y|)\,dxdy \\ &\leqslant \frac{CmM}{R^{N-a}} + \frac{m^2}{R^a} + R^a \int_{\mathbb{R}^N} \rho_n(x) \int_{|x-y|<R} \rho_n(y)\,dydx \\ &\leqslant \frac{CmM}{R^{N-a}} + \frac{m^2}{R^a} + R^a\, m \sup_{x\in\mathbb{R}^N} \left(\int_{|x-y|<R} \rho_n(y)\,dy \right)\end{aligned}$$

where C is positive constant depending only on a and N, M is the uniform bound on ρ_n and m is the mass of ρ_n, as before.

Since ρ_n vanishes by Lemma 2.2 (ii), we get that as $n \to \infty$ the last term in the above inequality is zero; hence,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy \leqslant \frac{CmM}{R^{N-a}} + \frac{m^2}{R^a}.$$

Letting $R \to \infty$, since $0 < a < N$, this yields that

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_n(x)\rho_n(y)\,dxdy \leqslant 0,$$

contradicting (2.14). Thus “vanishing” does not occur.

To show that “dichotomy” does not occur, first we need to prove a subadditivity condition similar to the one in [45]. As in [6, Lemma 1], we can prove a weaker subadditivity condition which states that

$$\text{for } m_1 > m_2 \text{ we have } I_{m_1} < I_{m_2}, \tag{2.15}$$

where, as above, I_{m_i} denotes the infimum of E over $\mathcal{A}_{m_i}$ with mass constraint $\int_{\mathbb{R}^N} \rho(x)\,dx = m_i$. Here we choose to display the dependence of the admissible class $\mathcal{A}$ on the mass by using the notation $\mathcal{A}_{m_i}$ to avoid confusion. Indeed, let $\{\rho_n\}_{n\in\mathbb{N}}$ be a minimizing sequence such that $\lim_{n\to\infty} E(\rho_n) = I_{m_2}$ and, by the scaling argument above, let $v \in C_c^\infty \cap \mathcal{A}_{m_1-m_2}$ be such that $E(v) < 0$ with $\int_{\mathbb{R}^N} v(x)\,dx = m_1 - m_2$.

By a density argument, we can assume further that ρ_n are compactly supported. For some fixed unit vector $z \in \mathbb{R}^N$, let

$$\tilde\rho_n(x) := \rho_n(x + nz).$$

Note that as $n \to \infty$ the distance $d_n := \operatorname{dist}(\operatorname{supp}(\tilde\rho_n), \operatorname{supp}(v))$ goes to $+\infty$. Moreover, since the energy is translation invariant we have that $E(\tilde\rho_n) = E(\rho_n)$.

Define

$$w_n(x) = v(x) + \tilde\rho_n(x),$$

and note that $\int_{\mathbb{R}^N} w_n(x)\,dx = m_1$. Consider $E(w_n)$:

$$E(w_n) = E(v) + E(\rho_n) + 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\tilde\rho_n(x)v(y)\,dxdy. \tag{2.16}$$

In this regime both attractive and repulsive terms of the interaction potential are of the form $1/|x|^a$ for some $0 < a < N$, with appropriate signs which make them attractive or repulsive. For such interactions we have that for any $\delta > 0$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\tilde{\rho}_n(x)v(y)}{|x-y|^a}\,dxdy \leqslant C\delta^a + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\tilde{\rho}_n(x)v(y)}{|x-y|^a}\chi_{\{d_n<|x|<\delta^{-1}\}}(|x-y|)\,dxdy$$
$$\leqslant C\delta^a + m_2(m_1-m_2)\left(\frac{1}{d_n}\right)^a.$$

As $n \to \infty$, the distance $d_n \to \infty$, hence we find from (2.16),

$$\liminf_{n\to\infty} E(w_n) \leqslant \liminf_{n\to\infty} E(\rho_n) + E(v) + C\delta^a,$$

with $a > 0$. Since $E(v) < 0$, one can choose δ small enough to conclude the weak subadditivity result

$$I_{m_1} < I_{m_2}.$$

Now, suppose "dichotomy" occurs, that is, there exists $\alpha \in (0,m)$ such that for all $\epsilon > 0$, there exist $k_0 \geqslant 1$ and $\rho_{1,k},\ \rho_{2,k} \in L^1_+(\mathbb{R}^N)$ satisfying for $k \geqslant k_0$

$$\|\rho_{n_k} - (\rho_{1,k}+\rho_{2,k})\|_{L^1(\mathbb{R}^N)} \leqslant \epsilon,$$
$$\big|\|\rho_{1,k}\|_{L^1(\mathbb{R}^N)} - \alpha\big| \leqslant \epsilon, \quad \big|\|\rho_{2,k}\|_{L^1(\mathbb{R}^N)} - (m-\alpha)\big| \leqslant \epsilon,$$

and

$$\operatorname{dist}(\operatorname{supp}(\rho_{1,k}), \operatorname{supp}(\rho_{2,k})) \to \infty \text{ as } k \to \infty.$$

Furthermore, after defining $v_k := \rho_{n_k} - (\rho_{1,k}+\rho_{2,k})$ we can assume that

$$0 \leqslant \rho_{1,k}, \rho_{2,k}, v_k \leqslant \rho_{n_k} \text{ and } \rho_{1,k}\rho_{2,k} = \rho_{1,k}v_k = \rho_{2,k}v_k = 0 \text{ a.e.}$$

We have that, for any $0 < a < N$,

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{n_k}(x)\rho_{n_k}(y)}{|x-y|^a}\,dxdy &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{1,k}(x)\rho_{1,k}(y)}{|x-y|^a}\,dxdy \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{2,k}(x)\rho_{2,k}(y)}{|x-y|^a}\,dxdy + 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{1,k}(x)\rho_{2,k}(y)}{|x-y|^a}\,dxdy \\ &+ 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho_{n_k}(x)v_k(y)}{|x-y|^a}\,dxdy - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v_k(x)v_k(y)}{|x-y|^a}\,dxdy.\end{aligned} \tag{2.17}$$

The last two terms above vanish as $k \to \infty$ using the integrability of the kernel around zero, the uniform bound on ρ_{n_k} and the fact that $\|v_k\|_{L^1(\mathbb{R}^N)} \to 0$. The third term on the right hand side of (2.17) goes to zero with $k \to \infty$ following similar steps as in establishing the weak subadditivity condition above, using the assumption $\operatorname{dist}(\operatorname{supp}(\rho_{1,k}), \operatorname{supp}(\rho_{2,k})) \to \infty$ as $k \to \infty$.

Since $\operatorname{dist}(\operatorname{supp}(\rho_{1,k}), \operatorname{supp}(\rho_{2,k})) \to \infty$ as $k \to \infty$, one of the components (say $\rho_{2,k}$) spreads to infinity. Hence, by the radial symmetry of ρ_n and by the fact that the kernel $K(|x|)$ approaches zero as $|x| \to \infty$, we get that

$$\lim_{k\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho_{2,k}(x)\rho_{2,k}(y)\,dxdy = 0.$$

These observations combined with (2.17) imply that

$$
\begin{aligned}
I_m &= \lim_{k\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho_{n_k}(x)\rho_{n_k}(y)\,dxdy \\
&\geqslant \liminf_{k\to 0} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)\rho_{1,k}(x)\rho_{1,k}(y)\,dxdy \qquad (2.18)\\
&= I_\alpha,
\end{aligned}
$$

which contradicts the weak subadditivity condition (2.15). Thus "dichotomy" does not occur.

As in the first case, "tightness up to translation" is the only possibility. Therefore the weak limit ρ_0 of the translated sequence $\overline{\rho}_n$ satisfies the mass constraint and hence, is a member of $\mathcal{A}$.

The weak lower semicontinuity in this regime follows directly from Lemma 2.4 as both attractive and repulsive terms of the energy are of the form considered in the lemma and by (2.9) the assumptions of the lemma are satisfied. We conclude that the minimization problem (2.1) has a solution when $-N < p < q < 0$. □

Remark 2.6. The concentration-compactness principle suffices to establish a weaker form compactness so that we can pass to a weak limit in the sequence ρ_n. However, the sequence does not necessarily convergence strongly to ρ in any $L^s(\mathbb{R}^N)$. Indeed, strong convergence can fail due to mass leaking out at infinity and/or because of oscillations. By the tightness of the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ the former does not happen; but, we cannot rule out the oscillations of ρ_n.

On the other hand, we note that for functionals which contain a term that is convex in ρ (cf. Remark 1.2), one can further show that the convergence of $\{\rho_n\}_{n\in\mathbb{N}}$ is strong (cf. [6, 45]).

Remark 2.7. Note that the proof of existence of minimizers also applies for potentials of the form

$$K(x) = f(|x|) - \frac{1}{p}|x|^p,$$

where $f(|x|) \to \infty$ as $|x| \to \infty$ and $-N < p < 0$.

Remark 2.8. When $N = 2$ the Newtonian potential is given by $-\frac{1}{2\pi}\log|x|$. Either considering the logarithmic term as the repulsion in

$$K(x) = \frac{1}{q}|x|^q - \log|x|, \qquad q > 0, \quad x \in \mathbb{R}^2$$

or as the attraction in

$$K(x) = \log|x| - \frac{1}{p}|x|^p, \qquad -2 < p < 0, \quad x \in \mathbb{R}^2$$

the proof of Theorem 2.5 applies since the properties of the interaction potential (singularity at the origin and blow-up at infinity) remain the same as in higher dimensions.

3. Characterization of Critical Points of E for $p < 0$. We begin by noting that equipartition of energy is a necessary condition for criticality. Indeed, the next proposition can be viewed as a weak formulation of the Euler-Lagrange equation.

PROPOSITION 3.1 (Equipartition of Energy). *If ρ_0 is a critical point of E, then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^q \rho_0(x)\rho_0(y)\,dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |x-y|^p \rho_0(x)\rho_0(y)\,dxdy \tag{3.1}$$

Proof. Let $\rho_0 \in \mathcal{A}$ be a critical point of (2.1) and consider the rescaled function ρ_λ given by

$$\rho_\lambda(x) := \frac{1}{\lambda^N}\rho_0\left(\frac{x}{\lambda}\right)$$

as before. Then a simple change of variables reveals that

$$E(\rho_\lambda) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\frac{\lambda^q}{q}|x-y|^q - \frac{\lambda^p}{p}|x-y|^p\right)\rho_0(x)\rho_0(y)\,dxdy.$$

Since ρ_0 is a critical point, we have $\frac{d}{d\lambda}\big|_{\lambda=1} E(\rho_\lambda) = 0$ which in turn yields that the equation (3.1) needs to be satisfied by ρ_0 as a necessary condition of criticality. ◻

Next we derive the another version of Euler–Lagrange equations. The same equations were formally obtained in [10] and derived in the context of minimization with respect to the Wasserstein distance in [4]. Here we take a more direct and elementary approach.

PROPOSITION 3.2. *Let $\rho_0 \in \mathcal{A}$ be a minimizer of the energy E. Then we have*

$$\begin{aligned} \Lambda(x) &\geqslant \mu \quad \textit{a.e. on the set} \quad \{x : \rho_0(x) = 0\} \\ \Lambda(x) &= \mu \quad \textit{a.e. on the set} \quad \{x : \rho_0(x) > 0\} \end{aligned} \tag{3.2}$$

where

$$\Lambda(x) := 2\int_{\mathbb{R}^N}\left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right)\rho_0(y)\,dy, \tag{3.3}$$

and μ is a constant.

Proof. Proceeding as in [44], let ρ_0 be a minimizer of E and let $\zeta \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be an arbitrary function. For $0 \leqslant \epsilon \leqslant 1$, consider

$$\rho_\epsilon(x) := \rho_0(x) + \epsilon\left(\zeta(x) - \frac{\int_{\mathbb{R}^N}\zeta(x)\,dx}{m}\rho_0(x)\right). \tag{3.4}$$

Clearly $\int_{\mathbb{R}^N}\rho_\epsilon(x)\,dx = m$. Also, we have $\rho_\epsilon \geqslant 0$ provided

$$\zeta(x) \geqslant -\frac{\rho_0(x)}{2} \quad \text{and} \quad \int_{\mathbb{R}^N}\zeta(x)\,dx \leqslant \frac{m}{2}. \tag{3.5}$$

By minimality of ρ_0 we have

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0^+} E(\rho_\epsilon) \geqslant 0. \tag{3.6}$$

Plugging ρ_ϵ into E, taking the derivative with respect to ϵ and evaluating at $\epsilon = 0$ gives

$$\begin{aligned}\frac{d}{d\epsilon}\bigg|_{\epsilon=0^+} E(\rho_\epsilon) &= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right)\rho_0(y)\zeta(x)\,dxdy \\ &\quad - 2\left(\frac{\int_{\mathbb{R}^N}\zeta(x)\,dx}{m}\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{1}{q}|x-y|^q - \frac{1}{p}|x-y|^p\right)\rho_0(x)\rho_0(y)\,dxdy \\ &= \int_{\mathbb{R}^N}\Lambda(x)\zeta(x)\,dx - \mu\int_{\mathbb{R}^N}\zeta(x)\,dx \\ &= \int_{\mathbb{R}^N}(\Lambda(x)-\mu)\zeta(x)\,dx,\end{aligned}$$

where Λ was defined in (3.3), and

$$\mu := \frac{\int_{\mathbb{R}^N}\Lambda(x)\rho_0(x)\,dx}{m}.$$

Hence, we get that

$$\int_{\mathbb{R}^N}(\Lambda(x)-\mu)\zeta(x)\,dx \geqslant 0. \tag{3.7}$$

The inequality (3.7) above holds, in particular, for all nonnegative functions $\zeta \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N}\zeta(x)\,dx \leqslant \frac{m}{2}.$$

This, in turn, implies that

$$\Lambda(x) - \mu \geqslant 0 \quad \text{a.e.}$$

Moreover, note that μ is the average of Λ with respect to the measure $\rho_0(x)dx$. Then, the condition $\Lambda(x) \geqslant \mu$ a.e. implies that $\Lambda(x) = \mu$ for a.e. x where $\rho_0(x) > 0$. Thus, we establish (3.2). □

Note that in Prop. 3.2, we actually do not need ρ_0 to be a minimizer but simply a critical point (vanishing first variation) in the sense of (3.6) holding for all ζ satisfying (3.4) and (3.5).

Next we consider the special case $p = 2-N$, for which the repulsion component of (2.1) reduces to the Coulomb energy (see Chapter 9 in [44]). In three dimensions, $p = -1$ corresponds to the well-known electrostatic potential energy or the Newton's gravitational potential. Case $p = 2-N$ was investigated in the context of the evolution equation (1.3) in [29, 28]. There, the authors focused on the existence of symmetric, bounded and compactly-supported steady states and they showed that for any attraction component $q > 2-N$, a unique such steady state exists. Moreover, numerical experiments suggest that these equilibrium solutions are global attractors for solutions of (1.3).

In particular, for $q = 2$, the steady state considered in [29] consists in a uniform density in a ball. It was shown in [13] that such uniform states (called patch solutions by the authors) are global attractors for the dynamics of (1.3). We study these steady states here from a variational point of view, and show that they are global minimizers of (2.1). Below, $\omega_N = \frac{\pi^{N/2}}{\Gamma(N/2+1)}$ denotes the volume of the unit ball in $\mathbb{R}^N$.

THEOREM 3.3. *For any $m > 0$ and $M \geqslant \frac{m}{\omega_N}$, the function $\rho(x) = \dfrac{m}{\omega_N}\chi_{B(0,1)}(x)$ is the global minimizer of the problem* (2.1) *when $q = 2$, $p = 2 - N$.*

Proof. First we check that ρ is a critical point of the functional E. When $q = 2$ the left-hand-side of (3.1) reduces to $m\int_{\mathbb{R}^N} |x|^2\rho(x)\,dx$ (see calculation leading to (2.3)) and hence,

$$2m\int_{\mathbb{R}^N} |x|^2\rho(x)\,dx = 2m^2N\int_0^1 r^{N+1}\,dr = \frac{2m^2N}{N+2}.$$

On the other hand, when $p = 2 - N$ the repulsive part is the Newtonian potential (i.e., $-\Delta_y(|x-y|^{2-N}) = N(N-2)\omega_N\delta_x$), and

$$\Phi(x) := \int_{B(0,1)} \frac{1}{N(N-2)\omega_N|x-y|^{N-2}}\,dy$$

solves the Poisson problem

$$-\Delta\Phi(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1,\\ 0 & \text{if } |x| > 1.\end{cases}$$

Since the right-hand side of the Poisson problem is radial, so is $\Phi(x)$. We use the expression of the Laplacian on $\mathbb{R}^N$ in hyper-spherical coordinates,

$$-\Delta\Phi(x) = -\frac{1}{r^{N-1}}\frac{d}{dr}\left(r^{N-1}\frac{d\Phi(r)}{dr}\right),$$

with $r = |x|$, and integrate once to get

$$\frac{d\Phi(r)}{dr} = \begin{cases} -\frac{r}{N} & \text{if } r \leqslant 1,\\ -\frac{1}{Nr^{N-1}} & \text{if } r > 1.\end{cases} \tag{3.8}$$

Now, as Φ solves the above Poisson problem, the repulsive term of the energy E is given by

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\rho(x)\rho(y)}{|x-y|^{N-2}}\,dxdy &= \frac{N(N-2)m^2}{\omega_N}\int_{\mathbb{R}^N} |\nabla\Phi|^2\,dx\\ &= \frac{N(N-2)m^2}{\omega_N}\int_{\mathbb{R}^N}\left(\frac{d}{d|x|}\Phi(|x|)\right)^2 dx.\end{aligned}$$

An explicit calculation yields

$$\frac{m^2}{\omega_N^2}\int_{B(0,1)}\int_{B(0,1)} \frac{1}{|x-y|^{N-2}}\,dxdy = \frac{2m^2N}{N+2}.$$

Hence ρ satisfies (3.1), and is a critical point of E.

Note that we can write the repulsive part in (2.3) using the H^{-1}-norm, and write the energy as

$$E(\rho) = m\int_{\mathbb{R}^N} |x|^2\rho(x)\,dx + N\omega_N\|\rho\|^2_{H^{-1}(\mathbb{R}^N)}.$$

Clearly both terms in the energy are strictly convex [2]. Since the energy is strictly convex in every direction and $\rho(x) = \frac{m}{\omega_N}\chi_{B(0,1)}(x)$ is a critical point, it is the global minimizer of the problem (2.1). ▯

REMARK 3.4. When $p = 2 - N$ the repulsive term is always strictly convex as it can be written as the square of the H^{-1}-norm of ρ; however, for $q > 2 - N$, $q \neq 2$, it is difficult to check the convexity of the attractive term due to cross-integral terms in the energy.

4. Binary Density Version. We consider the energy E where the density function ρ takes on only values $\{0, 1\}$. Then ρ is the characteristic function χ_A of the set $A := \{x \in \mathbb{R}^N : \rho(x) = 1\}$ and the problem (2.1) can be re-written using a set functional notation as

$$\text{minimize} \quad \mathcal{E}(A) = \int_A \int_A K(x - y)\, dxdy \tag{4.1}$$

over radial sets A of finite measure subject to the constraint

$$|A| = m.$$

Following the calculations in [26] we can find the first and second variations of $\mathcal{E}$. The idea there is to define an admissible perturbation of A, a family of sets $\{A_\epsilon\}_{\epsilon \in (-\tau,\tau)}$ for some $\tau > 0$, which satisfies the following three conditions: (i) $\chi_{A_\epsilon} \to \chi_A$ in $L^1(\mathbb{R}^N)$ as $\epsilon \to 0$, (ii) ∂A_ϵ is of class C^2, and (iii) $d/d\epsilon|_{\epsilon=0}|A_\epsilon| = 0$. For the calculation of the second variation the family needs to be modified to further satisfy the condition that it preserves mass up to second order, that is, the modified family $\{\tilde{A}_\epsilon\}$ needs to satisfy the extra condition: (iv) $d^2/d\epsilon^2|_{\epsilon=0}|\tilde{A}_\epsilon| = 0$.

For such an admissible family, χ_A is called a critical point of $\mathcal{E}$ if

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{E}(A_\epsilon) = 0,$$

and a critical point is stable if

$$\left.\frac{d^2}{d\epsilon^2}\right|_{\epsilon=0} \mathcal{E}(\tilde{A}_\epsilon) \geqslant 0.$$

Now we can state the result concerning the stable critical points of $\mathcal{E}(A)$.

PROPOSITION 4.1. *Let ρ be a critical point of $\mathcal{E}$ where $A := \{x \in \mathbb{R}^N : \rho(x) = 1\}$. Then*

$$\Lambda(x) = \lambda \quad \textit{for all } x \in \partial A, \tag{4.2}$$

where

$$\Lambda(x) = \int_A K(x - y)\, dy$$

and the Lagrange multiplier λ is a constant.

[2] A functional $E : \mathcal{A} \to \mathbb{R}$ is strictly convex if for all f and g in $\mathcal{A}$, $f \neq g$, and $t \in (0, 1)$ we have $E[tf + (1 - t)g] < tE[f] + (1 - t)E[g]$.

Moreover, if ρ is a stable critical point, then for any smooth function ξ on ∂A satisfying the condition

$$\int_{\partial A} \xi(x)\, d\mathcal{H}_x^{N-1} = 0,$$

we have that

$$\int_{\partial A}\int_{\partial A} K(x-y)\xi(x)\xi(y)\, d\mathcal{H}_x^{N-1} d\mathcal{H}_y^{N-1} + \int_{\partial A} (\nabla\Lambda(x)\cdot\nu(x))\xi^2(x)\, d\mathcal{H}_x^{N-1} \geqslant 0, \tag{4.3}$$

where $\mathcal{H}^{N-1}$ denotes the $N-1$-dimensional Hausdorff measure, and ν denotes the unit normal on ∂A pointing out of A.

The proof follows directly from the proof of Theorems 2.3 and 2.6 in [26]. In [26], the nonlocal isoperimetric functional (cf. Remark 1.3) consists of a perimeter term plus a repulsive interaction involving the Green's function for the Laplace operator on the flat N-torus. However, their calculation performed on each term separately applies to any radially symmetric kernel $K(|x|)$.

In Theorem 3.3 we showed that the global minimizer of the problem (2.1) for $q = 2$ and $p = 2 - N$ is a uniform density in a ball. It is clear that when we restrict the density functions to binary functions the result of Theorem 3.3 fails for any $m \neq \omega_N$. However, we can still prove explicitly for the binary density version that the ball is the global minimizer when $q = 2$ and $p = 2 - N$.

THEOREM 4.2. *For any $m > 0$ let $R := \left(\dfrac{m}{\omega_N}\right)^{1/N}$. Then the ball $B = B(0,R) \subset \mathbb{R}^N$, or rather its characteristic function χ_B, is the global minimizer of the problem (4.1) when $q = 2$ and $p = 2 - N$.*

Proof. Clearly χ_B satisfies the criticality condition (4.2). Now let $A \subset \mathbb{R}^N$ be any radially symmetric set of finite measure such that $|A| = m$. Using a similar idea to the one in [34], consider

$$\begin{aligned}
\mathcal{E}(\chi_A) - \mathcal{E}(\chi_B) &= m\int_{\mathbb{R}^N} |x|^2(\chi_A(x) - \chi_B(x))\, dx \\
&\quad + \frac{1}{N-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\chi_A(x)\chi_A(y)}{|x-y|^{N-2}}\, dxdy - \frac{1}{N-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\chi_B(x)\chi_B(y)}{|x-y|^{N-2}}\, dxdy \\
&= m\int_{\mathbb{R}^N} |x|^2(\chi_A(x) - \chi_B(x))\, dx + \frac{2}{N-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\chi_B(y)(\chi_A(x) - \chi_B(x))}{|x-y|^{N-2}}\, dxdy \\
&\quad + \frac{1}{N-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\chi_A(x) - \chi_B(x))(\chi_A(y) - \chi_B(y))}{|x-y|^{N-2}}\, dxdy \\
&\geqslant m\int_{\mathbb{R}^N} |x|^2(\chi_A(x) - \chi_B(x))\, dx + \frac{2}{N-2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\chi_B(y)(\chi_A(x) - \chi_B(x))}{|x-y|^{N-2}}\, dxdy \\
&= m\int_{\mathbb{R}^N} |x|^2(\chi_A(x) - \chi_B(x))\, dx + 2\int_B \Theta(y)\, dy,
\end{aligned}$$

where Θ is given by

$$\Theta(y) = \frac{1}{N-2}\int_{\mathbb{R}^N} \frac{\chi_A(x) - \chi_B(x)}{|x-y|^{N-2}}\, dx.$$

Note that for any point $x \in B = B(0,R)$ the function $\chi_A(x) - \chi_B(x)$ is either 0 or -1. Hence,

$$-\Delta\Theta = N\omega_N(\chi_A(x) - \chi_B(x)) \leqslant 0 \text{ on } B,$$

which means, Θ is subharmonic on B. Then, by the mean value property, we have that

$$\int_B \Theta(y)\,dy \geqslant m\Theta(0) = \frac{m}{N-2}\int_{\mathbb{R}^N} \frac{\chi_A(x) - \chi_B(x)}{|x|^{N-2}}\,dx$$

Thus, from the calculation above, we get that

$$\begin{aligned}
&\mathcal{E}(\chi_A) - \mathcal{E}(\chi_B) \geqslant m\int_{\mathbb{R}^N} |x|^2(\chi_A(x) - \chi_B(x))\,dx + \frac{2m}{N-2}\int_{\mathbb{R}^N} \frac{\chi_A(x) - \chi_B(x)}{|x|^{N-2}}\,dx \\
&= \int_{\mathbb{R}^N}\left(m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}}\right)(\chi_A(x) - \chi_B(x))\,dx \\
&= \int_A \left(m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}}\right)dx - \int_B \left(m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}}\right)dx, \\
&\geqslant 0
\end{aligned}$$

where the last inequality follows from the fact that A is a radially symmetric set. Indeed, writing the integral

$$\int_A \left(m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}}\right)dx = \int_{\mathbb{R}^N}\left(m|x|^2 + \frac{2m}{(N-2)|x|^{N-2}}\right)\chi_A(x)\,dx$$

in spherical coordinates yields

$$mN\omega_N\int_0^\infty \left(r^{N+1} + \frac{2}{N-2}r\right)\chi_A(r)\,dr.$$

Clearly, $r^{N+1} + \frac{2}{N-2}r$ is an increasing function; hence,

$$mN\omega_N\int_0^R \left(r^{N+1} + \frac{2}{N-2}r\right)dr \leqslant mN\omega_N\int_0^\infty \left(r^{N+1} + \frac{2}{N-2}r\right)\chi_A(r)\,dr$$

proving that the ball of radius $R = \left(\frac{m}{\omega_N}\right)^{1/N}$ is the global minimizer of the binary density problem. ▯

According to the computations in [29], we see that when one considers the Newtonian potential as the repulsive interaction, that is when $p = 2-N$, increasing the power of the attractive interaction, namely q, results in accumulation of the density ρ around the boundary of the support of ρ. Moreover, the resulting density profiles indicate the possibility of annulus shaped attractors. However, we will show that when we consider the binary minimization problem (4.1) in $\mathbb{R}^3$, spherical annuli are not even critical points for $p = -1$ and $q > 2$. Our arguments here rely on explicit calculations; hence, we chose to work in 3 dimensions. We believe that the calculations can be extended to N dimensions.

Without loss of generality take $m = 4\pi/3$, the volume of the unit ball in $\mathbb{R}^3$, and consider the spherical annulus

$$A := \{x \in \mathbb{R}^3 \colon R < |x| < (1+R^3)^{1/3}\}$$

for some $R > 0$. Recalling (4.2) let us define

$$\Lambda_{\mathrm{attr}}(x) := \int_A \frac{|x-y|^q}{q}\, dy \qquad \text{and} \qquad \Lambda_{\mathrm{rep}}(x) := \int_A \frac{1}{|x-y|^{N-2}}\, dy$$

for some $q > 2$. Let x_1 and $x_2 \in \partial A$ be two collinear points with the origin given in spherical coordinates by

$$x_1 = (R, 0, 0) \qquad \text{and} \qquad x_2 = ((1+R^3)^{1/3}, 0, 0),$$

that is, x_1 and x_2 are two points on the same ray from the origin such that x_1 lies on the inner boundary component of A whereas x_2 lies on the outer boundary component.

Assume, for a contradiction, that there exists a number $R > 0$, depending on q, such that A is a critical point of the minimization problem (4.1). Then the Euler–Lagrange equation (4.2) is satisfied at x_1 and x_2 simultaneously, and this implies that

$$(\Lambda_{\mathrm{attr}}(x_2) - \Lambda_{\mathrm{attr}}(x_1)) + (\Lambda_{\mathrm{rep}}(x_2) - \Lambda_{\mathrm{rep}}(x_1)) = 0. \tag{4.4}$$

Since Λ_{rep} solves the Poisson equation

$$-\Delta \Lambda_{\mathrm{rep}}(x) = 4\pi \chi_A(x) \quad \text{in} \quad \mathbb{R}^3$$

and is a radial function we can find the solution explicitly as

$$\frac{1}{4\pi}\Lambda_{\mathrm{rep}}(|x|) = \begin{cases} \frac{(1+R^3)^{2/3} - R^2}{2} & \text{if} \quad 0 \leqslant |x| < R, \\ -\frac{|x|^2}{6} - \frac{R^3}{3|x|} + \frac{(1+R^3)^{2/3}}{2} & \text{if} \quad R \leqslant |x| < (1+R^3)^{1/3}, \\ \frac{1}{3|x|} & \text{if} \quad |x| \geqslant (1+R^3)^{1/3}. \end{cases}$$

Thus the difference of the repulsive terms on two boundary components is given by

$$\frac{1}{4\pi}(\Lambda_{\mathrm{rep}}(x_2) - \Lambda_{\mathrm{rep}}(x_1)) = \frac{1}{3(1+R^3)^{1/3}} - \frac{(1+R^3)^{2/3} - R^2}{2}; \tag{4.5}$$

hence,

$$\lim_{R \to 0} \Lambda_{\mathrm{rep}}(x_2) - \Lambda_{\mathrm{rep}}(x_1) = -\frac{4\pi}{6} < 0.$$

Now referring back to (4.5) and taking the derivative of the right-hand side with respect to R we get that

$$\Lambda_{\mathrm{rep}}(x_2) - \Lambda_{\mathrm{rep}}(x_1)$$

is an increasing function of R for all $R > 0$. Moreover, $\Lambda_{\mathrm{rep}}(x_2) - \Lambda_{\mathrm{rep}}(x_1) < 0$ for all R and approaches zero as R increases.

Now, returning to the difference of attractive terms in (4.4) we see that by the choice of x_1 and x_2, using spherical coordinates, it can be expressed as

$$\Lambda_{\mathrm{attr}}(x_2) - \Lambda_{\mathrm{attr}}(x_1) = 2\pi \int_0^\pi \int_R^{(1+R^3)^{1/3}} \Bigg(\frac{((1+R^3)^{2/3} + r^2 - 2r(1+R^3)^{1/3}\cos\phi)^{q/2}}{q} - \frac{(R^2 + r^2 - 2rR\cos\phi)^{q/2}}{q} \Bigg) r^2 \sin\phi \, d\phi dr,$$

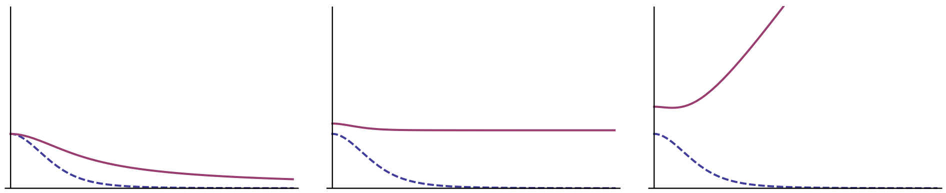


FIG. 4.1. *The graphs of $\Lambda_{attr}(x_2) - \Lambda_{attr}(x_1)$ (solid) and $-[\Lambda_{rep}(x_2) - \Lambda_{rep}(x_1)]$ (dashed) as a function of R with $q = 2, 3, 4$, respectively.*

which is a function of the single variable R.

First, note that, the above difference is an increasing function of q. Furthermore, calculating it explicitly for increasing values of $q > 2$ we see that $\Lambda_{\mathrm{attr}}(x_2) - \Lambda_{\mathrm{attr}}(x_1)$ does not intersect with $-[\Lambda_{\mathrm{rep}}(x_2) - \Lambda_{\mathrm{rep}}(x_1)]$ for any value of $R > 0$. However, this contradicts the equation (4.4); hence, there does not exist a positive number $R > 0$ such that the spherical annulus

$$A := \{x \in \mathbb{R}^3 \colon R < |x| < (1 + R^3)^{1/3}\}$$

is a critical point of the problem (4.1) in $\mathbb{R}^3$.

5. Existence of Minimizers for Positive-Power Repulsion. As we have already mentioned, the character of the interaction potential K is very different when p is positive. In this regime K does not have a singularity at zero and hence concentrations of densities on sets of dimension less than N is possible. We thus consider the class of admissible densities to be radially symmetric probability measures. A radially symmetric probability measures μ is defined via its Fourier transform $\hat{\mu} \in \mathcal{P}([0, \infty))$ as

$$\int_{r_1}^{r_2} d\hat{\mu}(r) = \int_{r_1 < |x| < r_2} d\mu(x) \quad \text{and} \quad \int_{0}^{r_2} d\hat{\mu}(r) = \int_{0 \leqslant |x| < r_2} d\mu(x)$$

for all $0 < r_1 < r_2$. Also, using the notation in [5], $\mu \in \mathcal{P}^r(\mathbb{R}^N)$ can be written as

$$\mu = \int_0^\infty \delta_r d\hat{\mu}(r),$$

where δ_r denotes a probability measure which is uniformly distributed on the spherical shell $\partial B(0, r)$. For $q > p > 0$, the variational problem is to

$$\text{(5.1)} \qquad \text{minimize} \quad E(\mu) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(\frac{1}{q}|x - y|^q - \frac{1}{p}|x - y|^p \right) d\mu(x) d\mu(y),$$

over radially symmetric probability measures $\mu \in \mathcal{P}^r(\mathbb{R}^N)$.

The main tool in establishing the existence of minimizers for (5.1) will be, again, the concentration-compactness principle. In particular, we will cite [49, Section 4.3] for the following version of Lemma 2.2.

LEMMA 5.1 (Concentration-compactness lemma for measures). *Let $\{\mu_n\}_{n \in \mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}^N$. Then there exists a subsequence $\{\mu_{n_k}\}_{k \in \mathbb{N}}$ satisfying one of the three following possibilities:*

(i) *(tightness up to translation) there exists* $y_k \in \mathbb{R}^N$ *such that for all* $\epsilon > 0$ *there exists* $R > 0$ *with the property that*

$$\int_{B(y_k,R)} d\mu_{n_k}(x) \geqslant 1 - \epsilon \qquad \text{for all } k.$$

(ii) *(vanishing)* $\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \mu_{n_k}(x) = 0$, *for all* $R > 0$;

(iii) *(dichotomy) there exists* $\alpha \in (0,1)$ *such that for all* $\epsilon > 0$, *there exist a number* $R > 0$ *and a sequence* $\{y_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$ *with the following property:*

Given $R' > R$ *there are non-negative measures* μ_k^1 *and* μ_k^2 *such that*

$$0 \leqslant \mu_k^1 + \mu_k^2 \leqslant \mu_{n_k},$$

$$\operatorname{supp}(\mu_k^1) \subset B(y_k, R),\ \operatorname{supp}(\mu_k^2) \subset \mathbb{R}^N \setminus B(y_k, R'),$$

$$\limsup_{k\to\infty} \left(\left| \alpha - \int_{\mathbb{R}^N} d\mu_k^1(x) \right| + \left| (1-\alpha) - \int_{\mathbb{R}^N} d\mu_k^2(x) \right| \right) \leqslant \epsilon.$$

Theorem 5.2. *There exists a minimum of the problem* (5.1) *when* $q > p > 0$.

Proof. Since for any $\mu \in \mathcal{P}^r(\mathbb{R}^N)$ we have that $\int_{\mathbb{R}^N} d\mu(x) = 1$. Also when $q > p > 0$ the interaction potential $K(|x|) \geqslant 1/q - 1/p$. Thus

$$\inf\{E(\mu) \colon \mu \in \mathcal{P}^r(\mathbb{R}^N)\} > -\infty.$$

Since $K(|x|) \leqslant 0$ when $0 \leqslant |x| \leqslant (q/p)^{1/(q-p)}$ we see that the above infimum is negative.

Now let $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}^r(\mathbb{R}^N)$ be a minimizing sequence of the problem (5.1). Then by the concentration compactness lemma for measures there is a subsequence of $\{\mu_n\}_{n\in\mathbb{N}}$, still indexed by n, which satisfies one of the three possibilities in Lemma 5.1.

Using the radial symmetry of the μ_n and the indefinite growth of the interaction potential K as $|x| \to \infty$ (see Figure 1.1(d)) we conclude as in the proof of Case 1 of Theorem 2.5 that "vanishing" and "dichotomy" do not occur since these two possibilities contradict the fact that $\{\mu_n\}_{n\in\mathbb{N}}$ is a minimizing sequence.

Therefore "tightness up to translation" is the only possibility. As in the case of Theorem 2.5 for $q > 0$, the centers y_k associated with the translation must be bounded. Hence we may assume the sequence of probability measures, $\{\mu_n\}_{n\in\mathbb{N}}$ is tight. Then, by the Prokhorov's theorem (cf. [15, Theorem 4.1]) there exists a further subsequence of $\{\mu_n\}_{n\in\mathbb{N}}$ which we still index by n, and a measure $\mu_0 \in \mathcal{P}^r(\mathbb{R}^N)$ such that

$$\mu_n \overset{\text{weak}-*}{\rightharpoonup} \mu_0$$

in $\mathcal{P}^r(\mathbb{R}^N)$ as $n \to \infty$.

To show weak lower semi-continuity of $E(\mu)$ we will proceed as in the proof of Theorem 2.5, paying attention to the fact that in this regime K becomes negative.

Since the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is tight, for any given $\epsilon > 0$ there exists $r > 0$ such that

$$\int_{B(0,r)} d\mu_0(x) \geqslant 1 - \epsilon.$$

Choose $R := \max\{(q/p)^{1/(q-p)} + 1, r\}$, and define

$$\tilde{G}_n(x) := \int_{B(0,R)} K(x,y)\, d\mu_n(y) \quad \text{and} \quad \tilde{G}_0(x) := \int_{B(0,R)} K(x,y)\, d\mu_0(y).$$

As $K(x,y)$ is continuous in x on $B(0,R)$, the sequence of functions $\tilde{G}_n$ converges uniformly to $\tilde{G}$ on $C(\overline{B(0,R)})$ by the Arzela–Ascoli theorem, using the compactness of the closed ball and the equicontinuity of $\tilde{G}_n$. Then, by the uniform convergence of $\tilde{G}_n$ and the weak-$*$ convergence of μ_n we get that

$$\liminf_{n\to\infty} \int_{B(0,R)}\int_{B(0,R)} K(x,y)\, d\mu_n(x) d\mu_n(y) = \int_{B(0,R)}\int_{B(0,R)} K(x,y)\, d\mu_0(x) d\mu_0(y).$$

Since $E(\mu_0) < \infty$, again, the energy on $\mathbb{R}^N \setminus B(0,R)$ is controlled and the above equality, as in (2.13), yields

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)\, d\mu_0(x) d\mu_0(y) &\leqslant \int_{B(0,R)}\int_{B(0,R)} K(x,y)\, d\mu_0(x) d\mu_0(y) + C\epsilon \\ &\leqslant \liminf_{n\to\infty} \int_{B(0,R)}\int_{B(0,R)} K(x,y)\, d\mu_n(x) d\mu_n(y) + C\epsilon \\ &\leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x,y)\, d\mu_n(x) d\mu_n(y) + C\epsilon.\end{aligned}$$

Sending ϵ to 0 gives the weak lower semi-continuity of E; hence, $\mu_0 \in \mathcal{P}^r(\mathbb{R}^N)$ is a solution of the minimization problem (5.1). $\square$

REFERENCES

[1] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[2] J.F.G. Auchmuty. Existence of axisymmetric equilibrium figures. *Arch. Rational Mech. Anal.*, 65(3):249–261, 1977.

[3] J.F.G. Auchmuty and R. Beals. Variational solutions of some nonlinear free boundary problems. *Arch. Rational Mech. Anal.*, 43:255–271, 1971.

[4] D. Balagué, J.A. Carrillo, T. Laurent, and G. Raoul. Dimensionality of local minimizers of the interaction energy. to appear in Arch. Rat. Mech. Anal.

[5] D. Balagué, J.A. Carrillo, T. Laurent, and G. Raoul. Nonlocal interactions by repulsive-attractive potentials: radial ins/stability. *Physica D*, 2013. DOI: 10.1016/j.physd.2012.10.002.

[6] J. Bedrossian. Global minimizers for free energies of subcritical aggregation equations with degenerate diffusion. *Appl. Math. Lett.*, 24(11):1927–1932, 2011.

[7] J. Bedrossian. Intermediate asymptotics for critical and supercritical aggregation equations and Patlak-Keller-Segel models. *Commun. Math. Sci.*, 9(4):1143–1161, 2011.

[8] J. Bedrossian, N. Rodríguez, and A.L. Bertozzi. Local and global well-posedness for aggregation equations and Patlak-Keller-Segel models with degenerate diffusion. *Nonlinearity*, 24:1683–1714, 2011.

[9] D. Benedetto, E. Caglioti, and M. Pulvirenti. A kinetic equation for granular media. *RAIRO Modél. Math. Anal. Numér.*, 31(5):615–641, 1997.

[10] A.J. Bernoff and C.M. Topaz. A primer of swarm equilibria. *SIAM J. Appl. Dyn. Syst.*, 10(1):212–250, 2011.

[11] A.L. Bertozzi, J.A. Carrillo, and T. Laurent. Blow-up in multidimensional aggregation equations with mildly singular interaction kernels. *Nonlinearity*, 22(3):683–710, 2009.

[12] A.L. Bertozzi and T. Laurent. Finite-time blow-up of solutions of an aggregation equation in $\mathbf{R}^n$. *Comm. Math. Phys.*, 274(3):717–735, 2007.

[13] A.L. Bertozzi, T. Laurent, and L. Flavien. Aggregation and Spreading via the Newtonian Potential: The Dynamics of Patch Solutions. *Math. Models Methods Appl. Sci.*, 22(Supp. 1):1140005, 2012.
[14] A.L. Bertozzi and D. Slepčev. Existence and uniqueness of solutions to an aggregation equation with degenerate diffusion. *Commun. Pure Appl. Anal.*, 9(6):1617–1637, 2010.
[15] P. Billingsley. *Weak convergence of measures: Applications in probability.* Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1971. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 5.
[16] A. Blanchet, J.A. Carrillo, and P. Laurençot. Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions. *Calc. Var. Partial Differential Equations*, 35(2):133–168, 2009.
[17] M. Bodnar and J.J.L. Velasquez. Derivation of macroscopic equations for individual cell-based models: a formal approach. *Math. Meth. Appl. Sci. (M3AS)*, 28:1757–1779, 2005.
[18] M. Bodnar and J.J.L. Velazquez. An integro-differential equation arising as a limit of individual cell-based models. *J. Differential Equations*, 222(2):341–380, 2006.
[19] M. Burger, V. Capasso, and D. Morale. On an aggregation model with long and short range interactions. *Nonlinear Anal. Real World Appl.*, 8-3:939–958, 2007.
[20] M. Burger and M. DiFrancesco. Large time behavior of nonlocal aggregation models with nonlinear diffusion. *Netw. Heterog. Media*, 3-4:749–785, 2008.
[21] M. Burger, M. DiFrancesco, and M. Franek. Stationary states of quadratic diffusion equations with long-range attraction. *Comm. Math. Sci.*, page in press, 2013.
[22] J. A. Carrillo, M. DiFrancesco, A. Figalli, T. Laurent, and D. Slepčev. Global-in-time weak measure solutions and finite-time aggregation for nonlocal interaction equations. *Duke Math. J.*, 156(2):229–271, 2011.
[23] J. A. Carrillo, M. DiFrancesco, A. Figalli, T. Laurent, and D. Slepčev. Confinement in nonlocal interaction equations. *Nonlinear Anal.*, 75(2):550–558, 2012.
[24] R. Choksi and M.A. Peletier. Small volume fraction limit of the diblock copolymer problem: I. sharp interface functional. *SIAM J. Math. Anal.*, 42-3:1334–1370, 2010.
[25] R. Choksi and M.A. Peletier. Small volume fraction limit of the diblock copolymer problem: Ii. diffuse interface functional. *SIAM J. Math. Anal.*, 43-2:739–763, 2011.
[26] R. Choksi and P. Sternberg. On the first and second variations of a nonlocal isoperimetric problem. *J. Reine Angew. Math.*, 611:75–108, 2007.
[27] K. Fellner and G. Raoul. Stable stationary states of non-local interaction equations. *Math. Models Methods Appl. Sci.*, 20(12):2267–2291, 2010.
[28] R. C. Fetecau and Y. Huang. Equilibria of biological aggregations with nonlocal repulsiveattractive interactions. *Physica D*, 2013. DOI: 10.1016/j.physd.2012.11.004.
[29] R. C. Fetecau, Y. Huang, and T. Kolokolnikov. Swarm dynamics and equilibria for a nonlocal aggregation model. *Nonlinearity*, 24(10):2681–2716, 2011.
[30] J.M. Haile. *Molecular Dynamics Simulation: Elementary Methods.* John Wiley and Sons, Inc., New York, 1992.
[31] D.D. Holm and V. Putkaradze. Aggregation of finite-size particles with variable mobility. *Phys Rev Lett.*, 95:226106, 2005.
[32] D.D. Holm and V. Putkaradze. Formation of clumps and patches in selfaggregation of finite-size particles. *Physica D.*, 220(2):183–196, 2006.
[33] Y. Huang and A.L. Bertozzi. Self-similar blowup solutions to an aggregation equation in $\mathbb{R}^n$. *SIAM J. Appl. Math.*, 70(7):2582–2603, 2010.
[34] V. Julin. Isoperimetric problem with a coulombic repulsive term. *Preprint*, 2012.
[35] H. Knuepfer and C.B. Muratov. On an isoperimetric problem with a competing non-local term. ii. the general case. *Comm. Pure Appl. Math.*, in press.
[36] H. Knuepfer and C.B. Muratov. On an isoperimetric problem with a competing non-local term. i. the planar case. *Comm. Pure Appl. Math.*, 66-7:1129–1162, 2013.
[37] T. Kolokolnikov, H. Sun, D. Uminsky, and A.L. Bertozzi. A theory of complex patterns arising from 2D particle interactions. *Phys. Rev. E, Rapid Communications*, 84:015203(R), 2011.
[38] C. Le Bris. Some results on the Thomas-Fermi-Dirac-von Weizsäcker model. *Diff. Int. Equ.*, 6:337353, 1993.
[39] C. Le Bris and P.-L. Lions. From atoms to crystals: a mathematical journey. *Bull. Amer. Math. Soc.*, 42, 2005.
[40] A.J. Leverentz, C.M. Topaz, and A.J. Bernoff. Asymptotic dynamics of attractive-repulsive swarms. *SIAM J. Appl. Dyn. Syst.*, 8(3):880–908, 2009.
[41] E.H. Lieb. Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *Studies in Appl. Math.*, 57:93–105, 1977.
[42] E.H. Lieb. Thomas-Fermi and related theories of atoms and molecules. *Rev. Mod. Phys.*,

52:603641, 1981.

[43] E.H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math. (2)*, 118(2):349–374, 1983.

[44] E.H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[45] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 1(2):109–145, 1984.

[46] J. Lu and F. Otto. Nonexistence of a minimizer for the Thomas-Fermi-Dirac-von Weizsäcker model. *Comm. Pure Appl. Math.*, in press.

[47] A. Mogilner and L. Edelstein-Keshet. A non-local model for a swarm. *J. Math. Biol.*, 38:534–570, 1999.

[48] H. Poincaré. Sur une théorème de M. Liapunoff rélatif a l'équilibre d'une masse fluide. *Comptes Rendus de L'Academie des Sciences*, 104:622–625, 1887.

[49] M. Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, third edition, 2000. Applications to nonlinear partial differential equations and Hamiltonian systems.

[50] C.M. Topaz, A.L. Bertozzi, and M.A. Lewis. A nonlocal continuum model for biological aggregation. *Bull. Math. Bio.*, 68:1601–1623, 2006.

[51] G. Toscani. One-dimensional kinetic models of granular flows. *M2AN Math. Model. Numer. Anal.*, 34(6):1277–1291, 2000.

[52] J. von Brecht, D. Uminsky, T. Kolokolnikov, and A.L. Bertozzi. Predicting pattern formation in particle interactions. *Math. Models Methods Appl. Sci.*, 22(Supp. 1):1140002, 2012.