Question 2

Without loss of generality, we assume that the n independent tosses are numbered 1 to n. Consider the outcome of first toss, denoted by X_1. If the first toss comes up heads, the total number of heads is even iff the number of heads in the remaining $n-1$ tosses is odd. Likewise, if the first toss comes up tails, the total number of heads is even iff the number of heads in the remaining $n-1$ tosses is even. Applying the law of total probability, we can write

$$\begin{aligned} P(\text{even \#heads in n tosses}) &= P(\text{odd \#heads in rem. n - 1 tosses} \mid X_1 = H) \cdot P(X_1 = H) \\ &+ P(\text{even \#heads in rem. n - 1 tosses} \mid X_1 = T) \cdot P(X_1 = T) \end{aligned}$$

Using the independence of remaining $n-1$ tosses from the first toss, we get

$$\begin{aligned} P(\text{even \#heads in n tosses}) &= P(\text{odd \#heads in rem. n - 1 tosses}) \cdot P(X_1 = H) \\ &+ P(\text{even \#heads in rem. n - 1 tosses}) \cdot P(X_1 = T) \end{aligned}$$

Rewriting this in terms of the given notation, we get

$$q_n = (1 - q_{n-1})p + q_{n-1}(1-p)$$

Applying some simplifications, we arrive at the recurrence relation

$$\begin{aligned} &= (1-2p)q_{n-1} + p \\ &= (1-2p)q_{n-1} + (-2p)\left(\frac{-1}{2}\right) - \frac{1}{2} + \frac{1}{2} \\ &= (1-2p)q_{n-1} + (1-2p)\left(\frac{-1}{2}\right) + \frac{1}{2} \\ q_n - \frac{1}{2} &= (1-2p)\left(q_{n-1} - \frac{1}{2}\right) \qquad (1) \end{aligned}$$

We will show by induction that

$$q_n - \frac{1}{2} = (1-2p)^i \left(q_{n-i} - \frac{1}{2}\right) \qquad (2)$$

We know from (1), this holds for $i = 1$. Assuming it holds for some $i > 1$, we will show that it holds for $i + 1$. By assumption,

$$q_n - \frac{1}{2} = (1-2p)^i \left(q_{n-i} - \frac{1}{2}\right)$$

Using (1) for $(q_{n-i} - \frac{1}{2})$, we get

$$= (1-2p)^{i+1} \left(q_{n-i-1} - \frac{1}{2}\right)$$

This resembles the exact form as (2) for $i + 1$, completing the proof by induction. Substituting $i = n$ in (2), we get

$$q_n - \frac{1}{2} = (1 - 2p)^n \left(q_0 - \frac{1}{2} \right)$$

It is trivial to see that the number of heads when no coin is tossed is zero, which is even. Thus, $q_0 = 1$.

$$q_n = \frac{1}{2} + (1 - 2p)^n \left(1 - \frac{1}{2} \right)$$
$$q_n = \frac{1 + (1 - 2p)^n}{2}$$

$\square$