CS1.404: Assignment 3

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KKT Condition

For the convex optimization problem,

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})$$
$$\text{s.t. } h_j(\mathbf{x}) \le 0 \qquad \forall j \in \{1,2,\ldots,l\}$$

$\mathbf{x}^*$ is an optimal solution, if there exists multipliers $\{\lambda_j\}_{j=1}^{l}$, such that the following equations hold.

$$\begin{aligned}
\nabla f(\mathbf{x}^*) + \sum_{j=1}^{l} \lambda_j \nabla h_j(\mathbf{x}^*) &= 0 \\
\lambda_j h_j(\mathbf{x}^*) &= 0 \qquad \forall j \in \{1,2,\ldots,l\} \\
\lambda_j &\ge 0 \qquad \forall j \in \{1,2,\ldots,l\} \qquad (1)
\end{aligned}$$

1 Trid Function

$$f(\mathbf{x}) = \sum_{i=1}^{d}(x_i-1)^2 - \sum_{i=2}^{d} x_{i-1}x_i$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 - x_2 - 2 \\ 2x_2 - x_1 - x_3 - 2 \\ 2x_3 - x_2 - x_4 - 2 \\ \vdots \\ 2x_{d-1} - x_{d-2} - x_d - 2 \\ 2x_d - x_{d-1} - 2 \end{bmatrix}$$

1.1 Test Case 0

$$h_1(\mathbf{x}) = x_1^2 - 2x_2$$
$$\nabla h_1(\mathbf{x}) = \begin{bmatrix} 2x_1 & -2 \end{bmatrix}^T$$

Substituting in (1), we get

$$2x_1 - x_2 - 2 + 2\lambda_1 x_1 = 0$$

$$2x_2 - x_1 - 2 - 2\lambda_1 = 0$$
$$\lambda_1(x_1^2 - 2x_2) = 0$$

1.2 Test Case 1

$$h_1(\mathbf{x}) = x_1^2 - x_2^2 + 1$$
$$\nabla h_1(\mathbf{x}) = \begin{bmatrix} 2x_1 & -2x_2 \end{bmatrix}^T$$

Substituting in (1), we get

$$2x_1 - x_2 - 2 + 2\lambda_1 x_1 = 0$$
$$2x_2 - x_1 - 2 - 2\lambda_1 x_2 = 0$$
$$\lambda_1(x_1^2 - x_2^2 + 1) = 0$$

1.3 Test Case 2

$$h_1(\mathbf{x}) = -1 - x_1$$
$$h_2(\mathbf{x}) = x_1 - 1$$
$$h_3(\mathbf{x}) = -1 - x_2$$
$$h_4(\mathbf{x}) = x_2 - 1$$
$$\nabla h_1(\mathbf{x}) = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$$
$$\nabla h_2(\mathbf{x}) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$
$$\nabla h_3(\mathbf{x}) = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$$
$$\nabla h_4(\mathbf{x}) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$$

Substituting in (1), we get

$$2x_1 - x_2 - 2 - \lambda_1 + \lambda_2 = 0$$
$$2x_2 - x_1 - 2 - \lambda_3 + \lambda_4 = 0$$
$$\lambda_1(x_1 + 1) = 0$$
$$\lambda_2(x_1 - 1) = 0$$
$$\lambda_3(x_2 + 1) = 0$$
$$\lambda_4(x_2 - 1) = 0$$

1.4 Test Case 3

$$h_1(\mathbf{x}) = -x_1$$
$$h_2(\mathbf{x}) = x_1 - 3$$
$$h_3(\mathbf{x}) = -x_2$$

$$h_4(\mathbf{x}) = x_2 - 3$$
$$\nabla h_1(\mathbf{x}) = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$$
$$\nabla h_2(\mathbf{x}) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$
$$\nabla h_3(\mathbf{x}) = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$$
$$\nabla h_4(\mathbf{x}) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$$

Substituting in (1), we get

$$2x_1 - x_2 - 2 - \lambda_1 + \lambda_2 = 0$$
$$2x_2 - x_1 - 2 - \lambda_3 + \lambda_4 = 0$$
$$\lambda_1 x_1 = 0$$
$$\lambda_2(x_1 - 3) = 0$$
$$\lambda_3 x_2 = 0$$
$$\lambda_4(x_2 - 3) = 0$$

1.5 Test Case 4

$$h_1(\mathbf{x}) = 3 - x_1$$
$$h_2(\mathbf{x}) = x_1 - 4$$
$$h_3(\mathbf{x}) = 3 - x_2$$
$$h_4(\mathbf{x}) = x_2 - 4$$
$$\nabla h_1(\mathbf{x}) = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$$
$$\nabla h_2(\mathbf{x}) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$
$$\nabla h_3(\mathbf{x}) = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$$
$$\nabla h_4(\mathbf{x}) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$$

Substituting in (1), we get

$$2x_1 - x_2 - 2 - \lambda_1 + \lambda_2 = 0$$
$$2x_2 - x_1 - 2 - \lambda_3 + \lambda_4 = 0$$
$$\lambda_1(x_1 - 3) = 0$$
$$\lambda_2(x_1 - 4) = 0$$
$$\lambda_3(x_2 - 3) = 0$$
$$\lambda_4(x_2 - 4) = 0$$

2 Matyas Function

$$f(\mathbf{x}) = 0.26(x_1^2 + x_2^2) - 0.48x_1x_2$$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 0.52x_1 - 0.48x_2 \\ 0.52x_2 - 0.48x_1 \end{bmatrix}$$

2.1 Test Case 5

$$\begin{aligned} h_1(\mathbf{x}) &= -x_1 \\ h_2(\mathbf{x}) &= x_1 - 1 \\ h_3(\mathbf{x}) &= -x_2 \\ h_4(\mathbf{x}) &= x_2 - 1 \\ \nabla h_1(\mathbf{x}) &= \begin{bmatrix} -1 & 0 \end{bmatrix}^T \\ \nabla h_2(\mathbf{x}) &= \begin{bmatrix} 1 & 0 \end{bmatrix}^T \\ \nabla h_3(\mathbf{x}) &= \begin{bmatrix} 0 & -1 \end{bmatrix}^T \\ \nabla h_4(\mathbf{x}) &= \begin{bmatrix} 0 & 1 \end{bmatrix}^T \end{aligned}$$

Substituting in (1), we get

$$\begin{aligned} 0.52x_1 - 0.48x_2 - \lambda_1 + \lambda_2 &= 0 \\ 0.52x_2 - 0.48x_1 - \lambda_3 + \lambda_4 &= 0 \\ \lambda_1 x_1 &= 0 \\ \lambda_2(x_1 - 1) &= 0 \\ \lambda_3 x_2 &= 0 \\ \lambda_4(x_2 - 1) &= 0 \end{aligned}$$

2.2 Test Case 6

$$\begin{aligned} h_1(\mathbf{x}) &= 1 - x_1 \\ h_2(\mathbf{x}) &= x_1 - 2 \\ h_3(\mathbf{x}) &= 1 - x_2 \\ h_4(\mathbf{x}) &= x_2 - 2 \\ \nabla h_1(\mathbf{x}) &= \begin{bmatrix} -1 & 0 \end{bmatrix}^T \\ \nabla h_2(\mathbf{x}) &= \begin{bmatrix} 1 & 0 \end{bmatrix}^T \\ \nabla h_3(\mathbf{x}) &= \begin{bmatrix} 0 & -1 \end{bmatrix}^T \\ \nabla h_4(\mathbf{x}) &= \begin{bmatrix} 0 & 1 \end{bmatrix}^T \end{aligned}$$

Substituting in (1), we get

$$\begin{aligned} 0.52x_1 - 0.48x_2 - \lambda_1 + \lambda_2 &= 0 \\ 0.52x_2 - 0.48x_1 - \lambda_3 + \lambda_4 &= 0 \\ \lambda_1(x_1 - 1) &= 0 \end{aligned}$$

$$\lambda_2(x_1 - 2) = 0$$
$$\lambda_3(x_2 - 1) = 0$$
$$\lambda_4(x_2 - 2) = 0$$

2.3 Test Case 7

$$h_1(\mathbf{x}) = -1 - x_1$$
$$h_2(\mathbf{x}) = x_1 + 0.5$$
$$h_3(\mathbf{x}) = -0.5 - x_2$$
$$h_4(\mathbf{x}) = x_2 - 0.5$$
$$\nabla h_1(\mathbf{x}) = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$$
$$\nabla h_2(\mathbf{x}) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$
$$\nabla h_3(\mathbf{x}) = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$$
$$\nabla h_4(\mathbf{x}) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$$

Substituting in (1), we get

$$0.52x_1 - 0.48x_2 - \lambda_1 + \lambda_2 = 0$$
$$0.52x_2 - 0.48x_1 - \lambda_3 + \lambda_4 = 0$$
$$\lambda_1(x_1 + 1) = 0$$
$$\lambda_2(x_1 + 0.5) = 0$$
$$\lambda_3(x_2 + 0.5) = 0$$
$$\lambda_4(x_2 - 0.5) = 0$$