Open Queueing Networks

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March 1, 2024

Queueing Networks

- A queueing network is made up of servers.
- Open Networks: An open queueing network has external arrivals and departures.

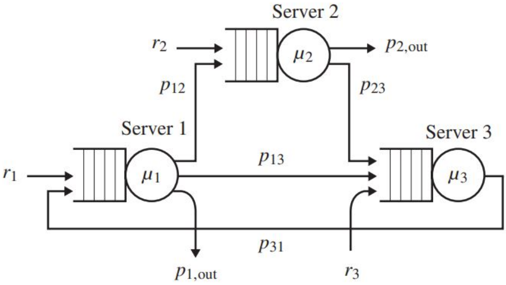
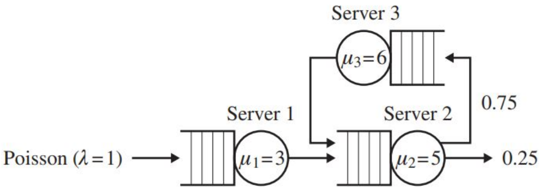


Figure: An open queueing network

Jackson Network

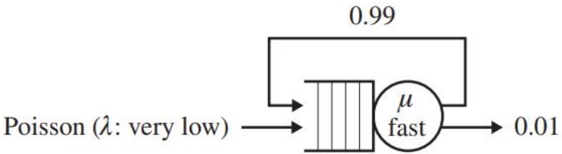
- General form of queueing network
  - Unbounded queues
  - FCFS (First Come First Serve) service
  - Arrivals from outside network $\sim$ Poisson(r_i)
  - Service rates $\sim$ exp(μ_i)
  - Probabilistic routing - P_{ij}
  - States: $(n_1, n_2, \ldots, n_k)$

$$\lambda_i = r_i + \sum_j \lambda_j P_{ji} \quad .$$



Independency?

- Consider a single server network with properties:
  - Low arrival rate
  - High service rate
  - High P_{ii}



Incremental increments property unsatisfied

Solving the Jackson

Rate of jobs leaving the state = Rate entering the state

$$\pi_{n_1,n_2,\ldots,n_k} \cdot \left[\sum_{i=1}^{k} r_i + \sum_{i=1}^{k} \mu_i (1 - P_{ii})\right] =$$

$$\underbrace{\sum_{i=1}^{k} \pi_{n_1,\ldots,n_i-1,\ldots,n_k} \cdot r_i}_{\text{outside arrival}} + \underbrace{\sum_{i=1}^{k} \pi_{n_1,\ldots,n_i+1,\ldots,n_k} \cdot \mu_i P_{i,\text{out}}}_{\text{departure to outside}}$$

$$+ \underbrace{\sum_{i=1}^{k} \sum_{j \neq i} \pi_{n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_k} \cdot \mu_j P_{ji}}_{\text{internal transition from server } j \text{ to server } i, j \neq i} .$$

Solving the Jackson(contd.)

- Very hard to solve
- Is there no better method?
- Local Balance Approach
  - No exact algorithm
  - Part of setting up equations is the "art"
  - Break both sides of the equation into k+1 matching components and then equate
  - First component - Outside arrivals/departures

Rate leaving state due to outside arrival = Rate entering state due to outside departure

Solving the Jackson(contd.)

$$A = A'$$

$$\sum_{i=1}^{k} \pi_{n_1,\ldots,n_i,\ldots,n_k} r_i = \sum_{i=1}^{k} \pi_{n_1,\ldots,n_i+1,\ldots,n_k} \mu_i P_{i,\text{out}}$$

- Can we make a guess for $\pi_{n_1,\ldots,n_k}$ such that it satisfies the above equality?
- Observe that $\pi_{n_1,\ldots,n_i,\ldots,n_k}$ term in A and the $\pi_{n_1,\ldots,n_i+1,\ldots,n_k}$ term in A' only differ in the n_i spot
- Let

$$\pi_{n_1,\ldots,n_i,\ldots,n_k} \cdot c_i = \pi_{n_1,\ldots,n_i+1,\ldots,n_k}$$

Solving the Jackson(contd.)

$$\sum_{i=1}^{k} \pi_{n_1,\ldots,n_i,\ldots,n_k} r_i = \sum_{i=1}^{k} \pi_{n_1,\ldots,n_i+1,\ldots,n_k} \mu_i P_{i,\mathrm{out}}$$

$$\sum_{i=1}^{k} \pi_{n_1,\ldots,n_i,\ldots,n_k} r_i = \sum_{i=1}^{k} \pi_{n_1,\ldots,n_k} \cdot c_i \cdot \mu_i P_{i,\mathrm{out}}$$

$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} (c_i \cdot \mu_i) P_{i,\mathrm{out}}.$$

Solving the Jackson(contd.)

Observe that if:

$$c_i \cdot \mu_i = \lambda_i$$

Then:

$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} \lambda_i P_{i,\text{out}}.$$

$$c_i = \frac{\lambda_i}{\mu_i} = \rho_i$$

Now,

$$\pi_{n_1}, \dots, \pi_{n_i}, \dots, \pi_{n_k} \cdot \rho_i = \pi_{n_1}, \dots, \pi_{n_i+1}, \dots, \pi_{n_k} \quad \forall i$$

Solving the Jackson(contd.)

Hence, it is reasonable to assume that:

$$\pi_{n_1}, \ldots, \pi_{n_i}, \ldots, \pi_{n_k} = C\rho^1_{n_1} \ldots \rho^k_{n_k}$$

where C is the normalizing constant. Now we will solve for:

$$B_i = B_i'$$

Here, B_i is the rate of rate of transitions leaving state $(n_1, n_2, \ldots, n_k)$ due to a departure from server i. Hence,

$$B_i = \pi_{n1}, \ldots, \pi_{nk} \cdot \mu_i (1 - P_{ii}).$$

Solving the Jackson(contd.)

And B_i' is the rate of transitions entering $(n_1, n_2, \ldots, n_k)$ due to an arrival at server i. Hence,

$$B_i' = \underbrace{\sum_{j \text{ s.t. } j \neq i} \pi_{n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_k} \mu_j P_{ji}}_{\text{internal transition from server } j \text{ to server } i (j \neq i)} + \underbrace{\pi_{n_1,\ldots,n_i-1,\ldots,n_k} r_i}_{\text{outside arrival}}.$$

Now, we will put:

$$\pi_{n_1}, \ldots, \pi_{n_i}, \ldots, \pi_{n_k} = C \rho^1_{n_1} \ldots \rho^k_{n_k}$$

Solving the Jackson(contd.)

$$B_i = B_i'$$

$$C\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k}\mu_i(1-P_{ii}) = \sum_{j\neq i} C\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k}\left(\frac{\rho_j}{\rho_i}\right)\mu_j P_{ji} + C\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_k^{n_k}\left(\frac{1}{\rho_i}\right)r_i$$

$$\mu_i(1-P_{ii}) = \sum_{j\neq i}\frac{\rho_j\mu_j P_{ji}}{\rho_i} + \frac{r_i}{\rho_i}$$

$$\rho_i\mu_i(1-P_{ii}) = \sum_{j\neq i}\rho_j\mu_j P_{ji} + r_i$$

$$\lambda_i(1-P_{ii}) = \sum_{j\neq i}\lambda_j P_{ji} + r_i$$

Solving the Jackson(contd.)

Lastly, we need to find the normalizing constant C:

$$\sum_{n_1,\ldots,n_k} \pi_{n_1,\ldots,n_k} = 1$$

$$C \sum_{n_1,\ldots,n_k} \rho_1^{n_1} \cdots \rho_k^{n_k} = 1$$

$$C \left(\sum_{n_1} \rho_1^{n_1} \right) \left(\sum_{n_2} \rho_2^{n_2} \right) \cdots \left(\sum_{n_k} \rho_k^{n_k} \right) = 1$$

$$C \left(\frac{1}{1-\rho_1} \right) \left(\frac{1}{1-\rho_2} \right) \cdots \left(\frac{1}{1-\rho_k} \right) = 1$$

Solving the Jackson(contd.)

Hence,

$$C = (1-\rho_1)(1-\rho_2)\cdots(1-\rho_k).$$

As a result,

$$\pi_{n_1,\ldots,n_k} = \rho_1^{n_1}(1-\rho_1)\rho_2^{n_2}(1-\rho_2)\cdots\rho_k^{n_k}(1-\rho_k).$$

Now, what about the jobs at server i? For server 1:

$$P\{n_1 \text{ jobs at server } 1\} = \sum_{n_2,\ldots,n_k} \pi_{n_1,\ldots,n_k}$$

Solving the Jackson(contd.)

$$= \sum_{n_2,\ldots,n_k} \rho_1^{n_1}(1-\rho_1)\rho_2^{n_2}(1-\rho_2)\cdots\rho_k^{n_k}(1-\rho_k) = \rho_1^{n_1}(1-\rho_1).$$

Likewise,

$$P\{n_i \text{ jobs at server } i\} = \rho_i^{n_i}(1-\rho_i).$$

Which means that all servers behave like M/M/1 queues in terms of their stationary queue length distributions even though their arrival processes are not usually Poisson.