Quantum Channels: Transition from Classical to Quantum Channels

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Introduction

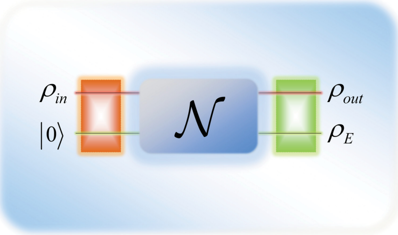
In this presentation, we will be discussing quantum channels and how they developed from classical channels. But before that, we need to define what quantum channels are and what they operate on. Quantum channels operate on Density Matrices, which are mathematical constructs that hold all information about a quantum state. They can be defined as:

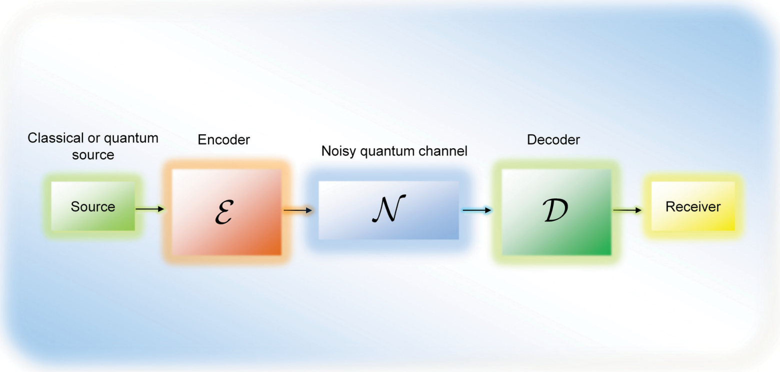
$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{1}$$

where ψ_i is the $i^t h$ state occuring with probability p_i

Introduction

A quantum channel acts as a linear *Completely Positive Trace Preserving (CPTP)* map from one density matrix to the other. This means that the product state of the system and the environment stays positive throughout the transformation, and the trace remains constant. These two properties are essential for a channel to be physically viable and make sense in the real world.
It is usually denoted with $\mathcal{N}$ and the state generated after the application of the channel is $\mathcal{N}(\rho)$.
Generally, this channel is second one of the three steps that happen in the process of sending information from A to B, the others being encoding and decoding. These three steps have been illustrated in the following figures.


ρ_{in}
$|0\rangle$
$\mathcal{N}$
ρ_{out}
ρ_E


Classical or quantum source
Encoder
Noisy quantum channel
Decoder
Source
$\mathcal{E}$
$\mathcal{N}$
$\mathcal{D}$
Receiver

Kraus Representation

Quantum channels are often represented in what is called the Kraus form (or the Choi-Kraus representation). In this form, quantum channels can be defined as linear combination in the following way:

$$\mathcal{N}(\rho) = \sum_i K_i \rho K_i^\dagger \tag{2}$$

where $\sum_i K_i^\dagger K_i = \mathbb{I}$

The Kraus representation is very useful in as a tool for analysing quantum channels, and makes several problens easier to solve. Kraus operators and their uses show up at many useful places throughout Quantum information theory. It makes channels easier to understand as well, by decomposing them into several ”sub-channels”.

Von Neumann Entropy

Quantum Relative Entropy

Quantum Mutual Information

Sample frame

The Holevo bound is the upper bound of information that can be extracted from a quantum system, characterized by:

$$\chi = S(\rho) - \sum_i p_i S(\rho_i).$$

To prove this, consider a quantum system that encodes classical information represented by a distribution with probability p_x for each input x, allowing us to define the classical state ρ_x as:

$$\rho_X := \sum_x p_x |x\rangle\langle x|,$$

where $|x\rangle$ (ket x) represents a quantum state and $\langle x|$ (bra x) represents the Hermitian conjugate of $|x\rangle$.

Since each input is mapped to a quantum state ρ_x, the combined state can be written as:

$$\rho_{XQ} := \sum_x p_x |x\rangle\langle x| \otimes \rho_x.$$

The received combined state is represented as:

$$\rho := \mathrm{tr}_X(\rho_{XQ}) = \sum_x p_x \rho_x,$$

where tr_X represents the trace over ρ_{XQ}.

To bound the maximum obtainable information, we need to bound the mutual information $S(X:Y)$ with $S(X:Q)$. From the monotonicity of quantum mutual information, we have:

$$S(X:Q') \leq S(X:Q),$$

and similarly:

$$S(X:Y) \leq S(X:Q'Y).$$

Combining these inequalities, we get:

$$S(X:Y) \leq S(X:Q).$$

Now, we simplify $S(X:Q)$ as follows:

$$\begin{aligned}
S(X:Q) &= S(X) + S(Q) - S(XQ) \\
&= S(X) + S(\rho) + \mathrm{tr}(\rho_{XQ} \log \rho_{XQ}) \\
&= S(\rho) + \sum_x p_x \,\mathrm{tr}(\rho_x \log \rho_x) \\
&= S(\rho) - \sum_x p_x S(\rho_x),
\end{aligned}$$

which gives the Holevo bound.

Sample Frame

Sample Frame

Private Capacity of a Wiretap Channel

The private capacity $P(\mathcal{N})$ of a classical wiretap channel $\mathcal{N} = p_{Y,Z|X}$ is defined as follows.

$$P(\mathcal{N}) = \max_{p_{U,X}(u,x)} [I(U;Y) - I(U;Z)]$$

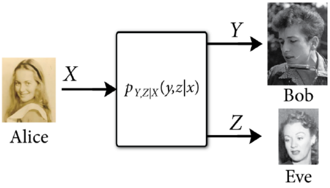


Figure: Classical wiretap channel.

Properties: Private Capacity of a Wiretap Channel

- Non-negativity

$$P(\mathcal{N}) \geq 0$$

- Additivity

$$P(\mathcal{N} \otimes \mathcal{M}) = P(\mathcal{N}) + P(\mathcal{M})$$

- Equivalence of asymptotic and single use capacity

$$\lim_{n \to \infty} \frac{1}{n} P(\mathcal{N}^{\otimes n}) = P(\mathcal{N})$$

Private Capacity of a Quantum Channel

Consider a classical-quantum state of the form given below.

$$\rho_{XA'} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho_{A'}^x$$

The private capacity $P(\mathcal{N})$ of a quantum channel $\mathcal{N}$ is defined as follows.

$$P(\mathcal{N}) = \max_{\rho_{XA'}} [I(X;B)_\rho - I(X;E)_\rho]$$

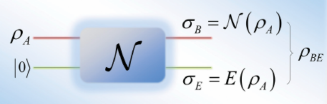


Figure: Private communication through a quantum channel.

Properties: Private Capacity of a Quantum Channel

- Non-negativity

$$P(\mathcal{N}) \geq 0$$

- Bounded by quantum capacity

$$P(\mathcal{N}) \leq Q(\mathcal{N})$$

- Non-additivity

$$P(\mathcal{N} \otimes \mathcal{M}) \neq P(\mathcal{N}) + P(\mathcal{M})$$

Quantum Capacity of a Quantum Channel

The quantity capacity $Q(\mathcal{N})$ of a quantum channel is defined as follows (where $|\psi\rangle = U^{\mathcal{N}}_{A'\to BE}|\phi\rangle_{AA'}$).

$$Q(\mathcal{N}) = \max_{\phi_{AA'}} [S(B)_\rho - S(AB)_\rho]$$
$$= \max_{\phi_{AA'}} \left[S(B)_\psi - S(E)_\psi\right]$$

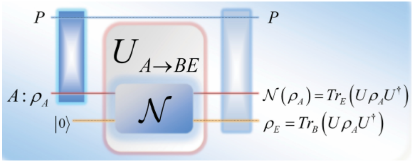


Figure: Quantum communication through a quantum channel.

Properties: Quantum Capacity of a Quantum Channel

- Non-negativity

$$Q(\mathcal{N}) \geq 0$$

- Non-additivity

$$Q(\mathcal{N} \otimes \mathcal{M}) \neq Q(\mathcal{N}) + Q(\mathcal{M})$$

- Relation with other capacities

$$Q(\mathcal{N}) \leq P(\mathcal{N}) \leq C(\mathcal{N})$$

Sample Frame

Sample Frame