MATRICES and DETERMINANTS

Prepared by Neha Agrawal Mathematically Inclined

I) MATRIX

- A set of mn numbers (real or imaginary) arranged in the form of a rectangular array. m: rows n: columns

$m \times n$: matrix
↓
cross

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$\boxed{a_{ij}}$ → row no. (i), column no. (j)

$A = [a_{ij}]$

- **SUBMATRIX**

Let $A_{m \times n}$ be a matrix. A matrix obtained by leaving some rows/columns or both.

II) TYPES OF MATRICES

1) **ROW**: A matrix having ONLY ONE ROW.

$A = [\quad]_{1 \times n}$

2) **COLUMN**: A matrix having ONLY ONE COLUMN

$A = [\quad]_{m \times 1}$

3) **SQUARE**: A matrix in which $m = n$

A matrix of order n.

4) **DIAGONAL ELEMENTS**:

Elements a_{ij} where $i = j$ in any matrix.

5) **DIAGONAL MATRIX**: A square matrix such that $\boxed{a_{ij} = 0 \ \forall i \neq j}$

diag $[a_{11} \ a_{22} \cdots a_{nn}]$

↓

SCALAR MATRIX: $a_{ij} = 0 \ \forall i \neq j$

$a_{ii} = K, K \neq 0$

↓

IDENTITY/UNIT MATRIX: $a_{ij} = 0 \ \forall i \neq j$

$a_{ii} = 1$

~~III)~~ 6) **NULL MATRIX**: All $a_{ij} = 0$ (any order)

7) **UPPER TRIANGULAR**:

A square matrix $A = [a_{ij}]$

$a_{ij} = 0 \ \forall i > j$ (DIAG. KE neeche All 0's)

8) **LOWER TRIANGULAR**:

A square matrix $A = [a_{ij}]$

$a_{ij} = 0 \ \forall i < j$ (DIAG. KE UPAR all 0's)

III) EQUALITY OF MATRICES   EQUIVALENT MATRICES

$A = B$ $A \sim B$

$A = [a_{ij}]_{m \times n}$ & $B = [b_{ij}]_{r \times s}$

EQUALITY	EQUIVALENT
1) $m = r$, $n = s$	1) $m = r$, $n = s$
2) $a_{ij} = b_{ij}$ corresponding elements	That's All!

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VI) TYPES of SQUARE MATRIX

1) **Symmetric**: $A=[a_{ij}]$ is Symmetric iff $A^T = A$
2) **Skew-Symmetric**: $A=[a_{ij}]$ is skew-symm iff $A^T = -A$ → $a_{ii}=0\ \forall i$

   Every sq. matrix $A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$
   $\quad\quad$ S $\quad\quad\quad\quad$ SS

3) **Idempotent**: $A^2 = A$
4) **Nilpotent**: A is nilpotent of order K, iff $A^K = 0$ $\quad K$: least +ve integer → INDEX
5) **Involutory**: $A^2 = I$
6) **Orthogonal**: $A^TA = AA^T = I$

VII) TRACE of a MATRIX

$A = [a_{ij}]_{m\times m}$: Square matrix

tr of $A = \sum_{i=1}^{n} a_{ii}$ (Sum of all Diagonal elts.)

VIII) DETERMINANT

Every Square matrix can be associated to a **UNIQUE** number/expression.

Denoted as $|A|$ or $\det A$

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IV) ALGEBRA OF MATRICES

OPERATION	INPUT	OUTPUT	PROPERTIES
SUM	$A_{m\times n} + B_{m\times n}$	$(A+B)_{m\times n}$	1) $A+B=B+A$ 2) $(A+B)+C = A+(B+C)$
DIFFERENCE	$A_{m\times n} - B_{m\times n}$	$(A-B)_{m\times n}$	NEGATIVE of a MATRIX $A_{m\times n}$ is $-A_{m\times n}$
SCALAR MULTIPLICATION	$K(A)_{m\times n}$ $K\neq 0$	$(KA)_{m\times n}$	K gets multiplied with every elt.
MATRIX MULTIPLICATION	$A_{m\times n} B_{p\times q}$ $n=p$	$(AB)_{m\times q}$	1) $AB \neq BA$ 2) $A(B+C) = AB+AC$ 3) $A(BC) = (AB)C$ 4) $A_{m\times m} I_{m\times m} = A_{m\times m} = I_{m\times m} A_{m\times m}$

V) TRANSPOSE of a MATRIX

SLEEP = WAKE UP

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{3\times 2}$ $\quad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}_{2\times 3}$

PROPERTIES

1) $(A')' = A$
2) $(KA)' = KA'$
3) $(A\pm B)' = A' \pm B'$
4) $(AB)' = B'A'$ REVERSAL LAW

XI) INVERSE of A MATRIX

$A_{m\times m}$, Then $\exists$ $B_{m\times m}$ such that

$AB = BA = I \Rightarrow B = A^{-1}$ or $A = B^{-1}$

$$\boxed{A^{-1} = \frac{1}{|A|}\,\text{adj}\,A}$$

$\rightarrow |A| \neq 0$ NON-SINGULAR

PROPERTIES of INVERSE

1) $(A^T)^{-1} = (A^{-1})^T$
2) $(AB)^{-1} = B^{-1}A^{-1}$
3) $(A^{-1})^{-1} = A$
4) $(KA)^{-1} = \frac{1}{K}A^{-1}, K \neq 0$
5) $AB = AC \Rightarrow B = C$ or $BA = CA \Rightarrow B = C$
6) $|A| \neq 0$, A is symm. $\Rightarrow A^{-1}$ is symm.

XII) PROPERTIES of DETERMINANTS

1) $D^T = D$
2) A det with all 0's in a row(col.) $D = 0$
3) A det $= 0$ if any 2 rows (cols.) are identical (proportional)
4) If any 2 rows (columns) are interchanged, $D_{new} = -D$
5) $D' = \begin{vmatrix} Ka_1 & Ka_2 & Ka_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = KD$

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IX) AREA of a TRIANGLE

Ar. of Δ with vertices (x_1,y_1), (x_2,y_2) & (x_3,y_3) is $\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \Delta$

If $\Delta = 0$, points are COLLINEAR.

X) MINORS (M_{ij})

MINORS (M_{ij}) — Determinant corres-ponding to the element.

$\downarrow$

CO-FACTORS (C_{ij}) — $C_{ij} = (-1)^{i+j} M_{ij}$ — EVEN +VE, ODD −VE

$\downarrow$

ADJOINT — $[C_{ij}]^T$

PROPERTIES of ADJOINT

1) $A(\text{Adj}A) = |A| I_n = (\text{adj}A)A$
2) $|\text{Adj}\,A| = |A|^{n-1}$
3) $|\text{Adj}(\text{Adj}\,A)| = |A|^{(n-1)^2}$
4) $\text{adj}(\text{adj}\,A) = |A|^{n-2}A$
5) $(\text{adj}\,A)^T = \text{adj}(A^T)$
6) $(\text{adj}\,AB) = (\text{adj}\,B)(\text{adj}\,A)$
7) $\text{adj}(A^{-1}) = (\text{adj}\,A)^{-1}$

(XV) DIFFERENTIATION of DETERMINANTS

$$\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

$\Delta'(x)$ = Diff. row-wise one by one or
col.-wise " " "

(XVI) SYSTEM OF LINEAR EQUATIONS

$\boxed{AX = B}$

- co-efficient matrix
- constants matrix
- Variables matrix

$\boxed{\text{CRAMER's RULE}}$ $AX=B$

$D \neq 0$

- At least one $D_1, D_2, D_3 \neq 0$
  Consistent
  UNIQUE (Non-Zero)
- $D_1=D_2=D_3=0$
  CONSISTENT
  Trivial (Zero)

$D = 0$

- $D_1 = D_2 = D_3 = 0$
  CONSISTENT
  Infinite
- Any one of $D_i \neq 0$
  Inconsistent
  (No soln.)

$\boxed{\text{HOMOGENEOUS SYSTEM}}$ $AX=0$

$|A| \neq 0$

$X=0$: Trivial soln.

$|A| = 0$

- Infinite solns. (Trivial & Non-Trivial)
- $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ if $x, y, z \neq 0$.

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6) SPLITTING Property
If each element of any row/column * can be expressed as a sum of 2 terms then det. can be expressed as sum of 2 det. * One at a time ONLY

7) For any det., $R_i \to R_i + mR_j$ or $C_i \to C_i + mC_j$
Then det stays the same.

8) det (skew-symm.) of odd order $= 0$

9) Det. of a diag. matrix = Prod. of its diagonal elts.

(XIII) TRANSPOSE of DET.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad D^T = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

SYMMETRIC

D is symm. iff $D = D^T$

$a_{ij} = a_{ji}$

SKEW-SYMMETRIC

$D^T = -D$

$a_{ji} = -a_{ij}$

(XIV) MULTIPLICATION of DET.

2x2 or 3x3 determinants multiplied same as 2 matrices