PH 142 Spring Semester 2025: Homework 9

Due Wednesday, April 30, 2025 at 11:59 pm

Instructions: There are 3 long problems for this assignment. Please upload your solutions to Canvas when completed. 10 points will be given for attempting all problems. One problem will be chosen randomly and graded in detail, out of 10 points. The sum of these scores will be the total grade, out of 20 points. Partial credit will be given. Please show all work.

1. Energy is emitted from an accelerating charge, q, with an acceleration, a, at the following rate:
$$\frac{dE}{dt} = \frac{q^2 a^2}{6\pi\epsilon_0 c^3},$$
where c is the speed of light.
   a. Does this equation seem reasonable? (HINT: One way to check if this expression makes sense is to analyze the dimensions.)
   b. The Antiproton Decelerator (AD) located at CERN is used to slow and eventually confine antiprotons (a proton with negative charge e) with a kinetic energy of 5 MeV. The radius of this circular particle decelerator is approximately 29 m. What fraction of the antiproton's energy is radiated during one round trip through the ring?
   c. If the AD were tuned to store positrons ("positive electrons") instead of antiprotons at the same kinetic energy, what fraction of the energy would be radiated during one round trip through the ring? (NOTE: since the positron is much lighter than the antiproton, the classical expression for the kinetic energy is no longer valid at 5 MeV of energy. Therefore, you will need to use the relativistic kinetic energy given by:
$KE = m_0 c^2 \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 \right)$, where m_0 is the rest mass of a positron $\approx 9.11 \times 10^{-31}$kg)
   d. The electron in a hydrogen-like $^{20}Ne^{9+}$ ion can be treated classically as being bound to the nucleus with a radius of 5.3×10^{-12} m and with a kinetic energy of 1360 eV. If atoms behaved classically, what fraction of the electron's energy would be radiated per second? What does this result tell us about the classical description of atoms and other subatomic particles (electrons, etc.)?

2. A laser with wavelength of $\lambda = 313$ nm and average power $P = 1$ mW is focused down to a beam radius of $w_0 = 50\ \mu$m and is used to cool a $^{9}Be^{+}$ ion inside a radiofrequency ion trap.
   a. What is the wavelength region of the E/M spectrum (visible, infrared, etc.) and the frequency of this laser light?
   b. For the beam parameters given above, calculate the electric field strength at the location of the ion (where the beam has been focused to a radius of $50\ \mu$m). NOTE: the laser intensity (power/area) is related to electric field by the following expression: $I = \frac{c\epsilon_0}{2}|E^2|$, where the laser intensity I is given in units of W/m^2.
   c. For typical ion traps, the maximum electric field strength needed for confinement is approximately 100 V/mm (at a point away from the equilibrium position). How does this value compare to the electric field value of the cooling laser described above?

① (a)

$$\frac{dE}{dt} = \frac{q^2 a^2}{6\pi\epsilon_0 c^3} \qquad \left[\frac{dE}{dt}\right] = \frac{J}{s} = W \qquad \left[\frac{q^2 a^2}{6\pi\epsilon_0 c^3}\right] = \frac{C^2\left(\cancel{m^2}/\cancel{s^4}\right)}{(F/\cancel{m})\left(\cancel{m^3}/\cancel{s^3}\right)}\left(\frac{1}{s}\right)$$

$$F = C/V = C/(J/C) = C^2/J \longrightarrow \frac{\cancel{C^2}}{\cancel{C^2}/J}\frac{1}{s} = \frac{J}{s} = W \qquad \boxed{\text{Units Match}}$$

(b) $c^3 \longrightarrow \frac{dE}{dt} \simeq 0 \rightarrow KE_0 - E_{Loss} \simeq KE_0 \longrightarrow \Delta v \simeq 0$

$$a_c = \frac{v^2}{r} \qquad K = \frac{1}{2}mv^2 \longrightarrow v = \sqrt{\frac{2K}{m}} \qquad dE = f(a)dt \xrightarrow{\int} E = f(a)t$$

$$a_c = \frac{2K}{rm} \qquad E_{Loss} = \left(\frac{2K}{rm}\right)^2 \frac{q^2}{6\pi\epsilon_0 c^3} t \qquad t = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{m}{2K}}$$

$$E_{Loss} = \left(\frac{2K}{\cancel{r}m}\right)^2 \frac{q^2}{\cancel{6}\cancel{\pi}\epsilon_0 c^3} \overset{1/3}{\cancel{2}\cancel{\pi}\cancel{r}}\sqrt{\frac{m}{2K}} = \frac{1}{3r}\left(\frac{2K}{m}\right)^2\left(\frac{m}{2K}\right)^{1/2}\frac{q^2}{\epsilon_0 c^3}$$

$$= \frac{1}{3r}\left(\frac{2K}{m}\right)^2\left(\frac{2K}{m}\right)^{-1/2}\frac{q^2}{\epsilon_0 c^3} = \frac{q^2}{\epsilon_0 c^3}\frac{1}{3r}\left(\frac{2K}{m}\right)^{3/2}$$

$$E_{Loss} = \frac{q^2}{\epsilon_0 c^3}\frac{1}{3r}\left(\frac{2K}{m}\right)^{3/2} \qquad \frac{E_{Loss}}{KE_0} = \frac{q^2}{\epsilon_0 c^3}\frac{1}{3r}\left(\frac{2K}{m}\right)^{3/2}\frac{1}{K}$$

$$\Rightarrow \frac{q^2}{\epsilon_0 c^3}\frac{1}{3r}\left(\frac{2}{m}\right)^{3/2} K^{3/2} K^{-1} = \frac{q^2}{\epsilon_0 c^3}\frac{1}{3r}\left(\frac{2}{m}\right)^{3/2}\sqrt{K}$$

$$= \boxed{4.627\times 10^{-20}}$$

(c) $$K = \frac{1}{2}mc^2\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right) \rightarrow \frac{2K}{mc^2} + 1 = \frac{1}{\sqrt{1 - v^2/c^2}} \longrightarrow$$

$$\sqrt{1 - v^2/c^2} = 1/\left(\frac{2K}{mc^2}+1\right) \rightarrow 1 - \frac{v^2}{c^2} = 1/\left(\frac{2K}{mc^2}+1\right)^2$$

$$1 - 1/\left(\frac{2K}{mc^2}+1\right)^2 = \frac{v^2}{c^2} \qquad c\sqrt{1 - 1/\left(\frac{2K}{mc^2}+1\right)^2} = v$$

$$E_{Loss} = \left(\frac{v^2}{r}\right)^2\frac{q^2}{6\pi\epsilon_0 c^3} t = \frac{2\pi}{r}\frac{q^2 v^3}{6\pi\epsilon_0 c^3} = \frac{1}{3r}\frac{q^2 v^3}{\epsilon_0 c^3} = \frac{1}{3r}\frac{q^2}{\epsilon_0}\left(\sqrt{1 - 1/\left(\frac{2K}{mc^2}+1\right)^2}\right)^3$$

$$t = \frac{2\pi r}{v} \qquad \frac{E_{Loss}}{KE} = \boxed{4.145\times 10^{-17}}$$

(d) $\frac{dE}{dt} = \frac{q^2 a^2}{6\pi \epsilon_0 c^3}$ $a = \frac{v^2}{r}$ $K = \frac{1}{2}mv^2$ $v = \sqrt{\frac{2K}{m}}$ $\frac{dE}{dt} \cdot 1s \approx E_{Loss}$

$$\frac{E_{Loss}}{K} \simeq \frac{q^2}{6\pi \epsilon_0 c^3} \frac{v^4}{r^2} \frac{1}{K} = \frac{2}{3} \frac{K}{\pi \epsilon_0 c^3 r^2} \left(\frac{q}{m}\right)^2 = \boxed{2.135 \times 10^{14}} = \underline{A}$$

$\frac{E_{Loss}}{K} = 100\%$ if $t = \frac{1}{A}$ = 4.68 femtoseconds!

This result implies that classical electrons will radiate their energy in femtoseconds before collapsing into the nucleus.

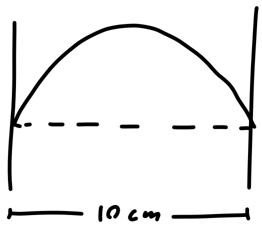
(2) (a) $\lambda = 313$ nm — $\boxed{\text{Near Ultraviolet}}$ $\lambda f = c \rightarrow f = \frac{c}{\lambda} = \boxed{957.8 \text{ THz}}$

(b) $I = \frac{c\epsilon_0}{2} E^2 = \frac{P}{A} = \frac{P}{\pi r^2}$ $r = 50 \mu m$ $P = 1 mW$

$$\frac{P}{\pi r^2} = \frac{c\epsilon_0}{2} E^2 \rightarrow E = \sqrt{\frac{2P}{\pi r^2} \frac{1}{c\epsilon_0}} = \boxed{9794.57 \frac{N}{C}}$$

(c) $100 \text{ V/mm} \rightarrow 100 \times 10^3 \text{ V/m} \rightarrow 100 \frac{kV}{m} \rightarrow 100 \frac{kN}{C}$ $\boxed{9.7 \frac{kN}{C} < 100 \frac{V}{mm}}$

3. A Fabry-Pérot cavity consists of two mirrors spaced 10.0 cm apart from each other. What is the lowest frequency mode of the standing waves supported in this cavity? NOTE: This value is commonly referred to as the free spectral range of the cavity.



$$\left.\begin{array}{l}\lambda f = v \longrightarrow f = \dfrac{v}{\lambda} \\ \lambda = 2 \cdot 10\,cm = 0.2\,m\end{array}\right\} f = \frac{299{,}792{,}458\,\frac{m}{s}}{0.2\,m} = \boxed{1.498\,GHz}$$

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