PH 142 Spring Semester 2025: Homework 3

Due Wednesday, February 12 2024, at 11:59 pm

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Instructions: There are 4 long problems for this assignment. Please upload your solutions to Canvas when completed. 10 points will be given for completing all problems. One problem will be chosen randomly and graded in detail, out of 10 points. The sum of these scores will be the total grade, out of 20 points. Partial credit will be given. Please show all work.

1. A charge q_1 is located at the origin. A second charge q_2 is located at a point a distance L away from the origin along the x axis. A third charge, q_3 will be placed in between these two charges (see Fig. 1 below). In this problem, you are asked to analyze the potential energy configuration of a few different cases.
   a. In the case where $q_1 = +4e$, $q_2 = -3e$, and $q_3 = +5e$, calculate the total potential energy of the system if q_3 is located at $x = L/2$.
   b. With charges q_1 and q_2 fixed, calculate the location where charge q_3 must be placed to minimize the total potential energy (i.e., the total potential energy equals zero).

a) $\boxed{u = \frac{e}{4\pi\epsilon_0}\left(\frac{4(5)}{L/2} - \frac{3(5)}{L/2} - \frac{3(4)}{L}\right)}$

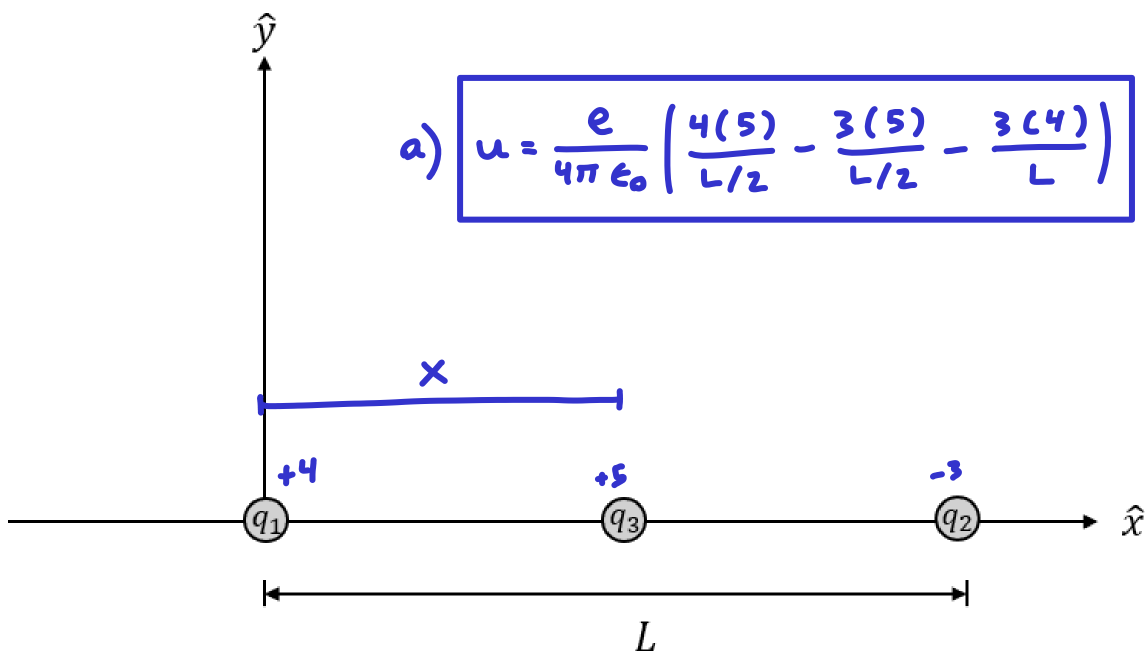


Fig. 1

b)

$$u = \frac{e}{4\pi\epsilon_0}\left(\frac{4(5)}{|x|} - \frac{3(5)}{|x-L|} - \frac{3(4)}{L}\right)$$

$$(x-L)(x-L) = x^2 - xL - xL + L^2$$

$$\frac{\partial u}{\partial x} = \frac{e}{4\pi\epsilon_0}\left(\frac{\cancel{3(5)}^{15}}{(x-L)^2} - \frac{\cancel{4(5)}^{20}}{x^2}\right) = \frac{e}{4\pi\epsilon_0}\frac{15x^2 - 20(x-L)^2}{x^2(x-L)^2}$$

Critical points:

$$0 = \cancel{\frac{e}{4\pi\epsilon_0}}\frac{15x^2-20(x-L)^2}{x^2(x-L)^2} = \frac{15x^2 - 20x^2 + 40Lx - 20L^2}{\cancel{x^4 - 2Lx^3 + L^2x^2}} \neq 0$$

$$0 = -5x^2 + 40Lx - 20L^2 \Rightarrow x = \frac{8L \pm \sqrt{8^2L^2 - 4(4L^2)}}{2}$$

$$= x^2 - 8Lx + 4L^2$$

$$= 4L \pm \frac{1}{2}\sqrt{64L^2 - 16L^2}$$

$$x_1 = \left(4 + \frac{\sqrt{48}}{2}\right)L = 7.464L \Leftarrow$$

$$= 4L \pm \frac{1}{2}\sqrt{48L^2}$$

$$x_2 = \left(4 - \frac{\sqrt{48}}{2}\right)L = 0.535L$$

$$= 4L \pm \frac{L}{2}\sqrt{48}$$

$$x_1 > x_2$$

$$u(x_1) = \frac{e}{4\pi\epsilon_0}\left(\frac{4(5)}{7.464L} - \frac{3(5)}{6.464L} - \frac{3(4)}{L}\right)$$

$$= \frac{e}{4\pi\epsilon_0}\frac{1}{L}\underbrace{\left(\frac{4(5)}{7.464} - \frac{3(5)}{6.464} - 3(4)\right)}_{-11.64} \checkmark$$

$$\boxed{x = \left(4 + \frac{\sqrt{48}}{2}\right)L}$$

$$u(x) = \frac{-11.64e}{4\pi\epsilon_0 L}$$

$$u(x_2) = \frac{e}{4\pi\epsilon_0}\left(\frac{4(5)}{0.535L} + \frac{3(5)}{1.535L} - \frac{3(4)}{L}\right)$$

$$= \frac{e}{4\pi\epsilon_0}\frac{1}{L}\underbrace{\left(\frac{4(5)}{0.535} + \frac{3(5)}{1.535} - 3(4)\right)}_{35.155} \times$$

2. A solid sphere of radius R, centered at the origin, carries a total charge Q, distributed evenly throughout the volume of the sphere. (see Fig. 2 below).
   a. Calculate the electric potential $V(r)$ everywhere. NOTE: Take $V = 0$ as $r \to \infty$. (HINT: recall in Ex. 22.9 from class we solved for the electric field everywhere. This could be helpful when calculating the electric potential.)

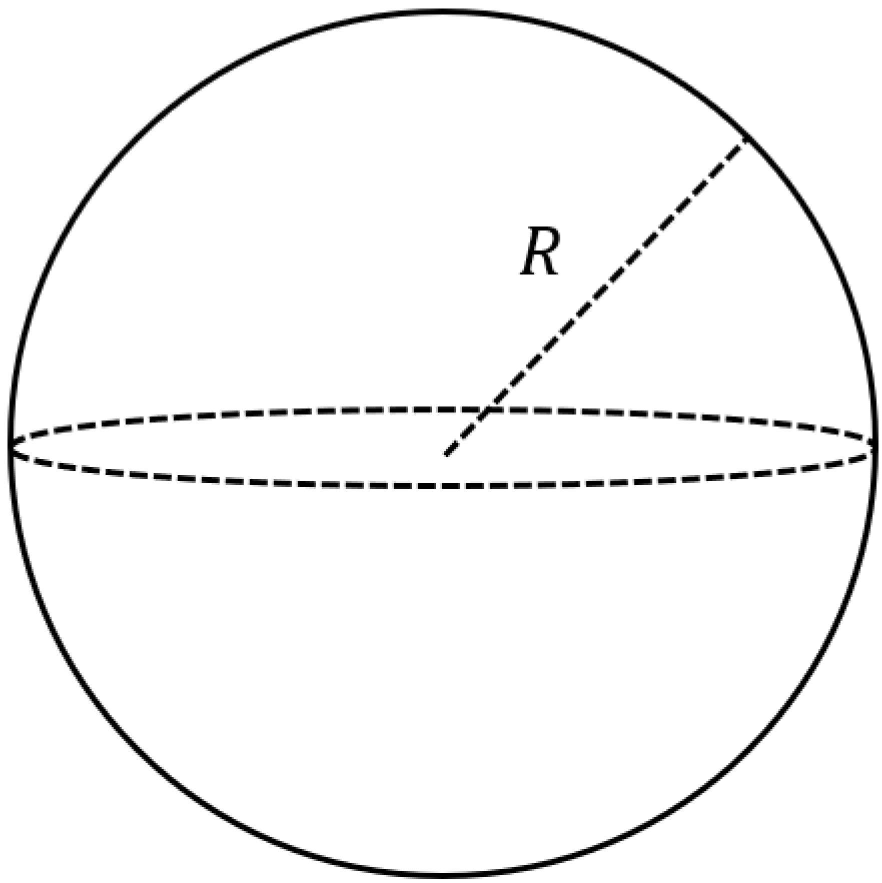


Fig. 2

a) $\vec{E} = -\nabla V$ $\quad \oint \vec{E} \cdot d\vec{A} = EA = \frac{Q_{enc}}{\epsilon_0}$ $\quad \rho V = Q \Longrightarrow \rho = \frac{Q}{4/3 \pi R^3}$

<u>Inside</u> $E(4\pi r^2) = \frac{\rho V}{\epsilon_0} = \frac{\rho(4/3 \pi r^3)}{\epsilon_0} = \frac{Q}{\cancel{4/3 \pi} R^3} \frac{(\cancel{4/3 \pi} r^3)}{\epsilon_0} = \frac{Q r^3}{R^3 \epsilon_0}$

$\Rightarrow E = \frac{Q r}{4\pi \epsilon_0} \frac{1}{R^3} \Rightarrow V(r) = \frac{-Q}{4\pi \epsilon_0} \frac{1}{R^3} \int_r^R r\, dr \Rightarrow \boxed{V(r) = \frac{-Q}{8\pi \epsilon_0} \frac{1}{R^3} \left(R^2 - r^2\right)}$

<u>Outside</u> $E(4\pi r^2) = \frac{Q}{\epsilon_0} \Rightarrow \frac{1}{4\pi \epsilon_0} \frac{Q}{r^2} \Rightarrow V(r) = \frac{-Q}{4\pi \epsilon_0} \int_0^r \frac{1}{r^2}\, dr$

$$\boxed{V(r) = \frac{Q}{4\pi \epsilon_0} \frac{1}{r}}$$

3. (Electrostatic bender). Highly charged argon ions with charge states $^{40}Ar^{8+}$, $^{40}Ar^{9+}$, and $^{40}Ar^{10+}$ are created inside an ion source and accelerated through a potential difference, $U = 2.5$ kV inside an ultrahigh vacuum chamber. The ions enter an electric field-free region which is 5 meters long. They then enter a device called an electrostatic bender consisting of two parallel plates with applied voltages $+V$ and $-V$. (NOTE: the magnitude of the applied voltages are equal, but the polarities are opposite.) The bender is in a region of the vacuum chamber that contains a 90-degree arc with radius, $R = 10$ cm, plate separation $d = 2$ cm. After leaving the bender, the ions continue to an ion capture trap 3 meters away. (See cross-sectional view in Fig. 3 below.)
    a. Calculate the kinetic energy of each of the three charge states after being accelerated out of the ion source.
    b. Calculate the time-of-flight for the three charge states to leave the source and enter the electrostatic bender field 5 meters away.
    c. Calculate the voltage V required to allow each charge state to pass through the 90-degree bend. Compare the results for the three different charge states.
    d. Calculate the total time-of-flight for the ions to travel from the source to the final capture trap.
    e. Suppose that we are only interested in transporting $^{40}Ar^{9+}$ ions from the source to the trap. Is it possible to use the static voltages on the bender to select one charge state? If so, explain. If not, is there some information gained from this problem that could allow us to do so? (Hint: Recall your answer from part (d). Do all three charge states arrive at the ion trap at the same time?)

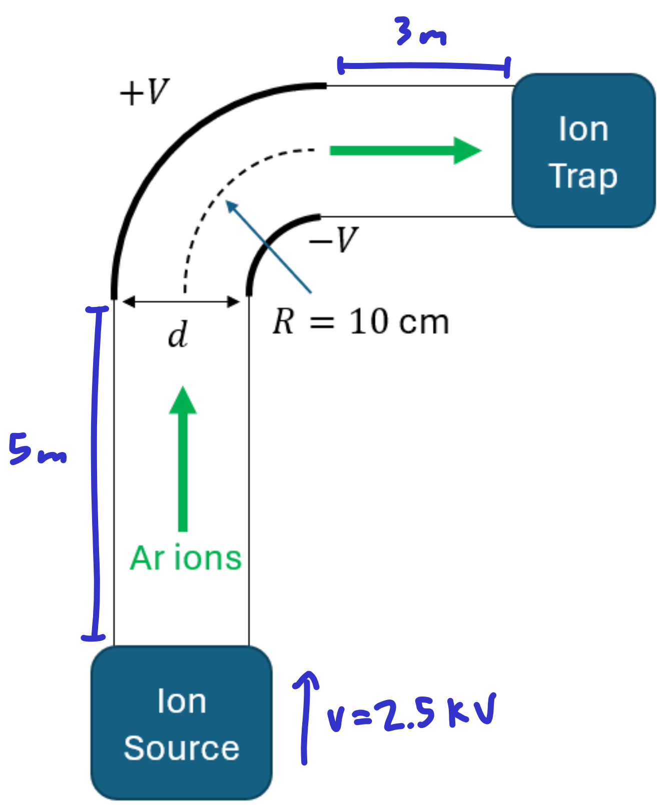


Fig. 3

a) $^{40}Ar^{8+}$, $q = 8e$, $8e \cdot 2500\,V =$ $\boxed{20{,}000\ eV}$
$^{40}Ar^{9+}$, $q = 9e$, $9e \cdot 2500\,V =$ $\boxed{22{,}500\ eV}$
$^{40}Ar^{10+}$, $q = 10e$, $10e \cdot 2500\,V =$ $\boxed{25{,}000\ eV}$ $\xrightarrow[J]{\times 1.602\times10^{-19}}$

b) $m = 40\ Da = 40\times1.66\times10^{-27}\,kg$ $\quad KE = \frac{1}{2}mv^2 \quad v = \sqrt{\frac{2KE}{m}}$

$^{40}Ar^{8+}$ $\quad v = \sqrt{\dfrac{2\cdot 20{,}000\cdot 1.602\times10^{-19}}{40\times1.66\times10^{-27}\,kg}} = 310{,}654.187\ m/s$

$^{40}Ar^{9+}$ $\quad v = \sqrt{\dfrac{2\cdot 22{,}500\cdot 1.602\times10^{-19}}{40\times1.66\times10^{-27}\,kg}} = 329{,}498.524\ m/s$

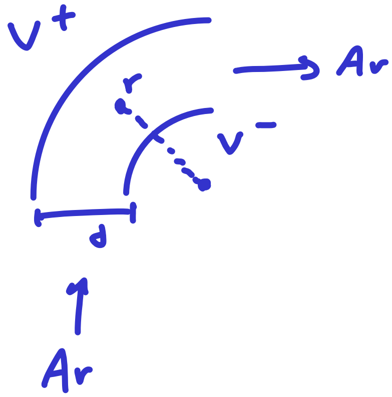
$^{40}Ar^{10+}$ $\quad v = \sqrt{\dfrac{2\cdot 25{,}000\cdot 1.602\times10^{-19}}{40\times1.66\times10^{-27}\,kg}} = 347{,}321.940\ m/s$

$^{40}Ar^{8+}$ $\quad 310{,}654.187\,\frac{m}{s}\cdot t = 5\,m \longrightarrow$ $\boxed{16.1\ \mu s}$
$^{40}Ar^{9+}$ $\quad 329{,}498.524\,\frac{m}{s}\cdot t = 5\,m \longrightarrow$ $\boxed{15\ \mu s}$
$^{40}Ar^{10+}$ $\quad 347{,}321.940\,\frac{m}{s}\cdot t = 5\,m \longrightarrow$ $\boxed{14.4\ \mu s}$

c)

$r = 10cm = 0.10m$
$d = 2cm = 0.02m$

$$F_T = m\frac{v^2}{r} \longrightarrow qE = m\frac{v^2}{r} \longrightarrow E = \frac{mv^2}{qr}$$



$$2V = -\int_0^d \frac{mv^2}{qr}\,dl = -\frac{mv^2}{qr}d$$

$$\longrightarrow V = -\frac{dmv^2}{2qr}$$

$m = 40 \times 1.66 \times 10^{-27}$ Kg

$^{40}Ar^{8+}$ — $310{,}654.187\ \frac{m}{s}$ — $8 \times 1.602 \times 10^{-19}$ C
$^{40}Ar^{9+}$ — $329{,}498.524\ \frac{m}{s}$ — $9 \times 1.602 \times 10^{-19}$ C
$^{40}Ar^{10}$ — $347{,}321.940\ \frac{m}{s}$ — $10 \times 1.602 \times 10^{-19}$ C

Calculator:

$^{40}Ar^{8+}$ — $|V| = 500V$
$^{40}Ar^{9+}$ — $|V| = 500V$
$^{40}Ar^{10}$ — $|V| = 500V$

d) $D = 5m + 3m + \frac{1}{4}2\pi(0.10m) = 8.157m$ $\qquad t = \frac{D}{v}$

Bend

$^{40}Ar^{8+}$ — $310{,}654.187\ \frac{m}{s}$ — $t = 26.26\ \mu s$
$^{40}Ar^{9+}$ — $329{,}498.524\ \frac{m}{s}$ — $t = 24.76\ \mu s$
$^{40}Ar^{10}$ — $347{,}321.940\ \frac{m}{s}$ — $t = 23.49\ \mu s$

Constant through bend

e) Although changing the static voltage doesn't help separate the ions, increasing the travel dist. could make the difference in arriving times great enough for a mechanism to allow ions in for a short time window.

4. A point charge, $+q$, is located at the origin. Find the electric field, in cartesian coordinates, using the definition of the potential gradient. (In other words, starting with the expression for the electric potential of a point charge, $V(x, y, z)$, find the corresponding electric field, $\vec{E}(x, y, z)$, for a positive point charge.) NOTE: Take $V = 0$ as $r \to \infty$.

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}} \qquad E = -\nabla V$$

$$\frac{\partial V}{\partial x} = \frac{q}{4\pi\epsilon_0}\frac{-2x}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} = -\frac{q}{4\pi\epsilon_0}x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$\frac{\partial V}{\partial y} = \frac{q}{4\pi\epsilon_0}\frac{-2y}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} = -\frac{q}{4\pi\epsilon_0}y\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$\frac{\partial V}{\partial z} = \frac{q}{4\pi\epsilon_0}\frac{-2z}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} = -\frac{q}{4\pi\epsilon_0}z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}$$

$$-\nabla V = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) = \vec{E}$$

$$\boxed{\vec{E} = \frac{q}{4\pi\epsilon_0}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}$$