$K = \frac{1}{4\pi \epsilon_0}$

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PH 142 Spring Semester 2025: Homework 2

Due Wednesday, February 5 2025, at 11:59 pm

Instructions: There are 4 long problems for this assignment. Please upload your solutions to Canvas when completed. 10 points will be given for completing all problems. One problem will be chosen randomly and graded in detail, out of 10 points. The sum of these scores will be the total grade, out of 20 points. Partial credit will be given. Please show all work.

1. An infinitely long, solid cylinder with radius R has volume charge density $\rho(r)$ (see Fig. 1 below).
    a. Calculate the electric field inside and outside of the cylinder for the case when the volume charge density $\rho(r)$ is constant ($\rho(r) = \rho$).
    b. Calculate the electric field inside and outside of the cylinder for the case when the volume charge density increases linearly with the distance from the axis of the cylinder, $\rho(r) = \beta r$, where β is some arbitrary constant.

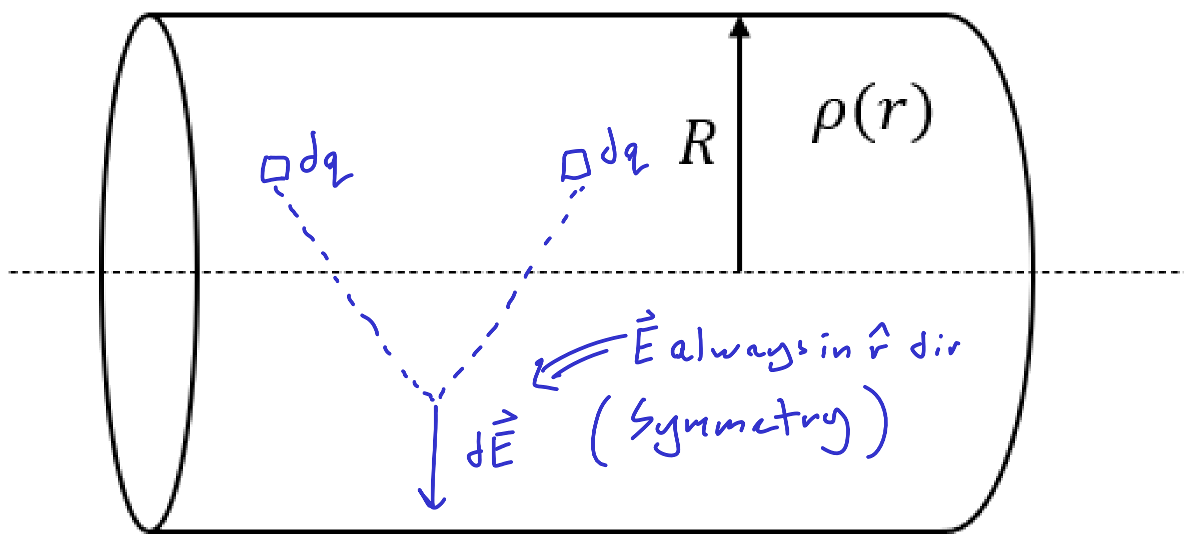
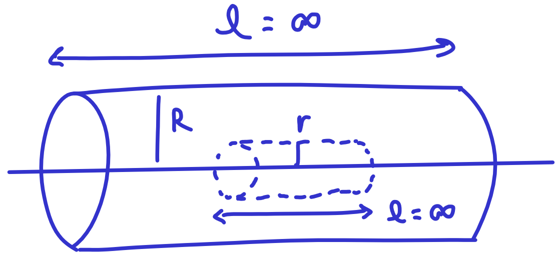


Fig. 1

a) Inside:



$$Q_{enc} = V\rho = \pi \underline{\underline{r}}^2 l \rho$$

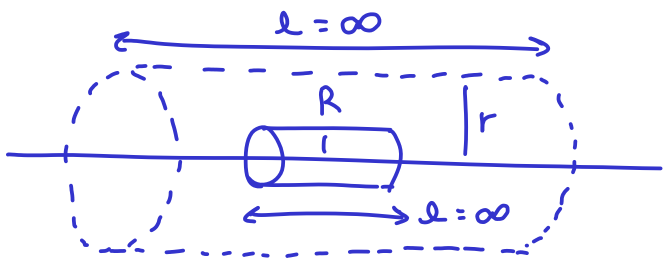
$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow EA = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E(2\pi r l + \cancel{2\pi r^2}) = \frac{\pi r^2 l \rho}{\epsilon_0}$$

constant at cons. radius

$$d\vec{A}_{caps} \cdot \vec{E} = 0$$

$$E(2\pi r l) = \frac{\pi r^2 l \rho}{\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{r \rho \hat{r}}{2 \epsilon_0}}$$

Outside:



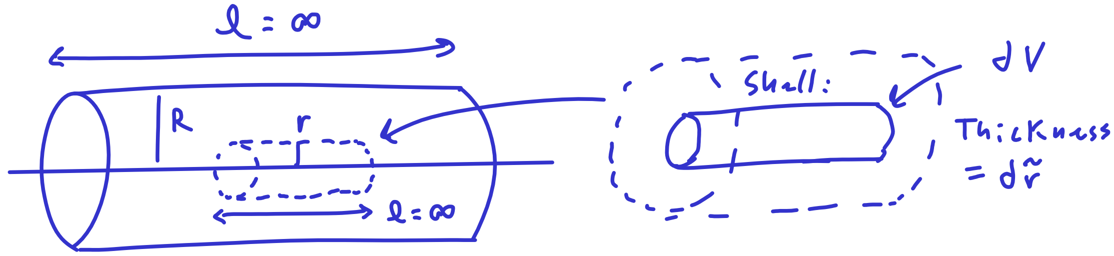
$$Q_{enc} = V\rho = \pi \underline{\underline{R}}^2 l \rho$$

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow EA = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E(2\pi r l + \cancel{2\pi r^2}) = \frac{\pi R^2 l \rho}{\epsilon_0}$$

$$d\vec{A}_{caps} \cdot \vec{E} = 0$$

$$E(2\pi r l) = \frac{\pi R^2 l \rho}{\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{R^2 \rho \hat{r}}{2 r \epsilon_0}}$$

b) <u>Inside:</u>



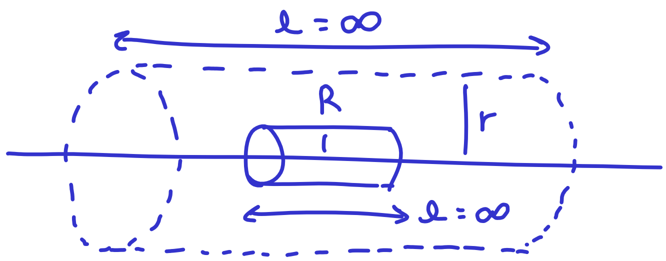
$$dQ_{enc} = dV\rho(r) = \underbrace{l\,2\pi\tilde{r}\,d\tilde{r}}_{dV}\,\underbrace{\beta\tilde{r}}_{\rho(r)} \Rightarrow Q_{enc} = 2\pi l\beta\int_0^r \tilde{r}^2\,d\tilde{r} = \underline{2\pi l\beta\frac{1}{3}r^3}$$

$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow EA = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E(2\pi r l + \cancel{2\pi r^2}) = \frac{2\pi l\beta r^3}{3\epsilon_0}$$

$d\vec{A}_{caps}\cdot\vec{E} = 0$

$$E(2\pi r l) = \frac{2\pi l\beta r^3}{3\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\beta r^2\hat{r}}{3\epsilon_0}}$$

<u>Outside:</u>



Same Q_{enc} w/ $r = R$:

$$Q_{enc} = 2\pi l\beta\frac{1}{3}R^3$$

$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow EA = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E(2\pi r l + \cancel{2\pi r^2}) = \frac{2\pi l\beta R^3}{3\epsilon_0}$$

$d\vec{A}_{caps}\cdot\vec{E} = 0$

$$E(2\pi r l) = \frac{2\pi l\beta R^3}{3\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\beta R^3\hat{r}}{3r\epsilon_0}}$$

2. A solid sphere of radius a, centered at the origin, carries a volume charge density of $\rho(r) = \alpha r^2$, where α is a constant (see Fig. 2 below).
   a. Calculate the electric field inside and outside of the sphere. (NOTE: For a spherically symmetric system, $dV = r^2 \sin(\theta)\, d\theta d\phi dr$. Therefore, the volume integral of some function $\rho(r)$ is given by $\int \rho(r) r^2 \sin(\theta)\, d\theta d\phi dr = 4\pi \int \rho(r) r^2 dr$)
   b. Plot the resulting electric field as a function of distance r from the origin.

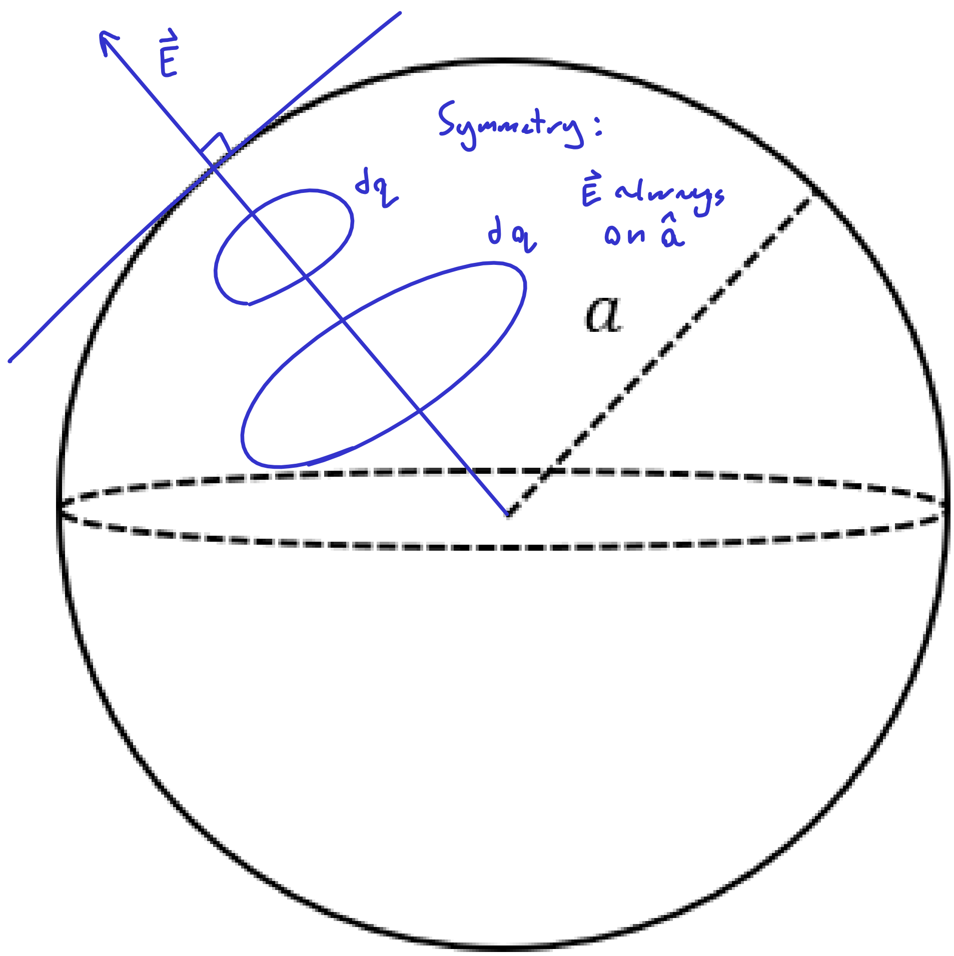


Fig. 2

a) **Inside:** $Q_{enc} = \overbrace{4\pi\int \rho(r)\tilde{r}^2\, d\tilde{r}}^{\text{Given}} \Rightarrow 4\pi\int_0^r \alpha\, \tilde{r}^2\tilde{r}^2\, d\tilde{r} = 4\pi\alpha\int_0^r \tilde{r}^4\, d\tilde{r}$

$= 4\pi\alpha\frac{1}{5}r^5$

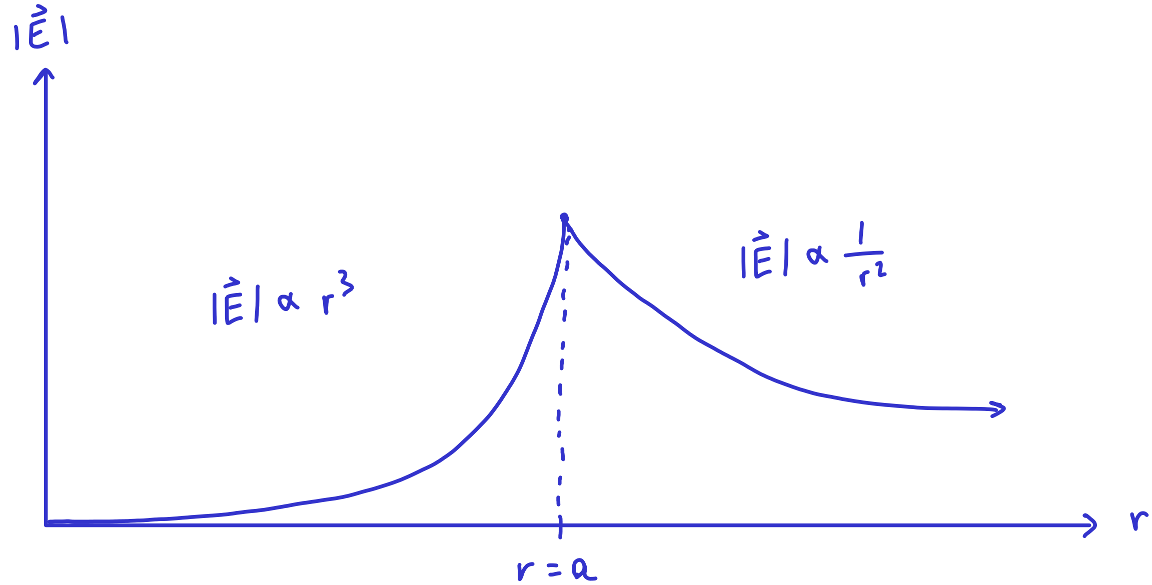
$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow EA = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E(4\pi r^2) = \frac{4\pi\alpha r^5}{5\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\alpha r^3}{5\epsilon_0}\hat{r}}$$

w/ a = r

Outside: $4\pi\alpha\int_0^a r^4\, dr = 4\pi\alpha\frac{1}{5}a^5$

$$\Phi_E = \oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow EA = \frac{Q_{enc}}{\epsilon_0} \Rightarrow E(4\pi r^2) = \frac{4\pi\alpha a^5}{5\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\alpha a^5 \hat{r}}{5 r^2\epsilon_0}}$$

b)



3. A neutral conducting sphere has two cavities with charges q_a and q_b, and radii a and b (see Fig. 3 below).
   a. Calculate σ_a, σ_b, and σ_R for this configuration.
   b. Find the electric field outside of the conductor.
   c. Find the electric field inside each cavity.
   d. What is the force on q_a and q_b.
   e. If we were to bring a third charge q_c near the conductor, which of your previous answers (a – d) would change, and how?

$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$ $\qquad E = \frac{\sigma}{\epsilon_0}$

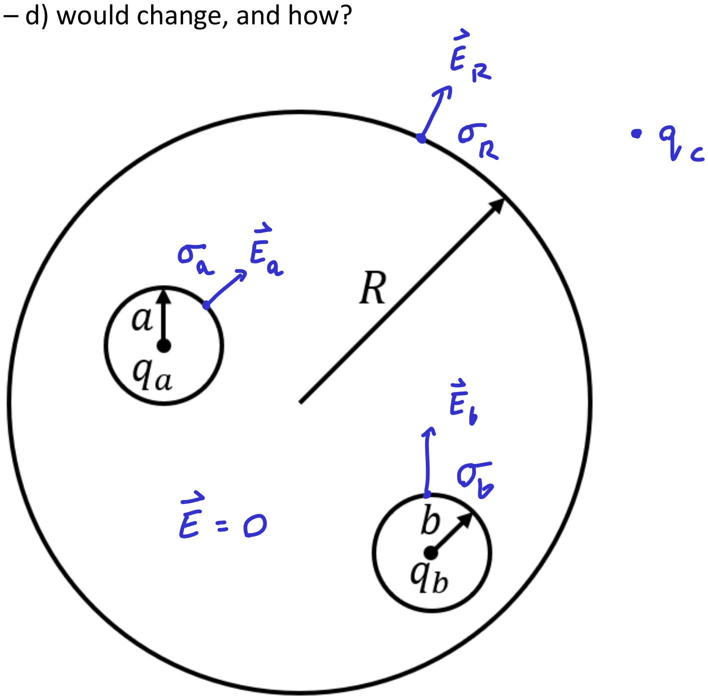


Fig. 3

4. The water molecule (H_2O) has a dipole moment of 6.17×10^{-30} C · m. A single H_20 molecule in the gas phase is placed in a region of electric field with magnitude $|\vec{E}| = 10^6$ N/C.
   a. Calculate the change in potential energy of the dipole when it changes it orientation, with respect to the electric field $\vec{E}$, from parallel to perpendicular.
   b. The thermal translational energy (kinetic energy) of a molecule in the gas phase is given as $KE = \frac{3}{2}k_bT$. Calculate the temperature T at which the thermal energy is equal to the change in the potential energy of the dipole. Is this temperature large or small compared to "room temperature" (approximately 293 K).
   c. Now the water molecule is oriented perpendicular to the electric field and held in place. When it is released (and allowed to rotate freely) calculate the torque induced on the dipole. Compare this answer to the peak torque specification of 500 N · m found on a 2025 Maserati Quattroporte sedan.

3)

a) σ_R) $E(4\pi R^2) = \frac{q_a + q_b}{\epsilon_0} \Rightarrow E = \frac{q_a + q_b}{(4\pi R^2)\cancel{\epsilon_0}} = \frac{\sigma_R}{\cancel{\epsilon_0}}$

$$\boxed{\sigma_R = \frac{q_a + q_b}{4\pi R^2}}$$

σ_a) Only dif. Q_{enc} and radius: $\boxed{\sigma_a = \frac{q_a}{4\pi a^2}}$

σ_b) 〃 $\boxed{\sigma_b = \frac{q_b}{4\pi b^2}}$

b) $E(4\pi r^2) = \frac{q_a + q_b}{\epsilon_0}$ $\boxed{\vec{E} = \frac{q_a + q_b}{\epsilon_0 (4\pi r^2)}\hat{r}}$

$r > R$

replace w/ q_b for inside $\vec{E}$ of b cavity

c) $E(4\pi r^2) = \frac{q_a}{\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{q_a\,\hat{r}}{\epsilon_0 (4\pi r^2)}}$

$r < a$

caused by conductor's rearranged charges

d) $\boxed{\vec{F} = 0}$ since both charges are in electrostatic equilibrium $(\vec{E} = 0)$

e) (a) doesn't change since Q_{enc} and the Gaussian surface don't change.

(b) Q_{enc} may include q_c.

(c) same as (a).

(d) the conductor maintains electrostatic equilibrium.

Change?	
a	✗
b	✓
c	✗
d	✗

4) a) $U = -\vec{p} \cdot \vec{E} \Rightarrow U_{\parallel} = -(6.7 \times 10^{-30}\ Cm)\, 10^{6}\ N/C$

$= -6.7 \times 10^{-30+6}\ J = -6.7 \times 10^{-24}\ J$

If $\vec{a}$ is $\parallel$ to $\vec{b}$

$\vec{a} \cdot \vec{b} = ab$

if $\perp$: $\vec{a} \cdot \vec{b} = 0$

$U_{\perp} = 0 \quad \therefore \quad \boxed{\Delta U_{\parallel \to \perp} = 6.7 \times 10^{-24}\ J}$

b) $KE = \frac{3}{2} k_b T \Rightarrow T = \frac{2KE}{3k_b} \rightarrow T = \frac{2(6.7 \times 10^{-24}\ J)}{3(1.38 \times 10^{-23}\ \frac{J}{K})}$

$k_b = 1.38 \times 10^{-23}\ \frac{J}{K}$

$\boxed{T = 0.323\ K}$

$T \ll$ room temp $(293K)$

c) $\vec{\tau} = \vec{p} \times \vec{E} \quad |\tau| = pE\underbrace{\sin\theta}_{=1}$

$\theta = 90° \quad \sin\theta = 1$

$\rightarrow |\tau| = pE$

$= (6.7 \times 10^{-30}\ Cm)\, 10^{6}\ N/C$

$\boxed{\tau = 6.7 \times 10^{-24}\ Nm}$

$\tau_{H_2O} \ll 500\ Nm$ [car]