

Plasma electrostatic wave dispersion solution for arbitrary ion distributions

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Abstract

A method of solving numerically the dispersion relation and hence investigating stability of arbitrary ion distributions to longitudinal electrostatic oscillations in the ion-acoustic range of velocities is described. Several subtleties arise that are bypassed in the historic analytic treatments. A dimensionless formulation is implemented that gives robust graphical interpretation and allows direct solution and intuitive display.

1 Background Theory

The longitudinal relative susceptibility to linear waves ($\propto \exp i(\mathbf{k} \cdot \mathbf{x} - \omega t)$) that are presumed to have real wave-vector \mathbf{k} and possibly complex frequency ω , for a uniform unmagnetized plasma or when the \mathbf{k} -direction is parallel to the magnetic field is

$$\chi = \sum \chi_j \quad \text{with} \quad \chi_j = -\frac{\omega_{pj}^2}{k^2} \int \frac{\partial f_j}{\partial v} \frac{1}{v - v_p} dv \quad (1)$$

where j denotes the species, $q_j = Z_j e$ is its charge, n_j its density, $\omega_{pj}^2 = n_j q_j^2 / \epsilon_0 m_j$ its plasma frequency squared, v the velocity in the parallel (\mathbf{k}) direction, f_j the *normalized* one-dimensional parallel velocity distribution function (such that $\int f_j dv = 1$), and v_p is the phase velocity

$$v_p \equiv \omega/k. \quad (2)$$

The integral must be performed using the standard Landau prescription along a contour below the pole at v_p in the complex v -plane.

It is convenient to introduce an energy scale expressed as an effective temperature T_0 . In terms of this temperature, the Debye length is $\lambda_D = \sqrt{\epsilon_0 T_0 / e^2 n_e}$ and the sound speed is $c_s = \sqrt{T_0 / m_i}$. At this stage one can think of these as corresponding to electron temperature $T_e = T_0$ and ion mass m_i . However, we will later consider T_e to be variable, independent of T_0 . In terms of these energy scaled parameters we can write

$$k^2 \lambda_D^2 \chi_j = -Z_j^2 \frac{m_i}{m_j} c_s^2 \int \frac{\partial f_j}{\partial v} \frac{1}{v - v_p} dv. \quad (3)$$

We note that the right hand side of this expression, for given plasma parameters, is a function only of the complex phase velocity v_p , not of k and ω independently. Expressed in this way, we can see that the dispersion relation is determined by the sum of distribution functions (differentiated and) multiplied by Z_j^2/m_j integrated over a single velocity space. One can illustrate this by plotting the sum of the distributions so weighted. An example is

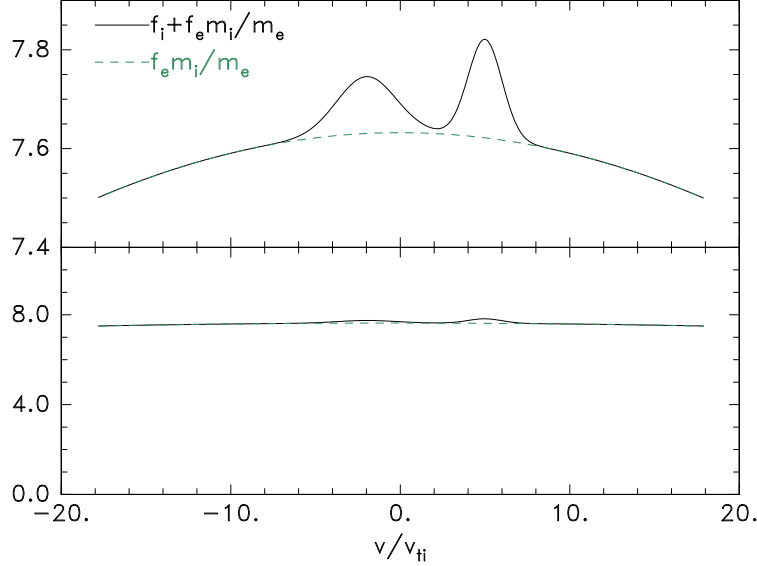


Figure 1: Combined velocity distribution weighted by $1/m_j$, which could be used to determine stability by the Penrose criterion.

Fig. 1. It has been plotted twice; the lower panel avoids the suppressed zero of the upper panel, so as to emphasize the fact that the ion susceptibility arises as a small perturbation on top of a nearly flat electron contribution. The stability of the combined distribution $f_c(v) = f_i(v) + f_e(v)m_i/m_e$ could be analysed by the Penrose criterion, namely that there are unstable modes if and only if there exists a real v_m such that $f_c(v_m)$ is a (local) minimum and $\int [f_c(v) - f_c(v_m)]/(v - v_m)^2 dv > 0$. The integral condition amounts to requiring the local minimum to be deep enough compared with the surrounding distribution. In the context of ion-ion instabilities, one can see that the higher the central f_e is, the less relatively deep a given ion distribution minimum is. Thus lower electron temperature (which makes f_e higher) tends to stabilize the combined distribution. This effect is *not* a result of electron Landau damping. We develop here a way to understand the problem more quantitatively, yet still giving an intuitive graphical rendering in velocity space.

The dispersion relation for (electrostatic) longitudinal waves is

$$\epsilon = 1 + \chi = 0, \quad \text{or} \quad \chi = -1. \quad (4)$$

Solving it can be approached by finding those phase velocities $v_p = \omega/k$ for which the imaginary part of $k^2 \lambda_D^2 \chi$ (and hence of χ) vanishes. If the corresponding real part, denoted $\Re(k^2 \lambda_D^2 \chi)$, for those v_p is negative, then there exists a real k for which $\Re(\chi) = -1$ and the

dispersion relation is solved. However, for v_p such that $\Re(k^2\lambda_D^2\chi) > 0$, no solution exists. If we are therefore able to evaluate the integrals to obtain $k^2\lambda_D^2\chi_j$ at relevant values of v_p , the above approach provides an algorithm for solving the dispersion relation.

In analytic treatments of this problem, convenient analytic representations of the ion distribution functions are generally used. They provide closed-form expressions for the integrals, often in terms of the (Fried and Comte) plasma dispersion function which arises for Gaussian distributions. The present work addresses the more general problem when the ion distribution functions are prescribed in arbitrary numerical terms.

2 Integrating the Susceptibility

2.1 Electron Susceptibility

In principle the electron distribution might be arbitrary as well as the ion distribution. However, the generally very different velocity scale of the electron distribution means that for the exploration of waves in the ion-acoustic frequency range the electrons' response is usually defined by the Debye length and the low velocity part of their distribution. Performing detailed numerical integrals over the electron distribution is then neither accurate nor appropriate. Therefore we regard the electrons here as having a Maxwellian distribution. As is well known, the electron susceptibility integrals can then be written

$$k^2\lambda_{De}^2\chi_e = \frac{1}{\sqrt{\pi}} \int \frac{ze^{-z^2}}{z-x} dz = 1 - 2xe^{-x^2} \int_0^x e^{z^2} dz + i\sqrt{\pi}xe^{-x^2} \quad (5)$$

where $x = \omega/k\sqrt{2T_e/m_e}$; the first integral is along the Landau contour. It should be emphasized that the Debye length here λ_{De} , is evaluated using the actual electron temperature T_e , not the scaling temperature T_0 (if it is different). This expression is an embodiment of the plasma dispersion function. In terms of standard functions it can be considered the derivative of $(-i\sqrt{\pi}/2)\exp(-x^2)\text{erfc}(-ix)$. In the ion-acoustic regime χ_e is well approximated by an expansion that takes v_p to be small compared with $\sqrt{2T_e/m_e}$, which yields

$$k^2\lambda_D^2\frac{T_e}{T_0}\chi_e = k^2\lambda_{De}^2\chi_e \approx 1 + i\sqrt{\frac{\pi m_e}{2T_e}} v_p. \quad (6)$$

For frequencies well below the electron gyro-frequency and $k\rho_e \ll 1$ (where ρ_e is the thermal gyro-radius) this one-dimensional expression can be readily extended to give the required longitudinal electron susceptibility for oblique propagation with respect to the magnetic field when the wave-vector has a component k_{\parallel} along the magnetic field, simply by replacing the definition of the normalized phase speed x with $x = \omega/k_{\parallel}\sqrt{2T_e/m_e}$ equivalent to replacing v_p with $v_{p\parallel}(= \omega/k_{\parallel})$. The k^2 factor multiplying χ_e , which arises from the Laplacian in Poisson's equation, remains the total k . However, the expansion eq. (6) becomes inappropriate for angles that enhance $v_{p\parallel}$ so much as to violate $|x| \ll 1$. Then a full numerical implementation of the Faddeeva function, $\exp(-x^2)\text{erfc}(-ix)$, in eq. (5) must be used.

2.2 Ion Susceptibility

The ion integrals of eq (3) must be evaluated numerically along the correct contour and performed so as to avoid large errors from the vicinity of the pole $v = v_p$. If a magnetic field is present, influencing the electron susceptibility, but the frequency is simultaneously much greater than the ion gyro-frequency, then the appropriate ion susceptibility remains unchanged. The distribution to be used is that in the k -direction.

For positive $\Im(v_p)$, the ion integral can be evaluated along the real v -axis, but the numerical errors that arise when $|v - v_p|$ is small are managed more effectively if an analytic integration by parts is first performed to give

$$-\int \frac{\partial f_j}{\partial v} \frac{1}{v - v_p} dv = \int \frac{\partial^2 f}{\partial v^2} \left\{ \frac{1}{2} \ln \left[\left(\frac{v - v_{pr}}{v_{pi}} \right)^2 + 1 \right] + i \arctan \left(\frac{v - v_{pr}}{v_{pi}} \right) \right\} dv, \quad (7)$$

where v_{pr} and v_{pi} are the real and imaginary parts of v_p . This transformation requires the derivatives to vanish at the end of the integration path, which must be observed in the numerical implementation, by using a sufficiently wide velocity range. The error introduced at low $|v_{pi}|$ by the steep behavior of $(v - v_{pr})/v_{pi}$ is first order in the velocity step size or smaller.

For exactly real v_p , the principal value of the integral can be evaluated after two integrations by parts as

$$\mathcal{P} \int \frac{\partial f_j}{\partial v} \frac{1}{v - v_p} dv = \int \frac{\partial^3 f}{\partial v^3} (v - v_p) (\ln |v - v_p| - 1) dv \quad (8)$$

giving rise to no singular contribution at $v = v_p$. The third derivative of f can use simple finite differences. Half of the residue,

$$2\pi i \left. \frac{\partial f}{\partial v} \right|_{v=v_p}, \quad (9)$$

at the pole must be added.

For negative imaginary part of v_p we require numerical analytic continuation of $f(v)$. If the integral path is, as before, along the real- v axis, then the full residue at the pole, eq (9), must be added to it, requiring $\left. \frac{\partial f}{\partial v} \right|_{v_p}$ to be known at complex values of v_p . Perhaps the most straightforward way to perform the continuation is by Fourier transforming f . If $f(v)$ is known along the real axis, then its Fourier transform can be evaluated:

$$F(p) = \int_{-\infty}^{\infty} e^{-ipv} f(v) dv. \quad (10)$$

Then the continuation of $f(z)$ to complex argument z is

$$f(z) = \int_{-\infty}^{\infty} e^{ipz} F(p) \frac{dp}{2\pi} \quad (11)$$

Analytic continuation is generally ill-conditioned. Successful numerical implementation requires the spatial frequency bandwidth to be constrained, otherwise short-wavelength errors grow exponentially into the continuation region. One can implement bandwidth limitation using a multiplicative function $B(p)$ that is 1 for small $|p|$ and becomes small for large $|p|$. Then the required derivative can be written

$$\left. \frac{\partial f}{\partial v} \right|_{v=v_p} = \int_{-\infty}^{\infty} ip e^{ipv_p} B(p) F(p) \frac{dp}{2\pi}. \quad (12)$$

This representation is obviously accurate only to the extent that the high-frequencies filtered out by $B(p)$, to suppress noise, are negligible.

It proves advantageous to use a bandwidth filter that depends upon the imaginary part of v_p , designed to limit the magnitude of the real part of the argument of the exponential: ipv_{pi} . This is done by putting in the inverse transform, eq (12):

$$e^{-pv_{pi}} B(p) = \exp \left(\frac{-pv_{pi}}{1 + (spv_{pi})^2} \right), \quad (13)$$

where s is a constant factor that for noiseless $f(v)$ is generally chosen to be $s \simeq 1/10$. This choice makes the maximum exponent value $1/2s = 5$. Increase of s might be required for noisy $f(v)$.

We suppose that the values of $f(v)$ (real v) are known on a uniform v -grid: $v_n = n\delta v$ with a total of N points. Then discrete fast fourier transforms (from the `fftpack` library) are used. The resulting frequencies are $p_n = n\delta p = n2\pi/(N\delta v)$ and the appropriate finite-duration form of the derivative replaces ip_n in eq (12) with a premultiplying factor $i \sin(n\delta p\delta v)/\delta v$. Thus the back-transform in full is the discrete equivalent of

$$\left. \frac{\partial f}{\partial v} \right|_{(v_{pr} + iv_{pi})} = \int \frac{i \sin(p\delta v)}{\delta v} e^{ipv_{pr}} \exp \left(\frac{-pv_{pi}}{1 + (spv_{pi})^2} \right) F(p) \frac{dp}{2\pi}. \quad (14)$$

3 Display and Solution

A simple graphical display allows one to read off immediately the solution of the dispersion relation, using a contour plot in complex v_p space of the value of $k^2\lambda^2\chi$, which is determined by the shape of the distribution function.

3.1 Ion Acoustic Waves

One simple way to display the solution for a single electron temperature $T_e = T_0$ is shown in Fig. 2. The velocity units are normalized to the cold ion sound speed $c_s = T_e/m_i$. The

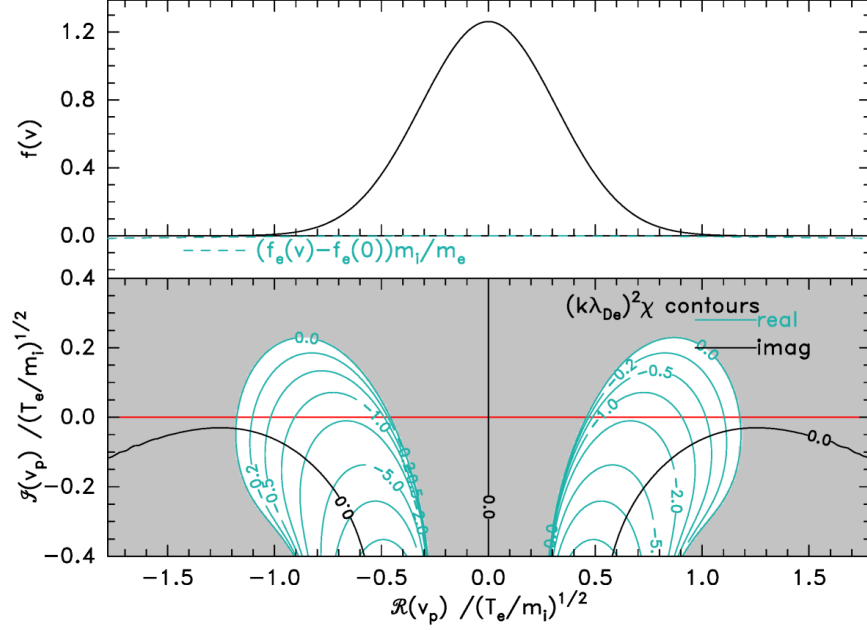


Figure 2: Contours of full susceptibility for $T_e = 10$, $T_i = 1$.

upper panel shows the ion distribution, a simple Maxwellian. A green dashed line is added (barely visible here) below the v -axis, which shows the deviation of the electron distribution from its peak value $(f_e(v) - f_e(0))m_i/m_e$ scaled by the mass ratio. It shows that the electron distribution ($T_e = 0$ here) hardly varies across the velocity range of the ion distribution. The lower panel plots contours of the full susceptibility in the form $(k\lambda_{De})^2\chi$ where $\chi = \chi_i + \chi_e$. The black contour is where the imaginary part is zero. The green contours are labelled with values of the real part. They are given only for negative values. In the gray shaded region the real part is positive and no solution of the dispersion relation exists. Since the dispersion relation is $\chi = -1$, the solution for a particular value of $k\lambda_{De}$ occurs at the intersection of the black contour with the green contour corresponding to a value $\Re(k^2\lambda_{De}^2\chi) = -(k\lambda_{De})^2$. All such intersections in this case are below the real axis of v_p , showing that ω/k has negative imaginary part and the waves are damped (k is real and ω has a negative imaginary part) in this case. We shall later encounter unstable cases where the black contour rises above the real v_p axis in the unshaded region. This display thus provides the entire dispersion relation for all real k , indicated by the contour value labels; in particular the zero contour is the long-wavelength ($k \rightarrow 0$) limit.

The contouring routine follows the black ($\Im(\chi) = 0$) contour, and as it does so, traces out values from small k , ($\Re(k^2\lambda_{De}^2\chi) \rightarrow 0$) to large k , ($\Re(k^2\lambda_{De}^2\chi) \rightarrow -\infty$). By collecting those values, and recognizing that $\omega = v_p k = v_p \sqrt{-\Re(k^2\lambda_{De}^2\chi)/\lambda_{De}}$, one can readily construct the corresponding dispersion relation and plot $\Re(\omega)$ and $\Im(\omega)$ as a function of k . Fig. 3 shows the result corresponding to Fig. 2. There are two branches corresponding to negative and positive phase velocity and $\Re(\omega)$ (k is taken positive). By symmetry they both have exactly the same damping factor $-\Im(\omega)$ which makes the two dashed lines plot exactly over one

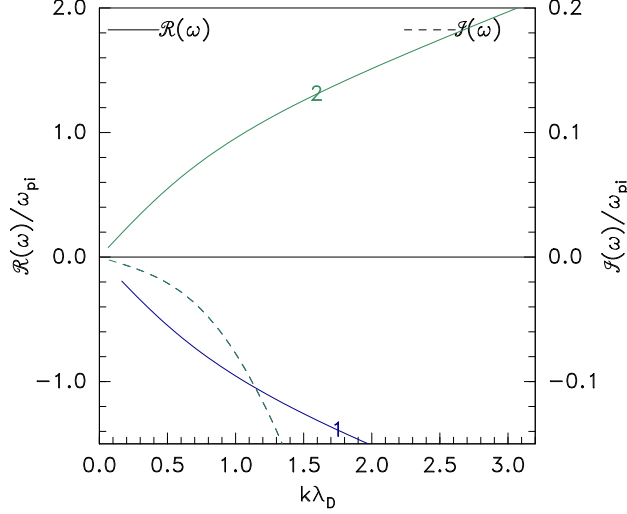


Figure 3: Dispersion plot of ω versus k for simple Maxwellian, $T_e = 10T_i$.

another.

An alternative way to display the approximate result for a range of electron temperatures not necessarily equal to T_0 is the following. We recognize that the dispersion relation can be written

$$k^2 \lambda_D^2 (\chi_i + 1) = -k^2 \lambda_D^2 \chi_e \approx -\frac{T_0}{T_e} \left(1 + i \sqrt{\frac{\pi m_e}{2T_e}} v_p \right). \quad (15)$$

The imaginary term in χ_e (electron Landau damping) is smaller by the square root of the inverse mass ratio, and significant mostly at high phase velocity. Neglecting it has the effect of raising the black contour (on which the imaginary part of χ is zero) toward zero at large $|v_p|$, but it has little effect where f_i is non-negligible. The real part of the dispersion relation is simply

$$k^2 \lambda_D^2 = -\Re(k^2 \lambda_D^2 \chi_i) - T_0/T_e. \quad (16)$$

If k is regarded as a free choice, the dispersion relation can be satisfied at any complex phase velocity at which the right hand side of (16) is positive: $\Re(k^2 \lambda_D^2 \chi_i)$ is more negative than T_0/T_e , and simultaneously $0 = \Im(k^2 \lambda_D^2 \chi) \simeq \Im(k^2 \lambda_D^2 \chi_i)$. The value of k^2 is given by eq. (16) for a root on a contour of known $\Re(k^2 \lambda_D^2 \chi_i)$ and T_0/T_e . Consequently, contours in the complex v_p plane of $\Re(k^2 \lambda_D^2 \chi_i)$ (which we've shown depends only on the shape of the ion distribution) correspond to contours of $k^2 \lambda_D^2$, if T_0/T_e is fixed. But alternatively we can consider a range of different possible T_0/T_e values, and regard the region of existence of solutions as being bounded by contours at which $\Re(k^2 \lambda_D^2 \chi_i) < -T_0/T_e$. Taking $T_e \rightarrow \infty$ gives the largest extent ($\Re(k^2 \lambda_D^2 \chi_i) < 0$) for which this can be satisfied, but for any other temperature ratio the contour equal to $-T_0/T_e$ bounds the existence region. For example when $T_0/T_e = 1$, $\Re(k^2 \lambda_D^2 \chi_i) < -1$.

Fig 4 shows what is obtained for the same simple Maxwellian as before, choosing $T_0 = 10T_i$. The light-gray shaded regions are those where $\Re(k^2 \lambda_D^2 \chi_i)$ is more positive than -1 . Those are regions where no solutions exist for $T_e = T_0$. For that electron temperature,

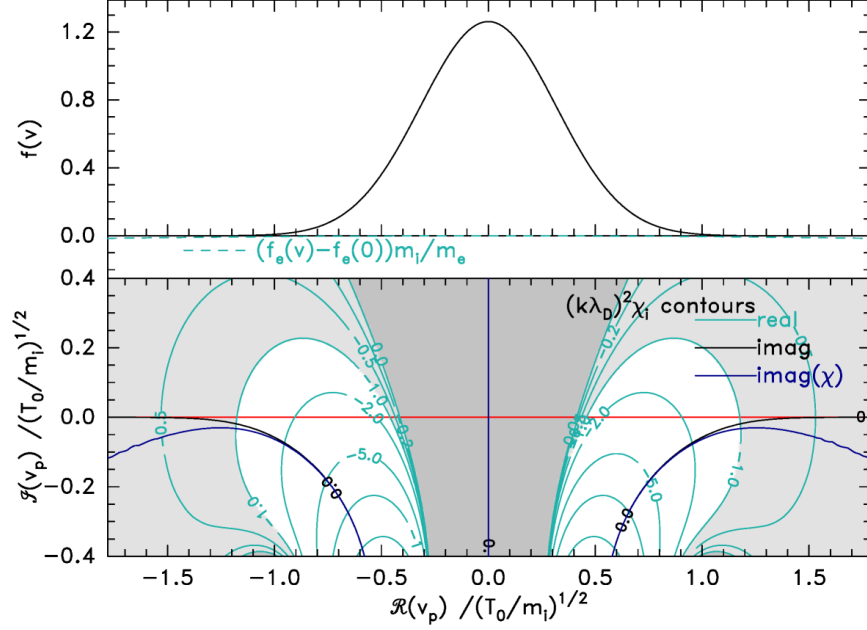


Figure 4: Contour Plot of ion susceptibility from which the dispersion relation can be read. The real-part contour heights are logarithmically spaced, being at minus 0., .2, .5, 1., 2., 5., 10. etc. These values can be interpreted as the T_0/T_e at which each contour is the $k \rightarrow 0$ boundary of solution existence. Simple Maxwellian ion distribution with $T_0 = 10T_i$.

the dispersion relation solutions lie (only) along those segments of the imaginary-part zero-contour that lie in unshaded regions. Those unshaded regions are actually identical to the unshaded regions of Fig. 2, even though the real contour values differ from $k^2\lambda_D^2\chi$ by -1 . Although the green and black contours are of χ_i , the actual dispersion relation requires $\Im(\chi) = 0$, using the full $\chi = \chi_i + \chi_e$. By plotting in addition the zero contour of $\Im(\chi)$ (for the case $T_e = T_0$): the blue line, one obtains the full solution (following the blue line and finding its intersections with the green) of the dispersion for that case. The black line is the $\Im(\chi) = 0$ contour ignoring electron Landau damping, which is exact only when $T_e \rightarrow \infty$. So the difference between the black and blue contours indicates the degree of approximation involved in ignoring electron Landau damping for $T_0/T_e = 1$. At any electron temperature between T_0 and ∞ the corresponding imaginary contour would lie between the blue and the black curves. At any point along such curves, the real part of $k^2\lambda_D^2\chi_i$ can be read from the green contours that intersect it. Knowing $k^2\lambda_D^2\chi_i$ one can immediately deduce the value of k from eq. (16).

Omission of electron Landau damping hardly affects the black contour in the unshaded region, but it raises it toward $v_{pi} = 0$ further out in the light shaded region. The smallest k solution for $T_e = T_0$: the long wavelength solution, occurs at the boundary with the (light) shaded region where $\Re(k^2\lambda_D^2\chi) = 0$ and $\Re(k^2\lambda_D^2\chi_i) = -1$. In Fig 4, that position is at $\Re(v_p) \approx \pm 1.15c_s$ because $T_i/T_0 = 0.1$ in this case, and the actual warm-ion acoustic speed

for such waves is approximately $\sqrt{(T_e + 3T_i)/m_i}$. The imaginary part of v_p is everywhere negative along the solution curve (blue) in this plot, indicating that these waves are damped. The damping rapidly becomes stronger as one moves away from the shaded edge because the real part becomes large (and negative) implying large k , and thus large ω_i (because $\omega_i = kv_{pi}$). The edge of the allowed solution region for a T_e different from T_0 is at the contour where $\Re(k^2\lambda_D^2\chi_i) = -T_0/T_e$. So this contour plot can be considered universal for any T_e . For example, if $T_e = 20T_i = 2T_0$, then the -0.5 contour is the allowed solution boundary and the long wavelength root has v_p just over 1.5. Or again, if $T_e = 5T_0$ the boundary is at the -2.0 contour.

A different case is shown in Fig 5, where a relatively long tail of slightly positive slope is

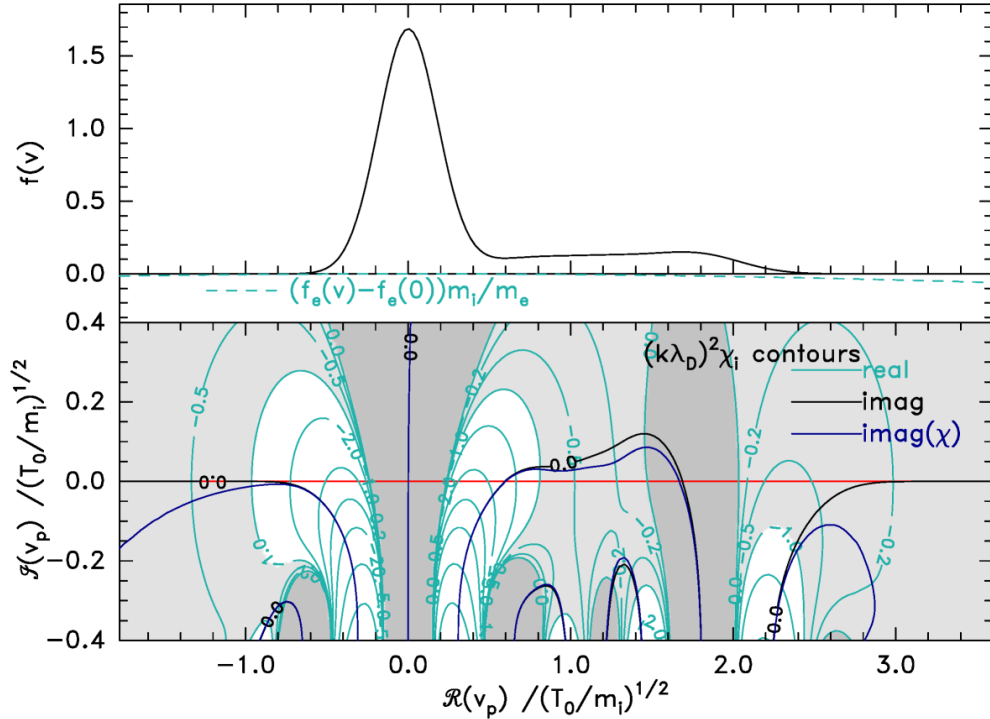


Figure 5: Contour Plot of ion susceptibility for an ion distribution with a long tail of slightly positive slope, showing unstable regions. $T_0 = 30T_i$.

added to the Maxwellian bulk (whose temperature is $T_0/30$). The tail induces an ion-acoustic (sometimes called ion-ion) instability. This is evidenced by the region where the $\Im(\chi_i) = 0$ contour rises above the real v_p -axis, corresponding to wave growth. The unstable solutions actually exist only near the low-velocity end of the tail for $T_e = T_0$. In the higher-velocity range, although the contour is still at positive v_{pi} it is in the light shaded v_p region where the real part of χ is positive and the dispersion relation cannot be satisfied. This observation is an important corrective to any improper transference of intuition derived from *electron* stability of Langmuir waves. Unstable Langmuir waves exist on an electron-distribution-tail

throughout most of the region where its slope is positive. That is not the case for ion-ion instability.

Actually a direct quantitative understanding of this difference can be derived from the figure. In the limit $T_e \rightarrow \infty$, the electrons contribute negligibly to the total susceptibility. Only the ion contribution need be considered. In that case the dispersion relation (along the black line) is *exactly* the same as it would be for Langmuir waves of electrons of the same distribution (scaling the velocities appropriately, see section 4.2). We see that then the solution region (outside the dark grey shading) extends almost (but actually not quite) to the upper end of the positive slope region. This is exactly the region of Langmuir wave instability for an electron distribution of this shape. However, as T_e is lowered, the electron contribution to χ becomes significant, and causes the solution region to contract towards the lower velocity end of the tail. If T_e becomes less than approximately $T_0/3$, the solution region (whose boundary is then on the ~ -3 contour) no longer contains any positive- $\Im(v_p)$ regions of the $\Im(\chi_i) = 0$ contour. The ion instability disappears. The last phase velocity to remain unstable is where the $\Im(\chi) = 0$ contour crosses the x-axis: $v_p \approx 0.5c_s$. This suppression of ion-ion instability by lowering of T_e is *nothing to do with electron Landau damping*, which is still negligible, as evidenced by the equality of the black and blue curves in this vicinity. It is caused by the real part (not the imaginary part) of the electron susceptibility.

In the opposite direction to the tail, a relatively weakly damped acoustic branch exists. A third branch, which is strongly damped unless T_e is very high, exists just beyond $2c_s$, and some other extremely heavily damped branches of little practical relevance also exist at $\Im(v_p) < -0.3$. Those heavily damped branches are omitted from the $\omega(k)$ plot of Fig. 6. The branch 2 is now unstable for $k\lambda_D \lesssim 1.4$.

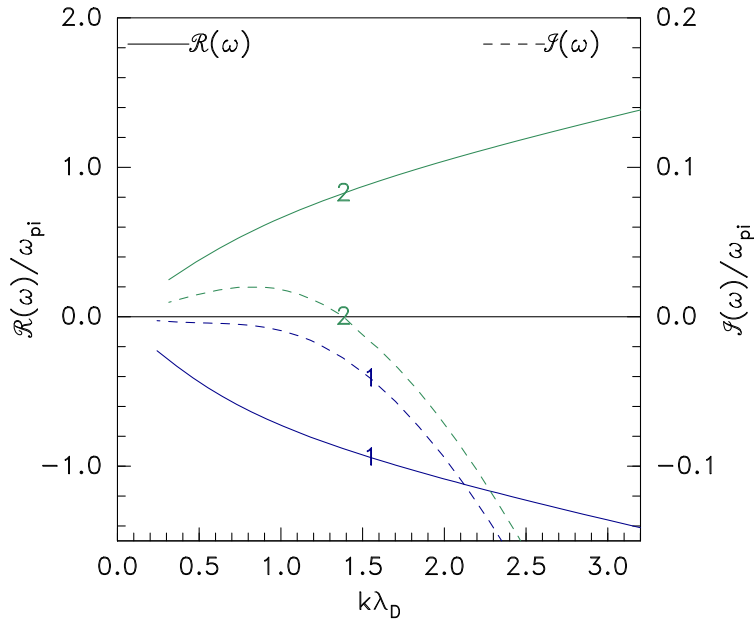


Figure 6: Dispersion curves for the $T_e = T_0 = 30T_i$ case of Fig. 5.

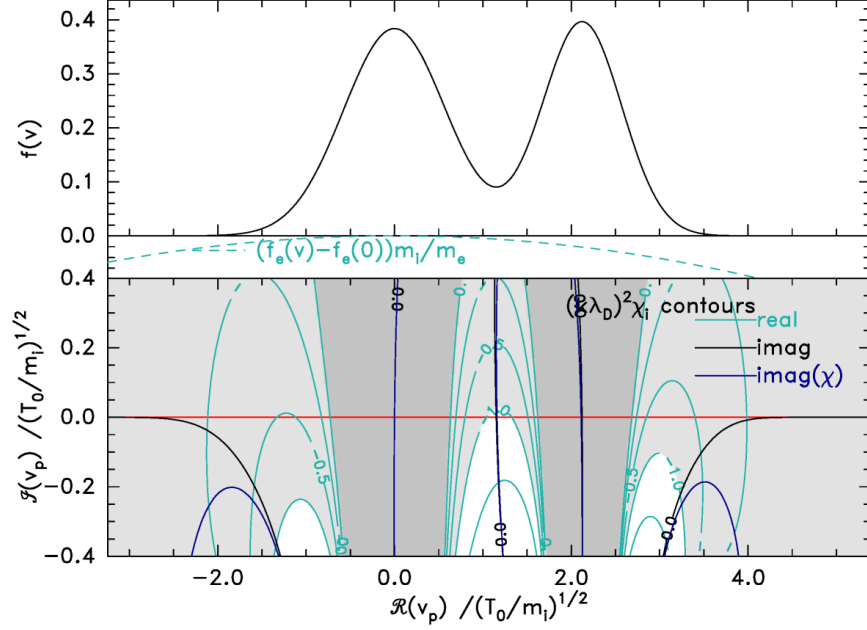


Figure 7: Contour Plot of ion susceptibility for an ion distribution with two humps. T_0 is 3 times the temperature of the unshifted hump.

Fig. 7 shows an ion distribution with two overlapping humps. The reference temperature is taken as 3 times the temperature of the unshifted hump (the other hump has width corresponding to a temperature 0.6 times lower). When $T_e = T_0$, waves are only just unstable in the distribution minimum near $v_p = 1.2$. Lower T_e would not be unstable. Higher T_e permits greater growth rate, again indicating the importance of T_e in determining stability, but again not arising from electron Landau damping. For five times higher T_e the maximum growth rate of the instability is greatly increased, yet the range of real part of the phase velocities that are unstable is extremely narrow, residing at the bottom of the valley between the humps, where electron Landau damping is unimportant.

Fig. 8 shows a case by contrast in which two Gaussian humps have negligible overlap. The stable ion sound branches for the two humps are clearly visible. The centers of the ion humps are separated by just over twice c_s . The result is that the allowed solution regions (unshaded) for $T_e = T_0$ merge in the middle, producing an unstable region along a section of the vertical $\Im(\chi) = 0$ branch. Lower temperatures (e.g. the -2.0 contour, $T_e = 0.5T_0$) don't merge and so do not have unstable solutions, higher temperatures (light gray) have greater extent and growth rate of instability. Generally speaking, humps separated by more than twice the ion sound speed don't lead to instability. Stringer¹ gives a criterion $1.3v_{ti} < \Delta v/2 < \sqrt{T_e/m_i}$ for ion-ion instability when equal Maxwellian ion distributions are separated by velocity Δv .

¹T E Stringer, Plasma Physics, **6**, 267-279 (1964).

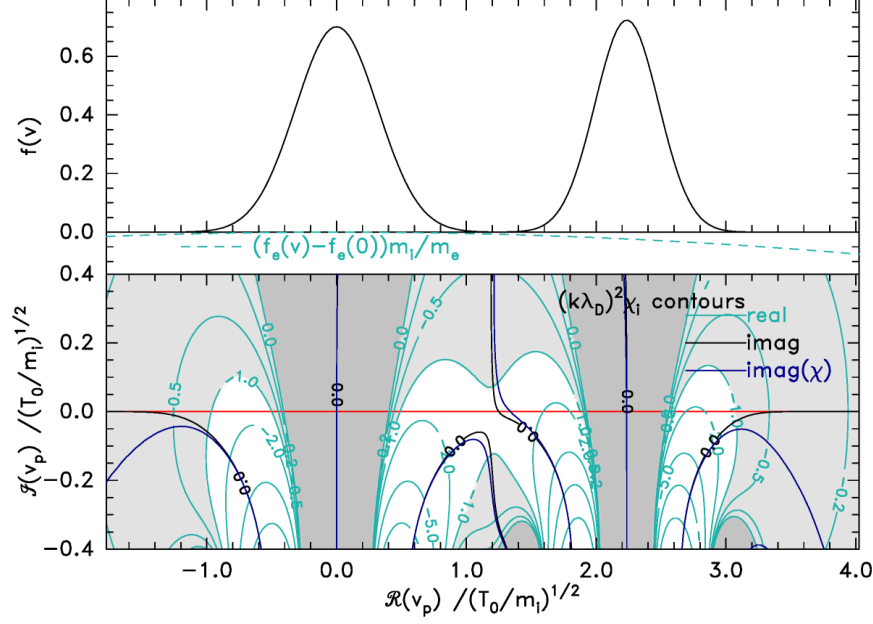


Figure 8: Contour Plot of ion susceptibility for an ion distribution with two well separated Gaussians. T_0 is 10 times the temperatures of the centered ion Gaussian.

4 Oblique Propagation

So far the analysis has been of electrostatic waves either in an unmagnetized plasma or propagating parallel to the magnetic field. Moreover here and in subsequent analyses the distribution function to be used has been the one dimensional distribution, i.e. integrated over velocities perpendicular to the wavevector \mathbf{k} .

4.1 Beam-like Ion distributions

If the ion distribution in three velocity directions were really represented by two shifted but otherwise isotropic finite width beams (e.g. shifted 3-D Gaussians) and the plasma were unmagnetized, then the relevant velocity shift between the beams as viewed in the one-dimensional distribution is equal to the velocity shift in 3-D, $|\Delta \mathbf{v}|$, times the cosine of the angle between \mathbf{k} and the direction of velocity shift i.e. $\mathbf{k} \cdot \Delta \mathbf{v} / (k \Delta v)$. If then Δv is large enough in one dimension to avoid ion-ion instability because $\Delta v \gtrsim 2\sqrt{T_e/m_i}$, that does *not* make an unmagnetized plasma stable. Instead we would require $\mathbf{k} \cdot \Delta \mathbf{v} / k \gtrsim 2\sqrt{T_e/m_i}$ for *all possible propagation directions*. Clearly that can never be satisfied, because we can make the wave vector and shift directions as near to perpendicular as we like.

The most common situation for oblique propagation of ion waves is that the electron cyclotron frequency is much higher than that of the waves, while the ion cyclotron frequency is much lower. Thus to a first approximation electrons are strongly magnetized, moving

only parallel to the magnetic field, while ions are unmagnetized. Since the magnetization of electrons changes their susceptibility only by replacing v , v_p in the integral by v_{\parallel} , $v_{p\parallel}$, or equivalently making the argument of eq. 5 $x = \omega/k_{\parallel}\sqrt{2T_e/m_e}$, oblique propagation mostly narrows the effective electron distribution, enhancing electron Landau damping at the low speeds typical of ion waves. However, as we have just seen, electron Landau damping is a relatively unimportant effect for controlling the ion-ion instability of separated beams. Electrons' more important influence is in setting the height of the combined distribution function and thereby affecting the real part of the susceptibility. That is not changed by propagation angle.

In general then, unmagnetized plasmas are hardly ever stabilized by having widely spaced beam distribution. Maybe a hollow sphere could do it.

4.2 Electron Langmuir Waves

The same display can also calculate the dispersion of electron Langmuir waves. Consider a high-frequency situation in which χ_i can be ignored and χ_e is the electron susceptibility for an arbitrary electron distribution function. Regard that distribution as being given by the upper frame of the plot (what in the ion acoustic analysis is the ion distribution function). Take the reference velocity to be $\sqrt{T_0/m_e}$. The dispersion relation is then $\chi_e = -1$. Since shape of the distribution function and the complex v_p together determine the value of $k^2\lambda_D^2\chi_e$ (eq. 3), the dispersion relation is satisfied for some real value of k at points in v_p space where the value of the real part of $k^2\lambda_D^2\chi_e$ is negative and the value of its imaginary part is zero, just as before. Therefore we exchange the roles of electrons and ions. We consider the plot to be for electron susceptibility. The solution of the dispersion relation is where the imaginary part is zero and the real part is negative. This reversal of roles helps one to understand the qualitative difference between ion acoustic stability and Langmuir stability. It lies purely in the fact that non-zero electron susceptibility shifts the real contours of the ion acoustic case. When $T_e/T_0 \rightarrow \infty$, and χ_e becomes negligible, the ion acoustic instability is exactly equivalent to the Langmuir stability with appropriately scaled velocity.

5 Group Velocity

The plots automatically give the phase velocity of the wave, since they are plots on the real and imaginary axes of v_p . The group velocity corresponding to waves on these solution contours can be found as follows.

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(v_p k) = v_p + \frac{k}{dk/dv_p} = v_p + \frac{k\lambda_D}{d(k\lambda_D)/dv_p} \quad (17)$$

We substitute the value $k^2\lambda_D^2$, from eq. (16), which is what is contoured in the plots. And we recognize that the second term by which v_g differs from v_p is simply half the scale length of the variation of $k^2\lambda_D^2$ as a function of v_p . Since (except for the zero level) the

contours are logarithmic in the ratios 1,2,5, the distance between the function's contours is approximately the scale length of the function.

Therefore the value of the group velocity is approximately half way between the contour of $-\Re(k^2\lambda_D^2\chi_i)$ on which it lies and the adjacent contour further from the shaded boundary. In most cases the shift from v_p is small enough to be inconsequential. There will be zero group-velocity solutions only if there are solutions with nearly zero phase velocity.