In 3-6 use the sample space given in example 9.1.1. Write each event as a set, and compute its probability.

3. The event that the chosen card is red and is not a face card.

$E = \{ AD, 2D, 3D, 4D, 5D, 6D, 7D, 8D, 9D, 10D$
$AH, 2H, 3H, 4H, 5H, 6H, 7H, 8H, 9H, 10H \}$

$$P(E) = \frac{n(E)}{n(\Omega)} = \frac{20}{52} = \frac{5}{13}$$

In 7-10, use the sample space given in Example 9.1.2. Write each of the following events as a set and compute its probability.

7. The event that the sum of numbers showing face up is 8.

$E = \{26, 35, 44, 53, 62\}$

$P(E) = \dfrac{n(E)}{n(\Omega)} = \dfrac{5}{36}$

In 7-10, use the sample space given in Example 9.1.2. Write each of the following events as a set and compute its probability.

9. The event that the sum of the numbers showing face up is at most 6.

$E = \{11, 12, 13, 14, 15, 21, 22, 23, 24, 31, 32, 33, 41, 42, 51\}$

$$P(E) = \frac{n(E)}{n(\Omega)} = \frac{15}{36}$$

12. Suppose that each child born is equally likely to be a boy or a girl. Consider a family with exactly three children. Let BBG indicate that the first two children born are boys and the third child is a girl, let GBG indicate that the first and third children are girls and the second is a boy, and so forth.

a. List the eight elements in the sample space whose outcomes are all possible genders of the three children.

$\Omega = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\}$

b. Write each of the events in the next column as a set and find its probability.

(i) The event that exactly on child is a girl.

$E = \{BBG, BGB, GBB\}$

$P(E) = \frac{m(E)}{m(\Omega)} = \frac{3}{8}$

(ii) The event that at least two children are girls.

$E = \{BGG, GBG, GGB, GGG\}$

$P(E) = \frac{m(E)}{m(\Omega)} = \frac{4}{8} = \frac{1}{2}$

iii) The event that no child is a girl

$E = \{BBB\}$

$P(E) = \frac{m(E)}{m(\Omega)} = \frac{1}{8}$

9.7.16

16. Two faces of a six-sided die are painted red, two are painted blue, and two are painted yellow. The die is rolled three times, and the colors that appear face up on the first, second, and third rolls are recorded.

– a. Let BBR denote the out come where the color appearing face up on the first and second rolls is blue, and the color appearing face up on the third roll is red. Because there are as many faces of one color as of any color, the outcome of this experiment are equally likely. List all 27 possible outcomes

Ω = { BBB, BBR, BBY, BRB, BYB, RBB, YBB, BRR, BYY, RBR, YBY, RRB, YYB,
RRR, RRY, RYR, YRR, RYY, YRY, YYR, YYY, BRY, BYB, RBY, RYB, YBR, YRB }

– b. Consider the event that all three rolls produce different colors. One outcome in this event is RBY and another is RYB. List all outcome in the event. What is the probability of the event?

E: { RYB, RBY, BRY, BYR, YRB, YBR }

$$P(E) = \frac{6}{27}$$

– c. Consider the event that two of the color that appear face up are the same. One outcome is RRB and another is RBR. List all outcomes. What is the probability of the event

E = { RRB, RBR, BRR, RRY, RYR, YRR, BBY, BYB, YBB, BBR, BRB, RBB,
YYR, YRY, RYY, YYB, YBY, BYY }

$$P(E) = \frac{n(E)}{n(\Omega)} = \frac{18}{27} = \frac{2}{3}$$

9.1.21

21. a) How many positive two-digit integers are multiple of 3?

10   12   15   ...   99

$3 \cdot 4$ $3 \cdot 5$ $3 \cdot 33$

$m = 33 - 4 + 1 = 30$

Answer: $m(E) = 30$

b) What is the probability that a randomly chosen positive two-digit integer is a multiple of 3?

$m(\Omega) = 99 - 10 + 1 = 90$

$P(E) = \frac{m(E)}{m(\Omega)} = \frac{30}{90} = \frac{1}{3}$

c) What is the probability that a randomly chosen positive two-digit integer is a multiple of 4?

10   12   16   ...   96

$4 \cdot 3$ $4 \cdot 4$ $4 \cdot 24$

$m(E) = 24 - 3 + 1 = 22$

$m(\Omega) = 99 - 10 + 1 = 90$

$P(E) = \frac{m(E)}{m(\Omega)} = \frac{22}{90} = \frac{11}{45}$

24. Suppose $A[1], A[2], \ldots A[n]$ is a one-dimentional array and $n \geq 2$. Consider the subarray

$$A[1], A[2], \ldots A[\lfloor \tfrac{n}{2} \rfloor]$$

a. How many elements in the subarray if (i) n is even and (ii) n is odd

i: # of elements $= \frac{n}{2}$

ii: # of elements $= \frac{n-1}{2}$

b. What is the probability that a randomly chosen array element is in the subarray (i) if n is even (ii) if n is odd

i: $P = \dfrac{\frac{n}{2}}{n} = \frac{1}{2}$

ii: $P = \dfrac{\frac{n-1}{2}}{n} = \frac{1}{2} - \frac{1}{n}$ $\quad$ ($\frac{1}{2}$ if $n >> 2$

9.1.28

28. If the larges of 56 consecutive integers is 279, what is the smallest?

$m = x$, $n = 279$, $n - m + 1 = 56$

$280 - m = 56$

$m = 280 - 56$

$= 224$

The smallest # is 224

9.1.32

32. A certain non-loap year has 365 days, and January 1 occurs on a Monday

$$\overset{1 = 7\cdot 0+1}{\text{Monday}} \underbrace{\qquad\qquad}_{+\,7\cdot 1} \overset{8 = 7\cdot 1+1}{\text{Monday}} \underbrace{\quad \cdots\cdots \quad}_{+\,7\cdot 52} \overset{365 = 7\cdot 52+1}{\text{Monday}}$$

a. How many Sundays are tw the year?

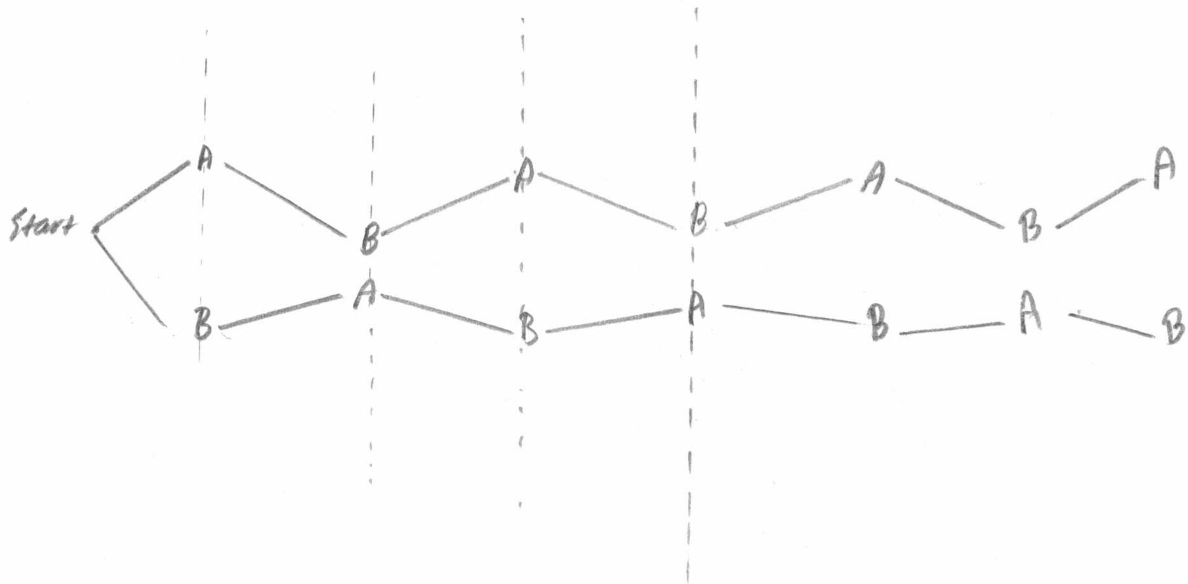
$$\#\text{ of Sundays} = 52-1+1 = 52$$

b. How many Mondays are tw the year?

$$\#\text{of Mondays} = 52-0+1 = 53$$

In 1-4, use the fact that in baseball's World Series, the first team to win four games wins the series.

4. How many ways can a World Series be played if no team wins two games in a row.



Answers 2 ways ABABABA & BABABAB

9.2.11

11 a. A bit string is a finite sequence of 0's and 1's. How many bit strings have length 8. ?

$2^8 = \underline{256}$

b. How many bit string of length 8 begin with three 0's?

$2^5 = \underline{32}$

c. How many bit strings of length 8 begin and end with a 1?

$2^6 = \underline{64}$

9.2.16

16. a. How many integers are there from 10 through 99?

\# of integers: $99 - 10 + 1 = \underline{\underline{90}}$

b. How many odd integers are there from 10 through 99?

\# of odd integers:

10 11 12 13 ... 99

6 +1 0 +0 +1 +n

$\frac{90}{2} = \underline{\underline{45}}$

c. How many integers from 10 through 99 have distinct integers?

\# of distinct integers: $\overset{1st}{9} \cdot \overset{2nd}{9} = 81$

d. How many odd integers from 10 through 99 have distinct digits?

\# of distinct odd integers: 40

Case 1st is even: $4 \cdot 5 = 20$

Case 1st is odd: $5 \cdot 4 = 20$

e. What is the probability that a randomly chosen two-digit integer has distinct digits? Has distinct digits and odd?

$P_D = \frac{m(D)}{m(\Omega)} = \frac{81}{90} = \frac{9}{10} \approx 90\%$

$P_{DO} = \frac{m(DO)}{m(\Omega)} = \frac{40}{90} = \frac{4}{9}$

20. Modify Example 9.2.4 by supposing that a PIN must not begin with any of the letters A-M and must end with a digit. Continue to assume that no symbol may be used more than once and that the total number of PINs is to be determined.

a. Find the error in the following "solution."

"Constructing a PIN is a four-step process.

step1: Choose the left-most symbol.

step2: Choose the second symbol from the left.

step3: Choose the third symbol from the left

step4: Choose the right-most symbol.

Because none of the thirteen letters from A to M may be chosen in step 1, there are $36-13=23$ ways to perform step1, there are 35 ways to perform step2 and 34 ways to perform step3 because previously used symbols may not be used. Since the symbol chosen in step 4 must be a previously unused digit there $10-3=7$ ways to perform step4. Thus there are $23 \cdot 35 \cdot 34 \cdot 7 = 191,590$ different PINs that satisfy the given conditions."

Error: The order of the steps makes the number of ways for step4 varies (not a constant), hence preventing the use of multiplication rule.

b. Reorder steps 1-4 in part (a) as follows:

step1: Choose the right-most symbol.

step2: Choose the left-most symbol

step3: Choose the second symbol from the left.

step4: Choose the third symbol from the left.

Use the Multiplication Rule to find the number of PINs that satisfy the given conditions.

# of PINs: $10 \cdot (36-13-1)(36-2)(36-3) = 10 \cdot 22 \cdot 34 \cdot 33 = 246,840$

In each of 24-28, determine how many times the innermost loop will be iterated when the algorithm segment is implemented and run. (Assume that m, n, p, a, b, c, and d are all positive integers).

```
24. for i := 1 to 30
        for j := 1 to 15
            [statement of the inner loop.
             None contain branching statements
             that lead outside the loop.]
        next j
    next i
```

Answer: $15 \cdot 30 = 450$ times

31. a. If p is a prime number and a is a positive integer, how many distinct positive divisor does p^a have.

$$p^a: \underbrace{1, p}_{1|p}, \underbrace{p, p^2}_{p|p^2}, \underbrace{\dots, p^a}_{p^{a-1}|p^a} \Rightarrow \text{Answer: } \underline{\underline{\overset{\text{ways}}{a+1}}}$$

b. If p and q are distinct prime numbers and a and b are positive integers, how many distinct positive divisors does $p^a q^b$ have

from part a: Answer $\overset{\text{ways}}{(a+1)} \cdot \overset{\text{ways}}{(b+1)}$

c. If p, q, and r are distinct prime numbers and a, b, and c are positive integers, how many distinct positive divisors does $p^a q^b r^c$ have?

from part a: Answer: $\overset{\text{ways}}{(a+1)} \cdot \overset{\text{ways}}{(b+1)} \cdot \overset{\text{ways}}{(c+1)}$

d. If $p_1, p_2, \dots, p_m$ are distinct primes and $a_1, a_2, \dots, a_m$ are positive integers, how many distinct divisors does $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ have?

from part a: Answer: $\overset{\text{ways}}{(a_1+1)} \overset{\text{ways}}{(a_2+1)} \cdots \overset{\text{ways}}{(a_m+1)}$

e. What is the smalles positive integers with exactly $\underline{\underline{12}}$ divisors?

Smalles Primes:
$2, 3, 5$

$12 = 6 \cdot 2 = 4 \cdot 3 = 3 \cdot 2 \cdot 2$

Case 3 | Case 2 | Case 1

Case 1: $(a+1)(b+1)(c+1) \Rightarrow a=2,\ b=1,\ c=1$

Case 2: $(a+1)(b+1) \Rightarrow a=3,\ b=2$

Case 3: $(a+1)(b+1) \Rightarrow a=5,\ b=1$

Case 1: $2^2 \cdot 3^1 \cdot 5^1 = 60 \Rightarrow$ Answer: $\underline{\underline{60}}$

Case 2: $2^3 \cdot 3^2 = 72$

Case 3: $2^5 \cdot 3^1 = 96$

35. Write all the 2-permutations of $\{w, x, y, z\}$

$P(4,2) = 12$

$P_{4,2} = \{wx, wy, wz, xw, xy, xz, yw, yx, yz, zw, zx, zy\}$

9.2.41

41. Prove that for all integers $n \geq 2$

$$P(n+1, 2) - P(n, 2) = 2P(n, 1)$$

Proof | Suppose $n \geq 2$

$$\begin{aligned}
P(n+1, 2) - P(n, 2) &= \frac{(n+1)!}{(n-1)!} - \frac{(n)!}{(n-2)!} \\
&= \frac{(n+1)(n)!}{(n-1)(n-2)!} - \frac{(n)!}{(n-2)!} \\
&= \frac{(n)!}{(n-2)!}\left(\frac{n+1}{n-1} - 1\right) \\
&= \frac{(n)!}{(n-2)!}\left(\frac{n+1-(n-1)}{n-1}\right) \\
&= \frac{(n)!}{(n-2)!}\left(\frac{2}{(n-1)}\right) \\
&= \frac{2(n)!}{(n-1)!} \\
&= 2P(n, 1) \quad \blacksquare
\end{aligned}$$

45. Prove theorem 9.2.2 by mathematical induction.

Theorem 9.2.28 For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$

Proof | Suppose $P(n) = n!$ for all integers $n \geq 1$ where n is the number of elements in a set

$$P(1) = 1! = 1$$ True for basis step.

Inductive Steps Let $P(k) = k!$ where k is an integer
Show $P(k+1)$ is true

$$P(k+1) = (k+1)P(k)$$
$$= (k+1)k!$$
$$P(k+1) = (k+1)!$$

$\therefore$ $P(n)$ is true by mathematical induction.

4. How many arrangements in a row of no more than three letters can be formed using the letters of the word NETWORK (with no repetition allowed)?

Answer: $7 \cdot 6 \cdot 5 + 7 \cdot 6 + 7 = 259$

9.3.11

11\. a. How many ways can the letters of the word QUICK be arranged in a row?

Answer: $5! = 120$

b\. How many aray can the letters ab the word QUICK be arranged in a row if the Q and the U must remain next to each other in the order QU?

Answer $= 4! = 24$

c\. How many ways can the letters ab the word QUICK be arranged in a row if the letter QU must remain together but may be in either the order QU or the order UQ

Answer: $4! + 4! = 48$

9.3.15

15. Identifier in a certain database language must begin with a letter, and then may be followed by other characters, which can be letters, digits, or underscores (_). However, 82 keywords (all consisting of 15 or fewer characters) are reserved and cannot be used as identifiers. How many identifiers with 30 or fewer characters are possible? (Write the answer using the summation notation and evaluate it using a formula from Section 5.2).

Answer: $\left(26\sum_{i=0}^{29} 37^{i}\right) - 82$

$= 26 \cdot \frac{(37^{30}-1)}{(37-1)} - 82 = 8.03 \times 10^{46}$

9.3.20

20. a. How many integers from 1 through 100,000 contain the digit 6 exactly once?

Answer: $5 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = \underline{32,805}$

b. How many integer from 1 through 100,000 contain the digit 6 at least once?

No 6: $9^5 + 1$

Answer: $10^5 - (9^5 + 1) = \underline{40950}$

c. If an integer is chosen randomly from 1 through 100,000, what is the probability that it contains two or more occurance of the digit 6?

$$P(E) = \frac{(40,950 - 32,805)}{100,000} = 8.146\ \%$$

9.3.25

25. Counting Strings:

a. Make a list of all bit strings of length zero, one, two, three, and four that do not contain the bit pattern 111.

Length 0: {ε}

Length 1: {0, 1}

Length 2: {00, 01, 10, 11}

Length 3: {000, 001, 010, 100, 011, 101, 110}

Length 4: {0000, 0001, 0010, 0100, 1000, 1001, 1010, 1100, 0101, 0110, 0011, 1011, 1101}

b. For each integer $n \geq 0$, let d_n = the number of bit strings of length n that do not contain the bit patter 111. Find d_0, d_1, d_2, d_3, and d_4.

$d_0 = 1$, $d_1 = 2$, $d_2 = 4$, $d_3 = 2^3 - 1 = 7$, $d_4 = 2^4 - 3 = 16$

c. Find a recurrence relation for $d_0, d_1, d_2, \ldots$

$d_n = d_{n-1} + d_{n-2} + d_{n-3} \quad n \leq 3 \qquad d_3 = 7,\ d_2 = 4$

d. Use the result of parts (b) and (c) to find the number of bit strings of length 5 that do not contain the pattern 111.

$d_5 = 4 + 7 + 13 = 24$

9.3.26

26. Counting Strings: Consider the set of all strings of a's, b's, and c's

a. Make a list of all of these strings of lengths zero, one, two, and three that do not contain the patter aa.

length 0: $\{\varepsilon\}$

length 1: $\{a, b, c\}$

length 2: $\{ab, ac, ba, bb, bc, ca, cb, cc\}$

length 3: $\{aba, abb, abc, aca, acb, acc, bab, bac, bba, bbb, bbc, bca, bcb, bcc, cab, cac, cba, cbb, cbc, cca, ccb, ccc\}$ 22

b. For each integer $n \geq 0$, let S_n = the number of strings of a's, b's, and c's of length n that do not contain the pattern aa. Find S_0, S_1, S_2, and S_3.

$S_0 = 1, \quad S_1 = 3, \quad S_2 = 8, \quad S_3 = 22$

c. Find a recurrence relation for $S_0, S_1, S_2, \ldots$.

$S_n = 2(S_{n-1} + S_{n-2})$ for $n \leq 2$

d. Use the results of parts (b) and (c) to find the number of strings of a's, b's, and c's of length four that do not contain the pattern aa.

$S_4 = 2(22 + 8) = 60$

e. Use the technique described in Section 5.8 to find an explicit formula for $S_0, S_1, S_2, \ldots$

31. Assume that birthdays are equally likely to occur in any one of the 12 months of the year

a. Given a group of four people A, B, C, and D, what is the total number of ways in which birth months could be associated with A, B, C, and D?

Answer: $12^4 = 20,736$

b. How many ways could birth months be associated with A, B, C, and D so that no two people would share the same birth month?

Answer: $12 \cdot 11 \cdot 10 \cdot 9 = 11,880$

c. How many ways could birth months be associated with A, B, C, and D so that at least two people would share the same birth month?

Answer: $12^4 - 12 \cdot 11 \cdot 10 \cdot 9 = 8,856$

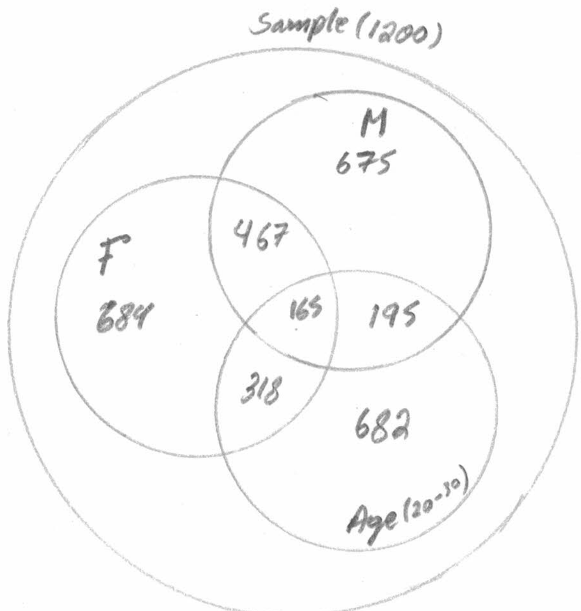
d. What is the probability that at least two people out of A, B, C, and D share the same birth month?

Answer $= \dfrac{8,856}{20,736} = 42.71\%$

e. How large must n be so that any group of n people, the probability that two or more people share the same birth month is at least 50% or more?

$$\frac{12^n - P(12,n)}{12^n} \geq \frac{1}{2} \Rightarrow \underline{\underline{n=6}} \text{ gives } 61.81\%$$

35. An interesting use of inclusion/exclusion rule is to check survey numbers for consistency. For example, suppose a public opinion polltaker reports that out of a national sample of 1,200, 675 are married, 682 are from 20 to 30 years old, 684 are female, 195 are married and from 20 to 30 years old, 476 are married females, 318 are females from 20 to 30 years old, and 165 are married females from 20 to 30 years old. Are the polltaker's figures consistent? Could they have occurred as a result of an actual sample survey?



$$n(\Omega) = 675 + 684 + 682 - 195 - 467 - 318 + 165$$
$$= 1226$$
$$1200 \neq 1226 \ !$$

Answer: The figures are inconsistent! → Not from an actual survey

For 40 and 41, use the definiton of the Euler phi function ϕ on page 396

40. Use the inclusion/exclusion principal to prove the following: If $n = pq$, where p and q are distinct prime numbers, then $\phi(n) = (p-1)(q-1)$

Proof | Suppose $n = pq$, where p and q are distinct primes

Let A be the set of positive integers such that $p \mid n$

& Let B be the set of positive integers such that $q \mid n$

clearly, $\phi(n) = (A \cup B)^c$

$$
\begin{aligned}
m((A \cup B)^c) &= m(U) - m(A \cup B) \\
&= m(U) - [m(A) + m(B) - m(A \cap B)] \\
&= n - (q + p - 1) \\
&= pq - q - p + 1 \\
&= q(p-1) - (p-1) \\
&= (p-1)(q-1)
\end{aligned}
$$

$\blacksquare$

$m(U) = n$
$m(A) = q$
$m(B) = p$
$m(A \cap B) = 1$

because p & q are primes

5. a. Given a set of four integers, must there be two that have the same remainder when divided by 3? Why?

Answer: Yes because there are only three remainders for 3 namely 0, 1, 2, and by the Pigeonhole Principal, 2 integers must have the same remainder.

b. Given a set of three integers, must there be two that have the same remainder when divided by 3? Why?

Answer: No. For example {0, 1, 2} is a set of 3 integers that do not share the same remainder when divided by 3.

9. a. If seven integers are chosen from between 1 and 12 inclusive, must at least on of them be odd? Why?

Answer: Yes because there are 6 even and 6 odd numbers between 1 and 12, chosing seven is guaranteed to have at least 1 odd number by the Pigeonhole Principal.

b. If ten integers are chosen from between 1 and 20 inclusive, must at least one of them be even? Why?

Answer: No. The set $\{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ contains ten integers between 1 and 20 and the set contains only odd numbers.

9.4.12

12. How many cards must you pick from a standard 52-card deck to be sure of getting at least 1 red card? Why?

Answer: 27 cards. Since there are 26 black cards in a standard deck, 27 must be picked to ensure 1 card is at least red.

9.4.17

17. How many integers must you pick in order to be sure that at least two of them have the same remainder when divided by 7?

Answer: 8

20. a. If repeated divisions by 20,483 are performed, how many distinct remainders can be obtained?

Answer: 20,483 (0 → 20,482)

b. When 5/20483 is written as a decimal, what is the maximum length of the repeating section of the representation?

Answer: 20,483

9.4.23

23. Is 56.5566555666555566 66... (where the strings of 5's and 6's become longer in each repeatition) rational or irrational?

Answer: Irrational

29. A certain college class has 40 students. All students in the class are known to be from 17 through 34 years of age. You want to make a bet that the class contains at least x students of the same age. How large can you make x and yet be sure to win the bet?

$n = 40$

age gap $= 34 - 17 + 1 = 18$

repeatition $= \frac{40}{18} = \frac{20}{9} = 2\frac{2}{9}$ $\because$ $2 < 2\frac{2}{9}$

Answer: 3