

Write the first four terms of the sequence defined by the formulas in 1-6

$$1. a_k = \frac{k}{10+k} \quad \text{for all } k \geq 1$$

$$\text{Answer 3 } a_1 = \frac{1}{11}$$

$$a_2 = \frac{2}{12} = \frac{1}{6}$$

$$a_3 = \frac{3}{13}$$

$$a_4 = \frac{4}{14} = \frac{2}{7}$$

5.1.8

Compute the first fifteen terms of each of the sequences in 8 and 9, and describe the general behavior of these sequences in word.
 (A definition of logarithm is given in Section 7.1).

$$8. g_n = \lceil \log_2 n \rceil \text{ for all integers } n \geq 1$$

$$\text{Answer: } g_1 = \log_2 1 = 0$$

$$g_2 = \log_2 2 = 1$$

$$g_3 = \log_2 3$$

$$g_4 = \log_2 4 = 2$$

$$g_5 = \log_2 5$$

$$g_6 = \log_2 6$$

$$g_7 = \log_2 7$$

$$g_8 = \log_2 8 = 3$$

$$g_9 = \log_2 9$$

$$g_{10} = \log_2 10$$

$$g_{11} = \log_2 11$$

$$g_{12} = \log_2 12$$

$$g_{13} = \log_2 13$$

$$g_{14} = \log_2 14$$

$$g_{15} = \log_2 15$$

$$\underbrace{a}_{\text{~~~~~}}^{\log_2 k} = k$$

Let $a_0 = 2$, $a_1 = 3$, $a_2 = -2$, $a_3 = 1$, $a_4 = 0$, $a_5 = -1$, and $a_6 = -2$.
 Compute each of the summations and products below.

a. $\sum_{i=0}^6 a_i = 1$

b. $\sum_{i=0}^0 a_i = 2$

c. $\sum_{j=1}^3 a_{3j} = -4$

d. $\prod_{k=0}^6 a_k = 0$

e. $\prod_{k=2}^2 a_{1k} = -2$

Compute the summations and products in 19-28

$$\begin{aligned} 27. \sum_{n=1}^{10} \left(\frac{1}{n} - \frac{1}{n+1} \right) &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{10} - \frac{1}{11} \\ &= \frac{10}{11} \end{aligned}$$

Evaluate the summation and products in 33-36 for indicated values of the variable.

35. $\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right)\dots\left(\frac{k}{k+1}\right) ; k=3$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

5. 1. 44

Write each of 43-52 using summation or product notation.

$$44. (1^3 - 1) + (2^3 - 1) + (3^3 - 1) - (4^3 - 1) + (5^3 - 1)$$

$$= \sum_{n=1}^5 (n^3 - 1) (-1)^{n+1}$$

Write each of 43-52 using summation or product notation.

$$51. n + (n-1) + (n-2) + \dots + 1$$

$$= \sum_{i=0}^{n-1} (n-i)$$

Transform each of 53 and 54 by making the change of variable $i = k+1$

$$\begin{aligned}
 53. \quad & \sum_{k=0}^5 k(k-1) \quad , \quad i = k+1 \Rightarrow k = i-1 \\
 & \text{Lower limit} \Rightarrow i=1 \\
 & \text{Upper limit} \Rightarrow i=6 \\
 = & \sum_{i=1}^5 (i-1)(i-2)
 \end{aligned}$$

5.1.61

Write each of 59-61 as a single summation or product.

$$61. \left(\prod_{k=1}^n \frac{k}{k+1} \right) \cdot \left(\prod_{k=1}^n \frac{k+1}{k+2} \right)$$

$$= \underbrace{\prod_{k=1}^n \frac{k}{k+2}}$$

5.1.75

Compute each of 62-76. Assume the value of the variables are restricted so that the expressions are defined.

$$75. \binom{n}{n-1} = \frac{n!}{(n-1)! (n-n+1)!} = \frac{n(n-1)!}{(n-1)! 1!} = \underline{\underline{n}}$$

5.1.88

Use the algorithm you developed for exercise 87 to convert the integers in 88-90 to hexadecimal notation.

88. 287

$$\begin{array}{r} 017 \\ 16 \overline{) 287} \\ 16 \\ \hline 127 \\ 112 \\ \hline 015 \end{array}$$

$$\begin{array}{r} 01 \\ 16 \overline{) 17} \\ 16 \\ \hline 1 \end{array}$$

$$\begin{array}{r} \xrightarrow{LE} \\ F11_{16} \\ \xleftarrow{BE} \\ 11F_{16} \end{array}$$

3. For each positive integer n , let $P(n)$ be the formula

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

a. Write $P(1)$. Is $P(1)$ true? $P(1) = 1 \Rightarrow$ True

b. Write $P(k)$.

$$P(k) = \frac{k(k+1)(2k+1)}{6}$$

c. Write $P(k+1)$

$$P(k+1) = \frac{(k+1)(k+2)(2k+3)}{6}$$

d. In a proof by mathematical induction that the formula holds for all integers $n \geq 1$, what must be shown in the inductive step?

Show that $P(k+1)$ is true.

Prove each statement in 6-9 using mathematical induction. Do not derive them from theorem 5.2.2 or theorem 5.2.3.

8. For all integers $n \geq 0$, $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$

Proof Let $P(n) = 2^{n+1} - 1$

$$P(0) = 2^{1+0} - 1 = 1 \quad \text{True}$$

$$P(k) = 2^{k+1} - 1$$

$$P(k+1) = 2^{k+2} - 1$$

$$2^{k+2} - 1 = 1 + 2 + 2^2 + \dots + 2^k + 2^{k+1}$$

$$= 2^{k+1} - 1 + 2^{k+1}$$

$$= 2^{k+1}(1+1) - 1$$

$$= 2^{k+1}(2) - 1$$

$$2^{k+2} - 1 = 2^{k+2} - 1$$

$\therefore P(k+1)$ is true & $P(0)$ is true

\therefore The statement is true by mathematical induction. \blacksquare

Prove each statement in 10-17 by mathematical induction.

14. $\sum_{i=1}^{n+1} i \cdot 2^i = n 2^{n+2} + 2$ for all integers $n \geq 0$

Proof | Let $P(n) = n 2^{n+2} + 2$

$$P(0) = 0 + 2 = 2 \quad \text{True}$$

$$P(k) = k 2^{k+2} + 2$$

$$P(k+1) = (k+1) 2^{k+3} + 2$$

Inductive Step *

$$\begin{aligned} (k+1) 2^{k+3} + 2 &= \sum_{i=1}^{k+2} i 2^i \\ &= \sum_{i=1}^{k+1} i 2^i + (k+2) 2^{k+2} \\ &= k 2^{k+2} + 2 + (k+2) 2^{k+2} \\ &= k 2^{k+2} (1+1) + 2 + 2^{k+3} \\ &= k 2^{k+3} + 2^{k+3} + 2 \end{aligned}$$

$$(k+1) 2^{k+3} + 2 = (k+1) 2^{k+3} + 2$$

$\therefore P(0)$ is true and $P(k+1)$ is true

\therefore The statement is true by mathematical induction. ■

Prove each statement in 10-17 by mathematical induction.

15. $\sum_{i=1}^n i(i)8 = (n+1)8 - 1$, for all integers $n \geq 1$

Proof | Let $P(n) = (n+1)8 - 1$

$$P(1) = 2 - 1 = 1 \quad \text{True}$$

$$P(k) = (k+1)8 - 1$$

$$P(k+1) = (k+2)8 - 1$$

$$\begin{aligned} (k+2)8 - 1 &= \sum_{i=1}^{k+1} i(i)8 \\ &= \sum_{i=1}^k i(i)8 + (k+1)(k+1)8 \\ &= (k+1)8 - 1 + (k+1)(k+1)8 \\ &= (k+1)8 (1+k+1) - 1 \\ &= (k+1)8 (k+2) - 1 \end{aligned}$$

$$(k+2)8 - 1 = (k+2)8 - 1$$

$\therefore P(1)$ is true & $P(k+1)$ is true

\therefore The statement is true by mathematical induction ■

5. 2. 20

Use the formula for the sum of the first n Integers and/or the formula for the sum of a geometric sequence to evaluate the sums in 20-29 or to write them in closed form.

$$20. \quad 4 + 8 + 12 + 16 + \dots + 200$$

$$= 4(1 + 2 + 3 + 4 + \dots + 50)$$

$$= 4 \cdot \frac{50(51)}{2}$$

$$= 2 \cdot 50 \cdot 51$$

$$= 100 \cdot 51$$

$$= 5100$$

5. 2. 23

Use the formulas for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sums in 20-29 or to write them in closed form.

23. $7 + 8 + 9 + 10 + \dots + 600$

$$= \frac{600(601)}{2} - \underline{\underline{6}} - \underline{\underline{5}} - \underline{\underline{4}} - \underline{\underline{3}} - \underline{\underline{2}} - \underline{\underline{1}}$$

$$= 300(601) - 21$$

$$= 180279$$

5.2.28

Use the formulas for the sum of the first n integers and for formula for the sum of a geometric sequence to evaluate the sums in 20-29 or to write them in closed form.

$$\begin{aligned}
 28. \quad & 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}, \text{ where } n \text{ is a positive integer} \\
 & = \frac{2^{-(n+1)} - 1}{2^{-1} - 1} \\
 & = -2 \cdot 2^{-n-1} + 2 \\
 & = -2^{-n} + 2 \\
 & = \frac{-1}{2^n} + 2 \\
 & = 2 - \frac{1}{2^n}
 \end{aligned}$$

30. Find a formula in n , a , m , and d for the sum $(a+md) +$

$$(a+(m+1)d) + (a+(m+2)d) + \dots + (a+(m+n)d),$$

where m and n are integers, $n \geq 0$, and a and d are real numbers. Justify your answer.

$$(a + (m+n)d) = a + md + nd$$

$$\begin{aligned} \sum_{n=0}^m (a + (mn)d) &= \sum_{n=0}^m (a + md + nd) \\ &= (a + md)(m+1) + \frac{m(m+1)d}{2} \\ &= (m+1)\left(a + md + \frac{nd}{2}\right) \end{aligned}$$

Justification

$$\text{Let } P(n) = (n+1)\left(a + md + \frac{nd}{2}\right)$$

$$P(0) = (a + md) \quad \text{True}$$

$$P(k) = (k+1)\left(a + md + \frac{kd}{2}\right)$$

$$P(k+1) = (k+2)\left(a + md + \frac{(k+1)d}{2}\right)$$

$$\text{Inductive Step*} \quad (k+2)\left(a + md + \frac{(k+1)d}{2}\right) = (k+1)\left(a + md + \frac{kd}{2}\right) + (a + (m+k+1)d)$$

$$= (k+1)\left(a + md + \frac{kd}{2}\right) + (a + md + (k+1)d)$$

$$= (k+1)\left(a + md + \frac{kd}{2} + cd\right) + a + md$$

$$= (k+1)\left(a + md + \frac{1}{2}d(k+2)\right) + a + md$$

$$(k+2)\left(a + md + \frac{(k+1)d}{2}\right) = (k+2)\left(a + md + \frac{1}{2}d(k+1)\right)$$

True by mathematical induction. \blacksquare

Find the mistakes in the proof fragments in 33-35

33. Theorem: For any integer $n \geq 1$,

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

"Proof (by mathematical induction): Certainly the theorem is true for $n=1$ because $1^2 = 1$ and $\frac{1(1+1)(2 \cdot 1 + 1)}{6} = 1$. So the basis step is true"

For the inductive step, suppose that for some integer $k \geq 1$,

$$k^2 = \frac{k(k+1)(2k+1)}{6} \text{. We must show that}$$

$$(k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} \text{.}$$

Mistakes: 1) The inductive step.

$$\text{It should be } 1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

2) What must be shown

$$\text{Should be } 1^2 + 2^2 + \dots + k^2 + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Find the first four terms of each of the recursively defined sequences in 1-8

3. $C_k = k(C_{k-1})^2$, for all integers $k \geq 1$ & $C_0 = 1$

$$C_0 = 1$$

$$C_1 = 1(1)^2 = 1$$

$$C_2 = 2(1)^2 = 2$$

$$C_3 = 3(2)^2 = 12$$

Find the first four term of each of recursively defined sequences in 1-8

7. $U_k = kU_{k-1} - U_{k-2}$, for all integers $k \geq 3$, $U_1 = 1$, $U_2 = 1$

$$U_1 = 1$$

$$U_2 = 1$$

$$U_3 = 3(1) - 1 = 2$$

$$U_4 = 4(2) - 1 = 7$$

5. 6. 13

13. Let t_0, t_1, t_2, \dots be defined by the formula $t_n = 2 + n$ for all integers $n \geq 0$. Show that the sequence satisfies the recurrence relation

$$t_k = 2t_{k-1} - t_{k-2}$$

$t_k = 2 + k$ $t_{k-1} = 1 + k$ $t_{k-2} = k$	$2 + k = 2(k+1) - k$ $2 + k = 2 + k$
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5. 6. 17

Tower of Hanoi with adjacency Requirement: Suppose that in addition to the requirement that they never move a larger disk on top of a smaller one, the priests who move the disks of the Tower of Hanoi are also allowed only to move disk one by one from one pole to an adjacent pole. Assume poles A and C are at the two ends of the row and pole B is in the middle. Let

$$a_n = \begin{bmatrix} \text{the minimum number of moves} \\ \text{needed to transfer a tower of } n \\ \text{disks from pole A to pole C} \end{bmatrix}$$



a. Find a_1 , a_2 , and a_3 .

b. Find a_4

c. Find a recurrence relation for a_1, a_2, a_3, \dots

Part a: $a_1 = 2$

$a_2 = 8$

$a_3 = 26$

Part b: $a_4 = 80$

$$\begin{aligned} a_2 &= 2 + 1 + 2 + 1 + 2 \\ &= 8 \end{aligned}$$

$$\begin{aligned} a_3 &= 8 + 1 + 8 + 1 + 8 \\ &= 26 \end{aligned}$$

$$\begin{aligned} a_4 &= 26 + 1 + 26 + 1 + 26 \\ &= 60 + 18 + 2 = 80 \end{aligned}$$

Part c: $a_n = 3a_{n-1} + 2$ for all integers $n \geq 0$

20. Tower of Hanoi Poles in a circle: Suppose that instead of being lined up in a row, the three poles for the original Tower of Hanoi are placed in a circle. The monks move disks one by one from one pole to another, but they may only move disk one over in a clockwise direction and she may never move a larger disk over a smaller one. Let C_n be the minimum number of moves needed to transfer a pile of n disk from one pole to the next adjacent pole in the clockwise direction

a. Justify the inequality $C_k \leq 4C_{k-1} + 1$ for all integers $k \geq 2$

b. The expression $4C_{k-1} + 1$ is not the minimum number of moves need to transfer a pile of k disks from one pole to another. Explain Why
 $C_3 \neq 4C_2 + 1$

Part a)

$$\begin{aligned} S_3 &\leq 21 \quad \& \quad S_2 = 5 \quad \& \quad S_1 = 1 \\ S_3 &\leq 4(5) + 1 \quad \Rightarrow \quad S_3 \leq 21 \\ S_2 &\leq 4(1) + 1 \quad \Rightarrow \quad S_2 \leq 5 \end{aligned}$$

$$C_k \leq 4C_{k-1} + 1 \text{ for all integers } k \geq 2$$

A	B	C
From A-B	From A-C	
$B_1 = 1$	$G = 2$	
$B_2 = 5$	$C_2 = 7$	
$B_3 = 15$	$C_3 = 21$	

Part b) Let $C_k = 4C_{k-1} + 1 \quad \& \quad C_1 = 1, C_2 = 5$

Case A-B $\therefore 15 = 4(5) + 1$

$$15 \neq 21$$

.....

$$\therefore C_k \neq 4C_{k-1} + 1$$

5.6.24

In 29-34 F_0, F_1, F_2, \dots is the Fibonacci sequence

29. Use the recurrence relation and values F_0, F_1, F_2, \dots given in example 5.6.6 to compute F_{13} and F_{14}

$$F_0 = 0$$

$$F_1 = 1$$

⋮

$$F_{11} = 144$$

$$F_{12} = 233$$

$$F_{13} = 377$$

$$F_{14} = 610$$

In 24-34, F_0, F_1, F_2, \dots is the Fibonacci sequence

27. Prove that $F_k^2 - F_{k-1}^2 = F_k F_{k+1} - F_{k+1} F_{k-1}$, for all integers $k \geq 1$

$$\begin{aligned}
 \text{Proof} | \quad F_k^2 - F_{k-1}^2 &= (F_k + F_{k-1})(F_k - F_{k-1}) \\
 &= (F_{k+1})(F_k - F_{k-1}) \\
 &= F_{k+1} F_k + F_{k+1} F_{k-1}
 \end{aligned}$$