Omar Alkendi

Chapter 2

March 25, 2020

Section 3.1

4 --------------------------------Done

8 --------------------------------Done

12 ------------------------------Done

16 ------------------------------Done

20 ------------------------------Done

24 ------------------------------Done

28 ------------------------------Done

32 ------------------------------Done

Section 3.2

6 --------------------------------Done

12 ------------------------------Done

18 ------------------------------Done

24 ------------------------------Done

30 ------------------------------Done

36 ------------------------------Done

42 ------------------------------Done

48 ------------------------------Done

Section 2.3

10 ------------------------------Done

20 ------------------------------Done

30 ------------------------------Done

40 ------------------------------Done

50 ------------------------------Done

60 ------------------------------Done

Section 3.4

4 --------------------------------Done

8 --------------------------------Done

12 ------------------------------Done

16 ------------------------------Done

20 ------------------------------Done

24 ------------------------------Done

28 ------------------------------Done

32 ------------------------------Done

4. Let $Q(n)$ be the predicate "$n^2 \leq 30$."

a. Write $Q(2)$, $Q(-2)$, $Q(7)$, and $Q(-7)$, and indicate which of these statements are true and which are false.

b. Find the truth set of $Q(n)$ if the domain of n is $\mathbb{Z}$, the set of all integers.

c. If the domain is set to $\mathbb{Z}^+$ of all positive integers, what is the truth set of $Q(n)$?

a. $Q(2) = 4$; true.

$Q(-2) = 4$; true.

$Q(7) = 49$; false.

$Q(-7) = 49$; false.

b. Truth set of $Q(n)$ is $\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$.

c. Truth set of $Q(n)$ is $\{1, 2, 3, 4, 5\}$.

8. Let $B(x)$ be "$-10 < x < 10$." Find the truth set of $B(x)$ for each of the following domains.

a. $\mathbb{Z}$ b. $\mathbb{Z}^+$ c. The set of all even integers

a. $\{-9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

b. $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

c. $\{-8, -6, -4, -2, 0, 2, 4, 6, 8\}$.

Find counter-examples to show the statements in 9-12 are false

12. $\forall$ real number x and y, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$

Let $x=1$, $y=1$

$\sqrt{1+1} = \sqrt{2} \neq \sqrt{1} + \sqrt{1} = 2$

$\sqrt{2} \neq 2$

$\therefore \forall x \in R$ & $\forall y \in R$, $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$ ■

16. Rewrite each of the following statements in the form "$\forall$ ___ x, ___."

a. All dinosaurs are extinct.

$\forall x \in$ Dinosaurs, x is extinct.

b. Every real number is positive, negative, or zero

$\forall x \in \mathbb{R}$, x is positive, negative, or zero.

c. No irrational number are integers

$\forall x \in \mathbb{Q}'$, $x \notin \mathbb{Z}$.

d. No logician is lazy

$\forall x \in$ logician, x is not lazy

e. The number 2,147, 581, 953 is not equal to the square of any integer

$\forall x \in \mathbb{Z}$, $x^2 \neq 2,147,581,935$.

f. The number -1 is not equal to the square of any real number.

$\forall x \in \mathbb{R}$, $x^2 \neq -1$.

20 Rewrite the following statement informally in at least two different ways without using variables or the symbol ∀ or the words "for all"

$\forall$ real numbers x, if x is positive, then the square root of x is positive.

- The square root of a positive real number is positive.
- A positive real number has a positive square root.

24. Rewrite the following statements in the two forms "$\exists$ ___ x such that ___" and "$\exists x$ such that ___ and ___".

a. Some hatters are mad.

- $\exists$ a hatter x such that x is mad.
- $\exists x$ such that x is hatter and x is mad.

b. Some questions are easy.

- $\exists$ a question x such that x is easy
- $\exists x$ such that x is a question and x is easy.

28. Let the domain of x be the set D of subjects discussed in mathematics courses, and let $Real(x)$ be "x is a real number," $Pos(x)$ be "x is a positive real number," $Neg(x)$ be "x is a negative real number," and $Int(x)$ be "x is an integer."

Write each statement without using quantifiers or variables. Indicate which are true and which are false, and justify your answer as best as you can.

---

a. $Pos(0)$

0 is a positive real number.

False; zero is neither positive or negative.

b. $\forall x, Real(x) \wedge Neg(x) \rightarrow Pos(-x)$

The opposite of a real negative number is a positive real number

True; $x = -(-x)$

c. $\forall x, Int(x) \rightarrow Real(x)$

All integers are real numbers

True; $\mathbb{Z} \subseteq \mathbb{R}$

d. $\exists x$ such that $Real(x) \wedge \sim Int(x)$

There is at least one real number that is not an integer.

True; $\mathbb{Q}' \subseteq \mathbb{R}$ and $\mathbb{Q}'$ is irrational.

32. Let $\mathbb{R}$ be the domain of the predicate variable x. Which of the following are true and which are false? Give counter examples for the statements that are false.

a. $x > 2 \rightarrow x > 1$; True

b. $x > 2 \rightarrow x^2 > 4$; True

c. $x^2 > 4 \rightarrow x > 2$; False

$-3 \not> 2$ but $(-3)^2 > 4$

d. $x^2 > 4 \leftrightarrow |x| > 2$; True.

6. Write the negation for each of the following statements.

a. Sets A and B do not have any points in common.

~a: There are some common points between sets A and B.

b. Towns P and Q are not connected by any road on the map

~b: There are some roads that connect towns P and Q.

In each of 11-14 determine whether the proposed negation is correct. If it is not, write a correct negation

12. Statement: The product of any irrational number and any rational number is irrational.

Proposed negation: The product of any irrational number and any rational number is rational.

Answer: The proposed negation is <u>incorrect</u>

Correct negation: The product of some irrational number and some rational number is rational.

In 16-23, write a negation for each statement.

18. $\forall x \in \mathbb{R}$, if $x(x+1) > 0$ then $x > 0$ or $x < -1$

$\exists x \in \mathbb{R} \mid x(x+1) >$ and $x \leq 0$ and $x \geq -1$

24. Rewrite the statements in each pair in if-then form an indicate the logical relationship between them.

a. All the children in Tom's family are female
All the females in Tom's family are children

a.1 If a person is a child in Tom's family, then the person is a female.
a.2 If a person is a female in Tom's family, then the person is a child.

a.2 is the converse of a.1.

b. All the integers that are greater than 5 and end in 1, 3, 7, or 9 are prime
All the integers that are greater than 5 and are prime end in 1, 3, 7, or 9.

b.1 If a number is greater than 5 and ends w/ 1, 3, 7, or 9, then it is a prime.
b.2 If a number is greater than 5 and is a prime, then it ends w/ 1, 3, 7, or 9.

b.2 is the converse of b.1

In 26-33, for each statement in the referenced excercise write the converse, inverse, and contrapositive. Indicate as best as you can which among the statement, its converse, its inverse, its contrapositive are true and which are false. Give a counter example for each that is false.

30. Excercise 20: $\forall$integers a, b and c, if a-b <u>and</u> b-c is even, then a-c is even.

Let $a-b = 2n$ & $b-c = 2m$

$a = 2n + b$ $\qquad -c = 2m - b$

$\therefore a - c = 2n + \cancel{b} + 2m - \cancel{b} = 2(m+n)$

$\because a - c = 2(m+n)$ which is a multiple of 2

$\therefore a - c$ is even

$\therefore$ The statement is true ■

Contrapositive: $\forall$integers a, b and c, if a-c is odd, then a-b <u>or</u> b-c is odd.

Converse : $\forall$integers a, b and c, if a-c is even, the a-b <u>and</u> b-c are even.

Inverse : $\forall$integers a, b and c, if a-b <u>or</u> a-c is not even, then a-c is not event.

Statement : True (See the proof)
Contrapositive : True (Syllogism)
Converse : False (Ex. $a=3$, $b=2$, $c=1 \Rightarrow a-c = 2$ (even), $a-b = 1$ ($\sim$even $\equiv$ odd) ■)
Inverse : False (Ex. $a=3$, $b=2$, $c=1 \Rightarrow a-b = 1$ (odd), $a-c=2$ (even) ■)

36. If $P(x)$ is a predicate and the domain of x is the set of all real numbers, let R be "$\forall x \in \mathbb{Z}, P(x)$," let S be "$\forall x \in \mathbb{Q}, P(x)$," and let T be "$\forall x \in \mathbb{R}, P(x)$."

a. Find a definition for $P(x)$ (but do not use "$x \in \mathbb{Z}$") so that R is true and both S and T are false.

Answer: $P(x)$ is "$2x \neq 1$"

True for R ($\because \frac{1}{2} \notin \mathbb{Z}$)

False for S ($\because 2(\frac{1}{2}) = 1$ & $\frac{1}{2} \in \mathbb{Q}$)

False for T ($\because 2(\frac{1}{2}) = 1$ & $\frac{1}{2} \in \mathbb{R}$)

b. Find a definition for $P(x)$ (but do not use "$x \in \mathbb{Q}$") so that R & S are true and T is false.

Answer: $P(x)$ is "$\frac{x^2}{2} \neq 1$"

True for R ($\because \sqrt{2} \notin \mathbb{Z}$)

True for S ($\because \sqrt{2} \notin \mathbb{Q}$)

False for T ($\because \frac{1}{2}(\sqrt{2})^2 = 1$ & $\sqrt{2} \in \mathbb{R}$)

Rewrite each statement of 39-42 in if-then form.

42. Passing a comprehensive exam is a necessary condition for obtaining a master's degree.

Answer: If a graduate student could not pass the comprehensive exam, the student can't obtain the master's degree.

48. A frequent-flyer club brochure states "You may select among carriers only if the offer the same lowest fare." Assuming "only if" has its formal, logical meaning, does this statement guarantee that if two carriers offer the same lowest fare, the customer will be free to choose between them? Explain.

Let q be "You are offered the same lowest fare,"
and p be "You may select among carriers."

only if mean $\sim q \to \sim p$ or equivalently $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

$\therefore$ Therefore, it possible not be able to select among carrier even though two carriers offered the same lowest fare. ∎

Answers No; see explanation.

10. This exercise refers to example 3.3.3. Determine wheather each of the following is true or false.

a. $\forall$ students S, $\exists$ a dessert D such that S chose D. True (Tim)

b. $\forall$ student S, $\exists$ a salad T such that S chose T. False (Yuen)

c. $\exists$ a desser D such that $\forall$ student S, S chose D. True (Pie)

d. $\exists$ a beverage B such that $\forall$ students D, D chose B. False.

e. $\exists$ an item I such that $\forall$ students S, S did not choose I. False.

f. $\exists$ a station Z such that $\forall$ students S, $\exists$ an Item I such that S chose I from Z. True (Pie)

20. Recall that reversing the order of the quantifiers in a statement with two different quantifiers may change the truth value of the statement—but it does not necessarily do so. All the statements in the pairs on the next page refer to the Tarski World of figure 3.3.1. In each pair, the order of the quantifiers is reversed but everything else is the same. For each pair, determine whether the statements have the same or opposite truth values. Justify your answers.

a. (1) For all squares y there is a triangle x such that x and y have different colors.
(2) There is a triangle x such that for all squares y, x and y have different colors.

- Statement 1 is true because for any square there exists a triangle with a different color.

Statement 2 is false because there are squares that have the same color as the triangles.

∴ Opposite truth values.

b. (1) For all circles y there is a square x such that x and y have the same color.
(2) There is a square x such that for all circles y, x and y have the same color.

Statement 1 is true because for any circle there is a square with a different color

Statement 2 is false because there is at least two circles with different colors,

∴ Opposite truth values.

For each of the statements in 29 and 30 (a) write a new statement by interchanging the symbols $\forall$ and $\exists$, and (b) state which is true: the given version, the version with the interchanged quantifiers, neither, or both.

30. $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}^-$ (the set of all negative real numbers) $x > y$.

a) $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}^-$ such that $x > y$

b) Both are true. Any non-negative is always > any negative.

40. In informal speach, most sentences of the form "There is ___ every ___" are intended to be understood as meaning "∀___ ∃ ___," even though the existential quantifier <u>there is</u> comes before the universal quantifier <u>every</u>. Note that this interpretation applies to the following well-known sentences. Rewrite them using quantifiers and variables.

a. There is a sucker born every minute.

∀ minutes m, ∃ a sucker s such that s was born at m.

b. There is a time for every purpose under heaven.

∀ purpose p under heaven, ∃ time t such that p occurs at t.

In 46-54, refer to the Tarski world given in figure 3.1.1, which is [not] printed again here for [Saving space]. The domains of all variables consist of all the objects in the Tarski World. For each statement, (a) indicate whether the statement is true or false and justify your answer, (b) write the given statement using the formal logical notation illustrated in Example 3.1.10, and (c) write the negation of the given statement using the formal logical notation of Example 3.3.10.

50. For every object x, there is an object y such that if $x \neq y$ then x and y have different colors.

a) True. There are 3 different colors, so there is at least one case in which $x \neq y$ means that x and y have different colors.

b) $\forall x(\exists y(x \neq y \to \sim \text{SameColorAs}(x,y)))$

c) $\exists x(\forall y(x \neq y \wedge \text{SameColorAs}(x,y)))$

In 59-61, find the answers Prolog would give if the following questions were added to the program given in Example 3.3.11.

60. a. ?isabove(w_1, g)

Prolog: No

b. ?color$(w_2$, blue$)$

Prolog: No

c. ?isabove(X, b_1)

Prolog: $X = g$

Use universal instantiation or universal modus ponens to fill in valid conclusions for the arguments in 2-4.

4. $\forall$ real numbers r, a, and b, if r is a positive, then $(r^a)^b = r^{ab}$
$r = 3$, $a = \frac{1}{2}$, and $b = 6$ are particular real numbers such that r is positive

$\therefore$ $r = 3^3 = 27$

Some of the arguments in 7 - 18 are valid by universal modus ponens or universal modus tollens; others are invalid an exhibit the converse or the inverse error. State which are valid and which are invalid. Justify your answer.

8. All freshman must take writing

Caroline is a freshman (More like a freshwoman)

∴ Caroline must take writing.

Valid by universal modus ponens.

Some of the arguments in 7-18 are valid by universal modus ponens or universal modus tollens; others are invalid and exhibit the converse or the inverse error. State which are valid and which are invalid. Justify your answer.

12. All honest people pay their taxes.

Darth is not honest. (Vader?)

∴ Darth does not pay his taxes.

Invalid; inverse error.

Some of the arguments in 7-18 are valid by universal modus ponens or universal modus tollens; others are invalid and exhibit the converse or the inverse error. State which are valid and which are invalid. Justify your answer.

16. If a number is even, then twice the number is even.

The number $2n$ is even, for a particular number n.

$\therefore$ The particular number n is even.

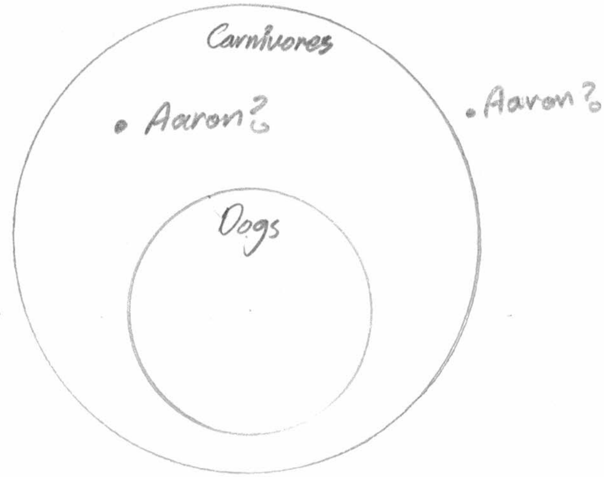
Invalid; converse error.

20. a) Use a diagram to show that the following argument can have true premise and a false conclusion.

All dogs are carnivorous

Aaron is not a dog

$\therefore$ Aaron is not a carnivorous.



b) What can you conclude about the validity or invalidity of the following argument from? Explain how the result from part (a) leads to this conclusion.

$\forall x$, if $P(x)$ then $Q(x)$

$\sim P(a)$ for a particular a

$\therefore \sim Q(a)$

The argument is invalid due to inverse error. The result from (a) indicate that $P(x)$ is a subset of $Q(x)$. However, the scope of $Q(x)$ is unknown, hence, invalid.

Indicate whether the argument in 21-27 are valid or invalid. Support your answers by drawing diagrams.

24. No vegetarians eat meat

All vegans are vegetarians

∴ No vegans eat meat.

<u>Valid</u>

In excercises 28-32 reorder the premises in each of the arguments to show that the conclusion follows as a valid consequence from the premises. It may be helpful to rewrite the statements in if-then form and replace some statements by their contrapositive. Excercises 28-30 refer to the kind of Tarski world discussed in Example 3.1.13 and 3.3.1. Excercises 31-32 are adapted from *Symbolic Logic* by Lewis Carroll.

28. 1. Every object that is to the right of all blue object is above all the triangles. $(rb \to a\triangle)$

2. If an object is a circle, then it is to the right of all the blue objects $(c \to rb) \equiv (\sim rb \to \sim c)$

3. If an object is not a circle, then it is not gray. $(\sim c \to \sim g) \equiv (g \to c)$

∴ All the gray objects are above all the triangles.

Reorder as proof (transivity)

3) gray → circle
2) circle → to the right of all blue object
1) to the right... → above all triangles

∴ gray → above all triangles by transivity ∎

32. 1. When I work a logic example without grumbling, you may be sure it is one I understand (~grumble → understand)

2. The arguments in these examples are not arranged in regular like the ones I am used to. (~ordered)

3. No easy examples make my head ache. (easy → ~ headache)

4. I can't understand examples if the arguments are not arranged in regular order like the ones I am used to. (~ordered → ~understand)

5. I never grumble at an example unless it gives me headache. (~ headache → ~ grumble)

∴ The examples are not easy (~ easy)

Reorder:

2) ~ ordered

4) ~ ordered → ~ understand

contrapositive 1) ~ understand → grumble

contrapositive 5) grumble → headache

contrapositive 3) headache → ~ easy

∴ ~ easy by transivity ∎