5. Use Theorem 9.5.1 to prove algebraically that $\binom{n}{r} = \binom{n}{n-r}$, for integers n & r with $0 \le r \le n$. (This can be done by direct calculation; it is not necessarily to use mathematical induction).

Theorem: $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Proof | Let n & r such that $0 \le r \le n$

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\frac{n!}{r!(n-r)!} = \text{"}$$

$$\frac{n!}{(n-(n-r))!\,(n-r)!} = \text{"}$$

$$\frac{n!}{(n-r)!(n-(n-r))!} = \binom{n}{n-r} \qquad \blacksquare$$

10\. a. Use Pascal's Triangl given in Table 9.7.1 to compute the value of $\binom{6}{2}$, $\binom{6}{3}$, $\binom{6}{4}$, $\binom{6}{5}$

$\binom{6}{2} = 15$ $\binom{6}{3} = 20$ $\binom{6}{4} = 15$ $\binom{6}{5} = 6$

b: Use results of part (a) and Pascal's Formula to compute

$$\binom{7}{3} = \binom{6}{3} + \binom{6}{2} = 35$$

$$\binom{7}{4} = \binom{6}{4} + \binom{6}{3} = 35$$

$$\binom{7}{5} = \binom{6}{5} + \binom{6}{4} = 21$$

c. Complete Pascal's Triangle that correspondes to $n=7$

$n=6$ $\substack{0\\0}$		1	6	15	20	15	6	1
$n=7$ $\substack{0\\0}$	1	7	21	35	35	21	7	1

9.7.15

15. Prove the following generalization of exercise 13: Let r be a fixed nonnegative integer. For all integers n w/ $n \geq r$

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$$

Proof | Let n & r s.t $0 \leq r \leq n$

Basis Step $n=r \Rightarrow P(r) = \sum_{i=r}^{r} \binom{r}{r} = 1$ True

Let $P(k) = \sum_{i=r}^{k} \binom{k}{r} = \binom{k+1}{r+1}$ w/ $k \geq r$

Show that $P(k+1)$ is true

$$P(k+1) : \sum_{i=r}^{k+1} = \sum_{i=1}^{k} \binom{k}{r} + \binom{k+1}{r}$$

$$= \binom{k+1}{r+1} + \binom{k+1}{r}$$

$$= \binom{k+1}{r+1} + \binom{k+1}{r+1-1}$$

$$= \binom{(k+1)+1}{r+1}$$

$\therefore P(k+1)$ is True

$\therefore \sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$ is true by M.I. ■

9.7.25

Use Binoimtal Theorem to expand expressions in 19-27

25. $\left(x + \frac{1}{x}\right)^5$ know: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

$$= \frac{5!}{0!(5-0)!} x^5 + \frac{5!}{1!(5-1)!} x^{\cancel{4}3} \cdot \frac{1}{\cancel{x}} + \frac{5!}{2!(5-2)!} x^{\cancel{3}} \frac{1}{\cancel{x^2}} + \frac{5!}{3!(5-3)!} \cancel{x^2} \frac{1}{x^{\cancel{3}}}$$

$$+ \frac{5!}{4!(5-4)!} \cancel{x} \frac{1}{x^{\cancel{4}3}} + \frac{5!}{5!(5-5)!} \frac{1}{x^5}$$

$$= x^5 + 5x^3 + 10x + 10 \frac{1}{x} + \frac{5}{x^3} + \frac{1}{x^5}$$

In 29-34, find the coefficient of the given term when the expression is expanded by the Binomial Theorem.

31. $a^5 b^7$ in $(a-2b)^{12}$ $\Rightarrow n = 12$ & $k = 7$

$C = 2^7 \cdot \binom{12}{7} = (-2)^7 \cdot \frac{12!}{7!(5)!} = (-2)^7 \cdot \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$C = -101{,}376$

8. Redo Exercise 7 assuming that $P(A) = 0.5$ and $P(B) = 0.4$

a. $P(A \cup B) = 0.9$

b. $P(C) = 1 - 0.9 = 0.1$

c. $P(A \cup C) = 0.6$

d. $P(A^c) = 1 - 0.5 = 0.5$

e. $P(A^c \cap B^c) = P(C) = 0.1$

f. $P(A^c \cup B^c) = 0.5 + 0.6 = 1.1$ $\because$ but can't be >1 $\therefore 1$

$= 1.$

9.8.16

16. A run contain four balls numbered 2, 2, 5, & 6. If a person a person selects two balls randomley, what is the expected value of the sum of the numbers on the balls?

	(x,y)	S	P
	2, 2	4	1/6
2x	2, 6	8	2/6
2x	2, 5	7	2/6
	6, 5	11	1/6

$$\langle S \rangle = \sum_i S_i \cdot P_i = 4 \cdot 1/6 + 8 \cdot \frac{2}{6} + 7 \cdot \frac{2}{6} + 11 \cdot 1/6$$

$$= 15 \cdot 1/6 + 15 \cdot 2/6$$

$$= 15 \cdot 3/6 = 15/2$$

9.9.5

5. Suppose that A and B are events in sample space S and that P(A), P(B), P(A|B) are known. Derive a formula for $P(A \mid B^c)$.

$$P(B^c) = 1 - P(B)$$

$$P(A \mid B) = P(A \cap B)/P(B) \Rightarrow P(A \cap B) = P(A \mid B) \cdot P(B)$$

$$P(A) = P(A \cap B) \overset{+}{\cup} P(A \cap B') \Rightarrow P(A \cap B') = P(A) - P(A \cap B)$$

$$P(A \mid B^c) = P(A \cap B') / P(B')$$

$$= \frac{P(A) - P(A \mid B) \cdot P(B)}{1 - P(B)} \quad \blacksquare$$

9.9.10

10. Prove the full version of Baye's Theorem

Proof | Let S be a sample space s.t. $S = B_1 \cup B_2 \ldots \cup B_n$
where $0 < k < n$
If all B_n are mutually disjoint $\Rightarrow S = \sum_1^n B_i$

Let A be an event such that $P(A) \neq 0$

The definition of probability is $P(B_k | A) = P(B_k \cap A) / P(A)$

or $P(B_k \cap A) = P(B_k | A) \cdot P(A)$
Let $A = A \cap S$
We know that $A \cap S = (A \cap B_1) \cup (A \cap B_2) \ldots \cup (A \cap B_n)$
which are mutualy disjoint since B_n are mutualy disjoint

$\therefore P(A) = P[(A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n)]$

$$= \sum_i^n P(A \cap B_i) = \sum_i^n (A | B_i)(B_i)$$

$\therefore$ By the definition of $P(B_k | A) = \dfrac{P(B_k \cap A)}{P(A)}$

$$= \frac{P(A \cap B_k)}{P(A)}$$

$$= \frac{P(A | B_k) P(B_k)}{\sum_i^n P(A | B_i) P(B_i)} \quad \blacksquare$$

9.10.15

15. Two different factories both produce a certain automobile part. The probability that a component from the first factory is defective is 2% and 5% for the second factory. In a supply of 180 parts, 100 were obtained from the first factor, and 80 from the second factory.

a. What is the probability that a part choosen at random from the 180 is from the first factory

$$P(F_1) = \frac{100}{180} = \frac{5}{9} = 55.56\ \%$$

b. What is the probability that a part choosen at random from the 180 is from the second factory?

$$P(F_2) = \frac{80}{180} = \frac{4}{9} = 44,45\ \%$$

c. What is the probability that a part choosen from the 180 is defective?

$$P(d) = \frac{0{,}02 \times 100 + 0{,}05 \cdot 80}{180} = \frac{6}{180} = \frac{1}{30} = 3.33\%$$

d. If it was defective, what is the probability it came from the first factory.

$$P(d_1) = \frac{0{,}02 \cdot 100}{180} \Big/ \frac{6}{180} = \frac{2}{6} = \frac{1}{3} = 33{,}34\ \%$$

9.9.25

25. A coin is loaded so that the probability of heads is 0.7 and the probability of tails is 0.3. Suppose that the coin is tossed twice and that the results of the tosses are independent

a. What is the probability of obtaining two heads?

$$P(HH) = \frac{7}{10} \cdot \frac{7}{10} = \frac{49}{100} = 0.49 \approx 49\%$$

b. What is the probability ob obtaining exactly one head?

$$P(HT) + P(TH) = 2 \cdot \frac{7}{10} \cdot \frac{3}{10} = \frac{42}{100} = 42\%$$

c. What is the probability ob obtaining no heads?

$$P(TT) = \frac{3}{10} \cdot \frac{3}{10} = \frac{9}{100} = 9\%$$

d. What is the probability ob obtaing at least one head?

$$P(HH) + P(HT) + P(TH) = 0.42 + 0.49 = 0.91 = 91\%$$

31. Empirical data indicate that approximately 103 out of every 200 chidren born are male. Hence the probability of a new born being male is about 51.5%. Suppose that a family has six childnen, and suppose that the gender of all the children are mutually independent.

a. What is the probability that non of children is male?

$P(F_6) = \left(\frac{97}{200}\right)^6 = 1.3\%$

b. What is the probility that at least one of the children is male?

$P(F_6^c) = 1 - \left(\frac{97}{200}\right)^6 = 98.3\%$

c. What is the probability that exactly five of all the childnen are male?

$P(M_5) = 6\left(\frac{97}{200}\right)\left(\frac{103}{200}\right)^5 = 10.5\%$

For each graph on 8 and 9, finds

(i) All edges that are incident on v_1

$\{e_1, e_2, e_3\}$

(ii) All vertices that are adjacent to v_3.

$\{v_1, v_2, v_3\}$

(iii) All edges that are adjacent to e_1.

$\{e_2, e_3, e_8, e_9\}$

(iv) All loops

$\{e_6, e_7\}$

(v) All parallel edges

$\{e_8, e_9, e_5, e_4\}$

(vi) All isolated vertices

$\{v_6\}$

(vii) Degree of v_3

5

(viii) Degree of the graph

20

16. A graph has vertices of degrees 1, 1, 4, 4, & 6. How many edges does the graph have?

$$\#E = \frac{16}{2} = 8$$

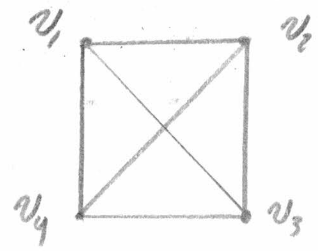
10.1.24

In each of 17-25, either draw a graph with the specified properties or explain why no such graph exits.

24. Simple graph w/ 6 edges and all vertices w/ degree 3

Total degree $= 12$

# of vertices $= 12/3 = 4$



32. Deduce from exercise 31 that for any positive integer n, if there is a sum of n odd integers that are even, then n is even.

Proof: Suppose $n, m, k, l, a, b \in \mathbb{Z}^+$

Let $m = 2k+1$ "odd"

Le $n = 2l+1$ "odd"

be $a = 2b$ "even"

$$a = \sum_{1}^{n} m = nm$$

$$2b = (2k+1)(2l+1)$$

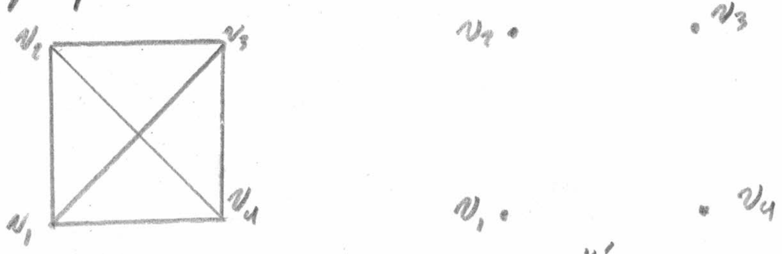
$$= 4kl + 2k + 2l + 1$$

$$2b = 2(2kl + k + l) + 1$$

$$b = (2kl + k + l) + \frac{1}{2} \iff \text{Contradiction "n can't be odd"}$$

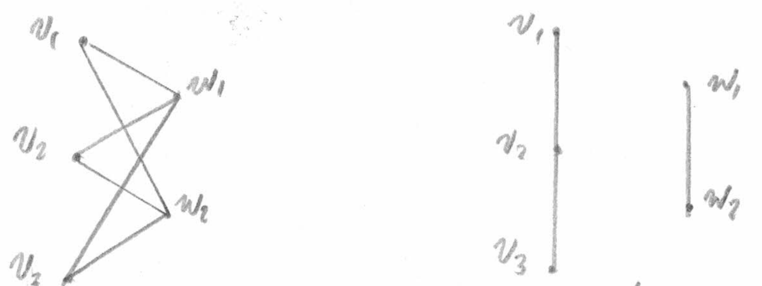
$\therefore$ If the sum of n odd integers is even $\to$ n is even

40. (a) Find the complement of the graph K_4, the complete graph on four vertices.



K_4 K_4'

(b) Find the complement of the graph $K_{3,2}$, the complete bipartite on (3, 2) vertices.

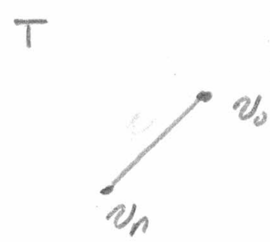


$K_{3,2}$ $K_{3,2}'$

5. Extend the argument given in the proof of Lemma 10.5.1 to show that a tree with more than one vertex has at least two vertices of degree 1.

Proof | Let T be a tree with at least two vertices

Since T is a tree, there should be no isolated vertices



While $\deg(v_x) > 1$ branch on e_x that is incident on v_x and terminates at v_y

Reapeat the last step untill leafs are reached

There should be at least two vertices of degree 1

Ex)

$\deg(v_0) = 1$
$\deg(v_n) = 1$

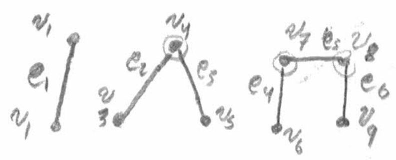
10.5.10

In each of 8-21, either draw a graph with the given specification or explain why no such graph exists.

10. Graph, circuit-free, nine vertices, six edges.

Total degree is 12 $\Rightarrow$ 1, 1, 1, 1, 1, 1, 2, 2, 2

Even # of odd

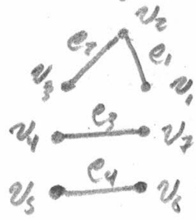


10.5, 15

15. Graph, circuit-free, seven versices, four-edges

Total degree = 8 $\Rightarrow$ 1, 1, 1, 1, 1, 1, 2

Even of odd

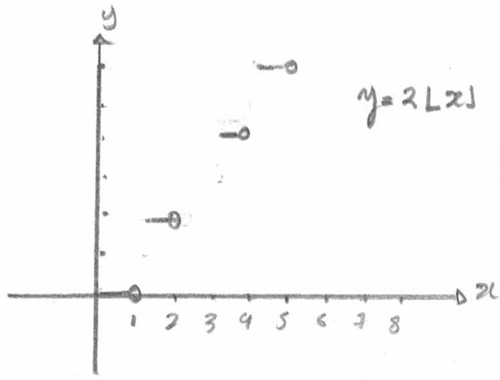


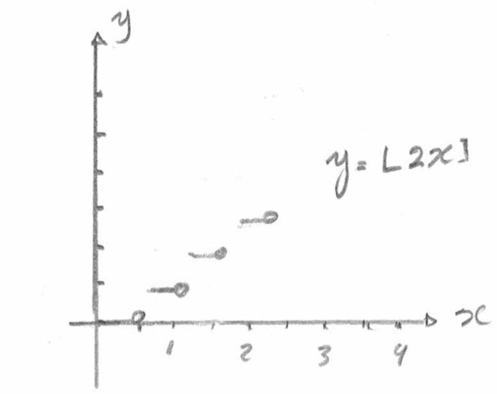
10.5.25

25. A graph has eight vertices and six edges. Is it connected? Why?

Answer: No, because for a graph of n vertices, it needs at least $n-1$ edges to be connected.

5. Draw the graph of $y = 2\lfloor x \rfloor$ & $y = \lfloor 2x \rfloor$ for all real numbers x. What can you conclude from these graphs?



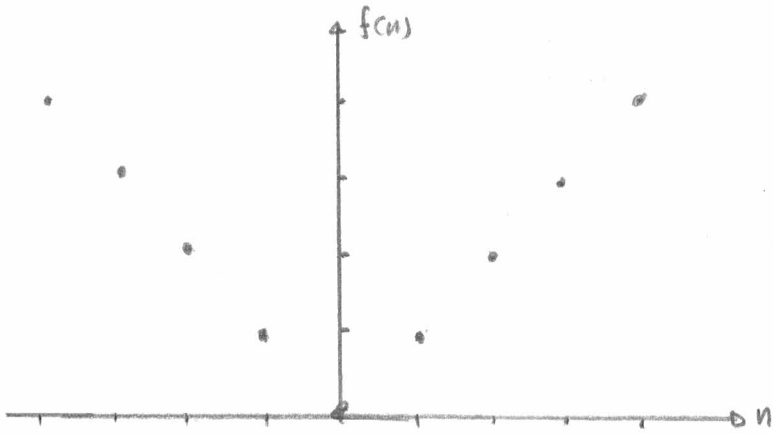


Conclusions $2\lfloor x \rfloor \neq \lfloor 2x \rfloor$

11.1.10

For each of 10-13, a function is defined on a set of integers. Graph each function

10. $f(n) = |n|$ for each integer n



11.1.20

20. Given real-valued functions f and g, with the same domain D, the sum of f and g, denoted $f+g$, is defined as follows:

For all real numbers x, $(f+g)(x) = f(x) + g(x)$

Show that if f and g are both increasing on set S, then $f+g$ is also increasing

Proof | Given $x_2 > x$ & $f(x_2) > f(x_1)$ & $g(x_2) > g(x_1)$

$$(f+g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2)$$

$\therefore$ $f+g$ is also increasing ■

11.1.25

28. If f be a real-value function of a real variable. Show that if f is increasing on a set S and M is any negative real number, then Mf is decreasing on S

Proof | Given $x_2 > x_1$ & $f(x_2) > f(x_1)$
Suppos m is a positive real number

$$mf(x_2) > mf(x_1)$$
$$-mf(x_2) > -mf(x_1)$$
$$mf(x_2) < mf(x_1)$$

$\because mf(x_1) > mf(x_2)$

$\therefore$ f is decreasing ∎

In each of 10-14 assume f and g are real-valued functions defined on the same set of nonnegative real numbers.

10. Prove that if $g(x)$ is $O(f(x))$ then $f(x)$ is $\Omega(g(x))$

Proof| Suppose $g(x)$ is $O(f(x))$ and B + b

$$|g(x)| \leq B|f(x)| \quad \forall x > b$$

$$\frac{1}{B}|g(x)| \leq |f(x)| \qquad \text{Let } A = \frac{1}{B} \text{ and } a = b$$

$$A|g(x)| = |f(x)| \quad \forall x > a$$

$$\Omega(g(x)) = |f(x)| \quad \blacksquare$$

11.2.40

Prove each of the statements in 40-47, assuming n is a variable that takes positive integer value.

40. $1^2+2^2+3^2+\dots+n^2$ is $\Theta(n^3)$

<u>Proof</u> | $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ which is $\Theta(n^3)$

$\therefore$ $1^2+2^2+3^2+\dots+n^2$ is $\Theta(n^3)$ ▨

31. Refer to the results of excercises 22 and 28 to find an order for $7x^4 - 95x^3 + 3$ from among the set of power functions.

From # 22: $|7x^4 - 95x^3 + 3| \le 105|x^4|$ for $x > 1$

From # 28: $7x^4 - 95x^3 + 3$ is $\Omega(x^4)$

$\therefore$ $7x^4 - 95x^3 + 3$ is $\Theta(x^4)$ ∎