Omar Alkendi

Chapter 4

April 06, 2020

Section 4.1

4 --------------------------------Done

5 --------------------------------Done

11 ------------------------------Done

12 ------------------------------Done

15 ------------------------------Done

20 ------------------------------Done

25 ------------------------------Done

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43 ------------------------------Done

Section 4.2

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Section 4.3

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Section 4.4

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Omar Alkendi

Chapter 4

April 06, 2020

Section 4.6

4 -----------------------------------Done

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21 ---------------------------------Done

22 ---------------------------------Done

27 ---------------------------------Done

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34 ---------------------------------Done

Prove the statements 4-10

4. There are integers m and n such that $m > 1$ and $n > 1$ and $\frac{1}{m} + \frac{1}{n}$ is an integer

Statement: $\exists m \in \mathbb{Z}^+$ and $\exists n \in \mathbb{Z}^+ \mid (\frac{1}{m} + \frac{1}{n}) \in \mathbb{Z}$ and $m > 1$ and $n > 1$

$\boxed{\text{Proof}}$ Let $m = 2$ and $n = 2$

$$\frac{1}{2} + \frac{1}{2} = \frac{2}{2}$$

$$= 1$$

$1 \in \mathbb{Z}$ ■

Prove the statements in 4-10

5. There are distinct integers m and n such that $\frac{1}{m} + \frac{1}{n}$ is an integer

Statements $\exists m \in \mathbb{Z}$ and $\exists n \in \mathbb{Z} \mid m \neq n$ and $(\frac{1}{m} + \frac{1}{n}) \in \mathbb{Z}$

$\boxed{\text{Proof}}$ Let m be any integer and $n = -m$

$$\frac{1}{m} - \frac{1}{m} = 0$$

$$0 \in \mathbb{Z} \quad \blacksquare$$

4.1.11

Disprove the statement in 11-13 by giving a counterexample.

11. For all real numbers a and b, if $a < b$ then $a^2 < b^2$

Statement: $\forall a \in \mathbb{R}$ and $\forall b \in \mathbb{R}$, if $a < b$ then $a^2 < b$

Counter Example | Let $a = -2$ and $b = 1$

$\therefore a < b$

So, $a^2 < b^2$

$(-2)^2 < (1)^2$

$4 \nless 1$ $\blacksquare$

Disprove the statements in 11-13 by giving a counter example.

12. For all integers n, if n is odd then $\frac{n-1}{2}$ is odd

Statement: $\forall n \in \mathbb{Z}$, if n is odd then $\frac{n-1}{2}$ is odd

Counter Example: Let $n=5$

Clearly, 5 is odd or $5 \bmod 2 = 1$

$$\therefore \frac{5-1}{2} \bmod 2 = 1$$

$$2 \bmod 2 = 1$$

$$0 \neq 1 \quad \blacksquare$$

In 14-16, determine whether the property is true for all integers, true for no integers, or true for some integers and true for other integers. Justify your answers.

15. $-a^n = (-a)^n$

Answer: True for odd integers n but false for even integers n.

Proof: Let a be a non-negative real number and n be an integer

n can be either even or odd

If n is even, then $n = 2k$ (Def. of even numbers)

if n is odd, then $n = 2k+1$ (Def. of odd numbers)

Case n is odd:

$$
\begin{aligned}
-a^{2k+1} &= (-a)^{2k+1} \\
&= (-a)^{2k+1} && \text{(Order of operations).} \\
&= (-a)^{2k}(-a) && \text{(Multiplication rule).} \\
&= a^{2k}(-a) && \text{(Power of product rule).} \\
&= -a^{2k+1} && \text{(Multiplication rule).}
\end{aligned}
$$

$-a^{2k+1} = -a^{2k+1}$ ■

Case n is even:

$$
\begin{aligned}
-a^{2k} &= (-a)^{2k} \\
&= ((-1)a)^{2k} && \text{(factor -1)} \\
&= (-1)^{2k} a^{2k} && \text{(Power of product)} \\
&= 1 \cdot a^{2k} && \text{(Multiplying by 1 property)}
\end{aligned}
$$

$-a^{2k} \neq a^{2k}$ ■

4.1.20

Each of the statements in 20-23 is true. For each, (a) rewrite the statement with the quantification implicit as If ___, then ___, and (b) write the first sentence of a proof (the "starting point") and the last sentence of the proof (the "conclusion to be shown"). Note that you don't need to understand the statement in order to be able to do these exercises.

20. For all integers m, if $m > 1$ then $0 < \frac{1}{m} < 1$

a) If an integer is greater than 1, then its reciprocal is between 0 and 1.

b) Start: Let m be an integer > 1

Conclusion: $0 < \frac{1}{m} < 1$

Prove the statements in 24-34. In each case use only the definition of the terms and the assumption listed on page 146, not any previously established properties of odd and even numbers. Follow the directions given in this section for writing proofs of universal statements.

25. The difference of any even integer minus any odd integer is odd .

Proof | Let a be an even integer

" b " " " "

" c " " odd "

$a = 2k$, $b = 2s$, $c = 2m + 1$

$a - b = 2(k - s)$

$(a-b) - c = 2(k-s) - (2m+1)$

$= 2(k - s - m) - 1$

$\therefore$ The difference of any even integer minus any odd integer is odd. ∎

Prove the statements in 24-34. In each case use only the definitions of the terms and the assumptions listed on page 146, not any previously established properties of odd and even integers. Follow the directions given in this section for writing proofs of universal statements.

31. If k is any odd integer and m is any even integer, then $k^2 + m^2$ is odd

Proof | Let $K = 2r+1$

" $m = 2s$

$$K^2 + m^2 = (2r+1)^2 + (2s)^2$$
$$= 4r^2 + 4r + 1 + 4s^2$$
$$= 2(2r^2 + 2r + 2s^2) + 1$$

$\therefore$ If k is any odd integer and m is any even integer, then $k^2 + m^2$ is odd.

∎

Find the mistakes in the "proofs" shown in 38-42

38. Theorem: For all integers k, if $k > 0$ then $k^2 + 2k + 1$ is composite

Proof: For $k = 2$, $k^2 + 2k + 1 = 2^2 + 2 \cdot 2 + 1 = 1$. But $9 = 3 \cdot 3$, and so 9 is composite. Hence the theorem is true.

Mistake: Arguing from example.

In 43-60, determine whether the statement is true or false. Justify your answer with a proof or a counterexample, as appropriate. In each case use only the definition of the terms and the Assumptions listed on page 146 not any previously established properties.

43. The product of any two odd numbers is odd

Proof | Let a be an odd integer
" b " " " "

$\therefore$ $a = 2k+1$, where k is an integer
$b = 2m+1$, " m " " "

$\therefore$ $a \cdot b = (2k+1)(2m+1)$
$= 4km + 2k + 2m + 1$
$= 2(2km + k + m) + 1$
$= 2(\text{Some integer}) + 1$

$\therefore$ The product of any two odd numbers is odd ■

The numbers in 1-7 are all rational. Write each number as the ratio of two integers.

4. 0.373737...

$$0.373737\ldots = x$$
$$37.373737\ldots = 100x$$
$$37.0 = 99x$$

$$0.373737 = \frac{37}{99}$$

The numbers in 1-7 are all rational. Write each number as a ratio of two integers

7. $52.4\overline{6721}$

$$52.4\overline{6721} = x$$

$$524.\overline{6721} = 10x$$

$$5246721.\overline{6721} = 100{,}000x$$

$$5246197 = 99990x$$

$$\therefore \quad 52.4\overline{6721} = \frac{5246197}{99990}$$

4.2.10

10. Assume that m and n are both integers and that $n \neq 0$. Explain why $(5m + 12n)/4n$ must be a rational number.

$$(5m + 12n)/(4n) = \frac{5m}{4n} + \frac{12n}{4n}$$

$$= \frac{5m}{4n} + \frac{3}{1}$$

$\because$ The sum is of two ratios

$\therefore$ $(5m+12n)/(4n)$ must be rational. $\blacksquare$

4.2.14

14. Consider the statements: The square of any rational number is a rational number.

a) Write the statement formally using a quantifier and a variable.

b) Determine whether the statement is true or false and justify your answer.

a: $\forall x,\ x \in \mathbb{Q} \rightarrow x^2 \in \mathbb{Q}$

b: Suppose $x = \frac{a}{b}$ where $x \in \mathbb{Q}$ and $a, b \in \mathbb{Z}$

$$x^2 = \left(\frac{a}{b}\right)^2$$

$$= \frac{a^2}{b^2}$$

$$= \frac{a \cdot a}{b \cdot b}$$

$$= \frac{a}{b} \cdot \frac{a}{b}$$

$\because$ The product of two integers is an integer

$\therefore$ x^2 is a ratio of two integers

$\therefore$ x^2 is rational.

■

4.2. 15

Determine which of the statements in 15-20 are true and which are false. Prove each true statement directly from the definitions, and give a counterexample for each false statement. In case the statements is false, determine whether a small change would make it true. If so, make the change and prove the new statement. Follow the directions for writing proofs on page 154.

15. The product of any two rational number is a rational number. $\Rightarrow$ True

Proof | Consider $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where a, b, c, and d are integers & $b \neq d \neq 0$

Clearly, x and y are rational numbers.

$$x \cdot y = \frac{a}{b} \cdot \frac{c}{d}$$

$$= \frac{ac}{bd}$$

$\because$ The product of two integers is an integer

$\therefore$ $x \cdot y$ is a ratio of two integers

$\therefore$ The product of any two rational numbers is a rational number.

$\blacksquare$

Determine which of the statements in 15-20 is true and which are false. Prove each statement directly from the definition, and give a counterexample for each false statement. In case the statement is false, determine whether a small change would make it true. If so, make the change and prove the new statement. Follow the directions for writing proofs on page 154.

20. Given any two rational numbers r and s with $r<s$, there is another rational number between r and s. → <u>True</u>

<u>Proof</u> | Let r and s be rational numbers where $r<s$

$$r+r < r+b < b+b$$

$$2r < r+b < 2b$$

$$\frac{\cancel{2}r}{\cancel{2}} < \frac{r+b}{2} < \frac{2b}{2}$$

$$r < \frac{r+b}{2} < b$$

∵ r and b are rational numbers
∵ the sum of r and b is
∴ $(r+b)$ is rational
∴ $\frac{(r+b)}{2}$ is rational

∵ $r < \frac{r+b}{2} < b$

∴ Between any two rational numbers r and b exists another rational number.

Derive the statements in 24-26 as corollaries of Theorem 4.2.1, 4.2.2, and the results of exercises 12, 13, 14, 15, and 17.

24. For any rational numbers r and s, $2r+3s$ is rational.

<u>Proof</u> Let r and s are rational numbers

$$2r+3s = r+r+s+s+s$$

$\because$ $2r+3s$ is the sum of rational numbers
$\therefore$ By Theorem 4.2.2, $2r+3s$ is rational.

∎

27. It is a fact that if n is nonnegative integer, then

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{1-(1/2^{n+1})}{1-(1/2)}$$

(A more general form of this statement is proved in Section 5.2) Is the right hand of this equation rational? If so, express it as ratio of two integers

$$x = \frac{1-\left(\frac{1}{2^{n+1}}\right)}{1-\frac{1}{2}} = \frac{\frac{2^{n+1}-1}{2^{n+1}}}{2^{-1}} = \frac{2^{n+1}-1}{2^n}$$

$\because$ 2^{n+1}, -1, and 2^n are integers, and $2^n \neq 0$

$\therefore$ $\dfrac{1-\left(\frac{1}{2^{n+1}}\right)}{1-\frac{1}{2}}$ is a rational number. ∎

In 35-39 find the mistakes in the "proofs" that the sum of any two rational numbers is a rational number.

37. "Proof: Suppose r and s are rational numbers. By definion of of rational, $r = a/b$ for some integers a and b with $b \neq 0$, and $s = a/b$ for some integers a and b with $b \neq 0$. Then

$$r + s = \frac{a}{b} + \frac{a}{b} = \frac{2a}{b}$$

Let $p = 2a$. Then p is an integer since it is the product of two integers. Hence $r+s = p/b$, where p and b are integers and $b \neq 0$. Thus $r+s$ is a rational number by the definition of rational. This is what was to be shown."

Mistakes The proof used the same letters to mean two different thing. By setting $r = \frac{a}{b}$ & $s = \frac{a}{b} \rightarrow r = s$, which make r & s not arbitrary, hence, violating that requirement.

Give a reason for your answer in each of 1-13. Assume that all variables represent integers.

4. Does 3 divide $(3k+1)(3k+2)(3k+3)$

$$= 3(k+1)(3k+1)(3k+2)$$

Answer: yes because the integer is a multiple of 3.

Give a reason for your answer in each of 1-13. Assume that all variables represent integers.

9. Is 4 a factor of $2a \cdot 34b$

$$= 2(2 \cdot 17)ab$$

$$= 4 \cdot 17 \cdot ab$$

Answer: Yes because the integer is a multiple of 4.

4.3.10

Give a reason for your answer in each of 1-13. Assume all variables represents integers.

10. Does $7 \mid 34$?

$7 \mid 2 \cdot 17$?

Answer: No because 7 is not a factor of 34.

4.3.13

Give a reason for your answer in each of 1-13. Assume all variables represent integers.

13. If $n = 4k+3$, does 8 divide n^2-1?

$$= (4k+3)^2 - 1$$

$$= 16K^2 + 24k + 9 - 1$$

$$= 8(2k^2 + 3K + 1)$$

Answer: Yes because 8 is a factor of n^2-1.

4.3.15

Prove statements 15 and 16 directly from the definition of divisibility.

15. For all integers a, b, and c if $a \mid b$ and $a \mid c$ then $a \mid (b+c)$.

Proof | Let a, b, and c be integers such that $a \mid b$ and $a \mid c$

$a \mid b \rightarrow b = a \cdot k$ where k is some integer

$a \mid c \rightarrow c = a \cdot l$ where l is some integer

$\because b + c = ak + al$
$= a(k + l)$

$\therefore a \mid b+c$ by the definition of divisibility. ■

4.3.22

For each statement in 19-31, determine whether the statement is true or false. Prove the statements directly from the definition of divisibility if it is true, and give a counter example if it is false.

22. A necessary condition for an integer to be divisible by 6 is that it be divisible by 2.

Answer: True.

Proof | Let a be an integr such that $6 \mid a$

By the definition of divisibility $a = 6 \cdot k$ where k is some integer

$= 2 \cdot 3 \cdot k$

$\therefore$ By the definition of divisibility, 2 is a factor of a which is a factor of 6.

$\blacksquare$

For each statement in 19-31, determine whether the statement is true or false. Prove the statement directly by the definition of divisibility if it is true, and give a counterexample if it is false.

23. A sufficient condition for an integer to be divisible by 8 is that it be divisible by 16

Answer: True

Proof | Let a be an integer such that $16 \mid a$

$\therefore$ $a = 16 \cdot k$, where k is some integer, by the definition of divisibility

$a = 8 \cdot 2k$

$\therefore$ $16 \mid a$ is a sufficient condition for $8 \mid a$ by the definition of divisibility.

$\blacksquare$

For each statement in 19–31, determine whether the statement is true or false. Prove the statement directly from the definitions if it is true, or give a counterexample if it is false.

27. For all integers a, b, and c, if $a \mid (b+c)$ then $a \mid b$ or $a \mid c$.

Answer: false

Counterexample | Let $a=3$, $b=2$, and $c=1$

Clearly, $3 \mid (2+1)$

But $3 \nmid 2$ nor $3 \nmid 1$

$\therefore$ The statement is false.

32. A fast-food chain has a contest in which a card with numbers on it is given to each customer who makes purchases. If some of of the numbers on the card add up to 100, then the customer wins \$100. A certain customer receives a card containing the numbers

72, 21, 15, 36, 69, 81, 9, 27, 42, and 63

Will the customer win \$100? Why or why not.

Answer: No because all the numbers are multiples of 3, and 100 is not a multiple of 3.

40. a) If a and b are integers and $12a = 25b$, does $12 \mid b$? does $25 \mid a$? Explain.

b) If x and y are integers and $10x = 9y$, does $10 \mid y$? does $9 \mid x$? Explain.

Solution: a) $\because$ $12a = 25b$

$2^2 \cdot 3 = 5^2 b$ and $2^2 \cdot 3 \neq 5^2$

$\because$ The standard factored for of $12a$ is the same as $25b$ by the unique factorization theorem

$\therefore$ $2^2 \cdot 3$ must occur in b and 5^2 must occur in a

$\therefore$ $25 \mid a$ and $12 \mid b$ ∎

b) $\because$ $10x = 9y$

$5 \cdot 2x = 3^2 y$ and $5 \cdot 2 \neq 3^2$

$\because$ By the unique factorization theorem, the standard form of $10x$ is the same as $9y$

$\therefore$ $5 \cdot 2$ must occur in y and 3^2 must occur in x

$\therefore$ $9 \mid x$ and $10 \mid y$ ∎

4.3.44

44. Prove that if n is any nonnegative integer whose decimal representation ends in 0, the $5 | n$.

Proof | Let n be a nonnegative integer whose decimal representation ends w/ 0.

$$\therefore \quad n = 10m + 0$$
$$= 10m$$
$$= 2 \cdot 5 \cdot m$$

$\because$ n is a multiple of 5

$\therefore$ n is divisible by 5. or $5 | n$ ■

For each of the values of n and d given in 1-6, find integers q and r such that $n = dq + r$ and $0 \leq r < d$

5. $n = -45$, $d = 11$

$n = (-5)11 + 10 \quad \Rightarrow q = -5 \quad \& \; r = 10$

Evaluate the expression in 7-10

8. 50 div 7 = 7

11. Check the correctness of formula (4.4.1) given in Eample 4.4.3 for the following values of DayT and N

a) DayT = 6 (Saturday) and N = 15
b) DayT = 0 (Sunday) and N = 7
c) DayT = 4 (Thursday) and N = 12

a. (6 + 15) % 7 = 0 ⇒ Sunday ⇒ Correct.

b. (0 + 7) % 7 = 0 ⇒ Sunday ⇒ Correct.

c. (4 + 12) % 7 = 2 ⇒ Tuesday ⇒ Correct.

4.4.16

16. Suppose d is a positive integer and n is any integer. I $d \mid n$, what is the remainder obtained when the quotient-remainder theorem is applied to n w/ divisor d?

$\because$ $d \mid n$ mean $n = d \cdot k$ where k is some integer

$\therefore$ $r = 0$.

4.4.20

20. Suppose a is an integer. If a mod 7 = 4, what is 5a mod 7?

$$5a \% 7 = (5 \cdot \cancel{7} \cdot d + 5 \cdot 4) \% 7$$

$$= 20 \% 7$$

$$= 6$$

4.4.22

Suppose c is an integer. If $c \bmod 15 = 3$, what is $10c \bmod 15$?

$$10c \,\%\, 15 = (15 \cdot \cancel{10}^{\,0} \cdot d + 10 \cdot 5) \,\%\, 15$$

$$= \quad 30 \,\%\, 15$$

$$= \quad 0$$

4.4.26

26. Prove that a necessary and a sufficient condition for a non negative integer n to be divisible by a positive integer d is that $n \bmod d = 0$

Proof | Let n and d be integers such that $d \mid n$

by definition, $d \mid n$ means $n = d \cdot k$ where k is some integer

by the quotient-remainder theorem, there exis integers $q = n$ div d
and $r = n \bmod d$

Clearly, $q = k$ and $r = 0$

$\therefore$ It is necessary and sufficient condition for n to be divisible by d is that the remainder $n \bmod d = 0$ ■

In 31-33, you may use properties listed in example 4.2.2.

31. a. Prove that for all integers $m+n$ and $m-n$ are either both odd or both even.

Case 1: Both m and n are even.

Property 4.2.3.1: The sum, product and difference of two even integers are even, hence

$\therefore$ $m+n$ and $m-n$ are both even

Case 2: Both m and n are odd

Property 4.2.3.2: The sum and differen of any two odd integers are even.

$\therefore$ $m+n$ and $m-n$ are both even.

Case 3: Either m or n is even, but not both.

Properties 4.2.3.5 and 4.2.3.6: The sum and difference of any odd integer and any even integer is odd.

$\therefore$ $m+n$ and $m-n$ are both odd

$\therefore$ $m+n$ and $m-n$ are either both odd or both even. $\blacksquare$

b.) Find all solutions to the equation $m^2 - n^2 = 56$ for which both m and n are positive.

Solution: $m^2 - n^2 = 56$

$(m+n)(m-n) = 56$ & $56 = 56\cdot 1,\ 28\cdot 2,\ 14\cdot 4,\ 7\cdot 8$

By part a) $m+n$ & $m-n$ can be either both even or both odd

$\therefore$ $28\cdot 2$ & $14\cdot 4$ are proper factors.

Using $14 = m+n$ & $4 = m-n \Rightarrow n = m-4$

$14 = m+m-4$

$18 = 2m$

$9 = m \Rightarrow n = 5$

Using $28 = m+n$ & $2 = m-n \Rightarrow n = m-2$

$28 = m+m-2$

$30 = 2m$

$15 = m \Rightarrow n = 13$

Answer: $(m,n) = \{(9,5),\ (15,13)\}$

c. Find all solutions to the equation $m^2 - n^2 = 8$ for which both m and n are positive integers.

$$m^2 - n^2 = 88$$

$(m+n)(m-n) = 88$ & $88 = 88 \cdot 1,\ 44 \cdot 2,\ 22 \cdot 4,\ 11 \cdot 8$

By part a., either both $m+n$ and $m-n$ are odd or both are even.

$\therefore$ $44 \cdot 2$ & $22 \cdot 4$ are proper factors

Using $44 \cdot 2$: $m+n = 44$ & $m-n = 2$

$2m = 46$ $\quad$ $m-2 = n$

$m = 23 \Rightarrow n = 21$

Using $22 \cdot 4$: $m+n = 22$ & $m-n = 4$

$2m = 26$ $\quad$ $m-4 = n$

$m = 13 \Rightarrow n = 9$

Answer: $(m,n) = \{(23, 21), (13, 9)\}$

Prove each of the statements in 35-46.

38. For any integer n, $n^2 + 5$ is not divisible by 4.

Proof | Let n be integer.
By the quotient-remainder theorem, n can be writen in one of the forms
$4q$ or $4q+1$ or $4q+2$ or $4q+3$
for some integer q.

Case 1: $n = 4q \Rightarrow n^2 = 16q^2 \Rightarrow n^2+5 = 16q^2+5$
$(n^2+5)\%4 = (16q^2+5)\%4$
$= 1$

$\because (n^2+5)\%4 \neq 0 \quad \therefore 4 \nmid n^2+5$

Case 2: $n = 4q+1 \Rightarrow n^2 = 16q^2+8q+1 \Rightarrow n^2+5 = 8(2q^2+q)+6$
$= 8(2q^2+q)+1$
$(n^2+5)\%4 = (8(2q^2+q)+6)\%4$
$= 2$

$\because (n^2+5)\%4 \neq 0 \quad \therefore 4 \nmid n^2+5$

Case 3: $n=4q+2 \Rightarrow n^2 = 16q^2+16q+4 \Rightarrow n^2+5 = 16(q^2+q)+9$
$= 16(q^2+q)+4$
$(n^2+5)\%4 = (16(q^2+q)+9)\%4$
$= 1$

$\because (n^2+5)\%4 \neq 0 \quad \therefore 4 \nmid n^2+5$

Case 4: $n=4q+3 \Rightarrow n^2 = 16q^2+24+9 \Rightarrow n^2+5 = 8(2q^2+3q+1)+6$
$= 8(2q^2+3q+1)+1$
$(n^2+5)\%4 = (8(2q^2+3q+1)+6)\%4$
$= 2$

$\because (n^2+5)\% \neq 0 \quad \therefore 4 \nmid n^2+5$

$\therefore$ By cases 1,2,3, and 4, $4 \mid n^2+5$ for any integer n.

Prove each of the statements in 35-46

44. For all real numbers x and y $|x| \cdot |y| = |xy|$

Proof | x has two cases $x \geq 0$ or $x < 0$

y has two cases $y \geq 0$ or $y < 0$

Case 1: $x \geq 0$ & $y \geq 0$

$$|x| \cdot |y| = (x) \cdot (y)$$
$$= xy$$
$$= |xy|$$

Case 2: $x \geq 0$ & $y < 0$

$$|x| \cdot |y| = x \cdot (-y)$$
$$= -xy$$
$$= |-xy|$$
$$= |xy|$$

Case 3: $x < 0$ & $y < 0$

$$|x| \cdot |y| = (-x)(-y)$$
$$= xy$$
$$= |xy|$$

Case 4: $x < 0$ & $y \geq 0$

$$|x||y| = (-x) \cdot y$$
$$= -xy$$
$$= |-xy|$$
$$|xy|$$

$\therefore$ for all real number x and y

$|x||y| = |xy|$ ∎

4. Use proof by contradiction to show that for all integers m, $7m+4$ is not divisible by 7.

Proof | suppose $7m+4$ is divisible by 7

By the definition of divisibility, $7 \mid 7m+4$

means $7m+4 = 7k$

$4 = 7k - 7m$

$4 = 7(k-m)$ where $k-m$ is some integer

But 4 is not a multiple of 7!

$\therefore$ $7m+4$ is not divisible by 7. ∎

Carefully formulate the negation of each statement in 5-7. Then prove each statement by contradiction.

5. There is no greatest even integer.

Negation: There is a greatest even integer.

Proof | Suppose there is a greatest even integer $N > n$ where n is all other even integers.

$N = 2k$ where k is some integer.

There exist a number m such that $m = 2(K+1)$
Hence, $m > N > n$

So N is the greatest and N is not the greatest.

$\therefore$ There is no greatest even integer. ∎

Prove each statement in 10 - 17 by contradiction.

10. The square root of any irrational number is irrational

Proof | Suppose that the square root of irrational number is rational, meaning it can be written as a quotient of two integers.

Let x be some irrational number and a and b some integers, and $b \neq 0$

$$(\sqrt{x})^2 = \left(\frac{a}{b}\right)^2$$

$$x = \frac{a^2}{b^2}$$

$\because \frac{a^2}{b^2}$ is a quotient of two integers,

$\therefore x$ is both rational and irrational !

$\therefore$ The squareroot of any irrational number is irrational. ■

4.6.12

Prove each statement in 10-17 by contradiction.

12. If a and b are rational numbers, $b \neq 0$, and r is an irrational number then $a+br$ is irrational

Proof | Suppose that the if a and b are rational number, $b \neq 0$, and r is an irrational nummber, then $a+br$ is rational.

Let $c = \frac{m}{n}$ where m and n are integers, and $n \neq 0$

Clearly c is a rational number by the definition of rational

Let $c = a + br$

$$\frac{c-a}{b} = r$$

$\therefore$ r is both rational and irrational ↯ Contradiction

$\therefore$ If a and b are rational numbers, $b \neq 0$, and r is an irrational number, then $a+br$ is irrational. ∎

Prove each statement in 10 - 17 by contradiction

15. If a, b, and c are integers and $a^2 + b^2 = c^2$, then at least one of a and b is even.

Proof | Suppose that if a, b, and c are integers and $a^2 + b^2 = c^2$, then both a and b are odd.

$a = 2k + 1$ where k is some integer
$b = 2m + 1$ where m is some integer

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (2k+1)^2 + (2m+1)^2 \\ &= 4k^2 + 4k + 1 + 4m^2 + 4m + 1 \\ &= 4(k^2 + k + m^2 + m) + 2 \\ &= 4n + 2 \\ &= 2(2n+1) \end{aligned}$$

Let $k^2 + k + m^2 + m = n$

$\Rightarrow \therefore c^2$ is an even number

Let $c = 2l$ where l is some integer

$c^2 = 4l^2 \longrightarrow 2(2n+1) = 4l^2$

$2n + 1 = 2l^2$

odd = even

$\therefore c^2$ is both even and odd & Contradiction

$\therefore$ If a, b, and c are integers and $a^2 + b^2 = c^2$, then at least one of a and b is even.

■

4.6.21

21. Consider the statement "For all integers n, if n^2 is odd then n is odd."
a. Write what would you suppose and what you need to show to prove this statement by contradiction.
b. Write what you would suppose and what you need to show to prove the statement by contraposition

a. Suppositions: For all integers n, if n^2 is odd then n is even.
Show : The supposition leads to a contradiction.

b. Suppositions: For all real integers n, if n is even then n^2 is eve.
Show : n^2 is even.

4.6.22

22. Consider the statement "For all real numbers r, if r^2 is irrational then r is irrational."

a. Write what would you suppose and what would you need to show to prove this statement by contradiction.

b. Write what would you suppose and what would you need to show to prove this statement by contraposition.

a. supposition: if r^2 is irrational, then r is rational.
Show : logical contradiction.

b. supposition: if r is rational, then r^2 is rational.
Show : r^2 is rational.

Prove each statement in 23-29 in two ways: (a) by contraposition and (b) by contradiction

27. For all integers m and n, if m+n is even then m and n are both even or m and n are both odd

a) Contraposition

Proof | For all integers m and n, if one is odd and the other is even, then m+n is odd

Let $m = 2k$ where k is some integer
$n = 2l+1$ " l " " "

Clearly, m is even and n is odd.

$m+n = 2(k+l)+1$
$= 2(\text{some integer})+1$

$\therefore$ m+n is odd ∎

b) Contradiction

Proof | For all integers m and n, if m+n is even, then either m or n is even while the other is odd

Let $m=2k$ wher k is some integer
$n=2l+1$ " l " " "

$m+n = 2(k+l)+1$

$\therefore$ m+n is even and odd ↯ contradiction

$\therefore$ For all integers m and n, if m+n is even, then m and n are both even or m and n are both odd. ∎

31. a. Prove by contraposition: For all positive integers n, r, and s, if $rs \le n$, then $r \le \sqrt{n}$ or $s \le \sqrt{n}$

Proof | Suppose for all positive integers n, r, and s, if $r > \sqrt{n}$ and $s > \sqrt{n}$, then $rs > n$

$r > \sqrt{n}$	$s > \sqrt{n}$
$rs > s\sqrt{n}$	$s\sqrt{n} > \sqrt{n}\sqrt{n}$
$rs > s\sqrt{n} > n$ (Transivity)	$s\sqrt{n} > n$

$\therefore rs > n$ ∎

31. b. Prove: For all integers $n > 1$, if n is not a prime, then there exits a prime number p such that $p \leq \sqrt{n}$ and n is divisible by p

Proof | Statement: $n \sim prime \rightarrow \exists p \mid p \leq \sqrt{n} \wedge p \mid n$

$p \mid n$ means $n = pk$ where $k > 1$

$n \leq pk$

$pk \geq n$ Rewrite

$p \geq \sqrt{n}$ or $k \geq \sqrt{n}$ by part a.

If k is not a prime, then $\exists m \mid m \mid k$

$\therefore m \mid k \rightarrow m \mid k$ and $m \mid n$ by transitivity.

If ... (repeats till a prime is found)

$\therefore \exists p \mid p$ is a prime $\wedge p \mid n$ ∎

Prime number $p = km$ where if $k = 1 \rightarrow m = p$ or if $k = p \rightarrow m \rightarrow 1$

$a \mid b$ means $b = al$

31. c. State the contrapositive of part (b)

Conrapositive: For all integers $n>1$, there exist a prime number p such that $p > \sqrt{n}$ or n is not divisible by p, then n is a prime.

34. Use the test for primality and the result of excercise 33 to determine whether the following numbers are prime.

a. 9,269. b. 9,103. c. 8,623. d. 7,917.

a. $\sqrt{9,269} \approx 96$

2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~	11	~~12~~	13
~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~	~~21~~	~~22~~	23	~~24~~	~~25~~
~~26~~	~~27~~	~~28~~	29	~~30~~	31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37
~~38~~	~~39~~	~~40~~	41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~
~~50~~	51	~~52~~	53	~~54~~	~~55~~	~~56~~	57	~~58~~	59	~~60~~	61
~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~	71	~~72~~	73
~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~	~~81~~	~~82~~	83	84	~~85~~
~~86~~	87	~~88~~	89	~~90~~	91	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	

9,269 / 13 = 713 $\Rightarrow$ 9,269 is not a prime

b. $\sqrt{9103} \approx 94$

none divides 9103 $\Rightarrow$ 9,103 is a prime

c. $\sqrt{8,623} \approx 92$

none divides 8623 $\Rightarrow$ 8623 is a prime

d. $\sqrt{7917} \approx 88$

7917 / 87 = 91 $\Rightarrow$ 7,917 is not a prime