MACRO Model Documentation

1 Questions

1.1 Relative prices of consumption

Looking at the Euler type equation of the optimal path of consumption (section 7, equation 40), the optimal path does not depend on the energy prices. An important feature of the model should be that energy price shocks have an effect on the consumption path. Is that already inherent in the optimization or is there a way to implement that?

Related to the above question do we also need a price for all other consumption goods C?

1.2 Capital and Macroeconomic Accounting, Investment

Both model versions (with and without macroeconomic accounting equation) need to be checked for economic and mathematic feasibility. Which version is feasible and to be preferred?

Can Y be interpreted as GDP as in the original model? This is related to the new modelling approach of taking the direct energy consumption of the household out of the production function and implementing it into the utility function.

1.3 Energy Accounting

Splitting the energy consumption of end-use services into inputs for the production function and direct consumption of the households in the utility function, makes changes in the energy determining equations necessary. Depending on the version with or without macroeconomic accounting, the energy cost equation EC has to be implemented.

1.4 Terminal Condition

Does the terminal condition suit the model?

1.5 Population vs. Labor Force

What is $L_{n,t}$? The labor force or the population? In the original model it is the labor force. Moreover, it is defined as exogenous labor supply growth (including both changes in labor force and labor productivity growth). In the calibration process the parameter is also calibrated with $L = L * (1 + grow)$, with $grow$ being changed until Y is equal to an exogenous GDP path. In that sense, L is assumed to incorporate increases in labor productivity.

1.6 Utility Function

Emmerling et al. (2025) does not use a log utility. Should I do that? For comparative reasons, I think it is best to use the same log function as in the old model. However, log utility is not so sensible to redistribution.

1.7 Saving rate

Is the saving rate by Emmerling et al. (2025) valid for us as well?

2 Main Features of the Model

The macroeconomic model is a Ramsey type intertemporal optimization model. A representative household is maximizing its utility (equation 14) in each macroeconomic region constrained by its wealth (capital) accumulation formulation (equation 15). See chapter "The Model".

The model and its respective optimization is derived from an upstream maximization problem of a household maximizing its utility, whose equilibrium solution is substituted back into the original utility function and capital formulation. This way we construct a single optimization problem, which is solvable using GAMS. See chapter "General Derivation of the Model".

2.1 New features:

First, the utility is not only dependent on consumption, but also on direct energy consumption, which is introduced in a Stone-Geary type formulation into the utility function (equation 1).

Second, household heterogeneity is introduced by implementing income deciles having different capital returns and labor income. In aggregation the households behave similar to the representative household.

Third, the introduction of income deciles allows for the implementation of Decent Living Energy (DLE) requirements for each decile, simply by constraining the model such that each decile has to have a minimum energy consumption according to the thresholds of the DLE requirements in each period.

At last, energy price shocks such as increasing carbon prices should affect lower income deciles relatively more strongly.

2.2 Future features:

The model should enable the implementation of redistributive policies in a next step.

3 Notation declaration

The following short notation is used in the mathematical description relative to the GAMS code:

Math Notation	GAMS set & index notation	Description
n	node (or node_active in loops)	spatial node corresponding to the macro-economic MESSAGE regions
t	year	year (2005, 2010, 2020, ..., 2100)
s	sector	sector corresponding to the (commercial) end-use demands of MESSAGE

3.1 Parameters (Exogenous)

A listing of all parameters exogenous to the model used in MACRO together with a description can be found in the table below.

Parameter	Description
period_t	Number of years in time period t (forward diff)
$\mathrm{total\_cost}_{n,t}$	Total system costs in region n and period t from MESSAGE model run
$\mathrm{enestart}_{n,s,t}$	Consumption level of (commercial) end-use services s in region n and period t from MESSAGE model run
$\mathrm{p}_{n,s,t}$	Shadow prices of (commercial) end-use services s in region n and period t from MESSAGE model run
$\mathrm{E}_{min,n,s,t}$	Subsistence level of direct energy consumption (end-use service) in region n, sector s and period t
$\mathrm{h}_{n,s,t}$	Share of the direct energy consumption of the total energy production in region n, sector s and period t
ϵ_n	Elasticity of substitution between capital-labor and total energy in region n
ρ_n	$\epsilon - 1/\epsilon$ where ϵ is the elasticity of substitution in region n
β_n	Consumption value share parameter in region n
$\sigma_{n,s}$	Direct energy consumption value share parameter in region n and of sector s
δ_n	Annual depreciation rate in region n
α_n	Capital value share parameter in region n
a_n	Production function coefficient of capital and labor in region n
$b_{n,s}$	Production function coefficients of the different end-use sectors in region n, sector s and period t
$\mathrm{udf}_{n,t}$	Utility discount factor in period year in region n and period t
$\mathrm{L}_{n,t}$	Labor force in region n and period t
$\mathrm{w}_{n,t}$	Wage rate in region n and period t
$\mathrm{grow}_{n,t}$	Annual growth rates of potential GDP in region n and period t
$\mathrm{aeei}_{n,s,t}$	Autonomous energy efficiency improvement (AEEI) in region n, sector s and period t

$\text{fin\_time}_{n,t}$	finite time horizon correction factor in utility function in region n and period t

3.2 Decision variables (Endogenous)

Variable	Definition	Description
$\mathrm{K}_{n,t}$	$\mathrm{K}_{n,t} \geq 0 \; \forall n, t$	Capital stock in region n and period t
$\mathrm{Y}_{n,t}$	$\mathrm{Y}_{n,t} \geq 0 \; \forall n, t$	Total production in region n and period t
$\mathrm{C}_{n,t}$	$\mathrm{C}_{n,t} \geq 0 \; \forall n, t$	Consumption in region n and period t
$\mathrm{I}_{n,t}$	$\mathrm{I}_{n,t} \geq 0 \; \forall n, t$	Investment in region n and period t
$\mathrm{PHYSENE}_{n,s,t}$	$\mathrm{PHYSENE}_{n,s,t} \geq 0 \; \forall n, s, t$	Physical end-use service use in region n, sector s and period t
$\mathrm{TE}_{n,s,t}$	$\mathrm{TE}_{n,s,t} \geq 0 \; \forall n, s, t$	Total value of end-use service in the production and utility function in region n, sector s and period t
$\mathrm{E}_{n,s,t}$	$\mathrm{E}_{n,s,t} \in [-\infty..\infty]$	Value of end-use service in the utility function in region n, sector s and period t
$\mathrm{YE}_{n,s,t}$	$\mathrm{YE}_{n,s,t} \in [-\infty..\infty]$	Value of end-use service in the production function in region n, sector s and period t
$\mathrm{EC}_{n,t}$	$\mathrm{EC} \in [-\infty..\infty]$	Approximation of system costs based on MESSAGE results
UTILITY	$\mathrm{UTILITY} \in [-\infty..\infty]$	Utility function (discounted log of consumption and direct energy consumption)

4 General Derivation of the Model

The model and its respective optimization is derived from an upstream maximization problem of a household maximizing its utility, whose equilibrium solution is substituted back into the original utility function and capital formulation. This way we construct a single optimization problem, which is solvable using GAMS.

4.1 Utility Function

The utility function, which is maximized, sums the discounted logarithm of consumption and direct energy consumption of end-use services of a single representative household over the entire time horizon of the model.

$L_{n,t}E_{min,s,n}$ is the amount of direct energy consumption of end-use services of each sector s to satisfy the subsistence level $E_{min,s,n}$, which is the per capita subsistence level for each sector. $L_{n,t}$ is the labor force for each region n.

$$UTILITY = \sum_{n}\sum_{t}^{T} udf_{n}^{t}\left(\log(C_{t,n}^{\beta}\prod_{s=1}^{3}(E_{t,s,n} - L_{n,t}E_{min,s,n})^{\sigma_{s,n}})\right) \tag{1}$$

$$UTILITY = \sum_{n}\sum_{t}^{T} udf_{n}^{t}\left(\beta\log(C_{t,n}) + \sum_{s=1}^{3}\sigma_{s,n}\log(E_{t,s,n} - L_{n,t}E_{min,s,n})\right) \tag{2}$$

4.2 Capital Dynamics

The household maximizes its utility subject to the constraint on wealth accumulation of capital in the sectors not represented in the energy model MESSAGE. The net capital (or wealth) formation $K_{n,t}$ is derived from the existing capital stock, returns on capital, labor income, minus the expenses for direct energy consumption and all other consumption goods, as well as depreciation of the previous capital stock.

$$K_{t+1,n} = (1-\delta_n)K_{t,n} + r_{t,n}K_{t,n} + w_{t,n}L_{t,n} - \sum_{s=1}^{3}(p_{t,s,n}E_{t,s,n}) - C_{t,n} + \pi_{t,n} \tag{3}$$

$$\pi_{t,n} = f(K_{t,n}, L_{t,n}, YE_{t,n}) - r_{t,n}K_{t,n} - w_{t,n}L_{t,n} - \sum_{s=1}^{3}(p_{t,s,n}YE_{t,s,n})$$

$$with \quad f(K_{t,n}, L_{t,n}, YE_{t,n}) = r_{t,n}K_{t,n} + w_{t,n}L_{t,n} + \sum_{s=1}^{3}(p_{t,s,n}YE_{t,s,n})\pi_{t,n} = 0$$

$$K_{t+1,n} = (1-\delta_n)K_{t,n} + r_{t,n}K_{t,n} + w_{t,n}L_{t,n} - \sum_{s=1}^{3}(p_{t,s,n}E_{t,s,n}) - C_{t,n} \tag{4}$$

$$C_{t,n} = (1-\delta_n)K_{t,n} + r_{t,n}K_{t,n} + w_{t,n}L_{t,n} - \sum_{s=1}^{3}(p_{t,s,n}E_{t,s,n}) - K_{t+1,n} \tag{5}$$

4.3 The Household's Maximization Problem

When substituting $C_{n,t}$ from equation (5) back into the utility function (equation (2)), the household's maximization problem reads as follows. We drop the region index here as each region maximizes the utility for itself.

$$\begin{aligned}\max_{E_s} = \sum_{t}^{T} udf^{t}\Bigg[&\beta\log\left((1-\delta_n)K_{t,n} + r_{t,n}K_{t,n} + w_{t,n}L_{t,n} - \sum_{s=1}^{3}(p_{t,s,n}E_{t,s,n}) - K_{t+1,n}\right)\\ &+\sum_{s=1}^{3}\sigma_s\log(E_{t,s} - L_{n,t}E_{min,t,s})\Bigg]\end{aligned} \tag{6}$$

$$\frac{\partial U}{\partial E_{t,s}} = -p_{t,s}\frac{\beta}{C_t} + \frac{\sigma_s}{E_{t,s} - L_{n,t}E_{min,t,s}} \stackrel{!}{=} 0$$

$$E_{t,s} - L_{n,t}E_{min,t,s} = \frac{\sigma_s}{\beta}\frac{C_t}{p_{t,s}} \tag{7}$$

$$E_{t,s} = L_{n,t}E_{min,t,s} + \frac{\sigma_s}{\beta}\frac{C_t}{p_{t,s}} \tag{8}$$

Substituting $E - L_{n,t}E_{min}$ from equation (7) into the utility function (equation 2):

$$UTILITY = \sum_n \sum_t^T udf_n^t \left(\log(C_{t,n}^{\beta} \prod_{s=1}^{3} (\frac{\sigma_{s,n}}{\beta}\frac{C_{t,n}}{p_{t,s,n}})^{\sigma_{s,n}}) \right) \tag{9}$$

$$UTILITY = \sum_n \sum_t^T udf_n^t \left(\beta \log(C_{t,n}) + \sum_{s=1}^{3} \sigma_{s,n} \log(\frac{\sigma_{s,n}}{\beta}\frac{C_{t,n}}{p_{t,s,n}}) \right) \tag{10}$$

$$UTILITY = \sum_n \sum_t^T udf_n^t \left(\beta \log(C_{t,n}) + \sum_{s=1}^{3} \sigma_{s,n} \log(C_{t,n}) - \sum_{s=1}^{3} \sigma_{s,n} \log(p_{t,s,n}) + \sum_{s=1}^{3} \sigma_{s,n} \log(\frac{\sigma_{s,n}}{\beta}) \right)$$

$$UTILITY = \sum_n \sum_t^T udf_n^t \left((\beta + \sum_{s=1}^{3} \sigma_{s,n}) \log(C_{t,n}) - \sum_{s=1}^{3} \sigma_{s,n} \log(p_{t,s,n}) + \sum_{s=1}^{3} \sigma_{s,n} \log(\frac{\sigma_{s,n}}{\beta}) \right) \tag{11}$$

Substituting E from equation 8 into the capital dynamics equation 4:

$$K_{t+1,n} = (1-\delta_n)K_{t,n} + r_{t,n}K_{t,n} + w_{t,n}L_{t,n} - \sum_{s=1}^{3}(p_{t,s,n}(L_{n,t}E_{min,t,s,n} + \frac{\sigma_{s,n}}{\beta}\frac{C_{t,n}}{p_{t,s,n}})) - C_{t,n} \tag{12}$$

$$K_{t+1,n} = (1-\delta_n)K_{t,n} + r_{t,n}K_{t,n} + w_{t,n}L_{t,n} - \sum_{s=1}^{3} p_{t,s,n}L_{n,t}E_{min,t,s,n} - \sum_{s=1}^{3} \frac{\sigma_{s,n}}{\beta}C_{t,n} - C_{t,n}$$

$$K_{t+1,n} = (1-\delta_n)K_{t,n} + r_{t,n}K_{t,n} + w_{t,n}L_{t,n} - \sum_{s=1}^{3} p_{t,s,n}L_{n,t}E_{min,t,s,n} - \frac{\beta + \sum_{s=1}^{3}\sigma_{s,n}}{\beta}C_{t,n} \tag{13}$$

Equations 11 and 13 now describe the basic optimization problem, which is then implemented into GAMS and is described in the next section.

5 The Optimization Model Implemented in GAMS

Please check which of these two versions are economically and mathematically feasible: with and without a macroeconomic accounting equation.

The original MACRO model follows a stock-based structure with adding a macroeconomic accounting equation

$$\mathrm{Y}_{n,y} = \mathrm{C}_{n,y} + \mathrm{I}_{n,y} + \mathrm{EC}_{n,y} \qquad \forall n, y$$

As we are changing the model into a flow-based logic, the utility function and the capital formulation should be sufficient to solve the model and determine C and K such that the accounting equation is not necessary. However, maybe the accounting equation is necessary to determine a solution corresponding to an exogenous GDP path.

That is why I introduce two versions. One simpler model without the accounting equation as well as one with a slightly modified accounting equation.

5.1 Without Macroeconomic Accounting Equation

The utility function and the capital formulation of the optimization problem are derived in the previous section (equations 11 and 13).

The rest of the model (production function and the energy accounting) are based on the original MACRO model.

5.1.1 Household Utility Maximization

To implement the model into GAMS we add the duration of a period to each time step as the model solves in 5 and 10 year time steps as well as a correction factor $\frac{1}{\mathrm{fin\_time}_{n,y}}$ reflecting the finite time horizon of the model.

$$\begin{aligned}
\max_{C_{t,n}} \quad & \sum_n \sum_t^{T-1} udf_n^t \Bigg[(\beta + \sum_{s=1}^{3} \sigma_{s,n}) \log(C_{t,n}) \\
& \qquad - \sum_{s=1}^{3} \sigma_{s,n} \log(p_{t,s,n}) + \sum_{s=1}^{3} \sigma_{s,n} \log\left(\frac{\sigma_{s,n}}{\beta}\right) \Bigg] \cdot period_t \\
& + \sum_n \sum_{T-1}^{T} udf_n^t \Bigg[(\beta + \sum_{s=1}^{3} \sigma_{s,n}) \log(C_{t,n}) \\
& \qquad - \sum_{s=1}^{3} \sigma_{s,n} \log(p_{t,s,n}) + \sum_{s=1}^{3} \sigma_{s,n} \log\left(\frac{\sigma_{s,n}}{\beta}\right) \Bigg] \cdot \left(period_{T-1} + \frac{1}{fin_time_{n,t}} \right)
\end{aligned} \tag{14}$$

5.1.2 Wealth (Capital) Accumulation Formulation

with

$$\begin{aligned}
K_{t+1,n} = & (1-\delta_n)^{period_t} K_{t,n} + ((1+r_{t,n})^{period_t} - 1) K_{t,n} \\
& + period_t \left(w_{t,n} L_{t,n} - \sum_{s=1}^{3} p_{t,s,n} L_{n,t} E_{min,t,s,n} - \frac{\beta + \sum_{s=1}^{3} \sigma_{s,n}}{\beta} C_{t,n} \right)
\end{aligned} \tag{15}$$

5.1.3 Production Function

We implement a nested constant elasticity of substitution (CES) production function with capital, labor, and the (commercial) end-use services represented in MESSAGE as inputs. Y should correspond to GDP.

$$\mathrm{Y}_{n,t} = \left(a_n \cdot \mathrm{K}_{n,t}^{(\rho_n \cdot \alpha_n)} \cdot \mathrm{L}_{n,t}^{(\rho_n \cdot (1-\alpha_n))} + \sum_s (b_{n,s} \cdot \mathrm{YE}_{n,s,t}^{\rho_n}) \right)^{\frac{1}{\rho_n}} \qquad \forall n, t > 1 \tag{16}$$

5.1.4 Energy Accounting

The original MACRO model is based on a stock-based structure. As the new model follows a flow-based logic, the energy equations, that is YE and E, also have to be changed.

In the original model there is only TE and $PHYSENE$. TE enters the production function as input. There is no energy in the utility function. As we split TE into YE and E, we also have to change the energy accounting.

Exogenous from MESSAGE:

Remember that 'PHYSENE' and 'p' are exogenous to the model coming endogenously from the iteratively linked energy model MESSAGE.

- 'PHYSENE' (physical energy demand)
- 'p' (energy prices)
- 'enestart' (reference energy consumption)

Endogenous from the macroeconomic model:

- 'TE' is determined by household optimization and production decisions
- 'E' (direct energy consumption) results from household optimization (Equation 8)
- 'YE' (productive energy use) results from the production function

Important difference from the old model: The new model no longer uses vintage-based accumulation of energy capital. Instead, 'TE' is determined directly in each period through optimal decisions of economic agents.

First, 'TE' is the sum of 'E' and 'YE':

$$\mathrm{TE}_{n,s,t} = \mathrm{E}_{n,s,t} + \mathrm{YE}_{n,s,t} \qquad \forall n, s, t \tag{17}$$

Second, 'E' is a fraction of 'TE' based on an exogenous share 'h' for each sector:

$$\mathrm{E}_{n,s,t} = \mathrm{h}_{n,s} * \mathrm{TE}_{n,s,t} \qquad \forall n, s, t \tag{18}$$

We implement 'equation 8' from the derivation of the model as a constraint into the optimization to determine 'E':

$$with \quad E_{t,s} = L_{n,t} E_{min,t,s} + \frac{\sigma_s}{\beta} \frac{C_t}{p_{t,s}}$$

$$TE_{t,s} = \frac{L_{n,t} E_{min,t,s} + \frac{\sigma_s}{\beta} \frac{C_t}{p_{t,s}}}{h_{n,s}} \tag{19}$$

5.1.5 Connection to MESSAGE Energy Model via PHYSENE

The relationship below establishes the link between physical energy $\mathrm{PHYSENE}_{n,s,y}$ as accounted in MESSAGE for the six commercial energy demands s and energy in terms of monetary value $\mathrm{TE}_{n,s,y}$ as specified in the production function of MACRO.

$$\mathrm{PHYSENE}_{n,s,t} \geq \mathrm{TE}_{n,s,t} \cdot \mathrm{aeei\_factor}_{n,s,t} \tag{20}$$

The cumulative effect of autonomous energy efficiency improvements (AEEI) is captured in:

$$\mathrm{aeei\_factor}_{n,s,y} = \mathrm{aeei\_factor}_{n,s,y-1} \cdot (1 - \mathrm{aeei}_{n,s,y})^{\mathrm{duration\_period}_y}$$

with $\mathrm{aeei\_factor}_{n,s,y=1} = 1$. Therefore, choosing the $\mathrm{aeei}_{n,s,y}$ coefficients appropriately offers the possibility to calibrate MACRO to a certain energy demand trajectory from MESSAGE.

5.1.6 Terminal Condition

We introduce a new terminal condition compared to the original model as we no longer use the investment parameter I.

$$\frac{\mathrm{C}_{n,t}}{\mathrm{K}_{n,t}} = \mathrm{r}_{n,t} + \delta_n - \mathrm{grow}_{n,t} \qquad \forall n, t = \text{last year} \tag{21}$$

5.1.7 Important Changes related to the original energy accounting

We drop the derivation of the energy cost necessary for the macroeconomic accounting equation as we do not need it.

$$\begin{aligned}
&\cancel{\mathrm{EC}_{n,y} = \mathrm{total_cost}_{n,y}} \\
&\cancel{+ \sum_s \mathrm{eneprice}_{s,y,n} \cdot (\mathrm{PHYSENE}_{n,s,y} - \mathrm{enestart}_{s,y,n})} \\
&\cancel{+ \sum_s \frac{\mathrm{eneprice}_{s,y,n}}{\mathrm{enestart}_{s,y,n}} \cdot (\mathrm{PHYSENE}_{n,s,y} - \mathrm{enestart}_{s,y,n})^2} \quad \forall n, y > 1
\end{aligned}$$

5.1.8 Main Question

Is the energy accounting, specifically YE and E, sufficiently determined?

5.2 The Model with macroeconomic accounting equation

5.2.1 Household Utility Maximization

$$
\begin{aligned}
\max_{C_{t,n}} \quad & \sum_n \sum_t^{T-1} udf_n^t \Bigg((\beta + \sum_{s=1}^{3} \sigma_{s,n}) \log(C_{t,n}) - \sum_{s=1}^{3} \sigma_{s,n} \log(p_{t,s,n}) \\
& \quad + \sum_{s=1}^{3} \sigma_{s,n} \log\left(\frac{\sigma_{s,n}}{\beta}\right)\Bigg) \cdot period_t \\
& + \sum_n \sum_{T-1}^{T} udf_n^t \Bigg((\beta + \sum_{s=1}^{3} \sigma_{s,n}) \log(C_{t,n}) - \sum_{s=1}^{3} \sigma_{s,n} \log(p_{t,s,n}) \\
& \quad + \sum_{s=1}^{3} \sigma_{s,n} \log\left(\frac{\sigma_{s,n}}{\beta}\right)\Bigg) \cdot \left(period_{T-1} + \frac{1}{fin\_time_{n,t}} \right)
\end{aligned} \tag{22}
$$

5.2.2 Wealth (Capital) Accumulation Formulation

Compared to the model without the accounting equation, we introduce I as part of the capital constraint.

$$
\begin{aligned}
K_{t+1,n} = {} & (1-\delta_n)^{period_t} K_{t,n} + ((1+r_{t,n})^{period_t} - 1) K_{t,n} \\
& + period_t \left(w_{t,n} L_{t,n} - \sum_{s=1}^{3} p_{t,s,n} L_{n,t} E_{min,t,s,n} - \frac{\beta + \sum_{s=1}^{3} \sigma_{s,n}}{\beta} C_{t,n} \right)
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
I_{n,t} = {} & ((1+r_{t,n})^{period_t} - 1) K_{t,n} \\
& + period_t \left(w_{t,n} L_{t,n} - \sum_{s=1}^{3} p_{t,s,n} L_{n,t} E_{min,t,s,n} - \frac{\beta + \sum_{s=1}^{3} \sigma_{s,n}}{\beta} C_{t,n} \right)
\end{aligned} \tag{24}
$$

$$
K_{t+1,n} = (1-\delta_n)^{period_t} K_{t,n} + I_{n,t} \tag{25}
$$

5.2.3 Production Function

$$
\mathrm{Y}_{n,t} = \left(a_n \cdot \mathrm{K}_{n,t}^{(\rho_n \cdot \alpha_n)} \cdot \mathrm{L}_{n,t}^{(\rho_n \cdot (1-\alpha_n))} + \sum_s (b_{n,s} \cdot \mathrm{YE}_{n,s,t}^{\rho_n}) \right)^{\frac{1}{\rho_n}} \qquad \forall n, t > 1 \tag{26}
$$

5.2.4 NEW Accounting Equation

Remember that in the original model the macroeconomic accounting equation is:

$$
\mathrm{Y}_{n,t} = \mathrm{C}_{n,t} + \mathrm{I}_{n,t} + \mathrm{EC}_{n,t} \qquad \forall n, t
$$

with EC being the total energy costs of households' and firms' energy consumption.

As we remove the energy consumption of households as inputs from the production function, the question arises whether Y is still interpretable as GDP.

If we assume that accounting in the original MACRO model is correct and that Y is no longer interpretable as GDP, we have to split the energy cost derivation into two equations and the total GDP would be:

$$\mathrm{Y}_{total,n,t} = \mathrm{Y}_{goods,n,t} + \mathrm{EC}_{household,n,t}$$

With:

- $\mathrm{Y}_{goods,n,t}$ being the production function
- $\mathrm{EC}_{household,n,t}$ is the cost of direct consumption of households

The accounting equation now is:

$$\mathrm{Y}_{total,n,t} = \mathrm{C}_{n,t} + \mathrm{I}_{n,t} + \mathrm{EC}_{total,n,t} \qquad \forall n, t$$

Or:

$$\mathrm{Y}_{goods,n,t} + \mathrm{EC}_{household,n,t} = \mathrm{C}_{n,t} + \mathrm{I}_{n,t} + \mathrm{EC}_{household,n,t} + \mathrm{EC}_{production,n,t}$$

Or simplified:

$$\mathrm{Y}_{goods,n,t} = \mathrm{C}_{n,t} + \mathrm{I}_{n,t} + \mathrm{EC}_{production,n,t} \qquad \forall n, t$$

5.2.5 Energy Accounting

The relation between 'TE', 'E' and 'YE' stays the same:

First, 'TE' is the sum of 'E' and 'YE':

$$\mathrm{TE}_{n,s,t} = \mathrm{E}_{n,s,t} + \mathrm{YE}_{n,s,t} \qquad \forall n, s, t \tag{27}$$

Second, 'E' is a fraction of 'TE' based on an exogenous share 'h' for each sector:

$$\mathrm{E}_{n,s,t} = \mathrm{h}_{n,s} * \mathrm{TE}_{n,s,t} \qquad \forall n, s, t \tag{28}$$

We implement 'equation 8' from the derivation of the model as a constraint into the optimization to determine 'E':

$$with \quad E_{t,s} = L_{n,t} E_{min,t,s} + \frac{\sigma_s}{\beta} \frac{C_t}{p_{t,s}}$$

$$TE_{t,s} = \frac{L_{n,t} E_{min,t,s} + \frac{\sigma_s}{\beta} \frac{C_t}{p_{t,s}}}{h_{n,s}} \tag{29}$$

However, we now need the equation determining EC and also have to split EC into $\mathrm{EC}_{household,n,t} + \mathrm{EC}_{production,n,t}$

$$\begin{aligned} \mathrm{EC}_{n,t} = \mathrm{total\_cost}_{n,t} & \\ + \sum_s \mathrm{eneprice}_{s,t,n} \cdot (\mathrm{PHYSENE}_{n,s,t} - \mathrm{enestart}_{s,t,n}) & \\ + \sum_s \frac{\mathrm{eneprice}_{s,t,n}}{\mathrm{enestart}_{s,t,n}} \cdot (\mathrm{PHYSENE}_{n,s,t} - \mathrm{enestart}_{s,t,n})^2 & \quad \forall n, t > 1 \end{aligned} \tag{30}$$

5.2.6 Connection to MESSAGE Energy Model via PHYSENE

The relationship below establishes the link between physical energy $\mathrm{PHYSENE}_{n,s,y}$ as accounted in MESSAGE for the six commercial energy demands s and energy in terms of monetary value $\mathrm{TE}_{n,s,y}$ as specified in the production function of MACRO.

$$\mathrm{PHYSENE}_{n,s,t} \geq \mathrm{TE}_{n,s,t} \cdot \mathrm{aeei\_factor}_{n,s,t}$$

The cumulative effect of autonomous energy efficiency improvements (AEEI) is captured in:

$$\mathrm{aeei\_factor}_{n,s,y} = \mathrm{aeei\_factor}_{n,s,y-1} \cdot (1 - \mathrm{aeei}_{n,s,y})^{\mathrm{duration\_period}_y}$$

with $\mathrm{aeei\_factor}_{n,s,y=1} = 1$. Therefore, choosing the $\mathrm{aeei}_{n,s,y}$ coefficients appropriately offers the possibility to calibrate MACRO to a certain energy demand trajectory from MESSAGE.

5.2.7 Terminal Condition

This is the terminal condition of the original model:

$$\mathrm{K}_{n,t} \cdot \left(\mathrm{grow}_{n,t} + \delta_n\right) \leq \mathrm{I}_{n,t} \qquad \forall n, t = \text{last year} \tag{31}$$

6 Implementation of Decent Living Energy (DLE) Requirements

Each representative household of each decile has a path of minimum direct energy consumption which is quantified by Kikstra et al. (2021, 2025). These DLE requirements are thresholds of end-use energy services for each sector s, which are necessary to achieve a decent standard of living and are given by:

$$e_{q,t,s,n} \geq DLE_{t,s,n} \tag{32}$$

7 Implementation of Income Deciles

New parameters and variables due to the implementation of income deciles and the respective new index q representing the specific income decile.

Parameter and Variables	Description
$\mathrm{l}_{q,n,t}$	Labor force of each decile q in region n and period t
$\mathrm{w}_{q,n,t}$	Wage rate of each decile q in region n and period t
$\mathrm{c}_{q,n,t}$	Consumption of each decile q in region n and period t
$\mathrm{a}_{q,n,t}$	Capital of each decile q in region n and period t
$\mathrm{s}_{q,n,t}$	Saving rate of each decile q in region n and period t
$\mathrm{DLE}_{n,t,s}$	Decent Living Energy Requirements in n and period t for each sector s
$\mathrm{pi}_{n,t}$	Capital return premium in region n and period t

In the next step, 10 income deciles will be introduced. The inequality features Q quantiles $q = 1..Q$, which add another index to the economic variables and distribute the total economic values across consumption/income quantiles. For each representative household in each quantile, the optimization is given by:

$$\begin{aligned}\max_{c_{q,t,n}} \quad & \sum_{n}\sum_{t=1}^{T-1} udf_n^t \Bigg((\beta + \sum_{s=1}^{3}\sigma_{s,n}) \log(c_{q,t,n}) - \sum_{s=1}^{3}\sigma_{s,n}\log(p_{t,s,n}) \\ & \quad + \sum_{s=1}^{3}\sigma_{s,n}\log\left(\frac{\sigma_{s,n}}{\beta}\right)\Bigg) \\ & + \sum_{n} udf_n^T \Bigg((\beta + \sum_{s=1}^{3}\sigma_{s,n}) \log(c_{q,T,n}) - \sum_{s=1}^{3}\sigma_{s,n}\log(p_{T,s,n}) \\ & \quad + \sum_{s=1}^{3}\sigma_{s,n}\log\left(\frac{\sigma_{s,n}}{\beta}\right)\Bigg) \cdot \left(period_{T-1} + \frac{1}{fin\_time_{n,T}} \right)\end{aligned} \tag{33}$$

Each household maximizes utility subject to the constraint on wealth accumulation, given by:

$$a_{q,t+1,n} = (1-\delta_n+r_{t,n})a_{q,t,n}+w_{q,t,n}l_{q,t,n}-\sum_{s=1}^{3} p_{t,s,n}l_{q,t,n}E_{min,t,s,n}-\frac{\beta+\sum_{s=1}^{3}\sigma_{s,n}}{\beta}c_{q,t,n} \tag{34}$$

where quantile-specific capital (or wealth), returns on capital, wages, and labor supplies are given by the macroeconomic model and micro-data-based estimates.

While labor decisions are taken as exogenous, the saving rate $s_{q,t,n}$ of each decile is endogenously determined and can be computed as:

$$\begin{aligned} s_{q,t,n} = & \frac{(1-\delta_n+r_{t,n})a_{q,t,n}+w_{q,t,n}l_{q,t,n}}{(1-\delta_n+r_{t,n})a_{q,t,n}+w_{q,t,n}l_{q,t,n}} \\ & - \frac{\sum_{s=1}^{3} p_{t,s,n}l_{q,t,n}E_{min,t,s,n} + \frac{\beta+\sum_{s=1}^{3}\sigma_{s,n}}{\beta}c_{q,t,n}}{(1-\delta_n+r_{t,n})a_{q,t,n}+w_{q,t,n}l_{q,t,n}} \end{aligned} \tag{35}$$

7.1 Utility Maximization Reads as Follows

$$
\begin{aligned}
L = &\sum_n \sum_t^{T-1} udf_n^t \left[(\beta + \sum_{s=1}^{3} \sigma_{s,n}) \log(c_{i,t,n}) - \sum_{s=1}^{3} \sigma_{s,n} \log(p_{t,s,n}) + \sum_{s=1}^{3} \sigma_{s,n} \log\left(\frac{\sigma_{s,n}}{\beta}\right) \right] \\
&+ \sum_n \sum_{T-1}^{T} udf_n^t \left[(\beta + \sum_{s=1}^{3} \sigma_{s,n}) \log(c_{i,t,n}) - \sum_{s=1}^{3} \sigma_{s,n} \log(p_{t,s,n}) + \sum_{s=1}^{3} \sigma_{s,n} \log\left(\frac{\sigma_{s,n}}{\beta}\right) \right] \\
&\cdot \left(period_{T-1} + \frac{1}{fin\_time_{n,t}} \right) \\
&+ \lambda_t \Bigg[-a_{i,t+1,n} + (1 - \delta_n + r_{t,n}) a_{i,t,n} + w_{i,t,n} l_{i,t,n} - \sum_{s=1}^{3} p_{t,s,n} l_{q,t,n} E_{min,t,s,n} \\
&- \frac{\beta + \sum_{s=1}^{3} \sigma_{s,n}}{\beta} c_{i,t,n} \Bigg]
\end{aligned} \tag{1}
$$

$$
\frac{\partial L}{\partial c_{i,t,n}} = udf_n^t \frac{(\beta + \sum_{s=1}^{3} \sigma_{s,n})}{c_{i,t,n}} - \lambda_t udf_n^t \frac{(\beta + \sum_{s=1}^{3} \sigma_{s,n})}{\beta} \stackrel{!}{=} 0 \tag{37}
$$

$$
\lambda_t = \frac{\beta}{c_{i,t,n}} \tag{38}
$$

$$
\frac{\partial L}{\partial a_{i,t,n}} = udf_n^t \lambda_t (1 - \delta_n + r_{t,n}) - udf_n^{t-1} \lambda_{t-1} \stackrel{!}{=} 0
$$

$$
udf_n^t \lambda_t (1 - \delta_n + r_{t,n}) = udf_n^{t-1} \lambda_{t-1} \tag{39}
$$

Such that the equilibrium condition is the following, where saving is implicitly decided from Euler condition (40) and the wealth constraint:

$$
udf_n^t \frac{\beta}{c_{i,t,n}} (1 - \delta_n + r_{t,n}) = udf_n^{t-1} \frac{\beta}{c_{i,t-1,n}}
$$

$$
c_{i,t,n} = udf_n^t (1 - \delta_n + r_{t,n}) c_{i,t-1,n} \tag{40}
$$

7.2 Income and Wealth Dynamics

Income and wealth dynamics are based on ongoing work by Emmerling et al. (2025). They introduce a skill premium for labour income as well as capital income correlating returns with wealth.

7.3 Income Dynamics

Wage levels $w_{q,t,n}$ by decile and hence income distribution are given exogenously based on empirical data for the specific regions n. To start, we assume that the development of income inequality over time follows the inequality assumptions of the different Shared Socioeconomic Pathways (SSPs). Thus, changes in wages per decile are given exogenously.

In a next step, we want to implement a skill premium based on education as proposed by Emmerling et al. (2025) as described below.

7.4 Wealth Dynamics

Emmerling et al. (2025) consider a decile-specific return on wealth,

$$r_{q,t,n} = \frac{10 r_{t,n}(1+\pi_{r,n})^q}{\sum_q (1+\pi_{r,n})^q}$$

where $\pi_{t,n}$ is the capital return premium, with which the higher income group receives higher returns. The average capital return equals the interest rate $r_{t,n}$ given exogenously.

7.5 Income Dynamics Emmerling et al. (2025)

The average wage is given exogenously. In order to obtain wages by quantiles, they use educational attainment data by quantile to compute education-category-dependent wage premiums $\pi_{q,t,n}$. Besides, they allow for a varying rate g_t in wage premiums over the years, considering that wage premiums across education years will increase in future education projections.

$$\pi_{q,t,n} = \pi_{q,t,n}^{t=0}(1+g_t)^t$$

The wage of each quantile is computed relative to the wage of the bottom quantile (q = D1),

$$w_{q,t,n} = w_{D1,t,n}(1+\pi_{q,t,n})^{EDU_{q,t,n}-EDU_{D1,t,n}}$$