マクロ計量経済学

第4回　講義ノート
マクロ動学モデルの解法　その4

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アウトライン

- Ch6　Sec 6.8 のモデルの合理的均衡解の
  Blanchard & Kahn (1980)による解法
  - Ch6_model_BK_2.m

  - Sec 6.3.1 のモデルのBlanchard & Kahn (1980)による解法
    Ch6_model_631_BK.m

Blanchard & Kahn (1980)

6.8.1 General Version

A linear model can be written (in what is known as a state space representation) as

State variables

$$B\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A\begin{bmatrix} x_t \\ y_t \end{bmatrix} + G\varepsilon_t, \tag{6.13}$$

Control variables

where x_t is an $(n \times 1)$ vector of predetermined variables at date t, y_t is an $(m \times 1)$ vector of non-predetermined variables at time t, $E_t y_{t+1}$ is the $(m \times 1)$ vector of expectations for the non-predetermined variables at date $t+1$, ε_t is a $(k \times 1)$ vector of stochastic shocks, A and B are $((n+m) \times (n+m))$ matrices, and G is an $((n+m) \times k)$ matrix. The difference between predetermined and non-predetermined variables is that the values of the predetermined variables at time $t+1$ do not depend on the values of the time $t+1$ shocks, while the values of the non-predetermined variables do depend on them. That is why, at

合理的期待モデルの固有値分解

6.8.2 Stochastic Shocks

When the economy has stochastic shocks, the solution is a bit different. We still consider only the case where B is invertible, so we can write the model in its stochastic version as

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = B^{-1}A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + B^{-1}G\,[\varepsilon_t]\,.$$

Using the exact same eigenvalue-eigenvector decomposition as before, one gets

$$\overset{\text{固有ベクトル}}{\overline{M}^{-1}} \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \overset{\text{固有値}}{\bar{\Lambda}}\,\overline{M}^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \overline{M}^{-1}B^{-1}G\,[\varepsilon_t]\,,$$

or

$$\overset{\text{固有ベクトル}}{\begin{bmatrix} \widehat{M}_{11} & \widehat{M}_{12} \\ \widehat{M}_{21} & \widehat{M}_{22} \end{bmatrix}} \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \overset{\text{固有値<1}}{\bar{\Lambda}_{11}} & 0_{12} \\ 0_{21} & \underset{\text{固有値>1}}{\bar{\Lambda}_{22}} \end{bmatrix} \overset{\text{固有ベクトル}}{\begin{bmatrix} \widehat{M}_{11} & \widehat{M}_{12} \\ \widehat{M}_{21} & \widehat{M}_{22} \end{bmatrix}} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \widehat{G}_1 \\ \widehat{G}_2 \end{bmatrix} [\varepsilon_t]\,,$$

where everything is as before except that we define the partition

$$\begin{bmatrix} \widehat{G}_1 \\ \widehat{G}_2 \end{bmatrix} = \overline{M}^{-1}B^{-1}G,$$

Blanchard & Kahn (1980)の合理的均衡解の解法

$$\begin{bmatrix} \widehat{M}_{11} & \widehat{M}_{12} \\ \widehat{M}_{21} & \widehat{M}_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \bar{\Lambda}_{11} & 0_{12} \\ 0_{21} & \bar{\Lambda}_{22} \end{bmatrix} \begin{bmatrix} \widehat{M}_{11} & \widehat{M}_{12} \\ \widehat{M}_{21} & \widehat{M}_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} \widehat{G}_1 \\ \widehat{G}_2 \end{bmatrix} [\varepsilon_t],$$

前頁の最後の式の2段目は　発散(不安定)解なので、合理的均衡解にする為に恒等式としてゼロにする。

$$\underbrace{\widehat{M}_{21} x_{t+1} + \widehat{M}_{22} E_t y_{t+1}}_{=0} = \underbrace{\bar{\Lambda}_{22} \left[\widehat{M}_{21} x_t + \widehat{M}_{22} y_t \right] + \widehat{G}_2 [\varepsilon_t]}_{=0}$$

$$E_t y_{t+1} = -\widehat{M}_{22}^{-1} \widehat{M}_{21} x_{t+1} \qquad y_t = -\widehat{M}_{22}^{-1} \widehat{M}_{21} x_t - \widehat{M}_{22}^{-1} \bar{\Lambda}_{22}^{-1} \widehat{G}_2 [\varepsilon_t].$$

前のページの最後の式の1段目は　安定解であり、 上の制約式を代入すると　次式が得られる。
この式が　均衡解　PとＱとなる。

$$x_{t+1} = \left[\widehat{M}_{11} - \widehat{M}_{12} \widehat{M}_{22}^{-1} \widehat{M}_{21} \right]^{-1} \bar{\Lambda}_{11} \left[\widehat{M}_{11} - \widehat{M}_{12} \widehat{M}_{22}^{-1} \widehat{M}_{21} \right] x_t$$
$$- \left[\widehat{M}_{11} - \widehat{M}_{12} \widehat{M}_{22}^{-1} \widehat{M}_{21} \right]^{-1} \left[\bar{\Lambda}_{11} \widehat{M}_{12} \widehat{M}_{22}^{-1} \bar{\Lambda}_{22}^{-1} \widehat{G}_2 - \widehat{G}_1 \right] [\varepsilon_t].$$

均衡解

$$x_t = P x_{t-1} + Q z_t,$$
$$y_t = R x_{t-1} + S z_t.$$

6.4節の HansenモデルのFOC

$$\bar{K}\tilde{K}_{t+1} = \bar{Y}\tilde{Y}_t - \bar{C}\tilde{C}_t + (1-\delta)\,\bar{K}\tilde{K}_t,$$

$$\tilde{\lambda}_t = \gamma\tilde{\lambda}_{t-1} + \varepsilon_t,$$

$$0 = \tilde{\lambda}_t - \theta\tilde{Y}_t + \theta\tilde{K}_t - (1-\theta)\,\tilde{C}_t,$$

$$0 = \tilde{K}_t + \tilde{r}_t - \tilde{Y}_t,$$

$$E_t\tilde{C}_{t+1} - \beta\bar{r}\,E_t\tilde{r}_{t+1} = \tilde{C}_t.$$

State variables

Control variables

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} \tilde{K}_{t+1} \\ \tilde{\lambda}_t \\ \tilde{Y}_t \\ E_t\tilde{C}_{t+1} \\ E_t\tilde{r}_{t+1} \end{bmatrix}$$

```
%Define variables
syms c  r  y  h  kp lam;  % control variables
syms cf rf yf hf kpf lamf; % future period
syms cb rb yb hb k  lamb; % state and lagged variables
```

```
%% Set FOC of Model: Page 100
eq1= y_ss*y - c_ss*c + k_ss*((1-DELTA)*k - kp);

% eq2 = y - h/(1-h_ss)-c;
eq2 = lam - GAMMA *lamb - eps;

% eq4 = lam + THETA*k + (1-THETA)*h - y ;
eq3 = lam - THETA*y + THETA*k - (1-THETA)*c  ;

eq4 = -y + k + r;

eq5 = -c + cf - BETTA*r_ss*rf ;
```

```
%% Create function f
f = [eq1;eq2; eq3; eq4; eq5];

xf = [kp, lam, y, cf, rf];
x =  [k, lamb, yb c, r];
z=    eps; % exogenous variables
```

QZ分解 (固有値が見つからない場合)

$$B\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A\begin{bmatrix} x_t \\ y_t \end{bmatrix} + G\varepsilon_t, \tag{6.13}$$

A generalized Schur decomposition takes a pair of square matrices (B and A) and decomposes them (usually using what is called a QZ algorithm) into the matrices T, S, Q, and Z, where

$$B = QTZ',$$

$$A = QSZ',$$

and Q and Z have the special properties that

$$QQ' = Q'Q = I = ZZ' = Z'Z$$

QZ分解 (固有値が見つからない場合)

$$B\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A\begin{bmatrix} x_t \\ y_t \end{bmatrix} + G\varepsilon_t, \tag{6.13}$$

$$QTZ'\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = QSZ'\begin{bmatrix} x_t \\ y_t \end{bmatrix}.$$

Premultiplying both sides by Q' (which removes Q, since $Q'Q = I$) and writing out Z' as a partitioned matrix gives

固有値に該当　　固有値に該当

$$\begin{bmatrix} T_{11} & T_{12} \\ 0_{21} & T_{22} \end{bmatrix}\begin{bmatrix} Z'_{11} & Z'_{12} \\ Z'_{21} & Z'_{22} \end{bmatrix}\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0_{21} & S_{22} \end{bmatrix}\begin{bmatrix} Z'_{11} & Z'_{12} \\ Z'_{21} & Z'_{22} \end{bmatrix}\begin{bmatrix} x_t \\ y_t \end{bmatrix},$$

```
%%  P139:  QZ  Decomposition  Sec

[T,S,q,z] = qz(B,A);   % upper tria
     stake=1;
[T,S,q,Z] = qzdiv(stake,T,S,q,z);
```

QZ分解を採用したBlanchard & Kahn (1980)

以下 の式の2段目は　発散(不安定)解なので、合理的均衡解にする為に恒等式としてゼロにする。

State variables

$$\begin{bmatrix} T_{11} & T_{12} \\ 0_{21} & T_{22} \end{bmatrix} \begin{bmatrix} Z'_{11} & Z'_{12} \\ Z'_{21} & Z'_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0_{21} & S_{22} \end{bmatrix} \begin{bmatrix} Z'_{11} & Z'_{12} \\ Z'_{21} & Z'_{22} \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix},$$

Control variables

$$T_{22} \left[Z'_{21} x_{t+1} + Z'_{22} E_t y_{t+1} \right] = S_{22} \left[Z'_{21} x_t + Z'_{22} y_t \right].$$

$$Z'_{21} x_t + Z'_{22} y_t = 0.$$

$$y_t = -\left(Z'_{22} \right)^{-1} Z'_{21} x_t = -N x_t,$$

```
% unstable  page 135
Z= Z';
N=inv(Z(4:5,4:5))*Z(4:5,1:3)
```

QZ分解を利用した合理的均衡解の解法

前頁の行列　−N　を代入

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ -Nx_{t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_t \\ -Nx_t \end{bmatrix}.$$

```
% stable
B11= B(1:3,1:3);
B12=B(1:3,4:5);
A11 = A(1:3,1:3);
A12 = A(1:3,4:5);

(B11-B12*N)
(A11-A12*N)

 RR  =  inv(B11-B12*N)*(A11-A12*N)
```

$$\left[B_{11} - B_{12}N\right] x_{t+1} = \left[A_{11} - A_{12}N\right] x_t,$$

$$x_{t+1} = \left[B_{11} - B_{12}N\right]^{-1} \left[A_{11} - A_{12}N\right] x_t.$$

均衡解

$$x_t = Px_{t-1} + Qz_t,$$

$$y_t = Rx_{t-1} + Sz_t.$$

計算例　行列SとTの比率が固有値

$$S = \begin{bmatrix} 0 & -6.0713 & 2.5534 & -5.6797 & -0.4798 \\ 0 & 5.2880 & -3.3924 & 6.2982 & 0.1563 \\ 0 & 0 & 0.7200 & 0.6793 & -0.0954 \\ 0 & 0 & 0 & 0.9103 & 0.5953 \\ 0 & 0 & 0 & 0 & .8228 \end{bmatrix}, \quad T = \begin{bmatrix} 1.6296 & -6.5107 & 3.6395 & -6.0515 & -0.1748 \\ 0 & 5.6147 & -2.9158 & 5.2866 & -0.2383 \\ 0 & 0 & 0.7579 & 0.6832 & -0.9014 \\ 0 & 0 & 0 & 0.8488 & 0.7907 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{eigenvalues} = \begin{bmatrix} 0/1.6296 \\ 5.2880/5.6147 \\ 0.7200/0.7579 \\ 0.9103/0.8488 \\ 0.8228/0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.9418 \\ 0.9500 \\ 1.0725 \\ \infty \end{bmatrix}.$$

計算例

$$
Z = \begin{bmatrix} 0 & 0.6779 & -0.3668 & 0.6371 & 0 \\ 0 & 0 & -0.53 & -0.3051 & 0.7912 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.3604 & -0.4318 & -0.6321 & -0.533 \\ 0 & -0.6407 & -0.631 & 0.3185 & -0.2998 \end{bmatrix}
$$

Z21

$$
y_t = -\left(Z'_{22}\right)^{-1} Z'_{21} x_t = -N x_t,
$$

$$
\begin{bmatrix} \tilde{C}_t \\ \tilde{r}_t \end{bmatrix} = -N \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix} = \begin{bmatrix} 0.5317 & 0.4468 & 0 \\ -0.9452 & 1.8445 & 0 \end{bmatrix} \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix}.
$$

$$B = \begin{bmatrix} 12.6695 & 0 & -1.2353 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & .36 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -.03475 \end{bmatrix} \qquad A = \begin{bmatrix} 12.353 & 0 & 0 & -.9186 & 0 \\ 0 & .95 & 0 & 0 & 0 \\ .36 & 0 & 0 & -.64 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\left[B_{11} - B_{12}N\right] \begin{bmatrix} \tilde{K}_{t+1} \\ \tilde{\lambda}_t \\ Y_t \end{bmatrix} = \left[A_{11} - A_{12}N\right] \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix}.$$

Using the matrices from our model, we get

$$\left[\begin{bmatrix} 12.67 & 0 & -1.24 \\ 0 & 1 & 0 \\ 0 & -1 & .36 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.53 & -0.447 & 0 \\ 0.945 & -1.845 & 0 \end{bmatrix} \right] \begin{bmatrix} \tilde{K}_{t+1} \\ \tilde{\lambda}_t \\ Y_t \end{bmatrix}$$

$$= \left[\begin{bmatrix} 12.353 & 0 & 0 \\ 0 & .95 & 0 \\ .36 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -.919 & 0 \\ 0 & 0 \\ -.64 & 0 \end{bmatrix} \begin{bmatrix} -0.532 & -0.447 & 0 \\ 0.9452 & -1.845 & 0 \end{bmatrix} \right] \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix},$$

$$\begin{bmatrix} \tilde{K}_{t+1} \\ \tilde{\lambda}_t \\ Y_t \end{bmatrix} = \begin{bmatrix} 12.669 & 0 & -1.235 \\ 0 & 1 & 0 \\ 0 & -1 & .36 \end{bmatrix}^{-1} \begin{bmatrix} 11.865 & -0.410 & 0 \\ 0 & 0.95 & 0 \\ 0.0197 & -0.286 & 0 \end{bmatrix} \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0.9418 & 0.1475 & 0 \\ 0 & 0.95 & 0 \\ 0.0548 & 1.8446 & 0 \end{bmatrix} \begin{bmatrix} \tilde{K}_t \\ \tilde{\lambda}_{t-1} \\ Y_{t-1} \end{bmatrix}.$$

図6.9の再現

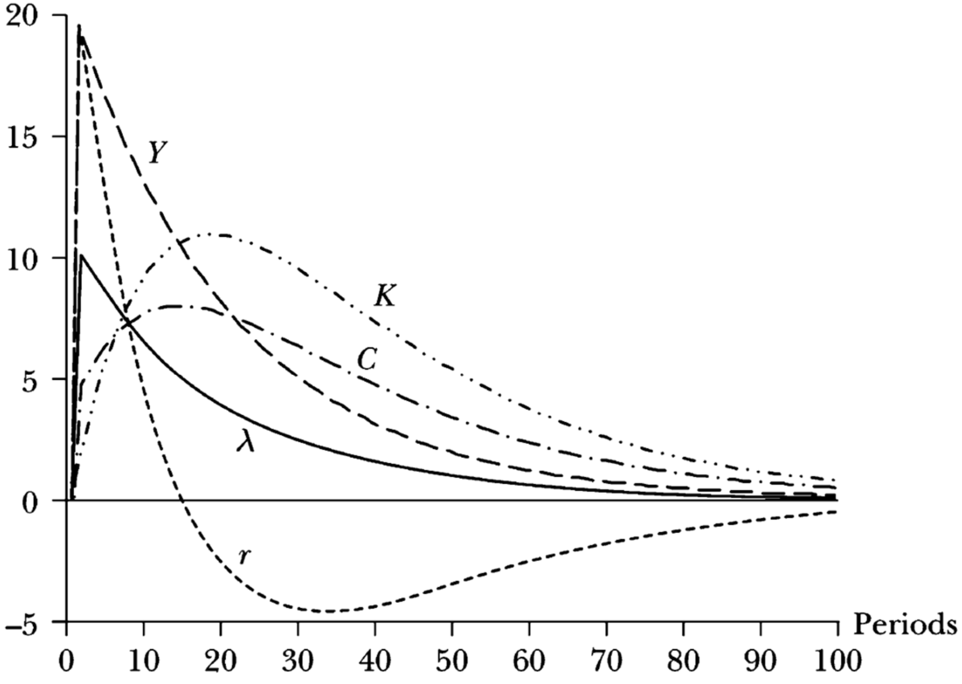

FIGURE 6.9 Responses for Hansen model solved using Schur