マクロ計量経済学

第5回　講義ノート
マクロ動学モデルの解法　その5

2025年5月07日

飯星　博邦

アウトライン

- Ch 7 Linear Quadratic Dynamic Programming
- 通常のベルマン方程式では
  価値関数Vは、数値でしか表現できないが、
  価値関数に2次形式を使うことで、解析的に表現できる。
- Matlab Code
  - Ch7_721.m
  - Ch7_731.m

関数F(x,y)のテイラー2次近似

The second-order Taylor expansion of the function $F(x_t, y_t)$ is

$$F(x_t, y_t) \approx F(\bar{x}, \bar{y}) + \begin{bmatrix} F_x(\bar{x}, \bar{y})' & F_y(\bar{x}, \bar{y})' \end{bmatrix} \begin{bmatrix} x_t - \bar{x} \\ y_t - \bar{y} \end{bmatrix}$$
$$+ \begin{bmatrix} \left(x_t - \bar{x}\right)' & \left(y_t - \bar{y}\right)' \end{bmatrix} \begin{bmatrix} \frac{F_{xx}(\bar{x}, \bar{y})}{2} & \frac{F_{xy}(\bar{x}, \bar{y})}{2} \\ \frac{F_{yx}(\bar{x}, \bar{y})}{2} & \frac{F_{yy}(\bar{x}, \bar{y})}{2} \end{bmatrix} \begin{bmatrix} x_t - \bar{x} \\ y_t - \bar{y} \end{bmatrix}.$$

効用関数の2次近似

The second-order Taylor expansion of the objective function is

$$
\begin{aligned}
u(k_t, k_{t+1}, h_t) \approx{} & \ln\left(f(\bar{k}, \bar{h}) - \delta\bar{k}\right) + A\ln(1-\bar{h}) \\
& + \frac{1}{\bar{c}}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)\right]\left(k_t - \bar{k}\right) - \frac{1}{\bar{c}}\left(k_{t+1} - \bar{k}\right) \\
& + \left[(1-\theta)\,\frac{1}{\bar{c}}\frac{\bar{y}}{\bar{h}} - \frac{A}{1-\bar{h}}\right]\left(h_t - \bar{h}\right) \\
& + \begin{bmatrix} \left(k_t - \bar{k}\right) \\ \left(k_{t+1} - \bar{k}\right) \\ \left(h_t - \bar{h}\right) \end{bmatrix}' \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \left(k_t - \bar{k}\right) \\ \left(k_{t+1} - \bar{k}\right) \\ \left(h_t - \bar{h}\right) \end{bmatrix},
\end{aligned}
$$

効用関数の2次近似のSymbolic mathでの解法

```
% Production Function
f = lam*k^THETA*h^(1-THETA);

% budget constraint
c = f + (1-DELTA)*k-kp;

% utility function
% eq1 = log(exp(k)^THETA*exp(h)^(1
 eq1 = log(c) + A*log(1-h);

% shock
 eq2 = -lam + GAMMA*lamb + eps ;

% variables
z = [ k,  lam, kp, h];

% Compute the first and second derivatives of f
fz=jacobian(eq1,z);      1階の微分

fzz=jacobian(fz',z);     2階の微分
```

目的関数からベルマン方程式への変換

2次形式の目的関数

$$\sum_{t=0}^{\infty} \beta^t \left[x_t' R x_t + y_t' Q y_t + 2 y_t' W x_t\right], \qquad (7.1)$$

State variables

目的関数

$$z_t' M z_t = [\, x_t' \quad y_t' \,] \begin{bmatrix} R & W' \\ W & Q \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix},$$

Control variables

ベルマン方程式

$$x_t' P x_t = \max_{y_t} \left[z_t' M z_t + \beta x_{t+1}' P x_{t+1}\right],$$

価値関数

目的関数

構造方程式(予算制約式・資本蓄積等)

$$x_{t+1} = A x_t + B y_t,$$

構造方程式を代入

$$x_t' P x_t = \max_{y_t} \left[x_t' R x_t + y_t' Q y_t + 2 y_t' W x_t + \beta \left(A x_t + B y_t\right)' P \left(A x_t + B y_t\right)\right].$$

目的関数の行列Mの設定

$$M = \begin{bmatrix} \boxed{m_{11} \quad m_{12} \quad m_{13} \quad m_{14} \quad m_{15}} \\ m_{21} & a_{11} & \widehat{a}_{1\lambda} & a_{12} & a_{13} \\ m_{31} & \widehat{a}_{\lambda 1} & \widehat{a}_{\lambda\lambda} & \widehat{a}_{\lambda 2} & \widehat{a}_{\lambda 3} \\ m_{41} & a_{21} & \widehat{a}_{2\lambda} & a_{22} & a_{23} \\ m_{51} & a_{31} & \widehat{a}_{3\lambda} & a_{32} & a_{33} \end{bmatrix},$$

0次(定数)と1次

2階微分

The second-order Taylor expansion of the function $F(x_t, y_t)$ is

$$F(x_t, y_t) \approx F(\bar{x}, \bar{y}) + [\, F_x(\bar{x}, \bar{y})' \quad F_y(\bar{x}, \bar{y})' \,] \begin{bmatrix} x_t - \bar{x} \\ y_t - \bar{y} \end{bmatrix}$$

$$+ [\, (x_t - \bar{x})' \quad (y_t - \bar{y})' \,] \begin{bmatrix} \frac{F_{xx}(\bar{x}, \bar{y})}{2} & \frac{F_{xy}(\bar{x}, \bar{y})}{2} \\ \frac{F_{yx}(\bar{x}, \bar{y})}{2} & \frac{F_{yy}(\bar{x}, \bar{y})}{2} \end{bmatrix} \begin{bmatrix} x_t - \bar{x} \\ y_t - \bar{y} \end{bmatrix}.$$

$$m_{11} = F(\bar{x}, \bar{y}) - \bar{x}' F_x(\bar{x}, \bar{y}) - \bar{y}' F_y(\bar{x}, \bar{y}) + \frac{\bar{x}' F_{xx}(\bar{x}, \bar{y}) \bar{x}}{2} + \bar{x}' F_{xy}(\bar{x}, \bar{y}) \bar{y} + \frac{\bar{y}' F_{yy}(\bar{x}, \bar{y}) \bar{y}}{2},$$

$$m_{12} = m'_{21} = \frac{F_x(\bar{x}, \bar{y})' - \bar{x}' F_{xx}(\bar{x}, \bar{y}) - \bar{y}' F_{yx}(\bar{x}, \bar{y})}{2}$$

$$m_{13} = m'_{31} = \frac{F_y(\bar{x}, \bar{y})' - \bar{x}' F_{xy}(\bar{x}, \bar{y}) - \bar{y}' F_{yy}(\bar{x}, \bar{y})}{2},$$

2階微分

$$m_{22} = \frac{F_{xx}(\bar{x}, \bar{y})}{2},$$

$$m_{23} = m'_{32} = \frac{F_{xy}(\bar{x}, \bar{y})}{2},$$

$$m_{33} = \frac{F_{yy}(\bar{x}, \bar{y})}{2}.$$

行列M　の設定

```
num_fzz = eval(fzz)/2; B=num_fzz;
disp('df/dzz^2 =');
disp(num_fzz);  2階微分
```

```
m(1,1)= eval(log(y-DELTA*k)+A*log(1-h)-z*fz'+z*fzz*z'/2);
m(1,2:5)= eval(fz/2-z*fzz/2);
m(2:5,1) = m(1,2:5)';
m(2:5,2:5)= num_fzz;   2階微分
```

行列Mの設定

0次(定数)
と1次

$$
M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & a_{11} & \widehat{a}_{1\lambda} & a_{12} & a_{13} \\ m_{31} & \widehat{a}_{\lambda 1} & \widehat{a}_{\lambda\lambda} & \widehat{a}_{\lambda 2} & \widehat{a}_{\lambda 3} \\ m_{41} & a_{21} & \widehat{a}_{2\lambda} & a_{22} & a_{23} \\ m_{51} & a_{31} & \widehat{a}_{3\lambda} & a_{32} & a_{33} \end{bmatrix},
$$

$$
\begin{aligned}
m_{11} = {} & \ln\left(f(\bar{k},\bar{h}) - \delta\bar{k}\right) + A\ln(1-\bar{h}) \\
& - \frac{1}{\bar{c}}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta) - 1\right]\bar{k} - \frac{\bar{y}}{\bar{c}}\bar{\lambda} \\
& - \left[(1-\theta)\frac{1}{\bar{c}}\frac{\bar{y}}{\bar{h}} - \frac{A}{1-\bar{h}}\right]\bar{h} \\
& + \begin{bmatrix} \bar{k} \\ \bar{\lambda} \\ \bar{k} \\ \bar{h} \end{bmatrix}' \begin{bmatrix} a_{11} & \widehat{a}_{1\lambda} & a_{12} & a_{13} \\ \widehat{a}_{\lambda 1} & \widehat{a}_{\lambda\lambda} & \widehat{a}_{\lambda 2} & \widehat{a}_{\lambda 3} \\ a_{21} & \widehat{a}_{2\lambda} & a_{32} & a_{32} \\ a_{31} & \widehat{a}_{3\lambda} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \bar{k} \\ \bar{\lambda} \\ \bar{k} \\ \bar{h} \end{bmatrix};
\end{aligned}
$$

$$
m_{12} = m_{21} = \frac{1}{\bar{c}}\left[\theta\frac{\bar{y}}{\bar{k}} + (1-\delta)\right] - [\,\bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{h}\,] \begin{bmatrix} a_{11} \\ \widehat{a}_{\lambda 1} \\ a_{21} \\ a_{31} \end{bmatrix},
$$

$$
m_{13} = m_{31} = \frac{\bar{y}}{\bar{c}} - [\,\bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{h}\,] \begin{bmatrix} \widehat{a}_{1\lambda} \\ \widehat{a}_{\lambda\lambda} \\ \widehat{a}_{2\lambda} \\ \widehat{a}_{3\lambda} \end{bmatrix}
$$

$$
m_{14} = m_{41} = -\frac{1}{\bar{c}} - [\,\bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{h}\,] \begin{bmatrix} a_{12} \\ \widehat{a}_{\lambda 2} \\ a_{22} \\ a_{32} \end{bmatrix},
$$

$$
m_{15} = m_{51} = \left[(1-\theta)\frac{1}{\bar{c}}\frac{\bar{y}}{\bar{h}} - \frac{A}{1-\bar{h}}\right] - [\,\bar{k} \quad \bar{\lambda} \quad \bar{k} \quad \bar{h}\,] \begin{bmatrix} a_{13} \\ \widehat{a}_{\lambda 3} \\ a_{23} \\ a_{33} \end{bmatrix}.
$$

行列Mの分割

$$M = \begin{bmatrix} -1.6374 & 1.0996 & -1.0886 & 1.9361 \\ 1.0996 & -0.6056 & 0.5986 & -1.3823 \\ -1.0886 & 0.5986 & -0.5926 & 1.4048 \\ 1.9361 & -1.3823 & 1.4048 & -6.6590 \end{bmatrix}$$

The matrices R, Q, and W come from the matrix M, where $M = \begin{bmatrix} R & W' \\ W & Q \end{bmatrix}$, so

$$R = \begin{bmatrix} -1.6374 & 1.0996 \\ 1.0996 & -0.6056 \end{bmatrix},$$

$$Q = \begin{bmatrix} -0.5926 & 1.4048 \\ 1.4048 & -6.6590 \end{bmatrix},$$

and

$$W = \begin{bmatrix} -1.0886 & 0.5986 \\ 1.9361 & -1.3823 \end{bmatrix}.$$

ベルマン方程式

$$x_t' P x_t = \max_{y_t} \left[x_t' R x_t + y_t' Q y_t + 2 y_t' W x_t + \beta \left(A x_t + B y_t \right)' P \left(A x_t + B y_t \right) \right].$$

構造方程式・予算制約式

$$x_{t+1} = A x_t + B y_t.$$

$$\begin{bmatrix} 1 \\ k_{t+1} \end{bmatrix} = A \begin{bmatrix} 1 \\ k_t \end{bmatrix} + B \begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix},$$

where for this particular problem

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

2次形式のベルマン方程式の1階微分

The first-order conditions[3] for the problem are

$$\left[Q + \beta B' P B \right] y_t = - \left[W + \beta B' P A \right] x_t,$$

which gives the policy function (matrix), F, where

政策関数

$$y_t = F x_t = - \left[Q + \beta B' P B \right]^{-1} \left[W + \beta B' P A \right] x_t. \qquad (7.4)$$

価値関数と政策関数の計算

$$P = R + \beta A' P A - \left(\beta A' P B + W'\right)\left[Q + \beta B' P B\right]^{-1}\left(\beta B' P A + W\right)$$

The matrix P can be found from an initial guess for P, for example, P_0 equals the identity matrix, and iterating on the matrix Ricotti equation,

添え字Kは計算回数

$$P_{k+1} = R + \beta A' P_k A - \left(\beta A' P_k B + W'\right)\left[Q + \beta B' P_k B\right]^{-1}\left(\beta B' P_k A + W\right). \quad (7.5)$$

The sequence of $\left\{P_k\right\}$, $k \to \infty$, converges to the desired P. Once P is approximated, the policy function, F, is found using equation 7.4. The matrix F gives a linear approximation of the optimal plan in the neighborhood of the stationary state.

価値関数 xPx と政策関数 F の計算

```
%% cal
 R=m(1:3,1:3);
 Q=m(4:5,4:5);
 W=m(1:3,4:5)';

  P= eye(3);

 for i=1:1000
     zinv=inv(Q+BETTA*B'*P*B);
     z2=BETTA*AA'*P*B+W';
     P=R+BETTA*AA'*P*AA-z2*zinv*z2';
 end

F=-zinv*(W+BETTA*B'*P*AA);
```

政策関数(行列)　F　の導出

$$y_t = F x_t = -\left[Q + \beta B' P B\right]^{-1} \left[W + \beta B' P A\right] x_t. \tag{7.4}$$

Control variables

State variables

$$\begin{bmatrix} 1 \\ k_{t+1} \end{bmatrix} = A \begin{bmatrix} 1 \\ k_t \end{bmatrix} + B \begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix},$$

Control variables

State variables

$$\begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix} = F \begin{bmatrix} 1 \\ k_t \end{bmatrix},$$

where for this particular problem

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 12.6695 \\ .3335 \end{bmatrix} = \begin{bmatrix} 0.5869 & 0.9537 \\ 0.4146 & -0.0064 \end{bmatrix} \begin{bmatrix} 1 \\ 12.6695 \end{bmatrix}.$$

価値関数 xPx の計算例

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and get

$$P_1 = \begin{bmatrix} -.7515 & .9987 \\ .9987 & -0.4545 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} -1.6909 & .8247 \\ .8247 & -0.1924 \end{bmatrix}.$$

After 200 iterations, the values in P have settled down to

$$P = \begin{bmatrix} -96.3615 & .8779 \\ .8779 & -0.0259 \end{bmatrix}.$$

図7.2の再現

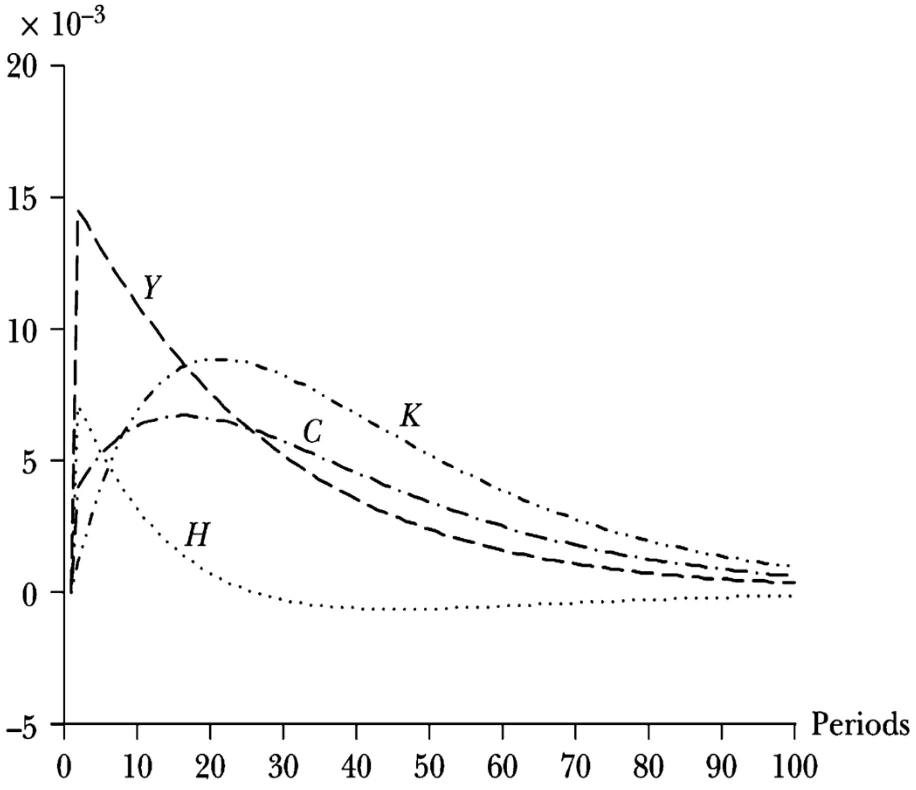
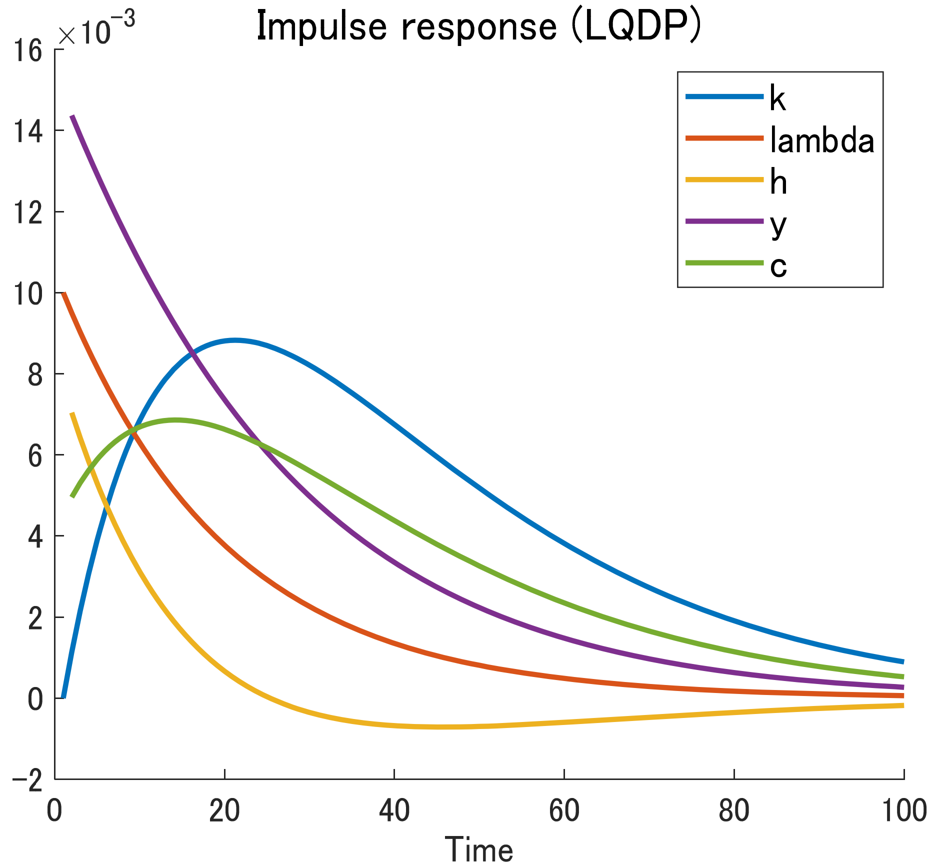


FIGURE 7.2 Responses found using the linear quadratic solution method



インパルス応答

$$F = \begin{bmatrix} -0.8470 & 0.9537 & 1.4340 \\ 0.1789 & -0.0064 & 0.2357 \end{bmatrix}.$$

Combining this policy function with the budget constraints, one gets

$$\begin{bmatrix} 1 \\ k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ .05 & 0 & .95 \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -0.8470 & 0.9537 & 1.4340 \\ 0.1789 & -0.0064 & 0.2357 \end{bmatrix} \begin{bmatrix} 1 \\ k_t \\ \lambda_t \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \varepsilon_{t+1},$$

```
%% Impulse response
eps = 0.01;
time = 100;
   yy= zeros(7,time);
   yy(1,1)=1;
   yy(2,1)=k_ss;
   yy(3,1)=1;
   yy(1:3,1) = yy(1:3,1) + C*eps;
% t >= 2
for  t =2:time

      yy(1,t)=1; % constant
      lam= GAMMA*yy(3,t-1)+(1-GAMMA)*1; % lam
     % capital
      k=F(1,:)*yy(1:3,t-1); % k
      kp=F(1,:)*[1,k,lam]'; % kp
     % labor
     h=F(2,:)*yy(1:3,t-1); % h
     % production function
     y= lam*k^THETA*h^(1-THETA); %y
     % consumption
     c= y + (1-DELTA)*k- kp; % c

     yy(2:7,t)=[k; lam; kp; h; y; c ];
end
```