Determinacy and Classification of Markov-Switching Rational Expectations Models *

Seonghoon Cho†

June 17, 2019

Abstract

In a general class of Markov-switching rational expectations models, this study derives necessary and sufficient conditions for determinacy, indeterminacy and the case of no stable solution. Contrary to linear rational expectations models, there is no known eigenvalue-eigenvector relation in this class of inherently non-linear models. Nevertheless, it is shown that the most stable solution in the mean-square stability sense plays the same role as what the generalized eigenvalues do for linear rational expectations models. It is this idea by which we establish the complete classification of Markov-switching as well as linear rational expectations models into three mutually disjoint and exhaustive sets mentioned above. The accompanying solution procedure is computationally efficient, and as tractable as standard solution methodologies for linear rational expectations models. The proposed methodology unveils several important implications for determinacy in the regime-switching framework that are absent in the linear rational expectations models.

JEL Classification: C62; D84; E3
Keywords: MOD Method, MOD Solution, Markov-switching, Determinacy, Forward method

*I am very grateful to Eric Leeper and Tao Zha, Jean Barthélemy, Oswaldo Costa, Bennett T. McCallum, Antonio Moreno, Tack Yun, Nigel McClung, Mikael Juselius, and seminar participants at University of Sydney, Bank of Finland, Seoul National University, Bank of Korea, Department of Mathematics at Yonsei University, and Korea DSGE Research Association for their helpful comments.

†School of Economics, Yonsei University, 50 Yonsei-ro, Seodaemun-gu, Seoul 03722, Republic of Korea. *E-mail:* sc719@yonsei.ac.kr. Tel.: +82-2-2123-2470, Fax: +82-2-393-1158.

1 Introduction

The modern economy has frequently witnessed structural breaks or changes in preferences, technology and policy stances in macroeconomics. For instance, monetary policy has recently been constrained by the zero lower bound in the U.S. Since then, the Federal Reserve now conducts a conventional monetary policy by adjusting the federal funds rate as before the financial crisis. The transition out of a zero lower bound regime is now a memory, and this history may be repeated in the future. In a general context, there has been no clear understanding about the equilibrium properties when the economy recurrently switches over different regimes. The present work contributes to the literature by providing a complete and tractable methodological foundation for the analysis of general Markov-switching rational expectations (MSRE) models.

The pioneering work of Davig and Leeper (2007) has already showed that the equilibrium property of a model with a regime-switching monetary policy stance can be quite different from what is understood in the standard fixed-regime counterpart. Since then, relatively few works have been done in this field of macroeconomics, possibly due to the lack of analytical tools comparable to those for the standard linear rational expectations (LRE) models. Most notably, the concept of determinacy – uniqueness of stable equilibrium – is not well-established. Some existing works appear to interpret the long-run Taylor principle of Davig and Leeper (2007) as determinacy in the bounded stability sense, although Farmer et al. (2010) show that the principle is not sufficient for determinacy. Others often presume that a unique stable minimum state variable solution leads to determinacy in a model with lagged variables, which is not true either because such a case can coexist with a continuum of stable sunspots in the MSRE models – a key difference from the LRE counterparts – as demonstrated in this paper. A condition for non-existence of stable sunspots to purely forward-looking models has been proposed by Farmer et al. (2009), which is hard to examine in practice as it requires one to search over a very high dimensional parameter space. Barthélemy and Marx (2017) propose conditions for a unique bounded solution, but their conditions are difficult to verify, particularly around the border of determinacy and indeterminacy regions.

The most important contribution of this study is to provide a complete classification result for MSRE models that encompasses virtually all of the existing economic models and beyond. First, both necessary and sufficient conditions are derived for a unique stable solution, multiple stable solutions and no stable solution in MSRE models. As a stability concept, mean-square stability is adopted following Farmer et al. (2009), Cho (2016) and Foerster et al. (2016). Second, the proposed methodology is as tractable as standard ones for LRE models.

The key finding of our approach is that one particular solution – the most stable solution among all solutions – is all of the information needed to establish the proposed methodology, despite the fact that there is no known eigenvalue-eigenvector system (eigensystem for short hereafter) in the MSRE models. This solution is known as the MOD (minimum of modulus) solution in the case of LRE models (see McCallum (2007)), and it will be used in the context of MSRE models for consistency. The intuition is that the properties of the MOD solution plays essentially the same role in MSRE models as the eigensystem in LRE models.

Specifically, the existence and uniqueness of a stable solution can be completely identified by just two simple functions of the MOD solution – the spectral radii of transition probability weighted matrices uniquely associated with the solution. The first one judges mean-square stability of the MOD solution itself, which has the form of minimum state variable (MSV) solution. The second one – the most crucial finding of this study – detects both non-existence of mean-square stable sunspot solutions *and* non-existence of other mean-square stable MSV solutions. These are the key properties of the MOD solution with which the classification result of the MSRE models is derived – partitioning the entire family of MSRE models into the three mutually disjoint and exhaustive cases: determinacy, indeterminacy and the case of no stable solution.

The proposed methodology – referred to as the MOD method hereafter – therefore boils down to the identification and computation of the MOD solution. Once the MOD solution is identified, the classification of MSRE models is complete. To do so, one can first consider the Gröbner basis approach of Foerster et al. (2016). This technique finds all the MSV solutions, thus the MOD solution is identified by examining their stability, completing our methodology. However, this approach is computationally very demanding in practice, as our experiments illustrate. Another novel feature of this paper is that our model classification result itself provides an identification condition for the MOD solution to MSRE models, again, just as the eigensystem does for LRE models. Therefore, the classification is complete if a solution is found to satisfy the identification condition by any solution technique, without having to compute all solutions. This identification condition is the key factor that makes our methodology tractable and efficient.

As a primary solution technique, we propose a generalized version of the forward method developed by Cho (2016), which is an extension of Cho and Moreno (2011) to MSRE models. This is basically a generalization of the traditional solve-the-model forward approach. While this approach appears primitive at first glance, as it yields only one solution known as the forward solution, it is computationally very efficient. More importantly, it is shown that the forward solution coincides with the MOD solution almost surely for the class of MSRE models considered in this paper and Foerster et al. (2016). The numerical search method of Farmer et al. (2011) or Maih (2015) can be viable alternatives to the forward method. While these can solve multiple MSV solutions, they may not always detect the MOD solution because their algorithms depend on the initial values and the number of MSV solutions to MSRE models is surprisingly large as we will show.

Since the proposed approach drastically differs from the existing methods, Section 2 starts with LRE models to illustrate intuition behind our methodology, although the class of LRE models is nested as a special case of MSRE models. It helps understand the equivalence of the MOD method and the eigensystem-based techniques for LRE models such as Uhlig (1997), Klein (2000) or the gensys algorithm of Sims (2002). Section 3 formally presents the main result, the MOD method for the class of MSRE models. In section 4, we apply the proposed methodology to simple economic examples and demonstrate some novel consequences for determinacy in the class of MSRE models. This section also demonstrates that uniqueness of a stable MSV solution, or the long-run Taylor principle does not imply determinacy in general. Section 5 concludes.[1]

2 Linear Rational Expectations Models

This section presents a class of LRE models and develops the MOD method to help understand 1) the role of the MOD solution in terms of the well-known generalized eigenvalues in LRE models, 2) the equivalence of the proposed methodology with standard ones, and 3) why the MOD method can be extended to regime-switching models.

[1]Computer codes for the MOD method, examples and a technical guide accompanying this paper can be found at https://sites.google.com/site/sc719en.

2.1 LRE Models and the Solution Set

While there are many different representations, we express the class of linear rational expectations models as:

$$x_t = AE_t x_{t+1} + Bx_{t-1} + Cz_t, \tag{1}$$

where x_t is an $n \times 1$ vector of endogenous variables and z_t is an $m \times 1$ vector of exogenous variables. Unlike the original representation of Blanchard and Kahn (1980), the model does not need to distinguish predetermined from non-predetermined variables, which is consistent with the representation of Sims (2002) in that x_t represents a vector of variables measurable at time t. The matrices, A, B and C are conformable and A may well be singular. Since exogenous variables do not influence determinacy as long as they are stationary, we exclude z_t in what follows.

Consistent with Farmer et al. (2009), any solution to a rational expectations model can then be written as a sum of a fundamental solution depending only on the minimum state variables (MSV) and a sunspot (or non-fundamental) component:

$$x_t = \Omega x_{t-1} + w_t, \tag{2}$$

$$w_t = FE_t w_{t+1}, \tag{3}$$

where (Ω, F) must satisfy the restrictions:

$$\Omega = (I_n - A\Omega)^{-1} B, \tag{4}$$

$$F = (I_n - A\Omega)^{-1} A. \tag{5}$$

$x_t = \Omega x_{t-1}$ is referred to as a MSV solution in the absence of w_t.[2] A sunspot component w_t is any stochastic process, but it must obey the restriction (3). Thus (2) with a non-zero w_t is called a sunspot solution. Notice that the matrix F is uniquely defined for each Ω. Therefore, the full set of solutions can be described in terms of Ω:

$$\mathcal{S} = \left\{\Omega \in \mathcal{C}^{n\times n} | \; r(\Omega^1) \leq r(\Omega^2) \leq ... \leq r(\Omega^N)\right\}, \tag{6}$$

where N is the number of MSV solutions, Ω^h satisfies (4) for all $h \in \{1, ..., N\}$ and $r(\cdot)$ is the spectral radius, i.e., maximum absolute eigenvalue of the argument matrix.

[2] When z_t is specified, a MSV solution has the form of $x_t = \Omega x_{t-1} + \Gamma z_t$. Refer to Cho and McCallum (2015), for instance.

Members of $\mathcal{S}$ are arranged in an increasing order of the spectral radius of Ω without loss of generality. The following particular solution in $\mathcal{S}$ is all of the information required to derive our methodology.

Definition 1 *A MSV solution $x_t = \Omega^1 x_{t-1}$ to a linear rational expectations model (1) is referred to as a MOD (minimum of modulus) solution if $r(\Omega^1) = \min r(\Omega)$ for all $\Omega \in \mathcal{S}$ in (6).*

While the meaning of the *MOD* solution is transparent, it should be stressed that its existence must be a part of the conditions for determinacy, consistent with Sims (2002). The corresponding F^1 as well as Ω^1 will play a pivotal role in establishing our methodology.

2.2 The MOD Method

The *MOD* method is two-fold: 1) identifying and computing the *MOD* solution and 2) classifying the LRE models into determinacy, indeterminacy and the case of no stable solution. For ease of exposition, the model classification is derived first, followed by the identification of the *MOD* solution.

2.2.1 Classification of LRE Models by the *MOD* Method

The model classification can be completely characterized by the *MOD* solution $\Omega^1 \in \mathcal{S}$ in (6) and its associated F^1. Two important roles of F^1 are as follows.

First, F^1 provides a simple condition for non-existence of stable sunspots. For any given solution Ω and its associated F subject to (3), Farmer et al. (2009) show that w_t has the following form:

$$w_t = \Lambda w_{t-1} + VV'\eta_t, \tag{7}$$

where Λ is an $n \times n$ matrix, V is an $n \times k$ matrix of which columns are orthonormal.[3] η_t is an arbitrary $n \times 1$ stochastic vector such that $E_t \eta_{t+1} = 0_{n\times 1}$ and

$$F\Lambda = FVV'\Lambda = VV'. \tag{8}$$

[3]A complex-valued MSV solution does not mean that the sunspot solutions are always complex-valued. For instance, quasi-differencing (2) using (7) yields a solution depending on more than the minimum set of the state variables such that $x_t = (\Lambda + \Omega)x_{t-1} - \Lambda\Omega x_{t-2} + VV'\eta_t$ and the coefficient matrices can be real-valued.

The central idea of our approach can be easily understood by vectorizing equation (8) such that:

$$(\Lambda' \otimes F)u = u, \tag{9}$$

where $u = vec(VV')/||vec(VV')||$ is an eigenvector associated with a unit root. Therefore, $r(\Lambda' \otimes F) = r(\Lambda)r(F) \geq 1$ for all Λ subject to (8). Moreover, it is well-known that there exists Λ_m such that $r(\Lambda_m) = \min r(\Lambda) = 1/r(F)$, which clarifies the role of F pertaining to existence of stable sunspots: there cannot be stable sunspot processes if and only if $r(F) \leq 1$. To help the reader compare this last result with the corresponding one in the MSRE model class, we report it as Result 1 – a special case of Proposition 1 in the following section – as follows.

Result 1 *Consider any process w_t in (7) subject to (3) where F is real-valued. Then, there is no stable sunspot component w_t if and only if $r(F) \leq 1$. That is,*

$$r(\Lambda)r(F) \geq 1, \tag{10}$$

for all Λ and there exists a Λ_m such that

$$r(\Lambda_m)r(F) = 1. \tag{11}$$

Proof. See Appendix A. ■

Since Result 1 holds for the MOD solution, the condition $r(F^1) \leq 1$ becomes necessary and sufficient for non-existence of stable sunspot components associated with the MOD solution. This is transparent in LRE models, but it is one of the most important contributions of this paper to develop a MSRE model counterpart, as will be shown in the following section.

Second, the metric $r(F^1)$ also contains a very important information for identifying determinacy without the information about the eigensystem, which is new to the literature. Specifically, the following result demonstrates the relation between all of the members of the fundamental solutions.

Result 2 *Consider a model (1) and the set of solutions (6) where Ω^1 is real-valued. Then, the following holds.*

1. For all $h \in \{2, ..., N\}$,

$$r(\Omega^h)r(F^1) \geq 1, \quad r(\Omega^1)r(F^h) \geq 1. \tag{12}$$

2. Ω^1 *is the unique MOD solution if*

$$r(\Omega^1)r(F^1) < 1. \tag{13}$$

Proof. See Appendix B. ■

While the formal proof is given in Appendix B, these results are derived from the relations among different MSV solutions such that $D^{hk} = F^k D^{hk} \Omega^h$ analogous to (8) where $D^{hk} = \Omega^h - \Omega^k$ for all $h, k = 1, ..., N$ with $h \neq k$. Vectorizing this relation yields:

$$((\Omega^h)' \otimes F^k)u^{hk} = u^{hk}, \tag{14}$$

where $u^{hk} = vec(D^{hk})/||vec(D^{hk})||$ becomes an eigenvector associated with a unit root. Therefore, $r((\Omega^h)' \otimes F^k) = r(\Omega^h)r(F^k) \geq 1$, which leads to Assertion 1. Note also that $r(\Omega^h)r(F^h) \geq r(\Omega^{h-1})r(F^h) \geq 1$ for all $h \geq 2$, implying Assertion 2. Consequently, Assertion 1 implies that when $r(F^1) \leq 1$, not just all of the sunspot components associated with Ω^1 are non-stationary from Result 1, but also all of the fundamental solutions other than Ω^1 are unstable too. Assertion 2 states that there is *at most* one solution satisfying (13) and if it exists, it must be the MOD solution. Therefore, $r(\Omega^1) < 1$ together with $r(F^1) \leq 1$ implies the existence and uniqueness of a stable solution, that is, determinacy. It is very important that determinacy *can* arise only under (13). For this reason, we partition the class of LRE models into the subset in which determinacy can arise and its complement. This partition does not just provide a complete classification conditions for each subset of models, but also the identification condition for the MOD solution. Formally,

Definition 2 *A linear rational expectations model (1) is referred to as determinacy-admissible if* $\Omega^1 \in S$ *in (6) is real-valued and* $r(\Omega^1)r(F^1) < 1$*. The model is referred to as determinacy-inadmissible if* Ω^1 *is complex-valued or* $r(\Omega^1)r(F^1) \geq 1$*.*

The most important technical aspect of determinacy-admissible models is that there is a unique real-valued MOD solution. This can be better understood by the following transformation of any LRE model (1):

$$x_t = \left(\frac{1}{\alpha}\right) AE_t x_{t+1} + \alpha B x_{t-1}, \tag{15}$$

where $\alpha > 0$. Then, it is straightforward to show that the MSV solution to this model is given by $\alpha\Omega$ and the corresponding matrix governing the sunspot component is given by F/α where (Ω, F) is a solution to the original model.[4] Therefore, the order of spectral radii of MSV solutions is the same as that of the original model because $r(\alpha\Omega) = \alpha r(\Omega)$. Then $r(\Omega^1)r(F^1) = r(\alpha\Omega^1)r(F^1/\alpha)$ under the original and the transformed model. Henceforth, any determinacy-admissible LRE model can be made determinate – $r(\alpha\Omega^1) < 1$ and $r(F^1/\alpha) \leq 1$ – by setting α such that $r(F^1) \leq \alpha < 1/r(\Omega^1)$ if and only if it is a determinacy-admissible model. Because the determinate solution is the MOD solution by construction, the identification condition for the MOD solution is given by the existence of a real-valued MSV solution such that $r(\Omega)r(F) < 1$ to any determinacy-admissible model.

It is also important to understand the likelihood of determinacy-inadmissible models. A complex-valued MOD solution implies non-uniqueness of MOD solution by definition, thus determinacy cannot arise.[5] The case $r(\Omega^1)r(F^1) > 1$ can arise only when the model contains completely decoupled equations, regardless of whether Ω^1 is real-valued or not. As will be shown in Section 2.3, virtually all economic models are determinacy-admissible. Results 1, 2 and the concept of determinacy-admissible models lead to the following complete classification result of LRE models.

Result 3 *Consider a given model (1) and the set of solutions $\mathcal{S}$ in (6). Then, necessary and sufficient conditions for determinacy, indeterminacy and the case of no stable solution are given in Table (1). Moreover, if there exists a real-valued solution such that $r(\Omega)r(F) < 1$, it is the unique MOD solution and the model is determinacy-admissible.*

---

[4] This can be seen by the solution set $\mathcal{S}$ to the transformed model: $\alpha\Omega = [I - (A/\alpha)(\alpha\Omega)]^{-1}(\alpha B) = \alpha(I - A\Omega)^{-1}B$, and $(1/\alpha)F = [I - (A/\alpha)(\alpha\Omega)]^{-1}(A/\alpha) = (1/\alpha)(I - A\Omega)^{-1}A$.

[5] In fact, the eigensystem in the following subsection shows that $r(\Omega^1)r(F^1) \geq 1$ for all complex-valued Ω^1, thus complex-valued Ω^1 is redundant in definition of determinacy-admissible models. But this may not be the case of MSRE models, thus we retain this condition for consistency of the concept of determinacy-inadmissible models.

Table 1: **Classification of LRE Models by the MOD Method**

	Determinacy-Admissible: Ω^1 is real-valued and $r(\Omega^1)r(F^1) < 1$	Determinacy-Inadmissible: Ω^1 is complex-valued or $r(\Omega^1)r(F^1) \geq 1$
Determinacy	$r(\Omega^1) < 1$, $r(F^1) \leq 1$	Impossible
Indeterminacy	$r(F^1) > 1$	$r(\Omega^1) < 1$
No Stable Solution	$r(\Omega^1) \geq 1$	$r(\Omega^1) \geq 1$

Proof. See Appendix C. ■

The result can be interpreted as follows. The model is determinate if and only if $r(\Omega^1) < 1$ and $r(F^1) \leq 1$, which arises only when the model is determinacy-admissible. In the case of a determinacy-admissible model, $r(F^1) > 1$ implies indeterminacy by Result 1 and the fact that $r(\Omega^1) < 1/r(F^1) < 1$. Next, suppose that the model is determinacy-inadmissible. When Ω^1 is real-valued, $r(\Omega^1) < 1$ implies that $r(F^1) > 1$ because $r(\Omega^1)r(F^1) \geq 1$. When it is complex-valued and $r(\Omega^1) < 1$, then the MOD solution is not unique, thus there are multiple stable MSV solutions. Therefore, the condition $r(\Omega^1) < 1$ implies indeterminacy. When $r(\Omega^1) \geq 1$, the model has no stable solution in any case. Since all cases are mutually disjoint and exhaustive, the conditions for each class are both necessary and sufficient.

Result 3 is the highlight of our MOD approach, the main classification result for LRE models. This is in fact a generalization of Proposition 2 of Cho and McCallum (2015). Our contribution is that we prove Result 3 without resorting to the eigensystem, whereas they derive the same result for the case of a unique real-valued MOD solution using the eigensystem. This idea enables us to extend Result 1 through 3 to MSRE models.

2.2.2 Implementation of the MOD Method

The ultimate purpose of the MOD method is the classification of the family of LRE models by the number of stable solutions: determinacy, indeterminacy and the case of no stable solution. The MOD approach is completed by identifying and computing the MOD solution. It is straightforward to do so in LRE models because one can check the existence and uniqueness of the MOD solution by use of the eigensystem.

While we do not need to implement the MOD method to determine determinacy in LRE models, we must do so in MSRE models because of lack of an eigensystem. In principle, one can identify the MOD solution by computing all the MSV solutions and evaluating their stability. However, this approach turns out to be very inefficient even for LRE models as Section 3 illustrates. To highlight the importance of the MOD method summarized in Result 3, let us suppose that the eigensystem is unknown for LRE models.

Then the proposed procedure is as follows. One needs to use an alternative solution technique prior to the compute-all-solutions approach. If a real-valued solution is found such that $r(\Omega)r(F) < 1$, then it is the MOD solution from Result 3. If a solution is obtained such that $r(\Omega)r(F) \geq 1$ and $r(\Omega) < 1$, the model is indeterminate and we do not need to check whether it is the MOD solution. Therefore, the computing-all-solutions approach is required to complete classification only in the case of $r(\Omega^1)r(F^1) \geq 1$ and $r(\Omega^1) \geq 1$, a subset of determinacy-inadmissible models. Fortunately, determinacy-inadmissible models would hardly be of interest in economics. Therefore, finding a real-valued MOD solution with the property $r(\Omega)r(F) < 1$ suffices almost surely to completely classify LRE models. The alternative solution technique for both LRE and MSRE models, which will be discussed in Section 3, is as efficient as standard solution techniques for LRE models.

2.3 Equivalence with Standard Methods

We have successfully classified the LRE models with a single MOD solution without the help of the eigensystem. This subsection demonstrates the equivalence of standard eigensystem approach and our method, justifying the extension of the MOD method to MSRE models. Additionally, it is shown that our approach identifies the conditions under which the so-called root counting approach fails in classifying LRE models, consistent with the gensys algorithm of Sims (2002).

The standard approach of identifying determinacy can be described by the eigensystem of a given model (1) and its full set of solutions (2) and (3). While there are several way of representing the eigensystem, we reformulate (1) such that[6]:

$$\tilde{B}y_t = \tilde{A}E_t y_{t+1} + \tilde{C}z_t, \tag{16}$$

[6]Our model representation (16) is consistent with Klein (2000) or Uhlig (1997) and more general than Blanchard and Kahn (1980) and King and Watson (1998) in that singular matrix A can be analyzed.

where $y_t = [x'_t \; x'_{t-1}]'$ and

$$\tilde{B} = \begin{bmatrix} I_n & -B \\ I_n & 0_{n\times n} \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A & 0_{n\times n} \\ 0_{n\times n} & I_n \end{bmatrix}. \tag{17}$$

Let $\xi_{A,B}$ be the set of generalized eigenvalues implied by the model. Formally,

$$\xi_{A,B} = \{\xi_i \in \mathcal{C} | 0 = |\tilde{A} - \xi_i \tilde{B}|, \; |\xi_1| \leq |\xi_i| \leq |\xi_{2n}|\}. \tag{18}$$

As is well-known, for any (Ω, F) in $\mathcal{S}$, Ω is associated with n out of $2n$ generalized eigenvalues, thus the maximum number of the MSV solutions is ${}_{2n}C_n$. Moreover, the corresponding F is associated with the inverses of the remaining n roots in $\xi_{A,B}$ as McCallum (2007) demonstrates.

The condition for determinacy-admissible models can then be stated using the eigensystem as follows:

$$\Omega(\xi_1, ..., \xi_n) \in \mathcal{S}, \; |\xi_n| < |\xi_{n+1}|. \tag{19}$$

This condition implies that the MOD solution is the one associated with the n smallest eigenvalues, $\Omega^1 = \Omega(\xi_1, ..., \xi_n)$, and emphasizes its *existence* and *uniqueness*, the core parts of the gensys algorithm, which shares the same eigensystem with the present one.[7] Strict inequality $|\xi_n| < |\xi_{n+1}|$ implies that the MOD solution is unique and thus real-valued. Since $F^1 = F(1/\xi_{n+1}, ..., 1/\xi_{2n})$, $r(\Omega^1)r(F^1) = |\xi_n|/|\xi_{n+1}| < 1$. Therefore, (19) is exactly the same as the condition for determinacy-admissible models. In fact, this is explicitly or implicitly assumed in most of literature. Then the condition for determinacy, indeterminacy and no stable solution can be respectively stated as $|\xi_n| < 1 \leq |\xi_{n+1}|$, $|\xi_{n+1}| < 1$ and $|\xi_n| \geq 1$. Thus it is obvious that these conditions coincide with those in Table 1. It is also straightforward to show that $\{\alpha\xi_i, i = 1, ..., N\}$ is the set of the generalized eigenvalues of the order-preserving transformation (15).

[7]Model (1) can be written as a gensys form by defining $k_t = E_t x_{t+1}$, $x_t = k_{t-1} + \eta_t$ and $\Gamma_0 \hat{y}_t = \Gamma_1 \hat{y}_{t-1} + \Pi z_t + \Psi \eta_t$ where $\hat{y}_t = [x'_t \; k'_t]$ and Γ_0, Γ_1, Π and Ψ are functions of A,B and C. Generalized eigenvalues of the matrix pencil $\Gamma_0 - \xi\Gamma_1$ are exactly the same as those of (18). It is conventional to define k_t manually to include only non-zero expectational variables in vector $\hat{y}_t$. But such a representation in gensys is model-dependent whereas ours is not. Importantly, non-zero eigenvalues are invariant to any type of gensys form.

Next the condition for determinacy-inadmissible models is then given by:

$$\Omega(\xi_1, ..., \xi_n) \notin \mathcal{S} \text{ or } |\xi_n| = |\xi_{n+1}|. \tag{20}$$

First, suppose that $\Omega(\xi_1, ..., \xi_n)$ does not exist. Then, $r(\Omega^1) = |\xi_{n+i}|$ for some $i \in \{1, ..., N\}$ because Ω^1 must contain at least one root larger than $|\xi_n|$. The corresponding F^1 must contain a root $1/|\xi_{n+i-j}|$ for some $j \geq 1$, implying that $r(\Omega^1)r(F^1) \geq 1$. Second, if $\Omega^1 = \Omega(\xi_1, ..., \xi_n)$ and $|\xi_n| = |\xi_{n+1}|$, the MOD solution is not unique and thus $r(\Omega^1)r(F^1) = |\xi_n|/|\xi_{n+1}| = 1$. In both cases, if Ω^1 is complex-valued, $r(\Omega^1)r(F^1) \geq 1$. Henceforth, (20) is also equivalent to the condition for determinacy-inadmissible models. Let ξ_{n+i} be the largest eigenvalue of the MOD solution for a determinacy-inadmissible model. Then, $i > 0$ if $\Omega(\xi_1, ..., \xi_n) \notin \mathcal{S}$ and $i = 0$ otherwise. Therefore, a model of this type is indeterminate (has no stable solution) if and only if $|\xi_{n+i}| < (\geq)1$. This condition is exactly the same as $r(\Omega^1) < (\geq)1$. All of these results for determinacy-admissible and -inadmissible models are shown in Table 2.

Table 2: **Classification of LRE Models by Eigensystem**

	Determinacy-Admissible: $\Omega(\xi_1, ..., \xi_n) \in S$ and $\lvert\xi_n\rvert < \lvert\xi_{n+1}\rvert$	Determinacy-Inadmissible: $\Omega(\xi_1, ..., \xi_n) \notin S$ or $\lvert\xi_n\rvert = \lvert\xi_{n+1}\rvert$
Determinacy	$\lvert\xi_n\rvert < 1 \leq \lvert\xi_{n+1}\rvert$	Impossible
Indeterminacy	$\lvert\xi_{n+1}\rvert < 1$	$r(\Omega^1) = \lvert\xi_{n+i}\rvert < 1,\ i \geq 0$
No Stable Solution	$\lvert\xi_n\rvert \geq 1$	$r(\Omega^1) = \lvert\xi_{n+i}\rvert \geq 1,\ i \geq 0$

To summarize, the equivalence between our MOD approach and the standard method relying on the eigensystem can be confirmed by comparing Tables 2 and 1.[8] A key advantage of the MOD method over the standard approach is that it is currently the only option that can be extended to MSRE models.

Finally, we address the implications, likelihood and economic relevance of determinacy-inadmissible models. First, it is difficult to recall any reasonable economic model with multiple MOD solutions, complex-valued or repeatedly real-valued. Second, suppose that $\Omega(\xi_1, ..., \xi_n) \notin \mathcal{S}$. This is precisely the case in which the usual root-counting approach fails: it would conclude that a model is determinate if $|\xi_n| < 1 \leq |\xi_{n+1}|$, i.e.,

[8]Results 2 can also be verified in terms of the eigenvalues.

the number of of unstable generalized eigenvalues is n. But such a model has no stable solution because $r(\Omega^1) \geq |\xi_{n+1}| \geq 1$. Root-counting is not intrinsically ill-designed. It fails just because the existence of the MOD solution is not examined.

Sims (2007) shows that models that a root-counting approach fails must contain completely unrelated equations. But it also reveals that any economic model would hardly contain completely decoupled equations. It is even harder for researchers to conclude that such models are determinate, without actually computing the determinate solution. Nevertheless, the existence of the MOD solution is crucial in correctly classifying LRE models, and this is a key feature of Sims (2002) and our approach. It is not clear whether there are other types of models in which the root-counting approach fails, but if so, all of such models are classified by our methodology as indeterminate or the case of no stable solution because they must be determinacy-inadmissible.

3 Markov-switching Rational Expectations Models

This section presents the MOD method in the class of general MSRE models. Unlike the LRE models, a particular type of stability must be adopted in order to define uniqueness of rational expectations solutions in this framework. While there are still several competing concepts, we adopt mean-square stability. We ask readers to refer to Farmer et al. (2009) and Cho (2016) for further discussion. As in the previous section, the classification of the MSRE models is derived, followed by the solution method.

3.1 MSRE Models and the Solution Set

Following Cho (2016), we present the class of MSRE models as:

$$x_t = E_t[A(s_t, s_{t+1})x_{t+1}] + B(s_t)x_{t-1} + C(s_t)z_t, \tag{21}$$

where x_t and z_t are respectively an $n \times 1$ vector of endogenous variables and an $m \times 1$ vector of exogenous variables. s_t is an ergodic Markov chain switching over S different regimes. The transition probability switching from regime i to j is denoted by $p_{ij} = \Pr(s_{t+1} = j | s_t = i)$ such that $\sum_{j=1}^{S} p_{ij} = 1$ for all $i, j \in \{1, 2, \ldots, S\}$. P is the transition probability matrix, for which the (i, j)-th element is p_{ij}. $A(\cdot), B(\cdot)$ and $C(\cdot)$ are regime-dependent coefficient matrices. In particular, the matrix A depends on the future state

s_{t+1} and therefore, it cannot be taken out of conditional expectations. As Foerster et al. (2016) and Cho (2016) have emphasized, this is a natural characteristic of standard dynamic stochastic general equilibrium models with microfoundation subject to regime-switching because agents take into account the possibility of future regime switching when deriving optimal decision rules.

As in the case of LRE models, we describe the complete family of solutions such that any solution is written as a sum of a MSV solution and a sunspot (or non-fundamental) component. In the context of MSRE models, the relevant state variables are not just x_{t-1} and z_t, but also the regime-switching variable s_t. The exogenous variables z_t is also ignored for simplicity of analysis as the classification result is independent of the presence of z_t as long as z_t is mean-square stable. Therefore, any solution can be written as:

$$x_t = \Omega(s_t)x_{t-1} + w_t, \tag{22}$$

$$w_t = E_t[F(s_t, s_{t+1})w_{t+1}], \tag{23}$$

where $(\Omega(s_t), F(s_t, s_{t+1}))$ must obey the following restrictions for all $s_t, s_{t+1} = 1, .., S$:

$$\Omega(s_t) = \{I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]\}^{-1}B(s_t), \tag{24}$$

$$F(s_t, s_{t+1}) = \{I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]\}^{-1}A(s_t, s_{t+1}). \tag{25}$$

$x_t = \Omega(s_t)x_{t-1}$ is referred to as a MSV solution in the absence of w_t. The corresponding $F(s_t, s_{t+1})$ is also uniquely defined per each $\Omega(s_t)$.[9]

To proceed, we define the following two types of $n^2S \times n^2S$ matrices that will be used to derive our proposed methodology:

$$\bar{\Psi}_{G\otimes H} = \left[p_{ji}G(j,i) \otimes H(j,i)\right], \tag{26}$$

$$\Psi_{G\otimes H} = \left[p_{ij}G(i,j) \otimes H(i,j)\right], \tag{27}$$

[9]One may suspect that our analysis consider only MSV solutions or there are other class of solutions beyond what is described by (22) and (23). However, w_t represents whatever process that cannot be written in the form of MSV solutions, thus the set of solutions characterized by (22) and (23) is exhaustive. For instance, substituting out w_t leads to the form of solutions depending on x_{t-1}, x_{t-2} as well as current and past regime past regimes s_t and s_{t-1}. A direct representation of the solution set in this way is similar to that of Sims (2002) in LRE models. But representing the solution space in terms of the MSV solutions and the sunspot components is the very idea with which we identify the MOD solution –one of the MSV solutions – and complete the classification of both LRE and MSRE models.

where $G(s_t, s_{t+1}) = G(i,j)$ and $H(s_t, s_{t+1}) = H(i,j)$ are $n \times n$ matrices and the expression in the squared bracket represents the (i,j)-th $n^2 \times n^2$ dimensional block for all $i, j = 1, ..., S$. Our results will be characterized by $\bar{\Psi}_{\Omega\otimes\Omega}$ and $\Psi_{F\otimes F}$ only where $\Omega = \Omega(s_{t+1})$ and $F = F(s_t, s_{t+1})$, but their derivations will need these matrices such as $\Psi_{\Omega'\otimes F}$.

Then, a solution $x_t = \Omega(s_t)x_{t-1}$ is referred to as mean-square stable if $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$. With this concept of stability, we define the full set of MSV solutions as:

$$\mathcal{S} = \left\{ \Omega(s_t) \in \mathcal{C}^{n\times n} | r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) \leq r(\bar{\Psi}_{\Omega^2\otimes\Omega^2}) \leq ... \leq r(\bar{\Psi}_{\Omega^N\otimes\Omega^N}) \right\}, \tag{28}$$

where N is the number of MSV solutions and $\Omega^h(s_t)$ satisfies (24) for all $h \in \{1, ..., N\}$. Note that the solutions in $\mathcal{S}$ are ordered in terms of spectral radii of $\bar{\Psi}_{\Omega\otimes\Omega}$. The model and the solution set nest the LRE counterparts as special cases in which (Ω, F) are independent of s_t because $r(\bar{\Psi}_{\Omega\otimes\Omega}) = r(\Omega)^2$ in the absence of Markov-switching. The MOD solution in the class of MSRE models can be defined as follows.

Definition 3 *A MSV solution $x_t = \Omega^1(s_t)x_{t-1}$ to a Markov-switching rational expectations model (21) is referred to as a MOD (minimum of modulus) solution in the mean-square stability sense if $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) = \min r(\bar{\Psi}_{\Omega\otimes\Omega})$ for all $\Omega(s_t) \in \mathcal{S}$ in (28).*

3.2 The MOD Method

This subsection formally develops our methodology for the classification of MSRE models and its implementation procedure using the properties of the MOD solution.

3.2.1 Classification of MSRE Models by the MOD Method

We first derive two important properties of $r(\bar{\Psi}_{F^1\otimes F^1})$, which are new findings in the literature. As shown by Cho (2016), a sunspot or non-fundamental component w_t subject to (23) has the following form:

$$w_{t+1} = \Lambda(s_t, s_{t+1})w_t + V(s_{t+1})V(s_{t+1})'\eta_{t+1}, \tag{29}$$

where the columns of $V(s_{t+1})$ form an orthonormal basis such that $E_t[V(s_{t+1})V(s_{t+1})'\eta_{t+1}] = 0_{n\times 1}$. w_t is mean-square stable if $r(\bar{\Psi}_{\Lambda\otimes\Lambda}) < 1$ where $\bar{\Psi}_{\Lambda\otimes\Lambda}$ is defined in Equation (26). w_t must satisfy the restriction (23) such that $w_t = E_t[F(s_t, s_{t+1})\Lambda(s_t, s_{t+1})]w_t$. This relation can be expanded in the following way:

$$\sum_{j=1}^{S} p_{ij} F(i,j)\Lambda(i,j) = \sum_{j=1}^{S} p_{ij} F(i,j) V(j) V(j)' \Lambda(i,j) = V(i)V(i)', \tag{30}$$

for $1 \leq i \leq S$, where the first equality is shown by Farmer et al. (2009). As Appendix B shows, vectorizing this relation and stacking it over $i = 1, ..., S$ yields the following crucial relation analogous to equation (9):

$$(\Psi_{\Lambda' \otimes F})\, u = u \tag{31}$$

for any $\Lambda(s_t, s_{t+1})$ where u is an eigenvector constructed by $V(i)V(i)'$ and $\Psi_{\Lambda' \otimes F}$ is defined in Equation (27). Using Equation (31), the following proposition formally states the role of $r(\bar{\Psi}_{F^1 \otimes F^1})$, which is a direct extension of Result 1.

Proposition 1 *Consider any process w_t in (29) subject to (23) where $F(s_t, s_{t+1})$ is real-valued at all states. Then, there is no mean-square stable sunspot component w_t if and only if $r(\Psi_{F \otimes F}) \leq 1$. That is,*

$$r(\Psi_{F \otimes F}) r(\bar{\Psi}_{\Lambda \otimes \Lambda}) \geq 1, \tag{32}$$

for all $\Lambda(s_t, s_{t+1})$ satisfying (30) and there exists a $\Lambda_m(s_t, s_{t+1})$ such that

$$r(\Psi_{F \otimes F}) r(\bar{\Psi}_{\Lambda_m \otimes \Lambda_m}) = 1. \tag{33}$$

A solution satisfying (33) can be constructed as follows. Let $X_{ij} = X(i,j)$ and $X_i = X(i)$, $i, j = 1, ..., S$ for any X. Suppose that $\xi_2 = r(\Psi_{F \otimes F})$. Let $\hat{p}_{ij} = 1$ if $p_{ij} \neq 0$ and 0 otherwise. Define $\hat{F}_{ij} = \sqrt{p_{ij}/\xi_2} F_{ij}$ and $\Psi_{\hat{F} \otimes \hat{F}} = \left[\hat{p}_{ij} \hat{F}_{ij} \otimes \hat{F}_{ij}\right]$. Then, $r(\Psi_{\hat{F} \otimes \hat{F}}) = r(\Psi_{F \otimes F})/\xi_2 = 1$. Extract an $n^2 S \times 1$ eigenvector $u = [u_1', ..., u_S']$ such that $u = \Psi_{\hat{F} \otimes \hat{F}} u$. Reshape u_i into an $n \times n$ matrix Q_i such that u_i is a vectorized Q_i Real Schur decomposition yields $Q_i = V_i D_i V_i'$ where D_i is a non-singular $k_i \times k_i$ matrix of which non-zero eigenvalues are those of Q_i and the $n \times k_i$ matrix V_i of which columns are the corresponding orthonormal bases. Then, $r(\bar{\Psi}_{\Lambda_m \otimes \Lambda_m}) = 1/\xi_2$ where

$$\begin{aligned}
\Lambda_m(i,j) &= V_j \Phi_{ij} V_i' \text{ if } p_{ij} \neq 0 \text{ where } \Phi_{ij} = \frac{1}{\xi_2} D_j V_j' F_{ij}' V_i D_i^{-1} \\
&= 0_{n \times n} \text{ if } p_{ij} = 0.
\end{aligned}$$

Proof. *See Appendix A.* ■

Just like Result 1 for the LRE models, Proposition 1 implies that there is no mean-square stable sunspot components w_t associated with a given solution $\Omega(s_t)$ and the corresponding $F(s_t, s_{t+1})$ if and only if $r(\Psi_{F\otimes F}) \leq 1$. It should be stressed that Proposition 1 is not merely an extension of Farmer et al. (2009). Instead, this is one of the most important contributions in the MSRE literature. First, Farmer et al. (2009) show that the determinacy condition can be expressed in terms of non-existence of stable $\Lambda(s_t, s_{t+1})$. However, their approach is difficult to implement in practice because it requires one to search for the entire family of sunspot processes – which is immensely large to identify – even for models without lagged variables, and $F(\cdot) = A(\cdot)$ depending only on the current state s_t. Assertion 2 of Proposition 1 proves the non-existence of stable sunspots by the analytical form of Λ_m as a function of P and $F(\cdot)$ to any general model of the form (21)[10]. Second, the existence of $\Lambda_m(\cdot)$ also implies that the numerical procedure of searching $\Lambda(\cdot)$ satisfying (33) proposed by Cho (2016) is no longer needed either in the case of indeterminacy. Specifically, Lemma 2 of Cho (2016) states that the condition $r(\Psi_{F\otimes F}) \leq 1$ is sufficient for non-existence of mean-square stable sunspot processes. Proposition 1 shows that the condition is necessary as well. Obviously, Proposition 1 applies to $F^1(\cdot)$ associated with the *MOD* solution $\Omega^1(s_t)$.

The condition $r(\Psi_{F^1\otimes F^1}) \leq 1$ also rules out the existence of mean-square stable MSV solutions other than $\Omega^1(s_t)$. Appendix B derives the relation among different MSV solutions such that $D^{hk}(s_t) = E_t[F^k(s_t, s_{t+1})D^{hk}(s_{t+1})]\Omega^h(s_t)$ where $D^{hk}(s_t) = \Omega^h(s_t) - \Omega^k(s_t)$ for all $h, k = 1, ..., N$ with $h \neq k$. Vectorizing this relation yields an important relation analogous to (14) such that:

$$\left(\Psi_{(\Omega^h)'\otimes F^k}\right) u^{hk} = u^{hk}. \tag{34}$$

where u^{hk} is an eigenvector associated with a unit root. Therefore, $r\left(\Psi_{(\Omega^h)'\otimes F^k}\right) \geq 1$, which implies that $r(\bar{\Psi}_{\Omega^h\otimes\Omega^h})r(\Psi_{F^k\otimes F^k}) \geq 1$ from Lemma 2 of Cho (2016). Using this fact, Proposition 2 formally presents the second implication of $F^1(s_t, s_{t+1})$, which extends Result 2 to the case of MSRE models.

Proposition 2 *Consider a model (21) and the set of solutions $\mathcal{S}$ in (28) where $\Omega(s_t)$ is*

[10] In fact, there are in general uncountably many $\Lambda(s_t, s_{t+1})$ satisfying (33) and $\Lambda_m(s_t, s_{t+1})$ is just one of them.

real-valued at all states. Then the following holds.

1. *For all $h \in \{2, ..., N\}$,*

$$r(\bar{\Psi}_{\Omega^h \otimes \Omega^h}) r(\Psi_{F^1 \otimes F^1}) \geq 1, \quad r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) r(\Psi_{F^h \otimes F^h}) \geq 1. \tag{35}$$

2. *$\Omega^1(s_t)$ is the unique MOD solution if*

$$r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) r(\Psi_{F^1 \otimes F^1}) < 1. \tag{36}$$

Proof. See Appendix B. ■

As in the LRE case, Proposition 1 and Assertion 1 of Proposition 2 imply that the condition $r(\Psi_{F^1 \otimes F^1}) \leq 1$ ensures non-existence of mean-square stable sunspot solutions associated with $\Omega^1(s_t)$ and non-existence of mean-square stable MVS solutions other than $\Omega^1(s_t)$. The following concept of determinacy-admissible models also plays a pivotal role in identifying the *MOD* solution as well as the classification of MSRE models.

Definition 4 *A Markov-switching rational expectations model (21) is referred to as determinacy-admissible if $\Omega^1(s_t) \in S$ in (28) is real-valued at all states, $s_t = 1, ..., S$ and $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) r(\Psi_{F^1 \otimes F^1}) < 1$. The model is referred to as determinacy-inadmissible if $\Omega^1(s_t)$ is complex-valued at least at one regime s_t or $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) r(\Psi_{F^1 \otimes F^1}) \geq 1$.*

The fact that there is a unique real-valued *MOD* solution for every determinacy-admissible model can also be understood by the order-preserving transformation of any MSRE model (21) (ignoring exogenous variables) as follows:

$$x_t = \left(\frac{1}{\alpha}\right) E_t[A(s_t, s_{t+1}) x_{t+1}] + \alpha B(s_t) x_{t-1}, \tag{37}$$

where $\alpha > 0$. Then $\alpha \Omega^1(s_t)$ is the *MOD* solution to the transformed model and $F^1(s_t, s_{t+1})/\alpha$ is the corresponding matrix governing the dynamics of the sunspot components where $\Omega^1(s_t)$ is the *MOD* solution to the original model. Then, it is straightforward to show that a model can be made determinate if and only if the model is determinacy-admissible by setting α such that $\sqrt{r(\Psi_{F^1 \otimes F^1})} \leq \alpha < 1/\sqrt{r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1})}$.[11] This explains

[11] To see this, note that $r(\bar{\Psi}_{\alpha\Omega^1 \otimes \alpha\Omega^1})\ r(\Psi_{F^1/\alpha \otimes F^1/\alpha}) = r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1})\ r(\Psi_{F^1 \otimes F^1}) < 1$, $r(\bar{\Psi}_{\alpha\Omega^1 \otimes \alpha\Omega^1}) < 1$ and $r(\Psi_{F^1/\alpha \otimes F^1/\alpha}) \leq 1$.

why finding the MOD solution to the original model is exactly the same as finding the determinate solution to the transformed determinate model.

We have demonstrated that it is difficult to encounter determinacy-inadmissible models in the LRE models, because they have complex-valued MOD solutions or decoupled equations. Such models would also be rare in the MSRE framework, but there might be other type of such models, for example, where the economic structure differs across the regimes. What matters, however, is that all of those models can never be determinate. Propositions 1 and 2 along with the concept of determinacy-admissibility lead to Proposition 3, the main classification result of the MSRE models in this paper.

Proposition 3 *Consider a MSRE model (21) and the set of solutions $\mathcal{S}$ in (28). Then, necessary and sufficient conditions for determinacy, indeterminacy and the case of no stable solution in the mean-square stability sense are given in Table (3). Moreover, if there exists a real-valued solution such that $r(\bar{\Psi}_{\Omega\otimes\Omega})r(\Psi_{F\otimes F}) < 1$, it is the unique MOD solution and the model is determinacy-admissible.*

Table 3: **Classification of MSRE Models**

	Determinacy-Admissible: $\Omega^1(s_t)$ is real-valued and $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1})r(\Psi_{F^1\otimes F^1}) < 1$	Determinacy-Inadmissible: $\Omega^1(s_t)$ is complex-valued or $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1})r(\Psi_{F^1\otimes F^1}) \geq 1$
Determinacy	$r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) < 1$, $r(\Psi_{F^1\otimes F^1}) \leq 1$	Impossible
Indeterminacy	$r(\Psi_{F^1\otimes F^1}) > 1$	$r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) < 1$
No Stable Solution	$r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) \geq 1$	$r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) \geq 1$

Proof. See Appendix C. ■

A general eigensystem of the MSRE models is unknown. Nevertheless, Proposition 3 completely classifies the MSRE models into three disjoint and exhaustive cases of determinacy, indeterminacy and no stable solution by using the properties of the MOD solution, which plays the same role as the generalized eigenvalues.[12] The result and the interpretation of Proposition 3 are fully analogous to those of Result 3 for the LRE models

[12]Notice that real-valuedness of $\Omega^1(s_t)$ must be explicitly included in the condition for determinacy-admissibility in the MSRE models. This is because it is possible that $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1})r(\Psi_{F^1\otimes F^1}) < 1$ in the MSRE models if $\Omega^1(s_t)$ is complex-valued, which implies indeterminacy.

and nest them as a special case. Specifically, model (21) is determinate in the mean-square stability sense if and only if $\Omega^1(s_t)$ is real-valued for all regimes, $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) < 1$ and $r(\Psi_{F^1 \otimes F^1}) \leq 1$. Suppose that the model is determinacy-admissible. Then it is indeterminate if and only if $r(\Psi_{F^1 \otimes F^1}) > 1$ from Proposition 1. When the model is determinacy-inadmissible, it is indeterminate if and only if $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) < 1$ because it implies that $r(\Psi_{F^1 \otimes F^1}) > 1$. In any case, the model has no stable solution if and only if $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) \geq 1$. The classification result can also be stated without partitioning the MSRE models by determinacy-admissibility. Then, necessary and sufficient conditions changes only in the case of indeterminacy into $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) < 1$ and $r(\Psi_{F^1 \otimes F^1}) > 1$.[13]

On Uniqueness of the Stable MSV Solution and Determinacy

Proposition 3 shows that the MOD method for MSRE models is essentially the same as that for LRE models, but with a very important distinction. In the determinacy-admissible LRE model, uniqueness of a stable MSV solution for models with lagged endogenous variables implies determinacy. In stark contrast, the uniqueness of a stable MSV solution can be compatible with indeterminacy in general in the MSRE framework. This is even so in the class of determinacy-admissible models, to which almost all economic models belong. This implies that one may falsely interpret a MSRE model with a unique stable MSV as determinate despite that it is in fact indeterminate.[14]

To illustrate the point, consider a determinacy-admissible model with multiple MSV solutions. In LRE models, a unique stable MOD solution exists if $r(\Omega^1) < 1 \leq r(\Omega^h)$ for all $h \geq 2$. From Result 2 and its proof, $r((\Omega^h)' \otimes F^1) = r(\Omega^h) r(F^1) \geq 1$. Moreover, the eigensystem implies the existence of a solution Ω^j for some $j \geq 2$ such that

$$r(\Omega^j) r(F^1) = 1. \tag{38}$$

Therefore, the unique stable MSV solution implies the non-existence of stable sunspots,

[13] Note that determinacy-admissibility is equivalent to $r(\Psi_{\Omega^{1\prime} \otimes F^1}) = r(\Omega^1) r(F^1) < 1$ in LRE models. This condition resembles one of the conditions for expectational stability of Evans and Honkapohja (2001). In contrast, $r(\Psi_{\Omega^{1\prime} \otimes F^1}) < 1$ is necessary for determinacy-admissibility in MSRE models. Therefore, it would be of interest to explore the relation among this condition, expectational stability and determinacy in the regime-switching context. This issue has recently been analyzed by McClung (2019).

[14] We emphasize that the argument here applies to models with lagged endogenous variables so that there are multiple MSV solutions. Obviously, uniqueness of a stable solution does not imply determinacy for a purely forward-looking model without lagged variables. This is because in such a case, the MOD solution is unique by construction and determinacy is identified by the non-existence of stable sunspots, $r(\Psi_{F^1 \otimes F^1}) \leq 1$ where $F(\cdot)$ collapses to $A(\cdot)$.

and thus determinacy because $r(\Omega^j) \geq 1$ implies that $r(F^1) \leq 1$.[15] In contrast, the corresponding condition for MSRE models is $r(\Psi_{\Omega^{h\prime} \otimes F^1}) \geq 1$ for all $h \geq 2$ from equation (34). We apply the Gröbner basis approach and find that there also exists a solution $\Omega^j(s_t)$ such that $r(\Psi_{\Omega^{j\prime} \otimes F^1}) = 1$, although there is no known proof. Even in this case, the following holds

$$r(\bar{\Psi}_{\Omega^j \otimes \Omega^j}) r(\Psi_{F^1 \otimes F^1}) \geq 1. \tag{39}$$

In fact, strict inequality holds for the latter in general in the MSRE models. An important implication of this result is that uniqueness of a stable MSV solution – $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) < 1 \leq r(\bar{\Psi}_{\Omega^h \otimes \Omega^h})$ for all $h \geq 2$ – may well be consistent with $r(\Psi_{F^1 \otimes F^1}) > 1$, implying indeterminacy. Henceforth, a valid classification of MSRE models requires to check not just the uniqueness of a stable MOD solution, but also non-existence of stable sunspot $r(\Psi_{F^1 \otimes F^1}) \leq 1$ as clearly stated in Proposition 3. The following corollary formally presents this result.

Corollary 1 *Consider a determinacy-admissible MSRE model (21) where $B(s_t) \neq 0_{n \times n}$ for at least one regime, and the set of solutions $\mathcal{S}$ in (28). The uniqueness of a mean-square stable MSV solution does not always imply determinacy in the mean-square stability sense.*

Proof. See Appendix D. ■

In Section 4, an example of this kind is provided: a baseline economic example with seemingly reasonable monetary and fiscal policy mixes has a unique stable MOD solution, but it turns out to be indeterminate. This is important because many existing papers have already analyzed regime-switching models similar to our example from the perspective of the fiscal theory of the price level. Validity of this approach critically hinges on determinacy because these models requires a unique equilibrium at which agents' beliefs are coordinated. For this reason, previous studies have tried to find a unique stable MOD solution and presumed that such a model is determinate. Therefore, if such models turn out to be indeterminate, then it would be difficult to justify that agents' expectations are anchored at that MOD solution.

[15]This is because $r(\Omega^j) = |\xi_{n+1}|$ and $r(F^1) = 1/|\xi_{n+1}|$.

3.2.2 Implementation of the MOD Method

From a theoretical perspective, the full set of MSV solutions including complex-valued ones in the MSRE models can be obtained by the Gröbner basis approach proposed by Foerster et al. (2016). This is very important because the MOD solution can be identified for all MSRE models, making our approach complete. However, it is computationally very demanding for the Gröbner basis technique to find all of the solutions even for a low dimensional model as we show below. Fortunately, the properties of the MOD solution summarized in Proposition 3 play almost the same role as the eigensystem for both LRE and MSRE models, which make our approach tractable. The implementation procedure of the MOD method utilizes these properties as follows.

1. Apply a primary solution technique. If the obtained solution $\Omega(s_t)$ is real-valued and $r(\bar{\Psi}_{\Omega\otimes\Omega})r(\Psi_{F\otimes F}) < 1$, then it is the MOD solution and the model is determinacy-admissible.

2. If $r(\bar{\Psi}_{\Omega\otimes\Omega})r(\Psi_{F\otimes F}) \geq 1$ and $r(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ for a solution, then the model is indeterminate, regardless of whether the solution is the MOD solution.

3. If $r(\bar{\Psi}_{\Omega\otimes\Omega})r(\Psi_{F\otimes F}) \geq 1$ and $r(\bar{\Psi}_{\Omega\otimes\Omega}) \geq 1$, or real-valued solution $\Omega(s_t)$ cannot be obtained, apply the Gröbner basis to identify the MOD solution.

Case 1 is self-sufficient, which would arise for almost all economic models. In Case 2, indeterminacy is confirmed without having to identify the MOD solution because it must be true that $r(\Psi_{F\otimes F}) \geq 1$. Case 3 is the only case the Gröbner basis approach is required, which would arise only if the model is indeed determinacy-inadmissible or the model is determinacy-admissible but the obtained solution differs from the MOD solution. For this sequential procedure to be implementable, a candidate for solution methodology should yield the MOD solution to determinacy-admissible models, and computation must be efficient.

Based on these criteria, we adopt a generalized version of the forward method of Cho (2016) as a primary solution technique – presented in Appendix E – which yields the forward solution and the computation time is comparable to standard techniques for LRE models. This approach is basically an extended version of a textbook-type solving-the-model-forward approach to MSRE models. The appendix also shows that the condition for the existence of the forward solution – absent in Cho (2016) – is weaker than those for determinacy-admissible models. This implies that once the forward solution exists for determinacy-admissible models, it is the MOD solution by Proposition 3. Indeed, we have not found any single case in which the equivalence of the two solutions breaks down for the entire class of MSRE models as long as the solution is real-valued. But it would be fair to state that the forward solution is highly likely to coincide with the MOD solution given that there is no known proof for the equivalence.[16] The numerical search method of Farmer et al. (2011) can also be a candidate. This method is efficient in terms of computational time, but it oftentimes fails to produce the MOD solution even for determinacy-admissible models because the search algorithm depends on the randomized initial guess and the number of solutions – to be shown below – is surprisingly large for MSRE models. Moreover, it is not applicable to the model in which the coefficient matrix of forward-looking variables, A depends on the future state s_{t+1}. The method proposed by Maih (2015) can be applied to our class of models, but it is also a search algorithm, hence, the MOD solution may not always be found.

To summarize, unless one is interested in determinacy-inadmissible models, the forward method is sufficient for implementing the MOD method because it yields the forward solution, conditions for determinacy, indeterminacy and no stable solution in one step as functions of the transition probabilities and the parameter matrices of a given model. The Gröbner basis technique would serve as the last resort to complete the MOD method only if the forward method or an alternative fails to identify the MOD solution.

3.2.3 Computation Procedure

The solution package accompanying this paper provides all of the codes implementing the MOD method. Matlab codes "fmmsre.m" and "gbmsre.m" implement the forward method and the Gröbner basis approach, respectively as a function of parameter matrices $P, A(s_t, s_{t+1})$ and $B(s_t)$. The latter controls the solution procedure in matlab, but uses a mathematics language referred to as "Singular" developed by Decker et al. (2019) which is based on C++. This is known as more than 100 times faster than matlab.

[16] The forward method originally proposed by Cho and Moreno (2011) for LRE models – as a solution refinement scheme – yields the forward solution consistent with the MOD solution except for only one type of models featuring a special block-recursive structure such as the one we analyze in Section 4. Cho (2016) adjusts the information set with which agents solve the model forward to recover the equivalence of the forward solution with the MOD solution. Appendix E formalizes this idea of information adjustment.

We have tested computation time for n dimensional atheoretical models with m number of lagged variables in a standard computer. For LRE models, it takes only around 10^{-3} second to implement the MOD method using either the gensys algorithm or the forward method even for large-scale models with $n = m = 10$. For MSRE models, the computation time is longer, but around 10^{-3} to 10^{-2} seconds. Therefore, there would be no difficulty even in estimating MSRE models using our proposed method as long as they are determinacy-admissible.

Only when the forward method cannot identify the MOD solution, – the third case in the implementation procedure above – one can apply the Gröbner basis technique. Basically this technique computes the polynomial bases as many as the number of solutions to the system of multivariate quadratic equations implied by (24), which contain n^2S number of unknowns in $n \times n$ matrices $\Omega(s_t)$. The problem is that the total number of solutions, N, is very large. To see this, suppose that P is an identity matrix. Then the model at each regime collapses to a LRE model. Let $N_s(n, m_s) = \frac{(n+m_s)!}{n!m_s!}$ be the number of solutions at regime s where m_s is the number of lagged variables in that regime.[17] In this case, $N = \Pi_{s=1}^{S} N_s(n, m_s)$. Because this is a very special case of a MSRE model, $N \geq \Pi_{s=1}^{S} N_s(n, m_s)$ in general. Our simulation indicates that the computation time increases roughly at the rate of 2^N. When $m_s = n$ for all regimes with $S = 2$, the Gröbner basis works fast for a univariate model, which has 4 solutions. For a bivariate model, it takes several minutes to yield 44 solutions. For a trivariate model, computation fails because the total number of solutions is at least 400 and the expected computation time would be longer than that for the bivariate model by a factor of 2^{356}. [18]

Despite these computational burdens, the Gröbner basis technique does work if the number of lagged variables is less than 2 in at least one regime for up to 5- or 6- dimensional models with two regimes. Indeed, Foerster et al. (2016) use mathematica to apply this technique to the models with only one lagged variable. More importantly, this technique can also be used to check the uniqueness of stable MSV solution to confirm Corollary 1 numerically, as well as all other propositions.

[17]For an n-dimensional LRE model, there are $2n$ generalized eigenvalues. But the number of finite eigenvalues is $n + m$. Any MSV solution is associated with n finite eigenvalues. Therefore, the total number of solutions is the same as the number of combinations of choosing n out of $n + m$ eigenvalues.

[18]For LRE models with $m = n$, the total number of MSV solutions is 20, 70 and 252 when $n = 3$, $n = 4$ and $n = 5$, respectively. Roughly speaking, this approach may work within a reasonably short time for both LRE and MSRE models if the number of solutions is less than 100. Refer to the solution package for more detail about computation time.

4 Examples

This section applies the MOD method to a highly stylized model with regime-switching monetary and fiscal policies from the perspective of the fiscal theory of the price level (FTPL). The choice of this model has several purposes. First, most of the classification results of the MOD method can be understood analytically. In particular, policy mixes inducing determinacy under a fixed policy regime are shown to be neither necessary nor sufficient for determinacy under regime-switching. Second, this model helps us to quantify the difference between determinacy and the long-run Taylor principle: the latter is shown to be necessary, but not sufficient for determinacy. Third, switching over seemingly standard policy mixes can lead to indeterminacy in spite of a unique stable MSV solution because they represents two very different Ricardian and non-Ricardian regimes. This is an example of Corollary 1, a new observation in the literature. The example starts with a LRE model in which all of the results can be interpreted analytically in terms of two parameters representing policy stances. Determinacy analysis is then conducted under several scenarios of regime-switching. A companion paper, Cho and Moreno (2019), provides a rigorous analysis of this kind of model in depth.

4.1 A Linear Model

Consider a New-Keynesian model that is augmented to include a government budget constraint that incorporates a tax policy. The essential feature of this type of model can be succinctly illustrated by a simple example. A monetary block representing a standard New-Keynesian model consists of a Fisher equation $i_t = E_t\pi_{t+1} + r_t$ and a Taylor-type rule $i_t = \alpha\pi_t$. i, r and π are nominal and real interest rate and inflation, respectively. Assuming the real interest rate to be exogenous and ignoring it, the monetary block can be written as $\alpha\pi_t = E_t\pi_{t+1}$. A fiscal block is a linearized government budget constraint augmented by a tax policy such that $b_t = 1/\beta b_{t-1} - \tau_t - c\pi_t$ where b_t is the government's debt to output ratio, and the tax policy is given by $\tau_t = \delta b_{t-1}$. β is the time discount factor and $c \neq 0$. By setting $\theta = 1/\beta - \delta$, the model can be represented as:

$$\pi_t = (1/\alpha)E_t\pi_{t+1}, \tag{40a}$$

$$b_t = \theta b_{t-1} - c\pi_t. \tag{40b}$$

This model is a simplified version of Leeper (1991) or Bhattarai et al. (2014) that abstracts from other important features to focus only on determinacy analysis. Following the latter, we specify $c = \bar{b}(1/\beta - \alpha)$ where $\bar{b}$ is debt to GDP ratio and β is the time discount factor. The first equation is by itself a well-defined rational expectations model. It is well-known that this monetary block, if treated in isolation, exhibits determinacy (indeterminacy) when the monetary policy is active (AM) with $\alpha \geq 1$ and passive (PM) with $\alpha < 1$. The dependent fiscal block is explosive with an active fiscal policy (AF) with $\theta \geq 1$, and stationary with a passive fiscal policy (PF) ($\theta < 1$).[19] For this reason, this type of model is block-recursive in which a non-zero c opens up the expectational channel through which agents may form inflation expectation based on the fiscal dynamics.

4.1.1 Determinacy Analysis

We conduct determinacy analysis of this model using the proposed MOD method and show its equivalence with the standard method. There are two MSV solutions in this model: a monetary solution Ω^M known as Ricardian in which inflation is independent of the fiscal block and a fiscal solution Ω^F (non-Ricardian equilibrium) where inflation does depend on the government debt. The analytical form of these solutions and their corresponding F are given by:

$$\Omega^M(\theta, 0) = \begin{bmatrix} 0 & 0 \\ 0 & \theta \end{bmatrix}, \quad F^M(1/\alpha, 0) = \begin{bmatrix} \frac{1}{\alpha} & 0 \\ -\frac{c}{\alpha} & 0 \end{bmatrix}, \tag{41}$$

$$\Omega^F(\alpha, 0) = \begin{bmatrix} 0 & \frac{\theta-\alpha}{c} \\ 0 & \alpha \end{bmatrix}, \quad F^F(1/\theta, 0) = \begin{bmatrix} \frac{1}{\theta} & 0 \\ -\frac{c}{\theta} & 0 \end{bmatrix}, \tag{42}$$

where the eigenvalues of each matrix are in parentheses. Therefore, $r(\Omega^M) = \theta$ and $r(F^M) = 1/\alpha$ while $r(\Omega^F) = \alpha$ and $r(F^F) = 1/\theta$. As is well-known, the model is determinate if and only if one policy is active and the other is passive. The equilibrium is monetary (fiscal) under the AM-PF policy (PM-AF) mix. In both cases, the MOD solution is the equilibrium at which expectations are coordinated. The model is indeterminate when both policies are passive and has no stable solution when both are active. PM-PF or AM-AF policy mixes are of less interest in the FTPL approach because the

[19] Following convention, the fiscal policy stance can also be stated in terms of the tax policy coefficient δ. The fiscal policy is known as active (passive) if $\delta \leq (>) 1/\beta - 1$.

equilibrium is not unique or unstable so that agents' beliefs are hard to be anchored at a particular solution.

MOD **Method** The classification of the *MOD* method is fully equivalent to the aforementioned taxonomy result. The model is determinacy-admissible if and only if $\alpha \neq \theta$ because $r(\Omega)r(F) = \theta/\alpha$ $(\alpha/\theta)$$< 1$ for the monetary (fiscal) solution. The *MOD* solution depends on the relative aggressiveness of the two policies: the monetary equilibrium Ω^M is the *MOD* solution if monetary policy is more aggressive than the fiscal counterpart $(\alpha > \theta)$. The fiscal equilibrium is the *MOD* solution if $\alpha < \theta$. In the former, the model is determinate if $r(\Omega^M) = \theta < 1$ and $r(F^M) = 1/\alpha \leq 1$, which is exactly the case of an AM-PF policy mix.[20] It is indeterminate if $r(F^M) > 1$, implying a PM-PF policy mix with $\theta < \alpha < 1$. The latter case is also determinate if $r(\Omega^F) = \alpha < 1$ and $r(F^F) = 1/\theta \leq 1$, corresponding to a PM-AF mix. There is another indeterminate case with PM-PF policy if $r(F^F) = 1/\theta > 1$, but the *MOD* solution is the fiscal one as $\alpha < \theta < 1$. In both cases, there is no stable solution if $r(\Omega^{MOD}) \geq 1$ because it is an AM-AF policy mix with $1 \leq \alpha, \theta$. The model becomes determinacy-inadmissible in the knife-edge case of $\alpha = \theta$ as $r(\Omega)r(F) = 1$ from Result 3. Therefore, the model can only be either indeterminate if both parameters are smaller than one, or has no stable solution otherwise. Typically, this particular case is not explicitly taken into account in the taxonomy of the model in terms of four different policy mixes. Our analysis indicates that this is not interesting because the model can never be determinate, the core property required for the perspective of the FTPL.

Eigensystem Approach The same result can also be confirmed by the eigensystem. The generalized eigenvalues are given by $(0, \alpha, \theta, \infty)$ with $\xi_1 = 0$ and $\xi_4 = \infty$. The solution with the two smallest generalized eigenvalues always exist, thus, the model is determinacy-admissible if and only if $\alpha \neq \theta$. Hence, the model is determinate if and only if there are exactly two eigenvalues strictly inside the unit circle. The AM-PF combination implies determinacy because $\xi_2 = \theta < 1 \leq \xi_3 = \alpha$, thus Ω^M is the determinate solution. The fiscal equilibrium is the *MOD* solution under the PM-AF combination as $\xi_2 = \alpha < 1$ $\leq \xi_3 = \theta$. PM-PF combinations are again the case in which $0 < \alpha, \theta < 1$ implying indeterminacy whereas AM-AF combination is associated with $0 < 1 \leq \alpha, \theta$. When

[20] Determinacy contains the case of $\alpha = 1$, therefore, strictly speaking, the model is determinacy under a non-passive monetary policy and a passive fiscal policy. Following convention, we keep using the terminology AM-PF in this case. The same is true for PM-AF policy mix.

$\xi_2 = \xi_3 = \alpha = \theta$, the MOD solution still exists but is not unique, thus the model is either indeterminate or has no stable solution. This can be easily analyzed by the gensys algorithm or root-counting with the existence of the MOD solution.[21]

4.2 A Markov-switching Model

We extend the same model analyzed above to allow the central bank and the fiscal authority to switch their policy stances. The monetary policy stance can switch over the two regimes $\alpha(1)$ and $\alpha(2)$ and the fiscal policy stance can also differ across the two regimes, $\theta(1)$ and $\theta(2)$.

$$\pi_t = (1/\alpha(s_t))E_t\pi_{t+1}, \tag{43a}$$

$$b_t = \theta(s_t)b_{t-1} - c(s_t)\pi_t. \tag{43b}$$

where $c(s_t) = \bar{b}(1/\beta - \alpha(s_t))$. The model can be cast into the canonical form of (21) where $x_t = [\pi_t \; b_t]'$ and $A(\cdot)$ and $B(\cdot)$ are given by:

$$A(s_t) = \begin{bmatrix} 1/\alpha(s_t) & 0 \\ -c(s_t)/\alpha(s_t) & 0 \end{bmatrix}, \quad B(s_t) = \begin{bmatrix} 0 & 0 \\ 0 & \theta(s_t) \end{bmatrix}. \tag{44}$$

We consider three among many possible scenarios for an expositional purpose. First, only monetary policy switches over two different regimes while the fiscal policy remains passive. Second, only fiscal policy switches over two regimes while monetary policy remains active. Finally, both policies switch over two policy combinations for which determinacy prevails under fixed regime, that is, AM-PF and PM-AF combinations. In all cases, the transition probabilities are fixed at $P(1,1) = 0.95$ and $P(2,2) = 0.9$ while $\beta = 0.99$ and $\bar{b} = 1$. The forward solution computed by the modified forward method is the MOD solution, completing the model classification in all examples below.

[21] The key economic linkage of this model is c, which enables agents form expectations of future inflation by taking account of the fiscal block. When this channel is shut down with $c = 0$, non-Ricardian equilibrium $\Omega^F(0,\alpha)$ never exists (See Equation (42)). The model is then a collection of completely decoupled equations. Interestingly, the generalized eigenvalues remain unaltered. Therefore, if $\alpha < 1 < \theta$, then judgment by the root-counting alone would conclude that the model is determinate, which is not correct because $\Omega^F(0,\alpha)$ no longer exists. In this case, $\Omega^M(0,\theta)$ is the MOD solution and the model is determinacy-inadmissible because $r(\Omega^M)r(F^M) = \theta/\alpha > 1$ and it has no stable solution as $r(\Omega^M) = \theta > 1$. This highlights the importance of checking the existence of the MOD solution in classification, fully in line with the gensys algorithm. But at the same time, this shows why determinacy-inadmissible models are hard to find in the literature.

4.2.1 Regime-Switching in Monetary Policy

Suppose that the fiscal policy is passive and fixed at both regimes such that $\theta < 1$. The task is to find the combinations of monetary policy stance ensuring determinacy for all possible combinations over $\alpha(s_t) > \theta$. The case in which $\alpha(s_t) < \theta$ for one or both regimes will be analyzed in the last example, which may induce a switching between the monetary and fiscal equilibria. We apply the forward method and find that the model is determinacy-admissible. Thus, the forward solution is the MOD solution, which is a monetary, Ricardian equilibrium, $\Omega^1(s_t) = \Omega^M(0, \theta)$ defined in (41). This Ricardian equilibrium is always mean-square stable as $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) = \theta^2 < 1$. Hence, model determinacy depends on the size of $r(\Psi_{F^1 \otimes F^1})$ where $F^1(s_t) = F^M(1/\alpha(s_t), 0)$. This is depicted as the white region in Panel A of Figure 1. The light grey area is the indeterminacy region of $r(\Psi_{F^1 \otimes F^1}) \geq 1$. The curve partitioning determinacy and indeterminacy is the set of $\alpha(1)$ and $\alpha(2)$ such that $r(\Psi_{F^1 \otimes F^1}) = 1$.

The result shows that a unique mean-square stable equilibrium allows the monetary policy to be temporarily passive in one regime and active in the other.[22] This resembles what the long-run Taylor Principle (LRTP) proposed by Davig and Leeper (2007) implies. However, the LRTP is not the condition for determinacy in boundedness or mean-square stability sense, as demonstrated by Farmer et al. (2010)[23]. Cho (2016) shows that the conditions for the LRTP coincide with $r(\bar{\Psi}_{\Omega^1}) < 1$ and $r(\Psi_{F^1}) \leq 1$, which may be referred to as "*mean-stability*". Therefore, mean-stability is necessary but not sufficient for mean-square stability. Panel A of Figure 1 clearly shows that the LRTP region is strictly larger than the determinacy counterpart: there is a sizable region with $r(\Psi_{F^1}) \leq 1 < r(\Psi_{F^1 \otimes F^1})$ in which the LRTP holds, but the model is indeterminate.

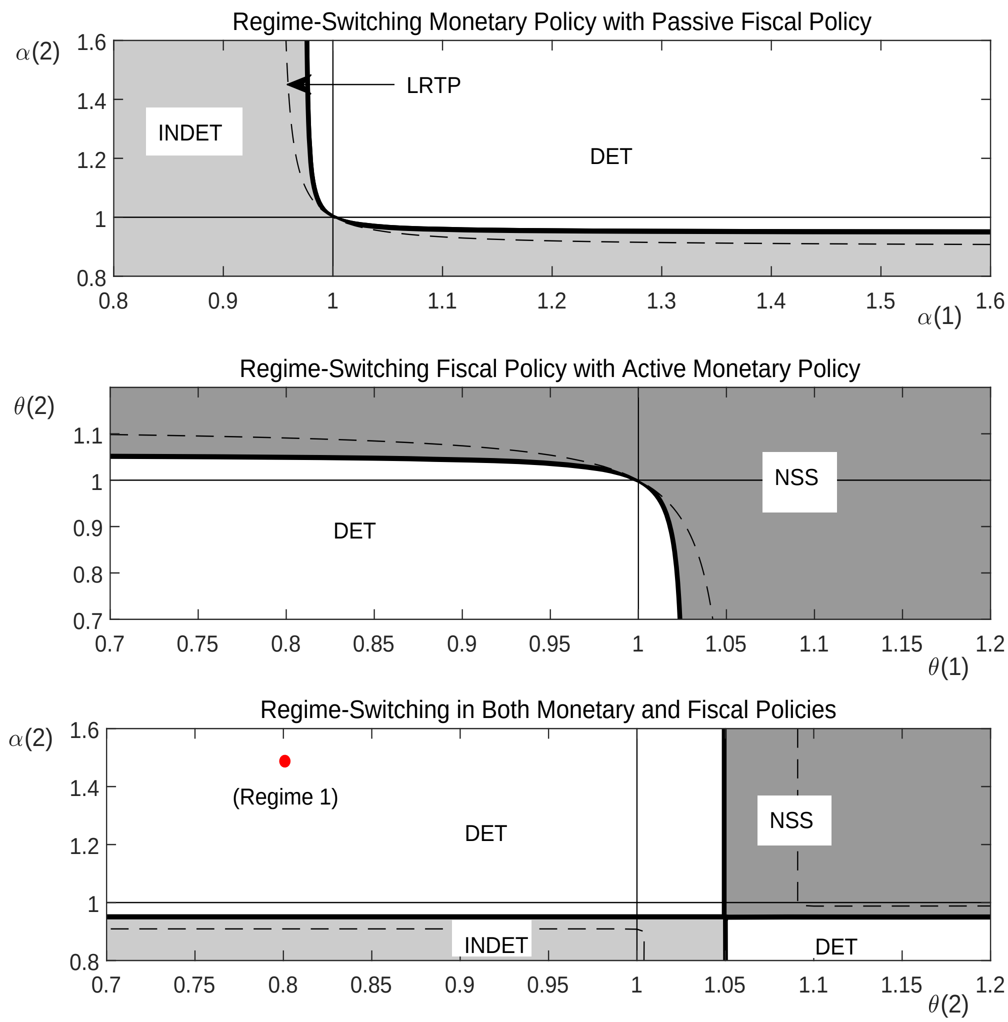
4.2.2 Regime-Switching in Fiscal Policy

A qualitatively similar implication to the model above can be drawn when the fiscal policy switches over two regimes $\theta(1)$ and $\theta(2)$ while monetary policy is active in both regimes. Interestingly, this case also has an analytical form of the MOD solution under

[22]For the regions $\alpha(s_t) < \theta$ in one or both regimes, indeterminacy arises as $r(\Psi_{F^1 \otimes F^1}) > 1$. This implies that a monetary policy can be temporarily passive but it should not be too passive for determinacy.

[23]Recently Barthélemy and Marx (2017) propose conditions for unique bounded solution, but their conditions are difficult to verify especially in the border of determinacy and indeterminacy regions. In contrast, our determinacy conditions can be straightforward to verify and identify regions for not just determinacy, but also indeterminacy and no stable solution in the mean-square stability sense.

Figure 1: Determinacy Analysis for Model (43) under Regime-Switching



This figure depicts the region for determinacy(DET) in white, indeterminacy(INDET) in light grey and no stable solution(NSS) in dark grey for model (43). Panel A considers monetary policy switching over two states $\alpha(1)$ and $\alpha(2)$ while holding the fiscal policy to be passive with $\theta = 0.8$. The region implied by the long-run Taylor Principle(LRTP) is also depicted by dashed line. Panel B is for fiscal policy switching over two states $\theta(1)$ and $\theta(2)$ while monetary policy is active with $\alpha = 1.5$. Panel C reports the result in terms of monetary and fiscal policy stances in regime 2 ($\alpha(2)$ $\theta(2)$) when the policy mix is AM-PF with $\alpha(1) = 1.5$ and $\theta(1) = 0.8 < 1$ in regime 1, as denoted by the red dot. Regions including determinacy from the dashed line in Panel B and C are the one implied by "mean stability" $r(\bar{\Psi}_{\Omega^1}) < 1$ and $r(\Psi_{F^1}) \leq 1$, analogous to the LRTP region.

determinacy when the monetary policy is active with $\alpha > 1$ and $\alpha > \theta(s_t)$ in both regimes. This is because the determinate solution is the monetary equilibrium, $\Omega^1(s_t) = \Omega^M(0, \theta(s_t))$ in (41). Determinacy is governed by $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) < 1$ because $r(\Psi_{F^1 \otimes F^1}) = 1/\alpha^2 < 1$ where $F^1(s_t) = F^M(0, 1/\alpha)$ under active monetary police. Panel B of Figure 1 depicts the determinacy region in this exercise with the locus $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) = 1$. The determinacy region is again larger than that under the fixed regime counterpart. That is, a temporarily active fiscal policy is admissible for determinacy if it is not too active.

However, there is an important difference between this example and the one with regime-switching monetary policy. The parameter area neighboring the determinacy region is not the indeterminacy region but the one – depicted in dark grey – in which no stable solution exists such that $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) \geq 1$. This is because the switching occurs over AM-PF and AM-AF combinations, i.e., the determinate region and a region with no stable solution under fixed regime.[24]

To our knowledge, there is no principle corresponding to the LRTP when lagged variables are present. But as shown above, a natural analogue would be the one implied by mean-stability. Since $r(\Psi_{F^1 \otimes F^1}) = 1/\alpha^2 < 1$, $r(\Psi_{F^1}) = 1/\alpha < 1$. There is a region in which $r(\bar{\Psi}_{\Omega^1}) < 1 \leq r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1})$. Therefore, the area analogous to the LRTP in this model contains an area in which the MOD solution is not mean-square stable. This is depicted by the region between the solid line and the dashed line.

4.2.3 Regime-Switching in Both Policies

Two policy combinations that have drawn much attention in the literature are AM-PF and PM-AF because it is well-known that these two cases lead to determinacy under fixed regime. But there has been no formal analysis about determinacy property regarding switching over these two policy mixes. Would the determinacy region be still larger than the one under fixed regime? To answer the question, suppose that the first regime is AM-PF with $\alpha(1) = 1.5$ and $\theta(1) = 0.8$. Then we can seek the policy combinations in regime 2, $\alpha(2)$ and $\theta(2)$, under which determinacy is ensured. Given that the policy combination is AM-PM type in regime 1, one can conjecture that AM-PM in regime 2 or

[24] In both examples, the monetary equilibrium prevails. By the same token, we test and find that the fiscal equilibrium prevails if the fiscal policy is active under regime-switching monetary policy and if the monetary policy is passive under regime-switching fiscal policy. In the former, the model is either determinate or has no stable solution. In the latter, the model is either determinate or indeterminate.

PM-AF in regime 1 may be associated with determinacy. The exact determinacy region is depicted by Panel C of Figure 1.

First, determinacy in the MSRE models is neither necessary nor sufficient for determinacy under the fixed regime-counterpart. This is first pointed out by Cho (2016) in the context of a standard New-Keynesian model with regime-switching monetary policy. Our analysis shows that such a phenomenon can be universal. In this example, determinacy prevails for all AM-PM policy combinations and some PM-PF, AM-AF or even PM-AF policy mixes. On the other hand, the PM-AF policy mix in regime 2 does not always ensure determinacy: it can lead to indeterminacy and even the case of no stable solution. In this case, the determinate solution depends on both $\alpha(s_t)$, $\theta(s_t)$ and the transition probabilities, and thus has no analytical form. Nevertheless, the MOD methodology can identify the determinacy region easily. Determinacy can be ensured with a very passive monetary policy and an aggressively active fiscal policy. But if fiscal policy is mildly active while monetary policy is very passive, it is closer to PM-PF policy mix, leading to indeterminacy. Similarly, if monetary policy is slightly passive, it is nearby an AM-AF policy mix, thus there is no stable solution (dark grey area). In contrast, when both policies are AM-PF or very close to it, determinacy is ensured as well.

Second, the equilibrium property of determinacy can be drastically different. In the top left determinacy region, the equilibrium is monetary as $\Omega^{MOD}(s_t)$ is analytically given by $\begin{bmatrix} 0 & 0 \\ 0 & \theta(s_t) \end{bmatrix}$: inflation is not affected by the government debt even when the policy mix is PM-AF in regime 2 as long as the policy stances are both mild. In contrast, the bottom-right determinacy region, the equilibrium is fiscal as inflation does depend on the government debt with the same but strongly PM-AF mix in regime 2. For instance, $\Omega^{MOD}(1) = \begin{bmatrix} 0 & 0.060 \\ 0 & 0.829 \end{bmatrix}$, $\Omega^{MOD}(2) = \begin{bmatrix} 0 & 1.032 \\ 0 & 0.883 \end{bmatrix}$ when $\alpha(2) = 0.8$ and $\theta(2) = 1.1$. [25]

Finally, an important result that has not been observed in the literature is unveiled.

[25] Recall that the model is determinacy-inadmissible when $\alpha = \theta$ in LRE case, which can be interpreted as the 45 degree line passing through $\alpha = \theta = 1$. In the MSRE models, there does exist a very tight area in which the model is determinacy-inadmissible. It is around the 45 degree line, but going through the point that separates the two different determinacy regions associated with the Ricardian and non-Ricardian equilibrium. These regions lie strictly inside the indeterminacy region and the region of no stable solution. The latter case is the one that the Gröbner basis technique is required. Indeed, this approach confirm that the forward solution coincide with the MOD solution.

The model can be indeterminate even when there is a unique stable MOD solution. For instance, suppose that the regime 1 is AM-PM with $\alpha(1) = 1.5$ and $\theta(1) = 0.95$ and a PM-AF policy mix $\alpha(2) = 0.95$ and $\theta(2) = 1.01$ is conducted in regime 2. Then $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) = 0.962$, $r(\Psi_{F^1 \otimes F^1}) = 1.0015$ and the product of the two is less than one. Therefore, the model is determinacy-admissible and indeterminate. The classification of the model is completed by the forward method. When we apply the Gröbner basis technique we find that there are four solutions and the MOD solution is indeed the forward solution. For the second most mean-square stable solution, $r(\bar{\Psi}_{\Omega^j \otimes \Omega^j}) = 1.011$ where $j = 2$. This implies that the MOD solution is the unique stable MSV solution, but the model is indeterminate. The parameter space over which the model is indeterminate with a unique stable MOD solution may be small. Nevertheless, this example highlights the importance of checking the non-existence of the stable sunspots to correctly identify the determinacy region, and this is exactly what our proposed methodology does.

5 Conclusion

From a mathematical point of view, determinacy implies that a given rational expectations model has a single stable solution. Not all economists would mechanically endorse the determinate solution as economically reasonable. Nevertheless, it is also true that determinacy has been regarded as one of the most important properties for most economic models under rational expectations. This paper contributes to the literature by providing a complete and tractable technical foundation for analyzing Markov-switching rational expectations models. Specifically, the proposed methodology computes the most stable solution in the mean-square stability sense to the set of general Markov-switching rational expectations models. Using this solution, it classifies the full set of MSRE models into determinacy, indeterminacy and no stable solution, and provides necessary and sufficient conditions for each case. The methodology is also computationally efficient and comparable to the standard solution techniques in the linear rational expectations models. Thus applied works should be easy to conduct in the context of Markov-switching rational expectations models.

Appendix

A Proof of Proposition 1 and Result 1

A.1 Proof of Result 1

It is proved in the main text that $r(\Lambda' \otimes F) = r(\Lambda)r(F) \geq 1$ for all Λ subject to (8). A solution Λ_m satisfying (11) can be constructed using the real Schur decomposition theorem: there exist an $n \times n$ matrix V of which columns are orthonormal and the upper block-triangular matrix D such that $F = VDV'$. Let $r(D) = r(F)$ and let V_m be the corresponding orthonormal vector. Then $\Lambda_m = V_m V_m'/r(F)$ satisfies (11).

A.2 Proof of Proposition 1

Lemma 2 of Cho (2016) shows that $r(\Psi_{F\otimes F})r(\bar{\Psi}_{\Lambda\otimes\Lambda}) \geq 1$ for all $\Lambda(s_t, s_{t+1})$ subject to (30). Therefore, the proof of Proposition 1 boils down to constructing a $\Lambda(s_t, s_{t+1})$ such that $r(\bar{\Psi}_{\Lambda\otimes\Lambda}) = 1/\xi_2$ when $r(\Psi_{F\otimes F}) = \xi_2$.

Step 1: Reformulation of the Restriction for w_t, (29). For a given $F\,(s_t, s_{t+1})$, consider any $\Lambda(s_t, s_{t+1})$ satisfying (30) and let $\tau_2 = r(\bar{\Psi}_{\Lambda\otimes\Lambda})$. In what follows, the subscript i and j denote the state s_t and s_{t+1}, respectively. Define $\hat{F}_{ij} = \sqrt{p_{ij}/\xi_2}F_{ij}$ and $\hat{\Lambda}_{ij} = \sqrt{p_{ij}\xi_2}\Lambda_{ij}$. Define $\hat{P} = [\hat{p}_{ij}]$ where $\hat{p}_{ij} = 1$ if $p_{ij} \neq 0$ and $\hat{p}_{ij} = 0$ otherwise for all i and j. $\hat{P}$ makes the algebra simpler although it is not a transition probability matrix. Then (30) can be written in terms of $\hat{P}$, $\hat{F}\,(\cdot)$ and $\hat{\Lambda}\,(\cdot)$ as:

$$\sum_{j=1}^{S} p_{ij}F_{ij}\Lambda_{ij} = \sum_{j=1}^{S} \hat{p}_{ij}\hat{F}_{ij}\hat{\Lambda}_{ij} = V_i V_i', \quad for \;\; 1 \leq i \leq S, \tag{45}$$

where V_i is to be constructed below. Note that $\Psi_{\hat{F}\otimes\hat{F}} = [\hat{p}_{ij}\hat{F}_{ij} \otimes \hat{F}_{ij}] = \frac{1}{\xi_2}\Psi_{F\otimes F}$ and $\bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}} = [\hat{p}_{ji}\hat{\Lambda}_{ji} \otimes \hat{\Lambda}_{ji}] = \xi_2\bar{\Psi}_{\Lambda\otimes\Lambda}$ where the expression inside the bracket denotes an ij-th $n^2 \times n^2$ matrix. Therefore, $r(\Psi_{\hat{F}\otimes\hat{F}}) = 1$, $r(\bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}}) = \xi_2\tau_2$. The goal is to find a particular $\hat{\Lambda}_{ij}$ such that $r(\bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}}) = 1$.

Step 2. Extracting Information from $\hat{F}(s_t, s_{t+1})$. Since $r(\Psi_{\hat{F}\otimes\hat{F}}) = 1$, one can easily find the $n^2S \times 1$ eigenvector $u = [u_1' \; ... \; u_S']'$ corresponding to the unit eigenvalue such that:[26]

$$u = \Psi_{\hat{F}\otimes\hat{F}} u. \tag{46}$$

By reshaping $n^2 \times 1$ subvector u_i into an $n \times n$ matrix Q_i, (46) can be written as:

$$\sum_{j=1}^{S} \hat{p}_{ij} \hat{F}_{ij} Q_j \hat{F}'_{ij} = Q_i, \tag{47}$$

where $u_i = vec(Q_i)$ for all i. In matrix form, this can be expressed as:

$$\begin{gathered} \hat{p}_{11}\hat{F}_{11}Q_1\hat{F}'_{11} + ... + \hat{p}_{1S}\hat{F}_{1S}Q_S\hat{F}'_{1S} = Q_1, \\ ... \\ \hat{p}_{S1}\hat{F}_{S1}Q_1\hat{F}'_{S1} + ... + \hat{p}_{SS}\hat{F}_{SS}Q_S\hat{F}'_{SS} = Q_S. \end{gathered} \tag{48}$$

Transposing each equation of (48) and vectorizing it must be the same as (46), which implies $u_i = vec(Q'_i)$, thus Q_i is symmetric.[27] Therefore, using the Schur decomposition theorem, we can construct a $k_i \times k_i$ diagonal matrix D_i where the diagonal elements are the non-zero eigenvalues of Q_i and $n \times k_i$ matrix V_i of which columns are orthonormal bases such that:

$$Q_i = V_i D_i V'_i. \tag{49}$$

Step 3. Constructing $\tilde{F}_{ij}$ and $\tilde{\Lambda}_{ij}$ such that $r(\Psi_{\tilde{F}\otimes\tilde{F}}) = r(\bar{\Psi}_{\tilde{\Lambda}'\otimes\tilde{F}}) = r(\bar{\Psi}_{\tilde{\Lambda}\otimes\tilde{\Lambda}}) = 1$. Define a $k_i \times k_j$ matrix $\tilde{F}_{ij}$ and a $k_j \times k_i$ matrix $\tilde{\Lambda}_{ij}$ as follows:

$$\tilde{F}_{ij} = V'_i \hat{F}_{ij} V_j, \quad \tilde{\Lambda}_{ij} = D_j \tilde{F}'_{ij} D_i^{-1}. \tag{50}$$

Then (48) can be written as the following three forms:

$$\sum_{j=1}^{S} \tilde{p}_{ij} \tilde{F}_{ij} D_j \tilde{F}'_{ij} = D_i, \quad \sum_{j=1}^{S} \tilde{p}_{ij} \tilde{F}_{ij} \tilde{\Lambda}_{ij} = I_{k_i}, \quad \sum_{i=1}^{S} \tilde{p}_{ij} \tilde{\Lambda}'_{ij} D_j'^{-1} \tilde{\Lambda}_{ij} = D_i'^{-1}, \tag{51}$$

by pre-multiplying V'_i and post-multiplying V_i to each equation of (48) with $\tilde{p}_{ij} = \hat{p}_{ij}$,

[26] The maximum eigenvalue of $\Psi_{\hat{F}\otimes\hat{F}}$ is real-valued and positive. This can be verified following Lemma 1 of Cho (2016) in which the variance of $z_t = x_t + v_t$ is governed by $\bar{\Psi}_{\Omega^h\otimes\Omega^h}$, $\Psi_{F^1\otimes F^1}$ and $\Psi_{\Omega^{h\prime}\otimes F^1}$ where $x_t = \Omega^h(s_t)x_{t-1}$ and $v_t = F^1(s_{t-1}, s_t)v_{t-1}$. From Proof of Proposition 2 below, the maximum eigenvalue of $\Psi_{\Omega^{h\prime}\otimes F^1}$ is positive and $r(\Psi_{\Omega^{h\prime}\otimes F^1}) \geq 1$. This implies that the maximum eigenvalue of $\bar{\Psi}_{\Omega^h\otimes\Omega^h}$ and $\Psi_{F^1\otimes F^1}$ must also be positive such that $r(\bar{\Psi}_{\Omega^h\otimes\Omega^h})r(\Psi_{F^1\otimes F^1}) \geq 1$.

[27] Otherwise, $\tilde{u}_i = \alpha vec(Q_i) + \beta vec(Q'_i)$ with arbitrary scalars α and β with some normalization must also be an eigenvector associated with unity, which is impossible.

post-multiplying D_i^{-1} and pre-multiplying $D_i'^{-1}$. Let $K = \sum_{i=1}^{S} k_i^2$ and $diag(V_i \otimes V_i)$ is $n^2S \times K$ matrix of which $i-th$ diagonal block is given by $V_i \otimes V_i$. Then the following relations hold:

$$\begin{aligned}
\Psi_{\tilde{F}\otimes\tilde{F}} &= [diag(V_i \otimes V_i)]'\Psi_{\hat{F}\otimes\hat{F}}[diag(V_i \otimes V_i)], \\
\Psi_{\tilde{\Lambda}'\otimes\tilde{\Lambda}'} &= [diag(D_i' \otimes D_i')]^{-1}\Psi_{\tilde{F}\otimes\tilde{F}}[diag(D_i' \otimes D_i')], \\
\Psi_{\tilde{\Lambda}'\otimes\tilde{F}} &= [diag(D_i' \otimes I_{k_i})]^{-1}\Psi_{\tilde{F}\otimes\tilde{F}}[diag(D_i' \otimes I_{k_i})],
\end{aligned}$$

where $diag(D_i' \otimes I_{k_i})$ and $diag(D_i' \otimes D_i')$ are $K \times K$ matrices. The first relation is directly implied by the definition of (50) and the property of Kronecker product. The second and third ones can be derived by vectorizing each of (51). Let the non-zero eigenvalues of a matrix M be $neig(M)$. Note that $diag(V_i \otimes V_i)$ is a subset of an orthonormal basis associated with the non-zero eigenvalues of for $\Psi_{\hat{F}\otimes\hat{F}}$, thus $neig(\Psi_{\tilde{F}\otimes\tilde{F}}) = neig(\Psi_{\hat{F}\otimes\hat{F}})$. Since $r(\Psi_{\hat{F}\otimes\hat{F}}) = 1$, $r(\Psi_{\tilde{F}\otimes\tilde{F}}) = 1$ as well. Similarly, $r(\bar{\Psi}_{\tilde{\Lambda}'\otimes\tilde{F}}) = r(\Psi_{\tilde{\Lambda}'\otimes\tilde{\Lambda}'}) = 1$.

Step 4. Constructing $\hat{\Lambda}(s_t, s_{t+1})$ such that $r(\Psi_{\hat{F}\otimes\hat{F}}) = r(\bar{\Psi}_{\hat{\Lambda}'\otimes\hat{F}}) = r(\bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}}) = 1$. $\hat{\Lambda}_{ij}$ can be recovered from $\tilde{\Lambda}_{ij}$ as:

$$\hat{\Lambda}_{ij} = V_j \tilde{\Lambda}_{ij} V_i' \tag{52}$$

First, this choice of $\hat{\Lambda}_{ij}$ fulfills equation (45). To see this, premultiply V_i and post-multiply V_i' to the second equation in (51). Then,

$$\sum_{j=1}^{S} \tilde{p}_{ij} V_i \tilde{F}_{ij} \tilde{\Lambda}_{ij} V_i' = \sum_{j=1}^{S} \tilde{p}_{ij} \hat{F}_{ij} \hat{\Lambda}_{ij} = V_i V_i', \quad for \quad 1 \leq i \leq S.$$

because $V_i V_i' \hat{F}_{ij} = \hat{F}_{ij}$ and $V_j V_j' \hat{\Lambda}_{ij} = \hat{\Lambda}_{ij}$. Note that

$$\begin{aligned}
\bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}} &= [diag(V_i \otimes V_i)]\bar{\Psi}_{\tilde{\Lambda}\otimes\tilde{\Lambda}}[diag(V_i \otimes V_i)]', \\
\bar{\Psi}_{\hat{\Lambda}'\otimes\hat{F}} &= [diag(V_i \otimes V_i)]\Psi_{\tilde{\Lambda}'\otimes\tilde{F}}[diag(V_i \otimes V_i)]'.
\end{aligned}$$

Again, columns of $diag(V_i \otimes V_i)$ are orthonormal, thus $neig(\bar{\Psi}_{\tilde{\Lambda}\otimes\tilde{\Lambda}}) = neig(\bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}})$, implying $r(\bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}}) = 1$. Similarly, $r(\bar{\Psi}_{\hat{\Lambda}'\otimes\hat{F}}) = 1$.

Step 5. Constructing $\Lambda(s_t, s_{t+1})$ such that $r(\Psi_{F\otimes F}) = \xi_2$, $r(\bar{\Psi}_{\Lambda'\otimes F}) = 1$ and $r(\bar{\Psi}_{\Lambda\otimes\Lambda}) = 1/\xi_2$. Finally, we recover $\Lambda_m(s_t, s_{t+1})$ as follows. Define $\Lambda_{m,ij}$ as:

$$\begin{aligned}\Lambda_{m,ij} &= \hat{\Lambda}_{ij}/\sqrt{p_{ij}\xi_2,} = V_j\Phi_{ij}V_i' \text{ if } p_{ij} > 0, \\ &= 0_{n\times n}, \text{ if } p_{ij} = 0,\end{aligned}$$

where $\Phi_{ij} = \frac{1}{\xi_2}D_jV_j'F_{ij}'V_iD_i^{-1}$. Then, Λ_m fulfils the restriction (45) with the properties, $\bar{\Psi}_{\Lambda_m\otimes\Lambda_m} = \xi_2^{-1}\bar{\Psi}_{\hat{\Lambda}\otimes\hat{\Lambda}}$, $\Psi_{F\otimes F} = \xi_2\Psi_{\hat{F}\otimes\hat{F}}$. Henceforth, $r(\bar{\Psi}_{\Lambda_m\otimes\Lambda_m}) = 1/\xi_2$ and $r(\bar{\Psi}_{\Lambda_m'\otimes F}) = 1$ when $r(\Psi_{F\otimes F}) = \xi_2$. All of the results above holds in the case of LRE models with $S = 1$. *Q.E.D.*

B Proof of Proposition 2 and Result 2

Proof of Assertion 1 Denote the difference of any two solutions by $D^{h1}(s_t) = \Omega^h(s_t) - \Omega^1(s_t)$ for any $h = 2, ..., N$ at $s_t = 1, ..., S$. First, suppose that $\Omega^h(s_t) \neq \Omega^1(s_t)$, for at least one state. We show that the following holds:

$$D^{h1}(s_t) = E_t[F^1(s_t, s_{t+1})D^{h1}(s_{t+1})]\Omega^h(s_t)]. \tag{53}$$

Consider an equilibrium path following a solution $x_t = \Omega^h(s_t)x_{t-1}$ ignoring z_t. Then the model in which the expectational term formed with this solution can be expressed as:

$$\begin{aligned}x_t &= E_t[A(s_t, s_{t+1})\Omega^h(s_{t+1})]x_t + B(s_t)x_{t-1} \\ &= E_t[A(s_t, s_{t+1})\Omega^1(s_{t+1})]x_t + E_t[A(s_t, s_{t+1})D^{h1}(s_{t+1})]x_t + B(s_t)x_{t-1} \\ &= \{I_n - E_t[A(s_t, s_{t+1})\Omega^1(s_{t+1})]\}^{-1}\{E_t[A(s_t, s_{t+1})D^{h1}(s_{t+1})]x_t + B(s_t)x_{t-1}\} \\ &= E_t[F^1(s_t, s_{t+1})D^{h1}(s_{t+1})]x_t + \Omega^1(s_t)x_{t-1}. \\ &= E_t[F^1(s_t, s_{t+1})D^{h1}(s_{t+1})]\Omega^h(s_t)x_{t-1} + \Omega^1(s_t)x_{t-1} = \Omega^h(s_t)x_{t-1}.\end{aligned}$$

Since this must be true for any given x_{t-1}, (53) holds.

Now expand Equation (53) as:

$$D^{h1}(s_t = i) = \sum_{j=1}^{S} P_{ij}F^1(s_t = i, s_{t+1} = j)D^{h1}(s_{t+1} = j)\Omega^h(s_t = i)], \tag{54}$$

for all $i = 1, ..., S$. By vectorizing this, we have

$$\left(\Psi_{(\Omega^h(s_t))' \otimes F^1}\right) u^{h1} = u^{h1}. \tag{55}$$

The vector $u^{h1}/||u^{h1}||$ is the eigenvector associated with one, thus $r(\Psi_{(\Omega^h(s_t))' \otimes F^1}) \geq 1$. Note that the expression $\Omega^h(s_t)$ is used in equations (54) because it is measured at time t, whereas it is $\Omega^h(s_{t+1})$ in the definition of $\Psi_{(\Omega^h)' \otimes F^1}$. Analogous to Claim 1 of Cho (2016), equation (55) implies that $r(\Psi_{(\Omega^h(s_t))' \otimes \Omega^h(s_t)'}) r(\Psi_{F^1 \otimes F^1}) \geq 1$. Moreover, $r(\Psi_{(\Omega^h(s_t))' \otimes \Omega^h(s_t)'}) = r(\bar{\Psi}_{\Omega^h \otimes \Omega^h})$ from Proposition 3.4 of Costa et al. (2005). Therefore, the following result holds:

$$r(\bar{\Psi}_{\Omega^h \otimes \Omega^h}) r(\Psi_{F^1 \otimes F^1}) \geq 1. \tag{56}$$

Assertion 1 of Result 1 is a special case in which $S = 1$ and $P = 1$. In this case, Equation (55) collapses to $\left((\Omega^h)' \otimes F^1\right) u^{h1} = u^{h1}$. Since $r\left((\Omega^h)' \otimes F^1\right) = r(\Omega^h) r(F^1)$,

$$r(\Omega^h) r(F^1) \geq 1. \tag{57}$$

Proof of Assertion 2 From Assertion 1 and $r(\bar{\Psi}_{\Omega^h \otimes \Omega^h}) \geq r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1})$,

$$r(\bar{\Psi}_{\Omega^h \otimes \Omega^h}) r(\Psi_{F^h \otimes F^h}) \geq r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) r(\Psi_{F^h \otimes F^h}) \geq 1, \tag{58}$$

for all $h > 1$. Therefore, if $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) r(\Psi_{F^1 \otimes F^1}) < 1$, then $\Omega^1(s_t)$ is the unique MOD solution. In the case of LRE models, $r(\bar{\Psi}_{\Omega^h \otimes \Omega^h}) = r((\Omega^h)^2) = (r(\Omega^h))^2$ and $r(\Psi_{F^h \otimes F^h}) = r((F^h)^2) = (r(F^h))^2$. Therefore, $(r(\Omega^h))^2 (r(F^h))^2 = (r(\Omega^h) r(F^h))^2 \geq 1$ if and only if $r(\Omega^h) r(F^h) \geq 1$, proving Assertion 2 of Result 2. *Q.E.D.*

C Proof of Proposition 3 and Result 3

Suppose that the model is determinacy-admissible. Then $\Omega^1(s_t)$ is unique from Assertion 2 of Proposition 2. If $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) < 1$ and $r(\Psi_{F^1 \otimes F^1}) \leq 1$, then the MOD solution is mean-square stable and there is no mean-square stable sunspots or other MSV solutions from Proposition 1 and Assertion 1 of Proposition 2. Therefore, the model is determinate. If $r(\Psi_{F^1 \otimes F^1}) > 1$, then $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) = 1/r(\Psi_{F^1 \otimes F^1}) < 1$, implying the existence of a continuum of stable sunspot components from Proposition 1. Therefore, the model is indeterminate. If $r(\bar{\Psi}_{\Omega^1 \otimes \Omega^1}) \geq 1$, all of the MSV solutions as well as sunspot solutions associated with any one of MSV solution are unstable from Assertion 2 of Proposition 2. Next, suppose that the model is determinacy-inadmissible. Then determinacy is not possible. If $\Omega^1(s_t)$ is complex-valued and $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) < 1$, there are more than one stable MOD solution. Therefore, the model is indeterminate. If $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) \geq 1$, the model has no stable solution by the same reason as above. The solution set is exhaustively decomposed into these three mutually disjoint cases. Therefore, the conditions for the three cases are necessary as well as sufficient. In the case of LRE models, $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) = r((\Omega^1)^2) < 1$ if and only if $r(\Omega^1) < 1$, and $r(\Psi_{F^1\otimes F^1}) = r((F^1)^2) < 1$ if and only if $r(F^1) < 1$. Therefore, the proofs apply to Result 3 in Section 2 as well. $Q.E.D.$

D Proof of Corollary 1

Suppose that a MSRE model is determinacy-admissible such that $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1})r(\Psi_{F^1\otimes F^1}) < 1$. Uniqueness of a stable solution implies that it is a real-valued MOD solution and $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) < r(\bar{\Psi}_{\Omega^j\otimes\Omega^j})$ for all $j \geq 2$. Also, $r(\bar{\Psi}_{\Omega^j\otimes\Omega^j})r(\Psi_{F^1\otimes F^1}) \geq 1$ from Proposition 2. When the inequality is strict, it is possible to have $r(\bar{\Psi}_{\Omega^1\otimes\Omega^1}) < 1 \leq r(\bar{\Psi}_{\Omega^j\otimes\Omega^j})$ and $r(\Psi_{F^1\otimes F^1}) > 1$. Therefore, proof is completed by finding such an example, which is given in Section 4. Or one can always construct such an example. To do so, first generate an arbitrary determinacy-admissible model such that $r(\bar{\Psi}_{\Omega^j\otimes\Omega^j})r(\Psi_{F^1\otimes F^1}) > 1$ for $j = 2$. With a given such a model, let $\alpha = (r(\Psi_{F^1\otimes F^1})/r(\bar{\Psi}_{\Omega^j\otimes\Omega^j}))^{1/4}$ where $j = 2$. Define a transformed model (37) such that $\tilde{A}(\cdot) = A(\cdot)/\alpha$ and $\tilde{B}(\cdot) = \alpha B(\cdot)$. Let $\tilde{\Omega}$ and $\tilde{F}$ be the matrices corresponding to Ω and F. Then one can show that

$$
\begin{aligned}
r(\bar{\Psi}_{\tilde{\Omega}^1\otimes\tilde{\Omega}^1}) &= \left(\frac{r(\bar{\Psi}_{\Omega^1\otimes\Omega^1})}{r(\bar{\Psi}_{\Omega^j\otimes\Omega^j})}\right)^{1/2}[r(\bar{\Psi}_{\Omega^1\otimes\Omega^1})r(\Psi_{F^1\otimes F^1})]^{1/2} < 1 \\
&< r(\bar{\Psi}_{\tilde{\Omega}^j\otimes\tilde{\Omega}^j}) = [r(\bar{\Psi}_{\Omega^j\otimes\Omega^j})r(\Psi_{F^1\otimes F^1})]^{1/2}
\end{aligned}
$$

Therefore, the transformed model has a unique stable MOD solution. Nevertheless, such a transformed model has always a continuum of mean-square stable sunspot solutions because $r(\Psi_{\tilde{F}^1\otimes\tilde{F}^1}) = r(\bar{\Psi}_{\tilde{\Omega}^j\otimes\tilde{\Omega}^j}) > 1$. Therefore, uniqueness of a mean-square stable MSV solution does not imply determinacy in MSRE models. $Q.E.D.$

E Modified Forward Method

The original forward method can fail to identify the MOD solution on its own. Nevertheless, the only known example is a block recursive model with a particular structure such as the one analyzed in Section 4.[28] The modification is to recover the equivalence of the forward solution and the MOD solution. Prior to a formal treatment, it is instructive to understand the rationale behind the modified forward method via this example.

The original forward method always yields the monetary solution ignoring fiscal block, thus the forward solution cannot be the MOD solution when fiscal policy is more active than monetary policy. This amounts to solving the monetary block forward separately because it is insulated from the fiscal block behaviorally. Therefore, expected inflation is computed using the state variables in the monetary block only. Generally speaking, only a subset of state variables can be used in this type of model. Henceforth, the modification is to reformulate the model such that full information is used to form expectations regardless of the model structure. That is, $E_t\pi_{t+1}$ must be consistent with the expectational relation of the fiscal block $E_t b_{t+1} = \theta b_t - cE_t\pi_{t+1}$. The idea is to add this expectational relation with an auxiliary parameter h as a restriction so that the model can be rewritten as:

$$\pi_t = (1/a)E_t\pi_{t+1} + h(E_t b_{t+1} - \theta b_t + cE_t\pi_{t+1}), \tag{59a}$$

$$b_t = \theta b_{t-1} - c\pi_t. \tag{59b}$$

Then, b_{t-1} in the second equation is added to the information set with which the agents in the first equation form expectations. The reformulated model (59) becomes no longer block-recursive. Nevertheless, the MSV solutions to (59) must be invariant to this adjustment because they must be consistent with the original equation. The matrix F does depend on h, but their spectral radius remain unchanged as well. Applying the forward method to this adjusted system of equation (59a) and (59b) opens up the possibility that b_t affects π_t. Indeed, under this approach, the forward solution converges to Ω^F if and only if $\theta > \alpha$. Therefore, one may interpret that the original forward method is consistent with the view of monetary dominance whereas the modified forward method is consistent with the FTPL perspective.

[28] Obviously, the forward solution does not exist to a model with a complex-valued MOD solution – thus determinacy-inadmissible – because the solution is defined as a limiting value of a sequence of real-valued parameters of a given model.

E.1 Modified Forward Method for MSRE Models

A vast majority of macroeconomic models does not possess the block recursive structure. Therefore, it is efficient to apply the original forward method and only if it fails to identify the MOD solution, use the forward method under full information. This is the modified forward method.

Summary of Original Forward Method of Cho (2016) A model (21) can be solved forward yields as:

$$x_t = E_t[M_k(s_t, s_{t+1}, \ldots, s_{t+k})x_{t+k}] + \Omega_k(s_t)x_{t-1}, \tag{60}$$

where $\Omega_1(s_t) = B(s_t)$, $F_1(s_t, s_{t+1}) = M_1(s_t, s_{t+1}) = A(s_t, s_{t+1})$ and for $k \geq 2$,

$$\Omega_k(s_t) = \{I_n - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]\}^{-1}B(s_t), \tag{61a}$$

$$F_k(s_t, s_{t+1}) = \{I_n - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]\}^{-1}A(s_t, s_{t+1}). \tag{61b}$$

The formula for the forward solution is given by $x_t = \Omega^*(s_t)x_{t-1}$ where $\Omega^*(s_t) = \lim_{k\to\infty}\Omega_k(s_t)$. The corresponding $F^*(s_t, s_{t+1})$ can be computed by the formula in equation (25). Note that this is the only solution that satisfies the so-called no bubble condition $E_t[M_k(s_t, s_{t+1}, \ldots, s_{t+k})x_{t+k}] = 0_{n\times 1}$. If $r(\bar{\Psi}_{\Omega^*\otimes\Omega^*})r(\Psi_{F^*\otimes F^*}) < 1$, the forward solution is the MOD solution. Only when this does not hold, apply the forward method under full information as follows.

Forward Method under Full Information The modified forward method is not literally different from the original one. It is to adjust the model such that full information of the state variables must be used to form expectations of the original model. In fact, Cho (2016) has already discussed this idea informally. But this paper formalize this idea in general setup as follows. To do so, define the expectational variables as $k_t = E_t[A(s_t, s_{t+1})x_{t+1}]$. Forward the model (21) one period ahead and take expectations as:

$$E_t[x_{t+1}] = E_t[k_{t+1}] + E_t[B(s_{t+1})]x_t. \tag{62}$$

$E_t[B(s_{t+1})]$ at each regime s_t can be easily computed by $\sum_j^S P(i,j)B(s_{t+1} = j)$ at each $s(t) = i = 1, ..., S$. Then we can augment this expectational relation to the original model as a constraint as:

$$
\begin{aligned}
x_t &= k_t + B(s_t)x_{t-1}, && (63a)\\
k_t &= E_t[A(s_t, s_{t+1})x_{t+1}] + H(E_t[x_{t+1}] - E_t[k_{t+1}] - E_t[B(s_{t+1})]x_t), && (63b)
\end{aligned}
$$

where H is an $n \times n$ matrix in which every single element is arbitrary but non-zero. Applying the expectational relation of the whole model is innocuous because it must hold regardless of block-recursiveness of the model. Let $y_t = [x_t' \; k_t']'$ be the $2n \times 1$ vector. Notice that $E_t[B(s_{t+1})]$ is a function of s_t. Collecting the coefficient matrices yields:

$$
\begin{bmatrix} I_n & -I_n \\ HE_t[B(s_{t+1})] & I_n \end{bmatrix} y_t = E_t\left[\begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ A(s_t, s_{t+1}) + H & -H \end{bmatrix} y_{t+1}\right] + \begin{bmatrix} B(s_t) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} y_{t-1}.
$$

Finally, by multiplying the inverse of the coefficient matrix of y_t we have the following form of the model:

$$
y_t = E_t[A^y(s_t, s_{t+1})y_{t+1}] + B^y(s_t)y_{t-1}. \tag{64}
$$

The modified forward method is to apply the original forward method to (64) and obtain the solution for y_t.

$$
y_t = (\Omega^y(s_t))^* y_{t-1}. \tag{65}
$$

The solution $(\Omega^y(s_t))^*$ is $2n \times 2n$ and since x_t is the first $n \times 1$ subvector of y_t, the first $n \times n$ component of $(\Omega^y(s_t))^*$ is the forward solution of the original model under full information.

E.2 Existence of the Forward Solution

Both the original model (21) and the adjusted model (64) has the same form. Therefore, we can examine the existence condition for the forward solution using the same formula as (61a) and (61b). Under the modified forward method, $\Omega_k(s_t)$ can be interpreted as $\Omega_k^y(s_t)$ in (65). The condition for the existence of the forward solution can be understood by differentiating and vectoring (61a) such that:

$$
\begin{bmatrix} vec(d\Omega_k(1)) \\ \dots \\ vec(d\Omega_k(S)) \end{bmatrix} = \begin{bmatrix} \Psi_{\Omega_{k-1}' \otimes F_{k-1}} \end{bmatrix} \begin{bmatrix} vec(d\Omega_{k-1}(1)) \\ \dots \\ vec(d\Omega_{k-1}(S)) \end{bmatrix}. \tag{66}
$$

The expression in the right-hand side shows that there are two possibilities in convergence of Ω_k. Formally,

Proposition 4 *Consider the forward representation of a MSRE model, (60). Then the forward solution Ω^* exists in the following cases.*

1. $r(\Psi_{\Omega'_{k-1}\otimes F_{k-1}}) < 1$ *for all* $k > 1$.
2. $r(\Psi_{\Omega'_{k-1}\otimes F_{k-1}}) \geq 1$ *for some* $k > 1$ *and* $u'_k vec(d\Omega_{k-1}) = 0_{n^2S\times 1}$ *for every eigenvector* u_k *associated with an unstable root of* $\Psi_{\Omega'_{k-1}\otimes F_{k-1}}$.

Proof. In both cases, $\lim_{k\to\infty} d\Omega_k(s_t) = 0_{n^2\times 1}$ for all s_t, implying the existence of Ω^*. *Q.E.D.* ■

Under the original forward method, Proposition 4 states that the equivalence of the forward solution and the MOD solution may break down only in the second case. In that case, even if $\Psi_{\Omega'_{k-1}\otimes F_{k-1}}$ contains a root larger than unity, it does not affect Ω_k, thus the forward solution can still exist. This is precisely what happened in the FTPL model with a PM-AF policy mix. In the LRE case, $\Omega^* = \Omega^M$ with $(\Omega^M \otimes F^M)vec(d\Omega^M) = 0_{4\times 1}$ with $k = 1$, despite $r(\Omega^M) \otimes r(F^M) = \theta/\alpha > 1$. The forward method under full information is to adjust the model such that Case 2 of Proposition 4 does not arise in the class of determinacy-admissible models.[29] Then Case 1 implies that in the MSRE framework, the forward solution exists for a broader class of models including determinacy-inadmissible models because, $r(\Psi_{\Omega^{*\prime}\otimes F^*}) < 1$ can be consistent with $r(\bar{\Psi}_{\Omega^*\otimes\Omega^*})r(\Psi_{F^*\otimes F^*}) \geq 1$. That is, if $r(\bar{\Psi}_{\Omega^*\otimes\Omega^*})r(\Psi_{F^*\otimes F^*}) < 1$, then $r(\Psi_{\Omega^{*\prime}\otimes F^*}) < 1$, but the converse is not true in general. This implies that the condition for the existence of the forward solution is much weaker than its mean-square stability $r(\bar{\Psi}_{\Omega^*\otimes\Omega^*}) < 1$. An extensive experiment so far has never found a single case in which the forward solution is not the MOD solution in atheoretical and economic examples. Nevertheless, the equivalence to a model with a real-valued MOD solution is an open question to be explored in the future.

To summarize, the modified forward method would be sufficient for analysis of virtually all economic models in practice. Only when the forward solution does not exist or $r(\bar{\Psi}_{\Omega^*\otimes\Omega^*})r(\Psi_{F^*\otimes F^*}) \geq 1$, one may need to apply the Gröbner basis approach to identify the MOD solution.

[29] When a model has completely decoupled equations, Case 2 still exists even under the forward method under full information. The example in Section 4 when $c = 0$ is such a case. However, recall that in that case, the solution with the smallest generalized eigenvalues does not exists, thus the model is determinacy-inadmissible. Moreover, the forward solution coincides with the MOD solution.

References

Barthélemy, Jean and Magali Marx, "Solving endogenous regime switching models," *Journal of Economic Dynamics and Control*, 2017, *77*, 1–25.

Bhattarai, Saroj, Jae Won Lee, and Woong Yong Park, "Inflation dynamics: The role of public debt and policy regimes," *Journal of Monetary Economics*, 2014, *67*, 93–108.

Blanchard, Olivier J. and Charles M. Kahn, "The Solution of Linear Difference Models Under Rational Expectations," *Econometrica*, 1980, *48*, 1305–1311.

Cho, Seonghoon, "Sufficient conditions for determinacy in a class of Markov-switching rational expectations models," *Review of Economic Dynamics*, 2016, *21*, 182–200.

_ **and Antonio Moreno**, "The Forward Method as a Solution Refinement in Rational Expectations Models," *Journal of Economic Dynamics and Control*, 2011, *35* (3), 257–272.

_ **and** _ , "Has Fiscal Policy Saved the Great Recession?," *Working Paper*, 2019.

_ **and Bennett T McCallum**, "Refining linear rational expectations models and equilibria," *Journal of Macroeconomics*, 2015, *46*, 160–169.

Costa, Oswaldo Luiz V., Marcelo D. Fragoso, and Ricardo P. Marques, *Discrete Time Markov Jump Linear Systems*, Springer, New York, 2005.

Davig, Troy and Eric M. Leeper, "Generalizing the Taylor Principle," *American Economic Review*, 2007, *97* (3), 607–635.

Decker, Wolfram, Gert-Martin Greuel, Gerhard Pfister, and Hans Schönemann, "Singular 4-1-2 — A computer algebra system for polynomial computations," 2019.

Evans, George W. and Seppo Honkapohja, *Learning and Expectations in Macroeconomics*, Princeton University Press, 2001.

Farmer, Roger E.A., Daniel F. Waggoner, and Tao Zha, "Understanding Markov-Switching Rational Expectations Models," *Journal of Economic Theory*, 2009, *144* (5), 1849–1867.

_ , _ , and _ , "Generalizing the Taylor Principle: Comment," *American Economic Review*, 2010, *100* (1), 608–617.

_ , _ , and _ , "Minimal State Variable Solutions to Markov-Switching Rational Expectations Models," *Journal of Economic Dynamics and Control*, 2011, *35* (12), 2150–2166.

Foerster, Andrew, Juan F Rubio-Ramírez, Daniel F Waggoner, and Tao Zha, "Perturbation methods for Markov-switching dynamic stochastic general equilibrium models," *Quantitative Economics*, 2016, *7* (2), 637–669.

King, Robert G. and Mark W. Watson, "The Solution of Singular Linear Difference Systems Under Rational Expectations," *International Economic Review*, 1998, *39*, 1015–1026.

Klein, Paul, "Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model," *Journal of Economics Dynamics and Control*, September 2000, *24* (10), 1405–1423.

Leeper, Eric M., "Equilibria under 'active' and 'passive' monetary and fiscal policies," *Journal of Monetary Economics*, 1991, *27* (1), 129–147.

Maih, Junior, "Efficient perturbation methods for solving regime-switching DSGE models," *Norges Bank Working Paper*, 2015.

McCallum, Bennett T., "E-stability Vis-a-Vis Determinacy Results for a Broad Class of Linear Rational Expectations Models," *Journal of Economic Dynamics and Control*, 2007, *31* (4), 1376–1391.

McClung, Nigel, "E-stability vis-a-vis Determinacy in Markov-Switching DSGE Models," *Working Paper, Available at SSRN: https://ssrn.com/abstract=3393007*, 2019.

Sims, Christopher A., "Solving Linear Rational Expectations Models," *Computational Economics*, 2002, *20* (1), 1–20.

Sims, Christopher A, "On the genericity of the winding number criterion for linear rational expectations models," Technical Report, Citeseer 2007.

Uhlig, Harald, "A toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily in Ramón Marimón and Andrew Scott , Ed. *Computational Methods for the Study of Dynamic Economies*, Oxford Universtiy Press," 1997, pp. 30–61.