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Determinacy and classification of Markov-switching rational expectations models☆

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ABSTRACT

In a general class of Markov-switching rational expectations models, this study derives necessary and sufficient conditions for determinacy, indeterminacy and the case of no stable solution. Classification of the models into these three mutually disjoint and exhaustive subsets is completely characterized by only one particular solution known as the minimum of modulus solution in the mean-square stability sense. The rationale behind this result comes from the novel finding that the solution plays the same role as what the generalized eigenvalues do for linear rational expectations models. Moreover, the solution has its own identification condition that does not require examining the entire solution space. The accompanying solution procedure is therefore computationally efficient, and as tractable as standard solution methodologies for linear rational expectations models. The proposed methodology unveils several important implications for determinacy in the regime-switching framework that differ from the linear model counterpart.

1. Introduction

The Great Moderation, a period of lower macroeconomic volatilities, and the Zero Lower Bound era are symbolic examples of some of the well-known economic regimes observed in the modern economy. A large number of papers have studied the underlying forces of those regime shifts and the transitional dynamics using the standard time series models shaped by the seminal work of Hamilton (1989). This approach has also been complemented by Markov-switching rational expectations (MSRE) models, which provide deeper economic accounts of recurrent regime shifts in terms of changes in the behavior of private agents and policymakers, as well as volatilities of structural shocks (See Bianchi, 2013; Davig and Leeper, 2007; Sims and Zha, 2006; Svensson et al., 2008, and Baele et al., 2015 for instance). For the analysis of these economic models to be relevant, however, the existence and uniqueness of a stable rational expectations equilibrium – determinacy – must be clearly understood.

While there has been significant progress in the literature, necessary and sufficient conditions for determinacy have not been clearly identified for general regime-switching models. Moreover, different stability concepts lead to different determinacy results. Farmer et al. (2009) adopts mean-square stability and derives those conditions for purely forward-looking mod-

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els only. Using the same concept of stability, Cho (2016) derives the conditions for general models including lagged endogenous variables, but they are only sufficient. Barthélemy and Marx (2019) also derive a necessary and sufficient condition for a unique bounded solution when models are purely forward-looking, and a sufficient condition when predetermined variables are present in the model. Using an alternative stability concept based on the Lyapunov spectrum, Neusser (2019) proposes a condition for determinacy, but it is only necessary.

The present work adopts mean-square stability as a stability concept among alternatives following Farmer et al. (2009), on the grounds of tractability of the methodology and feasibility of econometric inference. Then we provide a complete classification result for a general class of MSRE models, encompassing the linearized system of dynamic stochastic general equilibrium (DSGE) models with regime-switching structural parameters. Specifically, we derive both necessary and sufficient conditions for a unique stable solution (determinacy), multiple stable solutions (indeterminacy) and no stable solution. The classification of MSRE models – partitioning into those three mutually disjoint and exhaustive subsets – is fully characterized by only one particular rational expectations solution, which is one of the minimum state variable (MSV) solutions. This is a very powerful result given that these regime-switching models are inherently non-linear and the full set of solutions is too large to identify.[1]

The novelty of our approach lies in the finding that the solution – referred to as the minimum of modulus (*MOD*) solution in the spirit of McCallum (2007) – to MSRE models plays an equivalent role as what generalized eigenvalues do for linear rational expectations (LRE) models. Specifically, it provides the conditions for the non-existence of stable sunspots and the uniqueness of a stable MSV (or fundamental) solution. These conditions are shown to completely classify all MSRE models. Another important feature is that there is a simple identification condition for the *MOD* solution, hence, the full set of solutions need not be computed for examining their stability. This is the key factor that makes our methodology including the solution technique as tractable as the standard methods for LRE models such as those of Blanchard and Kahn (1980), Uhlig (1997), Klein (2000) or Sims (2002). There exists, however, a subset of MSRE models for which the identification condition fails and in such cases, our solution procedure may not be implementable. Fortunately, it will be shown that the failure of the condition is equivalent to the statement that determinacy can never arise: a model is either indeterminate or it has no stable solution. Except for such cases, our proposed methodology is shown to be computationally very efficient.

Our methodology uncovers several new equilibrium properties of MSRE models. First, the parameter space implied by determinacy, indeterminacy and no stable solution is neither necessary nor sufficient for each of the LRE model counterparts. Second, a unique stable MSV solution does not imply determinacy in a regime-switching model with lagged endogenous variables. This is a general phenomenon arising from non-linearity of the model, highlighting that the non-existence of stable sunspots must also be examined to correctly identify determinacy. Third, the long-run Taylor principle of Davig and Leeper (2007), an important equilibrium characteristic in regime-switching models, is only necessary for determinacy in the mean-square stability sense. These findings are illustrated through a simple model allowing shifts in monetary and fiscal policy stances.

This study builds upon and complements previous research. Farmer et al. (2009) show that a purely forward-looking Markov-switching model is determinate if and only if there is no stable sunspot. Their approach to find all sunspots and examine their stability is, however, difficult to implement because the sunspot space is extremely large to identify. The uniqueness of a stable MSV solution must also be verified for models with lagged endogenous variables for determinacy. Farmer et al. (2011) and Maih (2015) propose efficient numerical solution techniques for computing some, but not necessarily all of MSV solutions. The Gröbner basis technique of Foerster et al. (2016) can identify the uniqueness of a stable MSV solution by computing all of the MSV solutions, but it is computationally very demanding even for modest dimensional models because the number of MSV solutions is surprisingly large. More importantly, the unique stable MSV solution alone does not imply determinacy as mentioned above. The determinacy and indeterminacy conditions proposed by Cho (2016) are very similar to ours, but they are only sufficient because the conditions are derived by using the forward solution, which does not always coincide with the *MOD* solution. Therefore, the method is silent about the model classification when these conditions are not met. Our approach – referred to as the *MOD* method in what follows – resolves all of these problems by using just two function values of the *MOD* solution, which is always well-defined.

This paper is organized as follows. Section 2 introduces a simple regime-switching model to provide an intuition behind our methodology. Section 3 presents the class of MSRE models, solutions and examines their mean-square stability. Section 4 formally develops our methodology and presents the main results for MSRE models, including an efficient solution procedure. In Section 5, we apply our methodology to analyze the economic example introduced in Section 2.[2] Section 6 concludes.

2. Example

This section introduces a highly stylized model with regime-switching monetary and fiscal policies from the perspective of the fiscal theory of the price level (FTPL). While very simple, this example is rich enough to have all three cases of determinacy, indeterminacy and no stable solution, and helps understand the major issues addressed above and the key equilib-

[1] Comparison of determinacy results across different stability concepts is beyond the scope of this paper. We leave it as a future research agenda.

[2] All of the computer codes for our proposed method, examples and a technical guide accompanying this paper can be found at https://sites.google.com/site/sc719en.

rium characteristics of regime-switching models. We sketch the key idea of our approach using this example. A rigorous determinacy analysis will be presented after the methodology is developed. In a companion paper, Cho and Moreno (2020) analyze the extended FTPL models to study the equilibrium properties with an emphasis on regime-switching to and from the zero lower bound.

Consider a New-Keynesian model that is augmented by a government budget constraint that incorporates a tax policy. The essential feature of this type of model can be succinctly illustrated by a simple example, abstracting from other important features to focus only on determinacy. A monetary block representing a standard New-Keynesian model consists of a Fisher equation $i_t = E_t\pi_{t+1} + r_t$ and a Taylor-type rule $i_t = \alpha\pi_t$. i, r and π denote the nominal and real interest rates, and inflation, respectively. $E_t(\cdot)$ is the mathematical expectation conditional on the information available at time t. Assuming the real interest rate to be *i.i.d.* and thus exogenous, the monetary block can be written as $\alpha\pi_t = E_t\pi_{t+1} + r_t$. A fiscal block is a linearized government budget constraint augmented by a tax policy such that $b_t = (1/\beta)b_{t-1} - \tau_t - c\pi_t$ where b_t is the government's debt to output ratio, and the tax policy is given by $\tau_t = \delta b_{t-1}$. β is the time discount factor and $c \neq 0$. The fiscal policy can be expressed as $b_t = (1/\beta - \delta)b_{t-1} - c\pi_t$ by substituting out the tax policy. This is a simplified version of Leeper (1991) or Bhattarai et al. (2014). Following the latter, we specify $c = \bar{b}(1/\beta - \alpha)$ where $\bar{b}$ is the steady state value of b_t.

We extend the model to allow the monetary and fiscal authorities to switch their policy stances over the two regimes in terms of the parameters $\alpha(s_t)$ and $\delta(s_t)$ where s_t is a Markov chain and the transition probability switching from regime $s_t = i$ to $s_{t+1} = j$ is denoted by p_{ij} such that $\sum_{j=1}^{2} p_{ij} = 1$ for $i, j = 1, 2$. The model can then be written as:

$$\pi_t = \frac{1}{\alpha(s_t)}E_t\pi_{t+1} + \frac{1}{\alpha(s_t)}r_t, \tag{1a}$$

$$b_t = \theta(s_t)b_{t-1} - c(s_t)\pi_t, \tag{1b}$$

where $\theta(s_t) = 1/\beta - \delta(s_t)$ and $c(s_t) = \bar{b}(1/\beta - \alpha(s_t))$ for all $t \geq 1$ and b_0 is initially given. The monetary block, (1a) is by itself a well-defined rational expectations model if treated in isolation because it is behaviorally independent of the fiscal variables. In contrast, b_t in the fiscal block, (1b), does depend on inflation. For this reason, this type of model has a block-recursive structure. However, the evolution of inflation may well be influenced by the fiscal policy in equilibrium through expected inflation: $c(s_t)$ opens up the expectational channel through which agents may form inflation expectation based on the fiscal dynamics.

The famous taxonomy of Leeper (1991) describes the classification result in terms of monetary and fiscal policy stances for the regime-independent case. To be more specific, assume that $\alpha, \theta \geq 0$. Monetary policy is referred to as active (AM) if $\alpha \geq 1$ and passive(PM) otherwise. Fiscal policy is called active (AF) if $\theta \geq 1$ and passive (PF) otherwise.[3] It is well-known that the model is determinate with an AM-PF policy mix or a PM-AF policy mix.[4] It is indeterminate when both policies are passive (PM-PF), and has no stable solution when both policies are active (AM-AF).

The idea of this paper can be sketched using this model as follows. Our strategy is to state and derive the same taxonomy result in terms of the *MOD* solution. To shed light on what it means by this particular solution, we first consider the solutions that depend on the minimum state variables b_{t-1} and r_t. It can be shown that the model has two such solutions as follows, irrespective of stability:

$$\begin{array}{ll} \text{Monetary Solution} & \text{Fiscal Solution} \\ \pi_t = \frac{1}{\alpha}r_t, & \pi_t = \frac{\theta-\alpha}{c}b_{t-1} + \frac{1}{\theta}r_t, \\ b_t = \theta b_{t-1} - \frac{c}{\alpha}r_t. & b_t = \alpha b_{t-1} - \frac{c}{\theta}r_t. \end{array} \tag{2}$$

The first one is monetary in the sense that inflation evolves independent of the fiscal variable. In contrast, the second one is fiscal as inflation does depend on the government debt to GDP ratio. By letting $x_t = [\pi_t \ b_t]'$, these solutions can be written as $x_t = \Omega x_{t-1} + \Gamma r_t$ where the coefficient matrices for each MSV solution are:

$$\Omega^M = \begin{bmatrix} 0 & 0 \\ 0 & \theta \end{bmatrix}, \quad \Gamma^M = \begin{bmatrix} \frac{1}{\alpha} \\ -\frac{c}{\alpha} \end{bmatrix} \tag{3a}$$

$$\Omega^F = \begin{bmatrix} 0 & \frac{\theta-\alpha}{c} \\ 0 & \alpha \end{bmatrix}, \quad \Gamma^F = \begin{bmatrix} \frac{1}{\theta} \\ -\frac{c}{\theta} \end{bmatrix} \tag{3b}$$

The superscripts M and F represent the monetary and fiscal solution, respectively. It is easy to see that stability of the two MSV solutions are governed by the magnitude of θ and α, respectively. The *MOD* solution in this example is the MSV solution that has the smallest maximum eigenvalue of Ω, $\min(\theta, \alpha)$. Note that it can be monetary or fiscal depending on the relative policy stances. It is evident that the unique stable solution must be the *MOD* solution if the model is determinate.

[3] Following convention, the fiscal policy stance can also be stated in terms of the tax policy coefficient δ: the fiscal policy is active (passive) if $\delta \leq (>)1/\beta - 1$.

[4] Leeper (1991) uses boundedness as a stability concept. But as shown by Farmer et al. (2009), mean-square stability lead to the same taxonomy result in LRE models as long as shocks are covariance stationary and bounded.

Next, we need to examine the non-existence of stable sunspot solutions to check determinacy. As will be shown in Section 3, any solution can be written as a sum of a MSV component and a sunspot process w_t such that $x_t = \Omega x_{t-1} + \Gamma r_t + w_t$ where w_t is subject to the restriction $w_t = FE_t w_{t+1}$ with F being uniquely defined for every single MSV solution. Since the restriction is by itself a purely forward-looking model, stability of sunspots can be identified by the largest eigenvalue of F. It will be shown that the matrix F associated with each MSV solution is given by:

$$F^M = \begin{bmatrix} \frac{1}{\alpha} & 0 \\ -\frac{c}{\alpha} & 0 \end{bmatrix}, \qquad F^F = \begin{bmatrix} \frac{1}{\theta} & 0 \\ -\frac{c}{\theta} & 0 \end{bmatrix}. \tag{4}$$

Determinacy can then be stated as follows. First, the *MOD* solution is stable. Second, there must be no stable sunspot process. Third, there must be no other stable MSV solutions. The novel finding of this paper is that F associated with the *MOD* solution answers the last two requirements for determinacy. For example, consider the case of $\alpha > \theta$. Then, the *MOD* solution is monetary and it is stable if $\theta < 1$ from (3). The largest eigenvalue of F^M is $1/\alpha$. Therefore, there is no stable sunspot associated with the monetary solution if $\alpha \geq 1$. Note that this condition also implies that the other MSV (fiscal) solution is explosive from (3b) (Later, it will be shown that all sunspot solutions associated with the fiscal solution are also unstable). Therefore, the model is determinate with $\theta < 1 \leq \alpha$, which is precisely the AM-PF policy mix case. These two parameter values can also identify indeterminacy and the case of no stable solution.[5] The remainder of this paper is to generalize this idea in the class of regime-switching models. The only difference is that the *MOD* solution is defined in the mean-square stability sense.

Model (1) also helps understand the equilibrium characteristics to be answered by our proposed methodology. First, a complete classification has not been identified even for this type of simple regime-switching model. As mentioned above, Cho (2016) and Barthélemy and Marx (2019) provide only sufficient conditions for determinacy and indeterminacy in mean-square stability and boundedness, respectively. We illustrate the classification of Model (1) by our proposed methodology. Second, uniqueness of a stable MSV solution does not imply determinacy for regime-switching models with lagged endogenous variables because there can still exist a continuum of stable sunspot solutions. In an extended FTPL model, Ascari et al. (2020) identify the policy mixes ensuring the unique mean-square stable MSV solution. But such policy mixes may be subject to indeterminacy. That is, a unique stable solution is necessary but not sufficient for determinacy. Indeed, we show that model (1) can be indeterminate with a unique stable MSV solution, despite that switching occurs over the two different policy mixes that would lead to determinacy under fixed regimes – an AM-PF policy mix and a PM-AF policy mix. Finally, the long-run Taylor principle (LRTP) is related to stability in the first moment, which is strictly weaker than mean-square stability under regime-switching. This explains why the LRTP is only necessary for determinacy. Using the example, we quantify the difference between determinacy and the LRTP.

3. Markov-switching rational expectations models

This section presents the class of regime-switching models and the set of rational expectations solutions. Then we introduce the notion of mean-square stability. Finally, the *MOD* solution and determinacy are formally defined under mean-square stability.

3.1. MSRE Models

The class of MSRE models considered in this paper is similar to that of Cho (2016). The model can be expressed as:

$$x_t = E_t[A(s_t, s_{t+1})x_{t+1}] + B(s_t)x_{t-1} + C(s_t)z_t, \tag{5a}$$

$$z_t = R(s_t)z_{t-1} + G(s_t)\epsilon_t, \tag{5b}$$

where x_t is an $n \times 1$ vector of endogenous variables, z_t is an $m \times 1$ vector of exogenous variables, ϵ_t is an $l \times 1$ vector of structural shocks and s_t is an ergodic Markov chain switching over S different regimes for all $t \geq 1$.[6] x_0 and z_0 are initially given. The transition probability switching from regime i to j is denoted by $p_{ij} = \Pr(s_{t+1} = j | s_t = i)$ such that $\sum_{j=1}^{S} p_{ij} = 1$ for all $i, j \in \{1, 2, \ldots, S\}$. P is the transition probability matrix, for which the (i, j)-th element is p_{ij}. $E_t[\cdot]$ is the mathematical expectation conditional on all of the information available at time t. All coefficient matrices are conformable with appropriate dimensions at all regimes. In general, $A(\cdot)$ may well be singular as not all of the endogenous variables are forward-looking. In our setup, any variable observed at time t is denoted by the subscript t, and it is not required to distinguish the

[5] An analogous result can also be obtained when $\alpha < \theta$, meaning that the *MOD* solution is fiscal. There are two identical *MOD* solutions when $\alpha = \theta$, thus the model cannot be determinate. For this reason, this type of models will be referred to as determinacy-inadmissible. Section 5 analyzes all of these cases.

[6] Model (5) is general and flexible enough to accommodate models with any number of leads and lags of endogenous and exogenous variables, because a family of vector autoregressive moving average processes of arbitrary finite orders can be transformed into a VAR(1) form. The model can also be written without separating the exogenous variables. For instance, it can be written in terms of $y_t = [x_t' \;\; z_t']'$ and the exogenous shock ϵ_t, but at the cost of increasing the model dimension.

predetermined and non-predetermined variables. A standard LRE model is nested as a special case of (5) in the absence of regime-switching.

The model can be interpreted as a first-order approximation of an otherwise standard dynamic stochastic general equilibrium (DSGE) model in which structural parameters are regime-switching, around the steady state of the endogenous and exogenous variables at each given regime. The model also encompasses linear rational expectations models with regime-switching coefficients. A Markov process s_t is an exogenous and additional source of uncertainty, independent of the structural shock ϵ_t. In this paper, Model (5) is referred to as quasi-linear in the sense that the effect of independent regime-switching occurs through the coefficients. This is consistent with Foerster et al. (2016), who provide a perturbation method to derive a first-order approximation of the original non-linear DSGE model while preserving the regime-switching nature of the underlying model.[7] Note also that the matrix $A(\cdot)$ may well depend on the future state s_{t+1}. As Foerster et al. (2016) and Cho (2016) have emphasized, this is a natural characteristic of DSGE models with microfoundations subject to regime-switching because agents take into account the possibility of future regime switching in their optimal decision rules. Specific examples in those studies include optimal consumption decisions of households at time t facing regime-switching real interest rate or intertemporal substitution at time $t+1$.

To complete the model and make it consistent with the perspective of quasi-linearity, we impose the following assumption.

Assumption 1. s_t is an ergodic Markov chain. x_0, z_0 and s_t are independent of ϵ_t for all $t \geq 1$. The structural shock ϵ_t is asymptotically covariance stationary.

Ergodicity of Markov chain is required for x_t and z_t to have well-defined steady states, which are assume to be zero in this model.[8] As will be discussed below, Assumption 1 is a prerequisite for our analysis to be consistent with mean-square stability. If alternative stability concept such as boundedness is adopted, boundedness must be assumed for the shock processes. Barthélemy and Marx (2017) also show that boundedness is required for their perturbation method.

3.2. Rational Expectations Solutions

In this study, a rational expectations solution is defined as any stochastic process that solves model (5), irrespective of its stability. Our strategy is to decompose every solution x_t into two parts as an identity, $x_t \equiv v_t + w_t$ where v_t is an MSV component and w_t is a sunspot process. As shown by Cho (2016) in his Proposition 1, any solution to (5) can be written as a sum of these two parts such that:

$$x_t = [\Omega(s_t)x_{t-1} + \Gamma(s_t)z_t] + w_t, \tag{6}$$

$$w_t = E_t[F(s_t, s_{t+1})w_{t+1}], \tag{7}$$

where $\Omega(s_t)$, $\Gamma(s_t)$ and $F(s_t, s_{t+1})$ must obey the following restrictions for all $s_t, s_{t+1} = 1, .., S,\ t \geq 1$,

$$\Omega(s_t) = \{I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]\}^{-1}B(s_t), \tag{8}$$

$$\Gamma(s_t) = \{I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]\}^{-1}[C(s_t) + A(s_t, s_{t+1})\Gamma(s_{t+1})R(s_{t+1})], \tag{9}$$

$$F(s_t, s_{t+1}) = \{I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]\}^{-1}A(s_t, s_{t+1}). \tag{10}$$

Unlike the constant-regime models, s_t is included in the set of the minimum state variables as well as x_{t-1} and z_t. Formally, we define a MSV solution as follows.

Definition 1. A stochastic process $x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)z_t$ given by (6), (8) and (9) with $w_t = 0_{n\times 1}$ is referred to as a minimum state variable (MSV) solution to the Markov-switching rational expectations model (5).

A MSV solution is also referred to as a fundamental solution (see Farmer et al., 2011 for instance) in the sense that the rational expectations $E_t[x_{t+1}]$ is completely pinned down by a function of the minimum state variables defined in the model. Put differently, a non-zero stochastic process w_t is referred to as a sunspot (or non-fundamental) component because the presence of a non-zero stochastic process w_t in (6) implies that $E_t[x_{t+1}]$ is affected by a stochastic sunspot process as well as the minimum state variables. Note that the invertibility of $I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]$ is included as a requirement for a MSV

[7] Their equation (A.9) in Proposition 4 can be cast in the form of (5) by observing that $E_t[A_i(\theta(s_t), \theta(s_{t+1}))]$ is a function of s_t for $i \geq 2$.

[8] The model can also accommodate regime-switching drift terms and non-zero steady states. Mean-square stability of rational expectations solutions is, however, invariant to the presence of a regime-switching drift. For this reason, we abstract from the drift term.

solution to be well-defined in our setup.[9] As shown by Foerster et al. (2016), solving for $\Omega(s_t)$ is equivalent to solving for a system of polynomials of all elements of $\Omega(s_t)$ at all regimes, thus there are finitely many MSV solutions. It is important to note that $F(\cdot)$ and $\Gamma(\cdot)$ are uniquely defined per each $\Omega(\cdot)$. Therefore, $\Omega(s_t)$ will be referred to as a MSV solution in what follows.

The sunspot component w_t it represents any stochastic process that cannot be written in the form of MSV solutions. Henceforth, the general solution (6) with a non-zero stochastic process w_t is called a sunspot solution. Since w_t satisfying (7) is defined per every single MSV solution, our solution set is exhaustive.

In the case of LRE models, the solution set may also be written alternatively, for example, following Lubik and Schorfheide (2004) or Sims (2002).[10] And determinacy conditions can be characterized directly with an appropriate eigenvalue-eigenvector decomposition of the structural parameter matrices. Unfortunately, there is no such a decomposition such as generalized Schur decomposition theorem in regime-switching context.[11] Representing the solution space in terms of a MSV solution and a sunspot component is the very idea for our approach, as demonstrated in what follows.

Our representation of the solution set is completed by solving for the sunspot process w_t. Notice that (7) is by itself a purely forward-looking and homogeneous rational expectations model with respect to w_t. Therefore, $w_t = 0_{n\times 1}$ is always a solution for all $t \geq 0$. Otherwise, the matrix $F(\cdot)$ in (7) imposes a recursive restriction on sunspots such that w_t depends on w_{t-1}. To make the analysis consistent with the model, we consider a set of quasi-linear solutions following Farmer et al. (2009) and Cho (2016) such that:

$$w_{t+1} = \Lambda(s_t, s_{t+1})w_t + \eta_{t+1}, \tag{11}$$

where w_0 is initially given for all $s_t = i, s_{t+1} = j \in \{1, ..., S\}$, $\Lambda(i,j)$ is an $n\times n$ matrix, η_{t+1} is an $n\times 1$ vector such that $E_t\eta_{t+1} = 0_{n\times 1}$. Therefore, it must be true that $w_t = E_t[F(s_t, s_{t+1})\Lambda(s_t, s_{t+1})]w_t$ for any sunspot process w_t. Specifically, let $V(i)$ be an $n\times k(i)$ matrix of which columns are orthonormal, with $0 \leq k(i) \leq n$ and $k(i) > 0$ for some i such that the column space of $V(i)$ is the span of the support of w_t when $s_t = i$ for all $t \geq 0$. Then the restriction for the sunspot process can be expressed as:

$$V(i)V(i)' = \sum_{j=1}^{S} P(i,j)F(i,j)\Lambda(i,j), \tag{12}$$

with the following properties: $w_{t+1} = V(j)V(j)'w_{t+1},\ \eta_{t+1} = V(j)V(j)'\eta_{t+1},\ \Lambda(i,j) = V(j)V(j)'\Lambda(i,j) = \Lambda(i,j)V(i)V(i)'$. It should be emphasized that there is no other restriction for η_t at all such as boundedness or mean-square stability as long as it solves (7). Therefore, an additional assumption is required as follows to make our analysis consistent with mean-square stability.

Assumption 2. The sunspot shock η_t is asymptotically covariance stationary. w_0 and s_t are independent of η_t for all $t \geq 1$.

Remark. It turns out that the sunspot processes characterized by (11) and (12) constitute only a subset of all possible quasi-linear sunspot processes because of two reasons. First, η_t may depend on regime such that $\eta_t = H(s_{t-1}, s_t)\varsigma_t$ where ς_t is independent of s_t. Second, w_t may well depend on the entire history of current and past regimes $s_0, ..., s_t$ through the coefficient matrices $\Lambda(\cdot)$ and $H(\cdot)$ as demonstrated by Barthélemy and Marx (2019).[12] Fortunately, all of regime-dependent sunspot processes can still be written in the recursive form of (11) and (12) by defining an extended regime variable and the corresponding transition probabilities. More importantly, the methodology developed in this paper remain unchanged even in this case. For this reason, we proceed with our exposition using the set of sunspots described by Eqs. (11) and (12). Appendix A formally presents the set of all quasi-linear sunspots, and all of the propositions developed below will be proven with respect to this set of all sunspots.

3.3. Mean-square stability and determinacy

In this subsection, we present the condition for mean-square stability of rational expectations solutions, which defines the *MOD* solution and uniqueness of mean-square stable solution (determinacy). We remark that other stability concepts

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9 To illustrate the point, consider a simple model: $x_t = AE_t x_{t+1} + z_t$ where A is non-singular and z_t is *i.i.d*, thus, $B = 0_{n\times n}$. But let us write the solution in general form, $x_t = \Omega x_{t-1} + \Gamma z_t + w_t$ where Ω must solve $(I - A\Omega)\Omega = B$. Under the invertibility of $I - A\Omega$, the only MSV solution is $\Omega = 0_{n\times n}$, $\Gamma = I$, that is, $x_t = z_t$. The general solution is $x_t = z_t + w_t$ such that $w_t = AE_t w_{t+1}$, which implies $w_t = A^{-1}w_{t-1} + \eta_t$ such that $\eta_t = w_t - E_{t-1}w_t$. Quasi-differencing the general solution yields $x_t = A^{-1}x_{t-1} - A^{-1}z_{t-1} + z_t + \eta_t$ with $\eta_t \neq 0$, which is not a MSV solution. Note that $\tilde{\Omega} = A^{-1}$ solves the aforementioned restriction with $I - A\tilde{\Omega}$ being singular. Nevertheless, the solution involving $\tilde{\Omega}$ is nested as a sunspot solution as just shown above.

10 For example, consider the model in the footnote above. A standard solution representation is to write $x_t = A^{-1}x_{t-1} - A^{-1}z_{t-1} + \eta_t^x$ where $\eta_t^x = x_t - E_{t-1}x_t$. In our representation, it is given by $x_t = A^{-1}x_{t-1} - A^{-1}z_{t-1} + z_t + \eta_t$. One can always write $\eta_t = \eta_t^x - z_t$, implying that our solution representation is equivalent to a standard one. And determinacy is the case of $\eta_t = 0$ and $\eta_t^x = z_t$.

11 Neusser (2019) shows that Lyapunov exponents provide an analogous role in constructing the explicit solution formulas in non-linear rational expectations models including ours as the eigenvalue-eigenvector relation for linear models. However, necessary and sufficient conditions for the classification of MSRE models are not completely identified by the Lyapunov exponents.

12 We are grateful to Jean Barthélemy and Magali Marx for pointing this out.

can also be relevant for studying determinacy, but as mentioned earlier, difference stability concepts lead to different determinacy results. Therefore, the methodology developed below is valid only with mean-square stability.[13]

Farmer et al. (2009) provide a formal definition of mean-square stability: an $n \times 1$ stochastic process x_t is said to be mean-square stable if its first and second moments converge as time tends to infinity: $\lim_{t\to\infty} E_0[x_t] = \mu$ and $\lim_{t\to\infty} E_0[x_t x_t'] = \Sigma$ where μ is an $n \times 1$ vector and Σ is an $n \times n$ matrix. The MSV solution, exogenous process z_t and the sunspot process w_t follow a vector autoregressive process with regime-dependent coefficient matrix. For such a process, mean-square stability is equivalent to asymptotic covariance stationarity under certain conditions. Assumptions 1 and 2 are those conditions, which are equivalent to the assumptions in Section 3.4 of Costa et al. (2005). Clearly, Assumptions 1 and 2 impose a restriction on the scope of this paper in that our analysis below is valid under those assumptions.[14] Nevertheless, asymptotic covariance stationarity is general enough for much of econometric analysis and inference, which is our main rationale for adopting it in our analysis (Refer to Farmer et al., 2009 for a formal treatment).[15]

Now we show that under Assumptions 1 and 2, mean-square stability of rational expectations and all of our results and their proofs can be characterized by the following two types of $n^2S \times n^2S$ matrices:

$$\bar{\Psi}_{G\otimes H} = \left[p_{ji}(G(j,i)')^T \otimes H(j,i)\right], \tag{13}$$

$$\Psi_{G\otimes H} = \left[p_{ij}(G(i,j)')^T \otimes H(i,j)\right], \tag{14}$$

where $G(i, j)$ and $H(i, j)$ are $n \times n$ matrices for a given P. ^{T} and $'$ denote the non-conjugate transpose and conjugate transpose operator, respectively, which differ only for complex-valued matrices. The expression in the squared bracket represents the (i, j)-th $n^2 \times n^2$ dimensional block for all $i, j = 1, ..., S$.

Equipped with (13), we first examine stability of the exogenous variable. Under Assumption 1, z_t in (5b) is mean-square stable if and only if its homogeneous part $z_t = R(s_t)z_{t-1}$ is mean-square stable, and the condition is given by $\rho(\bar{\Psi}_{R\otimes R}) < 1$ from Theorem 3.9 and 3.33 of Costa et al. (2005) where $\rho(\cdot)$ is the spectral radius – maximum absolute eigenvalue – of the argument matrix, and $\bar{\Psi}_{R\otimes R}$ is defined by (13) with $G = H = R$. Their theorem 3.33 also shows that mean-square stability is equivalent to asymptotic covariance stationarity. To proceed, z_t is assumed to be mean-square stable in what follows. A MSV solution $x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)z_t$ is mean-square stable if and only if $\rho(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$ from the fact that z_t is exogenous to x_t and is mean-square stable. In general, x_t is mean-square stable if and only if $\max(\rho(\bar{\Psi}_{\Omega\otimes\Omega}), \rho(\bar{\Psi}_{R\otimes R})) < 1$. A sunspot process w_t has exactly the same structure as the exogenous variables z_t under Assumptions 2. Henceforth, w_t is mean-square stable if and only if $\rho(\bar{\Psi}_{\Lambda\otimes\Lambda}) < 1$.[16] Finally, given that the homogeneous parts of z_t and w_t are autonomous processes and they are mean-square stable, a general sunspot solution (6) is mean-square stable if and only if $\max(\rho(\bar{\Psi}_{\Omega\otimes\Omega}), \rho(\bar{\Psi}_{\Lambda\otimes\Lambda})) < 1$. The logic is exactly the same as that for mean-square stability of a MSV solution above.

Finally, we are able to define the key concept of this paper, the *MOD* solution. Let $\mathcal{S}$ be the set of $\Omega(s_t) \in \mathcal{C}^{n\times n}$, $s_t = 1, ..., S$ satisfying (8) in the order of the spectral radius such that:

$$\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) \le \rho(\bar{\Psi}_{\Omega_2\otimes\Omega_2}) \le ... \le \rho(\bar{\Psi}_{\Omega_N\otimes\Omega_N}), \tag{15}$$

where N is the number of MSV solutions. $F_h(\cdot)$ and $\Gamma_h(\cdot)$ defined in (10) and (9) are uniquely associated with $\Omega_h(s_t)$, $h = 1, ..., N$. Formally, the *MOD* solution in the class of MSRE models can be defined as follows.

Definition 2. A MSV solution $x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)z_t$ to a Markov-switching rational expectations model (5) is referred to as a minimum of modulus (*MOD*) solution in the mean-square stability sense if $\Omega(s_t) \in \mathcal{S}$ in (15) and $\Omega(s_t) = \Omega_1(s_t)$.[17]

In what follows, stability will be interchangeably used with mean-square stability unless otherwise stated. In general, a rational expectations model is referred to as determinate if it has a unique stable solution. By utilizing the solution representation, determinacy can be more concretely stated by three requirements. First, the *MOD* solution is stable. Second, there must be no mean-square stable sunspot process w_t associated with the *MOD* solution. Third, all other MSV solutions are not stable, which also implies that all of the sunspot solutions associated with $\Omega_j(s_t)$ for $j \ge 2$ cannot be stable, regardless of whether their associated sunspot processes are stable or not. Therefore, determinacy can be formally defined as follows.

[13] Mean-square stability is neither necessary or sufficient for bounded as pointed out by Costa et al. (2005) and Farmer et al. (2009). In their Theorem 2, Francq and Zakoıan (2001) show that the second-order stationarity is necessary but not sufficient for mean-square stability. Using the stability based on the Lyapunov exponents, Neusser (2019) assumes boundedness for the rational expectations solutions directly, and it does not coincide with the boundedness condition of Barthélemy and Marx (2019).

[14] This does not mean that our analysis is valid only with Assumptions 1 and 2. In fact, Costa et al. (2005) show that their analysis can also be applied with covariance stationary shock processes or an alternative shock processes such as l_2 processes.

[15] One may additionally assume ϵ_t and η_t to be bounded as well as asymptotically covariance stationary, if one think of the model as a linearized system of the underlying non-linear MSRE model via perturbation method following Barthélemy and Marx (2017). Our analysis below will remain unchanged because such shock processes constitute a subset of all asymptotically covariance stationary processes.

[16] The sunspot shocks may well be (perfectly) correlated with the structural shocks. However, as long as they are asymptotically covariance stationary, they do not affect the stability of sunspots.

[17] In the case of LRE models, $\bar{\Psi}_{\Omega\otimes\Omega} = \Omega \otimes \Omega$ and $\rho(\Omega \otimes \Omega) < 1$ if and only if $\rho(\Omega) < 1$. Therefore, the solution set $\mathcal{S}$ can be written as the set of $\Omega \in \mathcal{C}^{n\times n}$, satisfying (8) in the absence of regime-switching such that $\rho(\Omega_1) \le \rho(\Omega_2) \le ... \le \rho(\Omega_N)$.

Definition 3. A Markov-switching rational expectations model (5) under Assumptions 1 and 2 is determinate in the mean-square stability sense if

(1) the *MOD* solution $x_t = \Omega_1(s_t)x_{t-1} + \Gamma_1(s_t)z_t$ is a mean-square stable MSV solution,
(2) there is no mean-square stable sunspot process associated with the *MOD* solution and
(3) there are no other mean-square stable MSV solutions.

Obviously, the *MOD* solution is the unique stable solution under determinacy. The *MOD* method developed in Section 4 is to derive the conditions for these three requirements for determinacy. As we will show, $\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) < 1$ and $\rho(\Psi_{F_1\otimes F_1}) \leq 1$ are the only two conditions for necessity and sufficiency of determinacy. The first condition directly implies the stability of the *MOD* solution. Surprisingly, $\rho(\Psi_{F_1\otimes F_1}) \leq 1$ ensures not only non-existence of stable sunspots but also non-existence of other stable MSV solutions. These properties have already been illustrated for the example in Section 2.

4. The MOD Method

This section formally develops our methodology for the classification of MSRE models and its implementation procedure using the *MOD* solution only, without consideration of other MSV solutions or sunspot components. It is very similar to Cho (2016). However, what makes our approach complete is that we work with the *MOD* solution, which always exists. In contrast, another MSV solution – the forward solution – used in Cho (2016) may not coincide with the *MOD* solution. This explains why the conditions for determinacy and indeterminacy in that study are theoretically only sufficient. We first derive two important properties of $\rho(\Psi_{F_1\otimes F_1})$, both of which are new findings in the literature.

4.1. Non-existence of Mean-square Stable Sunspots

Plugging Eqs. (11) into (7) yields a sort of reciprocal relation between $F(\cdot)$ and $\Lambda(\cdot)$ such that $w_t = E_t[F(s_t, s_{t+1})\Lambda(s_t, s_{t+1})]w_t$. Eq. (12) is the formal statement for the restriction for all $\Lambda(\cdot)$ given $F(\cdot)$. Using this equation, we derive the necessary and sufficient condition for non-existence of stable sunspots, the second requirement for determinacy. While the restriction appears to be complicated, intuition can be built up by considering a constant regime model. (12) then collapses into $VV' = FVV'\Lambda$. Vectorizing this relation yields that $(\Lambda'\otimes F)u = u$ where $u = vec(VV')/||vec(VV')||$ is an eigenvector associated with a unit root. Therefore, it follows that $\rho(\Lambda'\otimes F) = \rho(\Lambda)\rho(F) \geq 1$ for all possible Λ. Also, it is straightforward to show that there exists a Λ_m such that $\rho(\Lambda_m)\rho(F) = 1$. Henceforth, there is no stable sunspot process if and only if $\rho(F) \leq 1$.

Our task is to extend this intuition to the regime-switching models. Specifically, Appendix B shows that vectorizing (12) at each $s_t = i$, and stacking them for $i = 1, ..., S$, leads to the following relation:

$$(\Psi_{\Lambda'\otimes F})u = u, \tag{16}$$

where $\Psi_{\Lambda'\otimes F}$ is defined in (14) and u is an eigenvector associated with a unit root. Using Equation (16), we derive the following proposition, the first property of $\Psi_{F\otimes F}$.

Proposition 1. *Consider any sunspot process w_t described in (11) and (12) subject to (7). Then, the following holds.*

$$\rho(\Psi_{F\otimes F})\rho(\bar{\Psi}_{\Lambda\otimes\Lambda}) \geq 1, \tag{17}$$

for all $\Lambda(s_t, s_{t+1})$ satisfying (12) and there exists a $\Lambda_m(s_t, s_{t+1})$ such that:

$$\rho(\Psi_{F\otimes F})\rho(\bar{\Psi}_{\Lambda_m\otimes\Lambda_m}) = 1. \tag{18}$$

Therefore, there is no mean-square stable sunspot process w_t if and only if $\rho(\Psi_{F\otimes F}) \leq 1$.

Proof. See Appendix B. □

Obviously, Proposition 1 applies to $F_1(\cdot)$ associated with the *MOD* solution $\Omega_1(s_t)$. Appendix B constructs a $\Lambda_m(\cdot)$ as a function of F_1 and P analytically. Its existence is the key factor that makes Proposition 1 very powerful, resolving the problem of checking (non-) existence of stable sunspots reported in the literature by a single function, $\rho(\Psi_{F_1\otimes F_1})$. To be more specific, Farmer et al. (2009) is the first to demonstrate that the problem is important for establishing determinacy for a purely forward looking model in which $F_1(\cdot) = A(\cdot)$, thus the *MOD* solution is the unique mean-square stable MSV solution by construction. They illustrate how to check the existence of stable sunspots by searching for all $\Lambda(\cdot)$. Unfortunately, this approach is difficult to implement because $\Lambda(\cdot)$ is very high-dimensional as there are n^2S^2 unknowns in general. Proposition 1 states that $\rho(\Psi_{A\otimes A}) \leq 1$ is the necessary and sufficient condition for determinacy in purely forward-looking models.

For models with lagged endogenous variables, Lemma 2 of Cho (2016) proves the first assertion (17), thus $\rho(\Psi_{F\otimes F}) \leq 1$ is sufficient for non-existence of stable sunspots. Assertion 2 of Proposition 1 proves that the condition is necessary as well by showing the existence of $\Lambda_m(\cdot)$ satisfying (18) for a given $F(\cdot)$.

4.2. Uniqueness of Mean-square Stable MSV Solutions

The second role of the condition $\rho(\Psi_{F_1\otimes F_1}) \le 1$ is that it also rules out the existence of mean-square stable MSV solutions other than $\Omega_1(s_t)$. Appendix C derives the following relation among different MSV solutions such that:

$$D_{hk}(s_t) = E_t[F_k(s_t, s_{t+1})D_{hk}(s_{t+1})]\Omega_h(s_t), \tag{19}$$

where $D_{hk}(s_t) = \Omega_h(s_t) - \Omega_k(s_t)$ for all $h, k = 1, \ldots, N$ with $h \neq k$. Vectorizing Eq. (19) yields an important relation such that:

$$\left(\Psi_{\Omega_h'\otimes F_k}\right)u_{hk} = u_{hk}, \tag{20}$$

where u_{hk} is an eigenvector associated with a unit root. Notice the similarity between (20) and (16). This observation leads to the following Proposition 2, the second implication of $F_1(s_t, s_{t+1})$.

Proposition 2. *Consider a MSRE model (5) and the set of MSV solutions $\mathcal{S}$ in (15). Then the following holds.*

1. *For all $h \in \{2, \ldots, N\}$, $\rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h})\rho(\Psi_{F_h\otimes F_h}) \ge \rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1})\rho(\Psi_{F_h\otimes F_h}) \ge 1$ and,*

$$\rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h})\rho(\Psi_{F_1\otimes F_1}) \ge 1. \tag{21}$$

   Therefore, if $\rho(\Psi_{F_1\otimes F_1}) \le 1$, then $\rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h}) \ge 1$ for all $h \ge 2$.
2. *$\Omega(s_t)$ is the unique real-valued MOD solution if*

$$\rho(\bar{\Psi}_{\Omega\otimes\Omega})\rho(\Psi_{F\otimes F}) < 1. \tag{22}$$

Proof. See Appendix C. □

Assertion 1 of Proposition 2 directly leads to the third requirement for determinacy: if $\rho(\Psi_{F_1\otimes F_1}) \le 1$, there is no stable MSV solution other than the *MOD* solution. Henceforth, under this condition, there are only two cases: the model has a unique stable MSV solution (*MOD* solution) if $\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) < 1$, or no stable solution otherwise. Assertion 1 of Proposition 2 is new to the literature, and is also a very strong result that makes our approach tractable: non-existence of other stable MSV solutions can be identified by the value of $\rho(\Psi_{F_1\otimes F_1})$ without knowing the total number of MSV solutions N, which is surprisingly large and without solving for all of them. To be more specific, Equation (8) – the restriction for MSV solutions – contains n^2S number of unknowns in $n \times n$ matrices $\Omega(s_t)$. The total number of MSV solutions N is unknown, but $N \ge \Pi_{s=1}^{S} N_s(n, m_s)$ where $N_s(n, m_s) = \frac{(n+m_s)!}{n!m_s!}$ is the maximum number of solutions at regime s and m_s is the number of lagged variables if the regime is permanent.[18] For example, when $m_s = n$ for all regimes with $S = 2$, the number of solutions are at least 4, 36, 400 and 4900 for $n = 1, 2, 3$ and 4.

Assertion 2 is a direct consequence of Assertion 1. There is at most one MSV solution such that $\rho(\bar{\Psi}_{\Omega\otimes\Omega})\rho(\Psi_{F\otimes F}) < 1$ and if so, it must be the *MOD* solution. Since it is unique, it must be real-valued. Henceforth, Assertion 2 can serve as an identification condition for the *MOD* solution.

4.3. Classification of MSRE Models

Propositions 1 and 2 lead to the following complete classification of MSRE models:

1. Model (5) is determinate if and only if $\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) < 1$ and $\rho(\Psi_{F_1\otimes F_1}) \le 1$.
2. Model (5) is indeterminate if and only if $\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) < 1$ and $\rho(\Psi_{F_1\otimes F_1}) > 1$.
3. Model (5) has no stable solution if and only if $\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) \ge 1$.

Note that all of these conditions are mutually disjoint and exhaustive. Therefore, the conditions for all three cases are both necessary and sufficient. While the formal proof is given in Appendix D, it is easy to understand that these results are direct consequences of Propositions 1 and 2.

Our classification above, however, requires one to actually identify and compute the *MOD* solution.[19] Note also that determinacy can arise only under the condition (22) in Proposition 2. Moreover, it is an identification condition for the *MOD* solution if the condition holds, which is crucial in computing the solution in practice. For this reason, we partition the class of MSRE models by this condition as follows.

Definition 4. A Markov-switching rational expectations model (5) is referred to as determinacy-admissible if $\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1})\rho(\Psi_{F_1\otimes F_1}) < 1$ for $\Omega_1(s_t) \in \mathcal{S}$ in (15). Otherwise, it is referred to as determinacy-inadmissible.

[18] For an n-dimensional LRE model, there are $2n$ generalized eigenvalues. But the number of finite eigenvalues is $n+m$. Any solution Ω is associated with n finite eigenvalues. Therefore, the total number of solutions is the same as the number of combinations of choosing n out of $n+m$ eigenvalues.

[19] This is one limitation of our approach in classifying MSRE models because the classification can be done in terms of the generalized eigenvalues implied by the structural parameter matrices of the model in the case of LRE context, without a direct analysis of the solution space. Nevertheless, our approach is a significant improvement over the previous studies because we only need the information of the *MOD* solution.

Uniqueness of a real-valued *MOD* solution for every determinacy-admissible model can also be understood by the transformation of any MSRE model (5) that preserves the order of the MSV solutions (ignoring exogenous variables for simplicity) as follows:

$$x_t = \left(\frac{1}{\alpha}\right) E_t[A(s_t, s_{t+1})x_{t+1}] + \alpha B(s_t)x_{t-1}, \tag{23}$$

where $\alpha > 0$. Then the pair of matrices defined in (8) and (10) for this transformed model are given by $(\alpha\Omega(s_t), F(s_t, s_{t+1})/\alpha)$ where is $(\Omega(s_t), F(s_t, s_{t+1}))$ is the pair to the original model. Hence, $\alpha\Omega_1(s_t)$ is the *MOD* solution. Then, it is straightforward to show that a model can be made determinate if and only if the model is determinacy-admissible by setting α such that $\sqrt{\rho(\Psi_{F_1\otimes F_1})} \le \alpha < 1/\sqrt{\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1})}$.[20] Because the determinate solution is the unique real-valued *MOD* solution by definition, and the order of MSV solutions is invariant to the transformation of the model, a solution satisfying $\rho(\bar{\Psi}_{\Omega\otimes\Omega})\rho(\Psi_{F\otimes F}) < 1$ is the *MOD* solution, proving Assertion 2 of Proposition 2.

Finally, Propositions 1 and 2, along with the concept of determinacy-admissibility, lead to Proposition 3, the main classification result of the MSRE models in this paper.

Proposition 3. *Consider a MSRE model (5) and the set of solutions $\mathcal{S}$ in (15). Then, necessary and sufficient conditions for determinacy, indeterminacy and the case of no stable solution in the mean-square stability sense are given in Table 1. Moreover, if there exists a solution such that $\rho(\bar{\Psi}_{\Omega\otimes\Omega})\rho(\Psi_{F\otimes F}) < 1$, it is the unique real-valued MOD solution and the model is determinacy-admissible.*

Table 1
Classification of MSRE models.

	Determinacy-Admissible: $\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1})\rho(\Psi_{F_1\otimes F_1}) < 1$	Determinacy-Inadmissible: $\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1})\rho(\Psi_{F_1\otimes F_1}) \ge 1$
Determinacy	$\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) < 1,\ \rho(\Psi_{F_1\otimes F_1}) \le 1$	Impossible
Indeterminacy	$\rho(\Psi_{F_1\otimes F_1}) > 1$	$\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) < 1$
No Stable Solution	$\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) \ge 1$	$\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) \ge 1$

Proof. See Appendix D. □

There is no known system such as the generalized eigenvalue-eigenvector decomposition (eigensystem for short) that yields the complete classification result to MSRE models. Nevertheless, Proposition 3 completely classifies MSRE models – including LRE models – into three disjoint and exhaustive cases of determinacy, indeterminacy and no stable solution by using the properties of the *MOD* solution in the mean-square stability sense.

To better understand the role of the *MOD* solution, consider the classification result for LRE models reported in Table 2, a special subset of MSRE models in which there is only one regime and $P = 1$.[21]

Appendix E explains Table 2 and proves that this classification result is identical to standard one based on generalized eigenvalues. For example, $\rho(\Omega_1) < 1$ and $\rho(F_1) \le 1$ are precisely the same conditions for determinacy in terms of generalized eigenvalues: there exist exactly n stable roots out of $2n$ generalized eigenvalues – which is the usual root-counting condition – *and* the existence of a MSV solution associated with the n smallest generalized eigenvalues. These are exactly the same conditions as the existence and uniqueness conditions of the gensys algorithm of Sims (2002). Also, Table 2 is a generalization of Proposition 2 of Cho and McCallum (2015). Our contribution is that we prove it using the properties of the *MOD* solution without resorting to the eigensystem, whereas they derive the same result using the eigensystem. And this is precisely the idea that we can develop Proposition 3 for MSRE models in which the eigensystem does not exist.

Finally, we address the likelihood and economic relevance of determinacy-inadmissible models. The first type of such models is those with multiple *MOD* solutions. It is easier to think of the LRE case with the largest eigenvalue of Ω_1 being either complex-valued or repeatedly real-valued. In fact, these possibilities are often assumed away in the literature. An example of this kind is the knife-edge case of the model in Section 2 when both monetary and fiscal policies is both active or passive with exactly the same degree such that $\alpha = \theta$. As mentioned above, such a model cannot be determinate. Second, a model with completely decoupled systems with a particular structure in which the usual root-counting approach fails according to Sims (2007). The case of $c = 0$ in the example is of this kind, which is no longer a FTPL model because what makes our FTPL example being economically relevant is the linkage that the government debt b_t depends on inflation through a non-zero c. Section 5 and Appendix E discuss more technical detail on this issue. While there may be another

[20] To see this, note that the transformed model is determinacy-admissible because $\rho(\bar{\Psi}_{\alpha\Omega_1\otimes\alpha\Omega_1})\ \rho(\Psi_{F_1/\alpha\otimes F_1/\alpha}) = \rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1})\ \rho(\Psi_{F_1\otimes F_1}) < 1$. And it is determinate because $\rho(\bar{\Psi}_{\alpha\Omega_1\otimes\alpha\Omega_1}) < 1$ and $\rho(\Psi_{F_1/\alpha\otimes F_1/\alpha}) \le 1$.

[21] Note that $\bar{\Psi}_{\Omega_1\otimes\Omega_1} = \Omega_1 \otimes \Omega_1$ and $\rho(\Omega_1) < 1$ if and only if $\rho(\Omega_1 \otimes \Omega_1) = \rho(\Omega_1)^2 < 1$. Similarly, $\rho(\Psi_{F_1\otimes F_1})$ is replaced with $\rho(F_1)$ and the condition for determinacy admissibility collapses to $\rho(\Omega_1)\rho(F_1) < 1$.

Table 2
Classification of LRE Models by the MOD Method.

	Determinacy-Admissible: $\rho(\Omega_1)\rho(F_1) < 1$	Determinacy-Inadmissible: $\rho(\Omega_1)\rho(F_1) \geq 1$
Determinacy	$\rho(\Omega_1) < 1,\ \rho(F_1) \leq 1$	Impossible
Indeterminacy	$\rho(F_1) > 1$	$\rho(\Omega_1) < 1$
No Stable Solution	$\rho(\Omega_1) \geq 1$	$\rho(\Omega_1) \geq 1$

types of determinacy-inadmissible models, such models would also be rare in the MSRE framework by the same token. Moreover, they can never be determinate. Therefore, the classification of MSRE models within the determinacy-admissible models would be sufficient in practice for the economic analysis unless one is particularly interested in determinacy-inadmissible models with no stable solution. Note also that there cannot be a determinacy-inadmissible model if it is purely forward-looking: the *MOD* solution is unique by construction with $\Omega(\cdot) = 0_{n\times n}$ and $F(\cdot) = A(\cdot)$. Hence, only determinacy or indeterminacy arises depending on the value of $\rho(\Psi_{A\otimes A})$.

4.4. Implementation of the MOD Method

Proposition 3 is a complete theoretical classification result, but it is required to actually identify and compute the *MOD* solution for the proposed methodology to be implementable. As argued above, the primary goal is to design an efficient solution procedure for determinacy-admissible models by using the identification condition for the *MOD* solution (22). We compare the three types of solution methods available in the literature and design a solution procedure. We also identify the case in which our proposed method is not implementable.

First, the numerical search methods proposed by Svensson et al. (2008), Farmer et al. (2011) and Maih (2015) can be used. Basically, these methods compute efficiently the real-valued MSV solutions to Eq. (8), a system of multivariate quadratic equations belonging to a family of algebraic Riccati equations (See for instance, Bini et al., 2012). These search algorithms compute multiple MSV solutions by choosing various randomized initial parameter values. If a solution satisfying the condition for the *MOD* solution (22) is found, the model classification is completed. However, it may take longer to find the *MOD* solution for large-scale models: the number of MSV solutions explodes exponentially as the dimension of the model and the number of regimes increase. Therefore, the number of random searches may increase rapidly until the *MOD* solution is found as the size of the model increases.

Second, a generalized version of the forward method of Cho (2016) can also be used. This method – explained in Appendix F - - extends a textbook-type solving-the-model-forward approach to MSRE models,[22] which amounts to constructing a unique sequence of matrices from the model (5): $\Omega_{(1)}(s_t) = B(s_t)$ and,

$$\Omega_{(k)}(s_t) = \{I_n - E_t[A(s_t, s_{t+1})\Omega_{(k-1)}(s_{t+1})]\}^{-1}B(s_t). \tag{24}$$

for $k \geq 2$. If the sequence defined in Eq. (24) converges for every s_t, $\Omega^*(s_t) \equiv \lim_{k\to\infty}\Omega_{(k)}(s_t)$ solves the restriction (8), thus the forward solution $x_t = \Omega^*(s_t)x_{t-1}$ is a real-valued MSV solution, – ignoring the exogenous variable – the only one consistent with the usual transversality condition among all MSV solutions. This approach has two important advantages. First, the condition for the existence of the forward solution is closely related to that for the *MOD* solution and the equivalence of the two solutions hardly breaks down. In our extensive experiment, the forward solution coincides with the *MOD* solution without an exception at least for determinacy-admissible models, although there is no known proof for the equivalence. Second, the computational efficiency actually comes from the fact that, in contrast to the random search algorithms, the forward method computes only one out of all MSV solutions, regardless of dimension and the number of regimes. Even for a full 10 -dimensional model, it takes only around 10^{-3} seconds – using a standard personal computer – to identify the classification of the model and the *MOD* solution in the case of LRE model by using the forward method or the gensys algorithm, and around 10^{-3} to 10^{-2} seconds for a MSRE model with two regimes by the forward method.

Third, the Gröbner basis technique proposed by Foerster et al. (2016) is a very important contribution because it computes all of the MSV solutions, completing our methodology theoretically. This technique computes as many polynomial bases as the number of solutions to the system of multivariate quadratic equations implied by (8). Nevertheless, it is computationally costly. Specifically, the computational time increases roughly at the rate of 2^N according to our simulation. Recall that $N \geq 4$, 36 and 400 for the model with $n = m = 1$, 2 and 3. According to our experiment, the corresponding computational time is around 10^{-3} seconds for $n = 1$ and several minutes for $n = 2$. But the Gröbner basis approach does not work when $n = 3$ as the expected time would be 2^{364} times longer than the bivariate case, which is clearly not feasible. Fortunately, one would not need this technique in practice because almost all models would be determinacy-admissible and the first two methods are sufficient to identify the *MOD* solution.

[22] The forward method originally proposed by Cho and Moreno (2011) for LRE models – as a solution refinement scheme – yields the forward solution consistent with the *MOD* solution except for only one type of models featuring a special block-recursive structure such as the one we analyze in Section 2. Cho (2016) adjusts the information set with which agents solve the model forward to recover the equivalence of the forward solution with the *MOD* solution. Appendix F formalizes this idea of information adjustment.

To summarize, we propose the following sequential procedure on the ground of the likelihood of finding the *MOD* solution and computational efficiency as follows.[23]

1. Apply the modified forward method or a random search method as a primary solution technique. If a solution $\Omega(s_t)$ satisfying (22) is found, it is the *MOD* solution and the model is determinacy-admissible. Therefore, the classification is completed by Proposition 3.
2. If a solution $\Omega(s_t)$ does not satisfy Eq. (22), but $\rho(\bar{\Psi}_{\Omega\otimes\Omega}) < 1$, then $\rho(\Psi_{F\otimes F}) \geq 1$ and the model is indeterminate regardless of whether $\Omega(s_t) = \Omega_1(s_t)$.
3. If $\rho(\bar{\Psi}_{\Omega\otimes\Omega})\rho(\Psi_{F\otimes F}) \geq 1$ and $\rho(\bar{\Psi}_{\Omega\otimes\Omega}) \geq 1$ for all solutions, then a primary solution technique cannot identify the *MOD* solution. Use the Gröbner basis technique.

In practice, only Case 1 would emerge and a primary solution technique is sufficient for applied research such as estimating MSRE models as long as they are determinacy-admissible. The remaining cases indicate the possibility that the model is determinacy-inadmissible, for instance, the one that has complex-valued *MOD* solutions. But it should be remarked that our proposed methodology is not tractable if the model belongs to Case 3 with a larger dimension.

4.5. Uniqueness of a stable MSV Solution and Determinacy

Proposition 3 shows that the *MOD* method for MSRE models is essentially the same as that for LRE models, but with one very important distinction. Uniqueness of a stable MSV solution implies determinacy for most – in fact determinacy-admissible – LRE models with lagged endogenous variables. In stark contrast, uniqueness of a stable MSV solution does *not* imply determinacy in the MSRE framework. Therefore, one may falsely interpret a MSRE model with a unique stable MSV as determinate despite that it is in fact indeterminate. Here we show that such a case is a general phenomenon that was previously unforeseen in the literature.

To illustrate the point, consider a determinacy-admissible LRE model with multiple MSV solutions. The LRE model counterpart of Eq. (21) is given by:

$$\rho(\Omega_h^T \otimes F_1) = \rho(\Omega_h)\rho(F_1) \geq 1, \tag{25}$$

for $h \geq 2$ where the equality results from the property of the Kronecker product. Moreover, there exists Ω_j such that $\rho(\Omega_j)\rho(F_1) = 1$ from the eigensystem relation. Therefore, the condition $\rho(\Omega_1) < 1 \leq \rho(\Omega_j)$ ensures that $\rho(F_1) \leq 1$. Henceforth, the unique stable MSV solution implies determinacy. In contrast, the corresponding conditions for MSRE models to Eq. (25) are given by:

$$\rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h})\rho(\Psi_{F_1\otimes F_1}) \geq \rho(\Psi_{\Omega_h'\otimes F_1}) \geq 1. \tag{26}$$

In general, the first inequality in (26) holds strictly and if so, it is possible that $\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) < 1 \leq \rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h})$ for all $h \geq 2$ and $\rho(\Psi_{F_1\otimes F_1}) > 1$, implying that the model has a unique stable MOD solution, but has a continuum of stable sunspot solutions.[24] In fact, any model with $\rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h})\rho(\Psi_{F_1\otimes F_1}) > 1$ can be made to be indeterminate with a unique stable MSV solution by the order-preserving transformation (23). This finding is formalized as follows.

Corollary 1. *Consider a MSRE model (5) where $B(s_t) \neq 0_{n\times n}$ for at least one regime. The uniqueness of a mean-square stable MSV solution does not always imply determinacy in the mean-square stability sense.*

Proof. Consider a determinacy-admissible MSRE model such that the first inequality holds strictly in (26). Then, Proposition 2 implies that $\rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1}) < 1/\rho(\Psi_{F_1\otimes F_1}) < \rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h})$ for all $h \geq 2$. Consider a transformed model (23) such that $\tilde{\Omega}(\cdot) = \alpha\Omega(\cdot)$ and $\tilde{F}(\cdot) = F(\cdot)/\alpha$. Let $\alpha = (\rho(\Psi_{F_1\otimes F_1})/\rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h}))^{1/4}$ where $h = 2$. Then, $\rho(\bar{\Psi}_{\tilde{\Omega}_1\otimes\tilde{\Omega}_1}) < [\rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h})\rho(\Psi_{F_1\otimes F_1})]^{-1/2} < 1$ and $\rho(\Psi_{\tilde{F}_1\otimes\tilde{F}_1}) = \rho(\bar{\Psi}_{\tilde{\Omega}_h\otimes\tilde{\Omega}_h}) = [\rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h})\rho(\Psi_{F_1\otimes F_1})]^{1/2} > 1$. Therefore, the transformed model has a unique stable MOD solution with a continuum of mean-square stable sunspot solutions. *Q.E.D.* □

In Section 5, we illustrate this point using the FTPL model introduced in Section 2. This is important because some existing papers have already analyzed regime-switching models similar to our example and presume that a model with a unique stable *MOD* solution is determinate, which is clearly not true if $\rho(\Psi_{F_1\otimes F_1}) > 1$.

[23] The solution package accompanying this paper provides all of the codes implementing the *MOD* method. The matlab code "fmmsre.m" implements the forward method as a function of parameter matrices $P, A(s_t, s_{t+1})$ and $B(s_t)$. With the same input arguments, the code "gbmsre.m" implements the Gröbner basis approach. This code controls the solution procedure in matlab, but uses a mathematics language referred to as "Singular" developed by Decker et al. (2019) which is based on C++. This is known as more than 100 times faster than matlab.

[24] This may also have an important implication for expectational stability of Evans and Honkapohja (2001) for regime-switching models. Note that determinacy-admissibility is equivalent to $\rho(\Omega_1^T \otimes F_1) = \rho(\Omega_1)\rho(F_1) < 1$ in LRE models, which is one of the conditions for expectational stability. In contrast, $\rho(\Psi_{\Omega_1'\otimes F_1}) < 1$ is necessary, but not sufficient for determinacy-admissibility in MSRE models. McClung (2020) has recently analyzed this issue and Branch et al. (2013) studies determinacy in boundedness and expectational stability in a purely forward-looking model.

5. Determinacy Analysis: Examples

This section applies the *MOD* method to the example introduced in Section 2. The model (1) can be cast into the canonical form of (5) with $x_t = [\pi_t \, b_t]'$, $z_t = r_t$ and,

$$A(s_t) = \begin{bmatrix} 1/\alpha(s_t) & 0 \\ -c(s_t)/\alpha(s_t) & 0 \end{bmatrix}, \quad B(s_t) = \begin{bmatrix} 0 & 0 \\ 0 & \theta(s_t) \end{bmatrix}, \quad C(s_t) = \begin{bmatrix} 1/\alpha(s_t) \\ -c(s_t)/\alpha(s_t) \end{bmatrix}. \tag{27}$$

Much of the analysis can be understood analytically. Moreover, the FTPL model provides an economic intuition for determinacy - (in)admissibility. The fixed regime case is first considered, followed by the regime-switching case under several scenarios.

5.1. A Linear Model

First, we demonstrate the equivalence of our classification result with the standard taxonomy result for the case of LRE model. Then, we report that the numerical implementation procedure indeed confirms this theoretical finding.

In Section 2, the matrix F has been presented in (4) for monetary and fiscal MSV solutions. It can be derived from the formula is $F = (I - A\Omega)^{-1}A$ from (10) where A is defined in (27) with constant α and c. From Eqs. (3) and (4), the spectral radii of Ω and F for the two MSV solutions are given by:

$$\rho(\Omega^M) = \theta, \qquad \rho(F^M) = 1/\alpha, \tag{28}$$

$$\rho(\Omega^F) = \alpha, \qquad \rho(F^F) = 1/\theta. \tag{29}$$

First, suppose that the monetary policy is relatively more aggressive than the fiscal policy with $\alpha > \theta$. For the monetary solution, $\rho(\Omega^M)\rho(F^M) = \theta/\alpha < 1$ from (28) whereas $\rho(\Omega^F)\rho(F^F) = \alpha/\theta > 1$ for the fiscal solution from (29). Therefore, the model is determinacy-admissible and the *MOD* solution is monetary ($\Omega_1 = \Omega^M$) from Proposition 3. Indeed, $\rho(\Omega^M) = \theta < \rho(\Omega^F) = \alpha$. It follows that, from Table 2, the model is determinate if and only if $\rho(\Omega_1) = \theta < 1$ and $\rho(F_1) = 1/\alpha \leq 1$. We have shown that this is exactly the case of an AM-PF policy mix ($\theta < 1 \leq \alpha$).[25] Indeterminacy arises if and only if $\rho(\Omega_1) = \theta < 1$ and $\rho(F_1) = 1/\alpha > 1$, which is a PM-PF type policy mix ($\theta < \alpha < 1$). Finally, no stable solution exists if the *MOD* solution is unstable, i.e, $\rho(\Omega_1) = \theta \geq 1$. Therefore both policies are active ($1 \leq \theta < \alpha$).

Second, suppose that the fiscal policy is relatively more aggressive than the monetary policy with $\theta > \alpha$. Exactly the same argument can be made as above: the model is determinacy-admissible and the *MOD* solution is fiscal because $\rho(\Omega^F)\rho(F^F) = \alpha/\theta < 1$. Now it is straightforward to see that determinacy, indeterminacy and the case of no stable solution in terms of $\rho(\Omega^F)$ and $\rho(F^F)$ correspond to $\alpha < 1 \leq \theta$ (PM-AF), $\alpha < \theta < 1$ (PM-PF) and $1 \leq \alpha < \theta$ (AM-AF), respectively.

Finally, suppose that $\alpha = \theta$, which is typically assumed away in the literature. Our approach provides one explanation for this conventional assumption: the model is determinacy-inadmissible because $\rho(\Omega)\rho(F) = 1$ for both MSV solutions from Proposition 3. And the two solutions are identical *MOD* solutions from Eq. (2) in this knife-edge case. Therefore, the model can only be either indeterminate if both parameters are smaller than one, or has no stable solution otherwise. This particular case is out of consideration because a unique stable equilibrium is the very crucial requirement for the perspective of the FTPL. This example illustrates why determinacy-inadmissible model is of less interest in a general context.

To summarize, in the context of LRE models, our classification result is exactly the same as the taxonomy result of Leeper (1991). In general, the equivalence of our approach and the standard one using the eigensystem must hold for our methodology to be valid. We prove the equivalence in Appendix E, and reiterate the equivalence in this example.

We also perform a numerical exercise following the implementation procedure of the *MOD* method in Section 4.4. In the first step, we apply the forward method and confirm that the forward solution coincides with the *MOD* solution for arbitrary values, α and θ such that $\alpha \neq \theta$. Therefore, the model classification is completed numerically at this step. When $\alpha = \theta$, the second step is required because the *MOD* solution is not identified: the forward solution exists but $\rho(\Omega)\rho(F) = 1$. However, if $\alpha = \theta < 1$, it follows that $\rho(\Omega) < 1 < \rho(F)$, thus indeterminacy is confirmed, without identifying the *MOD* solution. If $\alpha = \theta \geq 1$, the third step is required: since $\rho(\Omega) \geq 1$ and $\rho(\Omega)\rho(F) = 1$ for the forward solution, we apply alternative solution method – the generalized Schur decomposition theorem for this LRE model – to solve for all MSV solutions. Indeed there are two identical *MOD* solutions, same as the forward solution. Thus the model has no stable solution.

5.2. A Markov-switching Model

We consider three among many possible scenarios for exposition. In our exercise, the terms “active” and “passive” will be used to represent the policy stances under the fixed regime. In all cases, the transition probabilities are fixed at $P(1,1) = 0.95$ and $P(2,2) = 0.9$ while $\beta = 0.99$ and $\bar{b} = 1$. In all examples, we applied the forward method, and the forward solution is the *MOD* solution, completing the model classification.

[25] Determinacy contains the case of $\alpha = 1$, therefore, strictly speaking, the model is determinate under a non-passive monetary policy and a passive fiscal policy. Following convention, we keep using the terminology AM-PF in this case. The same is true for PM-AF policy mix.

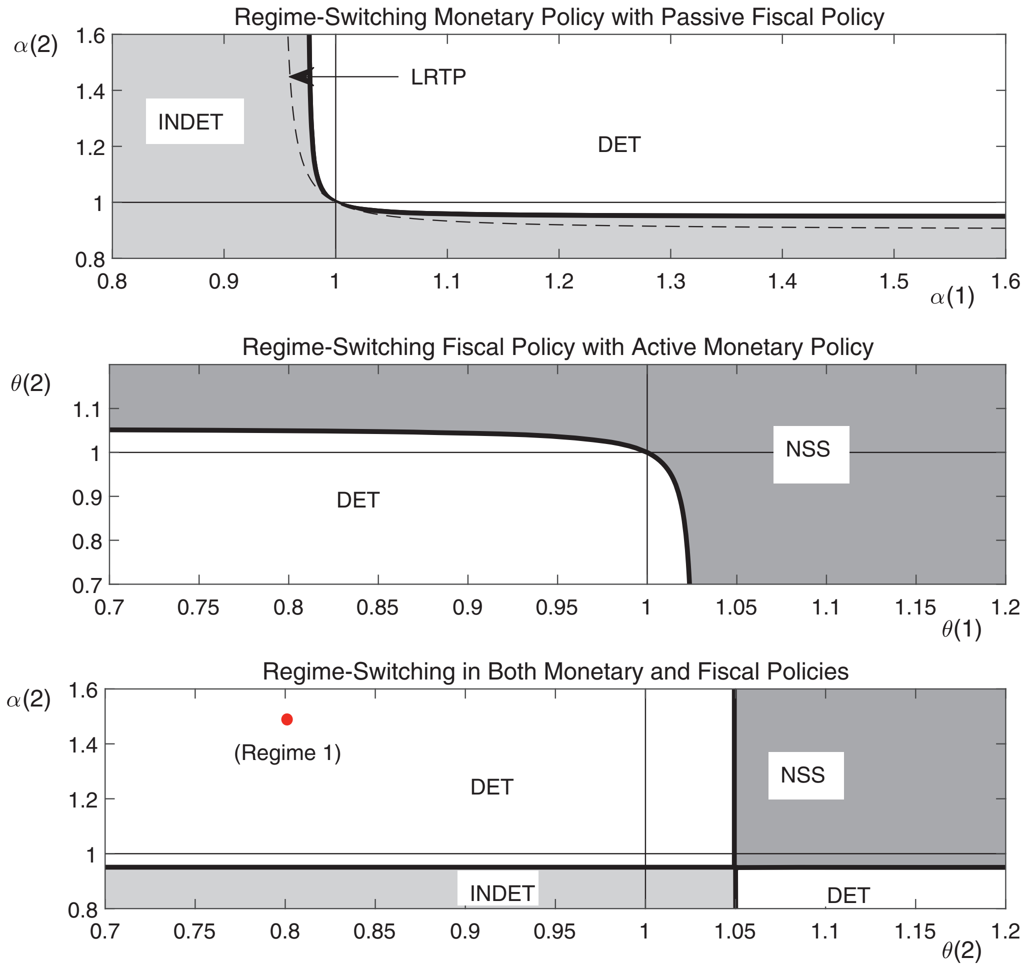

Fig. 1. Determinacy Analysis for Model (1) under Regime-Switching. This figure depicts the region for determinacy (DET) in white, indeterminacy (INDET) in light gray and no stable solution (NSS) in dark gray for model (1). Panel A considers monetary policy switching over two states $\alpha(1)$ and $\alpha(2)$ while holding the fiscal policy to be passive with $\theta = 0.8$. The region implied by the long-run Taylor Principle (LRTP) is also depicted in a dashed line. Panel B is for fiscal policy switching over two states $\theta(1)$ and $\theta(2)$ while monetary policy is active with $\alpha = 1.5$. Panel C reports the result in terms of monetary and fiscal policy stances in regime 2 ($\alpha(2)$ and $\theta(2)$) when the policy mix is AM-PF with $\alpha(1) = 1.5$ and $\theta(1) = 0.8$ in regime 1, as denoted by the red dot.

5.2.1. Regime-Switching in Monetary Policy

Suppose that the fiscal policy is passive ($\theta < 1$) and is fixed at both regimes. The task is to find the combinations of monetary policy stance ensuring determinacy for all possible combinations over $\alpha(s_t) > \theta$. The case in which $\alpha(s_t) < \theta$ for one or both regimes will be analyzed in the last example. We find that the *MOD* solution is a monetary equilibrium: $\Omega_1(s_t) = \Omega^M = \begin{bmatrix} 0 & 0 \\ 0 & \theta \end{bmatrix}$ defined in (3), and $F_1(s_t) = \begin{bmatrix} \frac{1}{\alpha(s_t)} & 0 \\ -\frac{c}{\alpha(s_t)} & 0 \end{bmatrix}$, which is F^M in (4) with $\alpha = \alpha(s_t)$. Therefore, the *MOD* solution is mean-square stable as $\rho(\bar{\Psi}_{\Omega_1 \otimes \Omega_1}) = \theta^2 < 1$. Hence, determinacy region is the combination of $\alpha(1)$ and $\alpha(2)$ such that $\rho(\Psi_{F_1 \otimes F_1}) \le 1$. This is depicted as the white region in Panel A of Fig. 1 when $\theta = 0.8$. The light gray area is the indeterminacy region of $\rho(\Psi_{F_1 \otimes F_1}) > 1$. The curve partitioning determinacy and indeterminacy is the set of $\alpha(1)$ and $\alpha(2)$ such that $\rho(\Psi_{F_1 \otimes F_1}) = 1$.

Panel A shows that a unique mean-square stable equilibrium allows the monetary policy to be temporarily passive in one regime and active in the other. Hence, the monetary policy can be described as “overall active” in a regime-switching context if $\alpha(1)$ and $\alpha(2)$ lie inside the determinacy region. This resembles what is implied by the long-run Taylor Principle (LRTP) proposed by Davig and Leeper (2007). However, the LRTP is not the condition for determinacy in boundedness, as demonstrated by Farmer et al. (2010).[26] Cho (2016) shows that the conditions for the LRTP coincide with $\rho(\bar{\Psi}_{\Omega_1}) = \theta < 1$

[26] Recently Barthélemy and Marx (2019) propose conditions for unique bounded solution, but their conditions are difficult to verify especially in the border of determinacy and indeterminacy regions. In contrast, our determinacy conditions are straightforward to verify and identify regions for not just determinacy, but also indeterminacy and no stable solution in the mean-square stability sense.

and $\rho(\Psi_{F_1}) \le 1$, which are necessary but not sufficient for mean-square stability.[27] Panel A of Fig. 1 clearly shows that the LRTP region is strictly larger than the determinacy counterpart: there is a sizable region with $\rho(\Psi_{F_1}) \le 1 < \rho(\Psi_{F_1 \otimes F_1})$ in which the LRTP holds, but the model is indeterminate.

5.2.2. Regime-switching in fiscal policy

A qualitatively similar implication to the model above can be drawn when the fiscal policy switches over two regimes $\theta(1)$ and $\theta(2)$ while monetary policy is active in both regimes. We consider the case in which the monetary policy is active with $\alpha > 1$ and $\alpha > \theta(s_t)$ in both regimes. The remaining case is also analyzed in the last example. The *MOD* solution is again the monetary equilibrium. $\Omega_1(s_t) = \Omega^M = \begin{bmatrix} 0 & 0 \\ 0 & \theta(s_t) \end{bmatrix}$ defined in (3) with $\theta = \theta(s_t)$, and $F_1(s_t) = F^M = \begin{bmatrix} \frac{1}{\alpha} & 0 \\ -\frac{c}{\alpha} & 0 \end{bmatrix}$ defined in (4). In this case, determinacy is identified by $\rho(\bar{\Psi}_{\Omega_1 \otimes \Omega_1}) < 1$ because $\rho(\Psi_{F_1 \otimes F_1}) = 1/\alpha^2 < 1$. Panel B of Fig. 1 depicts the determinacy region in this exercise with $\alpha = 1.5$, which is again larger than that under the fixed regime counterpart. That is, a temporarily active fiscal policy is admissible for determinacy if it is not too active. The policy mix in the regime-switching context can therefore be expressed as "overall AM-PF type".

However, the parameter area outside the determinacy region is the one – depicted in dark gray – in which no stable solution exists such that $\rho(\bar{\Psi}_{\Omega_1 \otimes \Omega_1}) \ge 1$. This is because the switching occurs over AM-PF and AM-AF combinations, i.e., the determinate region and a region with no stable solution under the fixed regime.[28]

5.2.3. Regime-Switching in Both Policies

Two policy combinations that have drawn much attention in the literature are AM-PF and PM-AF because it is well-known that these two cases lead to determinacy under fixed regime. But there has been no formal analysis about determinacy property regarding switching over these two policy mixes. Would the determinacy region be still larger than the one under the fixed regime? To answer the question, suppose that the first regime is AM-PF with $\alpha(1) = 1.5$ and $\theta(1) = 0.8$. Then we seek the policy combinations in regime 2, $\alpha(2)$ and $\theta(2)$, under which determinacy is ensured.

It turns out that the PM-AF type policy mix in regime 2 is not necessarily associated with determinacy. The exact determinacy region is depicted by Panel C of Fig. 1. In general, determinacy in MSRE models is neither necessary nor sufficient for determinacy under the fixed regime-counterpart. This was pointed out by Cho (2016) in the context of a standard New-Keynesian model with regime-switching monetary policy. Our analysis shows that such a phenomenon can be universal. If the policy mix is close to AM-PF as well in regime 2, the determinacy region is larger than the fixed regime counterpart, similar to the two previous examples: some other types of policy mixes (PM-PF, AM-AF, PM-AF) lead to determinacy. On the other hand, the PM-AF policy mix in regime 2 does not always ensure determinacy: the model can be indeterminate or even has no stable solution. Determinacy emerges with (1) both PM-AF policy stances are very strong or (2) very mild in regime 2. Indeterminacy (light gray area) emerges if fiscal policy is mildly active while monetary policy is strongly passive. The model has no stable solution (depicted in dark gray) when monetary policy is slightly passive and fiscal policy is strongly active in regime 2 as the policy mix is nearby an AM-AF region.

Discussion. Two additional findings from this exercise are noteworthy. First, the equilibrium property of determinacy can be drastically different. In the top left determinacy region, the equilibrium is monetary as inflation evolves independent of the fiscal aspect even when the fiscal policy is active in regime 2 (PM-AF) as long as the policy stances are both mild. Therefore, the overall policy mix is of AM-PF type. In contrast, the equilibrium in the bottom-right determinacy region is fiscal as inflation does depend on the government debt with strongly PM-AF mixes in regime 2: monetary policy is strongly passive and fiscal policy is significantly active in regime 2.[29] For instance, $\Omega_1(1) = \begin{bmatrix} 0 & 0.060 \\ 0 & 0.829 \end{bmatrix}$, $\Omega_1(2) = \begin{bmatrix} 0 & 1.032 \\ 0 & 0.883 \end{bmatrix}$ when $\alpha(2) = 0.8$ and $\theta(2) = 1.1$. This implies that a fiscally-led equilibrium can emerges only when a policy coordination is made to convince agents to believe that monetary policy will be very passive and fiscal policy will be very active once the economy switches to regime 2.

Second, the model can be indeterminate even when there is a unique stable *MOD* solution, confirming Corollary 1. For instance, suppose that the regime 1 is AM-PF with $\alpha(1) = 1.5$ and $\theta(1) = 0.95$ and regime 2 is a PM-AF policy mix with $\alpha(2) = 0.95$ and $\theta(2) = 1.01$. The model is determinacy-admissible with $\rho(\bar{\Psi}_{\Omega_1 \otimes \Omega_1}) = 0.962$, $\rho(\Psi_{F_1 \otimes F_1}) = 1.0015$. Therefore, the model is indeterminate from Proposition 3. We apply the Gröbner basis technique and find that there are four MSV solutions and confirm that the *MOD* solution is indeed the forward solution.[30] For the second MSV solution,

[27] Strictly speaking, the LRTP is defined in the monetary block with $\Omega_1 = 0$ and $F_1(s_t) = 1/\alpha(s_t)$. But as long as $\theta < 1$, it is straightforward to see that the LRTP result is identical.

[28] In both examples, the monetary equilibrium prevails. By the same token, we test and find that the fiscal equilibrium prevails if the fiscal policy is active under regime-switching monetary policy and if the monetary policy is passive under regime-switching fiscal policy. In the former, the model is either determinate or has no stable solution. In the latter, the model is either determinate or indeterminate.

[29] Recall that the model is determinacy-inadmissible when $\alpha = \theta$ in LRE case, which can be interpreted as the 45 degree line passing through $\alpha = \theta = 1$. In the MSRE models, there may exist such a region, but it should be strictly inside the indeterminacy region or the region of no stable solution. This is because such an area can never be associated with determinacy, as Proposition 3 indicates.

[30] One MSV solution is monetary, which has an analytical form such that $\Omega^M(s_t) = B(s_t)$, and the other three have no analytical forms but they are fiscal solutions in that inflation depends on b_{t-1} with a regime-dependent coefficient.

$\rho(\bar{\Psi}_{\Omega_2 \otimes \Omega_2}) = 1.011$. This implies that the *MOD* solution is the unique stable MSV solution, but the model is indeterminate. This example highlights the importance of checking the non-existence of the stable sunspots to correctly identify whether the model is determinate, and this is exactly what our proposed methodology summarized in Proposition 3 does.

6. Conclusion

From a mathematical point of view, determinacy implies that a given rational expectations model has a single stable solution. Not all economists would mechanically endorse the determinate solution as economically reasonable. Nevertheless, it is also true that determinacy has been regarded as one of the most important equilibrium properties for most economic models under rational expectations. This paper contributes to the literature by providing a complete and tractable technical foundation for analyzing general Markov-switching rational expectations models using the concept of mean-square stability. Specifically, the proposed methodology computes the *MOD* solution to the set of general Markov-switching rational expectations models. Using this solution, it classifies the full set of MSRE models into determinacy, indeterminacy and no stable solution, and provides necessary and sufficient conditions for each case. The methodology is also computationally efficient and comparable to the standard solution techniques in the linear rational expectations models as long as MSRE models are determinacy-admissible. Therefore, applied works should be easy to conduct in the context of Markov-switching rational expectations models.

Although our approach works for exogenous regime switching models, it may also provide a basic analysis for endogenous regime-switching models and pave the way for understanding and developing methodologies for endogenous regime-switching models.

Appendix A. Set of Sunspot Processes

This appendix presents a set of quasi-linear sunspot processes that may depend on past regimes. In this paper, we consider the following sunspot processes, generalizing (11):

$$w_{t+1} = \Lambda(s_0, ..., s_{t+1})w_t + \eta_{t+1}, \quad \eta_{t+1} = H(s_0, ..., s_{t+1})\varsigma_{t+1} \tag{30}$$

where $H(\cdot)$ is an $n \times n$ matrix and ς_t is an $n \times 1$ vector of stochastic processes independent of regime variables. As shown by Barthélemy and Marx (2019), sunspot processes may depend on past regimes. But we consider a set of the sunspot processes that may depend on past regimes from time 0 only, although $\Lambda(\cdot)$ and $H(\cdot)$ may also depend on $s_{-i},\ i = 1, 2, ...$ To be more explicit on the sunspot component, we extend Assumption 2 as follows.

Assumption 3. η_t in (30) is asymptotically covariance stationary. w_0 and $(s_0, ..., s_t)$ are independent of ς_t for all $t \geq 1$.

The key observation is, however, that (7) still imposes a recursive restriction between time t and $t+1$ on all sunspot processes under Assumptions 3. Therefore, the set of past regime-dependent sunspot processes can still be expressed as a recursive evolution. To see this, define an extended regime variable $s_{q,t} = (s_{t-q+1}, ..., s_t)$ for $q = 1, ..., t+1$, and for $t \geq 0$. An indexation rule for $s_{q,t}$ is that $s_{q,t} = 1, ..., S$ corresponds to $(s_{t-q+1}, ..., s_t) = (1, ..., 1, s_t)$ with $s_t = 1, ..., S$. $s_{q,t} = S+1$ through $2S$ when $(s_{t-q+1}, ..., s_t) = (1, ..., 2, s_t)$ and so on. Therefore, if w_{t+1} depends on $(s_{t-q+1}, ..., s_{t+1})$, the transition between time t and $t+1$ can be understood by the transition from $s_{q,t}$ to $s_{q,t+1}$. Then, $S_q = S^q$ is the number of regimes. The corresponding $S^q \times S^q$ transition probability matrix can be constructed as $P_q = (i_S \otimes I_{S^{q-1}} \otimes i_S')diag(i_{S^{q-2}} \otimes vec(P'))$ for all $q = 2, ..., t$, where i_S is an $S \times 1$ vector of ones, $vec(\cdot)$ is a vectorization operator and $diag(\cdot)$ is a diagonal matrix with the argument vector being the diagonal entries. When $q = 1$, $s_{q,t} = s_t$ and $P_q = P$. The following theorem is a description of all quasi-linear sunspot processes under Assumption 3.

Theorem 1. *The set of all quasi-linear sunspots w_t to (7) under Assumption 3 can be represented as follows.*

$$w_{t+1} = \Lambda(s_{q,t}, s_{q,t+1})w_t + \eta_{t+1}, \tag{31}$$

where $\Lambda(i, j)$ is an $n \times n$ matrix, η_{t+1} is an $n \times 1$ vector such that $E_t\eta_{t+1} = 0_{n\times 1}$ and $\eta_{t+1} = H(s_{q,t}, s_{q,t+1})\varsigma_{t+1}$ where $H(i, j)$ is an arbitrary $n \times h(j)$ matrix and ς_{t+1} is an arbitrary $h(j) \times 1$ vector of asymptotically covariance stationary sunspot shocks independent of regimes, $s_{q,t} = i,\ s_{q,t+1} = j \in \{1, \ldots, S^q\}$ for all $t \geq 0$ and all integers $q \geq 1$ with the following properties: w_0 is given exogenously, $\eta_{t+1} = V(j)V(j)'\eta_{t+1},\ \Lambda(i, j) = V(j)V(j)'\Lambda(i, j) = \Lambda(i, j)V(i)V(i)'$ and

$$V(i)V(i)' = \sum_{j=1}^{S^q} P_q(i, j)F(i, j)\Lambda(i, j), \tag{32}$$

where $F(s_t, s_{t+1}) = F(s_{q,t}, s_{q,t+1})$ without loss of generality, and $V(i)$ is an $n \times k(i)$ matrix of which columns are orthonormal, with $0 \leq k(i) \leq n$ and $k(i) > 0$ for some i.

Despite the complexity of the sunspot space characterized by (31) and (32), the recursive structure of sunspot processes for all $q = 2, ..., t$ is exactly the same as that for $q = 1$. That is, when $q > 1$, the only difference is that the Markov Chain and the transition probability matrix are defined as $s_{q,t}$ and P_q. Therefore, we only need to prove Theorem 1 for $q = 1$, which is identical to Proposition 2 of Cho (2016) except for $\eta_{t+1} = H(s_t, s_{t+1})\varsigma_{t+1}$. Note that the proof of his Proposition 2 applies to our Theorem 1 because it is independent of the presence of $H(\cdot)$ as long as ς_{t+1} is independent of the Markov chain, which is assumed in this paper.

Appendix B. Proof of Proposition 1

Vectorizing (12) directly implies that $\Psi_{\Lambda'\otimes F}u = u$ where $\bar{v}_i = vec(V(i)V(i)')$, $v = [\bar{v}(i)', ...\bar{v}(S)']'$, and $u = v/||v||$ is an eigenvector associated with a unit root. Therefore, $\rho(\Psi_{\Lambda'\otimes F}) \geq 1$. This inequality leads to $\rho(\Psi_{F\otimes F})\rho(\bar{\Psi}_{\Lambda\otimes\Lambda}) \geq 1$ for all $\Lambda(s_t, s_{t+1})$ subject to (12), from Lemma 2 of Cho (2016). Therefore, the proof of Proposition 1 boils down to constructing a $\Lambda_m(s_t, s_{t+1})$ such that $\rho(\bar{\Psi}_{\Lambda_m\otimes\Lambda_m}) = 1/\xi_2$ where $\rho(\Psi_{F\otimes F}) = \xi_2$. Define $\hat{F}(i, j) = F(i, j)/\sqrt{\xi_2}$. Since $\rho(\Psi_{\hat{F}\otimes\hat{F}}) = 1$ by construction, one can easily find an $n^2S \times 1$ eigenvector $\hat{u} = [\hat{u}(1)'...\hat{u}(S)']'$ corresponding to the unit eigenvalue such that:

$$\hat{u} = \Psi_{\hat{F}\otimes\hat{F}}\hat{u}. \tag{33}$$

Reshape the $n^2 \times 1$ subvector $\hat{u}(i)$ into an $n \times n$ matrix $Q(i)$ to write (33) as:

$$\sum_{j=1}^{S} p_{ij}\hat{F}(i, j)Q(j)\hat{F}(i, j)' = Q(i), \tag{34}$$

where $\hat{u}(i) = vec(Q(i))$ for all i. Transposing and vectorizing each equation of (34) implies that $\hat{u}(i) = vec(Q(i)') = vec(Q(i))$. Hence, $Q(i)$ is symmetric. Therefore, using the Schur decomposition theorem, one can construct a $k_i \times k_i$ diagonal matrix $D(i)$ where the diagonal elements are the non-zero eigenvalues of $Q(i)$ and $n \times k_i$ matrix $V(i)$ of which columns are orthonormal bases such that $Q(i) = V(i)D(i)V(i)'$.

Now we construct $\hat{\Lambda}(i, j)$ from (34) as:

$$\begin{aligned}\hat{\Lambda}(i, j) &= Q(j)\hat{F}(i, j)'[V(i)D(i)^{-1}V(i)'] \\ &= [V(j)D(j)V(j)']\hat{F}(i, j)'[V(i)D(i)^{-1}V(i)'].\end{aligned} \tag{35}$$

Then, it can be shown that $\rho(\Psi_{\hat{F}\otimes\hat{F}}) = \rho(\Psi_{\hat{\Lambda}'\otimes\hat{\Lambda}'}) = \rho(\Psi_{\hat{\Lambda}'\otimes\hat{F}}) = 1$ from (35) using the properties of the Kronecker product. Also, note that $Q(i)[V(i)D(i)^{-1}V(i)'] = V(i)V(i)'$. Therefore, post-multiplying $[V(i)D(i)^{-1}V(i)']$ to Eq. (34) and replacing $\hat{F}(i, j)$ with $F(i, j)/\sqrt{\xi_2}$ yields:

$$\sum_{j=1}^{S} p_{ij}\hat{F}(i, j)\hat{\Lambda}(i, j) = \sum_{j=1}^{S} p_{ij}F(i, j)\Lambda_m(i, j) = V(i)V(i)', \tag{36}$$

where $\Lambda_m(i, j)$ in (36) is defined as:

$$\Lambda_m(i, j) = \frac{1}{\xi_2}[V(j)D(j)V(j)']F(i, j)'[V(i)D(i)^{-1}V(i)']. \tag{37}$$

Consequently, $\Lambda_m(s_t, s_{t+1})$ in (37) is indeed a solution to (12) with the properties $\rho(\Psi_{\Lambda_m'\otimes\Lambda_m'}) = \rho(\bar{\Psi}_{\Lambda_m\otimes\Lambda_m}) = 1/\xi_2$. We have proved the case of $q = 1$ described in Theorem 1 in Appendix A. As in Theorem 1, the proof applies to all past regime-dependent sunspot shocks with $q = 2, ..., t + 1$ because one has only to replace s_t, S and P with $s_{q,t}$, S_q and P_q in the proof. *Q.E.D.*

Appendix C. Proof of Proposition 2

Consider an equilibrium path following a solution $x_t = \Omega_h(s_t)x_{t-1}$ ignoring z_t. Then the model in which the expectational term evaluated with this solution can be expressed as:

$$\begin{aligned}x_t &= E_t[A(s_t, s_{t+1})\Omega_h(s_{t+1})]x_t + B(s_t)x_{t-1} \\ &= E_t[A(s_t, s_{t+1})\Omega_k(s_{t+1})]x_t + E_t[A(s_t, s_{t+1})D_{hk}(s_{t+1})]x_t + B(s_t)x_{t-1} \\ &= \{I_n - E_t[A(s_t, s_{t+1})\Omega_k(s_{t+1})]\}^{-1}\{E_t[A(s_t, s_{t+1})D_{hk}(s_{t+1})]x_t + B(s_t)x_{t-1}\} \\ &= E_t[F_k(s_t, s_{t+1})D_{hk}(s_{t+1})]\Omega_h(s_t)x_{t-1} + \Omega_k(s_t)x_{t-1} = \Omega_h(s_t)x_{t-1},\end{aligned}$$

for any $\Omega_k(s_t)$ with $k \neq h$, and $D_{hk}(s_t) = \Omega_h(s_t) - \Omega_k(s_t)$. Therefore, $D_{hk}(s_t) = E_t[F_k(s_t, s_{t+1})D_{hk}(s_{t+1})]\Omega_h(s_t)]$, which can be written as:

$$D_{hk}(i) = \sum_{j=1}^{S} P_{ij}F_k(i, j)D_{hk}(j)\Omega_h(i). \tag{39}$$

By vectorizing (39), we have:

$$\left(\Psi_{\Omega_h'\otimes F_k}\right)u_{hk} = u_{hk}, \tag{40}$$

where $\bar{v}_{hk}(i) = vec(D_{hk}(i)D_{hk}(i)')$, $v_{hk} = [\bar{v}_{hk}(1)', ...\bar{v}_{hk}(S)']'$, and $u_{hk} = v_{hk}/||v_{hk}||$ is an eigenvector associated with a unit root. Eq. (40) implies that $\rho(\Psi_{\Omega_h'\otimes F_k}) \geq 1$. Therefore, $\rho(\Psi_{\Omega_h'\otimes\Omega_h'})\rho(\Psi_{F_k\otimes F_k}) \geq 1$ from Lemma 1 of Cho (2016). Moreover, $\rho(\Psi_{\Omega_h'\otimes\Omega_h'}) = \rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h})$ from Proposition 3.4 of Costa et al. (2005), proving Assertion 1. This directly results in Assertion 2: $\rho(\bar{\Psi}_{\Omega_h\otimes\Omega_h})\rho(\Psi_{F_h\otimes F_h}) \geq \rho(\bar{\Psi}_{\Omega_1\otimes\Omega_1})\rho(\Psi_{F_h\otimes F_h}) \geq 1$, for all $h \geq 2$. Therefore, if $\rho(\bar{\Psi}_{\Omega\otimes\Omega})\rho(\Psi_{F\otimes F}) < 1$ for an $\Omega(s_t)$, it is the unique *MOD* solution, implying that it is real-valued. *Q.E.D.*

Appendix D. Proof of Proposition 3

Proposition 3 is a direct consequence of Propositions 1 and 2, and the fact that any sunspot solution is stable if and only if $\max(\rho(\bar{\Psi}_{\Omega\otimes\Omega}), \rho(\bar{\Psi}_{\Lambda\otimes\Lambda})) < 1$. Since the conditions listed in Table 1 in Proposition 3 are mutually disjoint and exhaustive, those conditions for each case are both necessary and sufficient. *Q.E.D.*

Appendix E. The *MOD* method for linear rational expectations models

This appendix proves the equivalence between our approach and the existing methods that rely on the eigenvalue-eigenvector relation. Then it is shown that the equivalence applies to the FTPL model. We first remark that in LRE models, it follows that $\bar{\Psi}_{G\otimes H} = \Psi_{G\otimes H} = G \otimes H$ for matrices G and H. Also, $\rho(G \otimes H) = \rho(G)\rho(H)$. Note also that when $G = H$, $\rho(G \otimes G) = \rho(G)^2$. Therefore, $\rho(G)^2 < 1$ if and only if $\rho(G) < 1$. Therefore, one can replace $\rho(\bar{\Psi}_{\Omega_i\otimes\Omega_i})$, $\rho(\Psi_{F_j\otimes F_j})$ and $\rho(\bar{\Psi}_{\Lambda\otimes\Lambda})$ with $\rho(\Omega_i)$, $\rho(F_j)$ and $\rho(\Lambda)$, respectively for all $i, j = 1, ..., N$ in Propositions 1 through 3, Table 1 and Theorem 1. This proves our *MOD* method for LRE models – summarized in Table 2 – without the use of the eigensystem.

E1. Equivalence with Standard Methods

The standard approach for checking determinacy can be described by the generalized eigenvalues implied by the model (5) in the absence of regime-switching. While there are several ways of representing the eigensystem, we write the model such that:[31]

$$\tilde{B}y_t = \tilde{A}E_t y_{t+1} + \tilde{C}z_t, \tag{41}$$

where $y_t = [x_t' \ x_{t-1}']'$ and,

$$\tilde{B} = \begin{bmatrix} 0_{n\times n} & B \\ I_n & 0_{n\times n} \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} -A & I_n \\ 0_{n\times n} & I_n \end{bmatrix}. \tag{42}$$

Let $\xi_{A,B}$ be the set of generalized eigenvalues implied by the model. Formally,

$$\xi_{A,B} = \{\xi_i \in \mathcal{C} \mid 0 = |\tilde{A} - \xi_i\tilde{B}|, \ |\xi_1| \leq |\xi_i| \leq |\xi_{2n}|, i = 1, ..., 2n\}. \tag{43}$$

As is well-known, Ω is associated with n of $2n$ generalized eigenvalues, thus the maximum number of MSV solutions is $\binom{2n}{n}$. Moreover, the corresponding F is associated with the inverses of the remaining n roots in $\xi_{A,B}$ as McCallum (2007) demonstrates. The gensys algorithm also shares the same eigensystem with the present one.[32]

Now we show that the condition for determinacy-admissible models can be stated using the eigensystem as follows:

$$\Omega(\xi_1, ..., \xi_n) \in \mathcal{S} \text{ and } |\xi_n| < |\xi_{n+1}|. \tag{44}$$

This condition implies that the *MOD* solution is the one associated with the n smallest eigenvalues, $\Omega_1 = \Omega(\xi_1, ..., \xi_n)$, and it includes the *existence* and *uniqueness* of the *MOD* solution. A strict inequality $|\xi_n| < |\xi_{n+1}|$ implies that the *MOD* solution is unique and thus, it is real-valued. Since $F_1 = F(1/\xi_{n+1}, ..., 1/\xi_{2n})$, $\rho(\Omega_1)\rho(F_1) = |\xi_n|/|\xi_{n+1}| < 1$. Therefore, (44) is exactly the same as the condition for determinacy-admissible models. Then the condition for determinacy, indeterminacy and no stable solution can be respectively stated as $|\xi_n| < 1 \leq |\xi_{n+1}|$, $|\xi_{n+1}| < 1$ and $|\xi_n| \geq 1$. Thus it is obvious that these conditions coincide with those in Table 2. It is also straightforward to show that $\{\alpha\xi_i, i = 1, ..., 2n\}$ is the set of the generalized eigenvalues of the order-preserving transformation such that $x_t = \left(\frac{1}{\alpha}\right)AE_t[x_{t+1}] + \alpha Bx_{t-1}$, a special case of (23).

Next, the condition for determinacy-inadmissible models is given by:

$$|\xi_n| = |\xi_{n+1}| \text{ or } \Omega(\xi_1, ..., \xi_n) \notin \mathcal{S}. \tag{45}$$

[31] Our model representation (41) is consistent with Klein (2000) or Uhlig (1997) and more general than Blanchard and Kahn (1980) and King and Watson (1998) in that matrix A can be singular.

[32] Model (41) can be written in gensys form by defining $k_t = E_t x_{t+1}$, $x_t = k_{t-1} + \eta_t$ and $\Gamma_0\hat{y}_t = \Gamma_1\hat{y}_{t-1} + \Pi z_t + \Psi\eta_t$ where $\hat{y}_t = [x_t' \ k_t']'$. It can be shown that $\Gamma_0 = \tilde{A}J$, $\Gamma_1 = \tilde{B}J$ in (42) where J is a permutation matrix, and Π and Ψ are appropriately defined. Therefore, generalized eigenvalues of the matrix pencil $\Gamma_0 - \xi\Gamma_1$ are exactly the same as those of (43), as they are invariant to permutation. It is conventional to define k_t manually to include only non-zero expectational variables in vector $\hat{y}_t$. But such a representation in gensys form is model-dependent whereas ours is not. Importantly, non-zero eigenvalues are invariant to any type of gensys form.

Table E1
Classification of LRE models by eigensystem.

	Determinacy-Admissible: $\Omega(\xi_1, ..., \xi_n) \in \mathcal{S}$ and $\lvert\xi_n\rvert < \lvert\xi_{n+1}\rvert$	Determinacy-Inadmissible: $\Omega(\xi_1, ..., \xi_n) \notin S$ or $\lvert\xi_n\rvert = \lvert\xi_{n+1}\rvert$
Determinacy	$\lvert\xi_n\rvert < 1 \leq \lvert\xi_{n+1}\rvert$	Impossible
Indeterminacy	$\lvert\xi_{n+1}\rvert < 1$	$\rho(\Omega_1) = \lvert\xi_{n+i}\rvert < 1,\ i \geq 0$
No Stable Solution	$\lvert\xi_n\rvert \geq 1$	$\rho(\Omega_1) = \lvert\xi_{n+i}\rvert \geq 1,\ i \geq 0$

First, if $\Omega_1 = \Omega(\xi_1, ..., \xi_n)$ exists and $|\xi_n| = |\xi_{n+1}|$, the *MOD* solution is not unique – there exists another MOD solution, either complex or real-valued with the same $\rho(\Omega_1)$ –, thus $\rho(\Omega_1)\rho(F_1) = |\xi_n|/|\xi_{n+1}| = 1$. Second, if $\Omega(\xi_1, ..., \xi_n)$ does not exist, then $\rho(\Omega_1) = |\xi_{n+i}|$ for some $i\ \geq 1$. Therefore, the corresponding F_1 must contain a root $1/|\xi_{n+i-j}|$ for some $j \geq 1$, implying that $\rho(\Omega_1)\rho(F_1) \geq 1$. Henceforth, (45) is also equivalent to the condition for determinacy-inadmissible models. Moreover, it is also straightforward to see that a model of this type is indeterminate (has no stable solution) if and only if $\rho(\Omega_1) = |\xi_{n+i}| < (\geq)1$. The classification result by the eigensystem is presented in Table E.3:

To summarize, the *MOD* solution plays exactly the same role as the eigensystem, leading to the equivalence between our approach and standard methods relying on the eigensystem. This is the idea with which we extend the *MOD* method to MSRE models.

We discuss the likelihood and economic relevance of determinacy-inadmissible models in terms of the eigensystem. First, it is difficult to recall any reasonable economic model with multiple *MOD* solutions, complex-valued or repeatedly real-valued. We have demonstrated this feature using a simple FTPL model. Second, models with $\Omega(\xi_1, ..., \xi_n) \notin \mathcal{S}$ are also hard to find in economics. Sims (2007) shows that this case arises when models contain completely unrelated equations with a particular structure. This is precisely the model in which the usual root-counting approach fails: one may falsely conclude that a model is determinate if $|\xi_n| < 1 \leq |\xi_{n+1}|$, i.e., the number of unstable generalized eigenvalues is n. But such a model has no stable solution because $\rho(\Omega_1) \geq |\xi_{n+1}| \geq 1$. Root-counting is not intrinsically ill-designed. It fails just because the existence of the *MOD* solution is not examined. But it also reveals that any economic model would hardly contain completely decoupled equations. It is even harder for researchers to conclude that such models are determinate, without actually computing the determinate solution. To summarize, our method is equivalence to standard approach and our method for LRE models. Additionally, it is shown that our approach identifies the conditions under which the so-called root counting approach fails in classifying LRE models, consistent with the gensys algorithm of Sims (2002). But the *MOD* method is currently the only one that can be extended to MSRE models.

E2. Equivalence result for the FTPL model

We have shown that the classification result of the FTPL example is exactly the same as the well-known taxonomy of Leeper (1991). Here we also show the equivalence of our approach with the generalized eigensystem. The generalized eigenvalues implied by the model are given by $(0, \alpha, \theta, \infty)$ with $\xi_1 = 0$ and $\xi_4 = \infty$. The solution with the two smallest generalized eigenvalues always exists, thus, the model is determinacy-admissible if and only if $\alpha \neq \theta$. Hence, the model is determinate if and only if there are exactly two eigenvalues strictly inside the unit circle. The AM-PF combination implies determinacy because $\xi_2 = \theta < 1\ \leq \xi_3 = \alpha$, thus Ω^M is the determinate solution. The fiscal equilibrium is the *MOD* solution under the PM-AF combination as $\xi_2 = \alpha\ < 1\ \leq \xi_3 = \theta$. PM-PF combinations are again the case in which $0 < \alpha, \theta < 1$ implying indeterminacy, whereas the AM-AF combination is associated with $0\ < 1 \leq \alpha, \theta$. When $\xi_2 = \xi_3 = \alpha = \theta$, the *MOD* solution still exists but is not unique, thus the model is either indeterminate or has no stable solution.

Appendix F. Modified forward method

This appendix examines the existence condition of the forward solution, discusses its relation with the *MOD* solution and develops a modified forward method.

F1. Existence of the Forward Solution

We solve the model (5) forward, ignoring z_t, following Cho (2016) such that:

$$x_t = E_t[M_k(s_t, s_{t+1}, \ldots, s_{t+k})x_{t+k}] + \Omega_k(s_t)x_{t-1}, \tag{46}$$

where $\Omega_1(s_t) = B(s_t)$, $F_1(s_t, s_{t+1}) = M_1(s_t, s_{t+1}) = A(s_t, s_{t+1})$ and for $k \geq 2$,

$$\Omega_k(s_t) = \{I_n - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]\}^{-1}B(s_t), \tag{47a}$$

$$F_k(s_t, s_{t+1}) = \{I_n - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]\}^{-1}A(s_t, s_{t+1}). \tag{47b}$$

Cho (2016) shows that for the sequences defined in (47a,47b), if $\Omega^*(s_t) = \lim_{k\to\infty} \Omega_k(s_t)$, the corresponding $F^*(s_t, s_{t+1}) = lim_{k\to\infty} F_k(s_t, s_{t+1})$ exists, and the forward solution defined as $x_t = \Omega^*(s_t)x_{t-1}$ is a MSV solution. Moreover, it is the only solution satisfying the so-called no-bubble condition: $\lim_{k\to\infty} E_t[M_k(s_t, s_{t+1}, \ldots, s_{t+k})x_{t+k}] = 0_{n\times 1}$. This sequence is uniquely defined by the model and is real-valued. By construction, $\Omega^*(s_t)$ is real-valued and unique if it exists. To examine the existence condition of the forward solution, we differentiate (47a) with respect to $\Omega_k(s_t)$ at each regime and each $k > 1$, and vectorize it such that:

$$vec(d\Omega_k) = \left[\Psi_{\Omega'_{k-1}\otimes F_{k-1}}\right] vec(d\Omega_{k-1}), \tag{48}$$

where $d\Omega_k = [d\Omega_k'(1)...d\Omega_k'(S)]'$. Formally,

Proposition 4. *Consider the forward representation of a MSRE model, (46). Then the forward solution $\Omega^*(s_t)$ exists in the following cases.*

1. *There exists a small real number $\epsilon > 0$ and an integer K such that $\rho(\Psi_{\Omega'_{k-1}\otimes F_{k-1}}) < 1 - \epsilon$ for all $k \geq K > 1$.*
2. *$\rho(\Psi_{\Omega'_{k-1}\otimes F_{k-1}}) \geq 1$ for some $k > 1$ and $u_k' vec(d\Omega_{k-1}) = 0$ for every eigenvector u_k associated with an unstable root of $\Psi_{\Omega'_{k-1}\otimes F_{k-1}}$.*

Proof. In both cases, $\lim_{k\to\infty} d\Omega_k(s_t) = 0_{n^2\times 1}$ for all s_t, implying the existence of $\Omega^*(s_t)$ from (48). *Q.E.D.* □

In the first case, $\rho(\Psi_{(\Omega^*)'\otimes F^*}) < 1$ and there are two possibilities. 1) $\rho(\bar{\Psi}_{\Omega^*\otimes\Omega^*})\rho(\Psi_{F^*\otimes F^*}) < 1$. Then $\Omega^*(\cdot) = \Omega_1(\cdot)$ and the model is determinacy-admissible. 2) $\rho(\bar{\Psi}_{\Omega^*\otimes\Omega^*})\rho(\Psi_{F^*\otimes F^*}) \geq 1$. Our extensive experiment using the modified forward method has not found this case for determinacy-admissible models. For determinacy-inadmissible models, 2) must hold and the two solutions may differ. If so, it must be true that $\rho(\Psi_{\Omega'\otimes F}) < 1$ for both solutions. This is not possible in LRE models because there cannot be two solutions such that $\rho(\Psi_{\Omega'\otimes F}) = \rho(\Omega)\rho(F) < 1$ from Proposition 2. In short, the equivalence of the two solutions is highly likely at least for determinacy-admissible models. If the forward solution differs from the *MOD* solution, then one can apply a numerical search method to find the *MOD* solution, although there has been no such case.

In the second case, the forward solution still exists such that $\rho(\Psi_{(\Omega^*)'\otimes F^*}) \geq 1$ while the *MOD* solution can differ from the forward solution. An example – and the only type of examples known so far – is a special block recursive model such as the FTPL model with a PM-AF policy mix. The original forward method of Cho and Moreno (2011) computes $\Omega^* = \Omega^M$ with $(\Omega^M \otimes F^M)vec(d\Omega^M) = 0_{4\times 1}$ with $k = 1$, despite $\rho(\Omega^M) \otimes \rho(F^M) = \theta/\alpha > 1$. The *MOD* solution here is the fiscal solution and indeed $\rho(\Omega^F) \otimes \rho(F^F) = \alpha/\theta < 1$. The modification is to recover the equivalence of the forward solution and the *MOD* solution in such a case, eliminating the second case in general as follows.

F2. Modified forward method for MSRE models

The original forward method is equivalent to solving the block of a model separately if it is a well-defined model in isolation. Thus, the partial information is used in computing the forward solution. The modified forward method is to adjust the model such that full information of the state variables is used to form expectations of the original model. In fact, Cho (2016) has already used this idea but this paper formalizes it in general setup as follows. Define the expectational variables as $k_t = E_t[A(s_t, s_{t+1})x_{t+1}]$. Forward the model (5) one period ahead and take expectations as:

$$E_t[x_{t+1}] = E_t[k_{t+1}] + E_t[B(s_{t+1})]x_t. \tag{49}$$

$E_t[B(s_{t+1})]$ at each regime s_t can be easily computed by $\sum_j^S P(i,j)B(s_{t+1} = j)$ at each $s(t) = i = 1, ..., S$. Then we can augment this expectational relation, (49) to the original model as a constraint as:

$$x_t = k_t + B(s_t)x_{t-1}, \tag{50}$$

$$k_t = E_t[A(s_t, s_{t+1})x_{t+1}] + H(E_t[x_{t+1}] - E_t[k_{t+1}] - E_t[B(s_{t+1})]x_t), \tag{51}$$

where H is an $n \times n$ matrix in which every single element is arbitrary but non-zero. Applying the expectational relation of the whole model is innocuous because it must hold regardless of block-recursiveness of the model. Let $y_t = [x_t' \ k_t']'$ be the $2n \times 1$ vector. Notice that $E_t[B(s_{t+1})]$ is a function of s_t. Therefore, one can write (50) and (51) as:

$$y_t = E_t[A^y(s_t, s_{t+1})y_{t+1}] + B^y(s_t)y_{t-1}. \tag{52}$$

The modified forward method is to apply the original forward method to (52) and obtain the solution for y_t such that $y_t = (\Omega^y(s_t))^* y_{t-1}$. Finally, the first $n \times n$ component of $(\Omega^y(s_t))^*$ is the forward solution of the original model under full information.

To summarize, the forward solution is highly likely to be the *MOD* solution under the modified forward method as long as the model has a real-valued *MOD* solution, although there is no known proof for the equivalence between the two solutions in general.

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