Engineering Analysis & Design (Modelling & Simulation)

Course Code: CO207

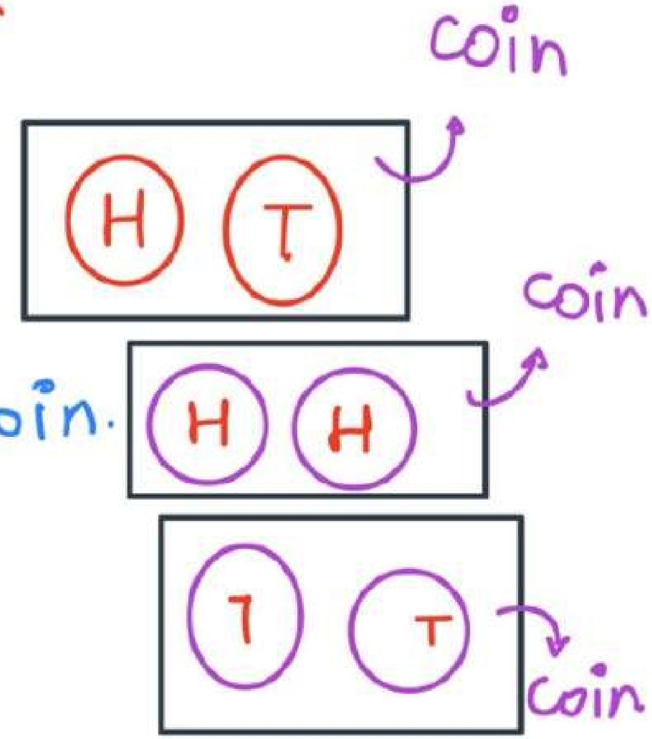
Unit -4:Evaluation of Simulation Output

Experiment

Experiment: Any process of observation is called as experminent.

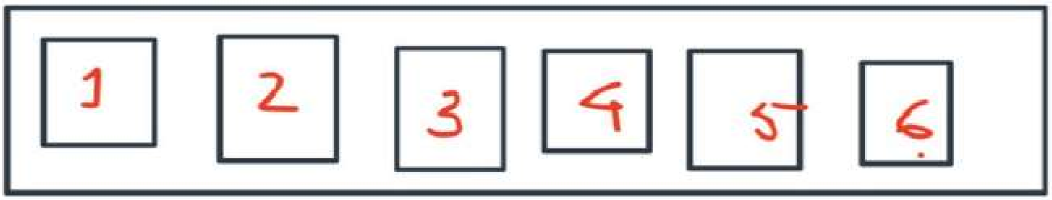
1) Tossing of Fair/Unbiased Coin.

2) Tossing of unfair/Biased coin.

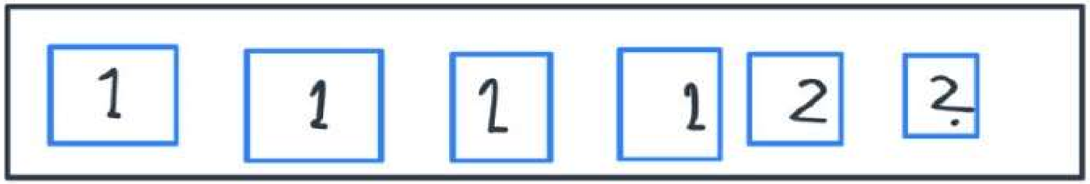


Experiment

3) Rolling of fair/unbiased dice:



4) Rolling of unfair/Biased dice.



Experiment

Outcome: Result of an experiment.

Experiment	outcome:	R.E.
Tossing of coin	H T	✓
Tossing of unfair coin	H H	not Random Experiment.
Rolling of dice.	1 2 3 4 5 6.	✓

Random Experiment

RANDOM Experiment:

→ outcomes cannot be predicted with certainity.

→ Outcomes are uncertain. or outcomes are having associated probability.

$$0 \leq P(E) \leq 1 \rightarrow \text{Random Experiment}$$

$$P(E) = 1 \rightarrow$$

$$P(E) = 0 \rightarrow$$

Non Random Experiment

Term Associated with Random Experiment

TERMS ASSOCIATED with Random Experiment:

1) **Sample Space:** "SET" of all outcomes:

Experiment	outcomes	Sample space
Tossing of fair coin	H, T	$S = \{H, T\}$
Rolling of fair dice	1, 2, 3, 4, 5, 6	$S = \{1, 2, 3, 4, 5, 6\}$

Random Experiment

Q. A coin is 3 times or 3 coins are Tossed simultaneously, form the sample space.

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

Sample Point

2) <u>Sample point:</u> Each element of sample space

or

Each individual outcome of R.E.

$\{1\}$

Representation of sample space

General Rep. of sample space:

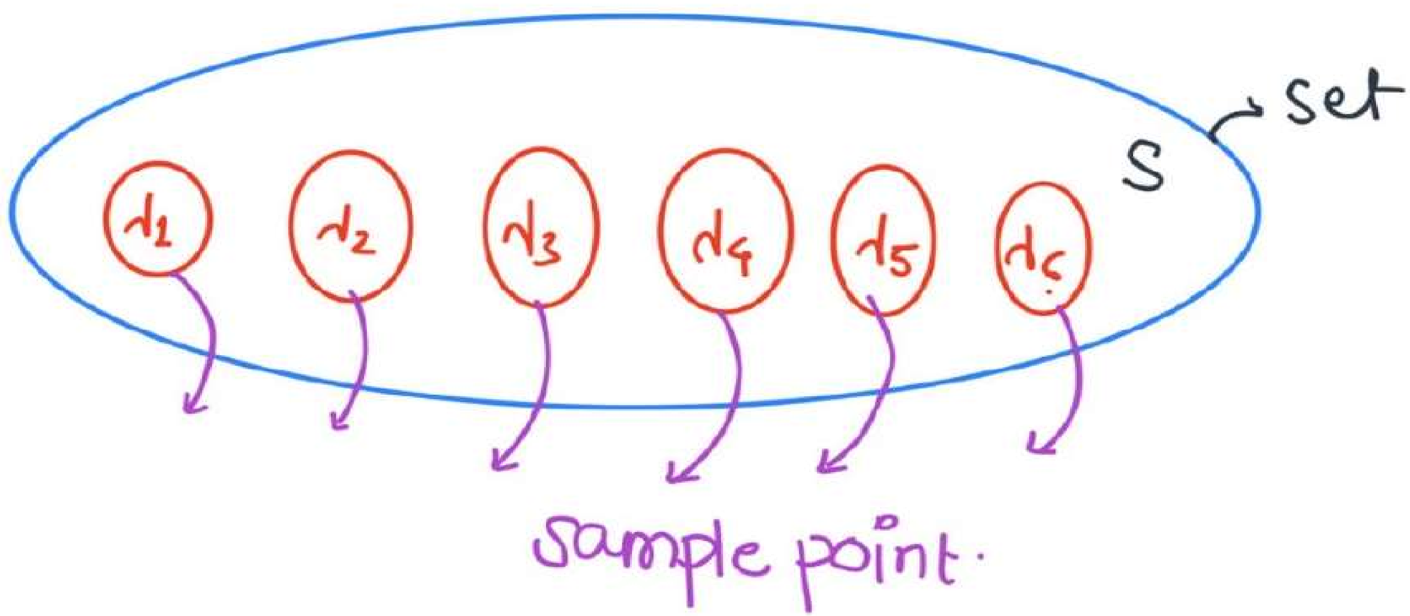
i) Set Representation

$$S = \{ d_1, d_2, d_3, d_4, d_5, d_6 \}$$

$$S = \{ d_i \}_{i=1}^{6}$$

Representation of sample space

ii) VENN DIAGRAM:



Event

$\boxed{\text{EVENT:}}$ → Any subset of sample space.

R.E. 2 coins are Tossed simultaneously.

$S = \{TT, TH, HT, HH\}$

sample point

$E_1 = \{TT\}$

$E_4 = \{TT, TH, HT, HH\}$

certain event

$E_2 = \{TT, TH\}$

$E_3 = \{\phi\}$ → Impossible Event

Event

$\boxed{\text{NOTE:}}$ → Every sample point is an event but converse is not necessarily True.

n: sample point $\Longrightarrow$ Total 2^n Events:

Random Variable

- The **range of all possible outcomes of an experiment** is known as the **"Sample Space (S)".**
- **Random variable:** Real value of random experiment is called random variable.
- A random variable X is a function (or rule) that assigns a real number R (any number greater than -∞ and less than +∞) to each point in the sample space S.

X: S→R

Example of Random Variable

- A typical example of a random variable is the outcome of a coin toss. Consider a probability distribution in which the outcomes of a random event are not equally likely to happen. If random variable Y, is the number of heads we get from tossing two coins, then Y could be 0, 1, or 2. This means that we could have no heads, one head, or both heads on a two-coin toss.

- However, the two coins land in four different ways: TT, HT, TH, and HH. Therefore, the $P(Y=0) = 1/4$ since we have one chance of getting no heads (i.e., two tails [TT] when the two coins are tossed). Similarly, the probability of getting two heads (HH) is $P(Y=2) = 1/4$. Notice that getting one head has a likelihood of occurring twice: in HT and TH. In this case, $P(Y=1) = 2/4 = 1/2$.

Types of Random Variable

There are two main types of random variables:

- ❑**Discrete random variable**
- ❑**Continuous random variable**

Discrete Random Variable

- A **discrete random variable** is a variable which can take on only finite number of values in a finite observation interval. So, we can say that discrete random variable has distinct values that can be counted.

- Ex.1: Number of steps to the top of the Eiffel tower.
- Ex.2: Number of customers arrived in restaurant.

Continuous Random Variable

- A **continuous random variable** that takes any value along a given interval of a number line such as length, depth, volume, time, weight and so on.

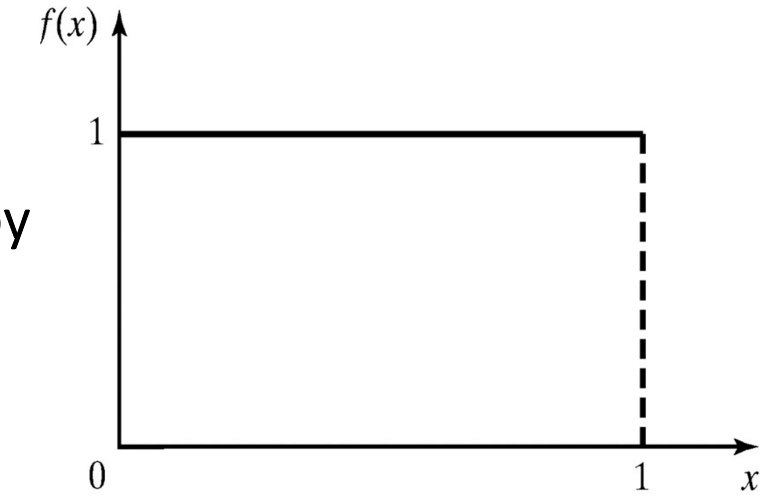
- Ex.1: The time a tourist stays at the top once he/she gets there.
- Ex.2: The weight of persons arrived in showroom.

Properties of Random Variables

- Uniformity
- Independence

- Each Random Number, R_i, must be an independently drawn sample from a uniform distribution with probability density function as

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$



- The mean(expected value) of each is given by

$$E(R) = \int_0^1 x\,dx = \frac{1}{2}$$

Generation of Random Numbers

- Random numbers are **widely used ingredient in the simulation** of almost all discrete systems.
- **Simulation languages generate random numbers** that are **used to generate event times** and other random variables.
- Random number generators have **applications in gambling, statistical sampling, computer simulation, cryptography, completely randomized design** and other areas where producing an unpredictable result is desirable.
- The generation of pseudo random numbers is an important and common task in computer programming.

Types of Random Numbers based on their Generation

True Random Numbers

- True random numbers are not produced by any user defined mathematical algorithm.
- These random numbers are generated from naturally occurring phenomenon such as radio active decay of isotopes.
- These numbers are generally used in cryptography.

Pseudo Random Numbers

- These random numbers are generated from user defined mathematical algorithm.
- These numbers are statistically independent and follow a uniform distribution.

Error or Departure of Random Numbers

- The generated random numbers might not be uniformly distributed.
- The mean of generated random numbers might be too high or too low.
- The variance of generated numbers might be too high or too low.
- There might be dependence
  - Autocorrelation between numbers
  - Numbers successively higher or lower than adjacent numbers
  - Several numbers above the mean followed by several numbers below the mean

Mean

- The mean is defined as the sum of the data divided by the number of data
  - The variable often used is μ, the Greek 'mu', or $\bar{x}$. Often μ is associated with a population and $\overline{x}$ is associated with a sample.
  - Symbolically, $\bar{x} = \frac{\sum x}{n}$, where $\sum x = x_1 + x_2 + \cdots + x_n$, and n is the number of data values. (The capital letter sigma, Σ ,represents summation.)
  - Example: Data is (1, 2, 3, 4, 5). The sum is 1+2+3+4+5=15. There are 5 data values, so the average is 15/5=3.

Median

- The median is the middle number when the data is listed in order.
- If there is an even number of data points, the median is the average of the two middle values.
  - Example: Data is (1,2,3,4,5). The median is 3
  - Example: Data is (1,2,3,4,5,6). The median is (3+4)/2=3.5
- Why is this quantity useful?
  - The median ignores outlying values. What if our data had been (1,2,3,4,1000)?
  - The mean is 202, which is not characteristic of any of the actual values.
  - The median is 3, which is more typical of most of the values.
  - The median is helpful when looking for a house to buy. The median house price is the typical price you'd pay, even though the millionaire's house at the corner of the block raises the mean of the house prices above the value most people paid for theirs.

Mode

- The mode represents the most populated class, or the group with the most members. This is yet another reasonable way of finding the middle of the data.
- Determining the mode is different for discrete data than it is for continuous data.
  - For discrete data, the mode is simply the number that appears the most times.
    - Data is (1, 1, 2, 3, 4, 4, 5, 5, 5). The mode is 5.
  - For continuous data, the mode is the center of the range of the class that has the most members in it.
    - Data is (1.1, 1.2, 1.3, 1.8, 2.0, 2.6, 3.1, 4.6, 4.8, 5.1). The class from 1-2 has the most members. The center of this range is 1.5, so the mode is 1.5. (Note: 1.5 does not even appear in the data.)
- In both cases, the mode can be quickly determined from the graph. The mode is the x-value that is at the center of the tallest bar in either the bar graph (discrete data) or histogram (continuous data).

Variance

- *Variance* (*var.* or σ^2 or s^2) is a measure of the spread of data about the average. We don't care which direction the difference is, so we will be ignoring the sign of the difference. In words, the variance is the sum of the squares of the differences divided by one less than the number of data values.
  - The equation is $var. = \frac{\sum(x-\bar{x})^2}{n-1}$

x	$\bar{x}$	$x-\bar{x}$	$(x-\bar{x})^2$
1	3	-2	4
2	3	-1	1
3	3	0	0
4	3	1	1
5	3	2	4
			10

- Example: Data is (1, 2, 3, 4, 5) and mean ($\bar{x}$) is 3.
  - Variance is 10/(5-1)=2.5
- If you are using a calculator, it is most likely that the calculator will compute the standard deviation (σ) instead. To get the variance from the standard deviation, simply find the square of the standard deviation:
  - $var = \sigma^2$

Standard Deviation

- *Standard deviation* (*std. dev.* or σ or *s*) is a measure of the spread of data about the average. We don't care which direction the difference is, so we will be ignoring the sign of the difference. In words, the standard deviation is the square root of (the sum of the squares of the differences divided by one less than the number of data values).
- The equation is $std.dev. = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} = \sqrt{var}$
- Example (from previous slide): Data is (`1, 2, 3, 4, 5`), mean ($\bar{x}$) is 3, and we previously found that the variance is var =2.5
- Since the standard deviation is the square root of variance,
  - Standard deviation is $\sigma = \sqrt{2.5} = 1.58$
- Question: Since standard deviation and variance differ by one keystroke, why do we need both?
  - The units of standard deviation are the same as the data. Variance has other direct uses (e.g. Analysis of Variance) and is also more easily computed.

Techniques for Generating Random Numbers

- ❑Linear Congruential Method (LCM).
- ❑Combined Linear Congruential Generators (CLCG).
- ❑Random-Number Streams.

Linear Congruential Method (LCM)

- To produce a sequence of integers, X_1, X_2, … between 0 and m-1 by following a recursive relationship:

$$X_{i+1} = (aX_i + c) \bmod m, \quad i = 0,1,2,\ldots$$

The multiplier

The increment

The modulus

- When c≠ 0 then form is called **mixed congruential method**
- When c=0 , the form is called **multiplicative congruential method**
- The selection of the values for a, c, m, and X_0 drastically affects the statistical properties and the cycle length.
- The random integers are being generated [0,m-1], and to convert the integers to random numbers

$$R_i = \frac{X_i}{m}, \quad i = 1,2,\ldots$$

Example: Linear Congruential Method (LCM)

- Use $X_0 = 27$, $a = 17$, $c = 43$, and $m = 100$.

$$R_i = \frac{X_i}{m}, \quad i = 1,2,\ldots$$

- The X_i and R_i values are:

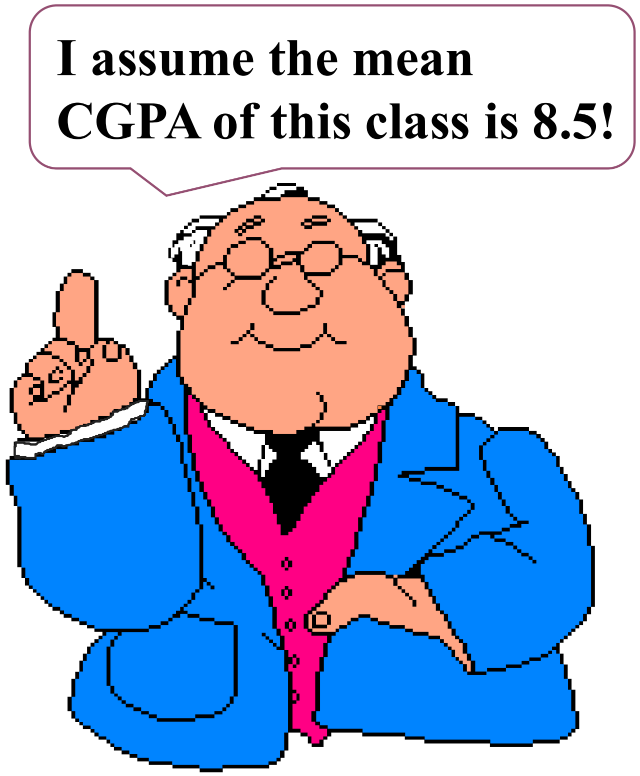
$X_1 = (17*27+43) \bmod 100 = 502 \bmod 100 = 2,$ $\quad R_1 = 0.02;$

$X_2 = (17*2+43) \bmod 100 = 77,$ $\quad R_2 = 0.77;$

$X_3 = (17*77+43) \bmod 100 = 52,$ $\quad R_3 = 0.52;$

What is a Hypothesis?

- A hypothesis is an assumption about the population parameter
  - A parameter is a Population mean or proportion
  - The parameter must be identified before analysis



What is Null Hypothesis(H_0)?

- States the assumption (numerical) to be tested

  e.g. The average # TV sets in Indian homes is at least 3 (H_0: $\mu \geq 3$)
- Begin with the assumption that the null hypothesis is TRUE.
- Similar to the notion of innocent until proven guilty.
- Refers to the Status Quo
- Always contains the ' = ' sign
- The Null Hypothesis may or may not be rejected.

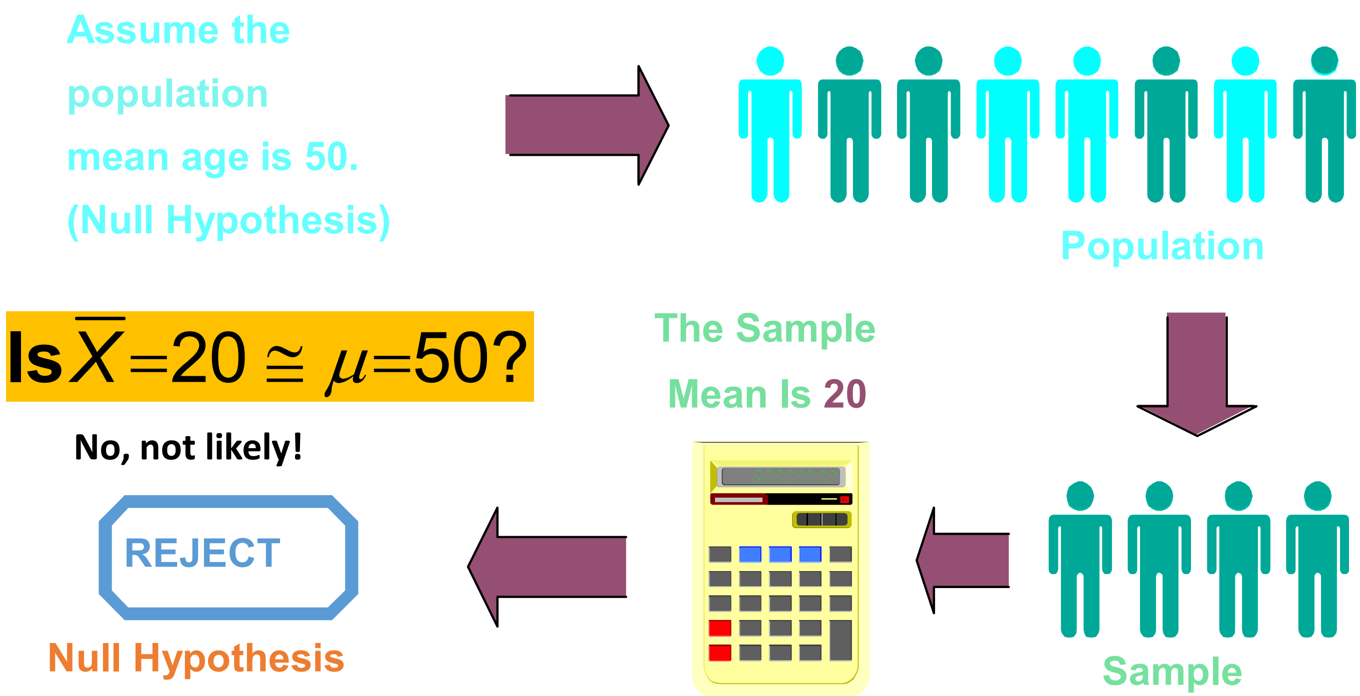
What is Alternative Hypothesis(H_1)?

- Is the opposite of the null hypothesis

e.g. The average # TV sets in Indian homes is less than 3 (H_1: $m < 3$)

- Challenges the Status Quo
- Never contains the '=' sign
- The Alternative Hypothesis may or may not be accepted

Hypothesis Testing Process



Test for Random Numbers

- Two categories:
- **Testing for uniformity:**

$$H_0: \quad R_i \sim U[0,1]$$
$$H_1: \quad R_i \nsim U[0,1]$$

- Failure to reject the null hypothesis H_0, means that evidence of non-uniformity has not been detected.
- **Testing for independence:**

$$H_0: \quad R_i \sim \text{independently}$$
$$H_1: \quad R_i \nsim \text{independently}$$

- Failure to reject the null hypothesis H_0, means that evidence of dependence has not been detected.

When to use these tests?

- If a **well-known simulation languages or random-number generators is used**, it is probably **unnecessary to test**
- **If the generator is not explicitly known or documented**, e.g., spreadsheet programs, symbolic/numerical calculators, **tests should be applied to many sample numbers**.

Types of Tests

- **Theoretical tests:** evaluate the choices of m, a, and c without actually generating any numbers
- **Empirical tests:** applied to actual sequences of numbers produced.

Goodness of Fit Test

- The goodness of fit of a statistical model **describes how well it fits a set of observations**.
- It is a statistical hypothesis test which **provides helpful guidance for evaluating the suitability of a potential input model**.
- Measures of Goodness of fit **typically summarize the discrepancy between observed values and the value expected under the model in question**.
- Two different methods:
  - **Kolmogorov-Smirnov test**
  - **Chi-square test**

Confidence Interval

- Suppose we are interested to know the average weight of 5000 apples.
- We can't measure all of them.
- We need to take a sample of the population to draw conclusion.
- Different samples of the population will give different results.
- A confidence interval refers to **the probability** that a **population parameter** will **fall between a set of values** for a certain proportion of times.
- It **communicates how accurate our estimate is likely to be**.
- We **use confidence interval to express the range** in which we are **pretty sure the population parameter lies**.

Confidence Interval

❑SAMPLE VARIATION

✓If **all the values** in the population **are almost the same** then our sample will have **little variation** and the **confidence interval will be smaller**.

✓Our **estimate will be close to the true population** value.

✓**Greater variation** in the population **leads to wider confidence interval**.

❑SAMPLE SIZE

✓**Sample size** also **affects** the **width of a confidence interval**.

✓**Small samples vary from each other** and **have less information**.

✓**Larger samples** will be **more similar to each other.**

Calculating Confidence Interval using Mean method

- Confidence Interval $= \bar{x} \pm t * \frac{s_x}{\sqrt{n}}$
  - ✓Where $\bar{x}$ is the mean of the sample.
  - ✓S_x is the standard deviation of the sample.
  - ✓n is the sample size.
  - ✓t-score is the number of standard deviations from the mean in a t-distribution
- Note: The confidence interval is mostly taken as 95%.

Numerical on Confidence Interval

Question: We need to know the mean weight of apples in an orchard. 15 apples have been taken as sample. The mean and standard deviation are calculated as 149.43 and 4.758 respectively. Determine the confidence interval with 95%.

Solution:

Mean =149.43, SD=4.758, n =15, t= 2.145 (as per T-distribution table)

Putting the values in the formula $\bar{x} \pm t * \frac{S_x}{\sqrt{n}}$

We get confidence interval = $\bar{x} \pm 2.636$

So the confidence interval will be 146.7 to 151.9.

This is not the interval that will hold the weights of 95% of the apples in the orchard

We can be 95% sure that the mean weight of the apples in the orchard is somewhere between 146.7 and 151.9 grams