Discrete Structure

Chapter 4

Relations/Function and Matrices

Outline

- Relations and their properties
- Representing Relations
- Closures of Relations
- Equivalence Relations
- Partial Orderings

Relations and Their Properties

※The most direct way to express a relationship between elements of two sets is to use ordered pairs.

For this reason, sets of ordered pairs are called **binary relations**.

Def:
Let A and B be sets. A binary relation from A to B is a subset R of $A\times B = \{ (a, b) : a\in A, b\in B \}$.

Example:

A : the set of students in your school.

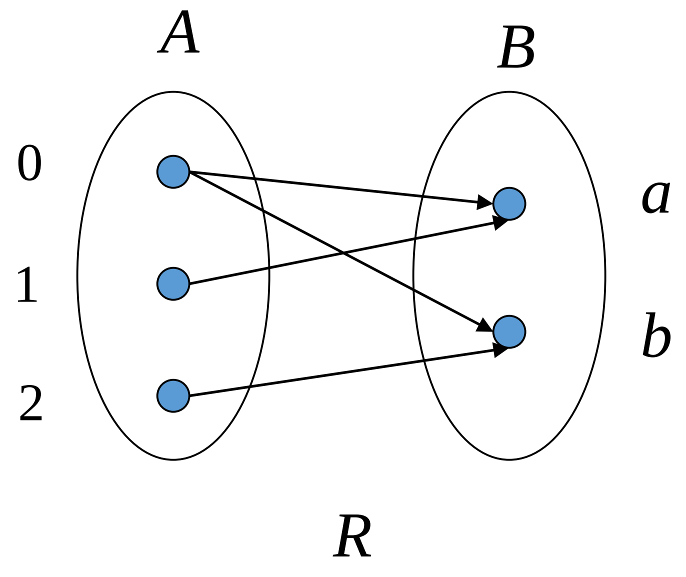
B : the set of courses.

$R = \{ (a, b) : a\in A, b\in B, a \text{ is enrolled in course } b \}$

Def: We use the notation aRb to denote that $(a, b)\in R$, and $a\not R b$ to denote that $(a,b)\notin R$.

Moreover, a is said to be related to b by R if aRb.

Example: Let $A=\{0, 1, 2\}$ and $B=\{a, b\}$, then $\{(0,a),(0,b),(1,a),(2,b)\}$ is a relation R from A to B. This means, for instance, that $0Ra$, but that $1\not R b$.



$$R \subseteq A\times B = \{ (0,a) , (0,b) , (1,a), \underbrace{(1,b)}_{\notin R} , \underbrace{(2,a)}_{\notin R} , (2,b)\}$$

- Binary relations represent relationships between the elements of two sets.
- n-ary relations express relationships among elements of more than two sets

Representing Binary Relations

- We can represent a binary relation R by a **table** showing (marking) the ordered pairs of R.

Example:

- Let A = {0, 1, 2}, B = {u,v} and R = { (0,u), (0,v), (1,v), (2,u) }
- **Table:**

R	u	v
0	x	x
1		x
2	x	

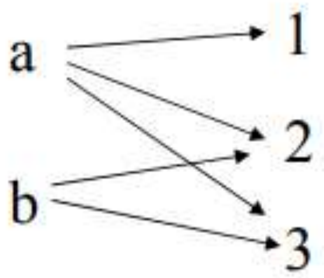
or

R	u	v
0	1	1
1	0	1
2	1	0

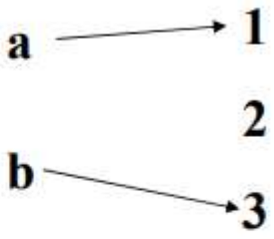
Relations vs. Functions

A relation can be used to express a 1-to-many relationship between the elements of the sets $\boldsymbol{A}$ and $\boldsymbol{B}$.

- **Example:**



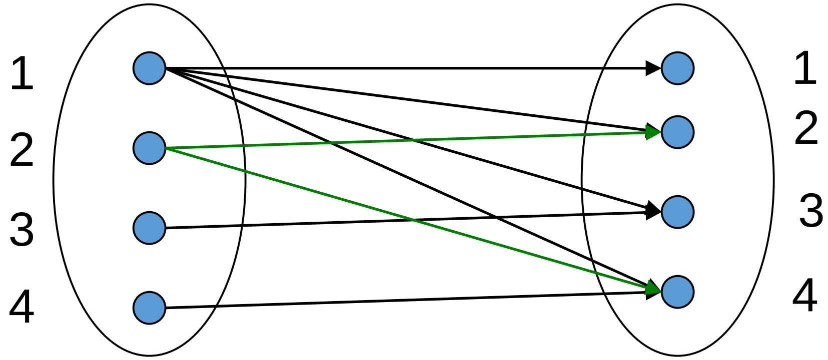
- A function defined on sets A,B A -> B assigns to each element in the domain set A exactly one element from B. So it is a special relation.



Example:

Let ***A*** be the set **{1, 2, 3, 4}**. Which ordered pairs are in the relation **R = { (a, b)| a** divides ***b*** **}**?

Sol :



$$R = \{ (1,1), (1,2), (1,3), (1,4), \\ (2,2), (2,4), \\ (3,3), \\ (4,4) \}$$

Example : Consider the following relations on **Z**.

$R_1 = \{ (a, b) \mid a \le b \}$

$R_2 = \{ (a, b) \mid a > b \}$

$R_3 = \{ (a, b) \mid a = b \text{ or } a = -b \}$

$R_4 = \{ (a, b) \mid a = b \}$

$R_5 = \{ (a, b) \mid a = b+1 \}$

$R_6 = \{ (a, b) \mid a + b \le 3 \}$

Which of these relations contain each of the pairs (1,1), (1,2), (2,1), (1,–1), and (2,2)?

Sol :

	(1,1)	(1,2)	(2,1)	(1,–1)	(2,2)
R_1	●	●			●
R_2			●	●	
R_3	●			●	●
R_4	●				●
R_5			●		
R_6	●	●	●	●	

Properties of Relations

Def. A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called reflexive if $(a,a) \in R$ for every $a \in A$.

Example: Consider the following relations on

$$\{1, 2, 3, 4\} :$$

$$R_2 = \{ (1,1), (1,2), (2,1) \}$$

$$R_3 = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$$

$$R_4 = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$$

which of them are reflexive ?

Sol :

$$R_3$$

Example: Which of the relations from Example 5 are reflexive?

$R_1 = \{ (a, b) \mid a \leq b \}$

$R_2 = \{ (a, b) \mid a > b \}$

$R_3 = \{ (a, b) \mid a = b \text{ or } a = -b \}$

$R_4 = \{ (a, b) \mid a = b \}$

$R_5 = \{ (a, b) \mid a = b+1 \}$

$R_6 = \{ (a, b) \mid a + b \leq 3 \}$

Sol : R_1, R_3 and R_4

Symmetric and Antisymmetric :

(1) A relation R on a set A is called symmetric if for $a, b \in A$,

$$(a, b) \in R \Rightarrow (b, a) \in R.$$

(2) A relation R on a set A is called antisymmetric if for $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$.

That is, a relation is symmetric if and only if a is related to b implies that b is related to a. A relation is antisymmetric if and only if there are no pairs of distinct elements a and b with a related to b and b related to a. That is, the only way to have a related to b and b related to a is for a and b to be the same element. The terms symmetric and antisymmetric are not opposites, because a relation can have both of these properties or may lack both of them

Example: Which of the relations from are symmetric or antisymmetric ?

$R_2 = \{ (1,1), (1,2), (2,1) \}$

$R_3 = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$R_4 = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

Sol :

R_2, R_3 are symmetric

R_4 are antisymmetric.

Example: Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Sol : It is not symmetric since $1|2$ but $2\nmid 1$.

It is antisymmetric since $a|b$ and $b|a$ implies $a=b$.

Note:

- Antisymmetric and symmetric can coexist

$$\forall (a, b) \in R,\ a \neq b \quad \begin{cases} \text{sym.} \Rightarrow (b, a) \in R \\ \\ \text{antisym.} \Rightarrow (b,a) \notin R \end{cases}$$

Therefore, if there is no (a, b) with a≠b in R, it can be satisfied at the same time

eg. Let $A = \{1,2,3\}$, give a relation R on A set. R is both symmetric and antisymmetric, but not reflexive.

Sol :

$$R = \{\ (1,1),(2,2)\ \}$$

Transitive:

A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called transitive if for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{A}$, $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}$ and $(\boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{R} \Rightarrow (\boldsymbol{a}, \boldsymbol{c}) \in \boldsymbol{R}$.

Example: Is the “divides” relation on the set of positive integers transitive?

Sol : Suppose $a|b$ and $b|c$

$\Rightarrow a|c$

$\Rightarrow$ transitive

Example: Which of the relations in $\boldsymbol{R_2}$, $\boldsymbol{R_3}$ and $\boldsymbol{R_4}$ are transitive ?

$\boldsymbol{R_2} = \{ (1,1), (1,2), (2,1) \}$

$\boldsymbol{R_3} = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$\boldsymbol{R_4} = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

Sol :

$\boldsymbol{R_2}$ is not transitive since

$(2,1) \in \boldsymbol{R_2}$ and $(1,2) \in \boldsymbol{R_2}$ but $(2,2) \notin \boldsymbol{R_2}$.

$\boldsymbol{R_3}$ is not transitive since

$(2,1) \in \boldsymbol{R_3}$ and $(1,4) \in \boldsymbol{R_3}$ but $(2,4) \notin \boldsymbol{R_3}$.

$\boldsymbol{R_4}$ is transitive.

Combining Relations

Example: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

The relation $R_1 = \{(1,1), (2,2), (3,3)\}$

and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain

$$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2), (3,3)\}$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

$$R_1 \oplus R_2 = \{(2,2), (3,3), (1,2), (1,3), (1,4)\}$$

symmetric difference, which is $(R_1 \cup R_2) - (R_1 \cap R_2)$

Def: Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a,c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by $S \circ R$.

- ✓ $S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S.
- ✓ For example, the ordered pairs (2, 3) in R and (3, 1) in S produce the ordered pair (2, 1) in $S \circ R$. Computing all the ordered pairs in the composite, we find

Example : What is the composite of the relations R and S, where R is the relation from{1, 2, 3} to {1, 2, 3, 4} with R = {(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)} and S is the relation from {1, 2, 3, 4} to {0, 1, 2} with S = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}?

Ans. $S \circ R$ is the relation from $\{1, 2, 3\}$ to $\{0, 1, 2\}$ with

$S \circ R = \{(1, 0), (1,1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

Def: Let $\boldsymbol{R}$ be a relation on the set $\boldsymbol{A}$. The powers $\boldsymbol{R}^n$, $\boldsymbol{n}$ = 1, 2, 3, …, are defined recursively by $\boldsymbol{R}^1 = \boldsymbol{R}$ and $\boldsymbol{R}^{n+1} = \boldsymbol{R}^n \circ \boldsymbol{R}$.

Example: Let $\boldsymbol{R} = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$.
Find the powers $\boldsymbol{R}^n$, $\boldsymbol{n}=2, 3, 4, \ldots$.

Sol. $\boldsymbol{R}^2 = \boldsymbol{R} \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

$\boldsymbol{R}^3 = \boldsymbol{R}^2 \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

$\boldsymbol{R}^4 = \boldsymbol{R}^3 \circ \boldsymbol{R} = \{(1, 1), (2, 1), (3, 1), (4, 1)\} = \boldsymbol{R}^3$.

Therefore $\boldsymbol{R}^n = \boldsymbol{R}^3$ for $\boldsymbol{n}=4, 5, \ldots$.

THEOREM: The relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is transitive if and only if $\boldsymbol{R}^n \subseteq \boldsymbol{R}$ for $\boldsymbol{n} = 1, 2, 3, \ldots$.

Representing Relations

1. Representing Relations using Matrices

2. Representing Relations using Diagraphs

1. Representing Relations using Matrices

A relation between finite sets can be represented using a zero–one matrix.

Suppose that R is a relation from $A=\{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2,\ldots, b_n\}$.

The relation R can be represented by the matrix

$M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i,b_j)\in R \\ 0, & \text{if } (a_i,b_j)\notin \mathrm{R} \end{cases}$$

Example: Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$

Let $R = \{(a, b) \mid a > b, a \in A, b \in B\}$.

What is the matrix M_R representing R?

Sol :

$$R = \{(2, 1), (3, 1), (3, 2)\}$$

$$\begin{array}{cc|cc} & & B & \\ & & 1 & 2 \\ \hline & 1 & 0 & 0 \\ A & 2 & 1 & 0 \\ & 3 & 1 & 1 \end{array}$$

$$\therefore M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Note. Different order of elements of A and B will produce different MR.
If A=B, the rows and columns should use the same order.

※ Let $A=\{a_1, a_2, \ldots, a_n\}$.

A relation R on A is reflexive iff $(a_i,a_i)\in R, \forall i$.

i.e.,

$$M_R = \begin{array}{c@{}c} & \begin{array}{ccccc} a_1 & a_2 & \cdots & \cdots & a_n \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{array} & \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \end{array}$$

All 1s on the diagonal.
Off Diagonal Elements can Be 0 or 1.

※ The relation R is symmetric iff $(a_i,a_j)\in R \Rightarrow (a_j,a_i)\in R$.

This means $m_{ij} = m_{ji}$ (i.e. MR is a symmetric matrix).

$$M_R = \begin{bmatrix} & 1 & \\ 1 & & 0 \\ & 0 & \end{bmatrix} = (M_R)^t$$

※ The relation R is antisymmetric iff

$$(a_i,a_j)\in R \text{ and } i \neq j \Rightarrow (a_j,a_i)\notin R.$$

This means that if $m_{ij}=1$ with $i\neq j$, then $m_{ji}=0$.

i.e.,

$$M_R = \begin{bmatrix} \ddots & 1 & & \\ 0 & \ddots & 0 & \\ & 1 & \ddots & 0 \\ & & 0 & \ddots \end{bmatrix}$$

※ The transitive property is not easy to judge directly from the matrix, and needs to be calculated

Example: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric ?

Sol :

Reflexive

Symmetric

Not Antisymmetric

Example: Suppose that $S=\{0, 1, 2, 3\}$. Let R be a relation containing (a, b) if $a \leq b$, where $a \in S$ and $b \in S$.
Is R reflexive, symmetric, antisymmetric ?

Sol :

$$M_R = \begin{array}{c|c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

∴ R is reflexive and antisymmetric, not symmetric.

Example: Suppose the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Sol :

$$M_{R_1 \cup R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad M_{R_1 \cap R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example: Find the matrix representing the relation $S \circ R$, where the matrices representing R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Sol :

$$M_{S \circ R} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

After $M_R \times M_S$ (matrix multiplication) change numbers

Example 6. Find the matrix representing the relation R^2, where the matrix representing R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Sol :

$$M_{R^2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

2. Representing Relations using Digraphs

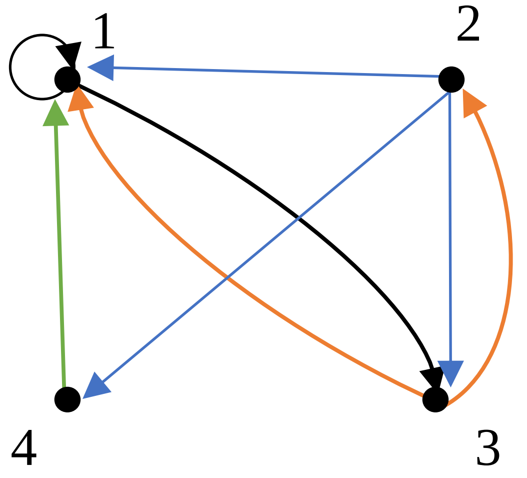
Def: A directed graph (digraph) consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs).

Example: Show the digraph of the relation $R=\{(1,1),(1,3),(2,1),(2,3),(2,4), (3,1),(3,2),(4,1)\}$ on the set $\{1,2,3,4\}$.

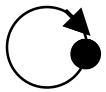
Sol :

Vertex : 1, 2, 3, 4

Edge : (1,1), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (4,1)

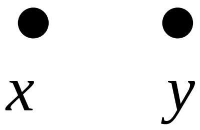


※ The relation R is **Reflexive** iff for every vertex,

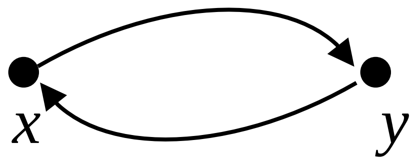


(Loop At Each Point)

※ The relation R is **Symmetric** iff for any vertices $x \neq y$, either

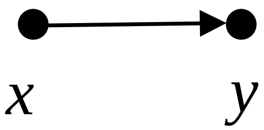


or



(If there is an edge between two points, it must be a pair of edges in different directions)

※ The relation R is **Antisymmetric** iff for any $x \neq y$,



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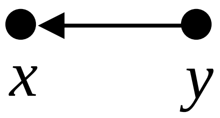

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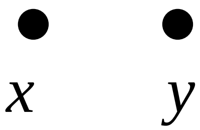
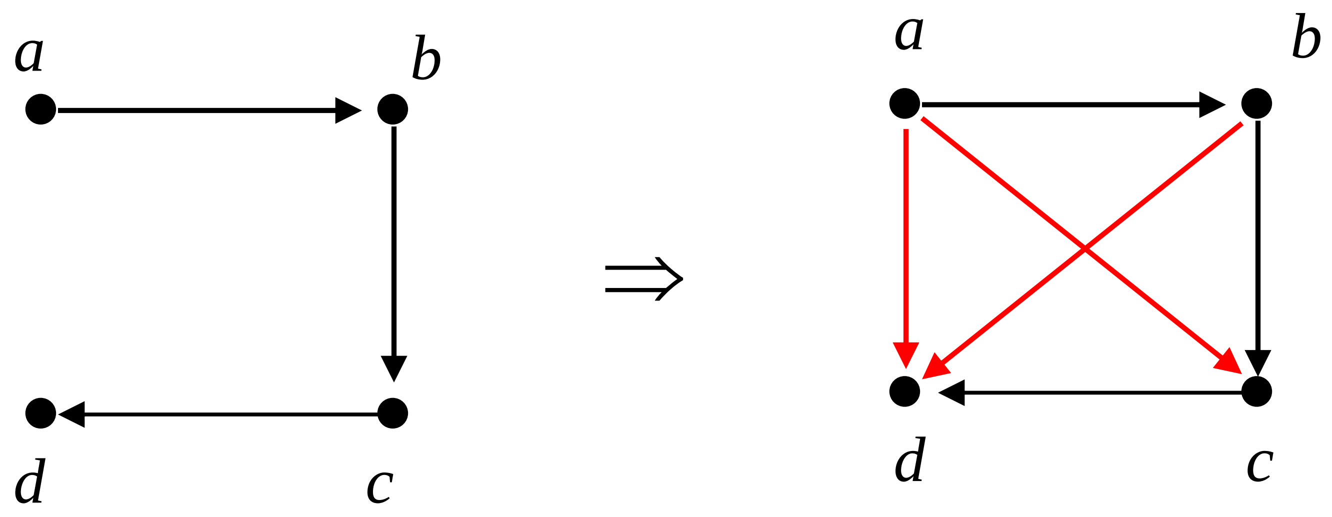


If there is an edge between two points, there must be only one edge

※ The relation R is Transitive iff for $\boldsymbol{a, b, c} \in \boldsymbol{A}$,
$(\boldsymbol{a, b}) \in \boldsymbol{R}$ and $(\boldsymbol{b, c}) \in \boldsymbol{R} \Rightarrow (\boldsymbol{a, c}) \in \boldsymbol{R}$.

This means:

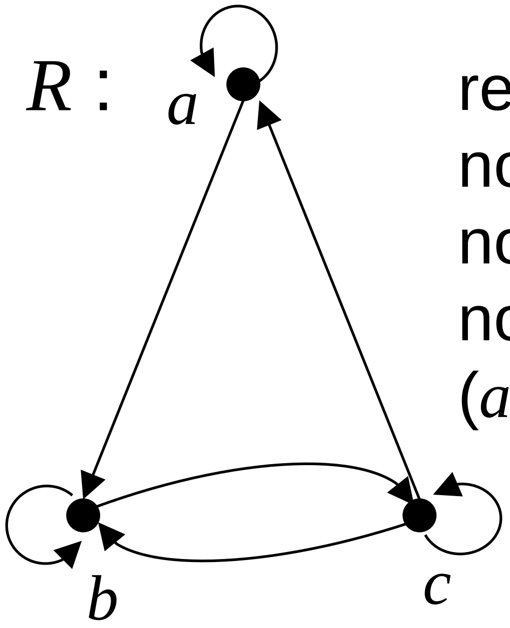


(As long as point x has a path to point y, x must have an edge directly connected to y)

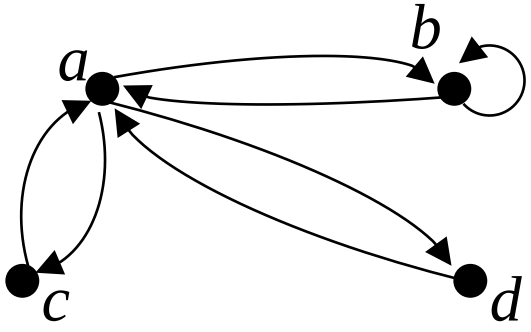
Example: Determine whether the relations R and S are reflexive, symmetric, antisymmetric, and/or transitive

Sol :

R :



reflexive,
not symmetric,
not antisymmetric,
not transitive
$(a \longrightarrow b,\ b \longrightarrow c,\ \ a \not\longrightarrow c)$



S

not reflexive,
symmetric
not antisymmetric
not transitive
$(b \longrightarrow a,\ a \longrightarrow c,\ \ b \not\longrightarrow c)$

Irreflexive : $(a,a) \notin R,\ \ \forall a \in A$

Closures of Relations

Closures of Relations

Def: The closure of a relation R with respect to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P.

- In terms of the digraph representation of R
  - To find the reflexive closure - add loops.
  - To find the symmetric closure - add arcs in the opposite direction.
  - To find the transitive closure - if there is a path from a to b, add an arc from a to b.

Closures of Relations

※ Closures

The relation R={(1,1), (1,2), (2,1), (3,2)} on the set A={1, 2, 3} is not reflexive.

Q: How to construct a smallest reflexive relation R_r such that $R \subseteq R_r$?

Sol: Let $R_r = R \cup \{(2,2), (3,3)\}$.

i. e., $R_r = R \cup \Delta$, where $\Delta = \{(a, a) | a \in A\}$.

R_r is a reflexive relation containing R that is as small as possible. It is called the reflexive closure of R.

Example: What is the reflexive closure of the relation $R=\{(a,b) \mid a < b\}$ on the set of integers ?

Sol : $R_r = R \cup \Delta = \{(a,b) \mid a < b\} \cup \{ (a, a) \mid a \in \mathbf{Z} \}$

$= \{ (a, b) \mid a \leq b, \quad a, b \in \mathbf{Z} \}$

Example :

The relation $R=\{ (1,1),(1,2),(2,2),(2,3),(3,1),(3,2) \}$ on the set $A=\{1,2,3\}$ is not symmetric. Find a smallest symmetric relation R_s containing R.

Sol : Let $R^{-1}=\{ (b, a) \mid (a, b) \in R \}$

Let $R_s= R \cup R^{-1}=\{ (1,1),(1,2),(2,1),(2,2),(2,3),$
$(3,1),(1,3),(3,2) \}$

R_s is called the Symmetric Closure of R.

Example: What is the symmetric closure of the relation $R=\{(a, b) \mid a > b\}$ on the set of positive integers ?

Sol :

The symmetric closure of R is the relation

R ∪ R−1 = {(a, b) | a>b}∪{(b, a) | a>b}={(a, b) | a = b}

OR

$$R \cup \{(b, a) \mid a > b\} = \{(c, d) \mid c \neq d\}$$

Def :

1.(Reflexive Closure of R on A)

R_r=the smallest reflexive relation containing R.

$R_r = R \cup \{ (a, a) \mid a \in A , (a, a) \notin R\}$

2.(Symmetric Closure of R on A)

R_s=the smallest symmetric relation containing R.

$R_s = R \cup \{ (b, a) \mid (a, b) \in R \text{ and } (b, a) \notin R\}$

3.(Transitive closure of R on A)

R_t=the smallest transitive relation containing R.

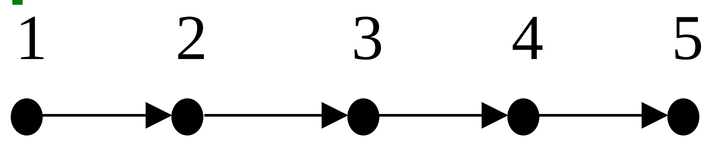
$R_t = R \cup \{(a, c) \mid (a, b) \in R_t \text{ and } (b, c) \in R_t\text{, but } (a, c) \notin R_t\}$(repeat)

Note. There is no antisymmetric closure, because if it is not antisymmetric, it means that there is $a \neq b$, and (a, b) & (b, a) are both $\in R$, then adding any pair cannot become antisymmetric.

Paths in Directed Graphs

Def: A path from a to b in the digraph G is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in G, where $n \in \mathbf{Z}^+$, and $x_0 = a$, $x_n = b$. This path is denoted by $x_0, x_1, x_2, \ldots, x_n$ and has length n.

Example:



A path from 1 to 5 of length 4

Theorem: Let R be a relation on a set A. There is a path of length n, where $n \in \mathbf{Z}^+$, from a to b if and only if $(a, b) \in R^n$.

Transitive Closures

Def: Let R be a relation on a set A. The connectivity relation $R*$ consists of pairs (a, b) such that there is a path of length at least one from a to b in R.

i.e., $$R^* = \bigcup_{i=1}^{\infty} R^i$$

Theorem: The transitive closure of a relation R equals the connectivity relation $R*$.

Lemma 1 Let R be a relation on a set A with $|A|=n$. then

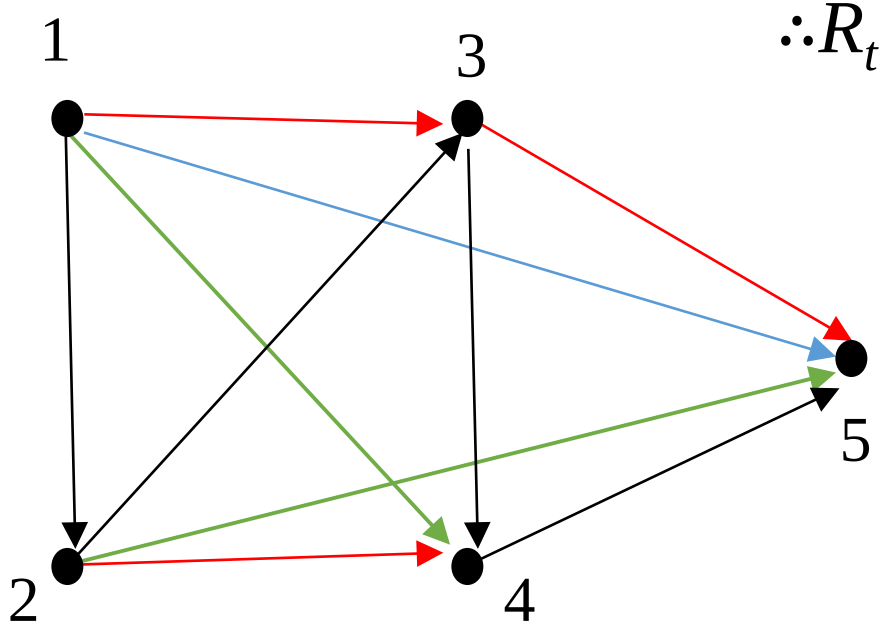
$$R^* = \bigcup_{i=1}^{n} R^i$$

Example: Let R be a relation on a set A, where

$A=\{1,2,3,4,5\}$, $R=\{(1,2),(2,3),(3,4),(4,5)\}$.

What is the transitive closure R_t of R ?

Sol :



$$\therefore R_t = R \cup R^2 \cup R^3 \cup R^4 \cup R^5$$
$$= \{(1,2),(2,3),(3,4),(4,5),$$
$$(1,3), (2,4), (3,5),$$
$$(1,4), (2,5),$$
$$(1,5)\}$$

Theorem: Let M_R be the zero-one matrix of the relation R on a set with n elements. Then the zero-one matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee \cdots \vee M_R^{[n]}.$$

Example: Find the zero-one matrix of the transitive closure of the relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Sol :

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Equivalence Relations

Equivalence Relations

Def: A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Example:

Let R be the relation on the set of integers such that aRb if and only if $a=b$ or $a=-b$. Then R is an equivalence relation.

Example:

Let R be the relation on the set of real numbers such that aRb if and only if $a-b$ is an integer. Then R is an equivalence relation.

Example: (Congruence Modulo m)

Let $m \in \mathbf{Z}$ and $m > 1$. Show that the relation

$R=\{ (a,b) \mid a\equiv b \pmod{m} \}$ is an equivalence relation on the set of integers.

(*a* is congruent to *b* modulo *m*, *a and b have the same remainder when divided by m)*

Sol : Note that $a\equiv b \pmod{m}$ iff $m \mid (a-b)$.

$\because$ ① $a\equiv a \pmod{m} \Rightarrow (a, a)\in R \Rightarrow$ **reflexive**

② If $a\equiv b \pmod{m}$, then $a-b=km,\ k\in\mathbf{Z}$

$\Rightarrow b-a= (-k)m \Rightarrow b\equiv a \pmod{m} \Rightarrow$ **symmetric**

③ If $a\equiv b \pmod{m}$, $b\equiv c \pmod{m}$

then $a-b=km,\ b-c=lm$

$\Rightarrow a-c=(k+l)m \Rightarrow a\equiv c \pmod{m} \Rightarrow$ **transitive**

$\therefore R$ is an equivalence relation.

Example:

Let $l(x)$ denote the length of the string x.
Suppose that the relation
$R=\{(a,b) \mid l(a)=l(b),\ a,b$ are strings of English letters $\}$
Is R an equivalence relation?

Sol :

$$\left.\begin{array}{lll} ① & (a,a)\in R\ \ \forall \text{string } a & \Rightarrow \text{reflexive} \\ ② & (a,b)\in R \Rightarrow (b,a)\in R & \Rightarrow \text{symmetric} \\ ③ & (a,b)\in R,(b,c)\in R \Rightarrow (a,c)\in R & \Rightarrow \text{transitive} \end{array}\right\} \quad \text{Yes.}$$

Example: Let R be the relation on the set of real numbers such that xRy if and only if x and y differ by less than 1, that is $|x-y| < 1$. Show that R is not an equivalence relation.

Sol :

① $xRx \ \forall x$ since $x-x = 0$ $\quad \Rightarrow$ reflexive

② $xRy \Rightarrow |x-y| < 1 \quad \Rightarrow |y-x| < 1 \quad \Rightarrow yRx$

$\Rightarrow$ symmetric

③ $xRy,\ yRz \quad \Rightarrow |x-y| < 1,\ |y-z| < 1 \quad \not\Rightarrow |x-z| < 1$

$\Rightarrow$ Not transitive

Take x = 2.8, y = 1.9, and z = 1.1,
so |x − y|=|2.8 − 1.9| = 0.9 < 1, |y − z|=|1.9 − 1.1| = 0.8 < 1, but |x − z|=|2.8 − 1.1| = 1.7 > 1. That is, 2.8R 1.9, 1.9R 1.1, but 2.8 R 1.1.

Equivalence Classes

Def:

Let R be an equivalence relation on a set A.

The equivalence class of the element $a \in A$ is

$$[a]_R = \{ s \mid (a, s) \in R \}$$

For any $b \in [a]_R$, b is called a representative of this equivalence class.

Note:

If $(a, b) \in R$, then $[a]_R = [b]_R$.

Example:

What are the equivalence class of 0 and 1 for congruence modulo 4 ?

Sol :

Let $R=\{ (a,b) \mid a \equiv b \pmod 4) \}$

Then $[0]_R = \{ s \mid (0,s) \in R \}$

$= \{ \ldots, -8, -4, 0, 4, 8, \ldots \}$

$[1]_{\mathrm{R}} = \{ t \mid (1,t) \in R \} = \{ \ldots, -7, -3, 1, 5, 9, \ldots \}$

Equivalence Classes and Partitions

Def.

A partition of a set S is a collection of disjoint nonempty subsets A_i of S that have S as their union.

In other words, we have $A_i \neq \varnothing, \forall i,$

$A_i \cap A_j = \varnothing$, for any $i \neq j$, and $\cup A_i = S$.

Example:

Suppose that $S=\{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1=\{1, 2, 3\}$, $A_2=\{4, 5\}$, and $A_3=\{6\}$ form a partition of S.

THEOREM:

Let R be an equivalence relation on a set A.
Then the equivalence classes of R form a partition of A.

Example:

List the ordered pairs in the equivalence relation R produced by the partition $A_1=\{1, 2, 3\}$, $A_2=\{4, 5\}$, and $A_3=\{6\}$ of $S=\{1, 2, 3, 4, 5, 6\}$.

Sol :

$$R=\{ (a, b) \mid a, b\in A_1\}\cup \{ (a, b) \mid a, b\in A_2\}$$
$$\cup \{ (a, b) \mid a, b\in A_3\}$$
$$=\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4,4), (4, 5), (5,4), (5, 5), (6, 6)\}$$

Example:

The equivalence classes of the congruence modulo 4 relation form a partition of the integers. OR

What are the sets in the partition of the integers arising from congruence modulo 4?

Sol : There are four congruence classes, corresponding to [0]4, [1]4, [2]4, and [3]4. They are the sets

$[0]_4 = \{ \ldots, -8, -4, 0, 4, 8, \ldots \}$

$[1]_4 = \{ \ldots, -7, -3, 1, 5, 9, \ldots \}$

$[2]_4 = \{ \ldots, -6, -2, 2, 6, 10, \ldots \}$

$[3]_4 = \{ \ldots, -5, -1, 3, 7, 11, \ldots \}$

Partial Orderings

Partial Orderings

Def: A relation R on a set S is called a partial ordering or partial order if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering R is called a partially ordered set, or poset, and is denoted by (S, R).

Example:

Show that the "greater than or equal" ($\geq$) is a partial ordering on the set of integers.

Sol :

① $x \geq x \ \forall x \in \mathbf{Z}$ $\Rightarrow$ reflexive

② If $x \geq y$ and $y \geq x$ then $x = y$. $\Rightarrow$ antisymmetric

③ $x \geq y,\ y \geq z \ \Rightarrow \ x \geq z$ $\Rightarrow$ transitive

Def:

The elements a and b of a poset $(S,\preccurlyeq)$ are called comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ or $b \preccurlyeq a$, a and b are called incomparable.

Example:

In the poset $(\mathbf{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Sol :

$3|9 \quad \Rightarrow$ comparable

$5 \nmid 7 \text{ and } 7 \nmid 5 \Rightarrow$ incomparable

Def: If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a totally ordered or linearly ordered set, and $\preccurlyeq$ is called a total order or a linear order. A totally ordered set is also called a chain.

Example:

The poset $(\mathbf{Z}, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.

Example:

The poset $(\mathbf{Z}^+, |)$ is not totally ordered.

Lexicographic Order:

The words in a dictionary are listed in alphabetic, or lexicographic, order, which is based on the ordering of the letters in the alphabet.

Def. Let $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$ be two posets. The lexicographic ordering $\preccurlyeq$ on $A_1 \times A_2$ is defined as

$(a_1, a_2) \prec (b_1, b_2)$ either if $a_1 \prec_1 b_1$ or

if both $a_1 = b_1$ and $a_2 \prec_2 b_2$

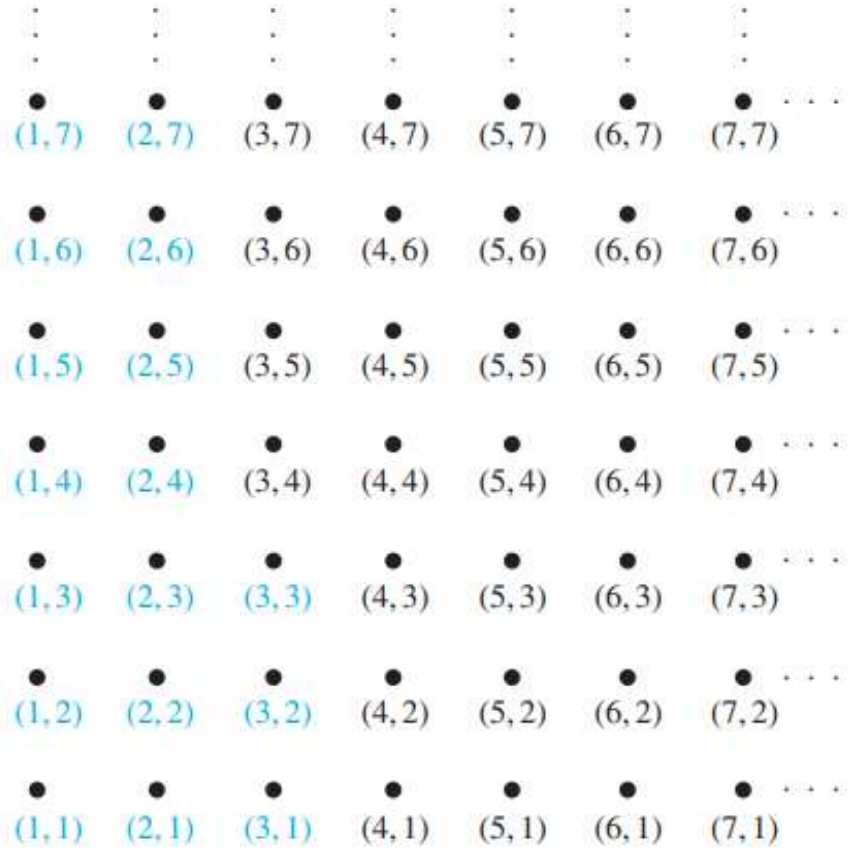
We obtain a partial ordering $\preccurlyeq$ by adding equality to the ordering $\prec$ on $A_1 \times A_2$.

Example:

In the poset $(\mathbf{Z}\times\mathbf{Z}, \preccurlyeq)$, where $\preccurlyeq$ is the lexicographic ordering constructed from the usual $\leq$ relation on $\mathbf{Z}$.

$$(3, 5) \prec (4, 8),$$
$$(3, 8) \prec (4, 5),$$
$$(4, 9) \prec (4, 11)$$

Sol: The Ordered Pairs Less Than (3, 4) in Lexicographic Order.



Hasse Diagrams:

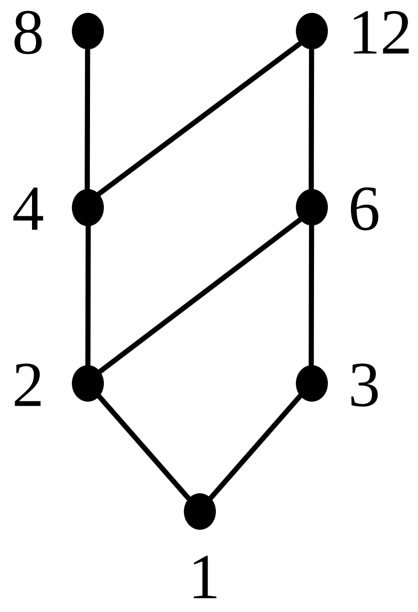
G*raphical representation* of the relation of elements of a **partially ordered set (poset)** with an implied *upward orientation*. A point is drawn for each element of the **poset** and joined with the line segment according to the following rules:

- If *p*<*q* in the poset, then the point corresponding to *p appears lower* in the drawing than the point corresponding to q.
- The two points p and q will be joined by line segment *if p is related to q*.

Example:

Draw the Hasse diagram representing the partial ordering $\{(a, b) \mid a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$.

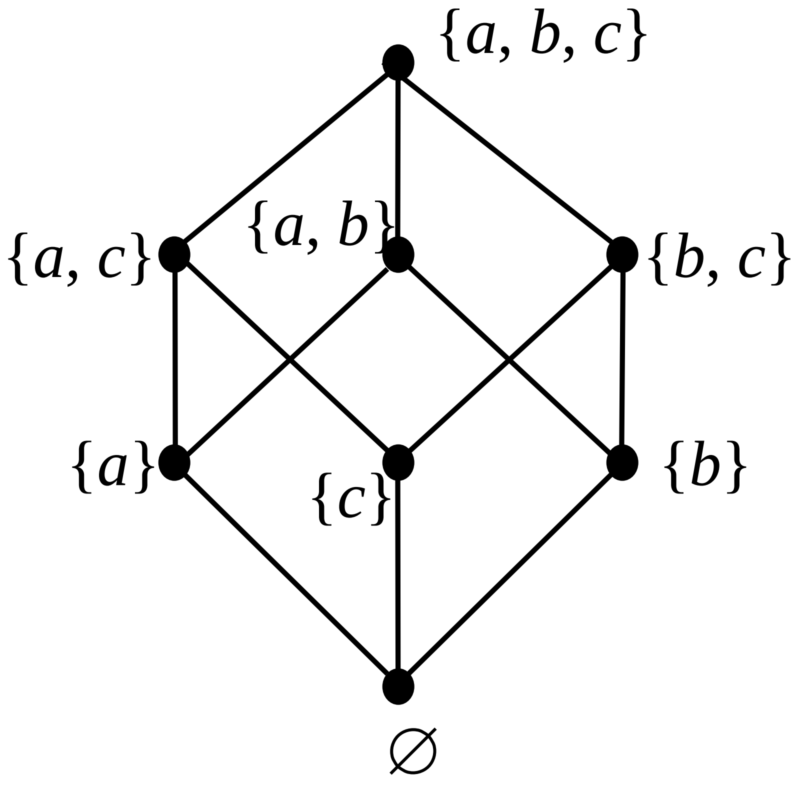
Sol :



Example:

Draw the Hasse diagram for the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set $P(S)$ where $S=\{a, b, c\}$.

Sol :



Maximal and Minimal Elements

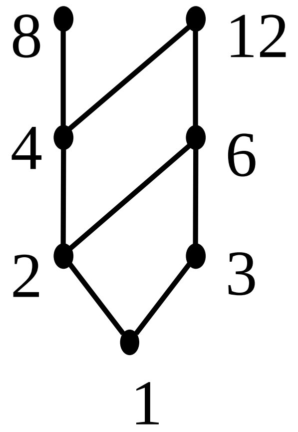
Def.

An element $a \in S$ is maximal in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$. Similarly, an element $a \in S$ is minimal if there is no $b \in S$ such that $b \prec a$.

a is the greatest element of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$. a is the least element of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$.

Example:

8, 12 are maximal,
1 is least and minimal,
no greatest element

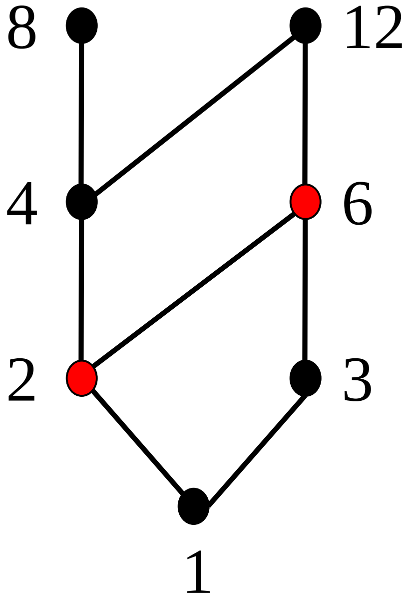


Def.

Let A be a subset of a poset $(S, \preccurlyeq)$. If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an upper bound of A.

If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called an lower bound of A.

Ex



$A=\{2, 6\}$

upper bound of A: 6, 12

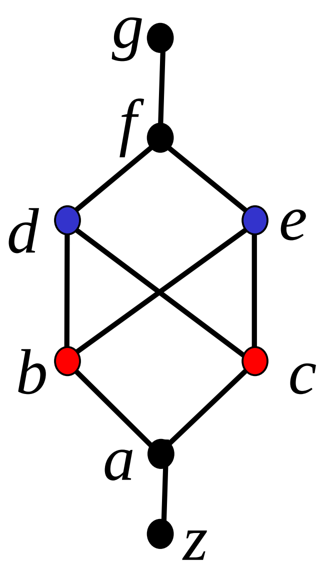
lower bound of A: 1, 2

Def.

Let A be a subset of a poset $(S, \preccurlyeq)$. An element x is called the least upper bound of A if x is an upper bound of A and $x \preccurlyeq z$ whenever z is an upper bound of A.

Let A be a subset of a poset $(S, \preccurlyeq)$. An element x is called the greatest lower bound of A if x is a lower bound of A and $y \preccurlyeq x$ whenever y is a lower bound of A.

Ex



$A_1=\{d, e\}$, $A_2=\{b, c\}$

least upper bound of $A_1 = f$

A_1 has no greatest lower bound

A_2 has no least upper bound

greatest lower bound of $A_2 = a$

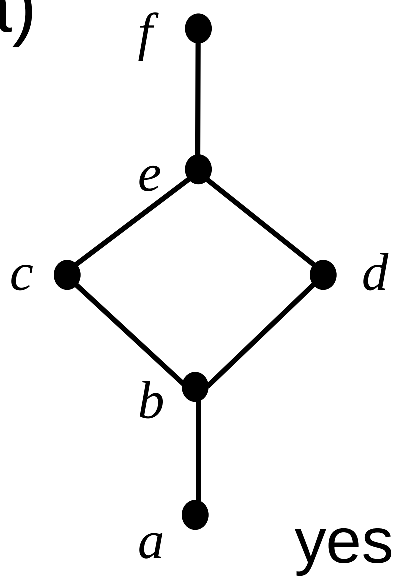
Lattices

Def. A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice.

Example:

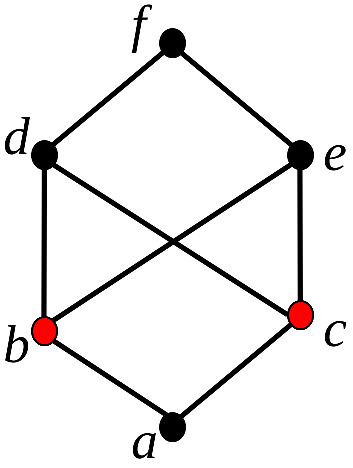
Determine whether the following posets are lattices.

(a)



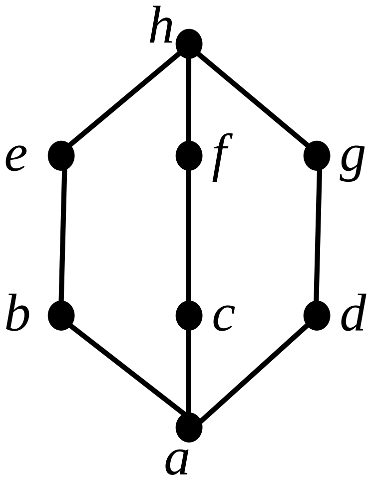
yes

(b)



(b,c) no least upper bound $\Rightarrow$ no

(c)



yes

Example:

Is the poset $(\mathbf{Z}^+, |)$ a lattice?

Sol : For any a, $b \in \mathbf{Z}^+$,

$gcd(a,b)$ is the greatest lower bound of a, b: least common multiple

$lcm(a,b)$ is the least upper bound of a, b: greatest common divisor

$\Rightarrow$ Yes

Example:

Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Sol : In $(\{1, 2, 3, 4, 5\}, |)$, 2 and 3 has no l.u.b. $\Rightarrow$ No.

In $(\{1, 2, 4, 8, 16\}, |)$,

any a, b has l.u.b. and g.l.b. $\Rightarrow$ Yes.

Topological Sorting

Suppose that a project is made up of 20 different tasks. Some tasks can be completed only after others have been finished. How can an order be found for these tasks?

Def.

A total ordering $\preccurlyeq$ is said to be compatible with the partial ordering R if $a \preccurlyeq b$ whenever aRb. Constructing a compatible total ordering from a partial ordering is called topological sorting.

Lemma 1.

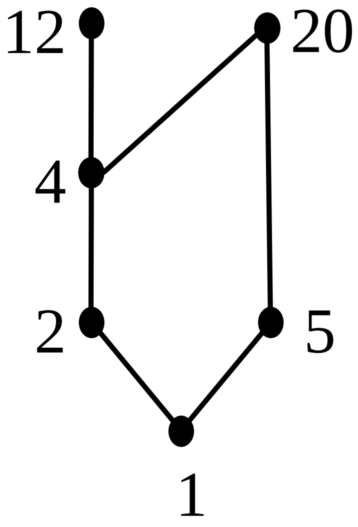
Every finite nonempty poset $(S, \preccurlyeq)$ has at least one minimal element.

Topological sorting: Output minimal element one by one, that is, get compatible total ordering from small to large

Example:

Find a compatible total ordering for the poset $(\{1, 2, 4, 5, 12, 20\}, |\ )$.

Sol :



$$1 \prec 2 \prec 5 \prec 4 \prec 12 \prec 20$$

The order of 2 and 5 can be swapped, as can 12 and 20