CO-205: Discrete Structures Tutorial #1

Summary

proposition	a statement that is true or false
propositional variable	a variable that represents a proposition
truth value	true or false
$\sim p$ (negation of p)	the proposition with truth value opposite to the truth value of p
logical operators	operators used to combine propositions
compound proposition	a proposition constructed by combining propositions using logical operators
truth table	a table displaying all possible truth values of propositions
$p \vee q$ (disjunction of p and q)	the proposition "p or q," which is true if and only if at least one of p and q is true
$p \wedge q$ (conjunction of p and q)	the proposition "p and q," which is true if and only if both p and q are true
$p \oplus q$ (exclusive or of p and q)	the proposition "p XOR q," which is true when *either* one of p or q is true
$p \rightarrow q$ (p implies q)	the proposition "if p, then q," which is false *if and only if* p is true and q is false
converse of $p \rightarrow q$	the conditional statement $q \rightarrow p$
Contrapositive of $p \rightarrow q$	the conditional statement $\sim q \rightarrow \sim p$
inverse of $p \rightarrow q$	the conditional statement $\sim p \rightarrow \sim q$
$p \leftrightarrow q$ (biconditional)	the proposition "p if and only if q," which is true if and only if p and q have the same truth value
bit	either a 0 or a 1
Boolean variable	a variable that has a value of 0 or 1
bit operation	an operation on a bit or bits
bit string	a list of bits
bitwise operations	operations on bit strings that operate on each bit in one string and the corresponding bit in the other string
logic gate	a logic element that performs a logical operation on one or more bits to produce an output bit
logic circuit	a switching circuit made up of logic gates that produces one or more output bits
tautology	a compound proposition that is always true
contradiction	a compound proposition that is always false
contingency	a compound proposition that is sometimes true and sometimes false
consistent compound propositions	compound propositions for which there is an assignment of truth values to the variables that makes all these propositions true
satisfiable compound proposition	a compound proposition for which there is an assignment of truth values to its variables that makes it true
logically equivalent compound propositions	compound propositions that always have the same truth values
predicate	part of a sentence that attributes a property to the subject
propositional function	a statement containing one or more variables that becomes a proposition when each of its variables is assigned a value or is bound by a quantifier
domain (or universe) of discourse	the values a variable in a propositional function may take
$\exists x\, P(x)$ (existential quantification of $P(x)$)	the proposition that is true if and only if there exists an x in the domain such that $P(x)$ is true
$\forall x P(x)$ (universal quantification of $P(x)$)	the proposition that is true if and only if $P(x)$ is true for every x in the domain
logically equivalent expressions	expressions that have the same truth value no matter which propositional functions and domains are used
free variable	a variable not bound in a propositional function
bound variable	a variable that is quantified
scope of a quantifier	portion of a statement where the quantifier binds its variable

Logical Equivalences

Equivalence	***Name***
$p \wedge T \equiv p$	Identity laws
$p \vee F \equiv p$	
$p \vee T \equiv T$	Domination laws
$p \wedge F \equiv F$	
$p \vee p \equiv p$	Idempotent laws
$p \wedge p \equiv p$	
$\sim(\sim p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$	Commutative laws
$p \wedge q \equiv q \wedge p$	
$(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative laws
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
$\sim(p \wedge q) \equiv \sim p \vee \sim q$	De Morgan's laws
$\sim(p \vee q) \equiv \sim p \wedge \sim q$	
$p \vee (p \wedge q) \equiv p$	Absorption laws
$p \wedge (p \vee q) \equiv p$	
$p \vee \sim p \equiv T$	Negation laws
$p \wedge \sim p \equiv F$	

Logical Equivalences involving Conditional / Bi-conditional Statements.

$p \to q \equiv \sim p \vee q$	$p \to q \equiv \sim q \to \sim p$
$p \vee q \equiv \sim p \to q$	$p \wedge q \equiv \sim(p \to \sim q)$
$\sim(p \to q) \equiv p \wedge \sim q$	
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$	
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$	
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$	
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$	
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$	
$p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$	
$p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$	$\sim(p \leftrightarrow q) \equiv p \leftrightarrow \sim q$

De Morgan's Laws for Quantifiers

Negation	*Equivalent Statement*	*When is Negation True?*	*When False?*
$\sim\exists x P(x)$	$\forall x \sim P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\sim\forall x P(x)$	$\exists x \sim P(x)$	*There is an x for which P(x) is false.*	*P(x) is true for every x.*

1. Which of these are propositions? What are the truth values of those that are propositions?
   a) Do not cross the road when the signal is red.
   b) What is the time now?
   c) There are no Windmills in the University.
   d) $3 + x = 12$.
   e) The moon is made of green cheese.
   f) $5x \geq 67$.

2. What is the negation of each of these propositions?
   a) Ram and Vinod are friends.
   b) There are 13 items in a baker's dozen.
   c) Bobby sent more than 50 text messages yesterday on his mobile phone.
   d) 144 is a perfect square.

3. Let p and q be the propositions
   p: I bought a lottery ticket this week.
   q: I won the million dollar jackpot.
Express each of these propositions as an English sentence.
(a) $\sim p$ (b) $p \vee q$ (c) $p \to q$ (d) $p \wedge q$ (e) $p \leftrightarrow q$
(f) $\sim p \to \sim q$ (g) $\sim p \wedge \sim q$ (h) $\sim p \vee (p \wedge q)$

4. Let *p* and *q* be the propositions
   p:You drive over 80km/h.
   q:You get a speeding ticket.
Write these propositions using *p* and *q* and known logical connectives.
a) You do not drive over 80km/h.
b) You drive over 80km/h, but you do not get a speeding ticket.
c) You will get a speeding ticket if you drive over 80km/h.
d) If you do not drive over 80km/h, then you will not get a speeding ticket.
e) Driving over 80km/h is sufficient for getting a speeding ticket.
f) You get a speeding ticket, but you do not drive over 80km/h.
g) Whenever you get a speeding ticket, you are driving over 80km/h.

5. Determine whether these bi-conditionals are true or false.
   a) $2 + 2 = 4$ if and only if $1 + 1 = 2$.
   b) $1 + 1 = 2$ if and only if $2 + 3 = 4$.
   c) $1 + 1 = 3$ if and only if monkeys can fly.
   d) $0 > 1$ if and only if $2 > 1$.

6. State the converse, contra-positive, and inverse of each of these conditional statements.
   a) If it snows tonight, then I will stay at home.
   b) I go to the beach whenever it is a sunny summer day.
   c) When I stay up late, it is necessary that I sleep until noon.

7. Construct a truth table for each of these compound propositions.
   a) $p \oplus p$ b) $p \oplus \sim p$
   c) $p \oplus \sim q$ d) $\sim p \oplus \sim q$
   e) $(p \oplus q) \vee (p \oplus \sim q)$ f) $(p \oplus q) \wedge (p \oplus \sim q)$

8. Explain, without using a truth table, why $(p \vee \sim q) \wedge (q \vee \sim r) \wedge (r \vee \sim p)$ is true when p, q, and r have the same truth value and it is false otherwise.

9. What is the value of x after each of these statements is encountered in a computer program, assume that x = 1 before the statement is reached?
   a) if $x + 2 = 3$ then $x := x + 1$
   b) if $(x + 1 = 3)$ OR $(2x + 2 = 3)$ then $x := x + 1$
   c) if $(2x + 3 = 5)$ AND $(3x + 4 = 7)$ then $x := x + 1$
   d) if $(x + 1 = 2)$ XOR $(x + 2 = 3)$ then $x := x + 1$
   e) if $x < 2$ then $x := x + 1$

10. Express the statement "You can see the movie only if you are over 18 years old or you have the permission of a parent." in terms of the prepositions –
   m: "You can see the movie,"
   a: "You are over 18 years old,"
   p: "You have the permission of a parent."

11. Express the statement "to use the wireless network in the airport you must pay the daily fee unless you

are a subscriber to the service" in terms of the propositions –

w: "You can use the wireless network in the airport"
d: "You pay the daily fee"
s: "You are a subscriber to the service"

12. Express these system specifications using the propositions given below and logical connectives (including negations)

p "The user enters a valid password,"
q "Access is granted,"
r "The user has paid the subscription fee"

a) "The user has paid the subscription fee, but does not enter a valid password."
b) "Access is granted whenever the user has paid the subscription fee and enters a valid password."
c) "Access is denied if the user has not paid the subscription fee."
d) "If the user has not entered a valid password but has paid the subscription fee, then access is granted."

13. Five friends have access to a chat room. Is it possible to determine who is chatting if the following information is known? Explain your reasoning.

- *Either Kevin or Heather, or both, are chatting.*
- *Either Randy or Vijay, but not both, are chatting.*
- *If Abby is chatting, so is Randy.*
- *Either both Vijay and Kevin are chatting or neither is chatting.*
- *If Heather is chatting, then Abby and Kevin are also chatting.*

14. Construct a combinatorial circuit using inverters, OR gates, and AND gates that produces the output $((\sim p \vee \sim r) \wedge \sim q) \vee (\sim p \wedge (q \vee r))$ from input bits p, q, and r.

15. Show that each of these conditional statements is a tautology.

a) $[\sim p \wedge (p \vee q)] \rightarrow q$
b) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
c) $[p \wedge (p \rightarrow q)] \rightarrow q$
d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

16. Is $(\sim p \wedge (p \rightarrow q)) \rightarrow \sim q$ is a tautology?
17. Is $(\sim q \wedge (p \rightarrow q)) \rightarrow \sim p$ is a tautology?
18. Is $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ a tautology?
19. Is $(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$ a tautology?

20. Show that

a) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$
b) $\sim(p \leftrightarrow q) \equiv p \leftrightarrow \sim q$
c) $p \rightarrow q \equiv \sim q \rightarrow \sim p$
d) $\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$
e) $\sim(p \oplus q) \equiv p \leftrightarrow q$
f) $\sim(p \leftrightarrow q) \equiv \sim p \leftrightarrow q$
g) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
h) $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
i) $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
j) $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$
k) $\sim p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \vee r)$
l) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
m) $p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$

21. Show that the following are not logically equivalent: -

a) $(p \rightarrow q) \rightarrow r$ *and* $p \rightarrow (q \rightarrow r)$
b) $(p \wedge q) \rightarrow r$ *and* $(p \rightarrow r) \wedge (q \rightarrow r)$
c) $(p \rightarrow q) \rightarrow (r \rightarrow s)$ *and* $(p \rightarrow r) \rightarrow (q \rightarrow s)$

22. Determine whether each of these compound propositions is satisfiable.

a) $(p \vee q \vee \sim r) \wedge (p \vee \sim q \vee \sim s) \wedge (p \vee \sim r \vee \sim s) \wedge (\sim p \vee \sim q \vee \sim s) \wedge (p \vee q \vee \sim s)$
b) $(\sim p \vee \sim q \vee r) \wedge (\sim p \vee q \vee \sim s) \wedge (p \vee \sim q \vee \sim s) \wedge (\sim p \vee \sim r \vee \sim s) \wedge (p \vee q \vee \sim r) \wedge (p \vee \sim r \vee \sim s)$
c) $(p \vee q \vee r) \wedge (p \vee \sim q \vee \sim s) \wedge (q \vee \sim r \vee s) \wedge (\sim p \vee r \vee s) \wedge (\sim p \vee q \vee \sim s) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee \sim q \vee s) \wedge (\sim p \vee \sim r \vee \sim s)$

23. Let $N(x)$ be the statement "*x* has visited Nainital," where the domain consists of the students in your school. Express each of these quantifications in English.

a) $\exists x N(x)$ b) $\forall x N(x)$ c) $\sim\exists x N(x)$
d) $\exists x \sim N(x)$ e) $\sim\forall x N(x)$ f) $\forall x \sim N(x)$

24. Translate these statements into English, where $R(x)$ is "*x* is a rabbit" and $H(x)$ is "*x* hops" and the domain consists of all animals.

a) $\forall x(R(x) \rightarrow H(x))$ b) $\forall x(R(x) \wedge H(x))$
c) $\exists x(R(x) \rightarrow H(x))$ d) $\exists x(R(x) \wedge H(x))$

25. Let $Q(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are the truth values of the following?

(a) $Q(0)$ (b) $Q(-1)$ (c) $Q(1)$ d) $\exists x Q(x)$
(e) $\forall x Q(x)$ (f) $\exists x \sim Q(x)$ (g) $\forall x \sim Q(x)$

26. Assuming that the domain of the propositional function $P(x)$ consists of −5, −3, −1, 1, 3, and 5. Express the statements given below using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\forall x((x \neq 1) \rightarrow P(x))$
d) $\exists x((x \geq 0) \wedge P(x))$
e) $\exists x(\sim P(x)) \wedge \forall x((x < 0) \rightarrow P(x))$

27. Translate these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) Something is not in the correct place.

b) All tools are in the correct place and are in excellent condition.

c) Everything is in the correct place and in excellent condition.

d) Nothing is in the correct place and is in excellent condition.

e) One of your tools is not in the correct place, but it is in excellent condition.

28. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.

a) $\forall x(x^2 \neq x)$ b) $\forall x(x^2 \neq 2)$ c) $\forall x(|x| > 0)$

29. Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) When there is less than 30 MB free space on the hard disk, a warning message is sent to all users.

b) No directories in the file system can be opened and no files can be closed when system errors have been detected.

c) The file system cannot be backed up if there is a user currently logged on.

d) Video on demand can be delivered when there are at least 8 MB of memory available and the connection speed is at least 256 kbps.

30. Suppose that Prolog facts are used to define the predicates *mother(M, Y)* and *father(F,X)*, which represent that *M* is the mother of *Y* and *F* is the father of *X*, respectively. Give a Prolog rule to define the predicate *grandfather(X, Y)*, which represents that *X* is the grandfather of *Y* . [*Hint:* You can write a disjunction in Prolog either by using a semicolon to separate predicates or by putting these predicates on separate lines.]

31. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "*x* is a clear explanation," "*x* is satisfactory," and "*x* is an excuse," respectively. Suppose that the domain for *x* consists of all English text. Express the following statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$: -

a) All clear explanations are satisfactory.

b) Some excuses are unsatisfactory.

c) Some excuses are not clear explanations.

d) Does (c) follow from (a) and (b)?

32. Translate these statements into English, where the domain for each variable consists of all real numbers.

a) $\exists x \forall y(xy = y)$

b) $\forall x \forall y(((x \geq 0) \land (y < 0)) \rightarrow (x - y > 0))$

c) $\forall x \forall y \exists z(x = y + z)$

33. Let $P(x, y)$ be the statement "Student *x* has taken class *y*," where the domain for *x* consists of all students in your class and for *y* consists of all computer science courses in your University. Express each of these quantifications in English.

a) $\exists x \exists y P(x, y)$ b) $\exists x \forall y P(x, y)$ c) $\forall x \exists y P(x, y)$ d) $\exists y \forall x P(x, y)$ e) $\forall y \exists x P(x, y)$ f) $\forall x \forall y P(x, y)$

34. Let $Q(x, y)$ be the statement "student *x* has been a contestant on quiz show *y*." Express each of these sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the domain for *x* consists of all students in the University and for *y* consists of all quiz shows on TV.

a) There is a student at your University who has been a contestant on a TV quiz show.

b) No student at your University has ever been a contestant on a TV quiz show.

c) There is a student in your University who has been a contestant on *KBC* and on *Mastermind India*.

d) Every TV quiz show has had a student from your University as a contestant.

e) At least two students from your University have been contestants on *KBC*.

35. Let $I(x)$ be the statement "*x* has an Internet connection" and $C(x, y)$ be the statement "*x* and *y* have chatted over the Internet," where the domain for the variables *x* and *y* consists of all students in your class. Use quantifiers to express each of these statements.

a) Jatin does not have an Internet connection.

b) Ruchi has not chatted over the Internet with Chitra.

c) Jerry and Shiva have never chatted over the internet.

d) No one in the class has chatted with Bobby.

e) Sanjay has chatted with everyone except Vijay.

f) Someone in your class does not have an Internet connection.

g) Not everyone in your class has an Internet connection.

h) Exactly one student in your class has an Internet connection.

i) Everyone except one student in your class has an Internet connection.

j) Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.

k) Someone in your class has an Internet connection but has not chatted with anyone else in your class.

l) There are two students in your class who have not chatted with each other over the Internet.

m) There is a student in your class who has chatted with everyone in your class over the Internet.

n) There are at least two students in your class who have not chatted with the same person in your class.

o) There are two students in the class who between them have chatted with everyone else in the class.

36. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

a) At least one console must be accessible during every fault condition.

b) The e-mail address of every user can be retrieved whenever the archive contains at least one message sent by every user on the system.

c) For every security breach there is at least one mechanism that can detect that breach if and only if there is a process that has not been compromised.

d) There are at least two paths connecting every two distinct endpoints on the network.

e) No one knows the password of every user on the system except for the system administrator, who knows all passwords.

37. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

a) $\forall x \exists y(x^2 = y)$

b) $\forall x \exists y(x = y^2)$

c) $\exists x \forall y(xy = 0)$

d) $\exists x \exists y(x + y \neq y + x)$

e) $\forall x(x \neq 0 \rightarrow \exists y(xy = 1))$

f) $\exists x \forall y(y \neq 0 \rightarrow xy = 1)$

g) $\forall x \exists y(x + y = 1)$

h) $\exists x \exists y(x + 2y = 2 \wedge 2x + 4y = 5)$

i) $\forall x \exists y(x + y = 2 \wedge 2x - y = 1)$

j) $\forall x \forall y \exists z(z = (x + y)/2)$

38. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y, where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.

a) $\forall x \forall y P(x, y)$ b) $\exists x \exists y P(x, y)$

c) $\exists x \forall y P(x, y)$ d) $\forall y \exists x P(x, y)$

39. Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\forall x \exists y \forall z T(x, y, z)$

b) $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$

c) $\forall x \exists y(P(x, y) \wedge \exists z R(x, y, z))$

d) $\forall x \exists y(P(x, y) \rightarrow Q(x, y))$

40. Find a counterexample, if possible, to these statements, where the domain for all variables consists of all integers.

a) $\forall x \forall y(x^2 = y^2 \rightarrow x = y)$

b) $\forall x \exists y(y^2 = x)$

c) $\forall x \forall y(xy \geq x)$