ICS141: Discrete Mathematics for Computer Science I

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Provided by McGraw-Hill

Lecture 6

Chapter 1. The Foundations

1.5 Rules of Inference

Previously…

- Rules of inference
  - Modus ponens
  - Modus tollens
  - Hypothetical syllogism
  - Disjunctive syllogism
  - Resolution
  - Addition
  - Simplification
  - Conjunction

Table 1 in pp.66

Resolution

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- $$\begin{array}{r} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$$

  Rule of ***Resolution***

  Tautology:

  $[(p \lor q) \land (\neg p \lor r)] \rightarrow (q \lor r)$

- When $q = r$:

  $[(p \lor q) \land (\neg p \lor q)] \rightarrow q$

- When $r = \mathbf{F}$:

  $[(p \lor q) \land (\neg p)] \rightarrow q$ (Disjunctive syllogism)

Resolution: Example

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$$\begin{array}{r} p \lor q \\ \neg p \lor r \\ \hline \therefore q \lor r \end{array}$$

- Example: Use resolution to show that the hypotheses "*Jasmine is skiing* or *it is not snowing*" and "*It is snowing* or *Bart is playing hockey*" imply that "*Jasmine is skiing* or *Bart is playing hockey*"

r $\neg p$ p q r q

$$(p \lor q) \land (\neg p \lor r) \rightarrow (q \lor r)$$

Formal Proofs

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- A formal proof of a conclusion C, given premises $p_1, p_2, \ldots, p_n$ consists of a sequence of *steps*, each of which applies some inference rule to premises or previously-proven statements to yield a new true statement (the *conclusion*).
- A proof demonstrates that *if* the premises are true, *then* the conclusion is true.

Formal Proof Example

- Suppose we have the following premises:
  "It is not sunny and it is cold."
  "We will swim only if it is sunny."
  "If we do not swim, then we will canoe."
  "If we canoe, then we will be home by sunset."

- Given these premises, prove the conclusion **"We will be home by sunset"** using inference rules.

Proof Example *cont.*

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- Step 1: Identify the propositions (Let us adopt the following abbreviations)
  - *sunny* = **"It is sunny"**; *cold* = **"It is cold"**; *swim* = **"We will swim"**; *canoe* = **"We will canoe"**; *sunset* = **"We will be home by sunset"**.
- Step 2: Identify the argument. (Build the argument form)

$$
\begin{array}{l}
\neg sunny \wedge cold \\
swim \rightarrow sunny \\
\neg swim \rightarrow canoe \\
canoe \rightarrow sunset \\
\hline
\therefore\ sunset
\end{array}
$$

It is not sunny and it is cold.

We will swim only if it is sunny.

If we do not swim, then we will canoe.

If we canoe, then we will be home by sunset.

We will be home by sunset.

Proof Example *cont.*

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- Step 3: Verify the reasoning using the rules of inference

$$\begin{array}{l} \neg sunny \land cold \\ swim \rightarrow sunny \\ \neg swim \rightarrow canoe \\ canoe \rightarrow sunset \\ \hline \therefore\ sunset \end{array}$$

Step	Proved by
1. $\neg sunny \land cold$	Premise #1.
2. $\neg sunny$	Simplification of 1.
3. $swim \rightarrow sunny$	Premise #2.
4. $\neg swim$	Modus tollens on 2 and 3.
5. $\neg swim \rightarrow canoe$	Premise #3.
6. $canoe$	Modus ponens on 4 and 5.
7. $canoe \rightarrow sunset$	Premise #4.
8. $sunset$	Modus ponens on 6 and 7.

Common Fallacies

- A ***fallacy*** is an inference rule or other proof method that is not logically valid.
  - A fallacy may yield a false conclusion!
- *Fallacy of affirming the conclusion*:
  - "$p \rightarrow q$ is true, and q is true, so p must be true." (No, because **F** $\rightarrow$ **T** is true.)
- Example
  - If David Cameron (DC) is president of the US, then he is at least 40 years old. ($p \rightarrow q$)
  - DC is at least 40 years old. (q)
  - Therefore, DC is president of the US. (p)

Common Fallacies (*cont'd*)

- *Fallacy of denying the hypothesis*:
  - "$p \to q$ is true, and p is false, so q must be false." (No, again because $\mathbf{F} \to \mathbf{T}$ is true.)
- Example
  - If a person does arithmetic well then his/her checkbook will balance. ($p \to q$)
  - I cannot do arithmetic well. ($\neg p$)
  - Therefore my checkbook does not balance. ($\neg q$)

Inference Rules for Quantifiers

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- $$\frac{\forall x\, P(x)}{\therefore P(c)}$$ **Universal instantiation**
  (substitute *any* specific member c in the domain)

- $$\frac{P(c)}{\therefore \forall x\, P(x)}$$ (for an arbitrary element c of the domain)
  Universal generalization

- $$\frac{\exists x\, P(x)}{\therefore P(c)}$$ **Existential instantiation**
  (substitute an element c for which $P(c)$ is true)

- $$\frac{P(c)}{\therefore \exists x\, P(x)}$$ (for some element c in the domain)
  Existential generalization

Example

- Every man has two legs. John Smith is a man. Therefore, John Smith has two legs.
- Proof
  - Define the predicates:
    - $M(x)$: x is a man
    - $L(x)$: x has two legs
    - J: John Smith, a member of the universe
  - The argument becomes

    1. $\forall x\, [M(x) \rightarrow L(x)]$
    2. $M(J)$

    $\therefore\ L(J)$

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Example *cont.*

$$\begin{array}{l} \forall x\,(M(x) \longrightarrow L(x)) \\ M(J) \\ \hline \therefore L(J) \end{array}$$

- The proof is
  1. $\forall x\,[M(x) \rightarrow L(x)]$ — *Premise 1*
  2. $M(J) \rightarrow L(J)$ — *U. I. from (1)*
  3. $M(J)$ — *Premise 2*
  4. $L(J)$ — *Modus Ponens from (2) and (3)*

- Note: Using the rules of inference requires lots of practice.
  - Try example problems in the textbook.

Another example

- Correct or incorrect: “At least one of the 20 students in the class is intelligent. John is a student of this class. Therefore, John is intelligent.”
- First: Separate premises from conclusion
  - *Premises*:
    1. At least one of the 20 students in the class is intelligent.
    2. John is a student of this class.
  - *Conclusion*: John is intelligent.

Answer

- Next, translate the example in logic notation.
  - *<u>Premise 1</u>*: At least one of the 20 students in the class is intelligent.

    Let the domain = all people

    $C(x)$ = "x is in the class"

    $I(x)$ = "x is intelligent"

    Then *Premise 1* says: $\boldsymbol{\exists x(C(x) \wedge I(x))}$
  - *<u>Premise 2</u>*: John is a student of this class.

    Then *Premise 2* says: **C(John)**
  - And the *Conclusion* says: ***I*(John)**

$$\begin{array}{l} \exists x\,(C(x) \wedge I(x)) \\ C(John) \\ \hline \therefore\ I(John) \end{array}$$

Answer (*cont'd*)

$$\begin{array}{l} \exists x\,(C(x) \wedge I(x)) \\ C(John) \\ \hline \therefore\ I(John) \end{array}$$

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- No, the argument is invalid; we can disprove it with a counter-example, as follows:

- Consider a case where there is only one intelligent student A in the class, and A ≠ John.
  - Then by existential instantiation of the premise **∃*x* (*C*(*x*) ∧ *I*(*x*)), *C*(A) ∧ *I*(A)** is true,
  - But the conclusion ***I*(John)** is false, since A is the only intelligent student in the class, and John ≠ A.
- Therefore, the premises *do not* imply the conclusion.

More Proof Examples

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- Is this argument correct or incorrect?
  - "All TAs compose easy quizzes.
    Mike is a TA.
    Therefore, Mike composes easy quizzes."
- First, separate the premises from conclusion:
  - *Premise 1*: All TAs compose easy quizzes.
  - *Premise 2*: Mike is a TA.
  - *Conclusion*: Mike composes easy quizzes.

Answer

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- Next, re-render the example in logic notation.
  - *Premise 1*: All TAs compose easy quizzes.
    - Let the domain = all people
    - Let $T(x)$ = "x is a TA"
    - Let $E(x)$ = "x composes easy quizzes"
    - Then *Premise 1* says: $\forall x(T(x) \rightarrow E(x))$
  - *Premise* 2: Mike is a TA.
    - Let M = Mike
    - Then *Premise 2* says: T(M)
  - And the *Conclusion* says: E(M)

$$\begin{array}{l} \forall x\, (T(x) \longrightarrow E(x)) \\ T(M) \\ \hline \therefore\ E(M) \end{array}$$

The Proof in Gory Detail

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- The argument is correct, because it can be reduced to a sequence of applications of valid inference rules, as follows:

$$\begin{array}{l} \forall x\,(T(x) \rightarrow E(x)) \\ T(M) \\ \hline \therefore\ E(M) \end{array}$$

- <u>Statement</u> <u>How obtained</u>

1. $\forall x(T(x) \rightarrow E(x))$ **(Premise #1)**
2. $T(\mathrm{M}) \rightarrow E(\mathrm{M})$ **(Universal Instantiation)**
3. $T(\mathrm{M})$ **(Premise #2)**
4. $E(\mathrm{M})$ **(*Modus Ponens* from #2 and #3)**

Another Example

- Prove that the sum of a rational number and an irrational number is always irrational.

- First, you have to understand exactly what the question is asking you to prove:
  - "For all real numbers x,y,
    if x is rational and y is irrational,
    then $x+y$ is irrational."
  - $\forall x,y$: Rational(x) $\wedge$ Irrational(y) $\rightarrow$ Irrational($x+y$)

Answer

- Next, think back to the definitions of the terms used in the statement of the theorem:
  - $\forall$ reals r : Rational(r) $\leftrightarrow$
    $\exists$ Integer(i) $\wedge$ Integer(j *with* $\neq 0$): $r = i/j$.
  - $\forall$ reals r : Irrational(r) $\leftrightarrow$ $\neg$Rational(r)

- You almost always need the definitions of the terms in order to prove the theorem!

- Next, let's go through one valid proof:

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What you might write

- **Theorem:**
  $\forall x,y : \text{Rational}(x) \wedge \text{Irrational}(y) \rightarrow \text{Irrational}(x+y)$
- **Proof:** Let x, y be any rational and irrational numbers, respectively. … (universal generalization)
- Now, just from this, what do we know about x and y? Think back to the definition of a rational number:
- … Since x is rational, we know (from the very definition of rational) that there must be some integers i and j such that $x = i / j$. So, let i_x, j_x be such integers …
- Notice that gave them the unique names i_x and j_x so we can refer to them later.

What next?

- What do we know about *y*? Only that *y* is irrational: $\neg\exists$ integers i,j: $y = i/j$.
- But, it's difficult to see how to use a direct proof in this case. So let's try to use proof by contradiction.
- So, what are we trying to show?
  Just that $x+y$ is irrational.
  That is, $\neg\exists i,j$: $(x + y) = i/j$.
- Now we need to hypothesize the negation of this statement!

More writing…

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- Suppose that $x+y$ were not irrational. Then $x + y$ would be rational, so ∃ integers i,j: $x + y = i / j$. So, let i_s and j_s be any such integers where $x + y = i_s / j_s$.
- Now, with all these things named, we can see what happens when we put them together.
- So, we have that $(i_x/j_x) + y = (i_s/j_s)$.
- Notice: We have enough information now to conclude something useful about *y*, by solving this equation for it!

Finishing the Proof

- Solving that equation for y, we have:

$$y = (i_s/j_s) - (i_x/j_x)$$
$$= (i_s j_x - i_x j_s)/(j_s j_x)$$

- Now, since the numerator and denominator of this expression are both integers, y is rational (by definition of a rational number).
- This contradicts the assumption that y is irrational. Therefore, our hypothesis that $x+y$ is rational must be false, and so the theorem is proved.

Example of a Wrong Answer

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- 1 is rational. $\sqrt{2}$ is irrational. $1+\sqrt{2}$ is irrational. Therefore, the sum of a rational number and an irrational number is irrational. (Attempting a direct proof.)

- Why does this answer deserve <u>no credit</u>?
  - We attempted to use an example to prove a universal statement.
    This is <u>always invalid</u>!
  - Even as an example, it's incomplete, because we never even proved that $1+\sqrt{2}$ is irrational!