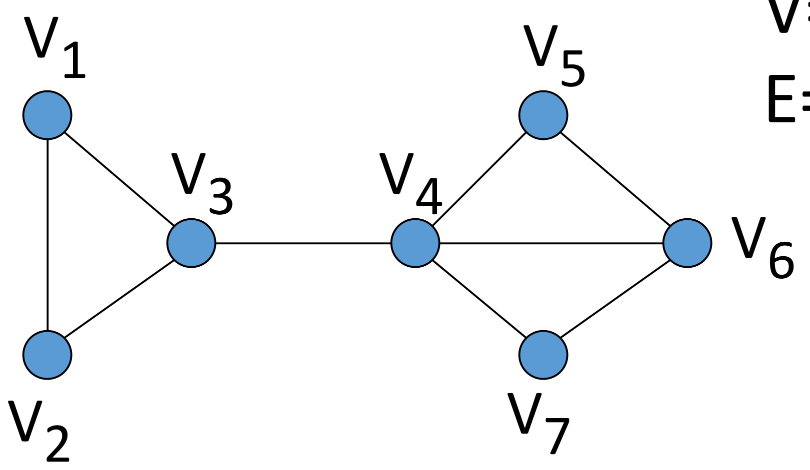
Discrete Mathematics

Chapter-6

Graphs

Introduction to Graphs

- **Def:** A (simple) graph G=(V,E) consists of a nonempty set V of vertices, and E, a set of unordered pairs of distinct elements of V called edges.
- Example:



G=(V,E), where

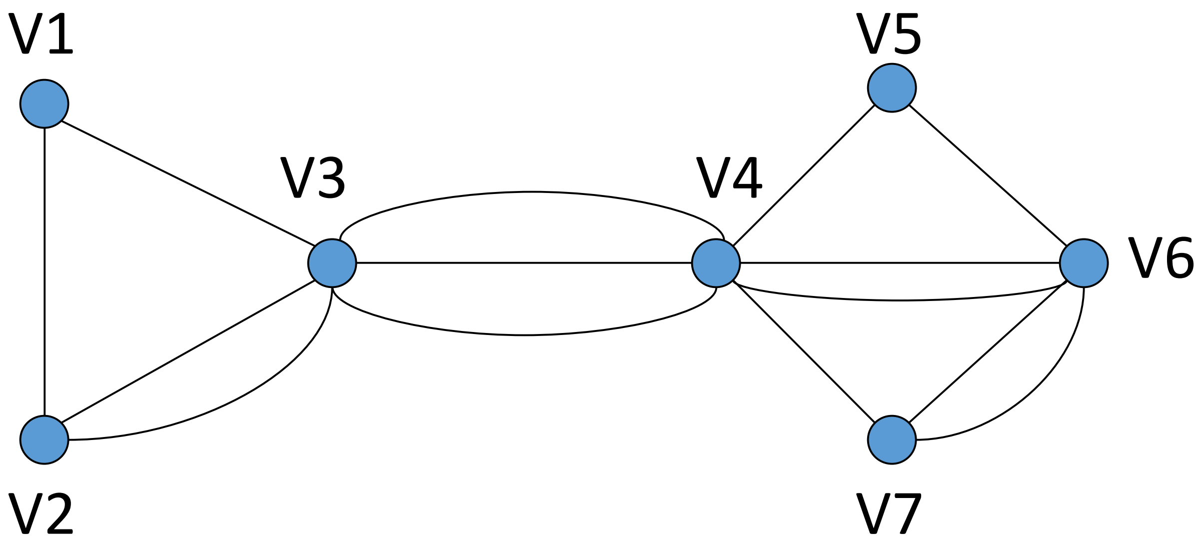
$V=\{ v_1,v_2,\ldots,v_7 \}$

$E=\{ \{v_1,v_2\}, \{v_1,v_3\}, \{v_2,v_3\}$
$\{v_3,v_4\}, \{v_4,v_5\}, \{v_4,v_6\}$
$\{v_4,v_7\}, \{v_5,v_6\}, \{v_6,v_7\} \}$

- **Multigraph :** Graphs that may have multiple edges connecting the same vertices, i.e.

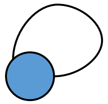
  Simple Graph + "allow multiple edges between two points"

Example:



Pseudo graph :

Simple Graph + Multiedge + Loop

loop -> 

Example:

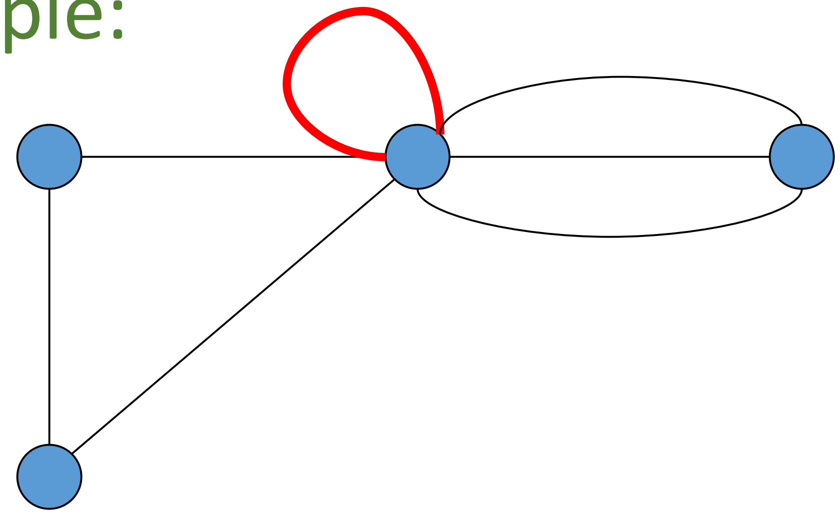


Table: Graph Terminology

Type	Edges	Multiple Edges	Loops
Simple Graph	Undirected	✗	✗
Multigraph		✓	✗
Pseudo graph		✓	✓
Directed graph	Directed	✗	✓
Directed multigraph		✓	✓

Graph Terminology (undirected)

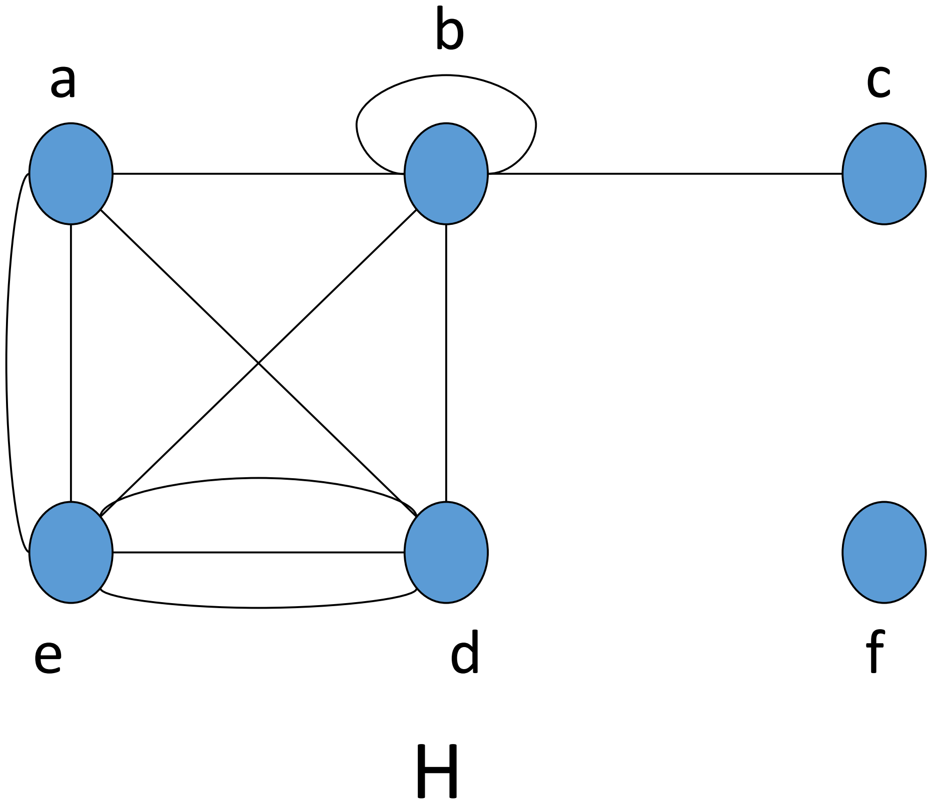
- **Def:** Two vertices u and v in an graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

- Def: The degree of a vertex v, denoted by deg(v), in an (undirected) graph is the number of edges incident with it.

(Note : loop needs to be counted twice)

Example 1. What are the degree of the vertices in the graph H ?

- **Sol :**



deg(a)=4

deg(b)=6

deg(c)=1

deg(d)=5

deg(e)=6

deg(f)=0

➢**Def:** A vertex of degree 0 is called isolated.

Example: “ f ” in Example 1.

THEOREM: The Handshaking Theorem: the sum of degrees of the vertices of a graph is twice the number of edges.

Let G=(V,E) be an undirected graph with n edges (i.e., |E|=n).

Then

$$\sum_{v \in V} \deg(v) = 2n$$

Each edge {u,v} will contribute a degree to u and v

Example 1. there are 11 edges, and

$$\sum_{v \in V} \deg(v) = 32$$

Example 2. How many edges are there in a graph with 10 vertices each of degree 6 ?

Sol :

$10 \cdot 6 = 2n \quad \Rightarrow \quad n=30$

- **Def :**

  G=(V,E) : directed graph

  $v \in V$

  $\deg^-(v)$: no of edges with v as a terminal vertex.

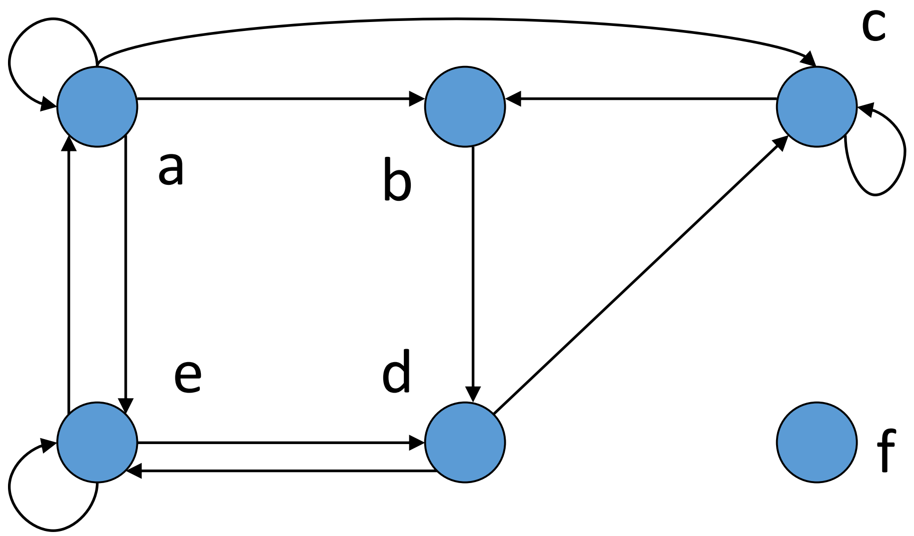
  (in-degree)

  $\deg^+(v)$: no of edges with v as a initial vertex.

  (out-degree)

Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

Example:



$deg^-(a)=2$, $deg^+(a)=4$

$deg^-(b)=2$, $deg^+(b)=1$

$deg^-(c)=3$ $deg^+(c)=2$

$deg^-(d)=2$ $deg^+(d)=2$

$deg^-(e)=3$ $deg^+(e)=3$

$deg^-(f)=0$, $deg^+(f)=0$

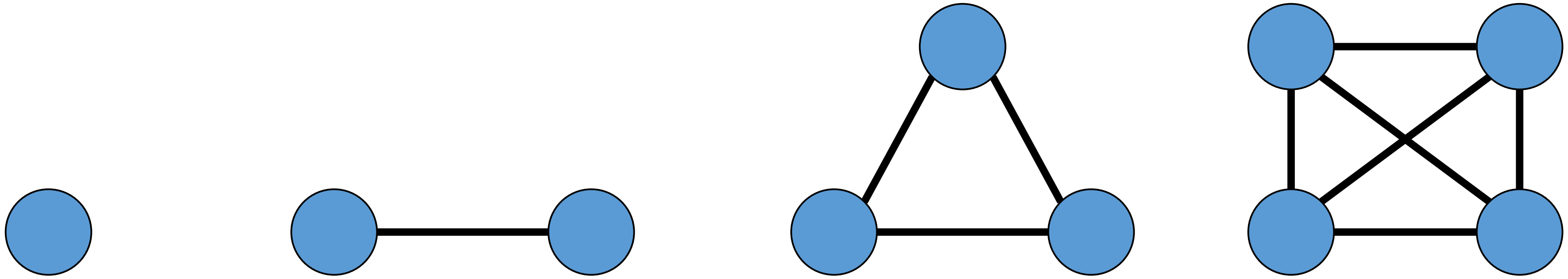
a is adjacent to b , b is adjacent from a

a : initial vertex of (a,b)

b : $\underbrace{\text{terminal}}_{\text{end}}$ vertex of (a,b)

- **Def :** The complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

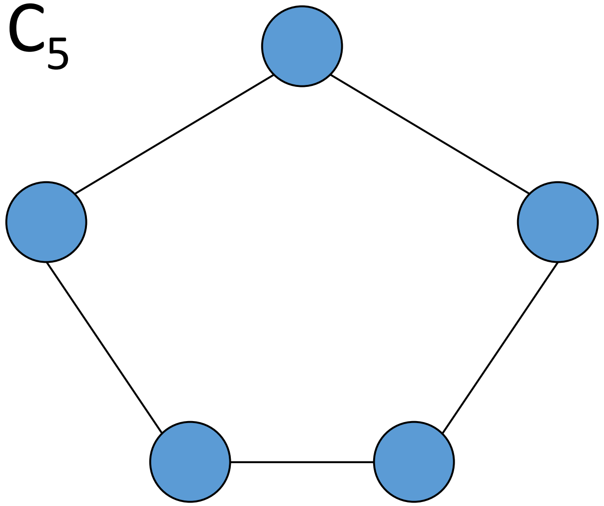
Example:



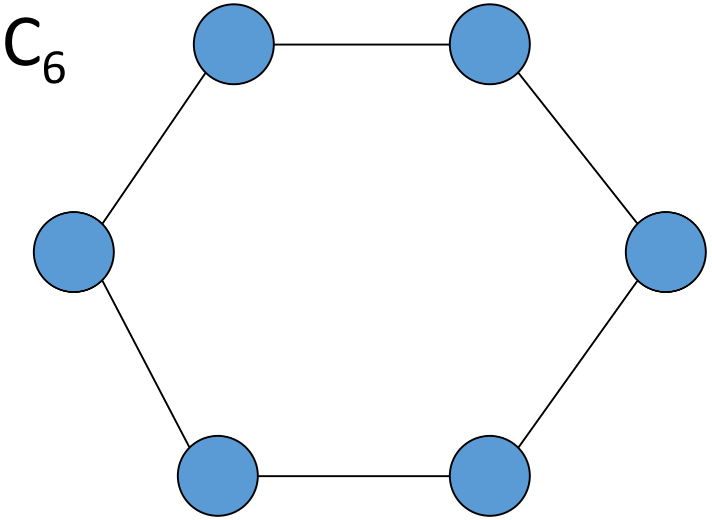
K_1 K_2 K_3 K_4

Example: The cycle C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1,v_2\}, \{v_2,v_3\}, \ldots, \{v_{n-1},v_n\}, \{v_n,v_1\}$.

C_5

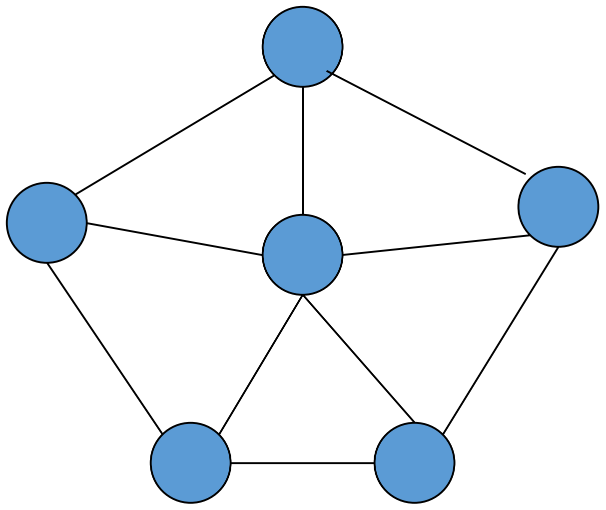


C_6

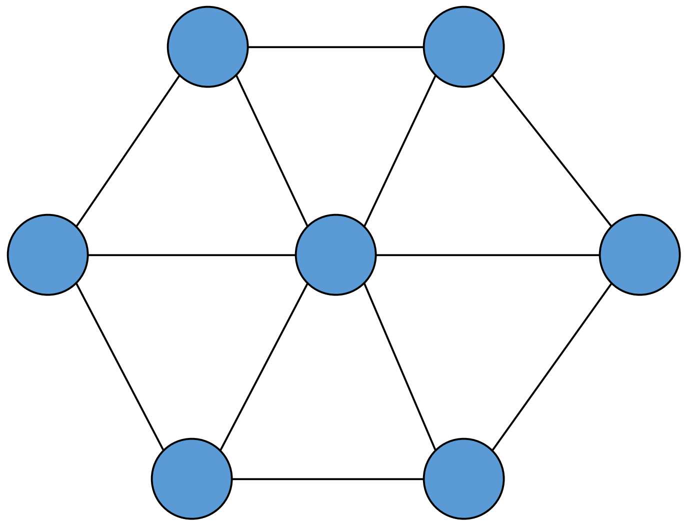


Example: Wheel (Wn), add one point to Cn and connect it to the other n points (n ≥ 3)

W_5



W_6

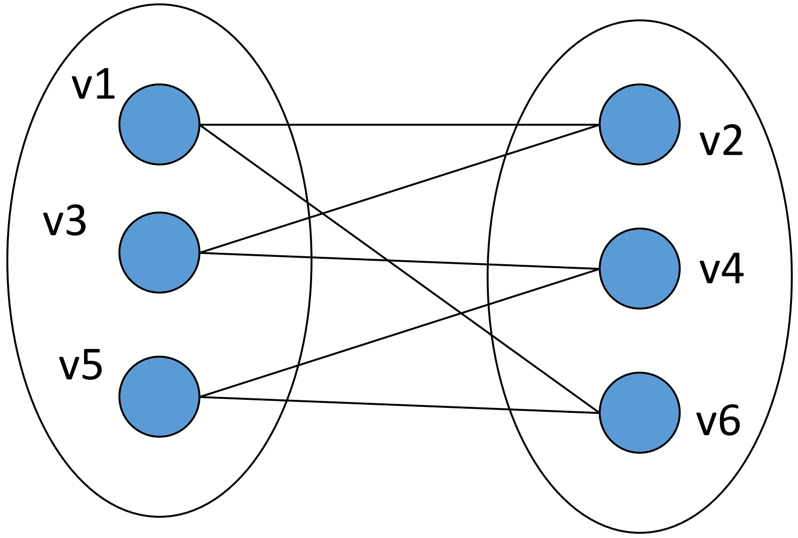


| V | = n + 1

| E | = 2n

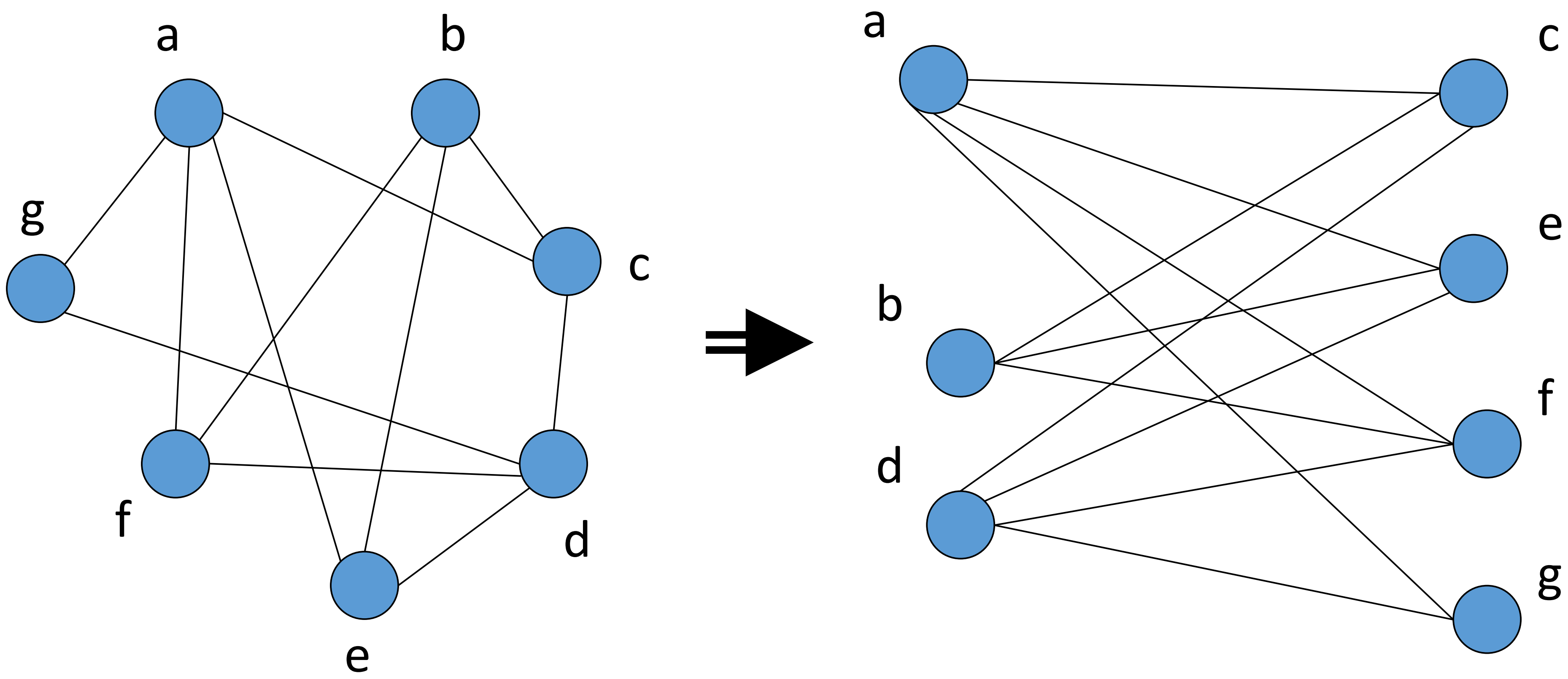
- **Def:** A simple graph $G=(V,E)$ is called bipartite if V can be partitioned into V_1 and V_2, $V_1 \cap V_2 = \varnothing$, such that every edge in the graph connect a vertex in V_1 and a vertex in V_2.

Example:



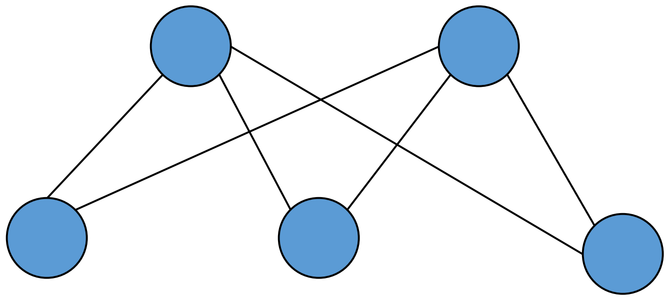
$\therefore C_6$ is bipartite.

Example: Is the graph G bipartite ?

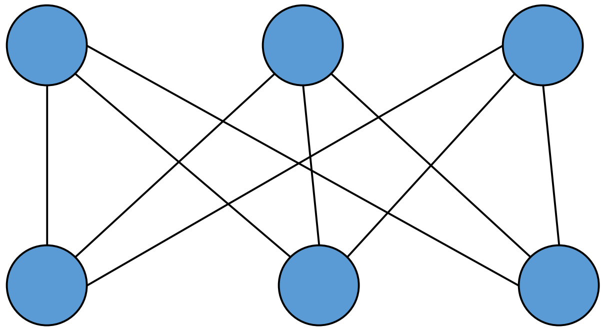


Yes !

Example: Complete Bipartite graphs ($k_{m,n}$)



$K_{2,3}$



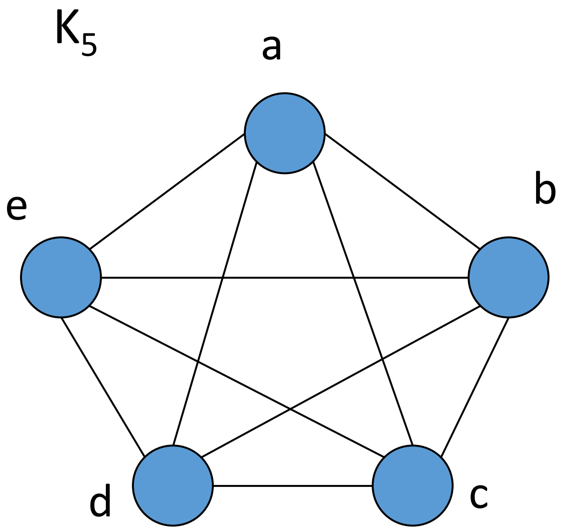
$K_{3,3}$

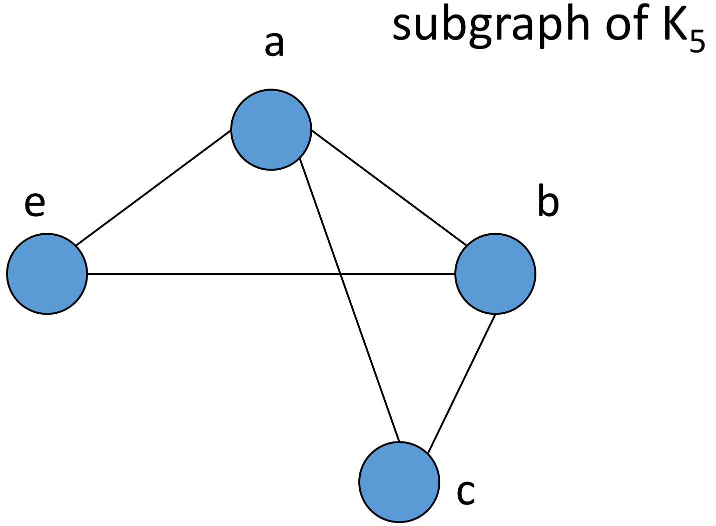
Note: $| E | = mn$

Def: A subgraph of a graph $G=(V,E)$ is a graph $H=(W,F)$ where $W \subseteq V$ and $F \subseteq E$.

A subgraph H of G is a proper subgraph of G if $H \neq G$

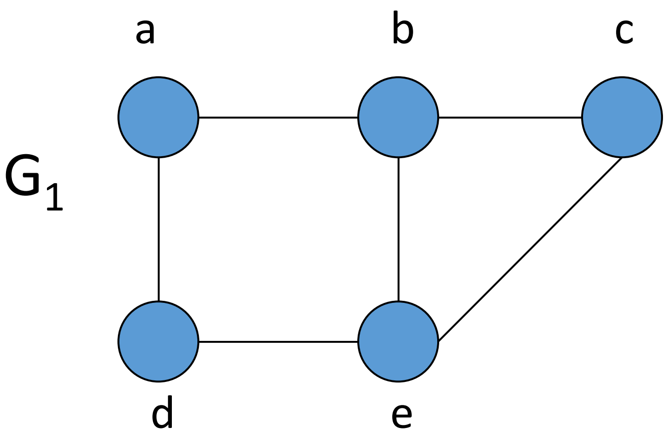
Example: A subgraph of K_5

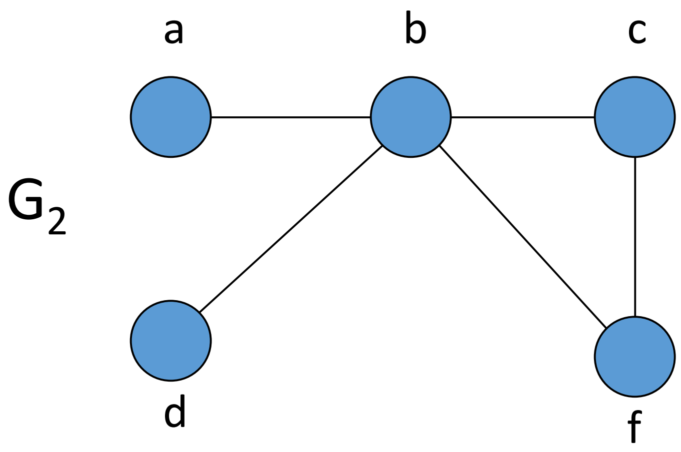




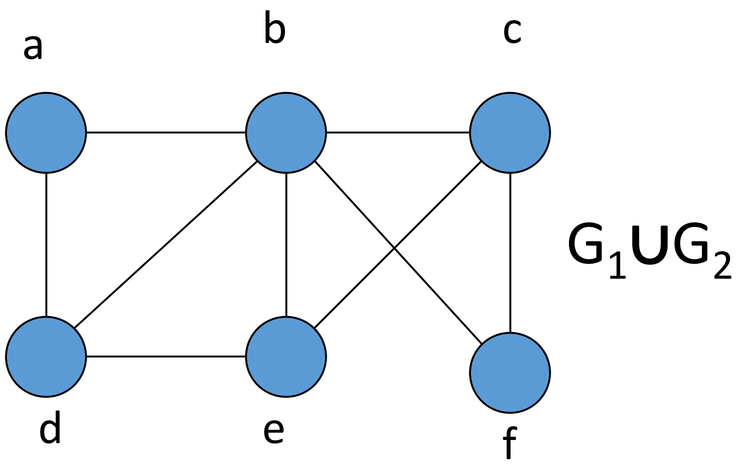
Def: The union of two simple graph $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is the simple graph $G_1 \cup G_2=(V_1 \cup V_2,E_1 \cup E_2)$

- **Example 15.**





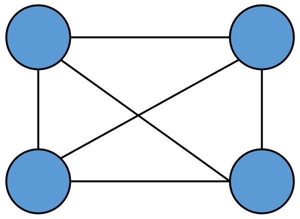
$\Longrightarrow$



Example: A simple graph $G=(V,E)$ is called **regular** if every vertex of this graph has the same degree. A regular graph is called **n-regular** if $\deg(v)=n$, $\forall v \in V$.

For example:

K_4 :

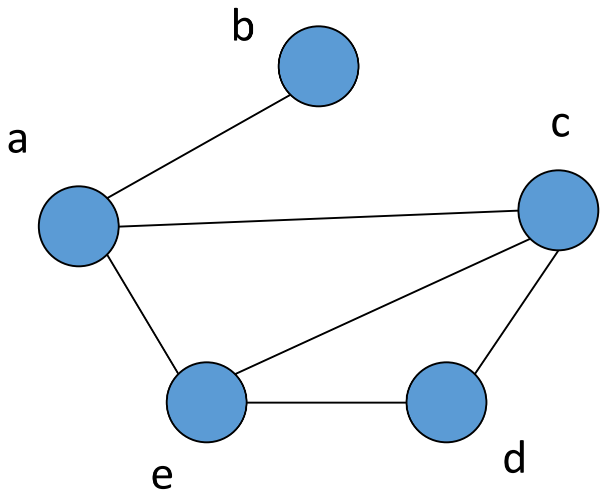


is 3-regular.

Representing Graphs and Graph Isomorphism

※Adjacency list

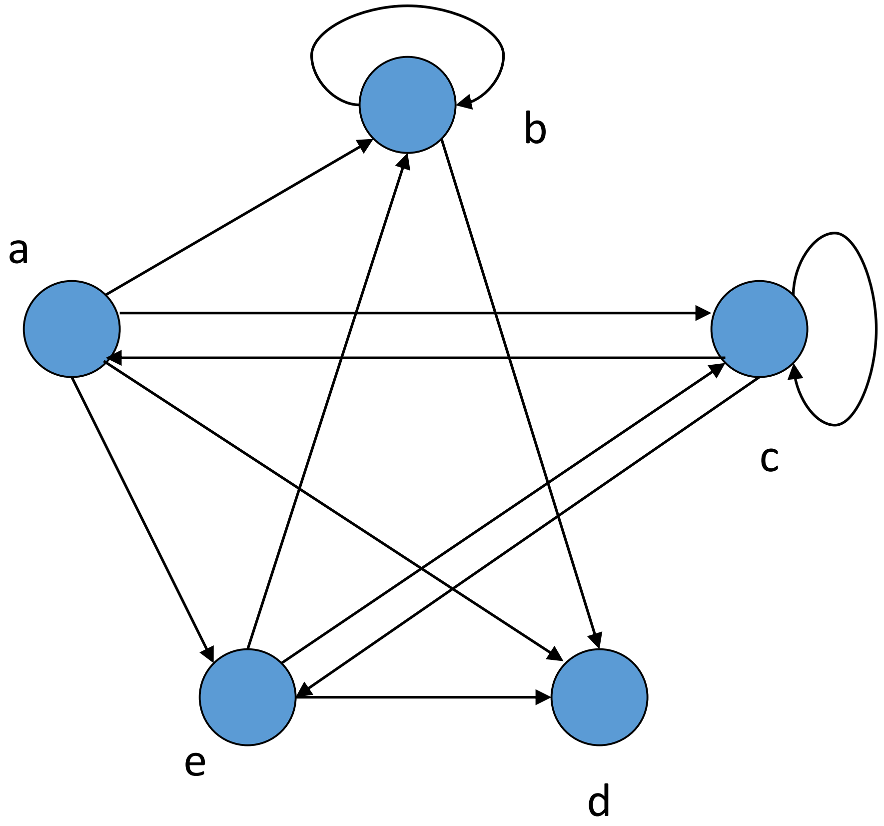
Example: Use adjacency list to describe the simple graph given below.



Sol :

Vertex	Adjacent vertices
a	b,c,e
b	a
c	a,d,e
d	c,e
e	a,c,d

Example: Digraph



Initial vertex	Terminal vertices
a	b,c,d,e
b	b,d
c	a,c,e
d	
e	b,c,d

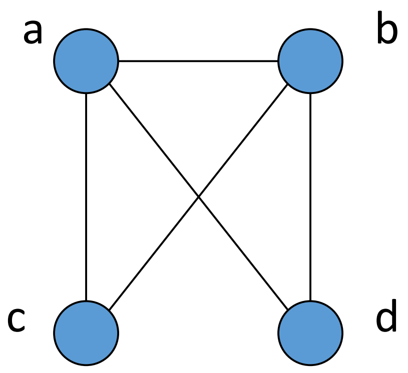
※Adjacency Matrices

Def. $G=(V,E)$: simple graph,

$V=\{v_1,v_2,\ldots,v_n\}$.

A matrix A is called the adjacency matrix of G if $A=[a_{ij}]_{nxn}$, where

Example:

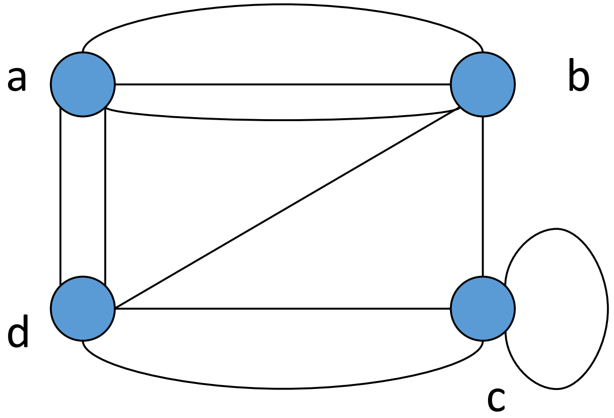


$$A_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \\ b \\ d \\ c \\ a \end{array} \begin{array}{c} \begin{array}{cccc} b & d & c & a \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

undirected graph

Example: Pseudograph : such matrices are not zero–one matrices



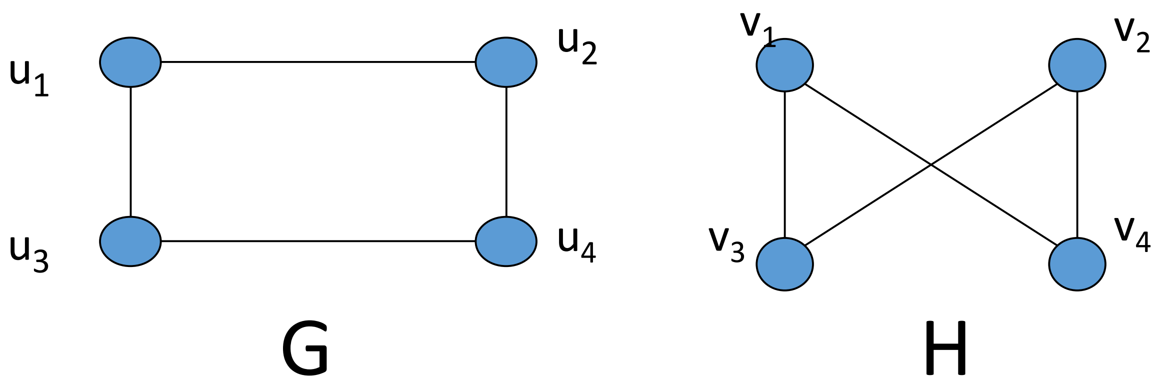
$$\Longrightarrow \quad A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Def. If $A=[a_{ij}]$ is the adjacency matrix for the directed graph, then

$$a_{ij} = \begin{cases} 1 & \text{, if } v_i \longrightarrow v_j \\ 0 & \text{, otherwise} \end{cases}$$

The matrix is not necessarily symmetrical

※Isomorphism of Graphs



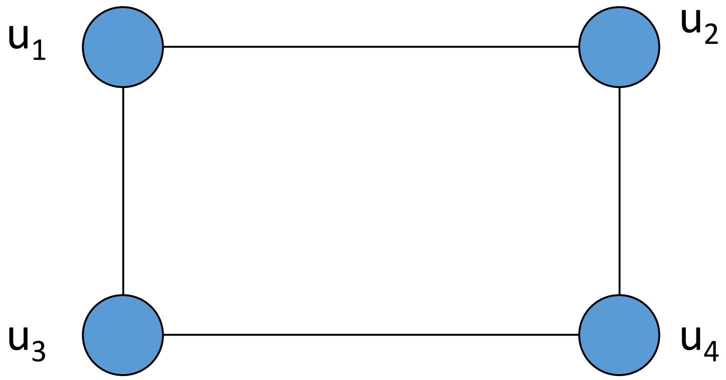
G is isomorphic to H

➢ **Def :**

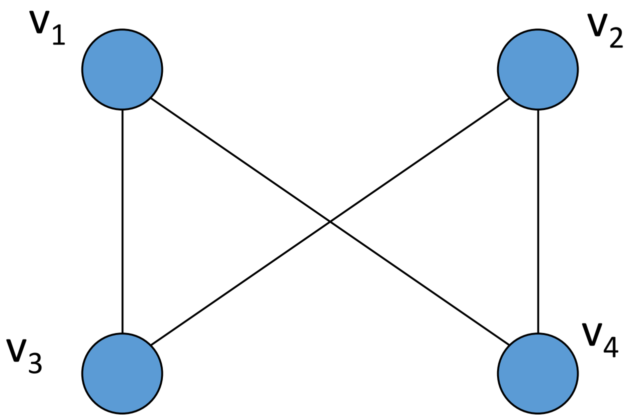
The simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are isomorphic if there is an one-to-one and onto function f from V_1 to V_2 with the property that $a \sim b$ in G_1 iff $f(a) \sim f(b)$ in G_2, $\forall a,b \in V_1$

f is called an isomorphism.

Example:



G=(V,E)



H=(W,F)

$f(u_1) = v_1$
$f(u_3) = v_3$
$f(u_2) = v_4$
$f(u_4) = v_2$

a one to-one correspondence between V and W

※ Isomorphism Graphs must have:

(1) The same points.

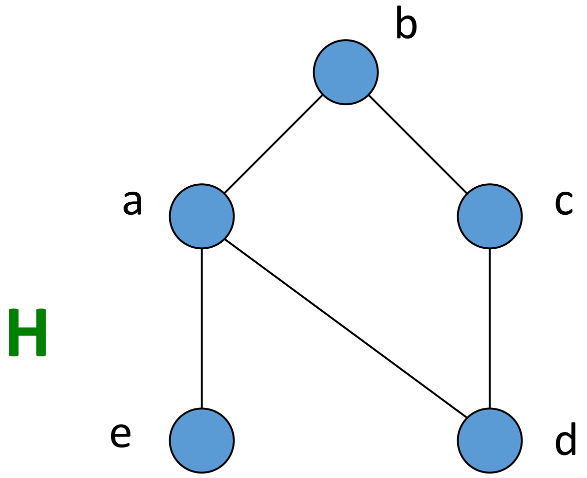
(2) The same number of edges.

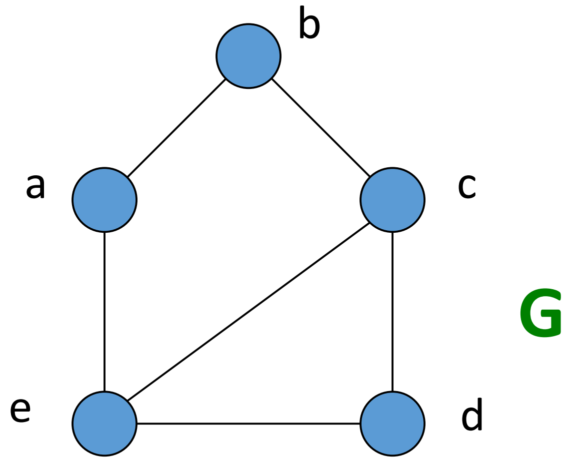
(3) The same degree distribution.

※ Given the second picture to determine whether they are Isomorphic is generally not easy to solve, and the answer is often negative.

Example:

Show that G and H are not isomorphic.

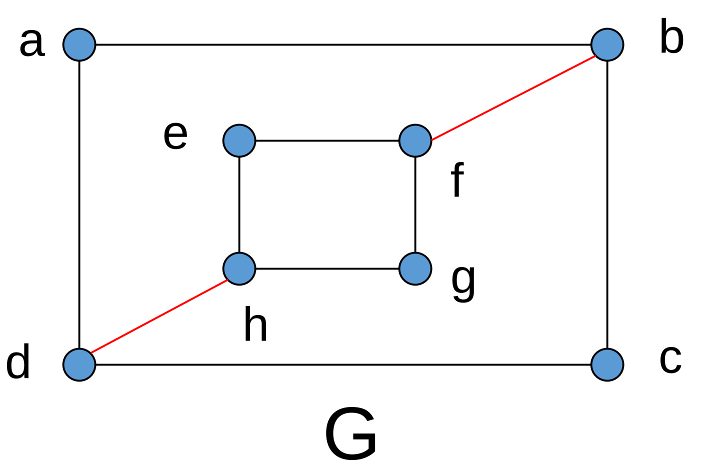


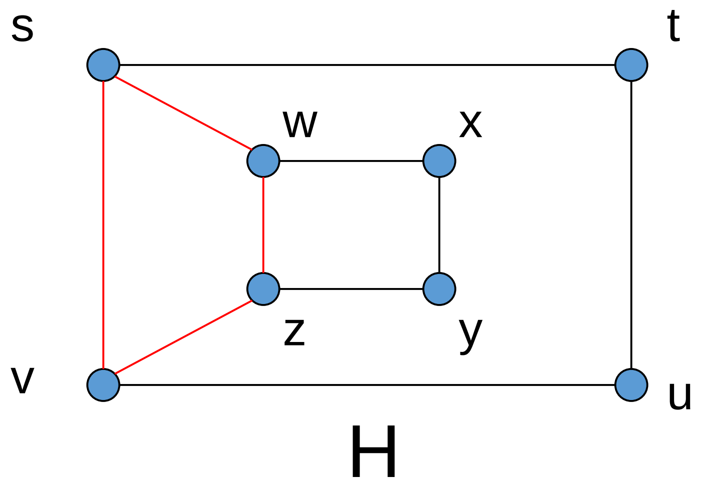


Sol :

∵H has a vertex of degree one, namely, e, whereas G has no vertices of degree one

Example: Determine whether G and H are isomorphic.





Sol :

∵ Points with degree 3 in G have d, h, f, b. They cannot be connected into 4-cycle

But the points with degree 3 in H have s, w, z, v. They can be connected into 4-cycle

∴ is not isomorphic.

Another method: The point in G with degree 3 is only connected to another point with deg = 3 next to it But there are 2 points with deg = 3 next to the points with deg = 3 in H.

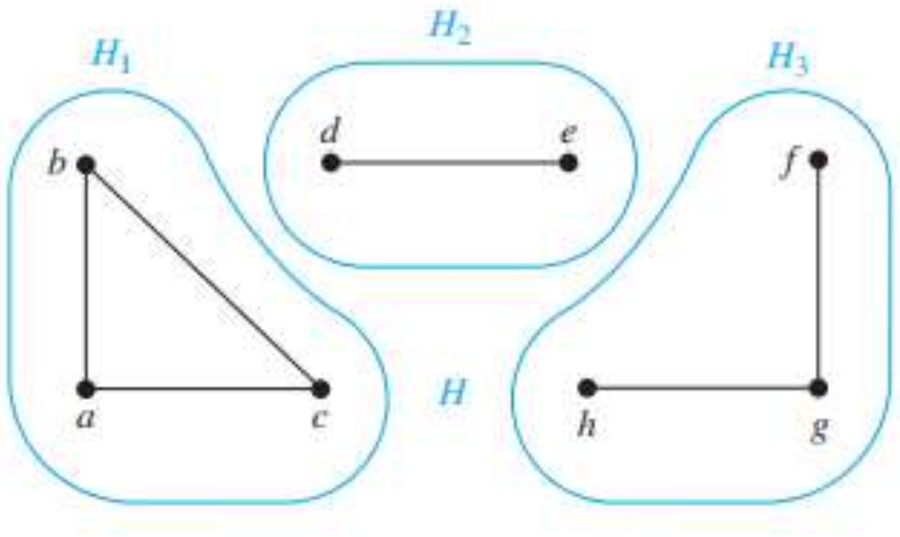
Connectivity

Def:

- In an undirected graph, a *path of length n from u to v* is a sequence of adjacent vertices going from vertex u to vertex v. (e.g., P: $u=x_0, x_1, x_2, \ldots, x_n=v$)
- Note: A path of length n has <u>n+1 *vertices*</u>，<u>n edges</u>
- A path is a *circuit* if $u=v$.
- A path *traverses* the vertices along it.
- A path or circuit is *simple* if it contains no vertex more than once. (simple circuit usually called *cycle*)

Connectedness

- **Def:** An undirected graph is *connected* iff there is a path between every pair of distinct vertices in the graph.
- **Def:** *Connected component*: Maximal connected subgraph. (A disconnected graph will have several component)

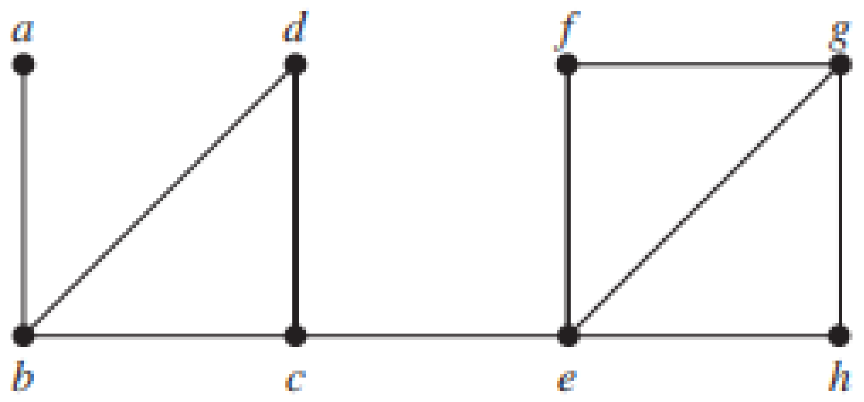


Connected Components H1, H2, and H3

Connectedness

Def: A *cut vertex* separates one connected component into several components if it is removed.

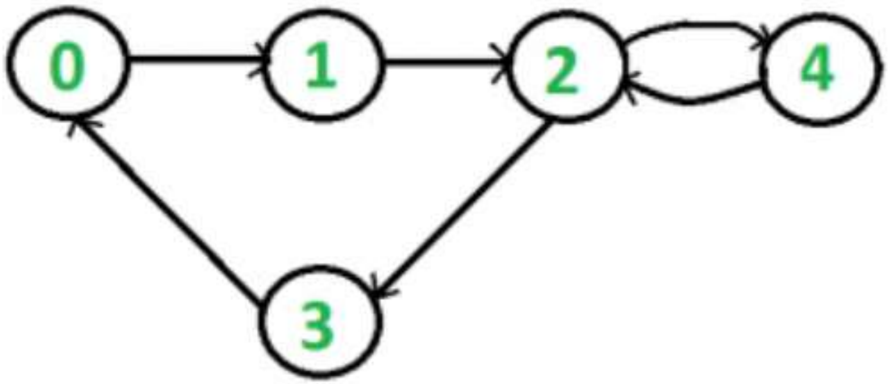
Def: A *cut edge* separates one connected component into two components if it is removed.



Connectedness in Digraphs

Def: A directed graph is **strongly connected** iff there is a path from a to b and from b to a whenever a and b are vertices in the graph.

Example:



Strongly Connected

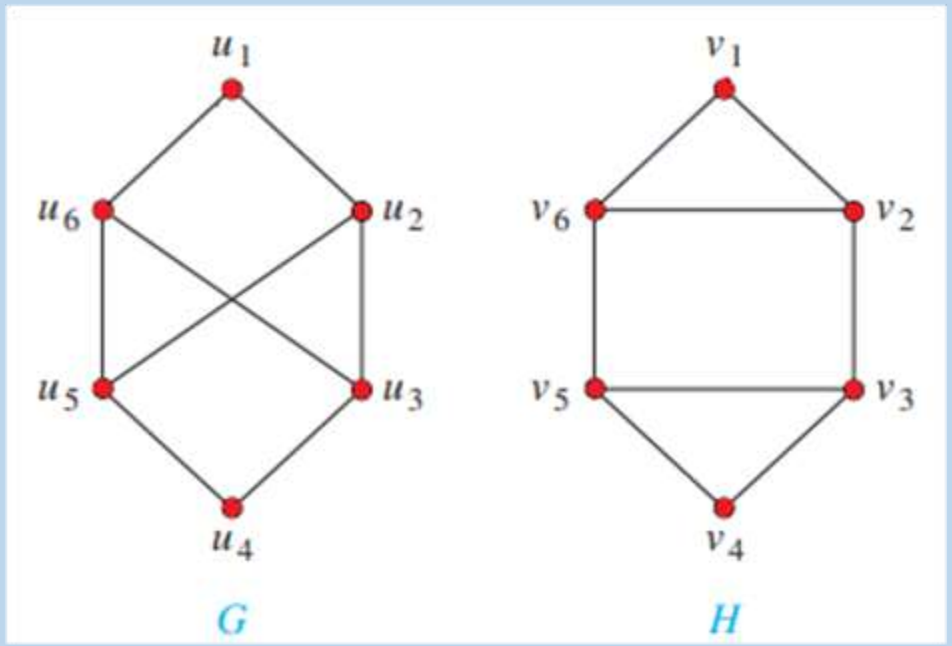
Not Strongly Connected

Note: *strongly* implies *weakly* but not vice-versa.

Paths & Isomorphism

- Note that connectedness, and the existence of a simple circuit of a particular length is a useful invariant that can be used to show that two graphs are not isomorphic.

Example: Determine whether the graphs G and H shown in Figure are isomorphic.

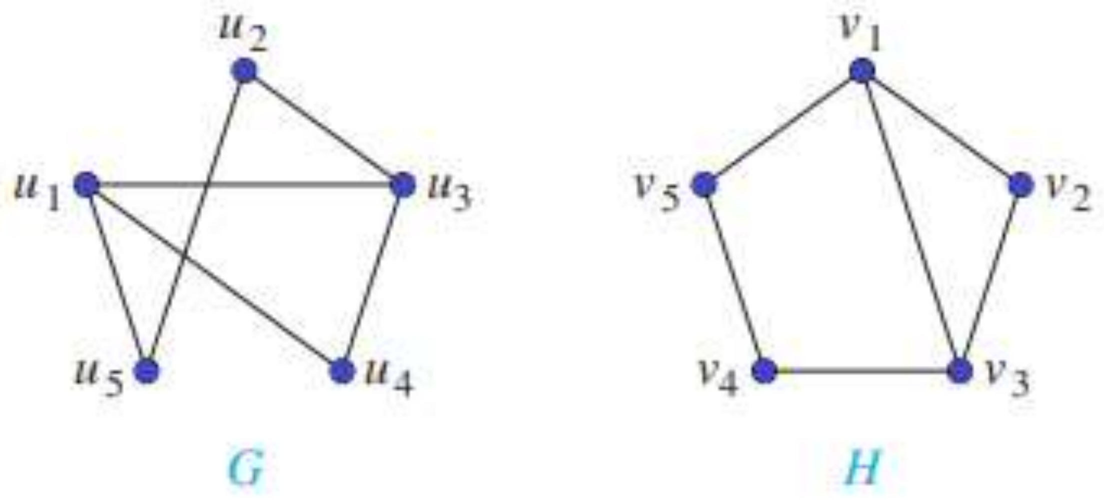


Sol: Both G and H have: 6 vertices and 8 edges.
4 vertices of degree 3, and 2 vertices of degree 2.
3 invariants—no. of vertices, no. of edges, and degrees of vertices—all agree for the two graphs.

H has a simple circuit of length 3, namely, v1, v2, v6, v1, whereas G has no simple circuit of length 3. The existence of a simple circuit of length three is an isomorphic invariant, G and H are not isomorphic.

Paths & Isomorphism

Example: Determine whether the graphs G and H shown in Figure are isomorphic.



Sol: Graphs G and H are Isomorphic

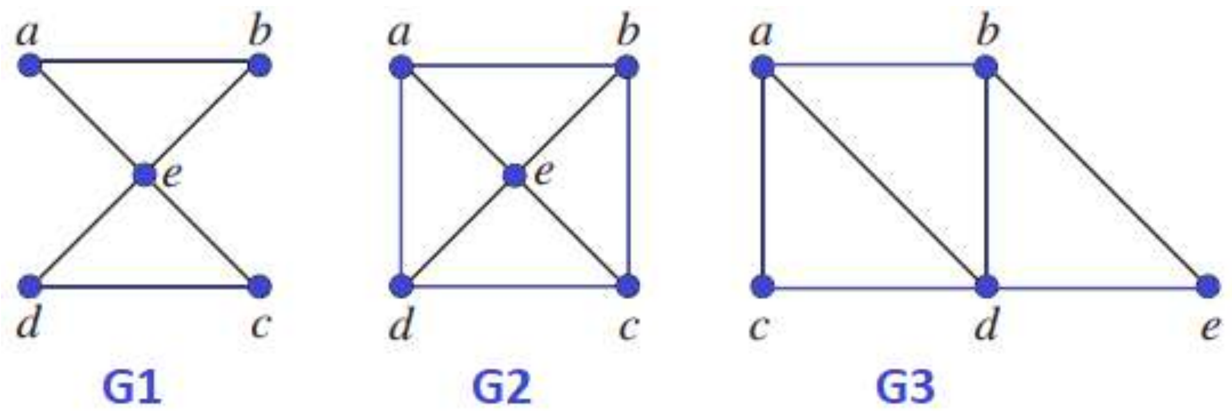
Euler & Hamilton Paths

- **Def:**

- An *Euler circuit* in a *G* is a simple circuit containing every edge of *G*.
- An *Euler path* in *G* is a simple path containing every edge of *G*.

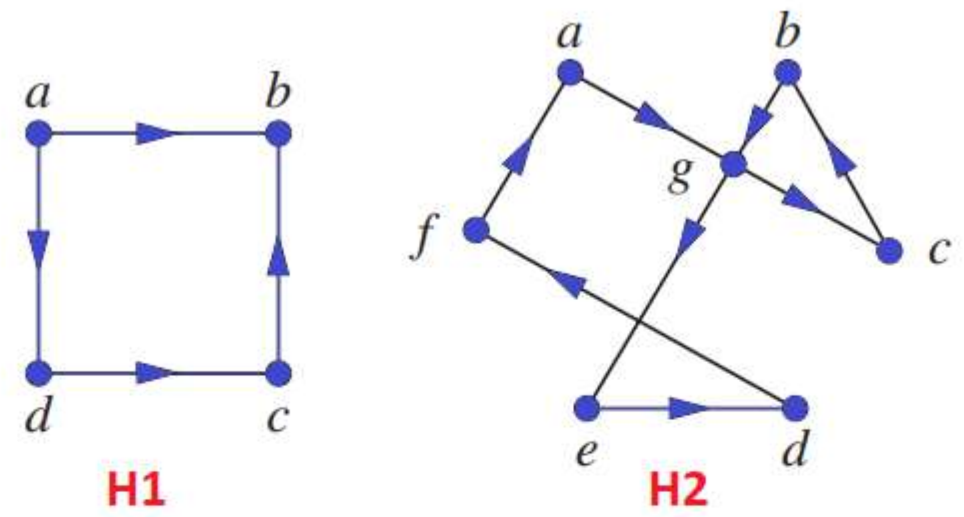
Example:

Which of the undirected graphs in Figure have an Euler circuit? Of those that do not, which have an Euler path?



Sol: G1 has an Euler circuit: a, e, c, d, e, b, a.
Neither of G2 or G3 has an Euler circuit.
G3 has an Euler path: a, c, d, e, b, d, a, b.
G2 does not have an Euler path.

Example: **Which of the directed graphs in Figure 4 have an Euler circuit? Of those that do not, which have an Euler path?**



Sol:

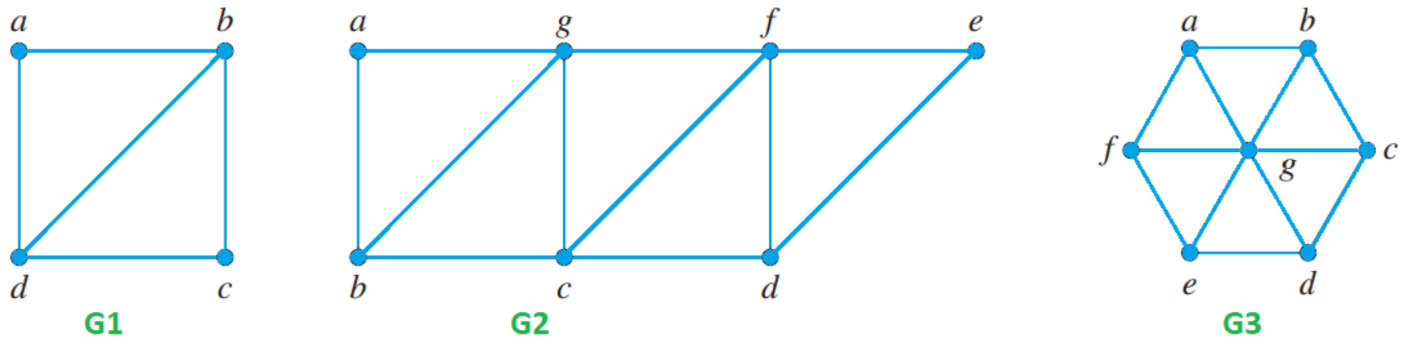
- **The graph H2 has an Euler circuit: a, g, c, b, g, e, d, f, a.**
- **Neither H1 nor H3 has an Euler circuit.**
- **H3 has an Euler path: c, a, b, c, d, b… And H1 does not.**

Useful Points

1. A connected multigraph has an Euler circuit iff each vertex has even degree.

2. A connected multigraph has an Euler path (but not an Euler circuit) iff it has exactly 2 vertices of odd degree.

Example: Which graphs shown in Figure have an Euler path?



Sol: G1: Exactly two vertices of odd degree: d, a, b, c, d, b

G2: Exactly two vertices of odd degree: b, a, g, f, e, d, c, g, b, c, f, d.

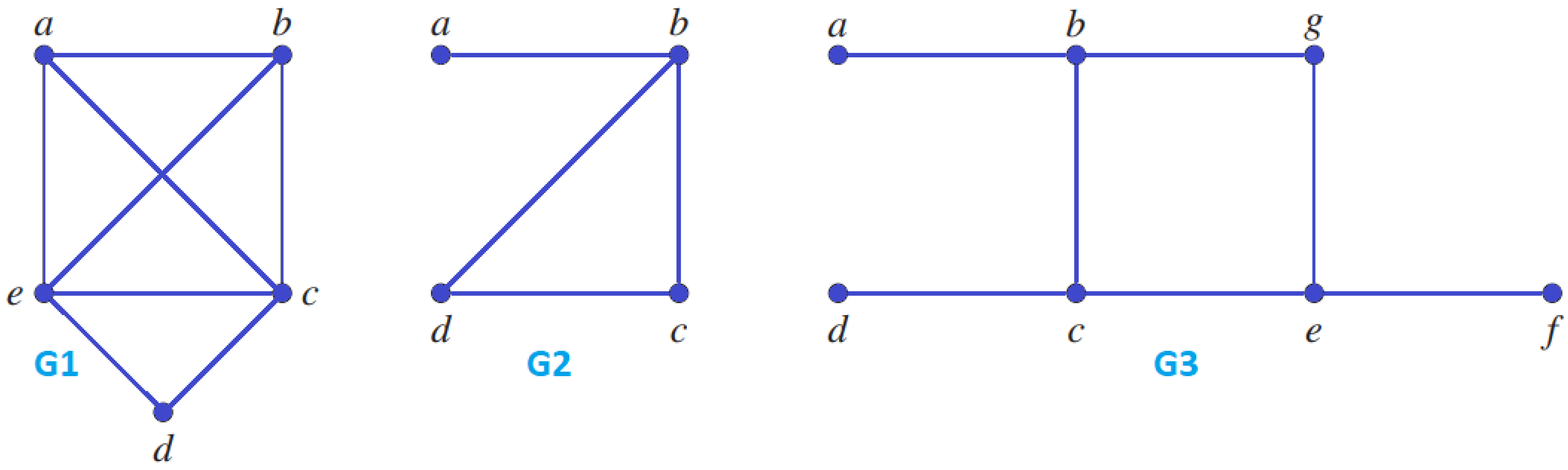
G3: Six vertices of odd degree, No Euler path.

Hamilton Paths

Def:

- A *Hamilton circuit* is a circuit that traverses each vertex in *G* exactly once.
- A *Hamilton path* is a path that traverses each vertex in *G* exactly once.
- Graph that contains Hamilton circuit is called *Hamilton Graph.*
- *If there is a vertax of degree one in a graph then it is impossible for it to have a Hamilton Circuit.*

Example: Which of the simple graphs in Figure have a Hamilton circuit or, if not, a Hamilton path?



Sol: G1: Hamilton circuit: a, b, c, d, e, a.

G2: No Hamilton circuit but have a Hamilton path: a, b, c, d.

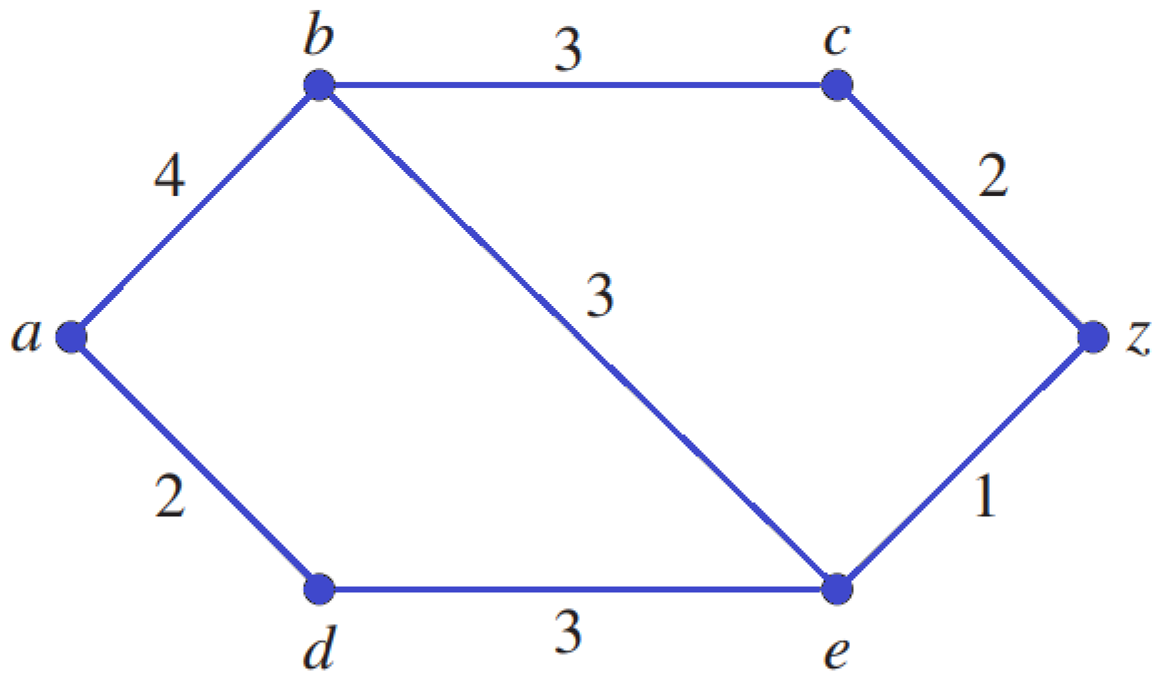
G3: Neither a Hamilton circuit nor a Hamilton path

Shortest Path Problems

- **Def:** Graphs that have a number assigned to each edge are called *weighted graphs*.
- **Shortest path Problem:**

  Determining the path of least sum of the weights between two vertices in a weighted graph.

Example: What is the length of a shortest path between a and z in the weighted graph shown in Figure?



Sol: Shortest Path: a, d, e, z of length 6

The Traveling Salesperson Problem

- The **traveling salesman problem** is one of the classical problems in computer science.
- A traveling salesman wants to visit a number of cities and then return to his starting point. Of course he wants to save time and energy, so he wants to determine the **shortest cycle** for his trip.
- We can represent the cities and the distances between them by a weighted, complete, undirected graph.
- The problem then is to find the **shortest cycle (of minimum total weight that visits each vertex exactly one)**.
- Finding the shortest cycle is different than Dijkstra's shortest path.

  It is much harder too, no polynomial time algorithm exists!

The Traveling Salesperson Problem

Example: **Suppose that the salesperson wants to visit New Delhi, Faridabad, Karnal, Chandigarh and Meerut City. In which order should he visit these cities to travel the minimum total distance?**

Sol:

Route	Total Distance in Miles
New Delhi–Faridabad–Karnal–Chandigarh–Meerut–New Delhi	610
New Delhi–Faridabad–Karnal–Meerut–Chandigarh–New Delhi	516
New Delhi–Faridabad–Meerut–Chandigarh–Karnal–New Delhi	588
New Delhi–Faridabad–Meerut–Karnal–Chandigarh–New Delhi	458
New Delhi–Faridabad–Chandigarh–Meerut–Karnal–New Delhi	540
New Delhi–Faridabad–Chandigarh–Karnal–Meerut–New Delhi	504
New Delhi–Chandigarh–Faridabad–Karnal–Meerut–New Delhi	598
New Delhi–Chandigarh–Faridabad–Meerut–Karnal–New Delhi	576
New Delhi–Chandigarh–Meerut–Faridabad–Karnal–New Delhi	682
New Delhi–Chandigarh–Karnal–Faridabad–Meerut–New Delhi	646
New Delhi–Karnal–Chandigarh–Faridabad–Meerut–New Delhi	670
New Delhi–Karnal–Faridabad–Chandigarh–Meerut–New Delhi	728

Planar Graphs

Def.: A graph is called *planar* if it can be drawn in the plane without any edge crossing. Such a drawing is called a *planar representation* of the graph.

Example:

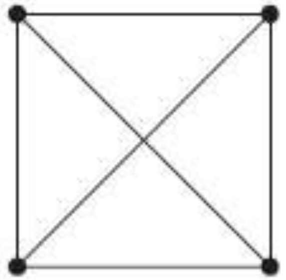


FIGURE 2 The Graph K_4.

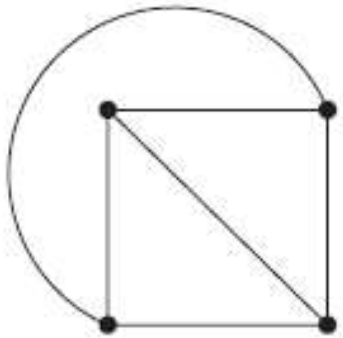


FIGURE 3 K_4 Drawn with No Crossings.

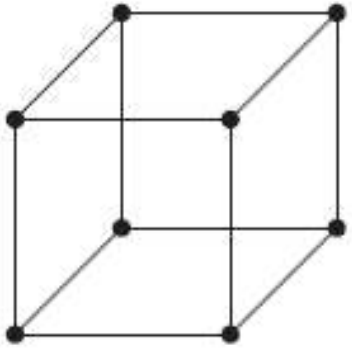


FIGURE 4 The Graph Q_3.

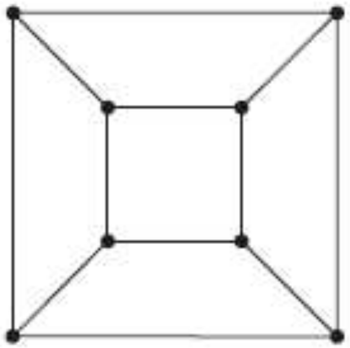


FIGURE 5 A Planar Representation of Q_3.

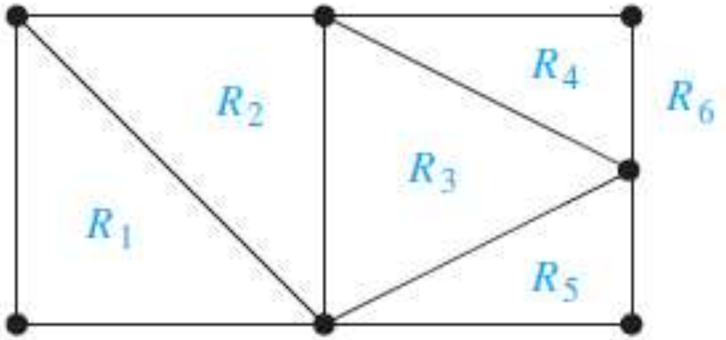
EULER'S FORMULA

- A planar representation of a graph splits the plane into **regions**, including an unbounded region.

- Let *G* be a connected planar simple graph with *e* edges and *v* vertices. Let *r* be the number of regions in a planar representation of *G*.

Then $r = e - v + 2$.

r=11-7+2 = 6



The Regions of the Planar Representation of a Graph.

Graph Coloring

Def:

A *coloring* of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

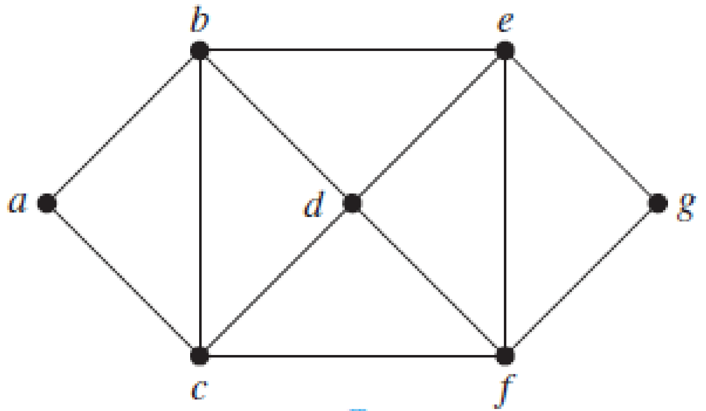
Def:

The *chromatic number* of a graph is the least number of colors needed for a coloring of the graph. (denoted by $\chi(G)$)

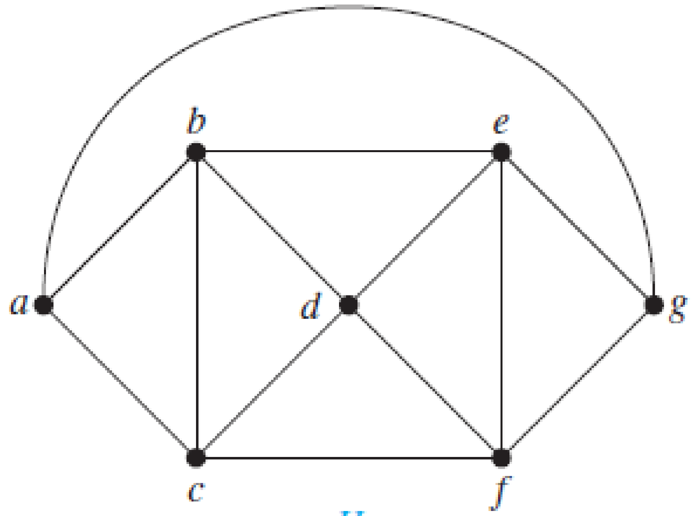
THE FOUR COLOR THEOREM

- **The Four Color Theorem:** The chromatic number of a planar graph is no greater than four.
- Note that the four color theorem applies only to planar graphs. Nonplanar graphs can have arbitrarily large chromatic numbers.
- Two things are required to show that the chromatic number of a graph is *k*.
  - First, we must show that the graph can be colored with *k* colors. This can be done by constructing such a coloring.
  - Second, we must show that the graph cannot be colored using fewer than *k* colors.

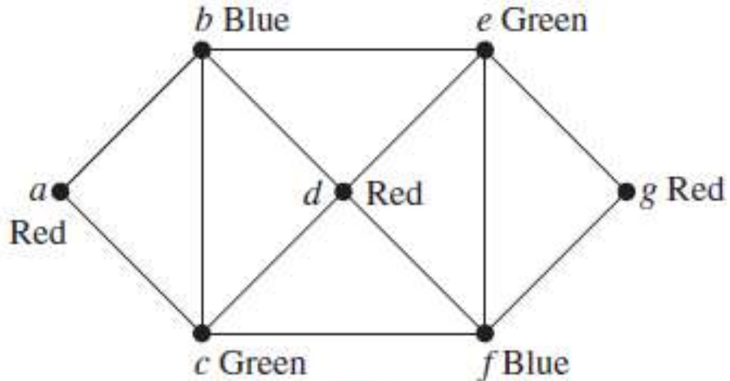
Example: What are the chromatic numbers of the graphs *G* and *H* shown in Figure



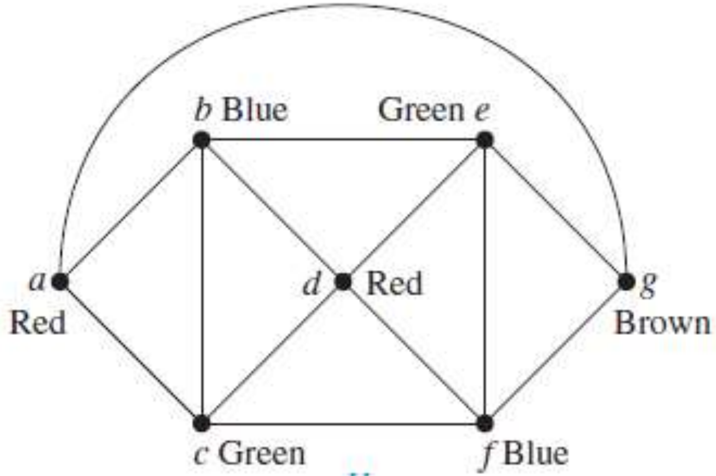
G



H



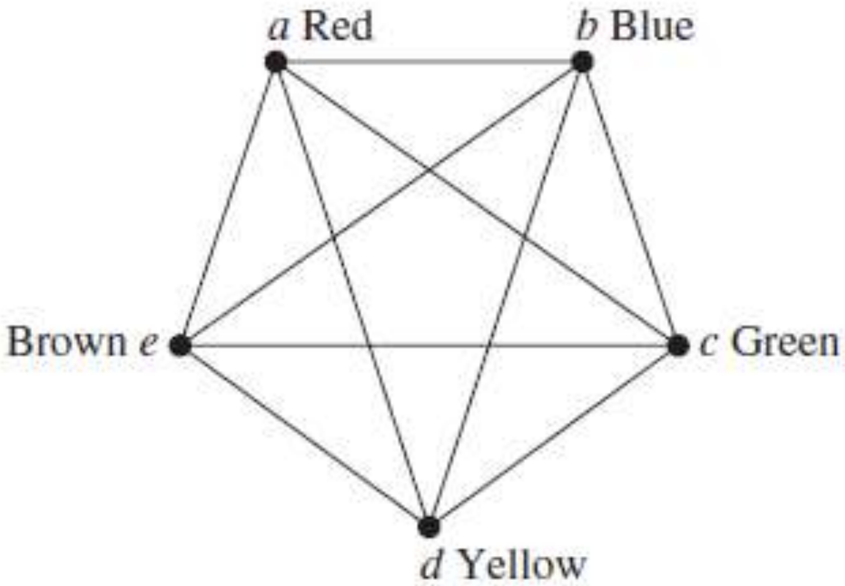
G



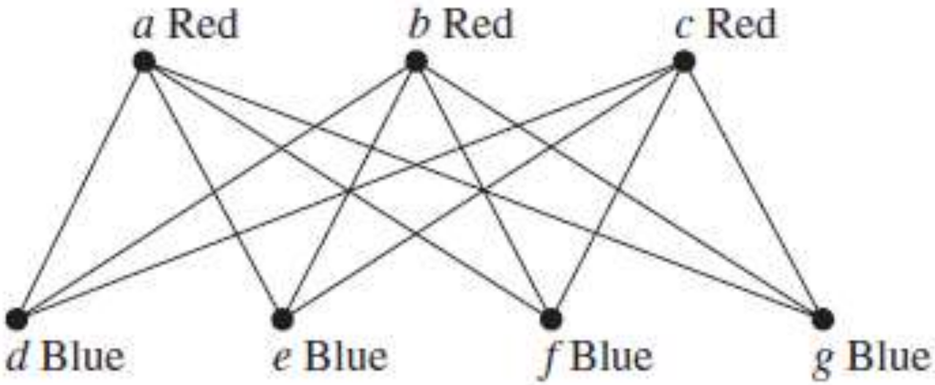
H

Colorings of the Graphs *G* and *H*.

Coloring of K_5 and $K_{3,4}$



A Coloring of K_5.



A Coloring of $K_{3,4}$.