

40. Explain, without using a truth table, why $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when p , q , and r have the same truth value and it is false otherwise.
41. Explain, without using a truth table, why $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$ is true when at least one of p , q , and r is true and at least one is false, but is false when all three variables have the same truth value.
42. What is the value of x after each of these statements is encountered in a computer program, if $x = 1$ before the statement is reached?
- if $x + 2 = 3$ then $x := x + 1$
 - if $(x + 1 = 3)$ OR $(2x + 2 = 3)$ then $x := x + 1$
 - if $(2x + 3 = 5)$ AND $(3x + 4 = 7)$ then $x := x + 1$
 - if $(x + 1 = 2)$ XOR $(x + 2 = 3)$ then $x := x + 1$
 - if $x < 2$ then $x := x + 1$
43. Find the bitwise OR, bitwise AND, and bitwise XOR of each of these pairs of bit strings.
- 101 1110, 010 0001
 - 1111 0000, 1010 1010
 - 00 0111 0001, 10 0100 1000
 - 11 1111 1111, 00 0000 0000
44. Evaluate each of these expressions.
- $1\ 1000 \wedge (0\ 1011 \vee 1\ 1011)$
 - $(0\ 1111 \wedge 1\ 0101) \vee 0\ 1000$
 - $(0\ 1010 \oplus 1\ 1011) \oplus 0\ 1000$
 - $(1\ 1011 \vee 0\ 1010) \wedge (1\ 0001 \vee 1\ 1011)$
- Fuzzy logic** is used in artificial intelligence. In fuzzy logic, a proposition has a truth value that is a number between 0 and 1, inclusive. A proposition with a truth value of 0 is false and one with a truth value of 1 is true. Truth values that are between 0 and 1 indicate varying degrees of truth. For instance, the truth value 0.8 can be assigned to the statement “Fred is happy,” because Fred is happy most of the time, and the truth value 0.4 can be assigned to the statement “John is happy,” because John is happy slightly less than half the time. Use these truth values to solve Exercises 45–47.
45. The truth value of the negation of a proposition in fuzzy logic is 1 minus the truth value of the proposition. What are the truth values of the statements “Fred is not happy” and “John is not happy?”
46. The truth value of the conjunction of two propositions in fuzzy logic is the minimum of the truth values of the two propositions. What are the truth values of the statements “Fred and John are happy” and “Neither Fred nor John is happy?”
47. The truth value of the disjunction of two propositions in fuzzy logic is the maximum of the truth values of the two propositions. What are the truth values of the statements “Fred is happy, or John is happy” and “Fred is not happy, or John is not happy?”
- *48. Is the assertion “This statement is false” a proposition?
- *49. The n th statement in a list of 100 statements is “Exactly n of the statements in this list are false.”
- What conclusions can you draw from these statements?
 - Answer part (a) if the n th statement is “At least n of the statements in this list are false.”
 - Answer part (b) assuming that the list contains 99 statements.
50. An ancient Sicilian legend says that the barber in a remote town who can be reached only by traveling a dangerous mountain road shaves those people, and only those people, who do not shave themselves. Can there be such a barber?

1.2 Applications of Propositional Logic

Introduction

Logic has many important applications to mathematics, computer science, and numerous other disciplines. Statements in mathematics and the sciences and in natural language often are imprecise or ambiguous. To make such statements precise, they can be translated into the language of logic. For example, logic is used in the specification of software and hardware, because these specifications need to be precise before development begins. Furthermore, propositional logic and its rules can be used to design computer circuits, to construct computer programs, to verify the correctness of programs, and to build expert systems. Logic can be used to analyze and solve many familiar puzzles. Software systems based on the rules of logic have been developed for constructing some, but not all, types of proofs automatically. We will discuss some of these applications of propositional logic in this section and in later chapters.

Translating English Sentences

There are many reasons to translate English sentences into expressions involving propositional variables and logical connectives. In particular, English (and every other human language) is

often ambiguous. Translating sentences into compound statements (and other types of logical expressions, which we will introduce later in this chapter) removes the ambiguity. Note that this may involve making a set of reasonable assumptions based on the intended meaning of the sentence. Moreover, once we have translated sentences from English into logical expressions we can analyze these logical expressions to determine their truth values, we can manipulate them, and we can use rules of inference (which are discussed in Section 1.6) to reason about them.

To illustrate the process of translating an English sentence into a logical expression, consider Examples 1 and 2.

EXAMPLE 1 How can this English sentence be translated into a logical expression?

“You can access the Internet from campus only if you are a computer science major or you are not a freshman.”



Solution: There are many ways to translate this sentence into a logical expression. Although it is possible to represent the sentence by a single propositional variable, such as p , this would not be useful when analyzing its meaning or reasoning with it. Instead, we will use propositional variables to represent each sentence part and determine the appropriate logical connectives between them. In particular, we let a , c , and f represent “You can access the Internet from campus,” “You are a computer science major,” and “You are a freshman,” respectively. Noting that “only if” is one way a conditional statement can be expressed, this sentence can be represented as

$$a \rightarrow (c \vee \neg f).$$

EXAMPLE 2 How can this English sentence be translated into a logical expression?

“You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old.”

Solution: Let q , r , and s represent “You can ride the roller coaster,” “You are under 4 feet tall,” and “You are older than 16 years old,” respectively. Then the sentence can be translated to

$$(r \wedge \neg s) \rightarrow \neg q.$$

Of course, there are other ways to represent the original sentence as a logical expression, but the one we have used should meet our needs.

System Specifications

Translating sentences in natural language (such as English) into logical expressions is an essential part of specifying both hardware and software systems. System and software engineers take requirements in natural language and produce precise and unambiguous specifications that can be used as the basis for system development. Example 3 shows how compound propositions can be used in this process.

EXAMPLE 3 Express the specification “The automated reply cannot be sent when the file system is full” using logical connectives.



Solution: One way to translate this is to let p denote “The automated reply can be sent” and q denote “The file system is full.” Then $\neg p$ represents “It is not the case that the automated

reply can be sent,” which can also be expressed as “The automated reply cannot be sent.” Consequently, our specification can be represented by the conditional statement $q \rightarrow \neg p$. ◀

System specifications should be **consistent**, that is, they should not contain conflicting requirements that could be used to derive a contradiction. When specifications are not consistent, there would be no way to develop a system that satisfies all specifications.

EXAMPLE 4 Determine whether these system specifications are consistent:

“The diagnostic message is stored in the buffer or it is retransmitted.”

“The diagnostic message is not stored in the buffer.”

“If the diagnostic message is stored in the buffer, then it is retransmitted.”

Solution: To determine whether these specifications are consistent, we first express them using logical expressions. Let p denote “The diagnostic message is stored in the buffer” and let q denote “The diagnostic message is retransmitted.” The specifications can then be written as $p \vee q$, $\neg p$, and $p \rightarrow q$. An assignment of truth values that makes all three specifications true must have p false to make $\neg p$ true. Because we want $p \vee q$ to be true but p must be false, q must be true. Because $p \rightarrow q$ is true when p is false and q is true, we conclude that these specifications are consistent, because they are all true when p is false and q is true. We could come to the same conclusion by use of a truth table to examine the four possible assignments of truth values to p and q . ◀

EXAMPLE 5 Do the system specifications in Example 4 remain consistent if the specification “The diagnostic message is not retransmitted” is added?

Solution: By the reasoning in Example 4, the three specifications from that example are true only in the case when p is false and q is true. However, this new specification is $\neg q$, which is false when q is true. Consequently, these four specifications are inconsistent. ◀

Boolean Searches



Logical connectives are used extensively in searches of large collections of information, such as indexes of Web pages. Because these searches employ techniques from propositional logic, they are called **Boolean searches**.

In Boolean searches, the connective *AND* is used to match records that contain both of two search terms, the connective *OR* is used to match one or both of two search terms, and the connective *NOT* (sometimes written as *AND NOT*) is used to exclude a particular search term. Careful planning of how logical connectives are used is often required when Boolean searches are used to locate information of potential interest. Example 6 illustrates how Boolean searches are carried out.

EXAMPLE 6 Web Page Searching Most Web search engines support Boolean searching techniques, which usually can help find Web pages about particular subjects. For instance, using Boolean searching to find Web pages about universities in New Mexico, we can look for pages matching *NEW AND MEXICO AND UNIVERSITIES*. The results of this search will include those pages that contain the three words *NEW*, *MEXICO*, and *UNIVERSITIES*. This will include all of the pages of interest, together with others such as a page about new universities in Mexico. (Note that in Google, and many other search engines, the word “AND” is not needed, although it is understood, because all search terms are included by default. These search engines also support the use of quotation marks to search for specific phrases. So, it may be more effective to search for pages matching “New Mexico” *AND* *UNIVERSITIES*.)



Next, to find pages that deal with universities in New Mexico or Arizona, we can search for pages matching (NEW AND MEXICO OR ARIZONA) AND UNIVERSITIES. (*Note:* Here the AND operator takes precedence over the OR operator. Also, in Google, the terms used for this search would be NEW MEXICO OR ARIZONA.) The results of this search will include all pages that contain the word UNIVERSITIES and either both the words NEW and MEXICO or the word ARIZONA. Again, pages besides those of interest will be listed. Finally, to find Web pages that deal with universities in Mexico (and not New Mexico), we might first look for pages matching MEXICO AND UNIVERSITIES, but because the results of this search will include pages about universities in New Mexico, as well as universities in Mexico, it might be better to search for pages matching (MEXICO AND UNIVERSITIES) NOT NEW. The results of this search include pages that contain both the words MEXICO and UNIVERSITIES but do not contain the word NEW. (In Google, and many other search engines, the word “NOT” is replaced by the symbol “-”. In Google, the terms used for this last search would be MEXICO UNIVERSITIES -NEW.)

Logic Puzzles



Puzzles that can be solved using logical reasoning are known as **logic puzzles**. Solving logic puzzles is an excellent way to practice working with the rules of logic. Also, computer programs designed to carry out logical reasoning often use well-known logic puzzles to illustrate their capabilities. Many people enjoy solving logic puzzles, published in periodicals, books, and on the Web, as a recreational activity.

We will discuss two logic puzzles here. We begin with a puzzle originally posed by Raymond Smullyan, a master of logic puzzles, who has published more than a dozen books containing challenging puzzles that involve logical reasoning. In Section 1.3 we will also discuss the extremely popular logic puzzle Sudoku.

EXAMPLE 7



In [Sm78] Smullyan posed many puzzles about an island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B . What are A and B if A says “ B is a knight” and B says “The two of us are opposite types?”

Solution: Let p and q be the statements that A is a knight and B is a knight, respectively, so that $\neg p$ and $\neg q$ are the statements that A is a knave and B is a knave, respectively.

We first consider the possibility that A is a knight; this is the statement that p is true. If A is a knight, then he is telling the truth when he says that B is a knight, so that q is true, and A and B are the same type. However, if B is a knight, then B 's statement that A and B are of opposite types, the statement $(p \wedge \neg q) \vee (\neg p \wedge q)$, would have to be true, which it is not, because A and B are both knights. Consequently, we can conclude that A is not a knight, that is, that p is false.

If A is a knave, then because everything a knave says is false, A 's statement that B is a knight, that is, that q is true, is a lie. This means that q is false and B is also a knave. Furthermore, if B is a knave, then B 's statement that A and B are opposite types is a lie, which is consistent with both A and B being knaves. We can conclude that both A and B are knaves.

We pose more of Smullyan's puzzles about knights and knaves in Exercises 19–23. In Exercises 24–31 we introduce related puzzles where we have three types of people, knights and knaves as in this puzzle together with spies who can lie.

Next, we pose a puzzle known as the **muddy children puzzle** for the case of two children.

EXAMPLE 8 A father tells his two children, a boy and a girl, to play in their backyard without getting dirty. However, while playing, both children get mud on their foreheads. When the children stop playing, the father says “At least one of you has a muddy forehead,” and then asks the children to answer “Yes” or “No” to the question: “Do you know whether you have a muddy forehead?” The father asks this question twice. What will the children answer each time this question is asked, assuming that a child can see whether his or her sibling has a muddy forehead, but cannot see his or her own forehead? Assume that both children are honest and that the children answer each question simultaneously.

Solution: Let s be the statement that the son has a muddy forehead and let d be the statement that the daughter has a muddy forehead. When the father says that at least one of the two children has a muddy forehead, he is stating that the disjunction $s \vee d$ is true. Both children will answer “No” the first time the question is asked because each sees mud on the other child’s forehead. That is, the son knows that d is true, but does not know whether s is true, and the daughter knows that s is true, but does not know whether d is true.

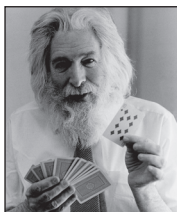
After the son has answered “No” to the first question, the daughter can determine that d must be true. This follows because when the first question is asked, the son knows that $s \vee d$ is true, but cannot determine whether s is true. Using this information, the daughter can conclude that d must be true, for if d were false, the son could have reasoned that because $s \vee d$ is true, then s must be true, and he would have answered “Yes” to the first question. The son can reason in a similar way to determine that s must be true. It follows that both children answer “Yes” the second time the question is asked. ◀

Logic Circuits

Propositional logic can be applied to the design of computer hardware. This was first observed in 1938 by Claude Shannon in his MIT master’s thesis. In Chapter 12 we will study this topic in depth. (See that chapter for a biography of Shannon.) We give a brief introduction to this application here.

A **logic circuit** (or **digital circuit**) receives input signals p_1, p_2, \dots, p_n , each a bit [either 0 (off) or 1 (on)], and produces output signals s_1, s_2, \dots, s_n , each a bit. In this section we will restrict our attention to logic circuits with a single output signal; in general, digital circuits may have multiple outputs.

In Chapter 12 we design some useful circuits.



RAYMOND SMULLYAN (BORN 1919) Raymond Smullyan dropped out of high school. He wanted to study what he was really interested in and not standard high school material. After jumping from one university to the next, he earned an undergraduate degree in mathematics at the University of Chicago in 1955. He paid his college expenses by performing magic tricks at parties and clubs. He obtained a Ph.D. in logic in 1959 at Princeton, studying under Alonzo Church. After graduating from Princeton, he taught mathematics and logic at Dartmouth College, Princeton University, Yeshiva University, and the City University of New York. He joined the philosophy department at Indiana University in 1981 where he is now an emeritus professor.

Smullyan has written many books on recreational logic and mathematics, including *Satan, Cantor, and Infinity*; *What Is the Name of This Book?*; *The Lady or the Tiger?*; *Alice in Puzzleland*; *To Mock a Mockingbird*; *Forever Undecided*; and *The Riddle of Scheherazade: Amazing Logic Puzzles, Ancient and Modern*. Because his logic puzzles are challenging, entertaining, and thought-provoking, he is considered to be a modern-day Lewis Carroll. Smullyan has also written several books about the application of deductive logic to chess, three collections of philosophical essays and aphorisms, and several advanced books on mathematical logic and set theory. He is particularly interested in self-reference and has worked on extending some of Gödel’s results that show that it is impossible to write a computer program that can solve all mathematical problems. He is also particularly interested in explaining ideas from mathematical logic to the public.

Smullyan is a talented musician and often plays piano with his wife, who is a concert-level pianist. Making telescopes is one of his hobbies. He is also interested in optics and stereo photography. He states “I’ve never had a conflict between teaching and research as some people do because when I’m teaching, I’m doing research.” Smullyan is the subject of a documentary short film entitled *This Film Needs No Title*.

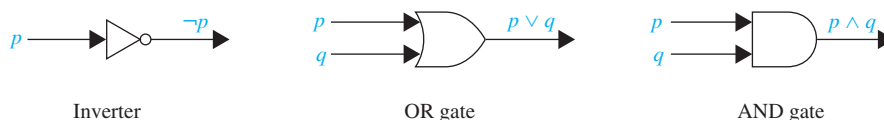


FIGURE 1 Basic logic gates.

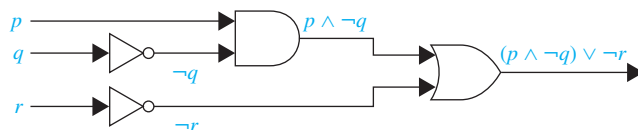


FIGURE 2 A combinational circuit.

Complicated digital circuits can be constructed from three basic circuits, called **gates**, shown in Figure 1. The **inverter**, or **NOT gate**, takes an input bit p , and produces as output $\neg p$. The **OR gate** takes two input signals p and q , each a bit, and produces as output the signal $p \vee q$. Finally, the **AND gate** takes two input signals p and q , each a bit, and produces as output the signal $p \wedge q$. We use combinations of these three basic gates to build more complicated circuits, such as that shown in Figure 2.

Given a circuit built from the basic logic gates and the inputs to the circuit, we determine the output by tracing through the circuit, as Example 9 shows.

EXAMPLE 9 Determine the output for the combinational circuit in Figure 2.

Solution: In Figure 2 we display the output of each logic gate in the circuit. We see that the AND gate takes input of p and $\neg q$, the output of the inverter with input q , and produces $p \wedge \neg q$. Next, we note that the OR gate takes input $p \wedge \neg q$ and $\neg r$, the output of the inverter with input r , and produces the final output $(p \wedge \neg q) \vee \neg r$. ◀

Suppose that we have a formula for the output of a digital circuit in terms of negations, disjunctions, and conjunctions. Then, we can systematically build a digital circuit with the desired output, as illustrated in Example 10.

EXAMPLE 10 Build a digital circuit that produces the output $(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))$ when given input bits p , q , and r .

Solution: To construct the desired circuit, we build separate circuits for $p \vee \neg r$ and for $\neg p \vee (q \vee \neg r)$ and combine them using an AND gate. To construct a circuit for $p \vee \neg r$, we use an inverter to produce $\neg r$ from the input r . Then, we use an OR gate to combine p and $\neg r$. To build a circuit for $\neg p \vee (q \vee \neg r)$, we first use an inverter to obtain $\neg p$. Then we use an OR gate with inputs q and $\neg r$ to obtain $q \vee \neg r$. Finally, we use another inverter and an OR gate to get $\neg p \vee (q \vee \neg r)$ from the inputs p and $q \vee \neg r$.

To complete the construction, we employ a final AND gate, with inputs $p \vee \neg r$ and $\neg p \vee (q \vee \neg r)$. The resulting circuit is displayed in Figure 3. ◀

We will study logic circuits in great detail in Chapter 12 in the context of Boolean algebra, and with different notation.

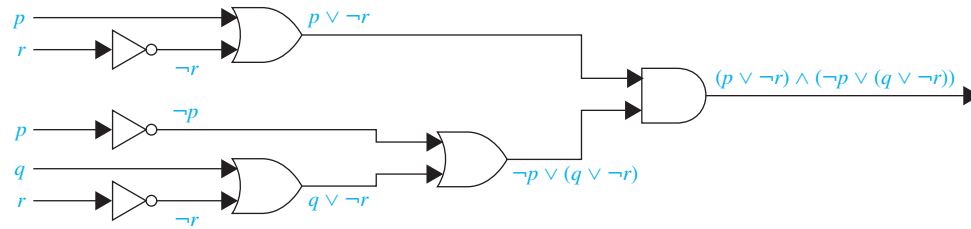


FIGURE 3 The circuit for $(p \vee \neg r) \wedge (\neg p \vee (q \vee \neg r))$.

Exercises

In Exercises 1–6, translate the given statement into propositional logic using the propositions provided.

1. You cannot edit a protected Wikipedia entry unless you are an administrator. Express your answer in terms of e : “You can edit a protected Wikipedia entry” and a : “You are an administrator.”
2. You can see the movie only if you are over 18 years old or you have the permission of a parent. Express your answer in terms of m : “You can see the movie,” e : “You are over 18 years old,” and p : “You have the permission of a parent.”
3. You can graduate only if you have completed the requirements of your major and you do not owe money to the university and you do not have an overdue library book. Express your answer in terms of g : “You can graduate,” m : “You owe money to the university,” r : “You have completed the requirements of your major,” and b : “You have an overdue library book.”
4. To use the wireless network in the airport you must pay the daily fee unless you are a subscriber to the service. Express your answer in terms of w : “You can use the wireless network in the airport,” d : “You pay the daily fee,” and s : “You are a subscriber to the service.”
5. You are eligible to be President of the U.S.A. only if you are at least 35 years old, were born in the U.S.A., or at the time of your birth both of your parents were citizens, and you have lived at least 14 years in the country. Express your answer in terms of e : “You are eligible to be President of the U.S.A.,” a : “You are at least 35 years old,” b : “You were born in the U.S.A.,” p : “At the time of your birth, both of your parents were citizens,” and r : “You have lived at least 14 years in the U.S.A.”
6. You can upgrade your operating system only if you have a 32-bit processor running at 1 GHz or faster, at least 1 GB RAM, and 16 GB free hard disk space, or a 64-bit processor running at 2 GHz or faster, at least 2 GB RAM, and at least 32 GB free hard disk space. Express your answer in terms of u : “You can upgrade your operating system,” b_{32} : “You have a 32-bit processor,” b_{64} :

“You have a 64-bit processor,” g_1 : “Your processor runs at 1 GHz or faster,” g_2 : “Your processor runs at 2 GHz or faster,” r_1 : “Your processor has at least 1 GB RAM,” r_2 : “Your processor has at least 2 GB RAM,” h_{16} : “You have at least 16 GB free hard disk space,” and h_{32} : “You have at least 32 GB free hard disk space.”

7. Express these system specifications using the propositions p “The message is scanned for viruses” and q “The message was sent from an unknown system” together with logical connectives (including negations).
 - a) “The message is scanned for viruses whenever the message was sent from an unknown system.”
 - b) “The message was sent from an unknown system but it was not scanned for viruses.”
 - c) “It is necessary to scan the message for viruses whenever it was sent from an unknown system.”
 - d) “When a message is not sent from an unknown system it is not scanned for viruses.”
8. Express these system specifications using the propositions p “The user enters a valid password,” q “Access is granted,” and r “The user has paid the subscription fee” and logical connectives (including negations).
 - a) “The user has paid the subscription fee, but does not enter a valid password.”
 - b) “Access is granted whenever the user has paid the subscription fee and enters a valid password.”
 - c) “Access is denied if the user has not paid the subscription fee.”
 - d) “If the user has not entered a valid password but has paid the subscription fee, then access is granted.”
9. Are these system specifications consistent? “The system is in multiuser state if and only if it is operating normally. If the system is operating normally, the kernel is functioning. The kernel is not functioning or the system is in interrupt mode. If the system is not in multiuser state, then it is in interrupt mode. The system is not in interrupt mode.”