DISCRETE STRUCTURES

IT-205

Course Description:

This course discusses concepts of basic logic, sets, combinational theory. Topics include Boolean algebra; set theory; symbolic logic; predicate logic, objective functions, equivalence relations, graphs, basic counting, proof strategies, set partitions, combinations, trees, summations, and recurrences.

?So, what's *this* class about

What are "discrete structures" anyway?

- "***Discrete***" ($\neq$ "discreet"!) - Composed of distinct, separable parts. (Opposite of *continuous*.)
  discrete:*continuous* :: *digital*:*analog*
- "***Structures***" - Objects built up from simpler objects according to some definite pattern.
- "***Discrete Mathematics***" - The study of discrete, mathematical objects and structures.

Discrete Structures We'll Study

- Propositions
- Predicates
- Proofs
- Sets
- Functions
- Orders of Growth
- Algorithms
- Integers
- Summations
- Sequences
- Strings
- Permutations
- Combinations
- Relations
- Graphs
- Trees
- Logic Circuits
- Automata

Some Notations We'll Learn

$\neg p$	$p \wedge q$	$p \oplus q$	$p \rightarrow q$	$p \Leftrightarrow q$	$\forall x\, P(x)$
$\exists x\, P(x)$	$\{a_1, \cdots, a_n\}$	$\mathbf{Z}, \mathbf{N}, \mathbf{R}$	$\therefore$	$\{x \mid P(x)\}$	$x \notin S$
$\varnothing$	$S \subseteq T$	$\lvert S \rvert$	$A \cup B$	$\overline{A}$	$\bigcap_{i=1}^{n} A_i$
$f : A \rightarrow B$	$f^{-1}(x)$	$f \circ g$	$\lfloor x \rfloor$	$\sum_{\alpha \in S} a_\alpha$	$\prod_{i=1}^{n} a_i$
O, Ω, Θ	$\min, \max$	$a \mid b$	$\gcd, \operatorname{lcm}$	$\bmod$	$a \equiv b \pmod{m}$
$(a_k \cdots a_0)_b$	$[a_{ij}]$	$\mathbf{A}^{\mathrm{T}}$	$\mathbf{A} \odot \mathbf{B}$	$\mathbf{A}^{[n]}$	$\binom{n}{r}$
$C(n; n_1, \cdots, n_m)$	$p(E \mid F)$	R^*	Δ	$[a]_R$	$\deg^+(v)$

?Why Study Discrete Math

- The basis of all of digital information processing is: *Discrete manipulations of discrete structures represented in memory.*
- It's the basic language and conceptual foundation for all of computer science.
- Discrete math concepts are also widely used throughout math, science, engineering, economics, biology, *etc.*, …
- A generally useful tool for rational thought!

Uses for Discrete Math in Computer Science

- Advanced algorithms & data structures
- Programming language compilers & interpreters.
- Computer networks
- Operating systems
- Computer architecture
- Database management systems
- Cryptography
- Error correction codes
- Graphics & animation algorithms, game engines, *etc.*…
- *I.e.*, the whole field!

Module #1:

Foundations of Logic

Module #1: Foundations of Logic (§§1.1-1.3, ~3 lectures)

Mathematical Logic is a tool for working with complicated *compound* statements. It includes:

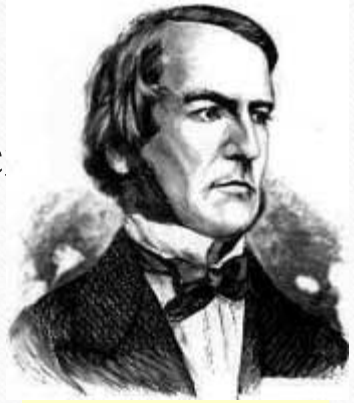
- A language for expressing them.
- A concise notation for writing them.
- A methodology for objectively reasoning about their truth or falsity.
- It is the foundation for expressing formal proofs in all branches of mathematics.

Foundations of Logic: Overview

- Propositional logic (§1.1-1.2):
  - Basic definitions. (§1.1)
  - Equivalence rules & derivations. (§1.2)
- Predicate logic (§1.3-1.4)
  - Predicates.
  - Quantified predicate expressions.
  - Equivalences & derivations.

Propositional Logic (§1.1)

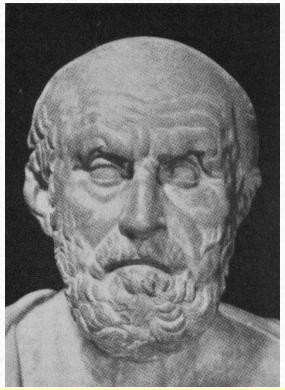
Propositional Logic is the logic of compound state[ments]
built from simpler statements
using so-called *Boolean connectives*.



George Boole
(1815-1864)

Some applications in computer science:

- Design of digital electronic circuits.
- Expressing conditions in programs.
- Queries to databases & search engines.



Chrysippus of Soli
(ca. 281 B.C. – 205 B.C.)

Definition of a *Proposition*

A *proposition* (*p*, *q*, *r*, …) is simply a *statement* (*i.e.*, a declarative sentence) *with a definite meaning*, having a *truth value* that's either *true* (T) or *false* (F) (**never** both, neither, or somewhere in between).

(However, you might not *know* the actual truth value, and it might be situation-dependent.)

[Later we will study *probability theory*, in which we assign *degrees of certainty* to propositions. But for now: think True/False only!]

Examples of Propositions

- "It is raining." (In a given situation.)
- "Beijing is the capital of China." • "1 + 2 = 3"

<u>But, the following are **NOT** propositions:</u>

- "Who's there?" (interrogative, question)
- "La la la la la." (meaningless interjection)
- "Just do it!" (imperative, command)
- "Yeah, I sorta dunno, whatever..." (vague)
- "1 + 2" (expression with a non-true/false value)

Operators / Connectives

An *operator* or *connective* combines one or more *operand* expressions into a larger expression. (*E.g.*, "+" in numeric exprs.)

Unary operators take 1 operand (*e.g.,* -3); *binary* operators take 2 operands (*eg* 3×4).

Propositional or *Boolean* operators operate on propositions or truth values instead of on numbers.

Some Popular Boolean Operators

Formal Name	**Nickname**	**Arity**	**Symbol**
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

The Negation Operator

The unary *negation operator* "¬" (*NOT*) transforms a prop. into its logical *negation*.

E.g. If p = "I have brown hair."

then $\neg p$ = "I do **not** have brown hair."

Truth table for NOT:

T :≡ True; F :≡ False
":≡" means "is defined as"

p	$\neg p$
T	F
F	T
Operand column	Result column

The Conjunction Operator

The binary *conjunction operator* "$\wedge$" (*AND*) combines two propositions to form their logical *conjunction*.

$\wedge$N
D

E.g. If p="I will have salad for lunch." and q="I will have steak for dinner.", then $p \wedge q$="I will have salad for lunch **and** I will have steak for dinner."

Remember: "$\wedge$" points up like an "A", and it means "$\wedge$ND"

Conjunction Truth Table

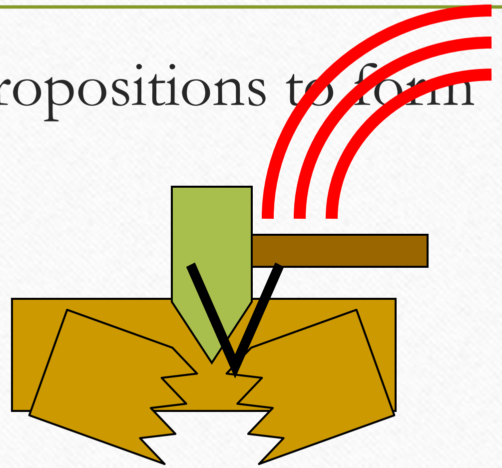
Operand columns

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

- Note that a conjunction $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ of n propositions will have 2^n rows in its truth table.
- Also: ¬ and ∧ operations together are suffi-cient to express *any* Boolean truth table!

The Disjunction Operator

The binary *disjunction operator* "$\vee$" (*OR*) combines two propositions to form their logical *disjunction*.



p="My car has a bad engine."

q="My car has a bad carburetor."

$p \vee q$="Either my car has a bad engine, **or** my car has a bad carburetor."

After the downward-pointing "axe" of "$\vee$" splits the wood, you can take 1 piece OR the other, or both.

Meaning is like "and/or" in English.

Disjunction Truth Table

- Note that $p \lor q$ means that p is true, or q is true, **or both** are true!
- So, this operation is also called *inclusive or,* because it **includes** the possibility that both p and q are true.
- "¬" and "$\lor$" together are also universal.

p	q	$p \lor q$
F	F	F
F	T	**T**
T	F	**T**
T	T	T

Note difference from AND

Nested Propositional Expressions

- Use parentheses to *group sub-expressions*:
  "I just saw my old friend, and either he's grown or I've shrunk." $= f \wedge (g \vee s)$
  - $(f \wedge g) \vee s$ would mean something different
  - $f \wedge g \vee s$ would be ambiguous
- By convention, "$\neg$" takes *precedence* over both "$\wedge$" and "$\vee$".
  - $\neg s \wedge f$ means $(\neg s) \wedge f$, **not** $\neg (s \wedge f)$

A Simple Exercise

Let p=“It rained last night”,
q=“The sprinklers came on last night,”
r=“The lawn was wet this morning.”

Translate each of the following into English:

$\neg p$ = “It didn’t rain last night.”

$r \land \neg p$ = “The lawn was wet this morning, and it didn’t rain last night.”

$\neg r \lor p \lor q$ = “Either the lawn wasn’t wet this morning, or it rained last night, or the sprinklers came on last night.”

The *Exclusive Or* Operator

The binary *exclusive-or operator* "$\oplus$" (*XOR*) combines two propositions to form their logical "exclusive or" (exjunction?).

p = "I will earn an A in this course,"

q = "I will drop this course,"

$p \oplus q$ = "I will either earn an A for this course, or I will drop it (but not both!)"

Exclusive-Or Truth Table

- Note that $p \oplus q$ means that p is true, or q is true, but **not both**!
- This operation is called *exclusive or,* because it **excludes** the possibility that both p and q are true.
- "¬" and "⊕" together are **not** universal.

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	**F**

Note difference from OR.

Natural Language is Ambiguous

Note that English "or" can be ambiguous regarding the "both" case!

"Pat is a singer or Pat is a writer." - $\vee$

"Pat is a man or Pat is a woman." - $\oplus$

Need context to disambiguate the meaning!

For this class, assume "or" means inclusive.

p	q	p "or" q
F	F	F
F	T	T
T	F	T
T	T	?

The *Implication* Operator

antecedent consequent

The *implication* $p \rightarrow q$ states that p implies q.

I.e., If p is true, then q is true; but if p is not true, then q could be either true or false.

E.g., let p = "You study hard."
q = "You will get a good grade."

$p \rightarrow q$ = "If you study hard, then you will get a good grade." (else, it could go either way)

Implication Truth Table

- $p \rightarrow q$ is **false** only when p is true but q is **not** true.
- $p \rightarrow q$ does **not** say that p causes q!
- $p \rightarrow q$ does **not** require that p or q **are ever true**!
- *E.g.* "(1=0) → pigs can fly" is TRUE!

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	**F**
T	T	T

The only False case!

Examples of Implications

- "If this lecture ends, then the sun will rise tomorrow." *True* or *False*?
- "If Tuesday is a day of the week, then I am a penguin." *True* or *False*?
- "If 1+1=6, then Bush is president." *True* or *False*?
- "If the moon is made of green cheese, then I am richer than Bill Gates." *True* or *False*?

Why does this seem wrong?

- Consider a sentence like,
  - “If I wear a red shirt tomorrow, then the U.S. will attack Iraq the same day.”
- In logic, we consider the sentence **True** so long as either I don’t wear a red shirt, or the US attacks.
- But in normal English conversation, if I were to make this claim, you would think I was lying.
  - Why this discrepancy between logic & language?

Resolving the Discrepancy

- In English, a sentence "if *p* then *q*" usually really *implicitly* means something like,
  - "<u>In all possible situations</u>, if *p* then *q*."
    - That is, "For *p* to be true and *q* false is *impossible*."
    - Or, "I *guarantee* that no matter what, if *p*, then *q*."
- This can be expressed in *predicate logic* as:
  - "For all situations *s*, if *p* is true in situation *s*, then *q* is also true in situation *s*"
  - Formally, we could write: $\forall s, P(s) \rightarrow Q(s)$
- This sentence is logically ***False*** in our example, because for me to wear a red shirt and the U.S. *not* to attack Iraq is a *possible* (even if not actual) situation.
  - Natural language and logic then agree with each other.

English Phrases Meaning $p \rightarrow q$

- "*p* implies *q*"
- "if *p*, then *q*"
- "if *p*, *q*"
- "when *p*, *q*"
- "whenever *p*, *q*"
- "*q* if *p*"
- "*q* when *p*"
- "*q* whenever *p*"
- "*p* only if *q*"
- "*p* is sufficient for *q*"
- "*q* is necessary for *p*"
- "*q* follows from *p*"
- "*q* is implied by *p*"

We will see some equivalent logic expressions later.

Converse, Inverse, Contrapositive

Some terminology, for an implication $p \rightarrow q$:

- Its *converse* is: $q \rightarrow p$.
- Its *inverse* is: $\neg p \rightarrow \neg q$.
- Its *contrapositive*: $\neg q \rightarrow \neg p$.
- One of these three has the *same meaning* (same truth table) as $p \rightarrow q$. Can you figure out which?

Contrapositive

How do we know for sure?

Proving the equivalence of $p \rightarrow q$ and its contrapositive using truth tables:

p	q	$\neg q$	$\neg p$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
F	F	T	T	T	T
F	T	F	T	T	T
T	F	T	F	F	F
T	T	F	F	T	T

The *biconditional* operator

The *biconditional* $p \leftrightarrow q$ states that p is true *if and only if (IFF)* q is true.

p = “Obama wins the 2008 election.”

q = “Obama will be president for all of 2009.”

$p \leftrightarrow q$ = “If, and only if, Obama wins the 2008 election, Obama will be president for all of 2009.”

Biconditional Truth Table

- $p \leftrightarrow q$ means that p and q have the **same** truth value.
- Note this truth table is the exact **opposite** of $\oplus$'s!
  - $p \leftrightarrow q$ means $\neg(p \oplus q)$
- $p \leftrightarrow q$ does **not** imply p and q are true, or cause each other.

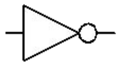
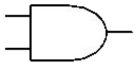
p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Boolean Operations Summary

- We have seen 1 unary operator (out of the 4 possible) and 5 binary operators (out of the 16 possible). Their truth tables are below.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
F	F	T	F	F	F	T	T
F	T	T	F	T	T	T	F
T	F	F	F	T	T	F	F
T	T	F	T	T	F	T	T

Some Alternative Notations

Name:	not	and	or	xor	implies	iff
Propositional logic:	$\neg$	$\wedge$	$\vee$	$\oplus$	$\rightarrow$	$\leftrightarrow$
Boolean algebra:	$\overline{p}$	pq	$+$	$\oplus$		
C/C++/Java (wordwise):	!	&&	||	!=		==
C/C++/Java (bitwise):	~	&	|	^		
Logic gates:						

Bits and Bit Operations

John Tukey
(1915-2000)

- A *bit* is a binary (base 2) digit: 0 or 1.
- Bits may be used to represent truth values.
- By convention:
  0 represents "false"; 1 represents "true".
- *Boolean algebra* is like ordinary algebra except that variables stand for bits, + means "or", and multiplication means "and".
  - See chapter 10 for more details.

Bit Strings

- A *Bit string* of *length n* is an ordered series or sequence of $n \geq 0$ bits.
    - More on sequences in §3.2.
- By convention, bit strings are written left to right: *e.g.* the first bit of "1001101010" is 1.
- When a bit string represents a base-2 number, by convention the first bit is the *most significant* bit. *Ex.* $1101_2 = 8+4+1 = 13$.

Counting in Bin

- Did you know that you can count to 1,023 just using two hands?
  - How? Count in binary!
    - Each finger (up/down) represents 1 bit.
- To increment: Flip the rightmost (low-order) bit.
  - If it changes 1→0, then also flip the next bit to the left,
    - If that bit changes 1→0, then flip the next one, *etc.*
- 0000000000, 0000000001, 0000000010, … …, 1111111101, 1111111110, 1111111111

Bitwise Operations

- Boolean operations can be extended to operate on bit strings as well as single bits.
- E.g.:
  01 1011 0110
  11 0001 1101
  11 1011 1111 Bit-wise OR
  01 0001 0100 Bit-wise AND
  10 1010 1011 Bit-wise XOR

End of §1.1

You have learned about:

- Propositions: What they are.
- Propositional logic operators'
  - Symbolic notations.
  - English equivalents.
  - Logical meaning.
  - Truth tables.
- Atomic vs. compound propositions.
- Alternative notations.
- Bits and bit-strings.
- Next section: §1.2
  - Propositional equivalences.
  - How to prove them.

Propositional Equivalence (§1.2)

Two *syntactically* (*i.e.*, textually) different compound propositions may be the *semantically* identical (*i.e.*, have the same meaning). We call them *equivalent*. Learn:

- Various *equivalence rules* or *laws*.
- How to *prove* equivalences using *symbolic derivations*.

Tautologies and Contradictions

A *tautology* is a compound proposition that is **true** *no matter what* the truth values of its atomic propositions are!

Ex. $p \vee \neg p$ [What is its truth table?]

A *contradiction* is a compound proposition that is **false** no matter what! *Ex.* $p \wedge \neg p$ [Truth table?]

Other compound props. are *contingencies*.

Logical Equivalence

Compound proposition p is *logically equivalent* to compound proposition q, written $p \Leftrightarrow q$, **IFF** the compound proposition $p \leftrightarrow q$ is a tautology.

Compound propositions p and q are logically equivalent to each other **IFF** p and q contain the same truth values as each other in <u>all</u> rows of their truth tables.

Proving Equivalence via Truth Tables

Ex. Prove that $p \vee q \Leftrightarrow \neg(\neg p \wedge \neg q)$.

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
F	F	F	T	T	T	F
F	T	T	T	F	F	T
T	F	T	F	T	F	T
T	T	T	F	F	F	T

Equivalence Laws

- These are similar to the arithmetic identities you may have learned in algebra, but for propositional equivalences instead.
- They provide a pattern or template that can be used to match all or part of a much more complicated proposition and to find an equivalence for it.

Equivalence Laws - Examples

- *Identity*: $p \wedge \mathbf{T} \Leftrightarrow p \quad p \vee \mathbf{F} \Leftrightarrow p$
- *Domination*: $p \vee \mathbf{T} \Leftrightarrow \mathbf{T} \quad p \wedge \mathbf{F} \Leftrightarrow \mathbf{F}$
- *Idempotent*: $p \vee p \Leftrightarrow p \quad p \wedge p \Leftrightarrow p$
- *Double negation:* $\neg\neg p \Leftrightarrow p$
- *Commutative:* $p \vee q \Leftrightarrow q \vee p \quad p \wedge q \Leftrightarrow q \wedge p$
- *Associative:* $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
  $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

More Equivalence Laws

- *Distributive*: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
  $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
- *De Morgan's*:
  $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
  $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
- *Trivial tautology/contradiction*:
  $p \vee \neg p \Leftrightarrow \mathbf{T}$ $\quad$ $p \wedge \neg p \Leftrightarrow \mathbf{F}$

Augustus De Morgan (1806-1871)

Defining Operators via Equivalences

Using equivalences, we can *define* operators in terms of other operators.

- Exclusive or: $p \oplus q \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$
  $p \oplus q \Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$
- Implies: $p \rightarrow q \Leftrightarrow \neg p \vee q$
- Biconditional: $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$
  $p \leftrightarrow q \Leftrightarrow \neg(p \oplus q)$

An Example Problem

- Check using a symbolic derivation whether $(p \land \neg q) \rightarrow (p \oplus r) \Leftrightarrow \neg p \lor q \lor \neg r$.

$(p \land \neg q) \underline{\rightarrow} (p \oplus r) \Leftrightarrow$

[Expand definition of $\rightarrow$] $\neg(p \land \neg q) \lor (p \underline{\oplus} r)$

[Defn. of $\oplus$] $\Leftrightarrow \neg(p \land \neg q) \lor ((p \lor r) \land \neg(p \land r))$

[DeMorgan's Law]

$\Leftrightarrow (\neg p \lor q) \lor ((p \lor r) \land \neg(p \land r))$

$\Leftrightarrow$ [associative law] *cont.*

Example Continued...

$(\neg p \lor q) \lor ((p \lor r) \land \neg(p \land r)) \Leftrightarrow$ [$\lor$ commutes]

$\Leftrightarrow \underline{(q \lor \neg p)} \lor ((p \lor r) \land \neg(p \land r))$ [$\lor$ associative]

$\Leftrightarrow q \lor \underline{(\neg p \lor} ((p \lor r) \land \neg(p \land r)))$ [distrib. $\lor$ over $\land$]

$\Leftrightarrow q \lor (((\underline{\neg p \lor} (p \lor r)) \land (\underline{\neg p \lor} \neg(p \land r)))$

[assoc.] $\Leftrightarrow q \lor (((\underline{\neg p \lor p) \lor r}) \land (\neg p \lor \neg(p \land r)))$

[trivail taut.] $\Leftrightarrow q \lor ((\underline{\mathbf{T}} \lor r) \land (\neg p \lor \neg(p \land r)))$

[domination] $\Leftrightarrow q \lor (\underline{\mathbf{T}} \land (\neg p \lor \neg(p \land r)))$

[identity] $\Leftrightarrow q \lor (\neg p \lor \neg(p \land r)) \Leftrightarrow$ *cont.*

End of Long Example

$q \vee (\neg p \vee \neg(p \wedge r))$

[DeMorgan's] $\Leftrightarrow q \vee (\neg p \vee (\neg p \vee \neg r))$

[Assoc.] $\Leftrightarrow q \vee ((\neg p \vee \neg p) \vee \neg r)$

[Idempotent] $\Leftrightarrow q \vee (\neg p \vee \neg r)$

[Assoc.] $\Leftrightarrow (q \vee \neg p) \vee \neg r$

[Commut.] $\Leftrightarrow \neg p \vee q \vee \neg r$

Q.E.D. (quod erat demonstrandum)

(Which was to be shown.)

Review: Propositional Logic (§§1.1-1.2)

- Atomic propositions: p, q, r, …
- Boolean operators: $\neg \ \wedge \ \vee \ \oplus \rightarrow \leftrightarrow$
- Compound propositions: $s :\equiv (p \wedge \neg q) \vee r$
- Equivalences: $p \wedge \neg q \Leftrightarrow \neg(p \rightarrow q)$
- Proving equivalences using:
  - Truth tables.
  - Symbolic derivations. $p \Leftrightarrow q \Leftrightarrow r$ …

Predicate Logic (§1.3)

- *Predicate logic* is an extension of propositional logic that permits concisely reasoning about whole *classes* of entities.
- Propositional logic (recall) treats simple *propositions* (sentences) as atomic entities.
- In contrast, *predicate* logic distinguishes the *subject* of a sentence from its *predicate*.
  - Remember these English grammar terms?

Applications of Predicate Logic

It is *the* formal notation for writing perfectly clear, concise, and unambiguous mathematical *definitions*, *axioms*, and *theorems* (more on these in chapter 3) for *any* branch of mathematics.

Predicate logic with function symbols, the "=" operator, and a few proof-building rules is sufficient for defining *any* conceivable mathematical system, and for proving anything that can be proved within that system!

Other Applications

Kurt Gödel
1906-1978

- Predicate logic is the foundation of the field of *mathematical logic*, which culminated in *Gödel's incompleteness theorem*, which revealed the ultimate limits of mathematical thought:
  - Given any finitely describable, consistent proof procedure, there will still be *some* true statements that can *never be proven* by that procedure.
- *I.e.*, we can't discover *all* mathematical truths, unless we sometimes resort to making *guesses.*

Practical Applications

- Basis for clearly expressed formal specifications for any complex system.
- Basis for *automatic theorem provers* and many other Artificial Intelligence systems.
- Supported by some of the more sophisticated *database query engines* and *container class libraries*
(these are types of programming tools).

Subjects and Predicates

- In the sentence "The dog is sleeping":
  - The phrase "the dog" denotes the *subject* - the *object* or *entity* that the sentence is about.
  - The phrase "is sleeping" denotes the *predicate*- a property that is true **of** the subject.
- In predicate logic, a *predicate* is modeled as a *function* $P(\cdot)$ from objects to propositions.
  - $P(x)$ = "x is sleeping" (where x is any object).

More About Predicates

- Convention: Lowercase variables x, y, z... denote objects/entities; uppercase variables P, Q, R… denote propositional functions (predicates).
- Keep in mind that the *result of applying* a predicate P to an object x is the *proposition* $P(x)$. But the predicate P **itself** (*e.g.* P=“is sleeping”) is **not** a proposition (not a complete sentence).
  - *E.g.* if $P(x)$ = “x is a prime number”, $P(3)$ is the *proposition* “3 is a prime number.”

Propositional Functions

- Predicate logic *generalizes* the grammatical notion of a predicate to also include propositional functions of **any** number of arguments, each of which may take **any** grammatical role that a noun can take.
  - *E.g.* let $P(x,y,z)$ = "x gave y the grade z", then if x="Mike", y="Mary", z="A", then $P(x,y,z)$ = "Mike gave Mary the grade A."

Universes of Discourse (U.D.s)

- The power of distinguishing objects from predicates is that it lets you state things about *many* objects at once.
- E.g., let $P(x)$="$x+1>x$". We can then say, "For *any* number x, $P(x)$ is true" instead of $(\mathbf{0}+1>\mathbf{0}) \land (\mathbf{1}+1>\mathbf{1}) \land (\mathbf{2}+1>\mathbf{2}) \land \ldots$
- The collection of values that a variable x can take is called x's *universe of discourse*.

Quantifier Expressions

- *Quantifiers* provide a notation that allows us to *quantify* (count) *how many* objects in the univ. of disc. satisfy a given predicate.
- "$\forall$" is the FOR$\forall$LL or *universal* quantifier.
  $\forall x\, P(x)$ means *for all* x in the u.d., *P* holds.
- "$\exists$" is the $\exists$XISTS or *existential* quantifier.
  $\exists x\, P(x)$ means there *exists* an x in the u.d. (that is, 1 or more) such that $P(x)$ is true.

The Universal Quantifier $\forall$

- Example:
  Let the u.d. of x be parking spaces at UF.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *universal quantification of* $P(x)$, $\forall x\, P(x)$, is the *proposition:*
  - "All parking spaces at UF are full."
  - *i.e.*, "Every parking space at UF is full."
  - *i.e.*, "For each parking space at UF, that space is full."

The Existential Quantifier $\exists$

- Example:
  Let the u.d. of x be parking spaces at UF.
  Let $P(x)$ be the *predicate* "x is full."
  Then the *existential quantification of* $P(x)$, $\exists x\, P(x)$, is the *proposition*:
  - "Some parking space at UF is full."
  - "There is a parking space at UF that is full."
  - "At least one parking space at UF is full."

Free and Bound Variables

- An expression like $P(x)$ is said to have a *free variable* x (meaning, x is undefined).
- A quantifier (either $\forall$ or $\exists$) *operates* on an expression having one or more free variables, and *binds* one or more of those variables, to produce an expression having one or more *bound variables*.

Example of Binding

- $P(x,y)$ has 2 free variables, x and y.

  y

  x
- $\forall x\, P(x,y)$ has 1 free variable, and one bound variable. [Which is which?]
- "$P(x)$, where x=3" is another way to bind x.
- An expression with <u>zero</u> free variables is a bona-fide (actual) proposition.
- An expression with <u>one or more</u> free variables is still only a predicate: $\forall x\, P(x,y)$

Nesting of Quantifiers

Example: Let the u.d. of x & y be people.

Let $L(x,y)$=“x likes y” (a predicate w. 2 f.v.’s)

Then $\exists y\, L(x,y)$ = “There is someone whom x likes.” (A predicate w. 1 free variable, x)

Then $\forall x\, (\exists y\, L(x,y))$ =
“Everyone has someone whom they like.”
(A ________ **Proposition** variables **0**

Review: Propositional Logic (§§1.1-1.2)

- Atomic propositions: p, q, r, …
- Boolean operators: $\neg \ \wedge \ \vee \ \oplus \rightarrow \leftrightarrow$
- Compound propositions: $s \equiv (p \wedge \neg q) \vee r$
- Equivalences: $p \wedge \neg q \Leftrightarrow \neg(p \rightarrow q)$
- Proving equivalences using:
  - Truth tables.
  - Symbolic derivations. $p \Leftrightarrow q \Leftrightarrow r$ …

Review: Predicate Logic (§1.3)

- Objects x, y, z, …
- Predicates P, Q, R, … are functions mapping objects x to propositions $P(x)$.
- Multi-argument predicates $P(x, y)$.
- Quantifiers: $[\forall x\, P(x)]$:≡ "For all x's, $P(x)$."
  $[\exists x\, P(x)]$:≡ "There is an x such that $P(x)$."
- Universes of discourse, bound & free vars.

Quantifier Exercise

If $R(x,y)$="x relies upon y," express the following in unambiguous English:

$\forall x(\exists y\, R(x,y))=$ Everyone has *someone* to rely on.

$\exists y(\forall x\, R(x,y))=$ There's a poor overburdened soul whom *everyone* relies upon (including himself)!

$\exists x(\forall y\, R(x,y))=$ There's some needy person who relies upon *everybody* (including himself).

$\forall y(\exists x\, R(x,y))=$ Everyone has *someone* who relies upon them.

$\forall x(\forall y\, R(x,y))=$ *Everyone* relies upon *everybody*, (including themselves)!

Natural language is ambiguous!

- "Everybody likes somebody."
  - For everybody, there is somebody they like, [Probably more likely.]
    - $\forall x \, \exists y \, Likes(x,y)$
  - or, there is somebody (a popular person) whom everyone likes?
    - $\exists y \, \forall x \, Likes(x,y)$
- "Somebody likes everybody."
  - Same problem: Depends on context, emphasis.

Game Theoretic Semantics

- Thinking in terms of a competitive game can help you tell whether a proposition with nested quantifiers is true.
- The game has two players, both with the same knowledge:
  - Verifier: Wants to demonstrate that the proposition is true.
  - Falsifier: Wants to demonstrate that the proposition is false.
- The Rules of the Game "Verify or Falsify":
  - Read the quantifiers from left to right, picking values of variables.
  - When you see "$\forall$", the falsifier gets to select the value.
  - When you see "$\exists$", the verifier gets to select the value.
- If the verifier can always win, then the proposition is true.
- If the falsifier can always win, then it is false.

Let's Play, "Verify or Falsify!"

Let $B(x,y)$:≡ "x's birthday is followed within 7 days by y's birthday."

Suppose I claim that among you:

$\underline{\forall x}\ \exists y\ B(x,y)$

Your turn, as falsifier:
You pick any x → *(so-and-so)*

$\underline{\exists y}\ B(\text{so-and-so},y)$

My turn, as verifier:
I pick any y → *(such-and-such)*

$B(\text{so-and-so,such-and-such})$

- Let's play it in class.
- Who wins this game?
- What if I switched the quantifiers, and I claimed that $\exists y\ \forall x\ B(x,y)$? Who wins in that case?

Still More Conventions

- Sometimes the universe of discourse is restricted within the quantification, *e.g.*,
  - $\forall x>0\ P(x)$ is shorthand for
    "For all x that are greater than zero, $P(x)$."
    $= \forall x\ (x>0 \rightarrow P(x))$
  - $\exists x>0\ P(x)$ is shorthand for
    "There is an x greater than zero such that $P(x)$."
    $= \exists x\ (x>0 \land P(x))$

More to Know About Binding

- $\forall x \; \exists x \; P(x)$ - x is not a free variable in $\exists x \; P(x)$, therefore the $\forall x$ binding isn't used.
- $(\forall x \; P(x)) \wedge Q(x)$ - The variable x is outside of the *scope* of the $\forall x$ quantifier, and is therefore free. Not a proposition!
- $(\forall x \; P(x)) \wedge (\exists x \; Q(x))$ – This is legal, because there are 2 different x's!

Quantifier Equivalence Laws

- Definitions of quantifiers: If u.d.=a,b,c,…
  $\forall x\, P(x) \Leftrightarrow P(a) \wedge P(b) \wedge P(c) \wedge \ldots$
  $\exists x\, P(x) \Leftrightarrow P(a) \vee P(b) \vee P(c) \vee \ldots$
- From those, we can prove the laws:
  $\forall x\, P(x) \Leftrightarrow \neg \exists x\, \neg P(x)$
  $\exists x\, P(x) \Leftrightarrow \neg \forall x\, \neg P(x)$
- Which *propositional* equivalence laws can be used to prove this?

DeMorgan's

More Equivalence Laws

- $\forall x\ \forall y\ P(x,y) \Leftrightarrow \forall y\ \forall x\ P(x,y)$
  $\exists x\ \exists y\ P(x,y) \Leftrightarrow \exists y\ \exists x\ P(x,y)$
- $\forall x\ (P(x) \wedge Q(x)) \Leftrightarrow (\forall x\ P(x)) \wedge (\forall x\ Q(x))$
  $\exists x\ (P(x) \vee Q(x)) \Leftrightarrow (\exists x\ P(x)) \vee (\exists x\ Q(x))$
- Exercise:
  See if you can prove these yourself.
  - What propositional equivalences did you use?

Review: Predicate Logic (§1.3)

- Objects x, y, z, …
- Predicates P, Q, R, … are functions mapping objects x to propositions $P(x)$.
- Multi-argument predicates $P(x, y)$.
- Quantifiers: $(\forall x\, P(x))$ =“For all x’s, $P(x)$.”
  $(\exists x\, P(x))$=“There is an x such that $P(x)$.”

More Notational Conventions

- Quantifiers bind as loosely as needed: parenthesize $\forall x\ P(x) \wedge Q(x)$
- Consecutive quantifiers of the same type can be combined: $\forall x\ \forall y\ \forall z\ P(x,y,z) \Leftrightarrow$ $\forall x,y,z\ P(x,y,z)$ or even $\forall xyz\ P(x,y,z)$
- All quantified expressions can be reduced to the canonical *alternating* form $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \ldots P(x_1, x_2, x_3, x_4, \ldots)$

Defining New Quantifiers

As per their name, quantifiers can be used to express that a predicate is true of any given *quantity* (number) of objects.

Define $\exists!x\,P(x)$ to mean "$P(x)$ is true of *exactly one* x in the universe of discourse."

$$\exists!x\,P(x) \Leftrightarrow \exists x\,\big(P(x) \wedge \neg\exists y\,(P(y) \wedge y \neq x)\big)$$

"There is an x such that $P(x)$, where there is no y such that P(y) and y is other than x."

Some Number Theory Examples

- Let u.d. = the *natural numbers* 0, 1, 2, …
- "A number x is *even*, $E(x)$, if and only if it is equal to 2 times some other number."
  $\forall x\,(E(x) \leftrightarrow (\exists y\;\; x=2y))$
- "A number is *prime*, $P(x)$, iff it's greater than 1 and it isn't the product of two non-unity numbers."
  $\forall x\,(P(x) \leftrightarrow (x>1 \wedge \neg\exists yz\;\; x=yz \wedge y\neq 1 \wedge z\neq 1))$

Goldbach's Conjecture (unproven)

Using $E(x)$ and $P(x)$ from previous slide,

$$\forall E(x>2):\ \exists P(p),P(q): p+q = x$$

or, with more explicit notation:

$$\forall x\, [x>2 \wedge E(x)] \rightarrow$$
$$\exists p\ \exists q\ P(p) \wedge P(q) \wedge p+q = x.$$

"Every even number greater than 2
is the sum of two primes."

Calculus Example

- One way of precisely defining the calculus concept of a *limit*, using quantifiers:

$$\left(\lim_{x\to a} f(x) = L\right) \Leftrightarrow$$

$$\left(\begin{array}{l}\forall \varepsilon > 0 : \exists \delta > 0 : \forall x : \\ \left(|x-a| < \delta\right) \to \left(|f(x)-L| < \varepsilon\right)\end{array}\right)$$

Deduction Example

- Definitions:

  s :≡ Socrates (ancient Greek philosopher);
  $H(x)$:≡ "x is human";
  $M(x)$:≡ "x is mortal".

- Premises:

  $H(s)$ *Socrates is human.* $\forall x\, H(x) \rightarrow M(x)$ *All humans are mortal.*

Deduction Example Continued

Some valid conclusions you can draw:

$H(\mathrm{s}) \rightarrow M(\mathrm{s})$ **[Instantiate universal.]** *If Socrates is human then he is mortal.*

$\neg H(\mathrm{s}) \vee M(\mathrm{s})$ *Socrates is inhuman or mortal.*

$H(\mathrm{s}) \wedge (\neg H(\mathrm{s}) \vee M(\mathrm{s}))$

Socrates is human, and also either inhuman or mortal.

$(H(\mathrm{s}) \wedge \neg H(\mathrm{s})) \vee (H(\mathrm{s}) \wedge M(\mathrm{s}))$ **[Apply distributive law.]**

$\mathbf{F} \vee (H(\mathrm{s}) \wedge M(\mathrm{s}))$ **[Trivial contradiction.]**

$H(\mathrm{s}) \wedge M(\mathrm{s})$ **[Use identity law.]**

$M(\mathrm{s})$ *Socrates is mortal.*

Another Example

- Definitions: $H(x) :\equiv$ "x is human"; $M(x) :\equiv$ "x is mortal"; $G(x) :\equiv$ "x is a god"
- Premises:
  - $\forall x\, H(x) \rightarrow M(x)$ ("Humans are mortal") and
  - $\forall x\, G(x) \rightarrow \neg M(x)$ ("Gods are immortal").
- Show that $\neg \exists x\, (H(x) \wedge G(x))$ ("No human is a god.")

The Derivation

- $\forall x\, H(x) \rightarrow M(x)$ and $\forall x\, G(x) \rightarrow \neg M(x)$.
- $\forall x\, \neg M(x) \rightarrow \neg H(x)$ **[Contrapositive.]**
- $\forall x\, [G(x) \rightarrow \neg M(x)] \wedge [\neg M(x) \rightarrow \neg H(x)]$
- $\forall x\, G(x) \rightarrow \neg H(x)$ **[Transitivity of $\rightarrow$.]**
- $\forall x\, \neg G(x) \vee \neg H(x)$ **[Definition of $\rightarrow$.]**
- $\forall x\, \neg(G(x) \wedge H(x))$ **[DeMorgan's law.]**
- $\neg \exists x\, G(x) \wedge H(x)$ **[An equivalence law.]**

End of §1.3-1.4, Predicate Logic

- From these sections you should have learned:
  - Predicate logic notation & conventions
  - Conversions: predicate logic ↔ clear English
  - Meaning of quantifiers, equivalences
  - Simple reasoning with quantifiers
- Upcoming topics:
  - Introduction to proof-writing.
  - Then: Set theory –
    - a language for talking about collections of objects.