



Recurrence Relations

Discrete Mathematical Structures:
Theory and Applications

Learning Objectives

- ❑ Learn about recurrence relations
- ❑ Learn the relationship between sequences and recurrence relations
- ❑ Explore how to solve recurrence relations by iteration
- ❑ Learn about linear homogeneous recurrence relations and how to solve them
- ❑ Become familiar with linear nonhomogeneous recurrence relations

Sequences and Recurrence Relations

EXAMPLE 8.1.2

Consider the following two sequences:

$$S_1 : 3, 5, 7, 9, \dots$$

$$S_2 : 3, 9, 27, 81, \dots$$

We can find a formula for the n th term of sequences S_1 and S_2 by observing the pattern of the sequences.

$$S_1 : 2 \cdot 1 + 1, 2 \cdot 2 + 1, 2 \cdot 3 + 1, 2 \cdot 4 + 1, \dots$$

$$S_2 : 3^1, 3^2, 3^3, 3^4, \dots$$

For S_1 , $a_n = 2n + 1$ for $n \geq 1$, and for S_2 , $a_n = 3^n$ for $n \geq 1$. This type of formula is called an **explicit formula** for the sequence, because using this formula we can directly find any term of the sequence without using other terms of the sequence. For example, $a_3 = 2 \cdot 3 + 1 = 7$.

Sequences and Recurrence Relations

EXAMPLE 8.1.3

Let S denote the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \dots$$

For this sequence, the explicit formula is not obvious. If we observe closely, however, we find that the pattern of the sequence is such that any term after the second term is the sum of the preceding two terms. Now

$$\text{3rd term} = 2 = 1 + 1 = \text{1st term} + \text{2nd term}$$

$$\text{4th term} = 3 = 1 + 2 = \text{2nd term} + \text{3rd term}$$

$$\text{5th term} = 5 = 2 + 3 = \text{3rd term} + \text{4th term}$$

$$\text{6th term} = 8 = 3 + 5 = \text{4th term} + \text{5th term}$$

$$\text{7th term} = 13 = 5 + 8 = \text{5th term} + \text{6th term}$$

Hence, the sequence S can be defined by the equation

$$f_n = f_{n-1} + f_{n-2} \tag{8.1}$$

for all $n \geq 3$ and

$$\begin{aligned} f_1 &= 1, \\ f_2 &= 1. \end{aligned} \tag{8.2}$$

This sequence is called the Fibonacci sequence

Sequences and Recurrence Relations

EXAMPLE 8.1.4

Consider the function $f : \mathbb{N}^0 \rightarrow \mathbb{Z}^+$ defined by

$$\begin{aligned} f(0) &= 1, \\ f(n) &= nf(n-1) \quad \text{for all } n \geq 1. \end{aligned}$$

Then

$$\begin{aligned} f(0) &= 1 = 0!, \\ f(1) &= 1 \cdot f(0) = 1 = 1!, \\ f(2) &= 2 \cdot f(1) = 2 \cdot 1 = 2 = 2!, \\ f(3) &= 3 \cdot f(2) = 3 \cdot 2 \cdot 1 = 6 = 3!, \end{aligned}$$

and so on. Here $f(n) = nf(n-1)$ for all $n \geq 1$ is the recurrence relation, and $f(0) = 1$ is the initial condition for the function f . Notice that the function f is nothing but the factorial function, i.e., $f(n) = n!$ for all $n \geq 0$.

Sequences and Recurrence Relations

Let us consider the function f as given in (8.3). If we write $a_n = f(n)$, then (8.3) translates into the following equation:

$$a_n = 2a_{n-1} + a_{n-2} \quad \text{for all } n \geq 2.$$

That is, a_n is defined in terms of a_{n-1} and a_{n-2} . As remarked previously, such an equation is called a recurrence relation. Moreover, (8.4) translates into $a_0 = 5$ and $a_1 = 7$. These are called the initial conditions for the recurrence relation.

Sequences and Recurrence Relations

DEFINITION 8.1.5

A **recurrence relation** for a sequence $a_0, a_1, a_2, \dots, a_n, \dots$ is an equation that relates a_n to some of the terms $a_0, a_1, a_2, \dots, a_{n-2}, a_{n-1}$ for all integers n with $n \geq k$, where k is a nonnegative integer. The **initial conditions** for the recurrence relation are a set of values that explicitly define some of the members of $a_0, a_1, a_2, \dots, a_{k-1}$.

The equation

$$a_n = 2a_{n-1} + a_{n-2} \quad \text{for all } n \geq 2,$$

as defined above, relates a_n to a_{n-1} and a_{n-2} . Here $k = 2$. So this is a recurrence relation with initial conditions $a_0 = 5$ and $a_1 = 7$.

Sequences and Recurrence Relations

EXAMPLE 8.1.9

Number of subsets of a finite set. Let s_n denote the number of subsets of a set A with n elements, $n \geq 0$. In Worked-Out Exercise 9 (Chapter 2, page 144), we proved that

$$\begin{aligned}s_0 &= 1, \\ s_n &= 2s_{n-1}, \quad \text{if } n > 0\end{aligned}$$

Hence, a recurrence relation for the sequence $s_0, s_1, s_2, s_3, s_4, \dots$ is

$$s_n = 2s_{n-1}, \quad n \geq 1$$

and an initial condition is $s_0 = 1$.

Sequences and Recurrence Relations

EXAMPLE 8.1.10

Compound Interest. Sam received a yearly bonus and deposited \$10,000 in a local bank yielding 7% interest compounded annually. Sam wants to know the total amount accumulated after n years. Let A_n denote the total amount accumulated after n years. Let us determine a recurrence relation and initial conditions for the sequence $A_0, A_1, A_2, A_3, \dots$.

The amount accumulated after one year is the initial amount plus the interest on the initial amount. Now A_{n-1} is the amount accumulated after $n - 1$ years. This implies that the amount at the beginning of n th year is A_{n-1} . It follows that the total amount accumulated after n years is the amount at the beginning of the n th year plus the interest on this amount. Because the interest rate is 7%, the interest earned during the n th year is $(0.07)A_{n-1}$. Hence,

$$\begin{aligned}A_n &= A_{n-1} + (0.07)A_{n-1} \\&= 1.07A_{n-1}, \quad n \geq 1, \\A_0 &= 10000.\end{aligned}$$

Thus, we find that a recurrence relation and an initial condition for the sequence $\{A_n\}_{n=0}^{\infty}$ are

$$\begin{aligned}A_n &= 1.07A_{n-1}, \quad n \geq 1, \\A_0 &= 10000.\end{aligned}$$

Sequences and Recurrence Relations

❑ Tower of Hanoi

- ❑ In the nineteenth century, a game called the Tower of Hanoi became popular in Europe. This game represents work that is under way in the temple of Brahma.
- ❑ There are three pegs, with one peg containing 64 golden disks. Each golden disk is slightly smaller than the disk below it.
- ❑ The task is to move all 64 disks from the first peg to the third peg.

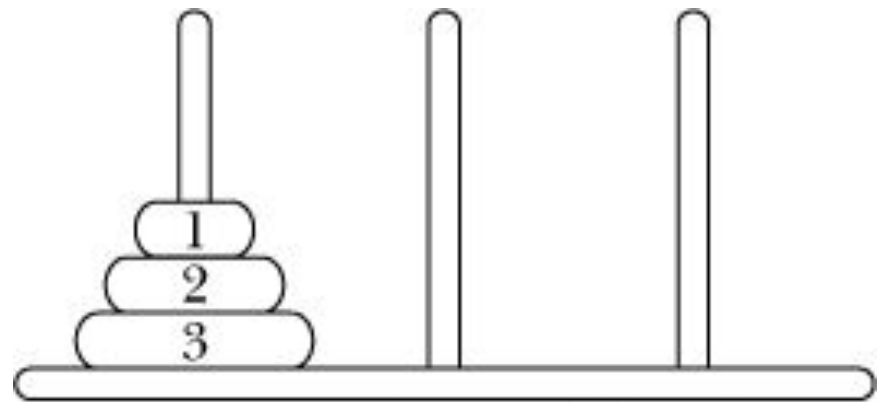


FIGURE 8.1 Tower of Hanoi problem with three disks

Sequences and Recurrence Relations

- ❑ The rules for moving the disks are as follows:
 1. Only one disk can be moved at a time.
 2. The removed disk must be placed on one of the pegs.
 3. A larger disk cannot be placed on top of a smaller disk.
- ❑ The objective is to determine the minimum number of moves required to transfer the disks from the first peg to the third peg.

Sequences and Recurrence Relations

- ❑ First consider the case in which the first peg contains only one disk.
 - ❑ The disk can be moved directly from peg 1 to peg 3.
- ❑ Consider the case in which the first peg contains two disks.
 - ❑ First move the first disk from peg 1 to peg 2.
 - ❑ Then move the second disk from peg 1 to peg 3.
 - ❑ Finally, move the first disk from peg 2 to peg 3.

Sequences and Recurrence Relations

- ❑ Consider the case in which the first peg contains three disks and then generalize this to the case of 64 disks (in fact, to an arbitrary number of disks).
- ❑ Suppose that peg 1 contains three disks. To move disk number 3 to peg 3, the top two disks must first be moved to peg 2. Disk number 3 can then be moved from peg 1 to peg 3. To move the top two disks from peg 2 to peg 3, use the same strategy as before. This time use peg 1 as the intermediate peg.
- ❑ Figure 8.2 shows a solution to the Tower of Hanoi problem with three disks.

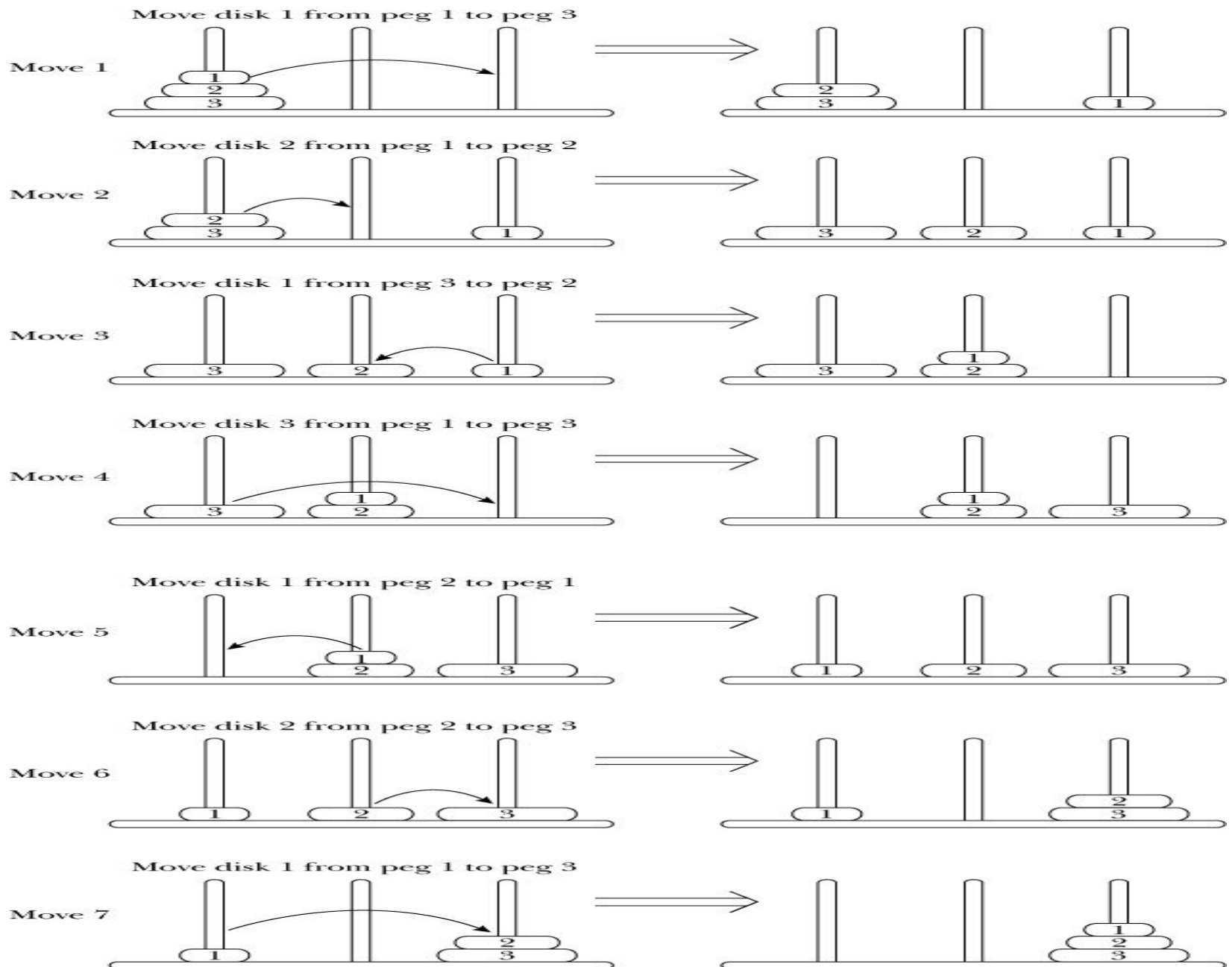


FIGURE 8.2 A solution to the Tower of Hanoi problem with three disks

- Generalize this problem to the case of 64 disks. To begin, the first peg contains all 64 disks. Disk number 64 cannot be moved from peg 1 to peg 3 unless the top 63 disks are on the second peg. So first move the top 63 disks from peg 1 to peg 2, and then move disk number 64 from peg 1 to peg 3. Now the top 63 disks are all on peg 2.
- To move disk number 63 from peg 2 to peg 3, first move the top 62 disks from peg 2 to peg 1, and then move disk number 63 from peg 2 to peg 3. To move the remaining 62 disks, follow a similar procedure.
- In general, let peg 1 contain $n \geq 1$ disks.
 1. Move the top $n - 1$ disks from peg 1 to peg 2 using peg 3 as the intermediate peg.
 2. Move disk number n from peg 1 to peg 3.
 3. Move the top $n - 1$ disks from peg 2 to peg 3 using peg 1 as the intermediate peg.

Sequences and Recurrence Relations

Let c_n denote the number of moves required to move n disks, $n \geq 0$, from peg 1 to peg 3. Step (1) requires us to move the top $n - 1$ disks from peg 1 to peg 2, which requires c_{n-1} moves. Step (2) requires us to move the n th disk from peg 1 to peg 3, which requires 1 move. Step (3) requires us to move $n - 1$ disks from peg 2 to peg 3, which requires c_{n-1} moves. Thus, it follows that

$$c_n = 2c_{n-1} + 1, \quad \text{if } n > 1, \quad (8.5)$$

and

$$c_1 = 1. \quad (8.6)$$

Now (8.5) is a recurrence relation for the sequence $\{c_n\}_{n=1}^{\infty}$ with the initial condition given by (8.6).

Sequences and Recurrence Relations

DEFINITION 8.1.13

Suppose a recurrence relation for a sequence $a_0, a_1, a_2, \dots, a_n, \dots$, is given. By a *solution of the recurrence relation* we mean to obtain an explicit formula for a_n , i.e., to find an expression for a_n that does not involve any other a_i .

EXAMPLE 8.1.14

Let S be the sequence $\{a_n\}_{n=0}^{\infty}$, where

$$a_n = 7a_{n-1} - 6a_{n-2} \quad \text{for all } n \geq 2. \quad (8.8)$$

Because a_n is defined in terms of the preceding terms a_{n-1} and a_{n-2} , Equation (8.8) is a recurrence relation.

Let us show that $a_n = 5 = 5 + 0 \cdot n$ is a solution of Equation (8.8). Here $a_0 = 5$, $a_1 = 5$, $a_2 = 5$, \dots , $a_n = 5$, and so on. Let us evaluate the right side of Equation (8.8), i.e.,

$$7a_{n-1} - 6a_{n-2} = 7 \cdot 5 - 6 \cdot 5 = 35 - 30 = 5 = a_n.$$

Hence, $a_n = 5$, $n \geq 0$ is a solution of the recurrence relation (8.8).

Now let $a_n = 6^n$. Here $a_0 = 6^0 = 1$, $a_1 = 6^1 = 6$, $a_2 = 6^2 = 36$, \dots , $a_{n-2} = 6^{n-2}$, $a_{n-1} = 6^{n-1}$, $a_n = 6^n$, and so on. Let us evaluate the right side of Equation (8.8), using the terms of this sequence. We have

$$\begin{aligned} 7a_{n-1} - 6a_{n-2} &= 7 \cdot 6^{n-1} - 6 \cdot 6^{n-2} \\ &= 7 \cdot 6^{n-1} - 6^{n-1} \\ &= (7 - 1) \cdot 6^{n-1} \\ &= 6 \cdot 6^{n-1} \\ &= 6^n \\ &= a_n. \end{aligned}$$

Therefore, $a_n = 6^n$, $n \geq 0$ is also a solution of the recurrence relation (8.8).

Note that the expression $a_n = 2^n$, $n \geq 0$ is not a solution of Equation (8.8).

Linear Homogenous Recurrence Relations

DEFINITION 8.2.1

Let $a_0, a_1, a_2, \dots, a_n, \dots$ be a sequence of numbers. A **linear homogeneous recurrence relation** of order k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}, \quad (8.31)$$

where $c_k \neq 0$ and c_1, c_2, c_3, \dots , and c_k are constants.

Linear Homogenous Recurrence Relations

EXAMPLE 8.2.2

Consider the following recurrence relations.

- (i) $a_n = 3a_{n-1} + a_{n-2}$
- (ii) $a_n = 3a_{n-1} + 5$
- (iii) $a_n = 3a_{n-1} + a_{n-2} \cdot a_{n-3}$
- (iv) $a_n = 3a_{n-1} + a_{n-2} + \sqrt{2}a_{n-3}$
- (v) $a_n = 3a_{n-1} + na_{n-2}$

Recurrence relations (i), (ii), (iii), and (iv) are recurrence relations with constant coefficients. Recurrence relation (v), $a_n = 3a_{n-1} + na_{n-2}$, is not a relation with constant coefficients. Notice that (i) is a linear homogeneous recurrence relation of order 2, (ii) is not a homogeneous recurrence relation because of the constant term 5, (iii) is not a linear recurrence relation because it contains $a_{n-2} \cdot a_{n-3}$, the product of terms a_{n-2} and a_{n-3} , and (iv) is a linear homogeneous recurrence relation of order 3.

Linear Homogenous Recurrence Relations

DEFINITION 8.2.3

A sequence $s_0, s_1, s_2, \dots, s_n, \dots$ is said to **satisfy** a linear homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \dots + c_k a_{n-k}, \quad c_k \neq 0 \quad (8.32)$$

of order k with constant coefficients if $s_n = c_1 s_{n-1} + c_2 s_{n-2} + c_3 s_{n-3} + \dots + c_k s_{n-k}$.

DEFINITION 8.2.4

If a sequence $s_0, s_1, s_2, \dots, s_n, \dots$ satisfies a linear homogeneous recurrence relation, then the sequence $s_0, s_1, s_2, \dots, s_n, \dots$ is also called a **solution** of that recurrence relation.

Linear Homogenous Recurrence Relations

EXAMPLE 8.2.5

Consider the recurrence relation $a_n = 3a_{n-1}$. This is a linear homogeneous recurrence relation of order 1. Let t be a nonzero number and suppose $a_n = t^n$ for all $n \geq 0$. Then $a_n = 3a_{n-1}$ implies that $t^n = 3t^{n-1}$. Therefore, $t = 3$. Thus, we find that $a_n = 3^n$. Hence, the sequence $1, 3, 3^2, 3^3, \dots, 3^n, \dots$ is a solution of the recurrence relation $a_n = 3a_{n-1}$.

Theorem 8.2.7: Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad c_2 \neq 0, \quad n > 1 \quad (8.34)$$

be a linear homogeneous recurrence relation with constant coefficients. Let t be a nonzero real number. Then the sequence $\{t^n\}$ satisfies the above recurrence relation if and only if

$$t^2 - c_1 t - c_2 = 0. \quad (8.35)$$

Linear Homogenous Recurrence Relations

DEFINITION 8.2.8

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, $c_2 \neq 0$, $n > 1$ be a linear homogeneous recurrence relation with constant coefficients. The equation

$$t^2 - c_1 t - c_2 = 0$$

is called the **characteristic equation** of the recurrence relation.

Theorem 8.2.9: Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad n > 1 \quad (8.37)$$

be a linear homogeneous recurrence relation of order 2, where c_1 and c_2 are constants and $c_2 \neq 0$

- (i) If the sequences $\{s_n\}$ and $\{p_n\}$ satisfy (8.37), then for any constants b and d , the sequence $\{bs_n + dp_n\}$ satisfies (8.37).
- (ii) Let r be a root of the characteristic equation

$$t^2 - c_1 t - c_2 = 0 \quad (8.38)$$

of (8.37). Then the sequence $\{r^n\}$ is a solution of (8.37).

Linear Homogenous Recurrence Relations

Theorem 8.2.10: Suppose that a sequence $\{d_n\}$ is a solution of the recurrence relation (8.37). If r_1 and r_2 are the distinct roots of the characteristic equation (8.38), then there exist constants b and d , which are to be determined, such that the solution of the recurrence relation (8.37) is

$$d_n = br_1^n + dr_2^n, \quad n = 0, 1, \dots$$

Corollary 8.2.11: Suppose that

$$a_0 = d_0, \quad a_1 = d_1$$

are the initial conditions for the recurrence relation (8.37), where d_0 and d_1 are constants. Further suppose that r_1 and r_2 are the roots of (8.38). If $r_1 \neq r_2$, then there exist constants b and d , which are to be determined by initial conditions, such that the solution of the recurrence relation (8.37) is

$$a_n = br_1^n + dr_2^n, \quad n = 0, 1, \dots$$

EXAMPLE 8.2.12

In this example, we solve the following linear homogeneous recurrence relation:

$$a_n = 7a_{n-1} - 10a_{n-2} \quad (8.41)$$

with initial conditions

$$a_0 = 1$$

$$a_1 = 8.$$

The characteristic equation of the given recurrence relation is:

$$t^2 - 7t + 10 = 0.$$

Next, we find the roots of this equation. Now,

$$t^2 - 7t + 10 = (t - 5)(t - 2)$$

and so

$$(t - 5)(t - 2) = 0.$$

This implies that the roots of the characteristic equation are $t = 5$, and $t = 2$. The roots are distinct. By Theorem 8.2.10, there exist constants c_1 and c_2 , which are to be determined from initial conditions, such that

$$a_n = c_1 5^n + c_2 2^n, \quad n \geq 0.$$

We substitute $n = 0$ and $n = 1$, respectively, to obtain

$$\begin{aligned}a_0 &= c_1 + c_2, \\a_1 &= 5c_1 + 2c_2.\end{aligned}$$

Using the initial conditions, we get

$$\begin{aligned}c_1 + c_2 &= 1, \\5c_1 + 2c_2 &= 8.\end{aligned}$$

Solving these equations for c_1 and c_2 , we get $c_1 = 2$ and $c_2 = -1$. Hence,

$$a_n = 2 \cdot 5^n - 2^n, \quad n \geq 0.$$

Hence, the sequence $\{2 \cdot 5^n - 2^n\}$ is the solution.

Linear Homogenous Recurrence Relations

Theorem 8.2.13: Suppose that a sequence $\{s_n\}$ is a solution of the recurrence relation (8.37). If r_1 and r_2 are the roots of the characteristic equation (8.38) such that $r_1 = r_2 = r$, then there exist constants b and d , which are to be determined, such that the solution of the recurrence relation (8.37) is

$$s_n = br^n + dnr^n, \quad n = 0, 1, \dots$$

Corollary 8.2.14: Suppose that

$$a_0 = d_0, \quad a_1 = d_1$$

are the initial conditions for the recurrence relation (8.37), where d_0 and d_1 are constants. Further suppose that r_1 and r_2 are the roots of (8.38) such that $r_1 = r_2 = r$. Then there exist constants b and d , which are to be determined from initial conditions, such that the solution of the recurrence relation (8.37) is

$$a_n = br^n + dnr^n, \quad n = 0, 1, \dots$$

EXAMPLE 8.2.15

In this example, we solve the following linear homogeneous recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial conditions

$$a_0 = 4$$

$$a_1 = 12.$$

The characteristic equation of this recurrence relation is the quadratic equation

$$t^2 - 4t + 4 = 0.$$

We find the roots of this equation. Now,

$$t^2 - 4t + 4 = (t - 2)(t - 2)$$

and so

$$(t - 2)(t - 2) = 0.$$

EXAMPLE 8.2.15

This implies that the roots of the characteristic equation are $t = 2$, and $t = 2$. The roots are not distinct. Therefore, by Theorem 8.2.13, there exist constants c_1 and c_2 , which are to be determined from initial conditions, such that

$$a_n = c_1 2^n + c_2 n 2^n, \quad n = 0, 1, \dots$$

We substitute $n = 0$ and $n = 1$, respectively, to obtain

$$a_0 = c_1$$

$$a_1 = 2c_1 + 2c_2.$$

Using the initial conditions, we get

$$c_1 = 4,$$

$$2c_1 + 2c_2 = 12.$$

Solving these equations for c_1 and c_2 , we get $c_1 = 4$ and $c_2 = 2$. Hence,

$$a_n = 4 \cdot 2^n + 2 \cdot n \cdot 2^n = 2 \cdot 2^{n+1} + n 2^{n+1} = (2 + n) 2^{n+1} = (n + 2) 2^{n+1}, \quad n \geq 0.$$

Thus, we find that the sequence $\{(n + 2) 2^{n+1}\}$ is the solution.

Linear Homogenous Recurrence Relations

Theorem 8.2.16: Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}, \quad c_k \neq 0 \quad (8.42)$$

be a linear homogeneous recurrence relation with constant coefficients. Let t be a nonzero real number. Then the sequence $\{t^n\}$ is a solution of the above recurrence relation if and only if

$$t^n - c_1 t^{n-1} - c_2 t^{n-2} - c_3 t^{n-3} - \cdots - c_k t^{n-k} = 0.$$

DEFINITION 8.2.17

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}$, $c_k \neq 0$ be a linear homogeneous recurrence relation with constant coefficients. The equation

$$t^k - c_1 t^{k-1} - c_2 t^{k-2} - c_3 t^{k-3} - \cdots - c_k = 0$$

is called the **characteristic equation** of this linear homogeneous recurrence relation.

Linear Homogenous Recurrence Relations

REMARK 8.2.18

To obtain the characteristic equation of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}$, $c_k \neq 0$, substitute $a_n = t^n$, $t \neq 0$, to get

$$t^n = c_1 t^{n-1} + c_2 t^{n-2} + c_3 t^{n-3} + \cdots + c_k t^{n-k}.$$

Thus,

$$\begin{aligned} t^n &= c_1 t^{n-1} + c_2 t^{n-2} + c_3 t^{n-3} + \cdots + c_k t^{n-k} \\ \Rightarrow t^n - c_1 t^{n-1} - c_2 t^{n-2} - c_3 t^{n-3} - \cdots - c_k t^{n-k} &= 0 \\ \Rightarrow t^{n-k}(t^k - c_1 t^{k-1} - c_2 t^{k-2} - c_3 t^{k-3} - \cdots - c_k) &= 0. \end{aligned}$$

Because $t \neq 0$, we have, $t^k - c_1 t^{k-1} - c_2 t^{k-2} - c_3 t^{k-3} - \cdots - c_k = 0$, which is the characteristic equation.

Linear Homogenous Recurrence Relations

Theorem 8.2.19: Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k} \quad (8.44)$$

be a linear homogeneous recurrence relation of order k , where $c_k \neq 0$ and c_1, c_2, c_3, \dots , and c_k are constants. Let

$$t^k - c_1 t^{k-1} - c_2 t^{k-2} - c_3 t^{k-3} - \cdots - c_k = 0$$

be the characteristic equation of (8.44).

- (i) If the sequences $\{s_n\}_{n=0}^{\infty}$ and $\{p_n\}_{n=0}^{\infty}$ are solutions of (8.44), then for any constants b and d , the sequence $\{bs_n + dp_n\}_{n=0}^{\infty}$ is a solution of (8.44).
- (ii) If r is a root of the characteristic equation, then the sequence $1, r, r^2, \dots, r^n, \dots$ is a solution of (8.44).
- (iii) If r_1, r_2, \dots, r_k are distinct roots of the characteristic equations, then there exist constants $b_1, b_2, b_3, \dots, b_k$, which are to be determined from initial conditions, such that a solution of (8.44) is given by

$$a_n = b_1 r_1^n + b_2 r_2^n + b_3 r_3^n + \cdots + b_k r_k^n,$$

Linear Homogenous Recurrence Relations

- (iv) If r is a root, of multiplicity m , of the characteristic equation, then $a_n = r^n$, $a_n = nr^n$, $a_n = n^2r^n, \dots$, and $a_n = n^{m-1}r^n$ are solutions of (8.44).
- (v) Suppose that

$$a_0 = d_0, a_1 = d_1, \dots, a_{n-1} = d_{n-1}$$

are the initial conditions for the recurrence relation (8.44), where d_0, d_1, \dots , and d_{n-1} are constants. If r_1, r_2, \dots , and r_t are t distinct roots of the characteristic equation with multiplicities m_1, m_2, \dots, m_t and $m_1 + m_2 + \dots + m_t = k$, then there exist constants c_{ij} , which are to be determined from the initial conditions, such that the solution of the recurrence relation (8.44) is

$$\begin{aligned} a_n = & (c_{00} + c_{01}n + \dots + c_{0m_1}n^{m_1-1})r_1^n \\ & + (c_{10} + c_{11}n + \dots + c_{1m_2}n^{m_2-1})r_2^n \\ & + \dots + (c_{t0} + c_{t1}n + \dots + c_{tm_t}n^{m_t-1})r_t^n, \quad n = 0, 1, \dots \end{aligned}$$

Linear Nonhomogenous Recurrence Relations

A **linear nonhomogeneous recurrence relation** with constants coefficients is a recurrence relation of the form

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n), \quad (8.55)$$

where $c_i, i = 1, 2, \dots, k$, are constants, $c_k \neq 0$, and $f(n)$ is a nonzero real-valued function.

If $f(n) = 0$, then (8.55) is a linear homogeneous equation (which we discussed in the previous section). There is no known general method for solving nonhomogeneous linear recurrence equations. However, we can develop a method for solving the special case

$$c_0 a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = b^n p(n), \quad (8.56)$$

where b is a constant and $p(n)$ is a polynomial in n .

Linear Nonhomogenous Recurrence Relations

EXAMPLE 8.3.2

Consider the recurrence

$$a_n + 5a_{n-1} + 6a_{n-2} = 3^n.$$

This is a nonhomogeneous recurrence relation of the form (8.56). Here $k = 2$, $b = 3$, and $p(n) = 1$.

EXAMPLE 8.3.3

Consider the recurrence

$$a_n + 5a_{n-1} + 6a_{n-2} = 3^n(n^2 + 6n + 5).$$

This is a nonhomogeneous recurrence relation of the form (8.56). Here $k = 2$, $b = 3$, and $p(n) = n^2 + 6n + 5$.

Linear Nonhomogenous Recurrence Relations

Theorem 8.3.5: Let

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n) \quad (8.62)$$

be a nonhomogeneous recurrence relation, where $c_i, i = 1, 2, \dots, k$, are constants, $c_k \neq 0$, and $f(n)$ is a nonzero real-valued function. Suppose $\{r_n\}$ is a particular solution of (8.62). Then $\{u_n\}$ is a solution of (8.62) if and only if $u_n = r_n + s_n$, for all n , and $\{s_n\}$ is a solution of the associated homogeneous part, $a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = 0$.

Linear Nonhomogenous Recurrence Relations

Theorem 8.3.6: Let

$$a_n - da_{n-1} = b^n u, \quad n \geq 1 \quad (8.67)$$

be a nonhomogeneous linear recurrence relation, with the initial condition

$$a_0 = e_0, \quad (8.68)$$

where d , b , u , and e_0 are constants, and b and u are nonzero. This nonhomogeneous linear recurrence relation can be transformed into the following linear homogeneous recurrence relation:

$$a_n - (b + d)a_{n-1} + bda_{n-2} = 0, \quad n \geq 2$$

with the initial conditions $a_0 = e_0$ and $a_1 = de_0 + bu$.

Moreover,

- (i) if $b \neq d$, then there exists a constant c_0 , which is to be determined from the initial condition, such that

$$a_n = c_0 d^n + \left(\frac{bu}{b-d} \right) b^n.$$

- (ii) if $b = d$, then there exists a constant c_0 , which to be is determined from the initial condition, such that

$$a_n = c_0 b^n + unb^n.$$

EXAMPLE 8.3.7

In this example, we use Theorem 8.3.6 to solve the recurrence relation

$$a_n - 4a_{n-1} = 8^n, \quad n \geq 1,$$

with the initial condition

$$a_0 = 1.$$

This is a recurrence relation of the form

$$a_n - da_{n-1} = b^n u,$$

where $d = 4$, $b = 8$, and $u = 1$. Because $b \neq d$,

$$\begin{aligned} a_n &= c_0 d^n + \frac{bu}{b-d} b^n \\ &= c_0 4^n + \frac{8}{4} 8^n \\ &= c_0 4^n + 2 \cdot 8^n \end{aligned}$$

for all $n \geq 0$, where c_0 is a constant satisfying the initial condition.

Now

$$1 = a_0 = c_0 4^0 + 2 \cdot 8^0 = c_0 + 2.$$

Hence, $c_0 = -1$. This implies that $a_n = -1 \cdot 4^n + 2 \cdot 8^n$ for all $n \geq 0$.

Linear Nonhomogenous Recurrence Relations

Theorem 8.3.10: Let

$$a_n - da_{n-1} = b^n(un + v), \quad n \geq 1, \quad (8.83)$$

be a nonhomogeneous linear recurrence relation, with the initial condition

$$a_0 = e_0, \quad (8.84)$$

where d, b, u, v , and e_0 are constants, and b and u are nonzero. This nonhomogeneous linear recurrence relation can be transformed into the following linear homogeneous recurrence relation:

$$a_n - (2b + d)a_{n-1} + b(2d + b)a_{n-2} - b^2da_{n-3} = 0, \quad n \geq 3 \quad (8.85)$$

with the initial conditions

$$a_0 = e_0 \quad \text{and} \quad a_1 = de_0 + b(u + v).$$

Moreover, the characteristic equation of (8.85) is

$$(t - d)(t - b)^2 = 0. \quad (8.86)$$

Linear Nonhomogenous Recurrence Relations

Let $\{r_n\}$ be a solution of (8.83).

(i) Suppose $b \neq d$. Then r_n is of the form

$$r_n = c_0 d^n + c_1 b^n + c_2 n b^n,$$

where c_0, c_1 , and c_2 are some constants.

(ii) Suppose $b = d$. Then $\{r_n\}$ is of the form

$$r_n = c_0 b^n + c_1 n b^n + c_2 n^2 b^n,$$

where c_0, c_1 , and c_2 are some constants.

EXAMPLE 8.3.11

Consider the recurrence relation

$$a_n - 3a_{n-1} = 2^n(4n + 3), \quad n > 1 \quad (8.94)$$

with initial conditions

$$a_0 = 0,$$

$$a_1 = 14.$$

This is a recurrence relation of the form

$$a_n - da_{n-1} = b^n(un + v).$$

Here $d = 3$, $b = 2$, $u = 4$, and $v = 3$.

We can solve this recurrence by using the technique of Theorem 8.3.10 and obtaining

$$a_n = c_0 3^n + c_1 2^n + c_2 n 2^n,$$

where c_0 , c_1 , and c_2 are constants, which are to be determined from the initial conditions.

Put $n = 2$ in (8.92) to get

$$a_2 - 3a_1 = 2^2(4 \cdot 2 + 3) = 44.$$

Because $a_1 = 14$, we get

$$a_2 = 3 \cdot 14 + 44 = 86.$$

Thus,

$$\begin{aligned} a_0 &= c_0 + c_1 = 0 \\ a_1 &= c_0 \cdot 3 + c_1 \cdot 2 + c_2 \cdot 2 = 14 \\ a_2 &= c_0 \cdot 3^2 + c_1 \cdot 2^2 + c_2 \cdot 2 \cdot 2^2 = 86 \end{aligned}$$

This implies that

$$\begin{aligned} c_0 + c_1 &= 0 \\ 3c_0 + 2c_1 + 2c_2 &= 14 \\ 9c_0 + 4c_1 + 8c_2 &= 86 \end{aligned}$$

We solve these equations for c_0 , c_1 , and c_2 to obtain $c_0 = 30$, $c_1 = -30$, and $c_2 = -8$. Thus, we find that

$$a_n = 30(3^n) - 30(2^n) - n2^{n+3}, \quad n \geq 0. \quad (8.95)$$

Linear Nonhomogenous Recurrence Relations

Theorem 8.3.13: Let

$$a_n + d_1 a_{n-1} + \cdots + d_k a_{n-k} = b^n p(n) \quad (8.96)$$

be a nonhomogeneous linear recurrence relation, where $p(n)$ is a polynomial of degree m . Then from this nonhomogeneous linear recurrence relation we can obtain a linear homogeneous recurrence that has following characteristic equation:

$$(t^k + d_1 t^{k-1} + \cdots + d_k)(t - b)^{m+1} = 0. \quad (8.97)$$

Moreover, a solution of (8.96) is also a solution of the linear homogeneous recurrence whose characteristic equation is given by (8.97).

□ Remark 8.3.14

□ There are two ways to solve a linear nonhomogeneous equation of the form

$$a_n + d_1 a_{n-1} + \cdots + d_k a_{n-k} = b^n p(n)$$

with some given initial conditions.

1. First find a particular solution and then add the particular solution to a solution of the associated linear homogeneous recurrence relation.
2. First obtain a linear homogeneous recurrence relation from the given linear nonhomogeneous recurrence relation, as shown in this section. Then a solution of the given linear nonhomogeneous recurrence relation is also a solution of the linear homogeneous equation obtained. Next find a solution of the homogeneous recurrence relation and use the initial conditions of the nonhomogeneous recurrence solution to find the constants. Finally, verify that the solution obtained satisfies the linear nonhomogeneous recurrence relation.