

39. How many license plates consisting of three letters followed by three digits contain no letter or digit twice?

A **circular  $r$ -permutation of  $n$**  people is a seating of  $r$  of these  $n$  people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table.

40. Find the number of circular 3-permutations of 5 people.
41. Find a formula for the number of circular  $r$ -permutations of  $n$  people.
42. Find a formula for the number of ways to seat  $r$  of  $n$  people around a circular table, where seatings are considered the same if every person has the same two neighbors without regard to which side these neighbors are sitting on.
43. How many ways are there for a horse race with three horses to finish if ties are possible? [Note: Two or three horses may tie.]
- \*44. How many ways are there for a horse race with four horses to finish if ties are possible? [Note: Any number of the four horses may tie.]
- \*45. There are six runners in the 100-yard dash. How many ways are there for three medals to be awarded if ties are possible? (The runner or runners who finish with the fastest time receive gold medals, the runner or runners who finish with exactly one runner ahead receive silver

medals, and the runner or runners who finish with exactly two runners ahead receive bronze medals.)

- \*46. This procedure is used to break ties in games in the championship round of the World Cup soccer tournament. Each team selects five players in a prescribed order. Each of these players takes a penalty kick, with a player from the first team followed by a player from the second team and so on, following the order of players specified. If the score is still tied at the end of the 10 penalty kicks, this procedure is repeated. If the score is still tied after 20 penalty kicks, a sudden-death shootout occurs, with the first team scoring an unanswered goal victorious.
- a) How many different scoring scenarios are possible if the game is settled in the first round of 10 penalty kicks, where the round ends once it is impossible for a team to equal the number of goals scored by the other team?
- b) How many different scoring scenarios for the first and second groups of penalty kicks are possible if the game is settled in the second round of 10 penalty kicks?
- c) How many scoring scenarios are possible for the full set of penalty kicks if the game is settled with no more than 10 total additional kicks after the two rounds of five kicks for each team?

## 6.4 Binomial Coefficients and Identities

As we remarked in Section 6.3, the number of  $r$ -combinations from a set with  $n$  elements is often denoted by  $\binom{n}{r}$ . This number is also called a **binomial coefficient** because these numbers occur as coefficients in the expansion of powers of binomial expressions such as  $(a + b)^n$ . We will discuss the **binomial theorem**, which gives a power of a binomial expression as a sum of terms involving binomial coefficients. We will prove this theorem using a combinatorial proof. We will also show how combinatorial proofs can be used to establish some of the many different identities that express relationships among binomial coefficients.

### The Binomial Theorem



The binomial theorem gives the coefficients of the expansion of powers of binomial expressions. A **binomial** expression is simply the sum of two terms, such as  $x + y$ . (The terms can be products of constants and variables, but that does not concern us here.)

Example 1 illustrates how the coefficients in a typical expansion can be found and prepares us for the statement of the binomial theorem.

**EXAMPLE 1** The expansion of  $(x + y)^3$  can be found using combinatorial reasoning instead of multiplying the three terms out. When  $(x + y)^3 = (x + y)(x + y)(x + y)$  is expanded, all products of a term in the first sum, a term in the second sum, and a term in the third sum are added. Terms of the form  $x^3$ ,  $x^2y$ ,  $xy^2$ , and  $y^3$  arise. To obtain a term of the form  $x^3$ , an  $x$  must be chosen in each of the sums, and this can be done in only one way. Thus, the  $x^3$  term in the product has a coefficient of 1. To obtain a term of the form  $x^2y$ , an  $x$  must be chosen in two of the three sums (and consequently a  $y$  in the other sum). Hence, the number of such terms is the number of 2-combinations of three objects, namely,  $\binom{3}{2}$ . Similarly, the number of terms of the form  $xy^2$  is the number of ways to pick one of the three sums to obtain an  $x$  (and consequently take a  $y$

from each of the other two sums). This can be done in  $\binom{3}{1}$  ways. Finally, the only way to obtain a  $y^3$  term is to choose the  $y$  for each of the three sums in the product, and this can be done in exactly one way. Consequently, it follows that

$$\begin{aligned}(x + y)^3 &= (x + y)(x + y)(x + y) = (xx + xy + yx + yy)(x + y) \\ &= xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy \\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

We now state the binomial theorem.

### THEOREM 1

**THE BINOMIAL THEOREM** Let  $x$  and  $y$  be variables, and let  $n$  be a nonnegative integer. Then

$$(x + y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

**Proof:** We use a combinatorial proof. The terms in the product when it is expanded are of the form  $x^{n-j} y^j$  for  $j = 0, 1, 2, \dots, n$ . To count the number of terms of the form  $x^{n-j} y^j$ , note that to obtain such a term it is necessary to choose  $n - j$   $x$ s from the  $n$  sums (so that the other  $j$  terms in the product are  $y$ s). Therefore, the coefficient of  $x^{n-j} y^j$  is  $\binom{n}{n-j}$ , which is equal to  $\binom{n}{j}$ . This proves the theorem.

Some computational uses of the binomial theorem are illustrated in Examples 2–4.

**EXAMPLE 2** What is the expansion of  $(x + y)^4$ ?



**Solution:** From the binomial theorem it follows that

$$\begin{aligned}(x + y)^4 &= \sum_{j=0}^4 \binom{4}{j} x^{4-j} y^j \\ &= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 \\ &= x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4.\end{aligned}$$

**EXAMPLE 3** What is the coefficient of  $x^{12} y^{13}$  in the expansion of  $(x + y)^{25}$ ?

**Solution:** From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13! 12!} = 5,200,300.$$

**EXAMPLE 4** What is the coefficient of  $x^{12} y^{13}$  in the expansion of  $(2x - 3y)^{25}$ ?

**Solution:** First, note that this expression equals  $(2x + (-3y))^{25}$ . By the binomial theorem, we have

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} (-3y)^j.$$

Consequently, the coefficient of  $x^{12}y^{13}$  in the expansion is obtained when  $j = 13$ , namely,

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!} 2^{12} 3^{13}.$$

We can prove some useful identities using the binomial theorem, as Corollaries 1, 2, and 3 demonstrate.

### COROLLARY 1

Let  $n$  be a nonnegative integer. Then

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

**Proof:** Using the binomial theorem with  $x = 1$  and  $y = 1$ , we see that

$$2^n = (1 + 1)^n = \sum_{k=0}^n \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^n \binom{n}{k}.$$

This is the desired result.

There is also a nice combinatorial proof of Corollary 1, which we now present.

**Proof:** A set with  $n$  elements has a total of  $2^n$  different subsets. Each subset has zero elements, one element, two elements,  $\dots$ , or  $n$  elements in it. There are  $\binom{n}{0}$  subsets with zero elements,  $\binom{n}{1}$  subsets with one element,  $\binom{n}{2}$  subsets with two elements,  $\dots$ , and  $\binom{n}{n}$  subsets with  $n$  elements. Therefore,

$$\sum_{k=0}^n \binom{n}{k}$$

counts the total number of subsets of a set with  $n$  elements. By equating the two formulas we have for the number of subsets of a set with  $n$  elements, we see that

$$\sum_{k=0}^n \binom{n}{k} = 2^n.$$

### COROLLARY 2

Let  $n$  be a positive integer. Then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0.$$

**Proof:** When we use the binomial theorem with  $x = -1$  and  $y = 1$ , we see that

$$0 = 0^n = ((-1) + 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^n \binom{n}{k} (-1)^k.$$

This proves the corollary.

**Remark:** Corollary 2 implies that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots.$$

### COROLLARY 3

Let  $n$  be a nonnegative integer. Then

$$\sum_{k=0}^n 2^k \binom{n}{k} = 3^n.$$

**Proof:** We recognize that the left-hand side of this formula is the expansion of  $(1 + 2)^n$  provided by the binomial theorem. Therefore, by the binomial theorem, we see that

$$(1 + 2)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} 2^k = \sum_{k=0}^n \binom{n}{k} 2^k.$$

Hence

$$\sum_{k=0}^n 2^k \binom{n}{k} = 3^n.$$



## Pascal's Identity and Triangle

The binomial coefficients satisfy many different identities. We introduce one of the most important of these now.

### THEOREM 2

**PASCAL'S IDENTITY** Let  $n$  and  $k$  be positive integers with  $n \geq k$ . Then

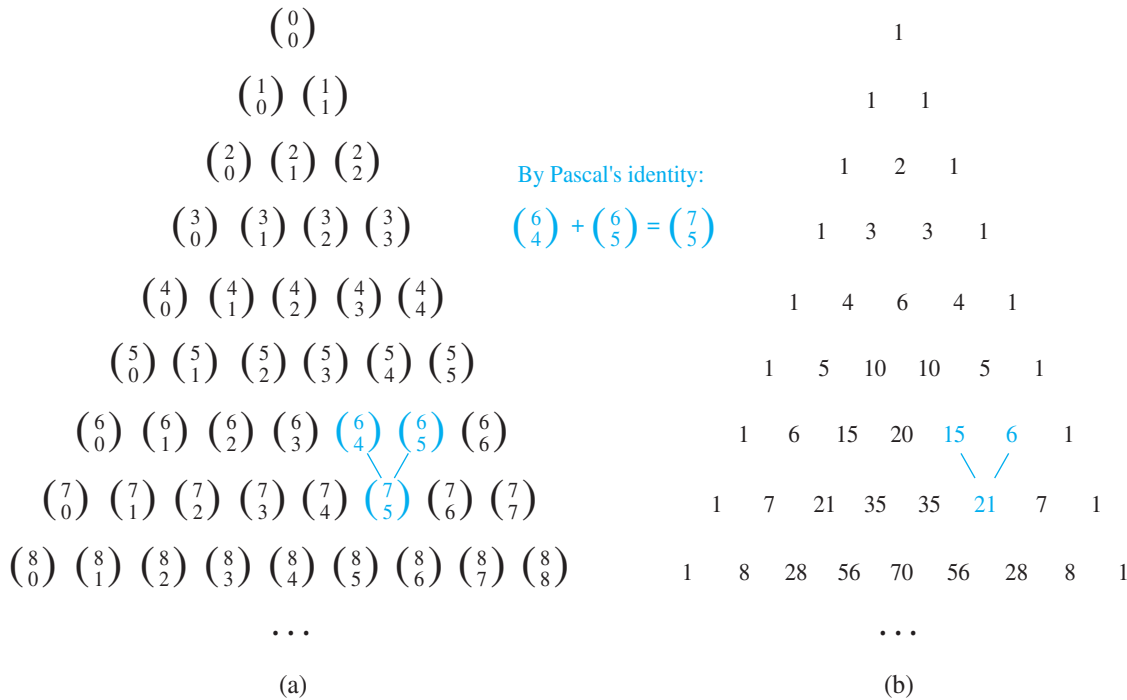
$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

**Proof:** We will use a combinatorial proof. Suppose that  $T$  is a set containing  $n + 1$  elements. Let  $a$  be an element in  $T$ , and let  $S = T - \{a\}$ . Note that there are  $\binom{n+1}{k}$  subsets of  $T$  containing  $k$  elements. However, a subset of  $T$  with  $k$  elements either contains  $a$  together with  $k - 1$  elements of  $S$ , or contains  $k$  elements of  $S$  and does not contain  $a$ . Because there are  $\binom{n}{k-1}$  subsets of  $k - 1$  elements of  $S$ , there are  $\binom{n}{k-1}$  subsets of  $k$  elements of  $T$  that contain  $a$ . And there are  $\binom{n}{k}$  subsets of  $k$  elements of  $T$  that do not contain  $a$ , because there are  $\binom{n}{k}$  subsets of  $k$  elements of  $S$ . Consequently,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$



**Remark:** It is also possible to prove this identity by algebraic manipulation from the formula for  $\binom{n}{r}$  (see Exercise 19).



**FIGURE 1** Pascal's Triangle.

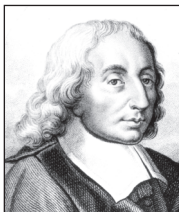
**Remark:** Pascal's identity, together with the initial conditions  $\binom{n}{0} = \binom{n}{n} = 1$  for all integers  $n$ , can be used to recursively define binomial coefficients. This recursive definition is useful in the computation of binomial coefficients because only addition, and not multiplication, of integers is needed to use this recursive definition.

Pascal's identity is the basis for a geometric arrangement of the binomial coefficients in a triangle, as shown in Figure 1.

The  $n$ th row in the triangle consists of the binomial coefficients

$$\binom{n}{k}, \quad k = 0, 1, \dots, n.$$

This triangle is known as **Pascal's triangle**. Pascal's identity shows that when two adjacent binomial coefficients in this triangle are added, the binomial coefficient in the next row between these two coefficients is produced.



**BLAISE PASCAL (1623–1662)** Blaise Pascal exhibited his talents at an early age, although his father, who had made discoveries in analytic geometry, kept mathematics books away from him to encourage other interests. At 16 Pascal discovered an important result concerning conic sections. At 18 he designed a calculating machine, which he built and sold. Pascal, along with Fermat, laid the foundations for the modern theory of probability. In this work, he made new discoveries concerning what is now called Pascal's triangle. In 1654, Pascal abandoned his mathematical pursuits to devote himself to theology. After this, he returned to mathematics only once. One night, distracted by a severe toothache, he sought comfort by studying the mathematical properties of the cycloid. Miraculously, his pain subsided, which he took as a sign of divine approval of the study of mathematics.

## Other Identities Involving Binomial Coefficients

We conclude this section with combinatorial proofs of two of the many identities enjoyed by the binomial coefficients.

### THEOREM 3

**VANDERMONDE'S IDENTITY** Let  $m$ ,  $n$ , and  $r$  be nonnegative integers with  $r$  not exceeding either  $m$  or  $n$ . Then


$$\binom{m+n}{r} = \sum_{k=0}^r \binom{m}{r-k} \binom{n}{k}.$$



**Remark:** This identity was discovered by mathematician Alexandre-Théophile Vandermonde in the eighteenth century.

**Proof:** Suppose that there are  $m$  items in one set and  $n$  items in a second set. Then the total number of ways to pick  $r$  elements from the union of these sets is  $\binom{m+n}{r}$ .

Another way to pick  $r$  elements from the union is to pick  $k$  elements from the second set and then  $r - k$  elements from the first set, where  $k$  is an integer with  $0 \leq k \leq r$ . Because there are  $\binom{n}{k}$  ways to choose  $k$  elements from the second set and  $\binom{m}{r-k}$  ways to choose  $r - k$  elements from the first set, the product rule tells us that this can be done in  $\binom{m}{r-k} \binom{n}{k}$  ways. Hence, the total number of ways to pick  $r$  elements from the union also equals  $\sum_{k=0}^r \binom{m}{r-k} \binom{n}{k}$ .

We have found two expressions for the number of ways to pick  $r$  elements from the union of a set with  $m$  items and a set with  $n$  items. Equating them gives us Vandermonde's identity. 

Corollary 4 follows from Vandermonde's identity.


### COROLLARY 4

If  $n$  is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k}^2.$$

**Proof:** We use Vandermonde's identity with  $m = r = n$  to obtain

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{n-k} \binom{n}{k} = \sum_{k=0}^n \binom{n}{k}^2.$$

The last equality was obtained using the identity  $\binom{n}{k} = \binom{n}{n-k}$ . 



**ALEXANDRE-THÉOPHILE VANDERMONDE (1735–1796)** Because Alexandre-Théophile Vandermonde was a sickly child, his physician father directed him to a career in music. However, he later developed an interest in mathematics. His complete mathematical work consists of four papers published in 1771–1772. These papers include fundamental contributions on the roots of equations, on the theory of determinants, and on the knight's tour problem (introduced in the exercises in Section 10.5). Vandermonde's interest in mathematics lasted for only 2 years. Afterward, he published papers on harmony, experiments with cold, and the manufacture of steel. He also became interested in politics, joining the cause of the French revolution and holding several different positions in government.

We can prove combinatorial identities by counting bit strings with different properties, as the proof of Theorem 4 will demonstrate.

**THEOREM 4**

Let  $n$  and  $r$  be nonnegative integers with  $r \leq n$ . Then

$$\binom{n+1}{r+1} = \sum_{j=r}^n \binom{j}{r}.$$


**Proof:** We use a combinatorial proof. By Example 14 in Section 6.3, the left-hand side,  $\binom{n+1}{r+1}$ , counts the bit strings of length  $n+1$  containing  $r+1$  ones.

We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with  $r+1$  ones. This final one must occur at position  $r+1, r+2, \dots$ , or  $n+1$ . Furthermore, if the last one is the  $k$ th bit there must be  $r$  ones among the first  $k-1$  positions. Consequently, by Example 14 in Section 6.3, there are  $\binom{k-1}{r}$  such bit strings. Summing over  $k$  with  $r+1 \leq k \leq n+1$ , we find that there are

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^n \binom{j}{r}$$

bit strings of length  $n$  containing exactly  $r+1$  ones. (Note that the last step follows from the change of variables  $j = k-1$ .) Because the left-hand side and the right-hand side count the same objects, they are equal. This completes the proof.  $\triangleleft$

## Exercises

- Find the expansion of  $(x+y)^4$ 
  - using combinatorial reasoning, as in Example 1.
  - using the binomial theorem.
- Find the expansion of  $(x+y)^5$ 
  - using combinatorial reasoning, as in Example 1.
  - using the binomial theorem.
- Find the expansion of  $(x+y)^6$ .
- Find the coefficient of  $x^5y^8$  in  $(x+y)^{13}$ .
- How many terms are there in the expansion of  $(x+y)^{100}$  after like terms are collected?
- What is the coefficient of  $x^7$  in  $(1+x)^{11}$ ?
- What is the coefficient of  $x^9$  in  $(2-x)^{19}$ ?
- What is the coefficient of  $x^8y^9$  in the expansion of  $(3x+2y)^{17}$ ?
- What is the coefficient of  $x^{101}y^{99}$  in the expansion of  $(2x-3y)^{200}$ ?
- \*10. Give a formula for the coefficient of  $x^k$  in the expansion of  $(x+1/x)^{100}$ , where  $k$  is an integer.
- \*11. Give a formula for the coefficient of  $x^k$  in the expansion of  $(x^2-1/x)^{100}$ , where  $k$  is an integer.
12. The row of Pascal's triangle containing the binomial coefficients  $\binom{10}{k}$ ,  $0 \leq k \leq 10$ , is:  
1 10 45 120 210 252 210 120 45 10 1  
Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.
13. What is the row of Pascal's triangle containing the binomial coefficients  $\binom{9}{k}$ ,  $0 \leq k \leq 9$ ?
14. Show that if  $n$  is a positive integer, then  $1 = \binom{n}{0} < \binom{n}{1} < \dots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \dots > \binom{n}{n-1} > \binom{n}{n} = 1$ .
15. Show that  $\binom{n}{k} \leq 2^n$  for all positive integers  $n$  and all integers  $k$  with  $0 \leq k \leq n$ .
16. a) Use Exercise 14 and Corollary 1 to show that if  $n$  is an integer greater than 1, then  $\binom{n}{\lfloor n/2 \rfloor} \geq 2^n/n$ .  
b) Conclude from part (a) that if  $n$  is a positive integer, then  $\binom{2n}{n} \geq 4^n/2n$ .
-  17. Show that if  $n$  and  $k$  are integers with  $1 \leq k \leq n$ , then  $\binom{n}{k} \leq n^k/2^{k-1}$ .
18. Suppose that  $b$  is an integer with  $b \geq 7$ . Use the binomial theorem and the appropriate row of Pascal's triangle to find the base- $b$  expansion of  $(11)_b^4$  [that is, the fourth power of the number  $(11)_b$  in base- $b$  notation].
19. Prove Pascal's identity, using the formula for  $\binom{n}{r}$ .
20. Suppose that  $k$  and  $n$  are integers with  $1 \leq k < n$ . Prove the **hexagon identity**  

$$\binom{n-1}{k-1} \binom{n}{k+1} \binom{n+1}{k} = \binom{n-1}{k} \binom{n}{k-1} \binom{n+1}{k+1},$$
which relates terms in Pascal's triangle that form a hexagon.