BLOCK III
FUNCTIONS

UNIT 7 FUNCTIONS - I

NOTES

Structure

7.0 INTRODUCTION

Functions are widely used in science, and in most fields of mathematics. Logically, a function is a process that associates each element of a set X, to a single element of a set Y. In mathematics, the term '*Function*' is referred as an expression, rule, or law that defines a relationship between one variable (the independent variable) and another variable (the dependent variable). Functions are ubiquitous in mathematics and are essential for formulating physical relationships in the sciences.

Basically, a function is a binary relation between two sets that associates every element of the first set to exactly one element of the second set. Typical examples are functions from integers to integers, or from the real numbers to real numbers. Functions elaborate the conditions of how a varying quantity depends on another quantity. For example, the position of a satellite is a *function* of

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time. Historically, the concept was elaborated with the infinitesimal calculus at the end of the 17th century, and until the 19th century, the functions that were considered were differentiable, i.e., they had a high degree of regularity. The concept of a function was formalized at the end of the 19th century in terms of set theory, and this greatly enlarged the domains of application of the concept. If the function is called f, this relation is denoted by $y = f(x)$ and which reads "f of x", where the element x is the *argument* or *input* of the function, and y is the *value of the function*, the *output*, or the *image* of x by f. The symbol that is used for representing the input is the variable of the function, for example f is a function of the variable x. Functions are also called *maps* or *mappings*, though some authors specify some distinction between 'Maps' and 'Functions'. A *function* is a rule that assigns each input exactly one output. We call the output the *image* of the input. The set of all inputs for a function is called the *domain*. The set of all allowable outputs is called the *codomain*. Functions are widely used in science, and in most fields of mathematics. It has been said that functions are, "The central objects of investigation" in most fields of mathematics.

An inverse function or anti-function is a function that 'Reverses' another function, for example if the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the result x, i.e., $g(y) = x$ if and only if $f(x) = y$. The inverse function of f is also denoted as f^{-1}.

A recursive definition of a function defines values of the function for some inputs in terms of the values of the same function for other (usually smaller) inputs. Properties of recursively defined functions and sets can often be proved by an induction principle that follows the recursive definition.

A binary function, also called bivariate function, or function of two variables, is a function that takes two inputs. Definitely stated, a function f is binary if there exists sets X, Y, Z such that $f: X \times Y \rightarrow Z$, where $X \times Y$ is the Cartesian product of X and Y. A n-ary function is a function with exactly n arguments or alternatively we can state that a n-ary function is a function which takes any number of arguments, or a variable number of arguments. Basically, the term n-ary means n operands or parameters. The arity of a relation (or predicate) is the dimension of the domain in the corresponding Cartesian product. A function of arity n thus has arity $n + 1$ considered as a relation. Arity is the number of arguments or operands taken by a function or operation in logic, mathematics, and computer science.

In this unit, you will study about the concept of functions, inverse function, recursively defined functions, functions with their limits, and binary and n-ary operations on functions.

7.1 OBJECTIVES

After going through this unit, you will be able to:

- Discuss the significance of functions in mathematics

- Give the definition of function
- Understand about the composition of functions
- Elaborate on the inverse function
- Explain what recursively defined functions are
- Describe the functions and their limits
- Comprehend on binary operations of a function
- Know about the n-ary operations of a function

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7.2 FUNCTIONS: DEFINITION AND COMPOSITION

In mathematics, a function is a binary relation between two sets that associates every element of the first set to exactly one element of the second set. Typical examples are functions from integers to integers, or from the real numbers to real numbers.

Functions were originally the idealization of how a varying quantity depends on another quantity. For example, the position of a planet is a *function of time*. Historically, the concept was elaborated with the infinitesimal calculus at the end of the 17th century, and, until the 19th century, the functions that were considered were differentiable, i.e., they had a high degree of regularity. The concept of a function was formalized at the end of the 19th century in terms of set theory, and this greatly enlarged the domains of application of the concept.

Typically, a function is a process or a relation that associates each element x of a set X, the domain of the function, to a single element y of another set Y (possibly the same set), the codomain of the function. It is customarily denoted by letters, such as f, g and h.

If the function is called f, this relation is denoted by $y = f(x)$ and which reads "f of x", where the element x is the *argument* or *input* of the function, and y is the *value of the function*, the *output*, or the *image* of x by f. The symbol that is used for representing the input is the variable of the function, for example f is a function of the variable x. Functions are also called *maps* or *mappings*, though some authors specify some distinction between 'Maps' and 'Functions'. A *function* is a rule that assigns each input exactly one output. We call the output the *image* of the input. The set of all inputs for a function is called the *domain*. The set of all allowable outputs is called the *codomain*. Functions are widely used in science, and in most fields of mathematics. It has been said that functions are, "The central objects of investigation" in most fields of mathematics.

Characteristically, a function is a relation from a set of inputs to a set of possible outputs where each input is related to exactly one output. This means that if the object x is in the set of inputs (called the *domain*) then a function f will map

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the object x to exactly one object $f(x)$ in the set of possible outputs (called the *codomain*).

As per Encyclopaedia Britannica, "Function, in mathematics, can be referred as an expression, rule, or law that defines a relationship between one variable (the independent variable) and another variable (the dependent variable)". Functions are ubiquitous in mathematics and are essential for formulating physical relationships in the sciences.

The modern definition of function was first given in 1837 by the German mathematician **Peter Dirichlet**, "If a variable y is so related to a variable x that whenever a numerical value is assigned to x, there is a rule according to which a unique value of y is determined, then y is said to be a **function** of the independent variable x".

This relationship is commonly symbolized as $y = f(x)$. In addition to $f(x)$, other abbreviated symbols, such as $g(x)$ and $P(x)$ are often used to represent functions of the independent variable x, especially when the nature of the function is unknown or unspecified.

7.2.1 Functions

A function or mapping from set A to set B is a 'Method' that pairs elements of set A with unique elements of set B and you denote $f : A \to B$ to indicate that f is a function from set A to set B.

B is called the *codomain* of the function f and A is called its *domain*. Also, for each element a of A, f defines an element b of B. Write it as $a \xrightarrow{f} f(a)$ or $a \xrightarrow{f} b$, $a \in A$, $b \in B$.

For example,

(*i*) The relation $f = \{(1, d), (2, c), (3, a)\}$ from $A = \{1, 2, 3\}$ to $B = \{a, c, d\}$ is a function from A to B. The domain of f is A and the codomain of f is B.

(*ii*) The relation $f = \{(a, b), (a, c), (b, d)\}$ from $A = \{a, b\}$ to $B = \{b, c, d\}$ is not a function.

Range of Function: Let $f : A \to B$ be a function. The range of the function $R(f)=\{f\,(a : a \in A\}$. (Note that $R\,(f\,) \subseteq B$).

Notes:

1. From above example: $R(f\,)$ is $\{d, c, a\}$,
2. Let $f : R \to R^+$ be $f(x) = x^2$ (R^+, the set of positive real numbers). Clearly, f is a function whose domain is the set of real numbers and the codomain is the set of positive real numbers.

$$R(f\,) \;=\; \{x^2 : x \in R\} = \{1, 4, 9,.....\}$$

Let $f : A \to B$ be a function f is said to be:

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- ***One-to-One* (1–1) *Function:*** If $x_1 \neq x_2$ then, $f(x_1) \neq f(x_2)$, $\forall\, x_1, x_2 \in A$.

  or

  Whenever $f(x_1) = f(x_2)$ then, $x_1 = x_2$. This function is also known as injective function.
- ***Onto Surjective Function:*** If for every element y in the codomain B, atleast one element x in the domain A such that $f(x) = Y$.

  or

  If $R(f)$ = Codomain B.
- ***Bijective Function:*** If f is both 1–1 and onto function.
- ***Constant Function:*** If every element of the domain is mapped to a unique element of the codomain or the codomain consists of only one element.
- ***Into Function:*** If atleast one element of the codomain is not mapped by any element of the domain.
- ***Identify Function:*** If $f(x) = x$, $\forall\, x \in B$, in this case $A \leq B$. Sometimes, it is defined as $f : A \to A$ and $f(x) = x$, $\forall\, x \in A$.

For example,

1. Let $f : R \to R$ be a function defined as $f(x) = 2(x + 2)$: Clearly, f is $1 - 1$ because if $2(x + 2) = 2\,(y + 2)$

$$\begin{aligned} &\Rightarrow\quad 2x + 4 = 2y + 4 \\ &\Rightarrow\quad 2x = 2y \Rightarrow x = y \\ &\therefore\quad f \text{ is } 1 - 1. \end{aligned}$$

2. Define $f : R \to R^+$ by $f(x) = e^x$, $\forall\, x \in R$. Clearly, f is $1 - 1$ because if $f(x_1) = f(x_2)$

$$\begin{aligned} \Rightarrow\quad e^{x_1} &= e^{x_2} \\ \Rightarrow e^{x_1 - x_2} &= 1 \\ \Rightarrow x_1 - x_2 &= 0 \\ \Rightarrow\quad x_1 &= x_2 \\ \therefore\quad f &\text{ is } 1 - 1. \end{aligned}$$

3. Let $A = \{5, 6, 7\}$ and $B = \{a, b\}$. Then the mapping $f : A \to B$ is defined as $f(5) = a$; $f(6) = b$; $f(7) = a$. Clearly, f is not $1 - 1$. But f is onto.
4. Consider the Example (2). If $f : R \to R^+$, defined by $f(x) = e^x$ is onto. Let x be any element IR^+, then $\log y \in IR$ such that $f(\log y) = e^{\log y} = y$.
5. Define $f : Z^+ \to Z^+$ as $f(n) = n^2$, $\forall\, n \in Z^+$. Clearly, f is an into mapping (not mapped by any element of Z^+) and $1 - 1$ mapping but f is not onto.
6. Define $f : Z \to Z$ by $f(n) = n + 1$ $\forall\, n \in Z$. Clearly, f is $1 - 1$ and onto. For if, (i) $f(n) = f(m) \Rightarrow n + 1, = m + 1 \Rightarrow n = m \therefore f$ is $1 - 1$.
7. If n is any element of Z, then $n - 1 \in Z$ such that $f(n - 1) = n - 1 + 1 = n$. Hence f is onto.

Note: A one-one mapping of a set S onto itself is sometimes called a permutation of the set S.

7.2.2 Inverse Function

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Let f be a bijective function from the set A to the set B. The inverse function of f is the function that is assigned to an element $b \in B$ the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence $f^{-1}(b) = a$, when $f(a) = b$ (Refer Figure 7.1).

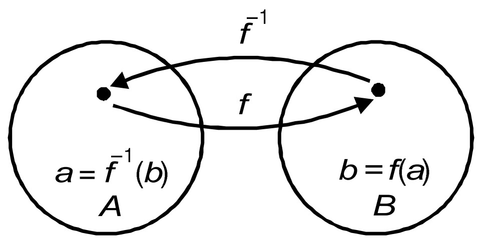

Fig. 7.1 *Inverse Function*

The function f^{-1} is the inverse function of f.

Note: A bijective function is called invertible since it can be defined as an inverse of this function.

Example 7.1:

(*i*) Define $f : Z \to Z$ by $f(n) = n + 1$. Is f invertible, and if it is what is its inverse?

Solution: The function f has an inverse, since it is a bijective function. Let y be the image of x, so that $y = x + 1$. Then $x = y - 1$, i.e., $y - 1$ is the unique element of Z that is sent to y by f. Hence $f^{-1} = y - 1$.

(*ii*) Let $A = \{a, b, c\}$, and $B = \{5, 6, 7\}$. Define $F: A \to B$ as $f(a) = 5; f(b) = 6; f(c) = 7$. Is f invertible, and if it is what is its inverse?

Solution: Clearly the given function is bijective. The inverse function f^{-1} of f is given as $f^{-1}(5) = a; f^{-1}(6) = b; f^{-1}(7) = c$.

(*iii*) Define $f : Z \to Z$ by $f(x) = x^2$. Is f invertible?

Solution: Since $f(-2) = f(2) = 4$, f is not 1–1. If an inverse function were defined, it would have to assign two elements to 2. Hence f is not invertible.

7.2.3 Compositions of Functions

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g denoted by $(f \circ g)$ is given in Figure 7.2 in such a way a that:

$$(f \circ g)(x) = f(g(x)) \quad \forall\, x \in A$$

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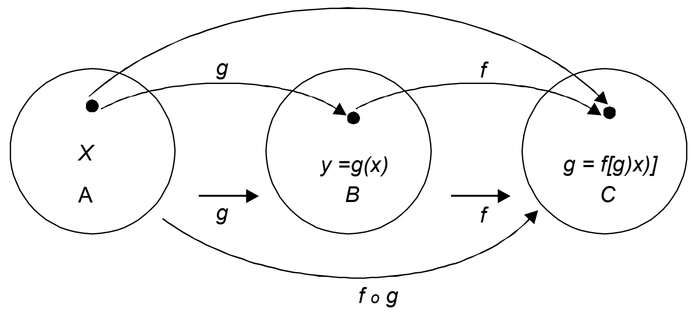

Fig. 7.2 *Composition of Function*

Example 7.2: Let $f : Z \to Z$ be a function defined by $f(x) = 2x + 3$. Let $g: Z \to Z$ be a function defined by $g(x) = 3x + 2$. Find (*i*) $f \; o \; g$ (*ii*) $g \; o \; f$.

Solution: Both $f \; o \; g$ and $g \; o \; f$ are defined. Further,

(*i*) $(f \; o \; g)\,(x) = f(g(x)) = f(3x + 2)$

$= 2(3x + 2) + 3 = 6x + 7$

(*ii*) $(g \; o \; f)\,(x) = g(f(x)) = g(2x + 3)$

$= 3(2x + 3) + 2 = 6x + 11$

Eventhough $f \; o \; g$ and $g \; o \; f$ are defined, $f \; o \; g$ and $g \; o \; f$ need not be equal, i.e., the commutative law does not hold for the composition of functions.

Example 7.3: Let $A = \{1, 2, 3\}$, $B = \{x, y\}$, $C = \{a\}$. Let $f: A \to B$ be defined by $f(1) = x$; $f(2) = y$; $f(3) = x$. Let $g: B \to C$ be defined by $g(x) = a$; $g(y) = a$.

Find (*i*) $f \; o \; g$, if possible (*ii*) $g \; o \; f$, if possible.

Solution: The solution is obtained as follows:

(*i*) $(f \; o \; g)\,(x) = f(g(x))$, but f cannot be applied on C and hence $f \; o \; g$ is meaningless.

(*ii*) $(g \; o \; f) : A \to C$ is meaningful. Now $(g \; o \; f)\,(x) = g(f(x))$, $\forall \; x \in A$.

$\therefore \quad (g \; o \; f)\,(1) = g(f(1)) = g(x) = a$

$(g \; o \; f)\,(2) = g(f(2)) = g(y) = a$

$(g \; o \; f)\,(3) = g(f(3)) = g(x) = a$

Result: If $f : A \to B$, $g: B \to C$ and $h: C \to D$, then $(h \; o \; g) \; o \; f = h \; o \; (g \; o \; f)$.

Proof: $[(h \; o \; g) \; o \; f]\,(x) = (h \; o \; g)\,(f(x))$

$= h[(g(f(x))]$ ...(7.1)

and $[h \; o \; (g \; o \; f)](x) = h[(g \; o \; f)\,(x)]$

$= h[g(f(x))]$ ...(7.2)

From Equations (7.1) and (7.2), $(h \; o \; g) \; o \; f = h \; o \; (g \; o \; f)$

Result: Let $f : A \to B$ and $g : B \to C$. Then,

(*i*) $g\ o\ f$ is onto, if both f and g are onto.

(*ii*) $g\ o\ f$ is $1 - 1$, if both f and g are $1 - 1$.

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Proof:

(*i*) Let $z \in C$. Since $g : B \to C$ is onto, an element $y \in B$ such that $g(y) = z$. Since $f : A \to B$ is onto, for an element $x \in B$, such that $f(x) = y$.

$\therefore\ (g\ o\ f)\ (x) = g(f(x)) = g(y) = z.$

$\therefore\ (g\ o\ f)$ is onto.

(*ii*) Let $x_1 \neq x_2$ be two elements in A. Since $f : A \to B$ is one-one, and $f(x_1) \neq (x_2)$, $g(f(x_1)) = g(f(x_2))$. Thus, $g\ o\ f$ is one-one. In B, since g: $B \to C$ is one-one and $f(x_1) \neq (x_2)$.

Check Your Progress

1. What do you understand by the function?
2. State the range of function.
3. What is one-to-one function?
4. Define onto surjective function and bijective function.
5. Give the definition of constant function.
6. What is into function?
7. What do you mean by the identify function?
8. State the inverse function.
9. Elaborate on the compositions of functions?

7.2.4 Some Important Functions

Greatest Integer Function: The greatest integer function is assigned to the real number x the largest integer that is less than or equal to x, and the values of this function are denoted by $[x]$ (or $\lfloor x \rfloor$). This is also known as floor function.

Ceiling Function: The ceiling function assigned to the real number x is the smallest integer that is greater than or equal to x. The value of this function is denoted by $\lceil x \rceil$.

For examples,

1. $\lfloor x \rfloor = \lfloor 1/2 \rfloor = 0$; $\lceil x \rceil = \lceil 1/2 \rceil = 1$
2. $\lfloor 5.6 \rfloor = 5$; $\lceil 5.6 \rceil = 6$
3. $\lfloor 4.1 \rfloor = 4$; $\lceil 4.1 \rceil = 5$
4. $\lfloor 3 \rfloor = 3$; $\lceil 3 \rceil = 3$

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Example 7.4: Data stored on a computer disk or transmitted over a data network are represented as a string of bytes. How many bytes are required to encode 500 bits of data?

Solution: To find the number of bytes needed, you determine the smallest integer that is atleast as large as the quotient when 500 is divided by 8, the number of bits in a byte.

$$\left\lceil \frac{500}{8} \right\rceil = \lceil 62.5 \rceil = 63 \text{ bytes are required.}$$

Example 7.5: In ATM (Asynchronous Transfer Mode), data are organized into cells of 53 bytes. How many ATM cells can be transmitted in 2 minutes over a connection that transmits data at the rate of 200 KB per second.

Solution: In 2 minutes, this connection can transmit 200000 × 60 × 2 = 2,40,00,000 bits, as each ATM cell is 53 bytes long, i.e., 53 × 8 = 424 bits long.

Number of ATM cells that can be transmitted in 2 minutes in the given connection is,

$$\left\lceil \frac{2,40,00,000}{424} \right\rceil = \lceil 56603.77 \rceil = 56604$$

Modulus Operator: If x is a non-negative integer and y is a positive integer, you define x mod y to be the remainder when x is divided by y.

Refer to the following examples.

1. 11 mod 2 = 1; 5 mod 1 = 0; 365 mod 7 = 1.
2. To find the day of the week after 365 days from Friday, it is calculated in following way:

   365 mod 7 = 1. Thus 365 days from Friday, it will be Saturday. Since 7 days after Friday is Friday. In general if $K > O$, $K \in Z$, after $7K$ days it is Friday again.
3. Another important application of mod operator is ISBN (International Standard Book Number). ISBN is a code of 10 characters separated by dashes such as 0-333-40736-7. It consists of four parts, a group code, a publisher code, a code that identifies the books among those books published by the particular publisher uniquely and a check character. This check character is used to validate an ISBN.

   For example, 0-333-40736-7, the group code is 0, which identifies the book as one from an English speaking country. The publisher code 333 identifies the book as one published by Macmillan. The code 40736 uniquely identifies the book among those published by Macmillan. The check character is n mod 11, where n is the sum of the 1st digit plus two times the 2nd digit plus three times the 3rd digit,... nine times the ninth digit. If the value is 10, the check characters is x. In our example,

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$n = 0 + 2.3 + 3.3 + 4.3 + 5.4 + 6.0 + 7.7 + 8.3 + 9.6 = 0 + 6 + 9 + 12 + 20 + 0 + 49 + 24 + 54 = 174$

$\therefore$ n mod 11 = 174 mod 11 = 7

4. ISBN is 0-07-003575-x. Here 0-stands for the book as one from an English-speaking country. The publisher code 07 identifies the book as one published by McGraw Hill. The code 003575 uniquely identifies the book as one published by McGraw Hill. The check character is n mod 11, where,

   $n = 0 + 2.0 + 3.7 + 4.0 + 5.0 + 6.3 + 7.5 + 8.7 + 9.5 = 0 + 0 + 21 + 0 + 0 + 18 + 35 + 56 + 45$

   $n = 175$

   $\therefore$ n mod 11=175 mod 11=10

   $\therefore$ The check character is 10, i.e., it is represented as x.

Hashing Function: The central computer (server) at an university maintains records of every student. How many memory locations can be alloted so that the student records can be retrieved fast? The solution to this problem is to use suitably chosen hashing function. Records are identified by using a key, which uniquely identifies each student's record. For example, student records are identified by using the register number/enrolment number of the student as the key. A hashing function h assigns memory location $h(p)$ to the record that has p as its key.

The most common hashing function is the function $h(p) = p$ mod q, where q is the number of available memory locations. The hashing function should be *onto* so that all memory allocations are possible. For example, $q = 11$, the record of the student with register number 15 is assigned to memory location 4, since $h(15) = 4$ mod 11. Similarly, since $h(122) = 122$ mod 11 = 1, the record of the student with register number 121 is assigned to memory location 1.

Since a hashing function is not 1–1, more than one file may be assigned to a memory location. When this happens, it is said that a collision occurs. To resolve a collision assign the first free location following the occupied memory location assigned by the hashing function. For example, after making two earlier assignments location 5 is assigned to the record of the student with register number 256.

7.2.5 Recursive Functions

In this section, first a class of functions are defined inductively and shown that such functions can be evaluated in a purely mechanical manner. You will restrict yourselves to only those functions whose arguments and values are natural numbers.

A function is said to be recursive if it can be obtained from the initial functions by a finite number of applications of the operations of composition, recursion and minimization over regular function.

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Total and Partial Functions: Any function $f : N^n = N \times N \times ... \rightarrow N$ is called *total function*, if it is defined for every n-tuple in N^n. If $f : D \rightarrow N$ is defined, where $D \leq N^n$, then f is called *partial function*.

For example, if $f : N^n \rightarrow N$ by $f(x, y) = x + y$, then f is a total function. If $g : N \times N \rightarrow N$ by $g(xy) = x - y$, then g is a partial function, since it is defined only for x, y when $x, y \in N$.

Total Function: A function $f(x_1, x_2,..., x_n)$ is said to be defined as total function $g(x_1, x_2,..., x_n, y)$ by minimization if,

$$f(x_1,\ x_2,\ ...,\ x_n) = \begin{cases} \mu_y\left(g(x_1, x_2, ..., x_n, y) = 0\right) \\ \text{Undefined} \end{cases}$$

Otherwise, if there exists such a y. Where μ_y means the least y greater than or equal to zero.

Partial Recursive Function: A function is said to be partial recursive, if it can be obtained from the initial functions by a finite number of applications of the operations of composition, recursion and minimization.

Zero Function: A function $Z : N \rightarrow N$ given by $Z(x) = 0$, for all $x \in N$, is called a *zero function*.

Successor Function: A function $S : N \rightarrow N$ given by $S(x) = x + 1$ for all $x \in N$, is called *successor function*.

Projection Function: A function $U_i^n : N^n \rightarrow N$ given by,

$U_i^n(x_1, x_2, ..., x_i, ... x_n) = x_i$, for all $x_i \in N$, $i = 1, 2, ... n$, is called *projection function*.

These three functions are known as *initial functions*.

Now, you extend the definition of composition of functions for more than one variable.

Let $f_1 : N \times N \rightarrow N, f_2 : N \times N \rightarrow N$ and $g : N \times N \rightarrow N$ be any three functions.

When $h: N \times N \rightarrow N$, the composition of g with f_1 and f_2 is given by,

$h(x, y) = g(f_1(x, y),\ f_2(x, y))$ for all $x, y \in N$.

Here, you assume that $R_{f_1} \times R_{f_2} \leq D_g$ and $D_h = D_{f_1} \cap D_{f_2}$

For example, define $f_1: N \times N \rightarrow N$ by $f_1(x, y) = x + y$,

$f_2: N \times N \rightarrow N$ given by $f_2(x, y) = xy + y^2$ and $g : N \times N \rightarrow N$ by $g(x,y) = xy$. Then $h : N \times N \rightarrow N$ is,

$$\begin{aligned} h(x, y) &= g(f_1(x, y)), (f_2(x, y)) \\ &= g(x + y, xy + y^2) = (x + y)(xy + y^2) \end{aligned}$$

Similarly, you can extend this for more variables.

Given a function $f(x_1, x_2,..., x_n)$ of n-variables, it is convenient to consider $n-1$ of the variables as fixed and vary only the remaining variable over the set of N. For example, suppose $f : N \times N \to N$ is a function defined by $f(x, y) = x + y$. To compute $f(x, y)$, x is treated as fixed and vary f. Suppose $f(2, 0) = 2$ is given, to find $f(2, 3)$, you proceed as:

$$
\begin{aligned}
f(2, 3) &= [(f(2, 0) + 1) + 1] + 1 \\
&= [(2 + 1) + 1] + 1 = [3 + 1] + 1 \\
&= 4 + 1 = 5
\end{aligned}
$$

In general, given a function $f(x_1, x_2,..., x_n)$ of n-variables, consider $n-1$ of the variables as fixed and vary only the remaining variable over N. The fixed $n-1$ variables are called parameters.

Recursive Function: Consider the known functions $g(x_1, x_2,...,x_n)$ and $h(x_1, x_2,..., x_n, y, z)$ of n and $n + 2$ variables. Define a function $f(x_1, x_2,..., x_n, y)$ of $n + 1$ variables by $f(x_1, x_2,..., x_n, 0) = g(x_1, x_2,...,x_n)$ and $f(x_1, x_2,..., x_n, y + 1) = h(x_1, x_2,..., x_n, y, f(x_1, x_2,..., x_n, y)$.

Then, f is called a *recursive function* or *recursion*.

Note: Here, the value of f at $y + 1$ is expressed in terms of the value of f at y. The variables $x_1, x_2,..., x_n$ are treated as parameters.

Primitive Recursive: A function f is called *primitive recursive*, if it can be obtained from the initial functions by a finite number of operations of composition and recursion.

Example 7.6: Show that the function $f(x, y) = x + y$ is primitive recursive. Use it to compute $f(2, 3)$.

Solution: Since,

$x + (y + 1) = (x + y) + 1, f(x, y + 1) = f(x, y) + 1 = S(f(x, y))$

Also, $f(x, 0) = x$

Now we define $f(x, y)$ as,

$f(x, 0) = x = \cup_1'(x)$

$f(x, y + 1) = S(\cup_3{}^3(x, y, f(x, y))$

Therefore, f is primitive recursive.

Now,

$$
\begin{aligned}
f(2, 3) &= S(f(2, 2)) \\
&= S(S(f(2, 1)) \\
&= S(S(S(f(2, 0)))) \\
&= S(S(S(2))) \quad \text{Since } f(2, 0) = 2. \\
&= S(S(3)) \\
&= S(4) \\
&= 5
\end{aligned}
$$

NOTES

4. Classification by Complexity: Another way to classify an algorithm is by the amount of time needed to complete it as compared to their input size. There are many varieties of algorithms that complete linear time as relative to its input size. Few do so in an exponential amount of time and some may take extra ordinary time and some never halt. A problem may require multiple algorithms of differing complexities.

5. Classification by Computing Power: An algorithm may be classified according to its computing power. This is done by considering some collection (class) of algorithms. Algorithms that run in polynomial time suffice for many important types of computation.

Check Your Progress

20. Give the definition of cryptology.
21. What do you understand by the decryption?
22. Define the exponential function.
23. State logarithmic function.
24. What are the uses of logarithms?

7.3 INVERSE FUNCTION

Let f be a function defined as $A \to B$, then an inverse function for f would be function in the opposite direction. This is denoted as f^{-1} and defined as $B \to A$. This has the property that says, a function $A \to B \to A$ (from B to A and back to B) returns each element of the initial set to itself. When an input x is put into the function f, it produces an output y. And putting y as input in f^{-1} produces x as output. Such functions are called *invertible*. Every function does not have an inverse. For a function to be inverse, it has to be bijective, i.e., one-one and onto.

Thus, f maps $X \to Y$, and f^{-1} maps $Y \to X$.

Domain of function f is X and its range, given by Y = Codomain.

The function f^{-1} having domain Y and range X = Codomain, is the inverse of f.

Thus, in other words, a function is invertible only when its inverse is also a function. Thus, an inverse relation is obtained by interchanging x and y everywhere. Following two conditions must be satisfied:

- Function should be one-to-one, i.e., an injection.
- Function should be onto, i.e., a surjection which means, Codomain = Range.

Real value functions are dealt with in elementary mathematics and for this reason domain is taken as a set of real numbers. Range is the image of the domain. Every function does not have an inverse, since bijection is not found in every function.

The function, defined as $f(x) = x^2$ may not be invertible since for two values of x (one positive and another negative) function has only one value. The function can be made invertible by restricting the domain. This can be understood by the following example:

In the relation $f(x) = x^2$, if $f(x)$ is denoted by y and it is written as $y = x^2$, you will find two values of x for a single value of y. Here, $x = \text{sqrt}(y)$ and this function would be valid only if y has positive values. Since $(-x)^2 = (x)^2$, it is not clear as to which value should be taken. Such conditions do not define an inverse. But if the definition is restricted and only positive values are taken for x, then the function becomes invertible. Thus, for a function like this, it is possible to define partial inverse only.

For trigonometric functions also, domain has to be restricted to define inverse of trigonometric functions (Refer Table 7.1). Let $f(x) = \sin x$. This function, called sine function, is not one-to-one. For more than one value of x, one value of $\sin x$ is there. For every given value of $\sin x$ value of x is not unique and the function is not one-to-one. Thus, the value of $\sin(x)$ is repeated and is a periodic function. But sine function is one-to-one in the interval from $[-\frac{\pi}{2}$ to $\frac{\pi}{2}]$. Thus, if domain is restricted, then its inverse can also be defined and such an inverse is called partial inverse. It is also denoted as arcsine. The restricted domain is the principal value of arcsine.

Table 7.1 *Trigonometric Functions*

Function	*Range of Usual Principal Value*
$\sin^{-1}$	$-\pi/_2 \leq \sin^{-1}(x) \leq \pi/_2$
$\cos^{-1}$	$0 \leq \cos^{-1}(x) \leq \pi$
$\tan^{-1}$	$-\pi/_2 < \tan^{-1}(x) < \pi/_2$
$\cot^{-1}$	$0 < \cot^{-1}(x) < \pi$
$\sec^{-1}$	$0 < \sec^{-1}(x) < \pi$
cosec^{-1}	$-\pi/_2 \leq \text{cosec}^{-1}(x) < \pi/_2$

Confusion might arise while writing inverse function. Expressions, $f^{-1}(x)$ and $f(x)^{-1}$ are not the same. The expression $f(x)^{-1}$ is the multiplicative inverse of $f(x)$ and $f^{-1}(x)$ is not. Same is the case with $(\sin x)^{-1}$ and $\sin^{-1}x$. For this reason it is preferred to write arcsine instead of $\sin^{-1}x$.

Properties of Inverse Functions

Uniqueness: Inverse function for a given function f, if exists, is unique.

Symmetry: A function and its inverse are symmetric to each other on the line of identity function, $y = x$.

NOTES

Inverse of a Composition

Figure 7.4 illustrates the concept of inverse of a composition.

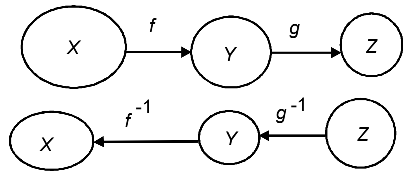

Fig. 7.4 *Inverse of a Composition*

If a composition of function is given by the formula $g \circ f$, then its inverse is,

$f^{-1} \circ g^{-1}$. Given mathematically, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

For example, a function defined as $f(t) = t + 7$, and let $g(t) = 2t$. Then, $f \circ g$ is the function, $f(g(t)) = f(2t) = 2t + 7$. Inverse of this is $(t - 7)/2$. To get its inverse, subtract 7 from this and divide it by 2. Composition is given by $(g^{-1} \circ f^{-1})\,(t)$.

$f^{-1}(t) = t - 7$

and $g^{-1}(t) = t/2$. $(g^{-1} \circ f^{-1})\,(t) = g^{-1}(f^{-1}) = g^{-1}(t - 7) = (t - 7)/2$.

This proves that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Self-Inverse: Let X be a set and $x \in X$. The identity function is defined by $id^{-1}(x) = id(x)$. Thus, identity function is an inverse of itself.

Inverses in Calculus: Calculus, dealing with single variable, deals with functions in which mapping is $R \to R$, i.e., domain and codomain both are a set of real numbers.

These are defined through formulas, like: $f(x) = (ax + b)^3$, where a and b are real numbers. A function $R \to R$ has an inverse as long as it is one-to-one, i.e., the graph of the function confirms to a horizontal line test. A horizontal line test is done by drawing a line parallel to the X-axis and to see that this line cuts the graph of the function at only one point. If it cuts the graph at only one point, it is one-to-one, otherwise not.

Table 7.2 lists few standard functions vis-à-vis their inverses:

Table 7.2 *Few Standard Functions*

Function $f(x)$	***Inverse*** $f^{-1}(y)$	***Notes***
$x + a$	$y - a$	
$a - x$	$a - y$	
mx	y / m	$m \neq 0$
$1 / x$	$1 / y$	$x, y \neq 0$
x^2		$x, y \geq 0$ only
x^3		no restriction on x and y
x^p	$y^{1/p}$	$x, y \geq 0$ in general, $p \neq 0$
e^x	$\log y$	$y > 0$
a^x	$\log_a y$	$y > 0$ and $a > 0$

NOTES

Finding the Inverse of a given Function: To find f^{-1}, if it exists, one should solve the equation $y = f(x)$ for x, as follows:

Let f be the function as given below:

$f(x) = (2x + 8)^3$

First, solve the equation $y = (2x + 8)^3$ for x

$$y = (2x+8)^3$$
$$\sqrt[3]{y} = 2x + 8$$
$$\sqrt[3]{y} - 8 = 2x$$
$$\frac{\sqrt[3]{y} - 8}{2} = x.$$

Thus, the inverse function f^{-1} is given by the formula $f^{-1}(y) = \dfrac{\sqrt[3]{y-8}}{2}$. By changing variable from y to x, you get $f^{-1}(x) = \dfrac{\sqrt[3]{x-8}}{2}$

Hence, to obtain inverse of a function, follow following steps:

Step 1. Put y for $f(x)$ and express x as a function of y, i.e., $x = f(y)$.

Step 2. Exchange variables, i.e., y in place of x and x in place of y to get $f^{-1}(x)$.

There are instances when a function is one-to-one, yet their inverse can not be given by a formula. To explain this, let us take a function defined as $f(x) = x + \sin x$. This is one-to-one, but it can not be solved algebraically for x and hence has no simple formulae for this.

Graph of the Inverse: If there is a function f having f^{-1} as its inverse, the graph of both these functions are identical except that the role of x and y has been reversed. Graph of f^{-1} can be obtained from that of f by interchanging x and y axes. Thus, an inverse of a function is the mirror image of that function on the line $y = x$ which is an identity function (Refer Figure 7.5).

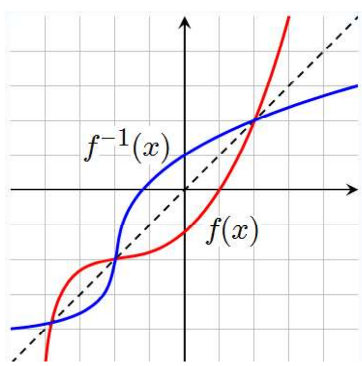

Fig. 7.5 *Graph of Inverse*

In this figure, graphs of $y = f(x)$ and $y = f^{-1}(x)$ is shown which is symmetrical about the dotted line $y = x$.

Inverses and Derivatives

NOTES

A function f is continuous and monotonic, i.e., it is either strictly increasing or decreasing without any local maxima or minima, then it is one-one and hence invertible. Let us take an example of the function, $y = x^3 + x$. This is invertible, as its derivative $f'(x) = 3x^2 + 1$ is always positive. An inverse function f^{-1} is differentiable, if the function f is differentiable provided $f'(x) \neq 0$.

$$\frac{d}{dy}\left[f^{-1}(y)\right] = \frac{1}{f'\left(f^{-1}(y)\right)}$$

If you set $x = f^{-1}(y)$, then the formula above can be written as,

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

Preimages

Let $f: X \rightarrow Y$ be a function, which may or may not be invertible and $y \in Y$, then the preimage of elements of set Y is the set of all elements of X that maps to Y. The preimage of y is the image of y under the full inverse of the function f.

In the same way, let S be a subset of Y. Then the preimage of S is the set of all elements of X, mapped to S.

If a single element y is taken, then $y \in Y$, its preimage is known as the fiber of y.

If $y \in$ R, then $f^{-1}(y)$ is referred to as a level set.

Example 7.33: A function is defined as $f(x) = 3x - 2$. Find its inverse.

Solution: $f^{-1}(x) = (x + 2)/3$.

Example 7.34: Find the inverse of the function defined as, $f(t) = -t^2 + 2$, $t <= 2$.

Solution: $f^{-1}(t) = \sqrt{2-t}$, $t <= 2$.

Example 7.35: Find the inverse of the quadratic function defined as,

$f(t) = t^2 - 2t$, $t >= -1$.

Solution: $f^{-1}(t) = 1 + \sqrt{t+1}$ where $t >= -1$.

Example 7.36: A function is defined as $f(x) = 2/x$, where $x \neq 0$. Find its inverse.

Solution: $f^{-1}(x) = 2/x$.

Example 7.37: Find the inverse of a function having definition:

$f(x) = (x + 1)/(x - 1)$, where $x \neq 1$.

Solution: $f^{-1}(x) = (x + 1)/(x - 1)$ and $x \neq 1$.

Proofs of the following results on limits are beyond the scope of this book and so they have been omitted.

NOTES

(A) If $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$, then

1. $\lim\limits_{x \to a} [f(x) \pm g(x)] = l \pm m$
2. $\lim\limits_{x \to a} [f(x)\, g(x)] = lm$
3. $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$, provided $m \neq 0$
4. $\lim\limits_{x \to a} [f(x)]^{g(x)} = l^m$, provided l^m is defined.

(B) If $f(x) < g(x)$ for all x, then $\lim\limits_{x \to a} f(x) \leq \lim\limits_{x \to a} g(x)$.

Example 7.56: Given $f(x) = 3 - 2x + x^2$. Find $f(0), f(-2), f(5), f(-1)$.

Solution: In $f(x)$, putting $x = 0$, $f(0) = 3 - 2 \times 0 + 0 = 3$

Similarly, $f(-2) = 3 - 2(-2) + (-2)^2 = 3 + 4 + 4 = 11$

$f(5) = 3 - 2 \times 5 + 25 = 18$

$f(-1) = 3 - 2(-1) + (-1)^2 = 6.$

Example 7.57: Let $y = f(x) = ax + b$ where a, b are constants.

Find $f(a), f(-a), f\left(\dfrac{a}{b}\right), f(k), f(0), f\left(-\dfrac{b}{a}\right)$

Solution:

$$f(x) = ax + b$$
$$f(a) = a \times a + b = a^2 + b$$
$$f(-a) = -a \times a + b = -a^2 + b$$
$$f\left(\frac{a}{b}\right) = a \cdot \frac{a}{b} + b = \frac{a^2}{b} + b$$
$$f(k) = a \times k + b = ka + b$$
$$f\left(-\frac{b}{a}\right) = -a \cdot \frac{b}{a} + b = -b + b = 0$$
$$f(0) = a \times 0 + b = b.$$

7.6 BINARY AND *n*-ARY OPERATIONS

Following are the properties and definitions of binary operations and n-ary operations that are used in the context of discrete mathematics.

NOTES

7.6.1 Binary Operations

In mathematics, a **binary operation**, also termed as dyadic operation, is a calculation that combines two elements (called operands) to produce another element. More formally, a **binary operation** is an operation of **arity two**.

More specifically, a binary operation on a set is an operation whose two domains and the codomain are the same set. Examples include the familiar arithmetic operations of addition, subtraction, and multiplication. Other examples are from different areas of mathematics, such as vector addition, matrix multiplication and conjugation in groups.

Definition: An operation of arity two that involves several sets is called a binary operation.

The scalar multiplication of vector spaces takes a scalar and a vector to produce a vector, and scalar product takes two vectors to produce a scalar. Such binary operations are called simply binary functions.

More precisely, a binary operation on a set *S* is a mapping of the elements of the Cartesian product *S* ´ *S* to *S*, such that:

$$f : S \times S \to S$$

Because the result of performing the operation on a pair of elements of *S* is again an element of *S*, therefore the operation is called a closed (or internal) binary operation on *S* or sometimes expressed as having the property of closure.

If *f* is not a function, but a partial function, then *f* is called a partial binary operation. For instance, division of real numbers is a partial binary operation, because one cannot divide by zero, for example $a/0$ is undefined for every real number a. In both universal algebra and model theory, binary operations are required to be defined on all of $S \times S$.

Properties of Binary Operation

Typical examples of binary operations are the addition (+) and multiplication (×) of numbers and matrices as well as composition of functions on a single set. For example,

- On the set of real numbers $\boldsymbol{R}$, $f(a, b) = a + b$ is a binary operation since the sum of two real numbers is a real number.
- On the set of natural numbers $\boldsymbol{N}$, $f(a, b) = a + b$ is a binary operation since the sum of two natural numbers is a natural number. This is a different binary operation than the previous one since the sets are different.
- On the set M(2,**R**) of 2×2 matrices with real entries, $f(A, B) = A + B$ is a binary operation since the sum of two such matrices is a 2×2 matrix.
- On the set M(2,**R**) of 2×2 matrices with real entries, $f(A, B) = AB$ is a binary operation since the product of two such matrices is a 2×2 matrix.

- For a given set C, let S be the set of all functions $h : C \to C$. Define $f : S \times S \to S$ by $f(h_1, h_2)(c)$ = (h_1 ‘ h_2) (c) = $h_1(h_2(c))$ for all $c \in C$, the composition of the two functions h_1 and h_2 in S. Then f is a binary operation since the composition of the two functions is again a function on the set C, i.e., a member of S.

Many binary operations of interest in both algebra and formal logic are commutative, satisfying $f(a, b) = f(b, a)$ for all elements a and b in S, or associative, satisfying $f(f(a, b), c) = f(a, f(b, c))$ for all a, b and c in S. Many also have identity elements and inverse elements.

On the set of real numbers **R**, subtraction, that is, $f(a, b)$ = a ” b, is a binary operation which is not commutative since, in general, a ” b “‘ b ” a.

It is also not associative, since in general, a ” (b ” c) ‘+ (a ” b) “ c; for example, 1 + (2 + 3) = 2 but (1 + 2) + 3 = +4.

Division (/), a partial binary operation on the set of real or rational numbers, is not commutative or associative.

Notation

Binary operations, in mathematics, are often written using infix notation, such as $a * b$, $a + b$, $a \bullet b$ or (by juxtaposition with no symbol) ab rather than by functional notation of the form $f(a, b)$. Powers are usually also written without operator, but with the second argument as superscript.

Binary operations sometimes use prefix or (probably more often) postfix notation, both of which dispense with parentheses. They are also called, respectively, Polish notation and reverse Polish notation.

Pair and Tuple

A binary operation, ab, depends on the ordered pair (a, b) and so $(ab)c$, where the parentheses here mean first operate on the ordered pair (a, b) and then operate on the result of that using the ordered pair $((ab), c)$, depends in general on the ordered pair $((a, b), c)$. Thus, for the general, non-associative case, binary operations can be represented with binary trees.

However,

- If the operation is associative, $(ab)c = a(bc)$, then the value of $(ab)c$ depends only on the tuple (a, b, c).
- If the operation is commutative, $ab = ba$, then the value of $(ab)c$ depends only on $\{\{a, b\}, c\}$, where braces indicate multisets.
- If the operation is both associative and commutative then the value of $(ab)c$ depends only on the multiset $\{a, b, c\}$.
- If the operation is associative, commutative and idempotent, $aa = a$, then the value of $(ab)c$ depends only on the set $\{a, b, c\}$.

7.6.2 *n*-Ary Operations

NOTES

The arity of a relation (or predicate) is the dimension of the domain in the corresponding Cartesian product. A function of arity n thus has arity $n + 1$ considered as a relation. Arity is the number of arguments or operands taken by a function or operation in logic, mathematics, and computer science.

Definition: A ***n*-ary function** is a function with exactly n arguments or alternatively we can state that a n-ary function is a function which takes any number of arguments, or a variable number of arguments.

Basically, the term ***n*-ary** means ***n* operands** or **parameters**.

The term 'Arity' is rarely employed in everyday usage. For example, rather than saying 'The arity of the addition operation is 2' or 'Addition is an operation of arity 2' we generally say that 'Addition is a binary operation'. In general, the naming of functions or operators with a given arity follows a convention similar to the one used for n-based numeral systems, such as binary and hexadecimal. One combines a Latin prefix with the **-ary ending**; for example:

- A **nullary function** takes **no arguments**.

  Example: $f( ) = 2$
- A **unary function** takes **one argument**.

  Example: $f(x) = 2x$
- A **binary function** takes **two arguments**.

  Example: $f(x, y) = 2xy$
- A **ternary function** takes **three arguments**.

  Example: $f(x, y, z) = 2xyz$
- An ***n*-ary function** takes ***n* arguments**.

  Example: $f(x_1, x_2, \ldots, x_n) = 2\prod_{i=1}^{n} x_i$

Check Your Progress

25. Give the definition of inverse function.
26. What do you mean by the functional recursion?
27. Explain binary operation.
28. What is arity and n-ary operation?

result adding new executables in a user's search path requires regeneration of the hash table using the rehash command before these programs can be executed without specifying the complete path.

NOTES

The central computer (server) at an university maintains records for each student. How many memory locations can be allotted so that the student records can be retrieved fastly? The solution to this problem is to use suitably chosen hashing function. Records are identified using a key, which uniquely identifies each student's record. For example, student records are identified using the Register number / Enrolment number of the student as the key. A hashing function H assigns memory location $H(p)$ to the record that has p as its key.

The most common hashing function is the function $H(p) = p \bmod q$, where q is the number of available memory locations. The hashing function should be *onto* so that all memory allocations are possible. For example, $q = 11$, the record of the student with register number 15 is assigned to memory, location 4, since $H(15) = 4 \bmod 11$. Similarly, since $H(122) = 122 \bmod 11 = 2$, the record of the student with register number 121 is assigned to memory location 2.

Since a hashing function is not 1–1, more than one file may be assigned to a memory location. When this happens, we say that a collision occurs. To resolve a collision is to assign the first free location following the occupied memory location assigned by the hashing function. For example, after making two earlier assignments we assign location 5 to the record of the student with register number 256.

8.4 COMPOSITE FUNCTIONS

A **composite function** is generally a function that is written inside another function. Composition of a function is done by substituting one function into another function.

For example, $f[g(x)]$ is the composite function of $f(x)$ and $g(x)$. The composite function $f[g(x)]$ is read as "f of g of x". The function $g(x)$ is called an inner function and the function $f(x)$ is called an outer function. Hence, we can also read $f[g(x)]$ as "the function g is the inner function of the outer function f".

The composite function can be solved by finding the composition of two functions. The symbol 'o' is used for defining the composition of a function. Following example illustrate how a composite function can be written or how to rewrite the composition of a function in a different form.

Examples for writing a composite function,

$(f \circ g)(x) = f[g(x)]$

$(f \circ g)(x) = f[g(x)]$

$(f \circ g)(x^2) = f[g(x^2)]$

For solving a composite function substitute the variable x that is in the outside function with the inside function. Then simplify the function.

NOTES

Note: The order in the composition of a function is important because $(f \circ g)(x)$ is NOT the same as $(g \circ f)(x)$.

If $f : X \to Y$ and $g : Y \to Z$, then the composite of functions f and g, written $g \circ f$, is the function which maps each element $x \in X$ into $(g \circ f)(x) = g(f(x)) \in Z$.

Thus, $g \circ f : X \to Z$ if $x \in X \Rightarrow g(f(x)) \in Z$. The functions obtained by means of such composition of functions, are called composite functions.

Example 8.1. Let $f : \mathbf{R} \to \mathbf{R}^+$ be defined by $f(x) = x^2 + 1$, $x \in \mathbf{R}$, and $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = 2x + 1 \;\forall\; x \in \mathbf{R}$ then the function $g \circ f : \mathbf{R} \to \mathbf{R}^+$ is defined by

$$\begin{aligned}(g \circ f)\, x &= g(f(x)) = g(x^2 + 1)\\ &= 2(x^2 + 1) + 1\\ &= 2x^2 + 3 \;\forall\; x \in \mathbf{R}\end{aligned}$$

Also, the function $f \circ g : \mathbf{R} \to \mathbf{R}^+$ exists and is defined by

$$\begin{aligned}(f \circ g)\, x &= f(g(x)) = f(2x + 1)\\ &= (2x + 1)^2 + 1\\ &= 4x^2 + 4x + 2, \;\forall\; x \in \mathbf{R}.\end{aligned}$$

The functions $g \circ f$ and $f \circ g$ defined on $\mathbf{R}$ are equal for $2x^2 + 3 = 4x^2 + 4x + 2$, *i.e.,* for $x = -1 \pm \sqrt{3/2}$ only.

Example 8.2. *If* $f : X \to Y$ and $g : Y \to Z$, then

(i) $g \circ f$ is onto if each of f and g is onto.

(ii) $g \circ f$ is univalent if each of f and g is univalent.

(iii) $g \circ f$ is one-one if each of f and g is one-one.

Solution.

(*i*) When each of f and g is onto, then $f(X) = Y$, $g(Y) = Z$ and so $g(f(X)) = Z$, i.e., $(g \circ f)(X) = Z$. Hence, $g \circ f$ is onto.

(*ii*) If f, g are univalent then $x_1, x_2 \in X$ and $x_1 \neq x_2$ imply $f(x_1) \neq f(x_2)$, and so, $g(f(x_1))\ g(f(x_2))$. Thus, $\forall\; x_1, x_2 \in X, x_1 \neq x_2 \Rightarrow (g \circ f)(x_1) \neq (g \circ f)(x_2)$, i.e., $g \circ f : X \to Z$ is univalent.

(iii) This result is the combination of (i) and (ii).

Associative Property

Example 8.3. If $f_1 : X_1 \Rightarrow X_2$, $\quad f_2 : X_2 \Rightarrow X_3$ and $f_3 : X_3 \Rightarrow X_4$ then $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$ on X_1 to X_4.

Solution. It is clear that both the functions $(f_3 \circ f_2) \circ f_1$ and $f_3 \circ (f_2 \circ f_1)$ map X_1 to X_4. If $x \in X_1$, then

$[(f_3 \circ f_2) \circ f_1](x) = (f_3 \circ f_2)\, f_1(x) = f_3(f_2(f_1(x)))$.