poset lattice

L subset

lub → join → join semilattice

glb → meet → meet semilattice

↓

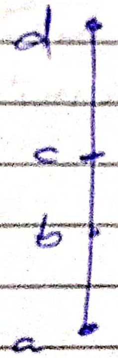
lattice

Join semilattice

Consider a poset L under the ordering ≤, let $a, b \in L$ then lub(a,b) or sup(a,b) is denoted by $a \vee b$ or $a \cup b$ and is called the join of a and b i.e. $a \vee b = \sup(a,b)$

other notations of join $\oplus$ or $+$

In a poset if join exists for every pair of elements, then poset is called join semi-lattice

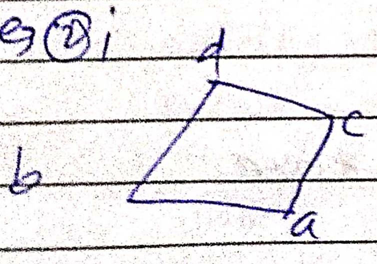


lub table

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

This is join semi-lattice.

eg ② i

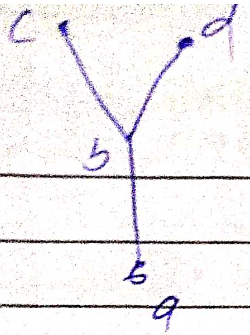


lub table

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	d	d
c	~~d~~c	d	c	d
d	d	d	d	d

→ join semi-lattice.

eg(3):



Lttices
Single top → join semi-lattice

(c, d) and (d, c) → ✗

→ not join semi-lattice

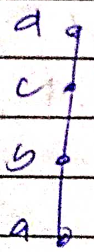
Meet Semi-lattice:-

→ consider a poset L under the ordering ≤. Let a, b ∈ L then glb(a, b) or inf(a, b) is denoted by a ∧ b or a ∩ b and is called the meet of a and b
i.e. a ∩ b = inf(a, b)

Other notations of meet are * or ·

In poset if meet exists for every pair of elements then poset is called meet semi-lattice.

eg(1):-

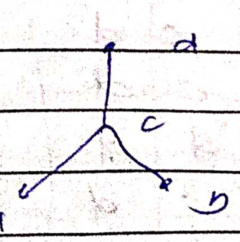


glb

∧	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

→ this is meet- ~~semi~~-lattice

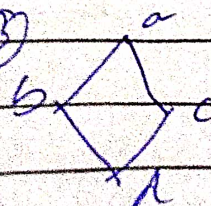
(2)



→ not meet semi lattice

→

(3)



→ meet-lattice

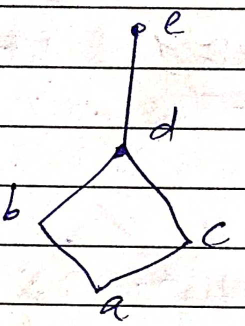
lattice:-

A poset $(P, \leq)$ is called a lattice if every 2-element subset of P has both a LUB and glb if lub(x, y) and glb(x, y) exist for every x and y in P.

we denote,

$$x \vee y = lub\{x, y\}$$
$$x \wedge y = glb\{x, y\}.$$

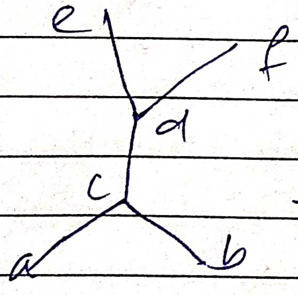
eg:- ①.



$\vee$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	d	d	e
c	c	d	c	d	e
d	d	d	d	d	e
e	e	e	e	e	e

→ both MSL, JSL
→ so it is lattice.

eg:-



→ is not MSL and JSL
→ not lattice

Properties of lattice:-

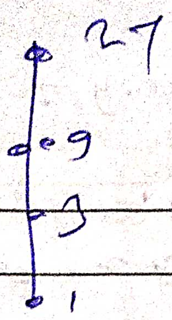
① Idempotent:
i) $a \vee a = a$
ii) $a \wedge a = a$

② Commutative
i) $a \vee b = b \vee a$
ii) $a \wedge b = b \wedge a$

③ Associative
i) $a \vee (b \vee c) = (a \vee b) \vee c$
ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

④ Absorption
i) $a \vee (a \wedge b) = a$

(1)



(i) $3 \vee 3 = \text{lub}\{3,3\}$
$= 3$

(ii) $3 \wedge 3 = \text{glb}\{3,3\} = 3$

(2) Com

$a = 1, b = 2$

(i) $1 \vee 3 = 3 \vee 1$
$3 = 3$

(ii) $1 \wedge 3 = 3 \wedge 1$
$1 = 1$

(3) $a = 1, b = 3, c = 9$

(i) $1 \vee (3 \vee 9) = (1 \vee 3) \vee 9$
$9 = (3 \vee 9) = 9$
$(9 = 9)$

(ii) (1) $1 \wedge (3 \wedge 9) = (1 \wedge 3) \wedge 9$
$1 \vee 3 = 3 \vee 1$
$\Rightarrow 3 = 3$

(2) $1 \wedge 3 = 3 \wedge 1$
$1 = 1$

(4) $a = 3, b = 9$

(i) $a \vee (a \wedge b) = a$
$= 3 \vee (3 \wedge 9)$
$= 3 \vee (3) = 3$

(ii) $3 \wedge (3 \vee 9) = 3$
$3 \wedge (9) =$
$= 3$

Sub lattice:-

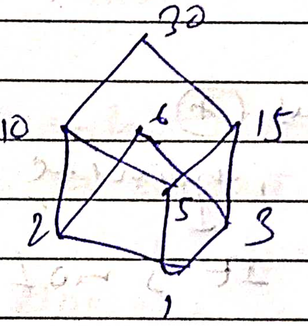
A non-empty subset L' of a lattice L is called a sublattice of L if. $a, b \in L' \Rightarrow$

$a \vee b, a \wedge b \in L'$ i.e. the algebra $(L', \vee, \wedge)$ is a sublattice of $(L, \vee, \wedge)$ iff L' is closed under both operations $\vee$ and $\wedge$.

$L' \subseteq L$
L, lattice.

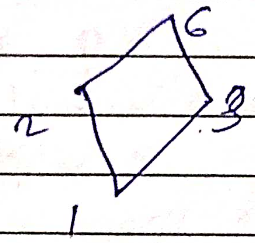
Eg:- If $L = D(30)$ having the Hasse diagram shown in fig.

divisor of 30.

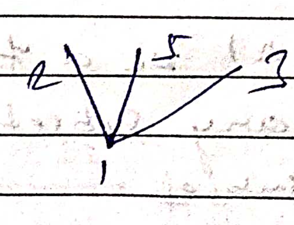


$L = \{1, 2, 3, 5, 6, 10, 15, 30\}$

i) $L_1 = \{1, 2, 3, 6\}$



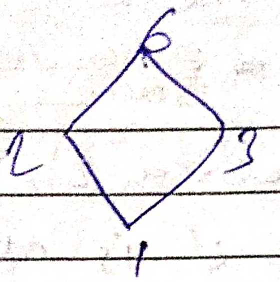
ii) $L_2 = \{1, 2, 5\}$



Complete Lattices:-

→ A lattice is called complete if each of its non-empty subsets has a least upper bound and a greatest lower bound.

Eg:- $L = \{1, 2, 3, 6\}$, the lattice $(L, |)$ is complete.



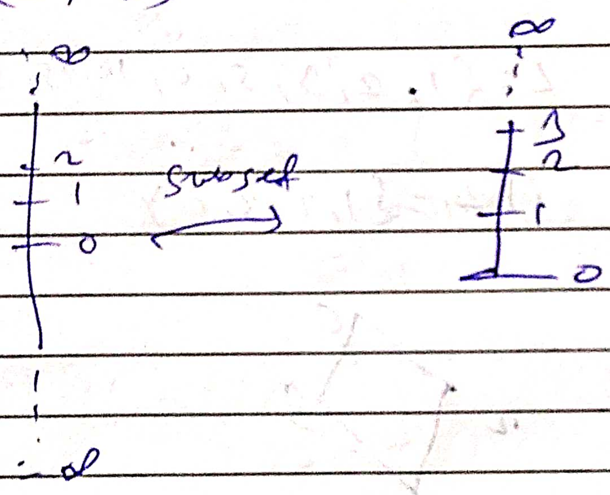
$S_1 = \{1, 2\}$

$S_2 = \{1, 2, 6\}$

$S_3 = \{1, 3, 6\}$

→ all subset ~~for all~~ are lattice so it's Complete lattice

Eg:- $(Z, \le)$ · set of integers



lub ($\nexists$)
infinite
↓
It is not lattice

Theorem:-

Prove that every finite lattice is complete.

Soln:- Let $(L, \vee, \wedge)$ be a finite lattice
let A be any subset of L, then A is also a finite subset.

Let $A = \{a_1, a_2, a_3 - - - a_n\}$ then

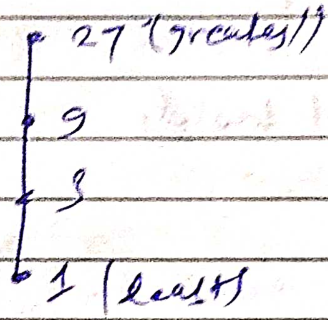
$a_1 \vee a_2 \vee a_3 \vee - - - \vee a_n$ and.
$a_1 \wedge a_2 \wedge a_3 \wedge a_4 - - \wedge a_n$ are the lub and glb of A in L.

Hence L is Complete lattice.

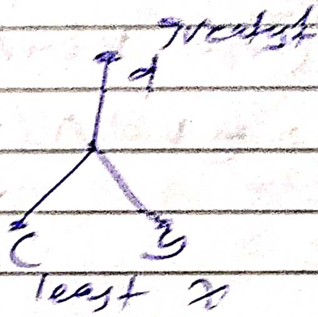
Bounded lattice

A lattice which has both elements least and greatest i.e. denoted by 0 and 1 respectively is called a bound lattice.

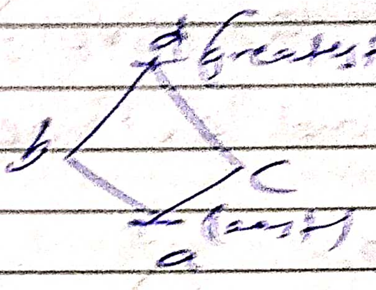
Eg:-



Bounded lattice



not bounded



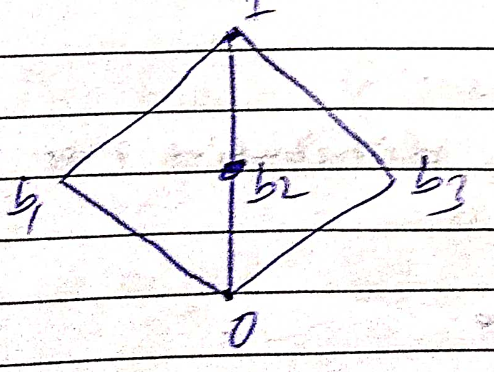
Bounded lattice

Complemented lattice-

In a bounded lattice $\{L, \vee, \wedge, 0, 1\}$ an element $b \in L$ is called a complement of an element $a \in L$ if $a \wedge b = 0$ and $a \vee b = 1$. Where 0 and 1 are lower and upper bound of L.

A bounded lattice is said to be complemented if every element has at least one complement in the lattice.

Eg:-



b_1 / b_2

$b_1 \wedge b_2 = 0$

$b_1 \vee b_2 = 1$

→ Complemented lattice

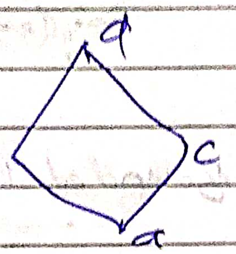
Distributive lattice

→ A lattice $(L, \vee, \wedge)$ is called a distributive lattice if for any $a, b, c \in L$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ holds.

Eg:-



i) $a, b, c \in L$ ii) a, b, d

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$\Rightarrow a \wedge (d) = a \vee a$

$= a = a$ (True)

ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$\Rightarrow a \vee a = b \wedge c$

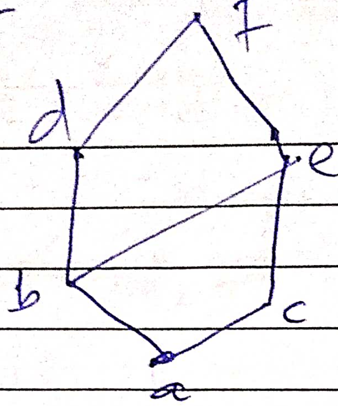
$\Rightarrow a = a$

iii) a, c, d
iv) b, c, d → all holds distributive prop.

→ This is distributive lattice.

Eg1 -

→ null or only one



Distributive lattice

↳ all element have one complement or null

$a^c = f$

$b^c = \times$

$c^c = d$

$d^c = c$

$e^c = \times$

$f^c = a$

$b \vee c = e$ (greatest)

$b \wedge c = a$ (least)

$d \wedge c = a$ (least)

$d \vee c = f$ (greatest)

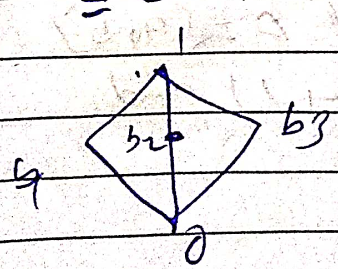
$e \wedge d = b$

$e \vee d = f$ (greatest)

Modular lattice:-

A lattice $(L, \leq)$ is said to be modular if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$ for all $a, b, c \in L$

eg:



(i) $0, b_1, b_2 \in L$
$a \quad b \quad c$

$a \vee (b \wedge c) = (a \vee b) \wedge c$

$0 \vee (b_1 \wedge b_2) = b_1 \wedge b_2$

$0 \vee 0 = 0$

(ii) $0, b_2, 1 \in L$
$a \quad b \quad c$

$0 \vee (b_2 \wedge 1) = (0 \vee b_2) \wedge 1$

$0 \vee b_2 = b_2 \wedge 1$

$\Rightarrow \quad b_2 = b_2$

modular lattice

Isomorphic lattice:-

→ Two lattices L_1 and L_2 are called isomorphic lattices if there is a bijection from L_1 and to L_2 i.e. $f:- L_1 \to L_2$ such that

$f(a \wedge b) = f(a) \wedge f(b)$ and
$f(a \vee b) = f(a) \vee f(b)$
→ for every element $a, b \in L_1$

Eg:- Let $L_1 = \{1, 2, 3, 6\}$ and $A = \{a, b\}$ then prove that the lattices $(L_1, /)$ and $(P(A), \subseteq)$ are isomorphic

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Sol:- Consider the mapping $f: L \to P(A)$
where $P(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$
and the mapping $f(1) = \phi$, $f(2) = \{a\}$, $f(3) = \{b\}$,
$f(6) = \{a,b\}$

A/q (a);
$f(a \wedge b) = f(a) \wedge f(b)$ and
$f(a \vee b) = f(a) \vee f(b)$ holds

$a = 1, b = 2$

$f(1 \wedge 2) = f(1) \wedge f(2)$

$f(1) = \phi \wedge \{a\}$

$\phi = \phi$