Recurrence Relation (Re-visited)

Ex. 1: How many binary strings (strings of 0's and 1's) of n bits have NO two consecutive zero's

$$a_n = \underbrace{1}_{\substack{\text{fix first} \\ \text{bit as} \\ \text{then solve} \\ \text{the same} \\ \text{problem} \\ \text{for } n-1 \text{ bit} \\ \text{argument}}}(a_{n-1}) + \underbrace{(0,1)}_{\substack{\text{if first} \\ \text{bit is fixed} \\ \text{as 0 then} \\ \text{second bit} \\ \text{must be} \\ \text{kept as 1} \\ \text{Leaving the} \\ \text{same problem} \\ \text{of size } n-2 \text{ to} \\ \text{be solved.}}}(a_{n-2})$$

$a_0 = 0$
$a_1 = 2$
$a_2 = 3$

fix first bit as → then solve the same problem ~~with~~ ~~for~~ $n-1$ bit argument

if first bit is fixed as 0 then second bit must be kept as 1 Leaving the same problem of size $n-2$ to be solved.

$$\Rightarrow \quad a_n = a_{n-1} + a_{n-2}$$

Ex. 2: I have buttons of four different colours. How many ways can we arrange n of them so that there are NO two consecutive blue ones.

Soln: $a_n = 3a_{n-1} + 3a_{n-2}$

Y G
R B

$$\therefore \quad a_n = \begin{pmatrix} Y \\ G \\ R \end{pmatrix}\left(a_{n-1}\right) + B\begin{pmatrix} Y \\ G \\ R \end{pmatrix}\left(a_{n-2}\right)$$

$$\Rightarrow a_n = 3a_{n-1} + 3a_{n-2}$$

Non-homogeneous Recurrence Relation

→ $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$ is called Non-Homogeneous if $f(n) \neq 0$

→ solution $= \underbrace{a_n^{(h)}}_{\text{Corresponding homogeneous RR soln}} + \underbrace{a_n^{(p)}}_{\text{Particular solution}}$

→

when $f(n)$ is	$a_n^{(p)}$ is of the form
c	A
cn	$An+B$
cn^2	An^2+Bn+C
$c\gamma^n$	$A\gamma^n$
$cn^2\gamma^n$	$\gamma^n(An^2+Bn+C)$

→ **Example 1.**

$$a_n = 2a_{n-1} + 7\times 5^n \quad \text{for } n \geq 1 \quad a_0 = 4$$

Step I:- $a_n^{(h)}$ calculation

$$r - 2 = 0 \Rightarrow r = 2$$

$$\Rightarrow a_n^{(h)} = \alpha_1 2^n$$

Step II:- $a_n^{(p)}$ calculation

Let's guess soln is of the form $C\cdot 5^n$

$$\Rightarrow C\cdot 5^n - 2C\,5^{n-1} = 7\times 5^n$$

for $n=1$.

$$\Rightarrow 5C - 2C = 35 \Rightarrow C = \frac{35}{3} \Rightarrow a_n^{(p)} = \left(\frac{35}{3}\right)\times 5^n.$$

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$$\frac{\text{Final solution}}{a_n} = \alpha_1 2^n + \left(\frac{35}{3}\right) \times 5^n.$$

Since $a_0 = 4$

$\Rightarrow 4 = \alpha_1 + \left(\frac{35}{3}\right)$

$\Rightarrow \alpha_1 = \cancel{\frac{35}{3}} -\frac{23}{3}$

Final soln $= \cancel{\frac{35 \times 2^n + 35}{3}}$

$= -\frac{23}{3} \times 2^n + \left(\frac{35}{3}\right) \times 5^n.$

Example 2: (Tower of Hanoi)

$a_n = 2a_{n-1} + 1$ $\qquad$ $\underline{a_1 = 1}$

Soln:

$\underline{a_n^{(h)} \text{ calculation}}$

$r - 2 = 0 \Rightarrow r = 2$

$a_n^{(h)} = \alpha_1 2^n$

$\underline{a_n^{(p)} \text{ calculation}}$

it will be of the for B

$\Rightarrow B = 2B + 1 \Rightarrow B = -1$

final **Soln** $a_n = \alpha_1 2^n - 1$

$\because a_1 = 1$

$\Rightarrow 1 = 2\alpha_1 - 1 \Rightarrow \alpha_1 = 1$

Soln: $2^n - 1.$

Master Theorem (subtract and conquer)

$$T(n) \le \begin{cases} c & \text{if } n \le 1 \\ aT(n-b)+f(n), & n > 1. \end{cases}$$

for some constants c, $a>0$, $b>0$, $d \geq 0$ and function $f(n)$.

if $f(n) = O(n^d)$

$$T(n) = \begin{cases} O(n^d), & \text{if } a < 1 \\ O(n^{d+1}), & \text{if } a = 1 \\ O(n^d a^{n/b}), & \text{if } a > 1. \end{cases}$$

Ex: $T(n) = 2T(n-1) + 1$

$a = 2 \quad b = 1$

$f(n) = O(n^0)$.

$$T(n) = O(n^0 2^{n/1})$$

$$= O(2^n)$$