CS 2050 Spring 2023 Homework 5

Due: February 24th

Released: February 17th

i. This assignment is due on **11:59 PM EST, Friday, February 24, 2023**. On-time submissions receive 2.5 points of extra credit. You may turn it in one day late for a 10 point penalty or two days late for a 25 point penalty. Assignments more than two days late will NOT be accepted. We will prioritize on- time submissions when grading before an exam.

ii. You will submit your assignment on **Gradescope**. Shorter answers may be entered directly into response fields, however longer answer must be recorded on a typeset (e.g. using LaTeX) or *neatly* written PDF.

iii. Ensure that all questions are correctly assigned on Gradescope. Questions that take up multiple pages should have all pages assigned to that question. Incorrect page assignments can lead to point deductions.

iv. You may collaborate with other students, but any written work should be your own. Write the names of the students you work with on the top of your assignment.

v. Always justify your work, even if the problem doesn't specify it. It can help the TA's to give you partial credit.

Author(s): David Teng, Richard Zhao

1. (15 points) For each of the following maps, where $f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$, determine whether f is:

   - onto and one-to-one
   - onto but not one-to-one
   - not onto but one-to-one
   - neither onto nor one-to-one
   - not a function

   a) $f(x, y) = x^2 + y + 1$

      onto but not one-to-one

   b) $f(x, y) = y! - 3x$

      not a function

   c) $f(x, y) = 9y^2 + 3y + 2$

      neither onto nor one-to-one

   d) $f(x, y) = \frac{x}{y} + y$

      not a function

   e) $f(x, y) = 30 * x + |y|$

      onto but not one-to-one

2. (9 points) Use the cashier's algorithm to make change using quarters, dimes, nickels, and pennies for the following amounts of money. You do not have to specifically show how you greedily formed change, but rather there are many ways to make this change and only the cashier's/greedy distribution will be accepted.

   a) 44 cents

      1 quarter, 1 dimes, 1 nickel, 4 pennies

   b) 74 cents

      2 quarter, 2 dimes, 4 pennies

   c) 93 cents

      3 quarter, 1 dimes, 1 nickel, 3 pennies

3. (8 points) Imagine that a new coin that is worth exactly 14 cents has been introduced to our existing currency system. Prove or disprove the statements (use an example or counterexample):

   (a) "The cashier's algorithm using quarters, dimes, nickels, 14-cent coins, and pennies can produce coins change using fewer coins than the algorithm without the 14 cent coin."

Proof using example with 14 cents. With 14-cent variant of cashiers algorithm, you can use a 1 14-cent coin.With regular cashiers algorithm, fewest coins is 1 dime and 4 pennies. 14 cents proves the statement as it can be done with 1 coin using the 14 cent cashier's algorithm which is less than 3 coins using the typical cashier algorithm.

(b) "The cashier's algorithm using quarters, dimes, nickels, 14-cent coins, and pennies and will produce change using the fewest coins possible for all coin values."

Proof by counterexample using 28 cents. With the 14-cent cashiers algorithm, it would take 4 coins (1 quarter 3 pennies) but it the least coins that could form the change would be 2 coins (2 14-cent coins). The counterexample of 28 cents disproves the statement because the modified cashiers algorithm does not produce change using the fewest coins possible as there is another combination that uses less coins.

4. (8 points) State whether the following is True or False and explain your reasoning for full credit:

a) Given two positive integers x and c, if $x + c < 5$, then $\lfloor \frac{x}{5} \rfloor = \lfloor \frac{x+c}{5} \rfloor$.

This is true because two positive integers must sum to a positive integer, so $0 < x + c < 5$. The division of any two positive integers with the denominator greater than the numerator will be between 0 and 1. Additionally, all numbers between 0 and 1 will have the same floor.

b) Given two functions $f(x)$ and $g(x)$, if $f(g(x))$ is undefined, then $g(f(x))$ must also be undefined.

False, assume $f(x) = \frac{1}{x}$ and $g(x) = 0$. $f(g(3)) = f(0) = \frac{1}{0} =$ undefined but $g(f(3)) = g(\frac{1}{3}) = 0$.

5. (20 points) For each part below, determine whether:

- $f(x)$ is $O(g(x))$
- $g(x)$ is $O(f(x))$
- $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$
- none of the above.

(a) $f(x) = 3x + \log(x)$, $g(x) = x + (\log(x))^2$
$g(x)$ is $O(f(x))$

(b) $f(x) = x!$, $g(x) = x^{2x}$
$g(x)$ is $O(f(x))$

(c) $f(x) = \frac{1}{x}$, $g(x) = \log(x)$
$f(x)$ is $O(g(x))$

(d) $f(x) = x^{2.0} + x^{1.9} + x^{1.8}...$, $g(x) = x^{2.1}$
$f(x)$ is $O(g(x))$

6. (10 points) List all numbers that you would compare 30 with while searching for the number 30 in the sequence $\{14\ 23\ 72\ 73\ 76\ 81\ 84\ 91\ 93\ 99\}$ using the binary search algorithm. You must use the version of the algorithm that was shown in class. Write all values compared against in the order the comparisons occur including all inequality comparisons and the final equality check. Note that if you compare against a number more than once, you must list it again for each additional comparison. e.g $\{1, 3, 7, 3\}$

$\{76, 23, 72\}$

7. (10 points) Prove or disprove the following statement.
$\log(x)(2x^4 + x^3 + 3x + \frac{9}{x})$ is $O(x^5)$. (If you choose to prove this statement, you must do so using witnesses).

Prove directly with $K = 1$ and $C = 4$ as witnesses,

1. $\log(x)(2x^4 + x^3 + 3x + \dfrac{9}{x})$ (Given)
2. $\log(x)2x^4 + \log(x)x^3 + \log(x)3x + \log(x)\dfrac{9}{x}$ (Distribute)
3. $2log(x)x^4 \leq x^5$ $\quad \forall x > 1$
4. $log(x)x^3 \leq x^5$ $\quad \forall x > 1$
5. $3log(x)x \leq x^5$ $\quad \forall x > 1$
6. $log(x)\dfrac{9}{x} \leq x^5$ $\quad \forall x > 1$
7. $2\log(x)x^4 + \log(x)x^3 + 3\log(x)x + \log(x)\dfrac{9}{x} \leq 4x^5 \quad \forall x > 1$ (Add (3),(4),(5),(6))

By $2\log(x)x^4 + \log(x)x^3 + 3\log(x)x + \log(x)\frac{9}{x} \leq 4x^5, \forall x > 1$ in (7), thus, with $K = 1$ and $C = 4$ as witnesses, $\log(x)(2x^4 + x^3 + 3x + \frac{9}{x})$ is $O(x^5)$ by definition of big-O.

8. (10 points) Let f and g be functions such that:

- $f(x) = 2x^2 \log(x^3)$, where $f : \mathbb{R}^+ \to \mathbb{R}$
- $g(x) = 3x^3$, where $g : \mathbb{R} \to \mathbb{R}$

(a) Determine whether $f(x)$ is $O(g(x))$. Justify your answer using witnesses. If $f(x)$ is not $O(g(x))$, then show an argument using a proof by contradiction using witnesses as to why $f(x)$ is not $O(g(x))$.

Proof directly using $K = 1$ and $C = 2$ as witnesses,

1.	$2x^2 \log(x^3)$	(Given $f(x)$)
2.	$2x^2 \leq 2x^2 \quad \forall x > 1$	
3.	$\log(x^3) \leq x \quad \forall x > 1$	
4.	$2x^2 \log(x^3) \leq 2x^3 \quad \forall x > 1$	
5.	$3x^3$	(Given $g(x)$)
6.	$3x^3 \leq 3x^3 \quad \forall x > 1$	

By $\quad 2x^2 \log(x^3) \leq 2x^3, \quad \forall x > 1$ in (4), thus, with $K = 1$ and $C = 2$ as witnesses, $\quad 2x^2 \log(x^3) \leq 2x^3$ is $O(x^3)$ by definition of big-O. By $3x^3 \leq 3x^3, \quad \forall x > 1$ in (4), thus, with $K = 1$ and $C = 3$ as witnesses, $2x^2 \log(x^3) \leq 2x^3$ is $O(x^3)$ by definition of big-O and therefore $f(x)$ is $O(g(x))$

(b) Determine whether $g(x)$ is $O(f(x))$. Justify your answer using witnesses.If $g(x)$ is not $O(f(x))$, then show an argument using a proof by contradiction using witnesses as to why $g(x)$ is not $O(f(x))$.

Proof through contradiction assuming $g(x)$ is $O(f(x))$. By definition of $O(n)$ there exists real numbers C, K such that $g(x) < Cf(x), \quad \forall x > K$

1.	$3x^3$	(Given $g(x)$)
2.	$2x^2 \log(x^3)$	(Given $f(x)$)
3.	$g(x) \leq Cf(x) \quad \forall x > K$	(Given)
4.	$3x^3 \leq C(2x^2 \log(x^3)) \quad \forall x > K$	(Substitute (1), (2) into (3))
5.	$3x^3 \leq 2x^2 C \log(x^3) \quad \forall x > K$	(Simplified)
6.	$3x \leq 2C \log(x^3) \quad \forall x > K$	(Divide x^2 from both sides)
7.	$3x \leq 2C \log(x^3) \quad \forall x > K$	(Divide x^2 from both sides)
8.	$\dfrac{3x}{\log(x^3)} \leq 2C \quad \forall x > K$	(Divide $\log(x^3)$ from both sides)

This inequality states at (8) that $\frac{3x}{\log(x^3)} \leq 2C \quad \forall x > K$ but x will continue to grow infinitely past C Thus this is a contradiction and $g(x)$ cannot be $O(f(x))$

9. (5 points) Determine the smallest time complexity of the following algorithm.

   $sum := 0$
   $x := 1$
   while $x < n$ **do**
       for $y = 1$ to n **do**
           $sum = sum + y$
       end for
       $x = 2 * x$
   end while
   $a := 0$
   while $a < n$ **do**
       $a = a + 1$
       $a* = 3$
   end while

   $O(n^2)$ because second-level nested for and while loops

10. (5 points) Determine the smallest time complexity of the following algorithm where n is an integer < y

   $prod := 1$
   $x := 1$
   $y = 100$
   while $x < x + 10$ **do**
       while $y > n$ **do**
           $prod* = y$
           $y/ = 2$
       end while
       $x = 2 * x$
   end while

   $O(log(n))$ because the first while loop is constant and the second while loop is logarithmic.