CS 2050 Spring 2023 Homework 4

Due: February 17th

Released: February 10th

i. This assignment is due on **11:59 PM EST, Friday, February 17, 2022**. On-time submissions receive 2.5 points of extra credit. You may turn it in one day late for a 10 point penalty or two days late for a 25 point penalty. Assignments more than two days late will NOT be accepted. We will prioritize on- time submissions when grading before an exam.

ii. You will submit your assignment on **Gradescope**. Shorter answers may be entered directly into response fields, however longer answer must be recorded on a typeset (e.g. using LaTeX) or *neatly* written PDF.

iii. Ensure that all questions are correctly assigned on Gradescope. Questions that take up multiple pages should have all pages assigned to that question. Incorrect page assignments can lead to point deductions.

iv. You may collaborate with other students, but any written work should be your own. Write the names of the students you work with on the top of your assignment.

v. Always justify your work, even if the problem doesn't specify it. It can help the TA's to give you partial credit.

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1. Determine whether the following are true or false. (3 points each)

   (a) $\emptyset \subset \emptyset$

   False

   (b) $\emptyset \subseteq \{a\}$

   True

   (c) $\{\emptyset\} \subset \{\{\emptyset, \emptyset\}\}$

   False.

   (d) $\{\emptyset\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

   False

2. Determine the cardinality of the following sets. (3 points each)

   (a) $\emptyset$

   0

   (b) $\{U\}$

   1

   (c) $\{a, \{b\}, b, c, \{b, c\}\}$

   5

   (d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \emptyset\}$

   3

3. Let A = {a, b, c, d, e}. Let B = {m, n, o}. Let C = {a, c, f}. Find the following: (3 points each)

   (a) $A \cup B \cup C$

   $\{a, b, c, d, e, f, m, n, o\}$

   (b) $B \cap A$

   $\emptyset$

   (c) B^2

   $\{\{m, m\}, \{m, n\}, \{m, o\}, \{n, m\}, \{n, n\}, \{n, o\}, \{o, m\}, \{o, n\}, \{o, o\}\}$

   (d) $\mathcal{P}(B)$

   $\{\emptyset, \{m\}, \{n\}, \{o\}, \{m, n\}, \{m, o\}, \{n, o\}, \{m, n, o\}\}$

   (e) $A \cup C \cap \emptyset$

   $\emptyset$

   (f) $|\mathcal{P}(C) \cup \mathcal{P}(B - \emptyset)|$

   15

   (g) $|\mathcal{P}(\mathcal{P}(\mathcal{P}(C)))|$

   $2^{2^{2^3}} = 2^{256}$

   (h) $A \cup B \cup \emptyset$

   $\{a, b, c, d, e, m, n, o\}$

4. Show that if X, Y, and Z are sets, then $\overline{(X \cap Y)} \cap Z = (\overline{X} \cap Z) \cup (\overline{Y} \cap Z)$. You cannot do this proof with set equivalencies. (10 points)

$$\begin{aligned}
&\overline{(X \cap Y)} \cap Z && \text{(Given)} \\
&= (\overline{X} \cup \overline{Y}) \cap Z && \text{(De Morgan's Law)} \\
&= Z \cap (\overline{X} \cup \overline{Y}) && \text{(Commutative Law)} \\
&= (Z \cap \overline{X}) \cup (Z \cap \overline{Y}) && \text{(Distributive Law)} \\
&= (\overline{X} \cap Z) \cup (\overline{Y} \cap Z) && \text{(Commutative Law)}
\end{aligned}$$

5. List all the elements of $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) - \mathcal{P}(\mathcal{P}(\emptyset)) \cup \mathcal{P}(\emptyset) - \emptyset$ (6 points)

$$\begin{aligned}
&\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) - \mathcal{P}(\mathcal{P}(\emptyset)) \cup \mathcal{P}(\emptyset) - \emptyset \\
&= \mathcal{P}(\mathcal{P}(\{\emptyset\}))) - \mathcal{P}(\{\emptyset\}) \cup \{\emptyset\} - \emptyset \\
&= \mathcal{P}(\{\emptyset, \{\emptyset\}\}))) - \{\emptyset, \{\emptyset\}\} \cup \{\emptyset\} - \emptyset \\
&= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} - \{\emptyset, \{\emptyset\}\} - \emptyset \\
&= \{\{\emptyset\}, \{\{\emptyset\}\}\}
\end{aligned}$$

6. Let the Universe be the set of all Georgia Tech students. Let T be the set of all students who are TAs, let S be the set of all students who are seniors, and let C be the set of all students who are in a club. Translate the following statements only using the sets given and set operations. (3 points each)

   (a) The set of all TAs who are both not seniors and in a club.
   $T \cap \overline{S} \cap C$

   (b) The set of all students who are a part of a club or are TAs and not both.
   $(C \cup T) - (C \cap T)$

   (c) The set of all students in a club who are not both TAs and a senior.
   $C \cap \overline{T \cap S}$

   (d) The set of students who are not seniors, not TAs, and are in a club.
   $\overline{S} \cap \overline{T} \cap C$

7. Find a set, if one exists, such that the set A meets the criteria where $|A| = 3$ and $A \in \mathcal{P}(\mathcal{P}(A))$? (5 points)

   $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$

8. Prove or disprove the following statements, for all sets A, B, and C such that each set A, B, and C are disjoint sets. (3 points each)

   (a) $A \cup (A \cap C) = A$

   Prove using direct proof, $A \cup (A \cap C) = A$

1.	$(A \cap C) = \emptyset$	(Given, Disjoint Sets definition)
2.	$A \cup (A \cap C)$	(Given)
3.	$= A \cup \emptyset$	(Substitution (1) into (2))
4.	$= A$	(Identity Law (3))

   By (2) and (4), we have proved using direct proof that $A \cup (A \cap C) = A$

   (b) $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$

   This proposition is not true because it does not work for all cases. For example, let $A = \{1\}$ and $B = \{2\}$

1.	$A = \{1\}$	(Given)
2.	$B = \{2\}$	(Given)
3.	$\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$	(Given)
4.	$\mathcal{P}(A) = \{\emptyset, \{1\}\}$	(Power Set (1))
5.	$\mathcal{P}(B) = \{\emptyset, \{2\}\}$	(Power Set (2))
6.	$A \cup B = \{1, 2\}$	(Union (1), (2))
7.	$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\} \{2\}\}$	(Union (4), (5))
8.	$\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$	(Power Set (6))
9.	$\{\emptyset, \{1\} \{2\}\} = \mathcal{P}(A \cup B)$	(Substitute (7) into (3))
10.	$\{\emptyset, \{1\} \{2\}\} \neq \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$	(Substitute (8) into (9))

   By (10), $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$ when $A = \{1\}$ and $B = \{2\}$, the statement does not this case therefore the statement false as it has to work for all cases.

9. Prove that:
(9 points)

$$\{9a + 17b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}$$

Prove using direct proof so that that $\{9a + 17b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}$

1. $a \in \mathbb{Z}$ (Given)
2. $b \in \mathbb{Z}$ (Given)
3. $\{9a + 17b\}$ (Given)
4. Let $a = 2k$, for $k \in \mathbb{Z}$ (Universal Instantiation (1), closed set under multiplication)
5. Let $b = -k$ (Universal Instantiation (2), closed set under multiplication)
6. $\{9(2k) + 17b\}$ (Substitute (4) into (3))
7. $\{9(2k) + 17(-k)\}$ (Substitute (5) into (6))
8. $\{k\}$ (Simplification (7))
9. $\mathbb{Z} \subseteq \{k\}$ (Universal Generalization (4) into (8))
10. $9a + 17b \in \mathbb{Z}$ (Closed set under addition and multiplication (1),(2))
11. $\{9a + 17b\} \subseteq \mathbb{Z}$ (Subset definition (10))
12. $k = \mathbb{Z}$ (Set equality (9), (11))

By (3) and (9), we have proved using direct proof that $\{9a + 17b \mid a, b \in \mathbb{Z}\} = \mathbb{Z}$.

Define sets A, B, C, and D such that:
(5 points)

$$|A - B - C - D| = |A| - |B| - |C| - |D|$$

$A = \emptyset$
$B = \emptyset$
$C = \emptyset$
$D = \emptyset$