CS 2050 Fall 2022 Homework 3

Due: February 3

Released: January 27

This assignment is due on **11:59 PM EST, Friday, February 3, 2023**. Early submissions receive 2.5 points of extra credit. You may turn it in one day late for a 10 point penalty or two days late for a 25 point penalty. Assignments more than two days late will NOT be accepted. We will prioritize on-time submissions when grading before an exam.

You should submit a typeset or *neatly* written pdf on Gradescope. The grading TA should not have to struggle to read what you've written; if your handwriting is hard to decipher, you will be required to typeset your future assignments.

Upon submission ensure that questions are correctly assigned on Gradescope. Questions that take up multiple pages should have all pages assigned to that question. Incorrect page assignments can lead to point deductions. You may collaborate with other students, but any written work should be your own. Write the names of the students you work with on the top of your assignment. Always justify your work, even if the problem doesn't specify it. It can help the TA's to give you partial credit.

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Common Proof Tools

This list serves as a reminder of proof justifications you may use throughout the homework. **We are providing the following here as hints for your homework, however, this list will not be provided on an exam**

1. x^n is even if and only if x is even
2. x^n is odd if and only if x is odd
3. Multiplication is closed under integers
4. Addition is closed under integers
5. The definition of a biconditional states that $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$
6. The list is not all inclusive and does not include other definitions learned in class and algebraic properties learned elsewhere

1. Using rules of inference, laws of logical equivalences, and other definitions taught in class, show that the premises conclude with “It is not true that “If I turn in my homework, then I will go to sleep early”” Be sure to define all propositional variables for full credit. Remember, it is possible that you will use all premises, but it is also possible that some are not needed. (14 points)

   a.) Today is Friday and Papa Johns is on fire.

   b.) I will win a chess match if today is a Friday or I do not watch a movie.

   c.) I sleep early only if I do not win a chess match.

   d.) I will get to class on time and turn in my homework if I do not sleep early.

   e.) Papa Johns is not on fire if and only if I do not turn in my homework.

   f.) Domino’s is not on fire today.

Let f be ”today is Friday”
Let p be ”Papa Johns is on fire”
Let c be ”I will win a chess match”
Let w be ”I will watch a movie”
Let s be ”I sleep early”
Let t be ”I will get to class on time”
Let h be ”I will turn in my homework”

	(1)	$f \wedge p$	Given (a)
	(2)	f	Simplification (1)
	(3)	$f \vee \neg m$	Addition (2)
	(4)	$(f \vee \neg m) \rightarrow c$	Given (b)
	(5)	c	Modus Ponens (3, 4)
	(6)	$s \rightarrow \neg c$	Given (c)
	(7)	$\neg s \vee \neg c$	Conditional Identity (6)
	(8)	$\neg s \vee (\neg c)$	Associative Law (7)
	(9)	$\neg\neg c$	Double Negation Law (5)
	(10)	$\neg(\neg c)$	Associative (9)
	(11)	$\neg s$	Disjunctive Syllogism (7, 10)
	(12)	$\neg s \rightarrow (t \wedge h)$	Given (d)
	(13)	$t \wedge h$	Modus Ponens (11, 12)
	(14)	h	Simplification (13)
	(15)	$h \wedge \neg s$	Conjunction (14, 11)
	(16)	$\neg\neg h \wedge \neg s$	Double Negation (15)
	(17)	$\neg(\neg h \vee \neg s)$	De Morgan’s Law (16)
$\therefore$	(18)	$\neg(h \rightarrow s)$	Conditional Identity (17)

2. Using rules of inference, laws of logical equivalences, and other definitions taught in class, show that the hypotheses below conclude with b. Give the reason for each step as you show that b is concluded. Each reason should be the name of a rule of inference and include which numbered steps are involved, For example, a reason for a step might be "Modus ponens using #2 and #3". (Hint: You may use the definition of biconditional and the commutative law). (16 points)

   1) $y \leftrightarrow x$
   2) $x \wedge (b \vee \neg d)$
   3) $x \wedge a \rightarrow \neg b$
   4) $(\neg y \vee x) \wedge c \rightarrow d$
   5) $y \rightarrow c$

	(6)	x	Simplification (2)
	(7)	$(y \rightarrow x) \wedge (x \rightarrow y)$	Conditional Law (1)
	(8)	$x \rightarrow y$	Simplification (7)
	(9)	y	Modus Ponens (6, 8)
	(10)	c	Modus Ponens (9, 5)
	(11)	$x \vee \neg y$	Addition (6)
	(12)	$\neg y \vee x$	Commutative Law (11)
	(13)	$(\neg y \vee x) \wedge c$	Conjugation (12, 10)
	(14)	d	Modus Ponen (13, 4)
	(15)	$b \vee \neg d$	Simplification (2)
	(16)	$b \vee (\neg d)$	Associative (15)
	(17)	$\neg\neg d$	Double Negation Law (14)
	(18)	$\neg(\neg d)$	Associative(17)
$\therefore$	(19)	b	Disjunctive Syllogism (16, 18)

3. Let x and y be integers. Use a direct proof to show that if $x + y$ is even, then $x^2y - y^3 + 2$ is even. Clearly state your reasoning for all statements and use a two-column proof for the body whenever possible. You should include an intro, body (in two column format), and a conclusion. (Hint: It might be beneficial to factor $x^2y - y^3$ in your scratch work) (10 Points)

   Let x and y be integers that $x + y$ is even.
   Since $x + y$ is even, by definition, $x + y = 2k$ for some integer k.
   Multiplying $x + y$ by any integer a gives

$$\begin{aligned} a(x+y) &= a(2k) \\ &= 2ka \end{aligned}$$

   Since k and a are both integers, ka is an integer. Let $k_2 = ka$, $2(ka) = 2k_2$ for some integer k_2. Therefore $a(x+y)$ and any integer multiple of $(x+y)$ is even. Therefore, $x^2y - y^3$ is when $(x + y)$ is even because

$$\begin{aligned} x^2y - y^3 &= y(x^2 - y^2) \\ &= y(x-y)(x+y) \end{aligned}$$

   Because $y(x - y)$ only contains subtraction and multiplication, $y(x + y)$ is an integer. As $a(x + y)$ for any arbitrary integer a is even, including $y(x + y)$, $x^2y - y^3$ is even. Since for every even integer i can be written as $2k$ for integer k, $i + 2$ is even because

$$\begin{aligned} i + 2 &= (2k) + 2 \\ &= 2k + 2 \\ &= 2(k+1) \end{aligned}$$

   Since $k+1$ is an addition of 2 integers, $k+1$ is an integer. Let $k_3 = k+1$, $2(k+1) = 2k_3$ for some integer k_3. Therefore $i+2$ is even for even integer i. And because $x^2y - y^3$ is even, $x^2y - y^3 + 2$ is even.

4. Let n be an integer. Prove the statement "If $n^2 + 2n + 10$ is odd, then n is odd." Make sure to include the introduction, body, and conclusion. Clearly state your reasoning for all statements and use a two-column proof for the body whenever possible. You should include an intro, body (in two column format), and a conclusion. (12 points each)

   a) Prove the statement using a proof by contrapositive.

Proof by contrapositive assuming n is even

n	s	r
(1)	n is even	Given
(2)	$n = 2k, k_1 \in Z$	Definition of even
(3)	$n + 2 = 2k_1 + 2$	Add 2 to both sides
(4)	$n + 2 = 2(k_1 + 1)$	Factorize
(5)	$n + 2 = 2k_2, k_2 \in Z$	Substitution and Declaration
(6)	$n + 2$ is even	Definition of even
(7)	$n(n + 2) = 2k_1(2k_2)$	Substitution using (5)
(8)	$n(n + 2) = 2(2k_1k_2)$	Associative and Commutative
(9)	$n(n + 2) = 2k_3, k_3 \in Z$	Substitution and Declaration
(10)	$n(n + 2)$is even	Definition of even
(11)	$n(n + 2) + 10 = 2k_3 + 10$	Substitution
(12)	$n(n + 2) + 10 = 2(k_3 + 5)$	Distributive
(13)	$n(n + 2) + 10 = 2k_4, k_4 \in Z$	Substitution and Declaration
(14)	$n(n + 2) + 10$ is even	Definition of even
(15)	$n^2 + 2n + 10 = n(n + 2) + 10$	Distributive
(16)	$n^2 + 2n + 10$ is even	Substitute (14) into (13)

Contrapositive because we established the contrapositive if n is even (1), $n^2 + 2n + 10$ must be even (16).

   b) Prove the statement using a proof by contradiction.

Proof by contradiction assuming $n^2 + 2n + 10$ is odd and n is even

n	s	r
(1)	$n^2 + 2n + 10$ is odd	Given
(2)	n is even	Given
(3)	$n = 2k, k_1 \in \mathrm{Z}$	Definition of even
(4)	$n + 2 = 2k_1 + 2$	Add 2 to both sides
(5)	$n + 2 = 2(k_1 + 1)$	Factorize
(6)	$n + 2 = 2k_2, k_2 \in \mathrm{Z}$	Substitution and Declaration
(7)	$n + 2$ is even	Definition of even
(8)	$n(n + 2)$	Declare
(9)	$n(n + 2) = 2k_1(2k_2)$	Substitution using (6)
(10)	$n(n + 2) = 2(2k_1k_2)$	Associative and Commutative
(11)	$n(n + 2) = 2k_3, k_3 \in \mathrm{Z}$	Substitution and Declaration
(12)	$n(n + 2)$is even	Definition of even
(13)	$n(n + 2) + 10 = 2k_3 + 10$	Substitution
(14)	$n(n + 2) + 10 = 2(k_3 + 5)$	Distributive
(15)	$n(n + 2) + 10 = 2k_4, k_4 \in \mathrm{Z}$	Substitution and Declaration
(16)	$n(n + 2) + 10$ is even	Definition of even
(17)	$n^2 + 2n + 10 = n(n + 2) + 10$	Distributive
(18)	$n^2 + 2n + 10$ is even	Substitute (17) into (16)

Contradiction because we established that $n^2+2n+10$ is odd (1) and n is even (2) but also showed that $n^2+2n+10$ is even (17) which is a contradiction as $n^2+2n+10$ cannot be simultaneously even and odd.

5. An irrational number is a number that cannot be expressed as the fraction of two integers. Prove that $\sqrt[n]{2}$ is irrational for any n $\geq$ 3. Clearly state your reasoning for all statements and use a two-column proof for the body whenever possible. You should include an intro, body (in two column format), and a conclusion.(12 points)

Proof by contradiction assuming $\sqrt[n]{2}$ is rational

n	s	r
(1)	$\sqrt[n]{2}$ is rational	Given
(2)	$\sqrt[n]{2} = \frac{p}{q}, (p, q \in \mathrm{Z}) \wedge (q \neq 0) \wedge (GCD(p,q) = 1)$	definition of rational
(3)	$2 = \frac{p^n}{q^n}$	power both sides by n
(4)	$2q^n = p^n$	multiply both sides by q^n
(5)	p^n is even	definition of even
(6)	p is even	p^n even iff p even
(7)	$p = 2k, k \in \mathrm{Z}$	definition of even
(8)	$2q^n = (2k)^n$	substitute (7) to (6)
(9)	$2q^n = 2^n k^n$	distribute exponent
(10)	$q^n = 2^{n-1} k^n$	divide both sides by 2
(11)	$q^n = 2^{(n \geq 2)} k^n$	given $n \geq 3$
(12)	$q^n = 2k_2, k_2 \in \mathrm{Z}$	Substitution and Declaration
(13)	q^n is even	definition of even
(14)	q is even	q^n even iff q even

Contradiction because we established in (2) that $GCD(p,q) = 1$ but if both p and q are even ((6), (14)) therefore $GCD(p,q) \geq 2$ and $2 \neq 1$

6. For each of the following conjectures, determine what method of proof would be the most efficient method for proving the statement. You need not write a formal proof (though you can) but you must give an explanation for why the method you chose would be most efficient. (12 points each)

   (a) Given an integer x greater than 2 such that $x^3 - x^2 + 1$ is even, prove that the x-th power of the x-th prime is always odd.

      Direct proof because every prime greater than 2 is odd and it is pretty simple to prove that all positive power of an odd number is odd (odd $\times$ odd = odd applied recursively to calculate prime to any power) while $x > 2$. $x^3 - x^2 + 1$ is extraneous information as it will be true for every x (odd - odd + odd = odd).

   (b) Prove that if x is a positive integer such that $\frac{x^4}{log(x)} > \sqrt{3}ln(x)$, then $x^3 + x > x^2 - x$.

      Proof by contrapositive because if the conclusion ($x^3 + x > x^2 - x$) is false, x has to be negative or zero as $x^3 + x \leq x^2 - x$ can be factored to $x(x^3 - x^2 + 2) \leq 0$ and $(x^3 - x^2 + 2x)$ is always positive. x cannot be negative as it directly contradicts the given, making the conclusion vacuous.

7. Assign the *On Time* question on Gradescope here.

8. Assign the *Matching* question on Gradescope here.