CS2050 Spring 2023 Homework 2

Due: January 27 @ 11:59 PM

Released: January 20

This assignment is due on **11:59 PM EST, Friday, January 27, 2023**. Submissions submitted at least 24 hours prior to the due date will receive 2.5 points of extra credit. On-time submissions receive no penalty. You may turn it in one day late for a 10-point penalty or two days late for a 25-point penalty. Assignments more than two days late will NOT be accepted. We will prioritize on- time submissions when grading before an exam.
You should submit a typeset or *neatly* written pdf on Gradescope. The grading TA should not have to struggle to read what you've written; if your handwriting is hard to decipher, you will be required to typeset your future assignments. A 5-point penalty will occur if pages are incorrectly assigned to questions when submitting.
You may collaborate with other students, but any written work should be your own. Write the names of the students you work with on the top of your assignment.
Always justify your work, even if the problem doesn't specify it. It can help the TA's to give you partial credit.
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1. Express each of these statements using predicates and quantifiers.(5 points each)

   a) All cats are mammals but not all mammals are cats.
   Let the domain for x be the set containing all animals.
   Let $C(x)$: x is a cat.
   Let $M(x)$: x is a mammal.

   $$\forall x(C(x) \rightarrow M(x)) \wedge \exists x(M(x) \wedge \neg C(x))$$

   c) All monkeys eat bananas.
   Let the domain for x be the set containing all animals.
   Let $M(x)$: x is a monkey.
   Let $E(x)$: x eats bananas.

   $$\forall x(M(x) \rightarrow E(x))$$

2. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers. If true, give a brief explanation for why. If false, give a counterexample (i.e. a set of values x,y that make the statement false. You must also explain why they make the statement false.) (4 points each)

   a) $\exists y \forall x((y > x^2) \rightarrow y > x)$

   True, if $y > x^2$, then $y > \pm x$ regardless if x is positive or minus as x will always be smaller. Because the $\exists y$, a example of a y that makes the statement true makes the statement true for $\exists y$. If $y = 0$, $y > x^2$ will always be false making the conditional and the statement true.

   b) $\forall y \exists ! x(\sqrt[x]{y} < 0)$

   False, given $y = -1$, $\sqrt[x]{y} < 0$ is true when x is any odd number, like when $x = 3$ $\{(\sqrt[3]{-1} = -1) < 0\}$ or $x = 5$ $\{(\sqrt[5]{-1} = -1) < 0\}$ therefore violating the $\exists !$ which only allows one x for each y value to satisfy the statement

   c) $\exists x \forall y((\sqrt[3]{x} \neq y^2) \rightarrow (x > 0))$

   True, as given the quantifier is $\exists x \forall y$ we only need to prove a singular x exists where all y values makes the statement true. In fact, the statement is true for all values of x where $x > 0$. Trivially, when $y = \sqrt[6]{x}$, the statement is true because it the condition $(\sqrt[3]{x} \neq y^2)$ is false. When $y \neq \sqrt[6]{x}$, the statement remains true because $x > 0$ stated already.

d) $\forall x \forall y(y > 0 \vee x < 0 \rightarrow x^y \geq y)$

False, as given the quantifier is $\forall x \forall y$ we only need to prove a singular x and y values that makes the statement false. An example is when $y = 1$ and $x = -3$ as the left side of the conditional $y > 0 \vee x < 0$ is true while the right side $x^y \geq y$ is false because $(-3)^1 = -3) \ngeq 1$

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e) $\forall x \forall y(x^2 = y^2 \leftrightarrow y^3 = x^3)$

False, as given the quantifier is $\forall x \forall y$ we only need to prove a singular x and y values that makes the statement false. An example is when $x = 1$ and $y = -1$ as the left side of the conditional $(1^2 = 1) = (-1^2 = 1)$ is true while the right side $\geq y$ is false because $(1^3 = 1) \neq ((-1)^3 = -1)$.

3. Express the statement $\exists ! x P(x)$ using only universal quantification, existential quantification, and logical operators.(10 points)

$\exists x, \forall y(P(x) \wedge (x \neq y \rightarrow \neg P(y)))$

4. Show that $\exists x \forall y((A(x) \to B(y)) \wedge (C(x,y) \to B(y)))$ is logically equivalent to $\exists x \forall y(\neg B(y) \to (\neg A(x) \wedge \neg C(x,y)))$. Make sure to cite all steps (e.g. De Morgan's law). You should neither combine steps nor skip steps. The only steps that can be combined on a single line are the associative and commutative laws.(10 points)

$$\exists x \forall y((A(x) \to B(y)) \wedge (C(x,y) \to B(y))) \qquad \text{(Given)}$$
$$= \exists x \forall y((\neg A(x) \vee B(y)) \wedge (C(x,y) \to B(y))) \qquad \text{(Conditional Identity)}$$
$$= \exists x \forall y((\neg A(x) \vee B(y)) \wedge (\neg C(x,y) \vee B(y))) \qquad \text{(Conditional Identity)}$$
$$= \exists x \forall y(((\neg A(x) \vee B(y)) \wedge \neg C(x,y)) \vee ((\neg A(x) \vee B(y)) \wedge B(y))) \qquad \text{(Distributive Law)}$$
$$= \exists x \forall y((\neg C(x,y) \wedge (\neg A(x) \vee B(y))) \vee (B(y) \wedge ((B(y) \vee \neg A(x)))) \qquad \text{(Commutative Law)}$$
$$= \exists x \forall y((\neg C(x,y) \wedge (\neg A(x) \vee B(y))) \vee B(y))) \qquad \text{(Absorption Law)}$$
$$= \exists x \forall y(((\neg C(x,y) \wedge \neg A(x)) \vee (\neg C(x,y) \wedge B(y))) \vee B(y)) \qquad \text{(Distributive Law)}$$
$$= \exists x \forall y((B(y) \vee (B(y) \wedge \neg C(x,y)) \vee (\neg C(x,y) \wedge \neg A(x))) \qquad \text{(Commutative/Associative Law)}$$
$$= \exists x \forall y(B(y) \vee (\neg C(x,y) \wedge \neg A(x))) \qquad \text{(Absorption Law)}$$
$$= \exists x \forall y(B(y) \vee (\neg A(x) \wedge \neg C(x,y))) \qquad \text{(Commutative Law)}$$
$$= \exists x \forall y(\neg\neg B(y) \to (\neg A(x) \wedge \neg C(x,y))) \qquad \text{(Double Negation Law)}$$
$$= \exists x \forall y(\neg B(y) \to (\neg A(x) \wedge \neg C(x,y))) \qquad \text{(Conditional Law)}$$

5. Rewrite each of these statements so that no negation is to the left of a quantifier. Push all negations in as far as they will go. This means past the conditionals and biconditionals as well.(5 points each)

   (a) $\forall x \neg \exists y \exists z(P(x,y) \leftrightarrow \neg S(y,z)))$

   $$\forall x \forall y \forall z((P(x,y) \wedge S(y,z) \vee (\neg S(y,z) \wedge \neg P(x,y)))$$

   (b) $\neg \forall y \neg \exists x(R(x,y) \to \exists z S(z,y))$

   $$\exists y \exists x(R(x,y) \to \exists z S(z,y))$$

6. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers. Otherwise, state that no counterexample exists. (5 points each)

   (a) $\forall x \forall y(y = x \to y^{\frac{1}{x}} \leq x^{-y})$

$x = 2$ and $y = 2$. $y = x$ but $(2^{\frac{2}{2}} = 2^1 = 2) \nleq (2^{-2} = \frac{1}{2^2} = \frac{1}{4})$

(b) $\exists x \forall y \exists z (x = y^z)$

No counterexample exists.

(c) $\forall x \forall y ((y \leq x) \rightarrow (x^{17} > y) \vee (y < 0))$

$x = \frac{1}{2}$ and $y = \frac{1}{4}$. $((y = \frac{1}{2}) \leq (x = \frac{1}{2}))$ but $((({\frac{1}{2}}^{17} = \frac{1}{2^{17}}) \ngtr (\frac{1}{4} = \frac{1}{2^2})) \vee (\frac{1}{4} \nless 0))$ so the conclusion is false so the statement is false.

7. Explain why $\forall x P(x) \vee \exists x Q(x)$ is logically equivalent to $\forall x \exists y (P(x) \vee Q(y))$. (A written English explanation may be easier than using logical equivalences.) (5 points)

It is equivalent because the statements on either side of the $\vee$ of $\forall x P(x) \vee \exists x Q(x)$ are referring to different x's. In the form $\forall x \exists y (P(x) \vee Q(y))$, a y is used instead of the second x. For the equation on the right, although $P(x)$ is under the quantifier $\exists y$, $P(x)$ has no y component so it does not get affected by it. Similar logic could be used with $\forall x$ and $Q(y)$

8. Let $C(x)$ be the statement "x has a pet cat," let $D(x)$ be the statement "x has a dog," let $H(x)$ be the statement "x has a horse," let $E(x)$ be the statement "x eats eggs," and let $A(x)$ be the statement "x has allergies to animals." Express each of these statements in terms of $C(x)$, $D(x)$, $H(x)$, $E(x)$, $A(x)$, quantifiers, and logical connectives. Let the domain consist of all people.(5 points each)

a) Any person with allergies to animals cannot own dogs, or cats but do eat eggs.

$$\forall x (A(x) \rightarrow (\neg D(x) \wedge \neg C(x) \wedge E(x)))$$

b) Nobody owns all three animals, but everyone owns a dog.

$$\neg \exists x ((C(x) \wedge D(x) \wedge H(x)) \vee \neg D(x))$$

c) Each type of the three animals is owned by at least one person who eats eggs.

$$\exists x \exists y \exists z ((E(x) \wedge C(x)) \wedge (E(y) \wedge D(y)) \wedge (E(z) \wedge H(z))$$