OM-S20-17: Duality - I

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http://preon.iiit.ac.in/om_quiz

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Introductory Example

$$Z = \min \;\; 6x_1 + 4x_2 + 2x_3$$

Subject to :

$$4x_1 + 2x_2 + x_3 \geq 5$$

$$x_1 + x_2 \geq 3$$

$$x_2 + x_3 \geq 4$$

$$x_i \geq 0, \;\; \forall i = 1, 2, 3.$$

- $6x_1 + 4x_2 + 2x_3 \geq 4x_1 + 2x_2 + x_3 \geq 5.$
- $6x_1 + 4x_2 + 2x_3 \geq (4x_1 + 2x_2 + x_3) + (x_1 + x_2) \geq 5 + 3 = 8.$
- $6x_1 + 4x_2 + 2x_3 \geq (4x_1 + 2x_2 + x_3) + 2(x_1 + x_2) \geq 5 + 2 \cdot 3 = 11.$
- $6x_1 + 4x_2 + 2x_3 \geq (4x_1 + 2x_2 + x_3) + (x_1 + x_2) + (x_2 + x_3) \geq 5 + 3 + 4 = 12.$

Receipie for Creating Dual Problems

PRIMAL	**DUAL**
$x_1, x_2, ..., x_n$	$y_1, y_2, ..., y_m$
A	A^T
b	c
c	b
$Max\ c^T X$	$Min\ b^T Y$
$\leq$	$y_i \geq 0$
$\geq$	$y_i \leq 0$
$=$	$y_i \in \mathbf{R}$
$x_j \geq 0$	$j^{th} constraint \geq$
$x_j \leq 0$	$j^{th} constraint \leq$
$x_j \in \mathbf{R}$	$j^{th} constraint =$

Examples/Problems

1. $\max \quad x_1 + x_2$
   Subject to
   $$x_1 + 2x_2 \leq 2$$
   $$2x_1 + x_2 \leq 2$$
   $$x_1 \geq 0; x_2 \geq 0$$
   What is the primal and dual optima?
2. $\min x_1 - x_2$
   Subject to:
   $$2x_1 + 3x_2 - x_3 + x_4 \leq 0$$
   $$3x_1 + x_2 + 4x_3 - 2x_4 \geq 3$$
   $$-x_1 - x_2 + 2x_3 + x_4 = 6$$
   $$x_1 \leq 0; x_2, x_3 \geq 0; x_4 \in R$$

Diet problem

- n food and m nutritions. Problem: Find a healthy diet of minimum cost. A is $m \times n$

$$\min c^T x$$

Subject to

$$Ax \leq b; \text{ and } x \geq 0$$

- Assume pills-seller has a way of supplying the nutrients directly. Seller wants to charge as much as he can for the nutrients. To be competitive with normal foods, the equivalent in pills of a food must cost less than the cost of the food.

$$\max b^T y$$

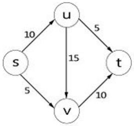
Subject to:

$$A^T y \leq c \text{ and } y \geq 0$$

MaxFlow-MinCut

$$
\begin{aligned}
Max \quad & x_{su} + x_{sv} \\
s.t \quad & x_{su} + 0 + 0 + 0 + 0 \leq 10 \\
& 0 + x_{sv} + 0 + 0 + 0 \leq 5 \\
& 0 + 0 + x_{uv} + 0 + 0 \leq 15 \\
& 0 + 0 + 0 + x_{ut} + 0 \leq 5 \\
& 0 + 0 + 0 + 0 + x_{vt} \leq 10 \\
& x_{su} - x_{uv} - x_{ut} = 0 \\
& x_{sv} + x_{uv} - x_{vt} = 0
\end{aligned}
$$

$$
\begin{aligned}
Min \quad & 10y_{su} + 5y_{sv} + 15y_{uv} + 5y_{ut} + 10y_{vt} \\
s.t. \quad & y_{su} + u_u \geq 1 \\
& y_{sv} + u_v \geq 1 \\
& y_{uv} - u_u + u_v \geq 0 \\
& y_{ut} - u_u \geq 0 \\
& y_{vt} - u_v \geq 0 \\
& y_i \geq 0 \\
& u_i \in \mathbf{R}
\end{aligned}
$$



u_u is 1 if u is in the cut with set S and 0 otherwise. Similarly u_v is a variable for vertex v.