classmate
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DUALITY - 1

$$\min.\quad 6x_1 + 4x_2 + 2x_3$$

$$4x_1 + 2x_2 + x_3 \geq 5$$

$x_i \geq 0$ $\quad x_1 + x_2 \geq 3$

$$x_2 + x_3 \geq 4$$

} primal problem.

$$\max.\quad 5y_1 + 3y_2 + 4y_3$$

*θ

$$4y_1 + y_2 \leq 6$$

$y_i \geq 0$ $\quad 2y_1 + y_2 + y_3 \leq 4$

$$y_1 + y_3 \leq 2$$

} dual problem

$\min\ c^T x$ $\qquad c = \begin{bmatrix} 6 & 4 & 2 \end{bmatrix}^T$

$Ax \leq b$ $\qquad A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $\qquad b = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$

$x \geq 0$

$\Downarrow$

$\max.\ b^T y$ $\qquad b^T = \begin{bmatrix} 5 & 3 & 4 \end{bmatrix}$

$A^T y \leq c$ $\qquad A^T = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\qquad c = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$

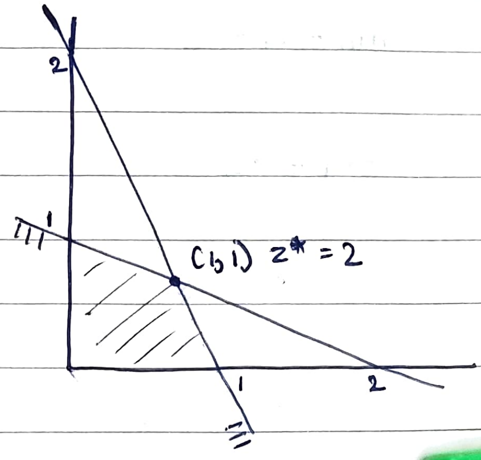
$y \geq 0$

$$\max.\ x_1 + x_2$$

$$x_1 + 2x_2 \leq 2$$

$$2x_1 + x_2 \leq 2$$

$$x_i \geq 0$$



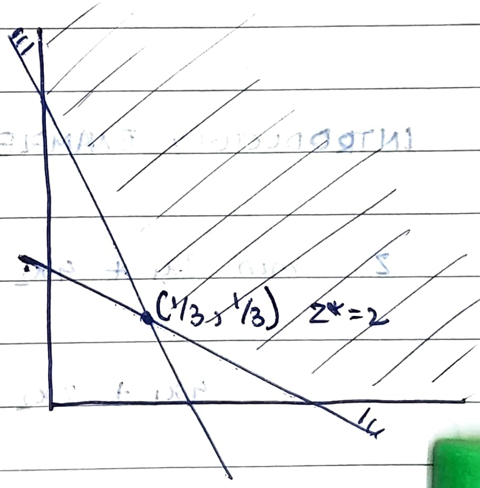
$\updownarrow$

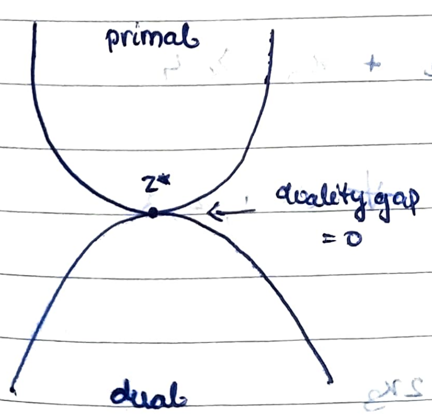
$$\min.\ 3y_1 + 3y_2$$

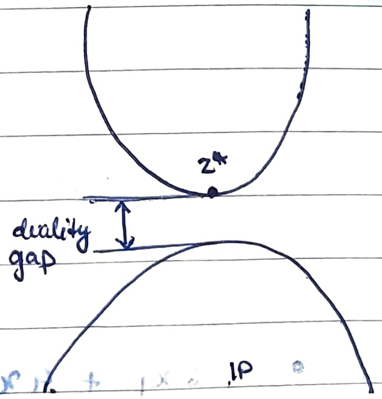
$$y_1 + 2y_2 \geq 1$$

$$2y_1 + y_2 \geq 1$$

$$y_i \geq 0$$







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$\min -x_1 - x_2 \quad \Leftrightarrow \quad \max.\ y_1 + 3y_2 + 2y_3$

$-x_1 + x_2 \leq 1 \qquad -y_1 + y_2 \geq -1$

$x_1 \leq 3 \qquad y_1 + y_3 \geq -1$

$x_2 \leq 2 \qquad y_i \geq 0$

$x_i \geq 0$

INTRODUCTORY EXAMPLE

$Z = \min\ 6x_1 + 4x_2 + 2x_3$

$4x_1 + 2x_2 + x_3 \geq 5$

$x_1 + x_2 \geq 3$

$x_2 + x_3 \geq 4$

$x_i \geq 0 \quad i = 1, 2, 3$

- $6x_1 + 4x_2 + 2x_3$

$\geq$

$4x_1 + 2x_2 + x_3$

$\geq$

5

- $6x_1 + 4x_2 + 2x_3$

  $\geq$

  $(4x_1 + 2x_2 + x_3) + (x_1 + x_2)$

  $\geq$

  $5 + 3 = 8$

- $6x_1 + 4x_2 + 2x_3$

  $\geq$

  $(4x_1 + 2x_2 + x_3) + 2(x_1 + x_2)$

  $\geq$

  $5 + 2 \cdot 3 = 11$

- $6x_1 + 4x_2 + 2x_3$

  $\geq$

  $(4x_1 + 2x_2 + x_3) + (x_1 + x_2) + (x_2 + x_3)$

  $\geq$

  $5 + 3 + 4 = 12$

RECIPE FOR CREATING DUAL PROBLEMS

primal	dual
$x_1, x_2, \ldots x_n$	$y_1, y_2, \ldots y_n$
A	A^T

primal	dual
b	c
c	b
$\max\ c^T x$	$\min.\ b^T y$
$\leq$	$y_i \geq 0$
$\geq$	$y_i \leq 0$
$=$	$y_i \in \mathbb{R}$
$x_j \geq 0$	j^{th} constraint $\geq$
$x_j \leq 0$	j^{th} constraint $\leq$
$x_j \in \mathbb{R}$	j^{th} constraint $=$

EXAMPLES / PROBLEMS

① $\max.\ x_1 + x_2$

st. $x_1 + 2x_2 \leq 2$

$2x_1 + x_2 \leq 2$

$x_1 \geq 0 \quad x_2 \geq 0$

what is the primal and dual optima?

- min. $x_1 - x_2$

st. $2x_1 + 3x_2 - x_3 + x_4 \leq 0$

$3x_1 + x_2 + 4x_3 - 2x_4 \geq 3$

$-x_1 - x_2 + 2x_3 + x_4 = 6$

$x_1 \leq 0 \quad x_2, x_3 \geq 0 \; ; \quad x_4 \in \mathbb{R}.$

DIET PROBLEM

- n food and m nutritions.

problem: find a healthy diet of min. cost.

A is $m \times n$

$$\min \; c^T x$$

$$Ax \leq b$$

$$x \geq 0$$

- assume pills-seller has a way of supplying the nutrients directly. seller wants to charge as much as he can for the nutrients. to be competitive with normal foods, the equivalent in pills of a food must cost less than the cost of the food.

$$\max \; b^T y \qquad \text{st.} \quad A^T y \leq c \;, \; y \geq 0$$

MAX FLOW - MIN CUT

max. $x_{su} + x_{sv}$

St. $x_{su} + 0 + 0 + 0 + 0 \leq 10$

$0 + x_{sv} + 0 + 0 + 0 \leq 5$

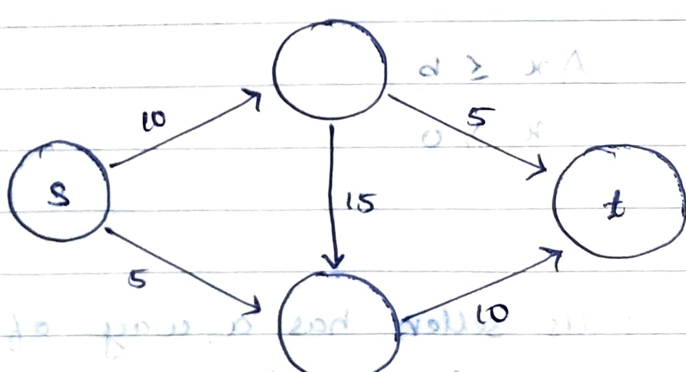
$0 + 0 + x_{uv} + 0 + 0 \leq 15$

$0 + 0 + 0 + x_{ut} + 0 \leq 5$

$0 + 0 + 0 + 0 + x_{vt} \leq 10$

$x_{su} - x_{uv} - x_{ut} = 0$

$x_{sv} + x_{uv} - x_{vt} = 0$



min. $10y_{su} + 5y_{sv} + 15y_{su} + 5y_{ut} + 10y_{vt}$

s.t. $y_{su} + u_u \geq 1$

$y_{sv} + u_v \geq 1$

$y_{uv} - u_u + u_v \geq 0$

$y_{ut} - u_u \geq 0$

$y_{vt} - u_v \geq 0$

$y_i \geq 0$

$u_i \in \mathbb{R}$

u_u is 1 if u is in cut with set S and 0 otherwise. Similarly u_v ~~for~~ is a variable for vertex v.