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# HW5

① prove the following using jensen's inequality

$$\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} > \frac{3}{x}$$

use this to show that

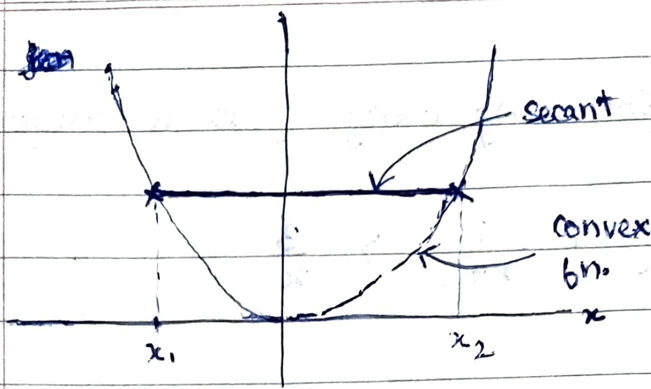
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

does not converge to a real no.

A) i wanted do reality check first. this is a javascript fn. to have some values.

```
function reciprocalSum (n) {
    return n>0? reciprocalSum(n-1) + 1/n
        : 0;
}
```

| | | | | |
|---|---|---|---|---|
| reciprocalSum (1) | | = | 1 | not converging |
| ... | 10 | = | 2.9 | |
| ... | 100 | = | 7.5 | |
| ... | 1000 | = | 9.8 | |




as we can see from above, between 2 points $x_1, x_2$ the secant is always above the convex fn.

i.e., for any point $x$ between $x_1$ and $x_2$

$$x = a x_1 + (1-a) x_2$$

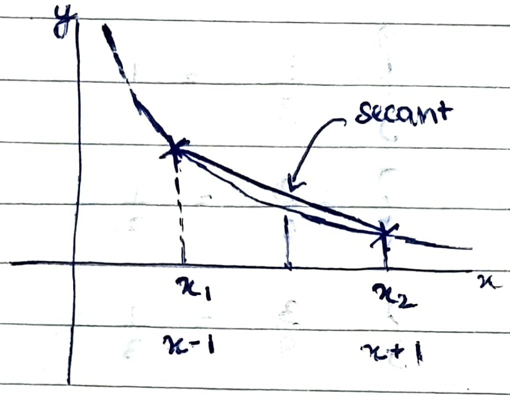
we can say that,

$$f_{secant}(x) \geq f_{convex}(x)$$

or, $a f(x_1) + (1-a) f(x_2) \geq f(a x_1 + (1-a) x_2)$

this is jensen's inequality.

now to look at $f(x) = \frac{1}{x}$




also convex fn.

also secant $\geq$ fn.

for the midpoint between $x_1 = x-1$
$x_2 = x+1$

$$\frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{1}{2}\left(\frac{1}{x+1}\right) \geq \frac{1}{x} \quad f(x)$$

$$\Rightarrow \frac{1}{x-1} + \frac{1}{x+1} \geq \frac{2}{x}$$

$$\Rightarrow \frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} \geq \frac{3}{x}$$

Since we know $f(x) = \frac{1}{x}$ is not a straight line,

$$\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} > \frac{3}{x} \qquad \text{(proved)}$$

$1$

$\left. + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right\} \quad \frac{3}{3} = 1$

$\left. + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \right\} \quad \frac{3}{6} = \frac{1}{2}$

$\left. + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \right\} \quad \frac{3}{9} = \frac{1}{3}$

$\left. + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} \right\} \quad \frac{3}{12} = \frac{1}{4}$

$+ \ldots$

So, if $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

we see that $S = 1 + 3 \times S$

- Since $S$ is positive, we can say that it is <u>not convergent</u>.

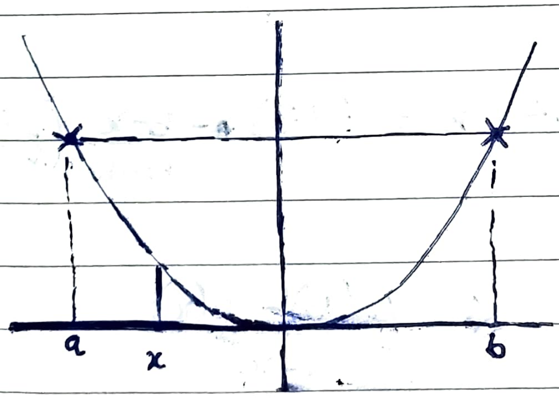
② prove the 3-chord lemma

if $f:[a,b] \to \mathbb{R}$ is convex, and

$a < x < b$, then

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(a)}{b-x}$$

first, lets do a reality check




here, $\frac{f(x)-f(a)}{x-a}$ is -ve

$\frac{f(b)-f(a)}{b-a}$ is 0

$\frac{f(b)-f(x)}{b-x}$ is +ve

satisfies..

using the rule,

secant $\geq$ convex fn. (at $x$)

$$f(a) + \frac{x-a}{b-a}\left(f(b) - f(a)\right) \geq f(x)$$

$$\Rightarrow \frac{x-a}{b-a}\left(f(b) - f(a)\right) \geq f(x) - f(a)$$

$$\Rightarrow \frac{f(b) - f(a)}{b-a} \geq \frac{f(x) - f(a)}{x-a} \quad \text{--- (1)}$$

also, secant at $x = f(b) - \frac{b-x}{b-a}\left(f(b) - f(a)\right)$

$$\Rightarrow \frac{f(b) - f(a)}{b-a} \leq \frac{f(b) - f(x)}{b-x} \quad \text{--- (2)}$$

so we can say,

$$\frac{f(x) - f(a)}{x-a} \leq \frac{f(b) - f(a)}{b-a} \leq \frac{f(b) - f(x)}{b-x}$$

(3) if $x, y, z$ are +ve real no. such that

$x + y + z = 1$,

show using jensen's inequality that

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 64.$$

attempt1:

$$\left(\frac{2x+y+z}{x}\right)\left(\frac{x+2y+z}{y}\right)\left(\frac{x+y+2z}{z}\right) \cancel{\geq 6}$$

$$= \left(2+\frac{y+z}{x}\right)\left(2+\frac{x+z}{y}\right)\left(2+\frac{x+y}{z}\right)$$

$\geq 8$

(but not good enough)

attempt2: (with help)

let $f(x) = \log\left(1+\frac{1}{x}\right)$ (convex fn.)

using the role

Secant $\geq$ convex fn. at K

where $K = \frac{1}{3}(x+y+z)$

$$\frac{1}{3}\{f(x)+f(y)+f(z)\} \geq f\left(\frac{x+y+z}{3}\right)$$

$$\frac{1}{3}\log\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq f(1/3)$$

$$\log\left(1+1/x\right)\left(1+1/y\right)\left(1+1/z\right) \geq 3\log\left(1+\frac{1}{1/3}\right)$$

$$= 3\log 4$$

$$= \log 64$$

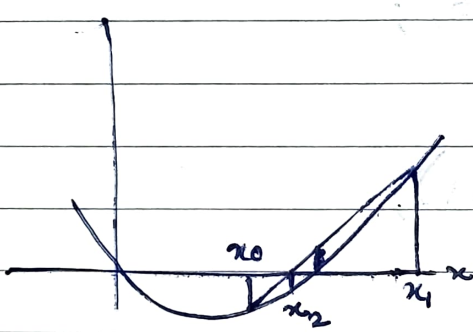
$$\Rightarrow \left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 64$$

(4) conduct 2 iterations of the secant method

$$x^3 - 2x - 5 = 0$$

$x_0 = 2 \qquad x_1 = 3$

secant method is a root finding iterative algorithm. it was developed long before newton's method.




* it is better ~~...~~a version of bisection method.

$$x_2 = x_1 - f(x_1) \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

derivation:

Solenoid

$$y_2 = 0$$

$$= \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_1) + f(x_1)$$

to make calculations simple, using javascript (nodejs).

```
f = x => x**3 - 2*x - 3;

x = (x1, x0) => x1 - f(x1) * ((x1-x0)/(f(x1)-f(x0)));

x0 = 2;
x1 = 3;

x2 = x(x1, x0);
```
↳ 1.94

```
x3 = x(x2, x1);
```
↳ 1.92

⑤ conduct 2 iterations of the bisection method for

$x e^{x} = 1$ where $x \in [0, 1]$

```
f = x => x * Math.exp(x) - 1;
```

f(0)
↳ -1

f(0.5)
↳ -0.18

f(1)
↳ 1.72

f(0.75)
↳ 0.59

f(0.625)
↳ 0.17

getting close

(6) find a real root of the equations

$x^2 - y^2 = 3$ — (1)

$x^2 + y^2 = 13$ — (2)

by doing 2 iterations of newton's method

$x_0 = y_0 = \sqrt{6.5}$

$f_1 = x^2 - y^2 - 3$

$\frac{\partial f_1}{\partial x} = 2x \qquad \frac{\partial f_1}{\partial y} = -2y$

$\therefore \quad x_1 = x_0 - \frac{f_1(x_0, y_0)}{2x_0}$

$y_1 = y_0 + \frac{f_1(x_0, y_0)}{2y_0}$

so no lets put into javascript:

$x_1 = 3.14$

$y_1 = 1.96$

$x_2 = 2.66$

$y_2 = 2.73$

$x_3 =$

$y_3 =$

$$f_2 = x^2 + y^2 - 13$$

$$\frac{\partial f_2}{\partial x} = 2x \qquad \frac{\partial f_2}{\partial y} = 2y$$

$$\therefore \quad x_1 = x_0 - \frac{f_2(x_0, y_0)}{2x_0}$$

$$y_1 = y_0 - \frac{f_2(x_0, y_0)}{2y_0}$$

so,

$$x_1 = 2.55$$
$$y_1 = 2.55$$

$$x_2 = 2.55$$
$$y_2 = 2.55$$

looks like already at root

7 do 2 iterations of the fixed point method with

$$2x - \cos x - 3 = 0$$

take $g(x) = \dfrac{\cos x + 3}{2}$ , $x_0 = \dfrac{\pi}{2}$

$$\left.\begin{aligned} x_1 &= 1.5 \qquad & f(x_1) &= -0.07 \\ x_2 &= 1.54 \qquad & f(x_2) &= 0.04 \end{aligned}\right\} \text{in nodejs}$$

its coming close to the root.

(8) write the pseudocode for the gradient ascent algorithm. your code must be vectorized. also run the algorithm to find the maximum of

$$6 - (x_1^2(x_1^2 - 16) + x_2^2(x_2^2 - 9)) \quad \text{from } (0,0)$$

in gradient descent,

$$x_1 = x_0 - \eta \nabla f(x_0)$$

$$\nabla f = \begin{bmatrix} -4x_1^3 + 32x_1 \\ \\ -4x_2^3 + 18x_2 \end{bmatrix}$$

```
function gradDes (gradf, x0, n=0.001) {
  do {
    var x1 = x0.map((v,i) => v-n*gradf[i](x0));
    var e = x1.reduce((acc,v,i) =>
                  acc + Math.abs(v-x0[i]),0);
    x0 = x1;
    console.log({x1,e});
  } while (e > 0.000001);
  return x1;
}
```

gradDes (gradf, x0) → [0,0]

} it looks like (0,0) could be maxima, as it appears for other values of $x_0$ as well.

classmate
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(9) many people prefer to use the gradient descent algorithm in batch mode. how and why do you think they do that?

most people actually use mini-batch gradient descent of several samples: it is computationally a whole lost faster, and does not require holding large datasets in RAM.

(10) give a proof of the convergence of the gradient descent algorithm. also, discuss how to choose and adjust the learning rate so that it converges.

assume that $f: \mathbb{R}^n \to \mathbb{R}$
is convex and differentiable, and

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x-y\| \quad \text{for any } x, y$$

i.e; $\nabla f$ is lipschitz continuous with constant $L > 0$

gradient descent with fixed step size $t \leq 1/L$ satisfies

$$f(x^{(k)}) - f(x^*) \leq \frac{\|x^{(0)} - x^*\|^2}{2tk}$$

i.e, gradient descent has convergence rate $O(1/k)$

i.e., to get $f(x^{(k)}) - f(x^*) \leq \epsilon$,

need $O(1/\epsilon)$ iterations.

$\nabla f$ lipschitz with constant $L \Rightarrow$

$$f(y) \leq f(x) + \nabla f(x)^T (y-x) + \frac{L}{2}\|y-x\|^2 \quad \forall_{x,y}$$

plugging in $y = x - t\nabla f(x)$

$$f(y) \leq f(x) - \left(1 - \frac{Lt}{2}\right) t \, \|\nabla f(x)\|^2$$

letting $x^+ = x - t\nabla f(x)$ and taking $0 < t \leq 1/L$

$$f(x^+) \leq f(x^*) + \nabla f(x)^T (x - x^*) - \frac{t}{2}\|\nabla f(x)\|^2$$

$$= f(x^*) + \frac{1}{2t}\left(\|x - x^*\|^2 - \|x^+ - x^*\|^2\right)$$

Summing over iterations

$$\sum_{i=1}^{K} \left(f(x^{(i)}) - f(x^*)\right)$$

$$\leq \frac{1}{2t}\left(\|x^{(0)} - x^*\|^2 - \|x^{(K)} - x^*\|^2\right)$$

$$\leq \frac{1}{2t}\|x^{(0)} - x^*\|^2$$

since $f(x^{(K)})$ is non-increasing

$$f(x^{(K)}) - f(x^*) \leq \frac{1}{K}\sum_{i=1}^{K}\left(f(x^{(i)}) - f(x^*)\right)$$

$$\leq \frac{\|x^{(0)} - x^*\|^2}{2tK}$$

When selecting a proper learning rate, we usually start with a large value like 0.1, then try exponentially lower values like 0.01, 0.001, ... Once we get a good learning rate, where the loss does not increase, we drop the learning rate by 1-2 orders of magnitude and use that instead, since it allow fine-grained learning.