FACTORIZATION OF MATRICES

Cholesky

every PD matrix A can be factorized as.

$$A = LL^T$$

where L is a lower Δ matrix.

$$\begin{bmatrix} a_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} l_{11} & L_{21}^T \\ 0 & L_{22}^T \end{bmatrix}$$

LU

A need not be PD; only needs to be non-singular.

$$\begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

$$Ax = b$$

~~COMPUTATION~~

$$\Rightarrow x = A^{-1}b$$

× not attractive, more complex

If we have a structure for A matrix, it could be done efficiently.

$$A[x_1 \dots x_n] = [b_1 \dots b_n]$$

I, PD, Tr. ‖ PD matrix, cholesky, LU

(1) $\underline{A}x = b$ factorization / decomposition (costly)

SVD > cholesky..

$$\downarrow$$

$\underline{LU}x = b$

(2a) $Lw = b$

easy solution

Δ matrix

(2b) $Ux = w$

if step (1) is used extensively, then ok.

COMPUTATION

cholesky

$$\begin{array}{ccc} & 1\times 1 & n-1\times 1 \\ A = & \left[\begin{array}{c|c} a_{11} & A_{21}^T \\ \hline A_{21} & A_{22}\end{array}\right] & \\ & 1\times n-1 & n-1\times n-1 \end{array} = \left[\begin{array}{c|c} l_{11} & 0 \\ \hline L_{21} & \underset{\text{lower } \Delta}{L_{22}} \end{array}\right] \left[\begin{array}{c|c} l_{11} & L_{21}^T \\ \hline 0 & L_{22}^T \end{array}\right]$$

$a_{11} = l_{11} \cdot l_{11}$

$\Rightarrow l_{11} = \sqrt{a_{11}}$ ——— (1).

$A_{21} = L_{21} \cdot \underline{l_{11}}$

$\Rightarrow L_{21} = \frac{1}{l_{11}} A_{21}$ ——— (2)

$$A_{22} = L_{21} L_{21}^T + L_{22} L_{22}^T$$

$$\Rightarrow L_{22} L_{22}^T = A_{22} - L_{21} L_{21}^T \quad \text{——— } (3)$$

another cholesky decomposition $(n-1)\times(n-1)$

★ $\frac{1}{3} n^3$ flops. $\langle T(n) \rangle$

LU.
$$\begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

$$A = L \quad U$$

$a_{11} = l_{11} u_{11}$ (assume $l_{11} = 1$) ——— (0).

$\Rightarrow u_{11} = a_{11}$ ——— (1).

$A_{12} = l_{11} U_{12}$

$\Rightarrow U_{12} = A_{12}$ ——— (2)

$A_{21} = L_{21} u_{11}$

$\Rightarrow L_{21} = \frac{1}{u_{11}} A_{21}$ ——— (3).

$A_{22} = L_{21} U_{12} + L_{22} U_{22}$

$\Rightarrow L_{22} U_{22} = A_{22} - L_{21} U_{12}$ ——— (4).

another LU decomposition

~~sing~~ ★ $\frac{2}{3} n^3$ flops. $\langle T(n) \rangle$

singular matrix should.
PD matrix: diagonal elements > 0? (check).

PROBLEMS.

(1). find cholesky of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 20 & 26 \\ 3 & 26 & 70 \end{bmatrix}$

$l_{11} = \sqrt{a_{11}} = 1$

$L \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$

$L_{21} = \frac{1}{l_{11}} A_{21} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$L_{22} L_{22}^T = A_{22} - L_{21} L_{21}^T$

$= \begin{bmatrix} 20 & 26 \\ 26 & 70 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 20 & 26 \\ 26 & 70 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 20 \\ 20 & 61 \end{bmatrix}$$

$$l'_{11} = \sqrt{a'_{11}} = 4$$

$$L'_{21} = \frac{1}{l'_{11}} A'_{21} = \frac{1}{4} \times 20 = 5$$

$$L'_{22} L'^{T}_{22} = A'_{22} - L'_{21} L'^{T}_{21} = 61 - 25 = 36$$

$$\Rightarrow L'_{22} = \sqrt{36} = 6.$$

$$\therefore L' = \begin{bmatrix} 4 & 0 \\ 5 & 6 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$$

(2) find LU decomposition of $\begin{bmatrix} 6 & 3 & 1 \\ 2 & 4 & 3 \\ 9 & 5 & 2 \end{bmatrix}$

Solve: $\begin{bmatrix} 6 & 3 & 1 \\ 2 & 4 & 3 \\ 9 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 17 \\ 29 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 3/2 & 1/6 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 6 & 3 & 1 \\ 0 & 3 & 8/3 \\ 0 & 0 & 1/18 \end{bmatrix}$$

$Lw = b \Rightarrow w = \begin{bmatrix} 18 & 11 & 1/6 \end{bmatrix}^T$

$\Rightarrow Ux = w \Rightarrow x = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^T$

QR DECOMPOSITION

- a square matrix Q is said to be orthogonal when:

$$QQ^T = Q^TQ = I$$

- QR decomposition : $A = QR$
  - Q is orthogonal.
  - R is an upper Δ matrix.

- why is this useful for us?
  - because QRx lets us write

$$Rx = Q^Tb$$ which is easy to solve for x.

- many problems can be solved faster (and reliably) by QR factorization.
  - least squares
  - least norm.

$$Ax = b$$

$\downarrow$ costly step

$$QRx = b$$

$$Rx = Q^T b$$

upper Δ matrix

$$\underset{n\times n}{A} = \begin{array}{cc} n\times 1 & n\times(n-1) \\ \left[\, a_1 \right. & \left. A_2 \,\right] \end{array} \qquad \underset{n\times n}{Q} = \begin{array}{cc} n\times 1 & n\times(n-1) \\ \left[\, q_1 \right. & \left. Q_2 \,\right] \end{array} \qquad \underset{n\times n}{R} = \begin{array}{cc} 1\times 1 & 1\times(n-1) \\ \left[\begin{matrix} r_{11} \\ 0 \end{matrix}\right. & \left.\begin{matrix} R_{12} \\ R_{22} \end{matrix}\right] \\ & (n-1)\times(n-1) \end{array}$$

$QQ^T = I$ (given).

notice that $Q^TQ = I$ implies

$$q_1^T q_1 = 1$$

$$q_1^T Q = 0$$

$$Q_2^T Q_2 = I$$

$$a_1 = q_1 r_{11}$$

what is this step?

taking norm gives us $r_{11} = a_1$

$$q_1 = \frac{1}{a_1} a_1 \quad \text{—— (1).}$$

$A_2 = q_1 R_{12} + Q_2 R_{22}$

$q_1^T A_2 = R_{12} + 0$

$\Rightarrow R_{12} = q_1^T A_2$ —— (2).

$\underset{n\times(n-1)}{Q_2}\ \underset{(n-1)\times(n-1)}{R_{22}} = A_2 - q_1 R_{12}$ —— (3).

$q_1 R_{12} = 0$?

another QR decomposition

★ $\frac{4}{3} n^3$ flops. $\langle T(n) \rangle$

LEAST SQUARES PROBLEM

$\min_x \| Ax - b \|$

$\min_x (Ax - b)^T (Ax - b)$

$\min_x (x^T A^T A x + b^T b - 2x^T A^T b)$

$2 A^T A x - 2A^T b = 0$

$A^T A x = A^T b.$