OM-S20-11: SVD, Least Squares and Least Norm

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14 Feb 2020

Singular Value Decomposition (SVD)

- Singular Value Decomposition (SVD) is a very powerful and popular matrix factorization.
- $A = UDV^T$
  - A is $m \times n$; U is $m \times n$; D is $n \times n$; V is $n \times n$
  - U is orthogonal
  - D is diagonal elements, D_{ii} the singular values.
  - V is orthogonal
  - $U^T U = V^T V = VV^T = I$

More on SVD

- Relationship to eigen values and eigen vectors

$$A^T A V = V D^2$$

$$A A^T U = U D^2$$

- Finding transpose

$$A^T = V D U^T$$

- Finding inverse

$$A^{-1} = V D^{-1} U^T$$

Least Square Problem and Solution

Given A and target vector b, Least square error (MSE) problem is:

$$\min_x ||Ax - b||$$

$$\min_x [Ax - b]^T [Ax - b]$$

$$\min_x \; (x^T A^T Ax + b^T b - 2x^T A^T b)$$

$$2A^T Ax - 2A^T b = 0$$

$$A^T Ax = A^T b$$

To obtain x, solve the above equation.

$$x = (A^T A)^{-1} A^T b$$

Least Norm Solution

$$\text{Minimize} ||x||$$
$$\text{Subject to} \quad Ax = b$$

$$\text{Soln:} \quad x^* = A^T(AA^T)^{-1}b$$

- x^* satisfy the $Ax = b$
- No other vector $x = x^* + (x - x^*)$ can be smaller.

$$||x||^2 = ||x^*||^2 + ||(x - x^*)||^2 + 2{x^*}^T[x - x^*]$$

  - Third term is zero

$$||x||^2 = ||x^*||^2 + ||(x - x^*)||^2$$
$$||x||^2 = ||x^*|| + \text{Positive Term}$$

Hence x^* is the smallest.

Efficient Solutions for Least Squaes and Least Norm

Least Squares

$$x^* = (A^T A)^{-1} A^T b$$

or Solve $A^T A x = A^T b$

Least Norm

$$x^* = A^T (A A^T)^{-1} b$$

How do we use:

- Cholesky
- LU
- QR
- SVD