OM-S20-18: Duality and Next Steps

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Duality Summary

PRIMAL	**DUAL**
$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
A	A^T
b	c
c	b
$Max\ c^T X$	$Min\ b^T Y$
$\leq$	$y_i \geq 0$
$\geq$	$y_i \leq 0$
$=$	$y_i \in \mathbf{R}$
$x_j \geq 0$	$j^{th} constraint \geq$
$x_j \leq 0$	$j^{th} constraint \leq$
$x_j \in \mathbf{R}$	$j^{th} constraint =$

Weak Duality

Let

$$P = \max\{c^T x | Ax \leq b, x \geq 0, x \in \mathbb{R}^n\}$$

$$D = \min\{b^T y | A^T y \geq c, y \geq 0, y \in \mathbb{R}^m\}$$

If x is a feasible solution of P and y is a feasible solution of D then

$$c^T x \leq b^T y$$

Why ?

$$c^T x = x^T c \leq x^T (A^T y) \leq (Ax)^T y \leq b^T y$$

Strong Duality for LP

Strong duality holds if

$$c^T x^* = b^T y^*$$

(Strong duality) If a linear programming problem has an optimal solution, so does its dual, and the respective optimal costs are equal.
Why?
Proof for Strong Duality is optional. Please go through yourself.

Duality Result for IP

If P and D are Primal-Dual pairs of a LP problem then one of the four cases occur:

1. Both are infeasible.
2. P is unbounded and D is infeasible.
3. D is unbounded and P is infeasible.
4. Both are feasible. There exists an optimal solution x^* and y^* for P and D such that $c^T x^* = b^T y^*$. As LP is a convex optimization problem, strong duality hold and hence when the problem is feasible, the optimal values of primal and dual are same.

Duality Result for IP

IP allows only weak duality. This can be easily proved as below.
Let $\bar{x}$ and $\bar{y}$ be the optima of the primal and dual integer programs. Let x^* and y^* are the optima of the relaxed primal and dual linear programs. Then:

$$\begin{aligned}
c^T\bar{x} &= max(c^Tx|Ax \leq b, x \geq 0, x \in \mathbf{Z}^n) && (1)\\
&\leq max(c^Tx|Ax \leq b, x \geq 0, x \in \mathbf{R}^n) && (2)\\
&= c^Tx^* && (3)\\
&= b^Ty^* && (4)\\
&= min(b^Ty|A^Ty \geq c, y \geq 0, y \in \mathbf{R}^n) && (5)\\
&\leq min(b^Ty|A^Ty \geq c, y \geq 0, y \in \mathbf{Z}^n) && (6)\\
&= b^T\bar{y} && (7)\\
& && (8)
\end{aligned}$$

Or

$$c^T\bar{x} \leq b^T\bar{y}$$

Duality Gap

Duality gap is the difference between the primal and dual solutions. If x^* is the optimal primal value and y^* is the optimal dual value then
Duality gap =

$$y^* - x^*$$

For weak duality : Duality Gap > 0 and for strong duality, Duality Gap $= 0$
Their difference is called the duality gap.
For convex optimization problems, the duality gap is zero

Next Set of Topics

- **Optimization in Manifold Learning and Spectral Methods** Familiarity with Graphs and Eigen vector based optimization
- **Nonlinear Optimization** First order and second order methods.
- **KKT and Lagrangian Duality** More on Duality. In more general optimization settings
- **Quadratic Optimization** Eg. Application in SVM

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Get ready ..

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