OM-M20-08: More on LP Relaxation

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Review-06 Link: `http://tiny.cc/64jajz`

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Graph Problems (Revisit)

Maximum weight bipartite matching Given a bipartite graph G(V,E) with $|X| = |Y|$

$$max \sum_{e \in E} w_e x_e \text{ Subject to}$$

$$\sum_{v \in e} x_e = 1 \forall v \in V; x_e \in \{0,1\} \forall e \in E$$

Min Vertex Cover Find $min|V'|, V' \subset V$ such that $\forall (u,v) \in e, e \in E$, either $u \in V'$ or $v \in V'$ or both.

$$\min \sum_{v \in V} x_v \text{ Subject to}$$

$$x_u + x_v \geq 1 \forall (u,v) \in E; x_u \in \{0,1\}$$

Maximum Independent Set $\max_{v \in V} x_v$ such that
$x_u + x_v \leq 1 \forall (u,v) \in E$, $x_u \in \{0,1\}$
ie Maximum size set such that no two vertices in it are connected by an edge.

Rounding for Other Two Problems

- **Minimum Vertex Cover**

$$S_{LP} = \{v \in V | x_v^* \geq \frac{1}{2}\}$$

$$|S_{LP}| = \sum_{v \in S_{LP}} 1 \leq \sum_{v \in S_{LP}} 2.x_v^* \leq \sum_{v \in V} 2.x_v^* \leq 2 \sum_{v \in V} y_v = 2|S_{OPT}| \quad (1)$$

Therefore,

$$|S_{OPT}| \leq |S_{LP}| \leq 2|S_{OPT}| \quad (2)$$

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- **Maximum Independent Set**
  - No useful bounds !!

LP Relaxation for 2 machines

- Two machines and J jobs. d_{ij} is the time for job j to run on machine i. We know the max time is the time taken by the machine ends last.

$$\min_{t, x_{ij}} t; Subjectto$$

$$\sum_j x_{ij} d_{ij} \leq t \forall i \in M$$

$$\sum_i x_{ij} = 1 \forall j \in J; x_{ij} \in \{0, 1\}$$

- Only one job will get split. **Why?** Let s be the job that gets split. We assign s to m_1 if $x_{1s} > x_{2s}$ otherwise, assign to m_2.
- If this leads to T_{approx} and T^* is the optimal solution then $T_{approx} \leq 2T^*$

$$T_{approx} \leq T' + T_s \text{ and } T' \leq T^*, T_s \leq T^*$$

$$T_{approx} \leq 2T^*$$

Where T' be optimal solytion before assigning s to one.

When does an LP Lead to Integer Solution?

- Note 1: The constraints of $Ax \leq b$ can be converted to $A^{'}x = b$ with additional variables.
- **Definition:** A matrix is Totally unimodular (TU) if all its square submatrices have a determinant in $\{-1, 0, 1\}$
- Note 2: If A is TU, $A' = [A, I]$ is also TU.
- If A is TU and b is integral, LP gives integral solutions.

When does an LP Lead to Integer Solution?

for BPG, A is TU

Consider a matrix A ($|V| \times |E|$). A has only two 1 in every column.

We can prove by Induction. Sketch.

- Q is 1×1. Q is TU Why?
- If sub-matrices of $(l-1) \times (l-1)$ are TU, Q then Q of $l \times l$ is also TU. **Why?**
  - Q has a column of no 1
  - Q has a solumn of one 1
  - All columns of Q have two 1s

Numerical Example