OM-M20-10: Factorization of Matrices

C. V. Jawahar

IIIT Hyderabad

http://preon.iiit.ac.in/om_quiz

11 Feb 2020

Cholesky And LU Decomposition

- **Cholesky** Every PD matrix A can be factorized as

$$A = LL^T$$

where L is a lower triangular matrix

$$\begin{bmatrix} a_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} l_{11} & L_{21}^T \\ 0 & L_{22}^T \end{bmatrix}$$

- **LU** A need not be PD; only needs to be non-singular.

$$\begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

Computation

Cholesky:

- $l_{11} = \sqrt{a_{11}}$ and $L_{21} = \frac{A_{21}}{l_{11}}$
- $L_{22}L_{22}^T = A_{22} - L_{21}L_{21}^T$; which can again be solved by Cholesky decomposition.
- $\frac{1}{3}n^3$ flops

LU

- $l_{11} = 1$ as L is a unit lower triangular matrix
- $u_{11} = a_{11}$
- $L_{21} = \frac{A_{21}}{a_{11}}$
- $U_{12} = A_{12}$
- $L_{22}U_{22} = A_{22} - \frac{1}{a_{11}}A_{21}A_{12}$; which can again be solved by LU decomposition.
- $\frac{2}{3}n^3$ flops.

Problems

Find Cholesky of

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 20 & 26 \\ 3 & 26 & 70 \end{bmatrix}$$

Find LU decomposition of

$$\begin{bmatrix} 6 & 3 & 1 \\ 2 & 4 & 3 \\ 9 & 5 & 2 \end{bmatrix}$$

QR Decomposition

- A square matrix Q is said to be orthogonal when $QQ^T = Q^TQ = I$
- QR Decomposition: $A = QR$
  - Q is orthogonal
  - R is an upper triangular matrix
- Why is this useful for us?
  - Because $QRx = b$ lets us write $Rx = Q^Tb$ which is easy to solve for x.
- Many problem can be solved faster (and reliably) by QR factorization
  - Least Squares
  - Least Norm
  - (more in the next lecture)

QR Decomposition

$$A = \begin{bmatrix} a_1 & A_2 \end{bmatrix}, Q = \begin{bmatrix} q_1 & Q_2 \end{bmatrix}, R \begin{bmatrix} r_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$

a_1 and q_1 are respectively first columns of A and Q. Notice that $Q^T Q = I$ implies

$$q_1^T q_1 = 1, q_1^T Q_2 = 0, Q_2^T Q_2 = I$$

$$\begin{bmatrix} a_1 & A_2 \end{bmatrix} = \begin{bmatrix} q_1 r_{11} & q_1 R_{12} + Q_2 R_{22} \end{bmatrix}$$

- $q_1 r_{11} = a_1$. Taking norm gives us $r_{11} = a_1$ and $q_1 = \frac{1}{a_1} a_1$
- $q_1 R_{12} + Q_2 R_{22} = A_2$. Multiply by q_1^T: $R_{12} = q_1^T A_2$
- $Q_2 R_{22} = A_2 - q_2 R_{12}$ which can be solved by QR decomposition.
- Cost $= \frac{4}{3} n^3$