OM-S20-12: $Ax = b$ to Constrained Optimization

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http://preon.iiit.ac.in/om_quiz

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Lagrangian and Constrained Optimization

$$\text{Max } f(x) \text{ subject to:} g(x) = 0$$

Lagrangian function:

$$L(x, \lambda) = f(x) - \lambda g(x)$$

$$\text{Maximize } \ f(x, y) \text{ Subject to: } g(x, y) = c$$
$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$$

Eg.

$$\text{Maximimize } \ x^2 y \text{ Subject to:} x^2 + y^2 = 1$$

$\frac{\partial L}{\partial x} = 0$ $\frac{\partial L}{\partial y} = 0$ and $\frac{\partial L}{\partial \lambda} = 0$ (Optimally conditions) leads to

$$2xy = 2x; x^2 = 2\lambda y \text{ and } x^2 + y^2 = 1$$

Ans: $(\pm\sqrt{\frac{2}{3}}, \pm\sqrt{\frac{1}{3}})$

$$\text{Maximimize } \ xy \text{ Subject to:} x + y = 10$$

Ans: 25

Example

Find the points on the circle $x^2 + y^2 = 80$ that is closes and farthest from (1,2)

$$\text{maximize } f(x, y) = (x-1)^2 + (y-2)^2$$

subject to:

$$x^2 + y^2 = 80$$

$\frac{\partial L}{\partial x} = 0$ $\frac{\partial L}{\partial y} = 0$ and $\frac{\partial L}{\partial \lambda} = 0$

$$\frac{(x-1)}{x} = \frac{(y-1)}{y} = \lambda$$

or $y = 2x$

$$x^2 + 4x^2 = 5x^2 = 80$$

$x = \pm 4$ and corresponding points are (4,8) and (-4,-8)

Two Examples

1. Least Norm

$$\text{Minimize } x^T x \text{ subject to } Ax = b$$

$$L(x, \lambda) = x^T x + \lambda^T [Ax - b]$$

$$2x + A^T \lambda = 0 \text{ and } Ax - b = 0$$

$$x = \frac{-A^T \lambda}{2} \text{ and } \frac{A(-A^T \lambda)}{2} - b = 0$$

$$\lambda = -2(AA^T)^{-1} b \text{ and } x = A^T (AA^T)^{-1} b$$

2. Solution as Eigen Vector

$$\text{Minimize } ||Ax||^2 \text{ Subject to:} ||x||^2 = 1$$

$$x^T A^T Ax - \lambda x^T x$$

or

$$A^T Ax = \lambda x$$

3. Homogeneous Equations: Ax = 0

$$Ax = 0$$

Trivial solution has the smallest norm and of no interest.

$$\text{Minimize}||Ax||^2 \text{ Subject to:}||x||^2 = 1$$

x is an eigen vactor of $A^T A$. Which one?
$x^T A^T Ax = \lambda^2$
Eigen vector corresponding to the which eigen value?
Solution using SVD: Which column of U/V?

Comments

SVD

- Pseudo inverse: $D_{ii}^{-1} = \frac{1}{D_{ii}}$ if $> \theta$ else zero.
- Rank of a matrix is equal to the number of nonzero singular values
- Low rank approximation using SVD.

Problem of Interest

1. Solve $Ax = b$ when A is square and full rank.
2. Least square solution for $Ax = b$
3. Least norm solution for $Ax = b$
4. Non-trivial solution to $Ax = 0$