

SVD, LEAST SQUARES AND LEAST NORM

singular value decomposition (SVD) is a very powerful and popular matrix factorization.

$$A = UDV^T$$

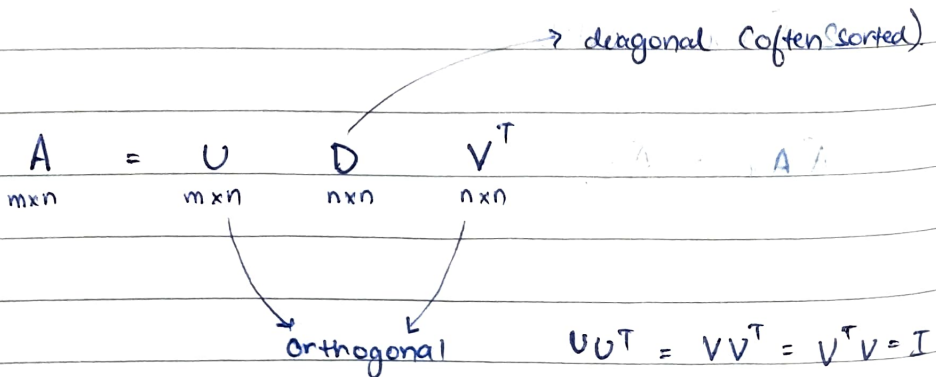
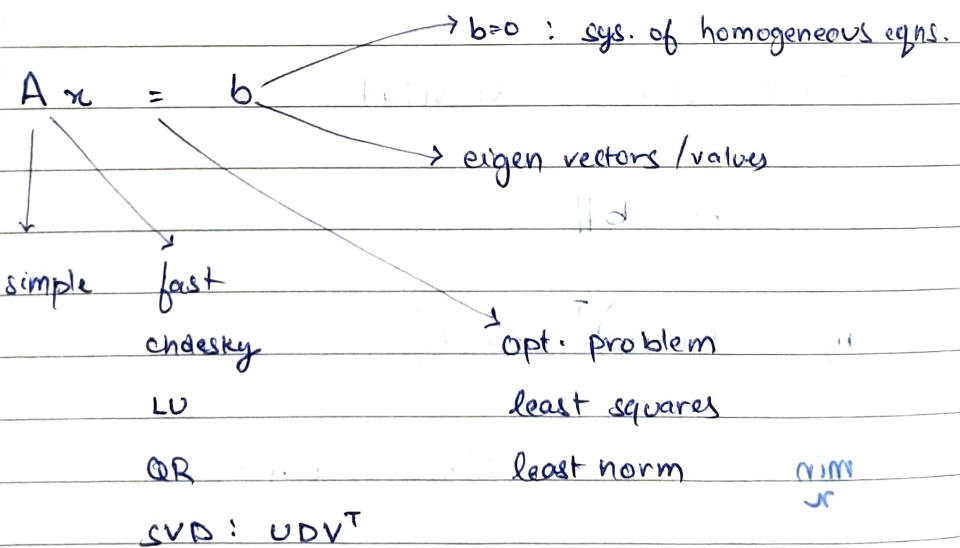
$$A: m \times n$$

$$U: m \times n \quad \text{orthogonal.}$$

$$D: n \times n \quad \text{diagonal (Di: the singular values)}$$

$$V: n \times n \quad \text{orthogonal.}$$

$$U^T U = V^T V = V V^T = I$$



$$A = LU$$

① $Ax = b$ factorize A on LU .

↓

$$LUx = b$$

②a $Lw = b$

②b $Ux = w$

$$A = UDV^T$$

① factorize A as UDV^T

②a compute $P = U^T b$

②b solve $Dw = P$

②c solve $V^T x = w$

$$A \Rightarrow x = Vw$$

MORE ON SVD

relationship to eigen values and eigen vectors.

$$A^T A \quad A^T A V = VD^2 \quad A A^T U = UD^2$$

finding transpose $A^T = VDU^T$

finding inverse $A^{-1} = VD^{-1}U^T$

$$A^{-1} = (UDV^T)^{-1} = VD^{-1}U^T$$

$$A^T = (UDV^T)^T = VDU^T$$

$$A^T A = (UDV^T)^T UDV^T$$

$$= \cancel{VDU^T} UDV^T = VD^2V^T$$

$$A^T A V = VD^2$$

↘ diagonal

$$= D^2 V$$

$$A^T A V_i = D_{ii}^2 V_i$$

V_i is the eigen vector,

(25)

D_{ii}^2 is the eigen value of $A^T A$

$$A^T A V_i = D_{ii}^2 V_i$$

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V_i is the eigen vector,

D_{ii}^2 is the eigen value of $A^T A$

sorting D_{ii} and other matrices won't change A .

$$A = UDV^T$$

$$A' = \begin{bmatrix} & & & \\ & & & \\ & & 0 & \\ & & & \end{bmatrix}$$

$$\min_{A'} \|A - A'\|$$

$$\text{rank}(A') = k$$

A' is the nearest rank k matrix to A .

$$A = A^{-1}$$

$(A + \lambda I)$: regularization

D_{ii} = eigen values of $A^T A$, $A A^T$

LEAST SQUARE PROBLEM AND SOLUTION

given A and target vector b . least square error (MSE) problem is:

$$\min_x \|Ax - b\|$$

$$\min_x [Ax - b]^T [Ax - b]$$

$$\min_x (x^T A^T A x + b^T b - 2x^T A^T b)$$

$$2A^T A x - 2A^T b = 0 \quad (\text{quadratic differentiation})$$

$$A^T A x = A^T b$$

to obtain x , solve the above equation:

$$x = (A^T A)^{-1} A^T b.$$

LEAST NORM SOLUTION

minimize $\|x\|$

st. $Ax = b.$

soln: $x^* = A^T (A A^T)^{-1} b$

x^* satisfy the $Ax = b$

no other vector $x = x^* + (x - x^*)$ can be smaller.

$$\|x\|^2 = \|x^*\|^2 + \|x - x^*\|^2 + 2x^{*T}[x - x^*]$$

third term is zero

$$\|x\|^2 = \|x^*\|^2 + \|x - x^*\|^2$$

$$\|x\|^2 = \|x^*\|^2 + \text{positive term}$$

hence x^* is the smallest.

EFFICIENT SOLUTIONS FOR LEAST SQUARES AND LEAST NORM

least squares

$$x^* = (A^T A)^{-1} A^T b$$

$$\text{or solve } A^T A x = A^T b \quad A^T$$

least norm

$$x^* = A^T (A^T A)^{-1} b$$

how do we use:

cholesky

LU

QR

SVD.

$$Ax = b$$

$m \times n$

$n \times 1$

$m \times 1$

$$m > n$$

(more rows)

LSE

MSE

$$\min \|Ax - b\|$$

$$x = (A^T A)^{-1} A^T b$$

$$\text{Solve } \underbrace{A^T A}_{\downarrow \text{PSD}} x = A^T b$$

PSD

$$m < n$$

(more columns)

LN

$$\min \|x\|$$

$$\text{s.t. } (Ax = b)$$

$$x = A^T (A A^T)^{-1} b$$