SOLVING AX = B

solving $Ax = b$

when A is non-singular square matrix

when A has some special properties.

avoid direct inversion

problems of the form: $Ax_i = b_i$ for $i = 1, 2$

least square problem when A has more rows than columns.

a solution that fits "best" all the equations.

least norm problem when A has more columns than rows.

among the many, a unique one such as with least norm.

SPECIAL SIMPLE CASES

$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

identity matrix

$A = I \quad x = A^{-1}b = b$

$\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}$

permutation matrix

A is a permutation matrix; exactly 1 entry is 1 in each row and each column, and 0s elsewhere.

$\begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$

diagonal matrix

A is a diagonal matrix

$\begin{bmatrix} 1 & & \\ 4 & 2 & \\ 5 & 6 & 3 \end{bmatrix}$

triangular matrix

A is a triangular matrix

class	complexity
I	$O(1)$
P	$O(1)$
D	$O(n)$
U/L Tr.	$O(n^2)$

$\max \; c^T x$ → $A'x = b$ | $x \in \mathbb{R}^n \rightarrow$ int. soln.

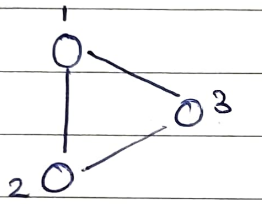
$Ax \leq b$

$x \geq 0$

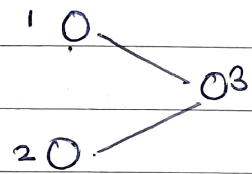
if A in TU & b is int.

$Ax = b$ | $TU \Rightarrow \{-1, 0, 1\}$

$x = A^{-1} b = \frac{1}{|A|} \, CF(A) \, [b]$



(E)



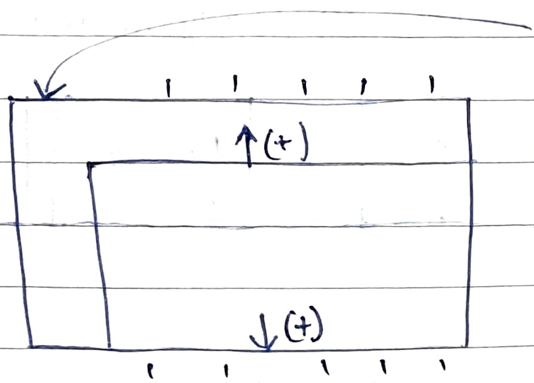
$$(v) \quad \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

A for bi-partite graph is TU. X | Y

all elements either 0 or 1.

all submatrices of 1×1? are TU.

if every n×n submatrices are TU
all (n+1) × (n+1) matrices are TU



0 1s

1 1s

2 1s

x

y

rows of this matrix are linearly independent

$Ax = b$ → least square

→ least norm.

→ sqrt non-singular.

$x = A^{-1}b.$

CHOLESKY AND LU DECOMPOSITION

Cholesky

every PD matrix A can be factorized as

$$A = LL^T$$

where L is a lower triangular matrix

$$\begin{bmatrix} a_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} l_{11} & L_{21}^T \\ 0 & L_{22} \end{bmatrix}$$

LU

- A need not be PD; only needs to be non-singular.

$$\begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

PD: positive ~~disparate~~ definite

$$\left[\begin{array}{c|c} a_{11} & 0 \\ \hline A_{21} & A_{22} \end{array}\right] \begin{bmatrix} x_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ B_2 \end{bmatrix}$$

$A_{21} \to (n-1)\times 1$ $\quad A_{22} \to (n-1)\times(n-1)$ $\quad X_2 \to (n-1)\times 1$ $\quad B_2 \to (n-1)\times 1$

$x_1 = \dfrac{b_1}{a_{11}}$; $A_{12}\, x_{11} + A_{22} X_2 = B_2$

lower Δ matrix.

$Ax = b$

$\downarrow$ ~~costly step~~

$LL^T x = b$

$$\left(\begin{array}{l} Lv = b \\ L^T x = v \end{array} \right)$$

$$A \begin{bmatrix} | & | & \\ x_i & x_i & \cdots \\ | & | & \end{bmatrix} = \begin{bmatrix} | & | & \\ b_i & b_i & \cdots \\ | & | & \end{bmatrix}$$

$A[x_i] = [b_i]$

max. independent set

\+ $\qquad = |V|$

min. vertex cover

COMPUTATION

cholesky

$I_{11} = \sqrt{a_{11}}$ and $L_{21} = \frac{A_{21}}{I_{11}}$

$L_{22} L_{22}^T = A_{22} - L_{21} L_{21}^T$

which can again be solved by cholesky decomposition

$\frac{1}{3} n^3$ flops.

LU

$I_{11} = 1$ as L is a unit lower Δ matrix

$u_{11} = a_{11}$

$L_{21} = \frac{A_{21}}{a_{11}}$

$U_{12} = A_{12}$

$L_{22} U_{22} = A_{22} - \frac{1}{a_{11}} A_{21} A_{12}$

which can again be solved by LU decomposition.

$\frac{2}{3} n^3$ flops

TIME COMPLEXITY OF SOLVING AX=B BY CHOLESKY LU

factorization step - cholesky

$\frac{1}{3}n^3$, LU: $\frac{2}{3}n^3$

forward substitution (n^2 flops):
Solve $Lw = b$ for z.

backward substitution (n^2 flops):
Solve $Ux = w$ for x.