Optimization Methods
Tutorial-1

Branch and Bound Technique

- Divide a problem into subproblems
- For each subproblem:
  - If it has no feasible solution, done;
  - If it has an integer optimal solution; done. Compare the optimal solution with the best solution we know till now.
  - If it has an optimal solution that is worse than the previous best solution, done.
  - If it has an optimal solution that are not all integer, better than the previous best solution, then we would have to divide this subproblem further and repeat.

$$Max\ Z = -x_1 + 4x_2$$

Such that

$$\begin{aligned} -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ x_1 &\leq 5 \end{aligned}$$

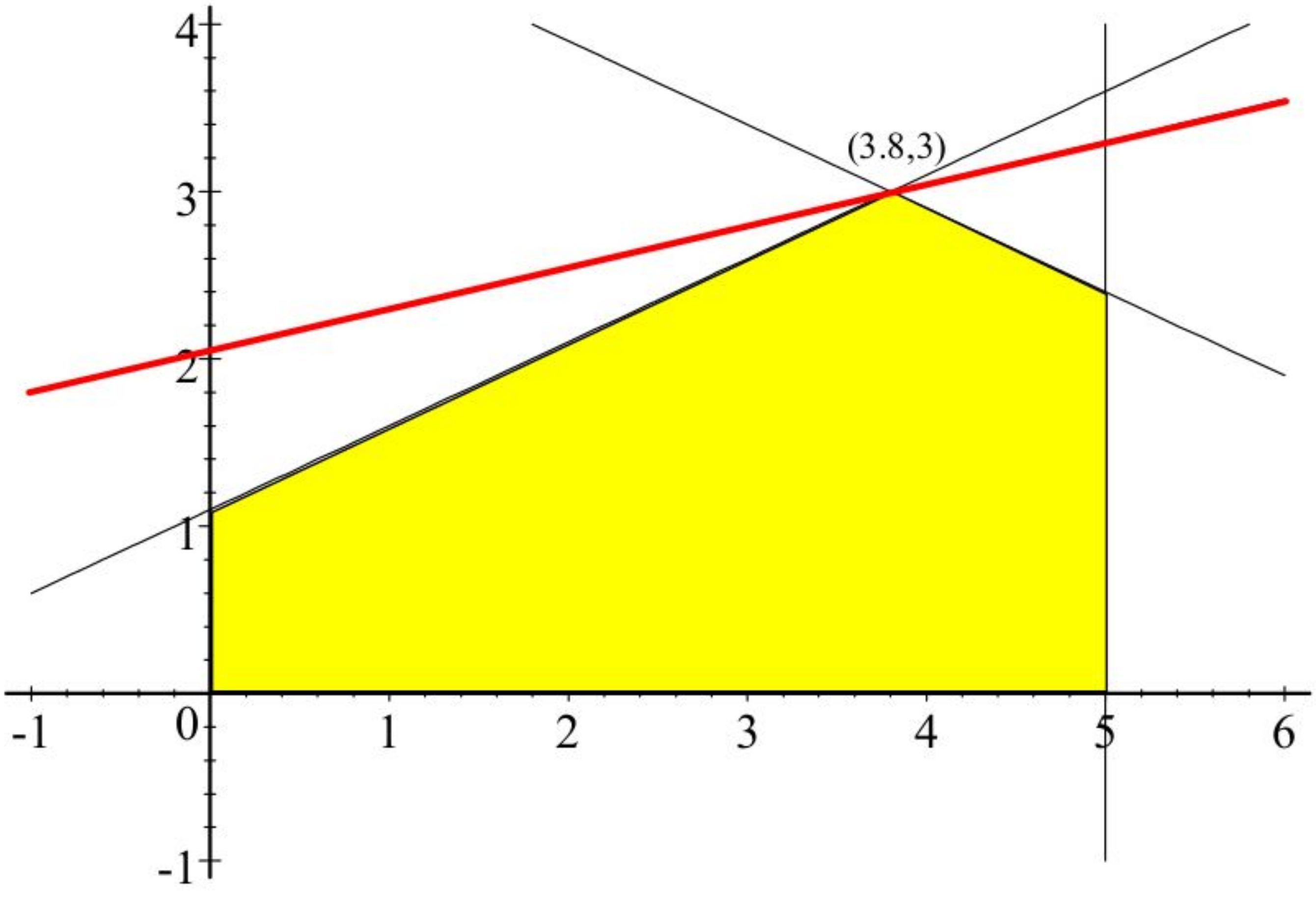
$$x_i \geq 0,\ \ x_i\text{'s are integers}$$

For the LP relaxation

$$MaxZ = -x_1 + 4x_2$$

$$\begin{aligned} \text{s. t. } -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ x_1 &\leq 5 \\ x_i &\geq 0 \end{aligned}$$

Optimal solution of the relaxation is $(3.8, 3)$ with $z = 8.2$. Then we consider two cases: $x_1 \geq 4$ and $x_1 \leq 3$.

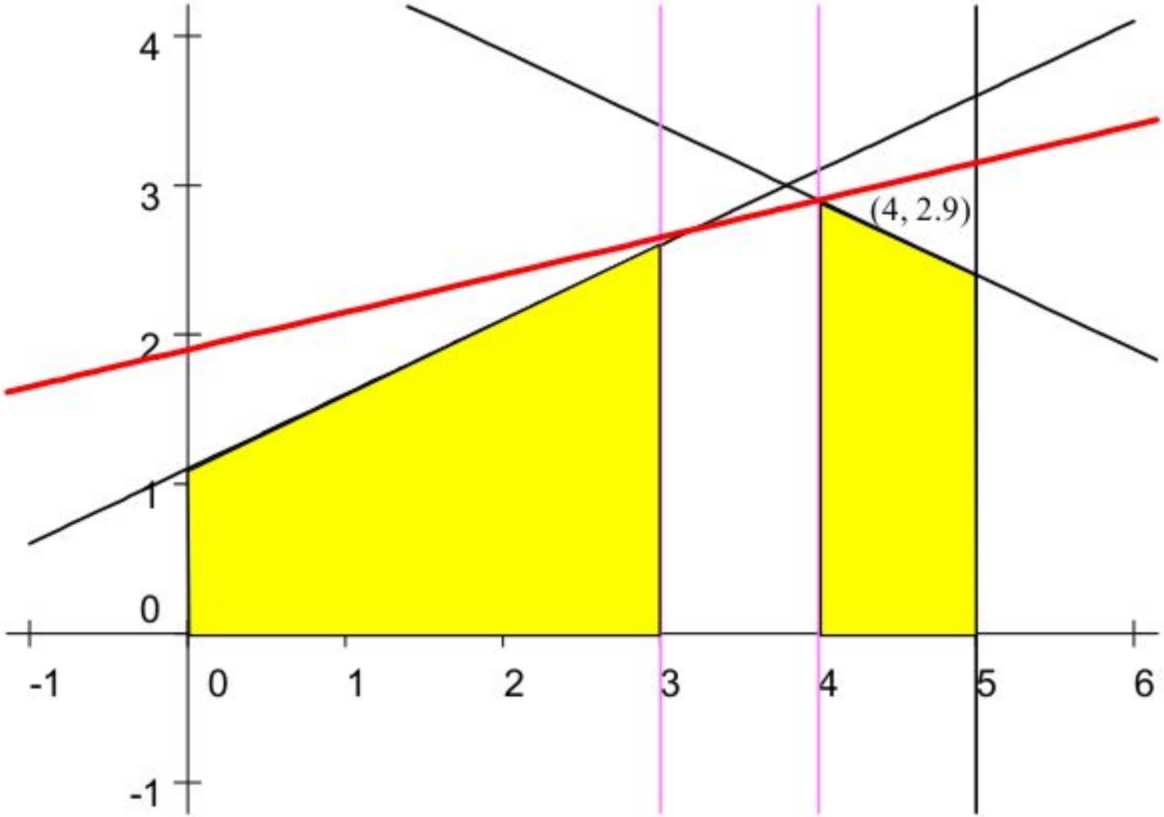


(3.8,3)
4
3
2
1
-1
-1
0
1
2
3
4
5
6

The linear programming relaxation

$$
\begin{aligned}
\text{Max } Z = -x_1 + 4x_2 & & \\
\text{s.t. } -10x_1 + 20x_2 &\leq& 22 \\
5x_1 + 10x_2 &\leq& 49 \\
x_1 &\leq& 5 \\
x_1 &\geq& 4 \\
x_2 &\geq& 0
\end{aligned}
$$

has optimal solution at $(4, 2.9)$ with $Z = 7.6$.


(4, 2.9)
4
3
2
1
0
-1
-1
0
1
2
3
4
5
6

The linear programming relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 4 &\leq x_1 \leq 5 \\ x_2 &\geq 3 \end{aligned}$$

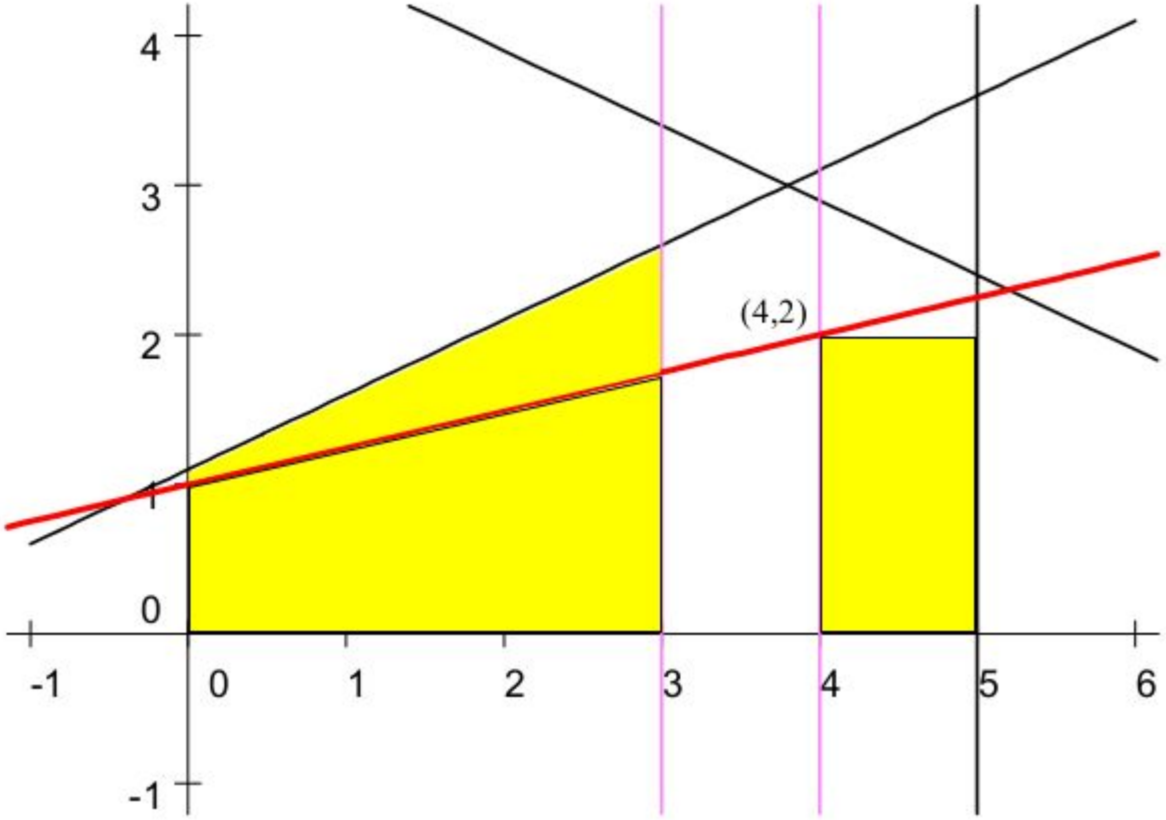
has no feasible solution $(5x_1 + 10x_2 \geq 50)$ so the IP has no feasible solution either.

The linear programming relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned}
\text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\
5x_1 + 10x_2 &\leq 49 \\
4 &\leq x_1 \leq 5 \\
0 &\leq x_2 \leq 2
\end{aligned}$$

has an optimal solution at $(4, 2)$ with $Z = 4$. This is the optimal solution of the IP as well. Currently, the best value of Z for the original IP is $Z = 4$.

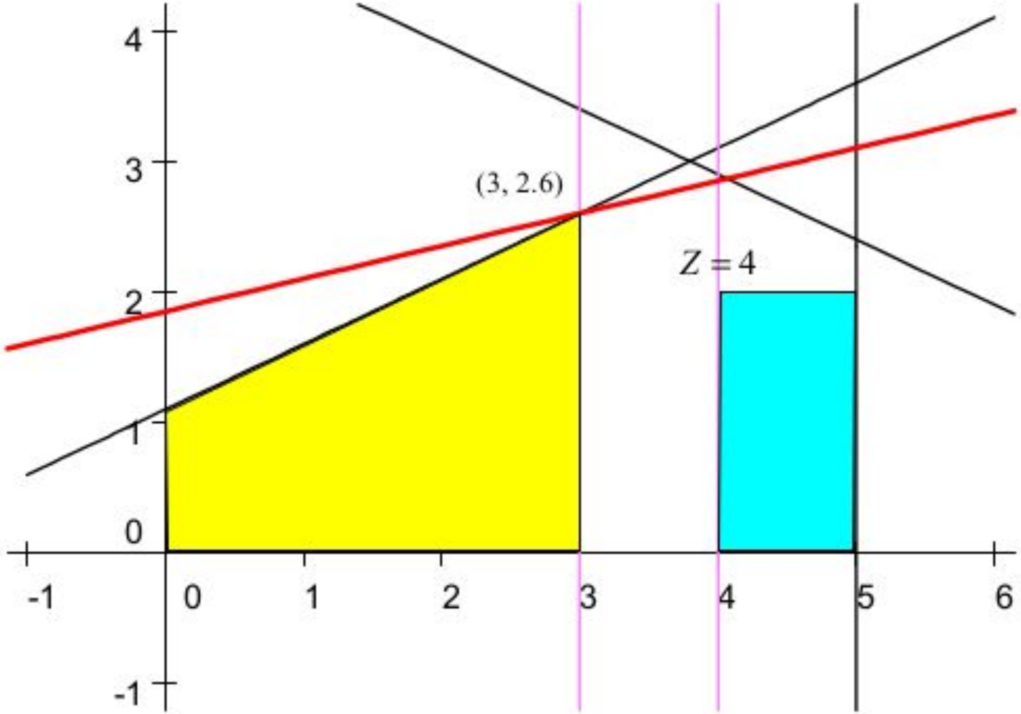

4
3
2
1
0
-1
-1
0
1
2
3
4
5
6
(4,2)

Now we consider the branch of $0 \le x_1 \le 3$. The LP relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t.} \quad -10x_1 + 20x_2 &\le 22 \\ 5x_1 + 10x_2 &\le 49 \\ x_1 &\le 3 \\ 0 &\le x_i \end{aligned}$$

has an optimal solution at $(3, 2.6)$ with $Z = 7.4$. We branch out further to two cases: $x_2 \le 2$ and $x_2 \ge 3$.



(3, 2.6)
Z = 4

The LP relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ x_1 &\leq 3 \\ x_2 &\geq 3 \end{aligned}$$

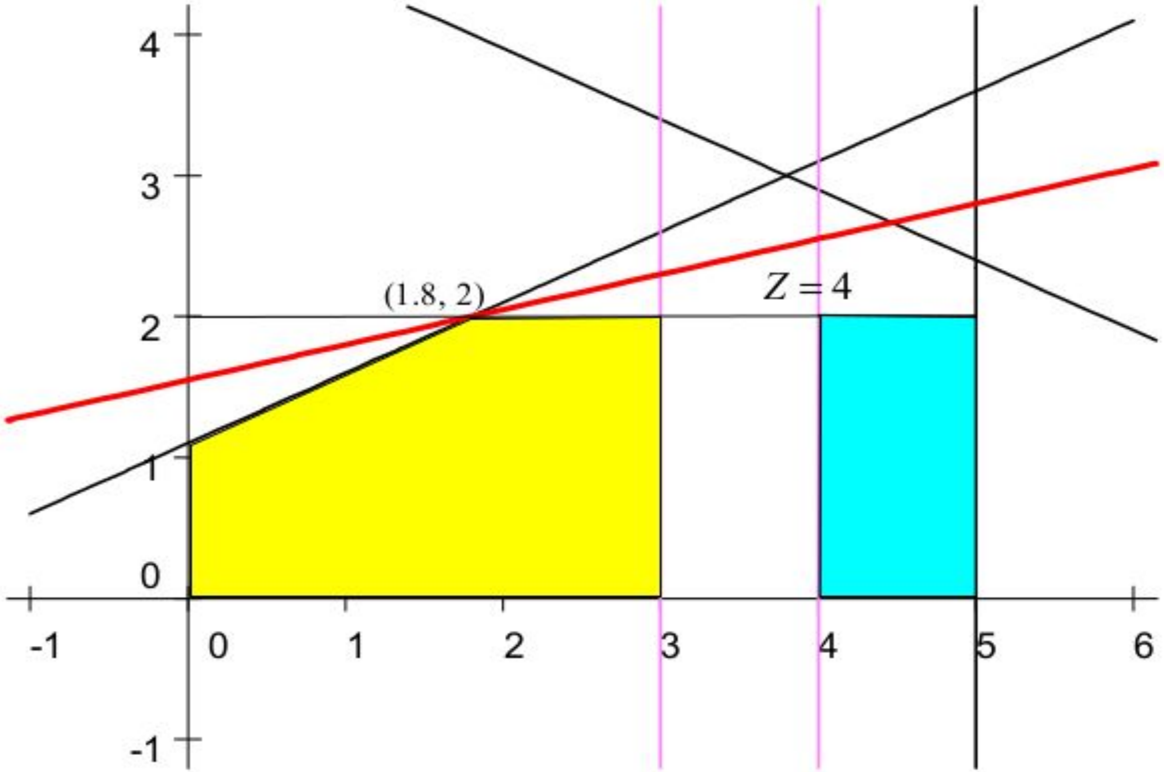
has no feasible solution ($-10x_1 + 20x_2 \geq 30$). The IP has no solution either.

Quiz

Solve:

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned} \text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 0 &\leq x_1 \leq 3 \\ 0 &\leq x_2 \leq 2 \end{aligned}$$



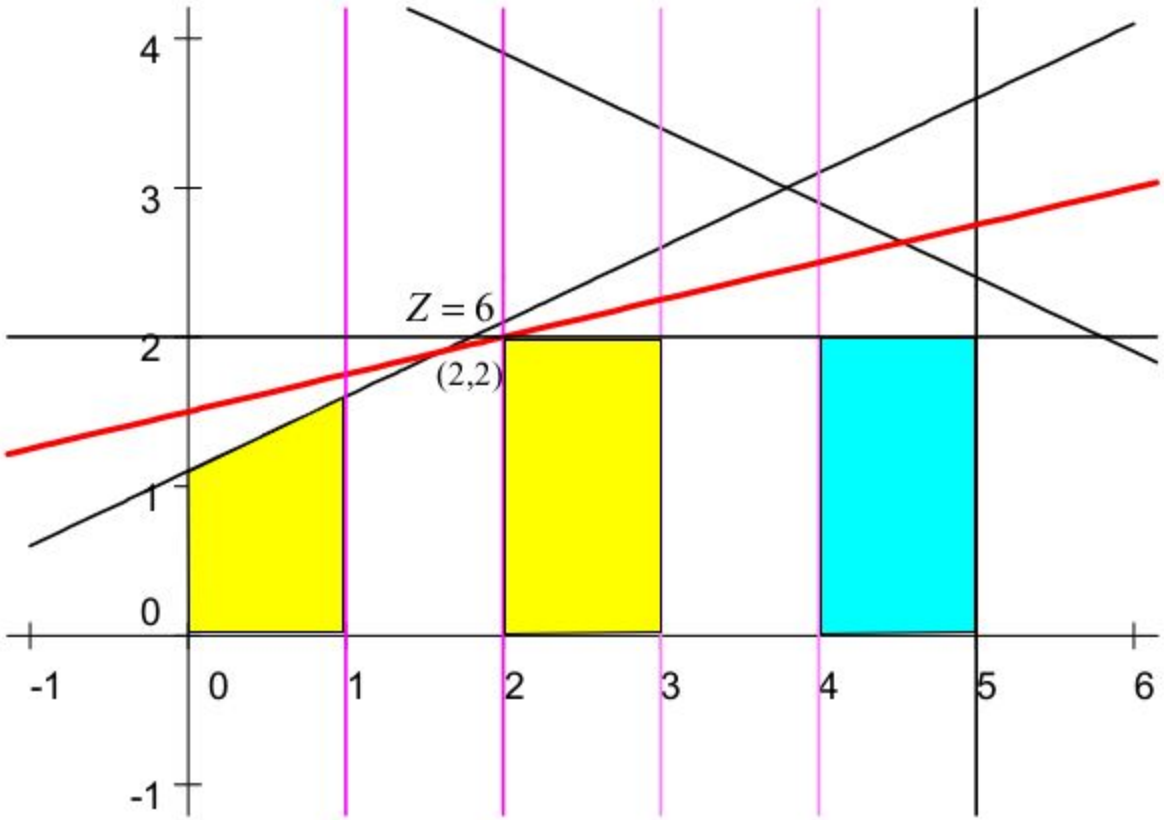
We branch further with two cases: $x_1 \geq 2$ or $x_1 \leq 1$ (we still have $0 \leq x_2 \leq 2$).

The LP relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned}\text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\ 5x_1 + 10x_2 &\leq 49 \\ 2 &\leq x_1 \leq 3 \\ 0 &\leq x_2 \leq 2\end{aligned}$$

has an optimal at $(2, 2)$, with $Z = 6$. Since this is better than the incumbent $Z = 4$ at $(4, 2)$, this new integer solution is our current best solution.



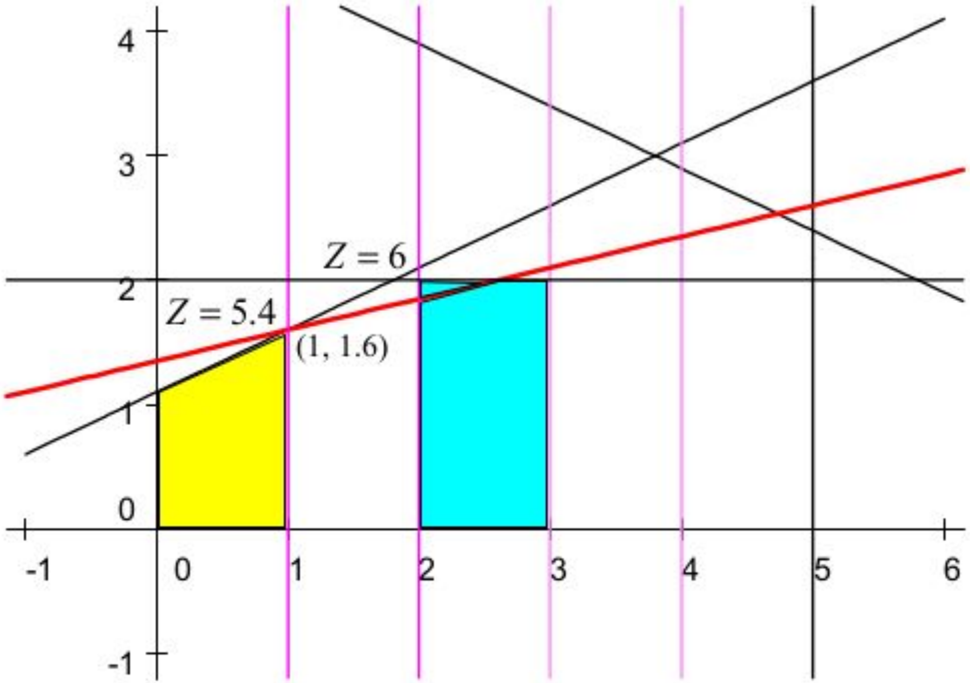
Z = 6
(2,2)
4
3
2
1
0
-1
-1
0
1
2
3
4
5
6

The LP relaxation

$$\text{Max } Z = -x_1 + 4x_2$$

$$\begin{aligned}
\text{s.t.} \quad -10x_1 + 20x_2 &\leq 22 \\
5x_1 + 10x_2 &\leq 49 \\
0 &\leq x_1 \leq 1 \\
0 &\leq x_2 \leq 2
\end{aligned}$$

has an optimal at $(1, 1.6)$ with $Z = 5.4$. Then any integer solution in this region can not give us a solution with the value of Z greater than 5.4. This branch is fathomed.



So, optimal solution is at x1 = 2, x2 = 2, and maximum value of Z is 6.