OM-S20-07: LP Relaxation

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Review

- LP and IP formulations
- Graphical Method to solve
- Branch and Bound for IP
- Bala's Method for BIP

Graph Problems (Revisit)

Maximum weight bipartite matching Given a bipartite graph G(V,E) with $|X| = |Y|$

$$max \sum_{e \in E} w_e x_e \ \text{ Subject to}$$

$$\sum_{v \in e} x_e = 1 \forall v \in V; x_e \in \{0, 1\} \forall e \in E$$

Min Vertex Cover Find $min|V'|, V' \subset V$ such that $\forall (u, v) \in e, e \in E$, either $u \in V'$ or $v \in V'$ or both.

$$\min \sum_{v \in V} x_v \ \text{ Subject to}$$

$$x_u + x_v \geq 1 \forall (u, v) \in E; x_u \in \{0, 1\}$$

Maximum Independent Set $\max_{v \in V} x_v$ such that $x_u + x_v \leq 1 \forall (u, v) \in E$, $x_u \in \{0, 1\}$
ie Maximum size set such that no two vertices in it are connected by an edge.

LP Relaxation

- Convert IP to LP.
- Call this as LP relaxation of original problem.
- $opt(LP) \leq opt(IP)$ (for a minimization problem).
- If $x_{LP} \in Z$ then we got lucky and in this case both $opt(LP)$ and $opt(IP)$ are same.
- Formulate a rounding procedure that transforms x_{LP} into an integral solution x' such that $cost(x') \leq c * cost(x_{IP})$.
- Then we say x' is a c-approximate optimal solution to the original problem.
- We give our final answer as x'.
- It's crucial to be able to get x', given x_{LP} and it is important that we understand how good the approximation is (c value)
- In case LP is infeasible, what does this tell us about feasibility of IP?

Rounding in BPG Matching

- What does it mean if infeasible? bound?
- Assume LP gives you $x_e \in [0, 1]$
- If an edge is not in $\{0, 1\}$, then there should be another edge in every vertex with the same situation.
- If we round 0.9 to 1.0, there there should be another edge that needs to reduce by 0.1 at the vertex.
- Let the edges change by ϵ, $x_i^* + \epsilon$, there will be a $x_j^* - \epsilon$
- There are cycles of such non-saturated edges. Let the new weights be y
- $w(y) = w(x^*) + \epsilon \sum_i (-1)^i w_{e_i} = w(x^*) + \epsilon\Delta$
- Where Δ is $\sum_{i=1}^{t} (-1)^i w_{e_i}$
- Since x^* is optimal, Δ has to be zero.
- Repeat this for all cycles. We will reach integer solution!!

Rounding for Other Two Problems

- **Minimum Vertex Cover**

$$S_{LP} = \{v \in V | x_v^* \geq \frac{1}{2}\}$$

$$|S_{LP}| = \sum_{v \in S_{LP}} 1 \leq \sum_{v \in S_{LP}} 2.x_v^* \leq \sum_{v \in V} 2.x_v^* \leq 2 \sum_{v \in V} y_v = 2|S_{OPT}| \quad (1)$$

Therefore,

$$|S_{OPT}| \leq |S_{LP}| \leq 2|S_{OPT}| \quad (2)$$

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- **Maximum Independent Set**
  - No useful bounds !!

`http://tiny.cc/b9j9iz`