OM-S20-15: Simplex method - I

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LP, Slack variables and BFS

Earlier we had formulated LP as:

$$\min_x c^T x, Ax \leq b, x \geq 0$$

where A is m x n matrix, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.
We add auxiliary/slack variables $x_{n+1}, x_{n+2}, ..x_{n+m}$ to turn inequalities in constraints to equations:

$$\min_x c^T x, Ax = b, x \geq 0$$

This leads to: $Ax = b$ where A is a m x (m+n) and $b \in \mathbb{R}^m$.

- A basic solution to a system of m linear equations in n unknowns ($n \geq m$) is obtained by setting $n - m$ variables to 0 and solving the resulting system to get the values of the other m variables.
- The variables set to 0 are called **nonbasic variables**; the variables obtained by solving the system are called **basic variables**.
- A basic solution that is feasible, is called **basic feasible solution**. They are the "vertices/corners" of the feasible regions.

Basic and Non-Basic Variables

Simplex algorithm (V0):

1. Start from any vertex as a feasible solution.
2. If a neighbouring vertex is better, solution is updated.
3. Repeat step2 until there is no better vertex.

From the set of n variables, we choose m basic variables as set B, $B \subseteq \{1, 2, ..n\}$ and set the rest to 0: $x_i = 0 \forall i \notin B$

$$[A_{B(1)}, A_{B(2)}, .., A_{B(m)}]x_B = b$$

$$\bar{B}x_B = b$$

where $\bar{B} = [A_{B(1)}, A_{B(2)}, .., A_{B(m)}]$ is a m x m matrix, $x_B \in \mathbb{R}^m, b \in \mathbb{R}^m$
This gives us a BFS: $x | x_B = \bar{B}^{-1}b, x_i = 0, i \notin B$
The smart thing to do is: Given $B_1^{-1}b$, how do we efficiently find $B_2^{-1}b$, and so on.

Basic Feasible Solution (BFS)

Let $x_{new} = x + \theta d$

Let j be the entering variable and l be the exiting variable. $d_j = 1$ and $d_i = 0$ for i not in the basic variabel set.

$$Ax_{new} = A(x + \theta d) = Ax = b \implies Ad = 0$$

$$\sum_{i=1}^{m} d_{B(i)} A_{B(i)} + A_j = 0, \implies \bar{B} d_B + A_j = 0, \implies d_B = -\bar{B}^{-1} A_j$$

which j to pick? For a given variable j, difference in cost due to j^{th} variable being basic is: $c_B^T d_B + c_j = c_j - c_B^T B^{-1} A_j$. We do for each j and select:

$$\bar{C} = [C_1, C_2, .., C_n]$$

$$\text{where } \bar{C}_j = C_j - C_B^T B^{-1} A$$

Simplex V1

Algorithm:

1. Start with an initial BFS
2. Repeat
   - Calculate $\bar{C}$, if all ≥ 0, then stop
   - Select a j for $\bar{C}$ s.t. $\bar{C}_j < 0$
   - $d_B = -B^{-1}A$
   - Find l and θ s.t. θ is increased till basic variable becomes 0:

     $$x_l + d_l\theta = 0$$

   - $B = B - \{l\} + \{j\}$
3. Current BFS is optima.

$$\textbf{Note}: \ \theta^* = \min_{i=1,\ldots,m | d_B(i)<0} \left(-\frac{x_{B(i)}}{d_{B(i)}} \right)$$

Next: Given B_1^{-1}, how do we find B_2^{-1} where B_2 is a new matrix whose column corresponding to l is replaced by that of j. i.e., $B_1^{-1}B_1 = I$. $B_1^{-1}B_2 = J$. Problem: Given J, use row transformations to convert to I.

Examples

1. Minimize $-x_1 - x_2$
   Subject to:
$$-x_1 + x_2 \leq 1; x_1 \leq 3; x_2 \leq 2; x_1, x_2 \geq 0$$
2. Minimize $z = x_1 + x_2$ Subject to
$$x_1 + 5x_2 \leq 5; 2x_1 + x_2 \leq 4; x_1, x_2 \geq 0$$
3. Minimize $z = -10x_1 - 12x_2 - 12x_3$ Subject to:
$$x_1 + 2x_2 + 2x_3 \leq 20$$
$$2x_1 + x_2 + 2x_3 \leq 20$$
$$2x_1 + 2x_2 + x_3 \leq 20$$
$$x_1, x_2, x_3 \geq 0$$