OM-M20-09: Solving Ax=b

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http://tiny.cc/i9lhjz

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Problems of Interest

- Solving $Ax = b$;
  - When A is non singular square matrix.
  - When A has some special properties.
  - Avoid direct inversion
  - Problems of the form: $Ax_i = b_i$ for $i = 1, 2,$
- Least square problem when A has more rows than columns.
  - A solution that fits "best" all the equations
- Least norm problem when A has more columns than rows.
  - Among the many, a unique one such as with least norm.

Special Simple Cases

- **Identity Matrix:** $A = I$; $x = A^{-1}b = b$
- **Permutation Matrix:** A is a permutation matrix: exactly one entry is 1 in each row and each column and 0s elsewhere
- **Diagonal Matrix:** A is diagonal matrix
- **Triangular Matrix** A is a triangular matrix
- Time Complexities:

Class	Complexity
I	O(1)
P	O(1)
D	O(n)
U/L Tr.	$O(n^2)$

Cholesky And LU Decomposition

- **Cholesky** Every PD matrix A can be factorized as

$$A = LL^T$$

where L is a lower triangular matrix

$$\begin{bmatrix} a_{11} & A_{21}^T \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} l_{11} & L_{21}^T \\ 0 & L_{22}^T \end{bmatrix}$$

- **LU** A need not be PD; only needs to be non-singular.

$$\begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

Note: A light read on PD matrices:
`https://www.math.utah.edu/~zwick/Classes/Fall2012_2270/Lectures/Lecture33_with_Examples.pdf`

Computation

Cholesky:

- $l_{11} = \sqrt{a_{11}}$ and $L_{21} = \frac{A_{21}}{l_{11}}$
- $L_{22}L_{22}^T = A_{22} - L_{21}L_{21}^T$; which can again be solved by Cholesky decomposition.
- $\frac{1}{3}n^3$ flops

LU

- $l_{11} = 1$ as L is a unit lower triangular matrix
- $u_{11} = a_{11}$
- $L_{21} = \frac{A_{21}}{a_{11}}$
- $U_{12} = A_{12}$
- $L_{22}U_{22} = A_{22} - \frac{1}{a_{11}}A_{21}A_{12}$; which can again be solved by LU decomposition.
- $\frac{2}{3}n^3$ flops.

Time Complexity of solving Ax=b by Cholesky/LU

- Factorization step - Cholesky: $(1/3)n^3$, LU: $(2/3)n^3$
- Forward Substitution(n^2 flops): Solve $Lw = b$ for z
- Backward Substitution(n^2 flops): Solve $Ux = w$ for x