AX=B TO CONSTRAINED OPTIMIZATION

LAGRANGIAN AND CONSTRAINED OPTIMIZATION

max $f(x)$ subject to $g(x) = 0$.

lagrangian function:

$$L(x, \lambda) = f(x) - \lambda g(x)$$

maximize $f(x,y)$ st. $g(x,y) = c$

e.g.

maximize $x^2 y$ st. $x^2 + y^2 = 1$

$\frac{\partial L}{\partial x} = 0$ $\quad \frac{\partial L}{\partial y} = 0$ $\quad \frac{\partial L}{\partial \lambda} = 0$

(optimally conditions) leads to

$$2xy = 2x$$

$$x^2 = 2\lambda y$$

and

$$x^2 + y^2 = 1$$

ans: $\left(\pm\sqrt{\frac{2}{3}}, \pm\sqrt{\frac{1}{3}}\right)$

maximize xy st. $x+y=10$

ans: 25.

$Ax = b$

LS — $m > n$ — $x^* = (A^TA)^{-1}A^Tb$

LN — $m < n$ — $x^* = A^T(AA^T)^{-1}b$

$\min \|x\|$

L0 — hard sparse x

L1

L2

EV soln. is an eigen vector

ex $xy - \lambda(x+y-10)$ || same for $L = xy + \lambda(x+y-10)$

$L = xy - \lambda x - \lambda y + \lambda 10$

$\frac{\partial L}{\partial x} = y - \lambda = 0$ —— (1)

$\frac{\partial L}{\partial y} = x - \lambda = 0$ —— (2)

$\frac{\partial L}{\partial \lambda} = -x - y + 10 = 0$ —— (3)

$x = y = \lambda$

$\therefore 2\lambda = 10 \Rightarrow x = y = \lambda = 5.$

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EXAMPLE

find the points on the circle $x^2+y^2=80$ that is closest and farthest from $(1,2)$.

maximize $f(x,y) = (x-1)^2 + (y-2)^2$

st. $x^2+y^2=80$

$$\frac{\partial L}{\partial x}=0 \qquad \frac{\partial L}{\partial y}=0 \qquad \frac{\partial L}{\partial \lambda}=0$$

$$\frac{x-1}{x} = \frac{y-1}{y} = \lambda$$

or $y = 2x$

$$x^2+4x^2 = 5x^2 = 80$$

$x = \pm 4$ and corresponding points are $(4,8)$, and $(-4,-8)$

TWO EXAMPLES

<u>least norm</u>

minimize $x^T x$ st. $Ax = b$

$$L(x, \lambda) = x^T x + \lambda^T [Ax - b]$$

$$2x + A^T \lambda = 0 \quad \& \quad Ax - b = 0$$

$$x = \frac{-A^T \lambda}{2} \quad \text{and} \quad \frac{A(-A^T \lambda)}{2} - b = 0$$

$$\lambda = -2(AA^T)^{-1} b \quad \text{and} \quad x = A^T (AA^T)^{-1} b.$$

Soln. as eigen vector

minimize $\|Ax\|^2$ st. $\|x\|^2 = 1$

$$x^T A^T A x - \lambda x^T x$$

or, $A^T A x = \lambda x$

$$L(x, \lambda) = x^T x + \lambda^T [Ax - b]$$

if you have multiple const. λ = vector
connected wrt. λ.

$2x + y = 1$

$$L(\lambda x) = x^T A^T A x - \lambda x^T x$$

$$\frac{\partial}{\partial x} = 0 \Rightarrow A^T A x = \lambda x$$

$$\Sigma x = \lambda x$$

matrix

eigen vector

eigen value

HOMOGENEOUS EQUATIONS : $AX = 0$

$Ax = 0$

trivial soln. has the smallest ~~soln~~. norm and of no interest.

many solutions with scale factor, so let norm of $x = 1$

minimize $||Ax||$ st. $||x|| = 1$

x is an eigen vector of A^TA. which one?

$x^T A^T A x = \lambda^2$

eigen vector corresponding to which eigen value?

Soln. using SVD : which column of U/V?

$Ax = \lambda x$

λ^2 to be zero - how you get smallest?

min $\|Ax\|$

Same as min $\|Ax\|^2$ —— EV.

$A = UDV^T$

(1) 1st form, A is non-singular & square.

(2). optimize least squares, least norm.

closed form soln.

global optimality. convex optimization problem.

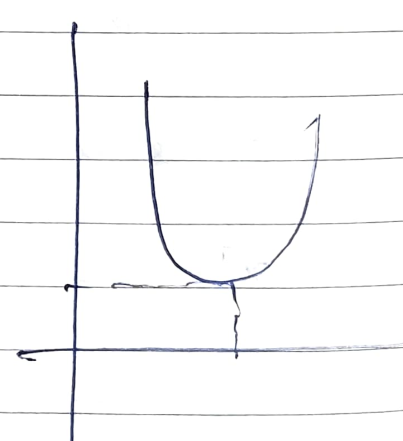
pseudo inverse

$x = A^+ b$

A not full rank

D_{ii} (SVD)

$D_{ii} < 0$ ...



(2) $\|Ax - b\| - \|x\|$

COMMENTS

SVD

- pseudo inverse : $D_{ii}^{-1} = \frac{1}{D_{ii}}$ if $> \theta$ else 0.
- rank of matrix is equal to the no. of non-zero singular values.
- low rank approximation using SVD.

Problems of interest

- Solve $Ax = b$ when A is square & full rank.
- least square soln. for $Ax = b$
- least norm soln. for $Ax = b$.
- non-trivial soln. ~~f~~ to $Ax = 0$.