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OM HW-1

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1.1

1. RTP $\|a+b\|^2 + \|a-b\|^2 = 2(\|a\|^2 + \|b\|^2)$

$$LHS = \|a+b\|^2 + \|a-b\|^2$$

$$= \sum_i (a_i+b_i)^2 + \sum_i (a_i-b_i)^2$$

$$= \sum_i (a_i^2 + b_i^2 - 2a_ib_i) + \sum_i (a_i^2 + b_i^2 - 2a_ib_i)$$

$$= \sum_i (2a_i^2 + 2b_i^2)$$

$$= 2\left\{\sum_i a_i^2 + \sum_i b_i^2\right\}$$

$$= 2(\|a\|^2 + \|b\|^2) = RHS \checkmark$$

2. RTP $(a+b)^T(a-b) = \|a\|^2 - \|b\|^2$

$$LHS = (a+b)^T(a-b)$$

$$= \sum_i (a_i+b_i)(a_i-b_i)$$

$$= \sum_i (a_i^2 - b_i^2)$$

$$= \sum_i a_i^2 - \sum_i b_i^2 = \|a\|^2 - \|b\|^2 = RHS \checkmark$$

1.2

RTP $B+B^T$ is symmetric (B is square matrix)

i.e., $B+B^T = (B+B^T)^T$

$$\text{RHS} = (B+B^T)^T$$
$$= B^T + (B^T)^T$$
$$= B^T + B = B + B^T = \text{LHS} \checkmark$$

RTP $(A^{-1})^T = (A^T)^{-1}$

we know that $I^T = I$ (identity matrix)

$\Rightarrow (AA^{-1})^T = A^{-1}A$

$\Rightarrow (A^{-1})^T A^T = A^{-1}A$

$\Rightarrow A(A^{-1})^T A^T = A$

$\Rightarrow A(A^{-1})^T = A(A^T)^{-1}$

$\Rightarrow (A^{-1})^T = (A^T)^{-1}$ ✓

1.3

RTP for finite dimensional vector space L1 & L2 norms are equivalent.

i.e.

$$c_1 \|x\|_2 \leq \|x\|_1 \leq c_2 \|x\|_2 \quad \forall x$$

where $c_1, c_2 \in \mathbb{R}$ and $0 < c_1 \leq c_2$

we know that

$$a^2 + b^2 \leq a^2 + b^2 + 2ab \quad \text{when } a, b \geq 0$$

$$\Rightarrow \sqrt{a^2+b^2} \leq a+b$$

or more generally

$$\Rightarrow \sqrt{\sum_i x_i^2} \leq \sum_i |x_i| \quad \text{where } x_i \in \mathbb{R}$$

$$\Rightarrow \|x\|_2 \leq \|x\|_1 \quad (c_1 = 1) \quad \text{--- (1)}$$

we also know that

$$(a+b)^2 \leq (a+b)^2 + (a-b)^2 \quad \text{where } a, b \geq 0$$

$$\Rightarrow (a+b)^2 \leq 2a^2 + 2b^2$$

$$\Rightarrow a+b \leq \sqrt{2}\sqrt{a^2+b^2}$$

or more generally

$$\Rightarrow \sum_i |x_i| \leq \sqrt{2}\sqrt{\sum_i x_i^2} \quad \text{where } x_i \in \mathbb{R}$$

$$\Rightarrow \|x\|_1 \leq \sqrt{2}\,\|x\|_2 \quad (c_2 = \sqrt{2}) \quad \text{--- (2)}$$

from inequalities (1) and (2), we can say that

$$c_1 \|x_2\| \leq \|x\|_1 \leq c_2 \|x\|_2 \quad ✓$$

st.

$$0 < \underset{\substack{\uparrow \\ 1}}{c_1} \leq \underset{\substack{\uparrow \\ \sqrt{2}}}{c_2} \qquad c_1, c_2 \in \mathbb{R}$$

2/

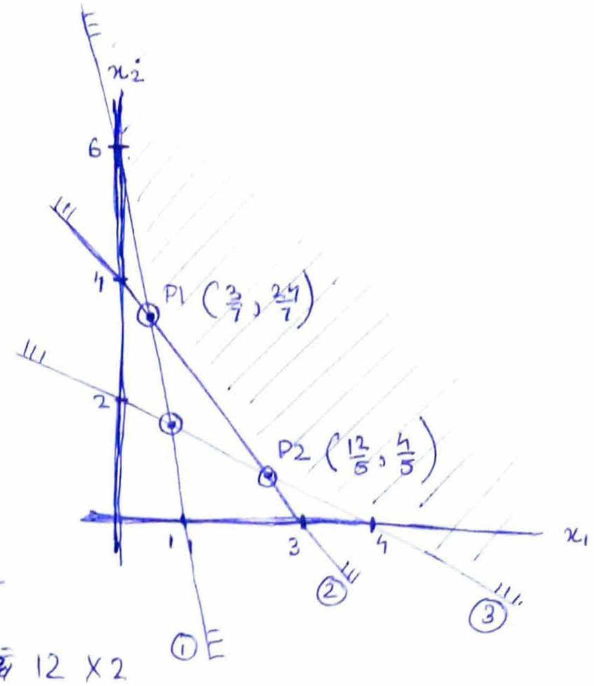
2.1

minimize $z = 5x_1 + 2x_2$

st. $6x_1 + x_2 \geq 6$ — ①

$4x_1 + 3x_2 \geq 12$ — ②

$x_1 + 2x_2 \geq 4$ — ③



finding P1

$6x_1 + x_2 = 6 \quad \times 3$

$4x_1 + 3x_2 = 12$

$14x_1 = 6$

$\Rightarrow x_1 = \frac{3}{7}$

$\Rightarrow x_2 = 6 - 6 \cdot x_1$

$= 6 - \frac{18}{7}$

$= \frac{42-18}{7} = \frac{24}{7}$

$\therefore P_1 = \left(\frac{3}{7}, \frac{24}{7}\right)$

finding P2

$4x_1 + 3x_2 = 12 \times 2$

$x_1 + 2x_2 = 4 \times 3$

$5x_1 = 12$

$\Rightarrow x_1 = \frac{12}{5}$

$\Rightarrow 2x_2 = 4 - x_1$

$= 4 - \frac{12}{5}$

$\Rightarrow x_2 = 2 - \frac{6}{5} = \frac{4}{5}$

$\therefore P_2 = \left(\frac{12}{5}, \frac{4}{5}\right)$

now,

for P1

$z = 5\left(\frac{3}{7}\right) + 2\left(\frac{24}{7}\right)$

$= \frac{15+48}{7} = \frac{63}{7}$

$= 9$

for P2

$z = 5\left(\frac{12}{5}\right) + 2\left(\frac{4}{5}\right)$

$= 12 + \frac{8}{5}$

$= 12 + 1.6 = 13.6$

Since this is a minimization problem, $Z^* = 9$, $x^* = \left(\frac{3}{7}, \frac{24}{7}\right)$

2.2

minimize $z = c_1x_1 + c_2x_2 + c_3x_3$

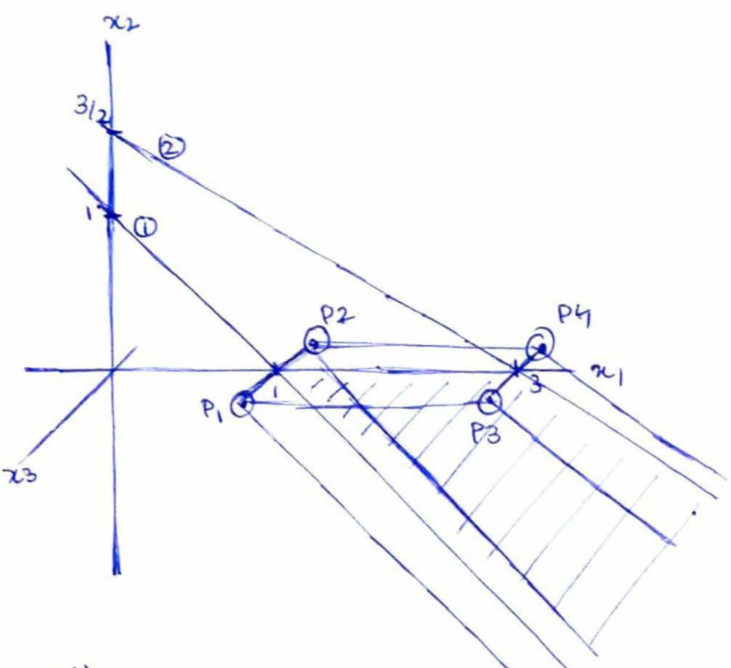
st. $x_1 + x_2 \geq 1$ —— ①

$x_1 + 2x_2 \leq 3$ —— ②

$x_1 \geq 0$

$x_2 \leq 0$

$-1 \leq x_3 \leq 1$



from figure,

$P_1 = (1,0,1)$ $P_3 = (3,0,1)$

$P_2 = (1,0,-1)$ $P_4 = (3,0,-1)$

(i) for $c = (-1,0,1)$

$z_1 = -1+0+1 = 0$

$z_2 = -1+0-1 = -2$

$z_3 = -3+0+1 = -2$

$z_4 = -3+0-1 = -4$

$\therefore z^* = -4$

$x^* = (3,0,-1)$ ✓

(ii) for $c = (0,1,0)$

$z_1 = 0+0+0 = 0$

$z_2 = 0+0+0 = 0$

$z_3 = 0+0+0 = 0$

$z_4 = 0+0+0 = 0$

$\therefore z^* = 0$ ✓

x^* = plane formed by P_1, P_2, P_3, P_4

(iii) for $c = (0,0,-1)$

$z_1 = -1$

$z_2 = 1$

$z_3 = -1$

$z_4 = 1$

$\therefore z^* = -1$

$\therefore x^*$ = plane formed by $P_1, P_3 \to \infty$ ✓

2.3

minimize $\sum_i \sum_j c_{ij} x_{ij}$

where $c = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & 3.4 & 2.2 & 2.9 \\ 2 & 3.4 & 2.4 & 2.5 \end{array}$

st.

$0 \leq \sum_j x_{1j} \leq 250$

$0 \leq \sum_j x_{2j} \leq 450$

$\sum_i x_{ia} = 200$

$\sum_i x_{ib} = 200$

$\sum_i x_{ic} = 200$

and x_{ij} = no. of cases shipped from cannery i to warehouse j.

✓

(note: equality can be written as 2 inequalities) minimize $\quad c_{1a}x_{1a} + c_{1b}x_{1b} + c_{1c}x_{1c} + c_{2a}x_{2a} + \cancel{x}\, c_{2b}x_{2b} + c_{2c}x_{2c}$

$$\Rightarrow x = \begin{bmatrix} x_{1a} \\ x_{1b} \\ x_{1c} \\ x_{2a} \\ x_{2b} \\ x_{2c} \end{bmatrix} \qquad \Rightarrow c = \begin{bmatrix} c_{1a} \\ c_{1b} \\ c_{1c} \\ c_{2a} \\ c_{2b} \\ c_{2c} \end{bmatrix}$$

$\therefore$ maximize $\; -c^T x$

st. $\; Ax \leq b$

st.

$$-x_{1a} - x_{1b} - x_{1c} \leq 0$$

$$-x_{2a} - x_{2b} - x_{2c} \leq 0$$

$$x_{1a} + x_{1b} + x_{1c} \leq 250$$

$$x_{2a} + x_{2b} + x_{2c} \leq 450$$

$$-x_{1a} - x_{2a} \leq -200$$

$$x_{1a} + x_{2a} \leq 200$$

$$-x_{1b} - x_{2b} \leq -200$$

$$x_{1b} + x_{2b} \leq 200$$

$$-x_{1c} - x_{2c} \leq -200$$

$$x_{1c} + x_{2c} \leq 200$$

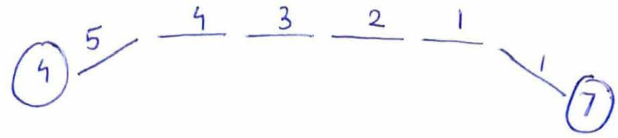
$$\therefore b = \begin{bmatrix} 0 \\ 0 \\ 250 \\ 450 \\ -200 \\ 200 \\ -200 \\ 200 \\ -200 \\ 200 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

3

3.1

$n = 7$ vertices



from picture it can be seen that no. of 5-length paths $= \underline{5!}$.

3.2

RTP $f \in O(g) \Rightarrow \log f \in O(\log g)$

LHS $= f \in O(g)$

$\Rightarrow f(n) \leq c_{\bullet} g(n) \quad \forall \; n \geq n_0$

$\Rightarrow \log f(n) \leq \log (c\, g(n))$

$\Rightarrow \log f(n) \leq \log c + \log g(n)$

$\Rightarrow \log f(n) \leq \log g(n)$

where $n \geq n_0 + f^{-1}(c)$

another constant.

$\Rightarrow \log f \in O(\log g)$

$\Rightarrow \underline{\text{RHS}}$.

.3

RTP $\log(n!) \in \Theta(n \log n)$

$n! \leq n^n$

$\Rightarrow \log n! \leq \log n^n$

$\Rightarrow \log n! \leq n \log n$

$\therefore \log n! \in O(n \log n)$ — ①

$n! \geq n \cdot (n-1) \cdots \frac{n}{2}$

$\Rightarrow n! \geq \frac{n}{2} \cdot \frac{n}{2} \cdots \frac{n}{2}$

$\Rightarrow n! \geq \left(\frac{n}{2}\right)^{n/2}$

$\Rightarrow \log n! \geq \log\left(\frac{n}{2}\right)^{n/2}$

$\Rightarrow \log n! \geq \frac{n}{2} \log \frac{n}{2}$

$\Rightarrow \log n! \geq \frac{n}{2}\{\log n - \log 2\}$

$\Rightarrow \log n! \geq \frac{n}{2} \log n - \frac{n}{2} \log 2$

$\Rightarrow \log n! \geq \frac{n}{4} \log n$

$\therefore \log n! \in \Omega(n \log n)$ — ②

from ① and ②

$\log n! \in \Theta(n \log n)$