1.4

$$\max.\ 5x_1 + 2x_2 + 12x_3 + 10x_4$$

$$\text{st.}\ 3x_1 + 6x_2 + 5x_3 + 5x_4 \leq 12$$

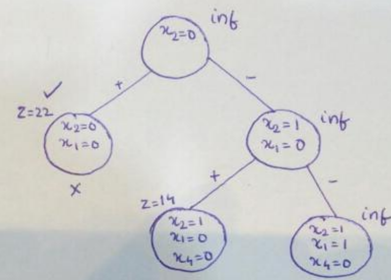
$$4x_1 + 9x_2 - 2x_3 + x_4 \leq 25$$

$$\Downarrow$$

$$\max.\ 2x_2 + 5x_1 + 10x_4 + 12x_3$$

$$\text{st.}\ 6x_2 + 3x_1 + 5x_4 + 5x_3 \leq 12$$

$$9x_2 + 4x_1 + x_4 - 2x_3 \leq 25$$



$$Z^* = 22$$

$x_1 = 0 \quad x_2 = 0 \quad x_3 = 1 \quad x_4 = 1$

2.1

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

$$\text{min. } OPT_{ILP} = \sum_{i \in V} x_i$$

$$\text{st. } x_i \in \{0, 1\} \ \forall\, i \in V,$$

$$x_i + x_j \geq 1 \ \forall\, \{i, j\} \in E$$

$$\text{min. } OPT_{LP} = \sum_{i \in V} x_i$$

$$\text{st. } 0 \leq x_i \leq 1 \ \forall\, i \in V,$$

$$x_i + x_j \geq 1 \ \forall\, \{i, j\} \in E$$

$$x_i^* = \begin{cases} 1 & \text{if } x_i \geq \frac{1}{2} \\ 0 & \text{if } x_i < \frac{1}{2} \end{cases}$$

$$OPT_{ROUND} = \sum_{i \in V} x_i^*$$

(a) to show $OPT_{LP} \leq OPT_{ILP}$

LP soln. is always better than IP solution.

(it is better than any soln. in the feasible region).

(b) to show $S^* = \{ i \in V \mid x_i^* = 1 \}$

still produces valid set cover.

indeed it does, selecting all vertices is also a valid vertex cover, as all vertices cover all the edges.

(c) approx. factor of LPR.

max. value of $\dfrac{OPT_{ROUND}}{OPT_{ILP}}$

$$OPT_{ROUND} = \sum_{v \in ROUND} 1$$

$$\leq \sum_{v \in ROUND} 2 x_v^*$$

$$\leq \sum_{v \in V} 2 x_v^*$$

$$\leq 2 \sum_{v \in V} y_v$$

$$= 2\,OPT_{ILP}$$

$$\therefore \text{max. } \frac{OPT_{ROUND}}{OPT_{ILP}} = 2$$

3.3

$$\begin{bmatrix} 2 & 2 & 7 \\ 3 & \lambda+9 & \lambda+11 \\ 3 & \lambda+4 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 10 \end{bmatrix} \xrightarrow{①} \left[\begin{array}{ccc|c} 2 & 2 & 7 & 7 \\ 0 & 5 & \lambda & 10 \\ 3 & \lambda+4 & 11 & 10 \end{array}\right]$$

$$\xrightarrow{②} \left[\begin{array}{ccc|c} 6 & 6 & 21 & 21 \\ 0 & 5 & \lambda & 10 \\ 6 & 2\lambda+8 & 22 & 20 \end{array}\right] \xrightarrow{③} \left[\begin{array}{ccc|c} 6 & 6 & 21 & 21 \\ 0 & 5 & \lambda & 10 \\ 0 & 2\lambda+2 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{④} \left[\begin{array}{ccc|c} 6 & 6 & 21 & 21 \\ 0 & 5 & \lambda & 10 \\ 0 & 0 & a_{33} & b_3 \end{array}\right]$$

⑤ for step ④ to work

$$2\lambda + 2 - 5K = 0 \quad \text{—— ①}$$

$$a_{33} = 1 - \lambda K \neq 0 \quad \text{—— ②}$$

where $K \neq 0$.

⑥

$$1 - \lambda K \neq 0$$

$$\Rightarrow \lambda K \neq 1$$

$$\Rightarrow K \neq \frac{1}{\lambda}$$

(to avoid)

⑦ putting in eq. ①

$$2\lambda + 2 - \frac{5}{\lambda} = 0$$

$$\Rightarrow 2\lambda^2 + 2\lambda - 5 = 0$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{4 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2}$$

$$= \frac{-2 \pm \sqrt{44}}{4}$$

$$\Rightarrow \lambda = 1.16, -2.16$$

⑧

∴ for given system of equations to have a unique soln., λ should not be 1.16 or -2.16.

4.2

min. $z = x_1 + x_2$

$-2x_1 + 2x_2 \leq 1$

$-8x_1 + 10x_2 \leq 13$

$x_1, x_2 \geqslant 0$

$x_1, x_2 \in \mathbb{R}$

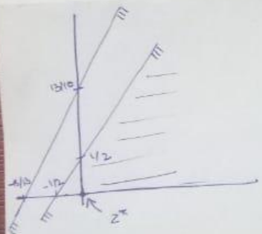
notice that for $x_1 = 0$, $x_2 = 0$ both constraints are satisfied.

hence the trivial soln is the optimal soln. here.

$Z^* = 0$

$x_1 = 0 \quad x_2 = 0$.

this can be seen graphically.



in case the problem was that of maximization, there would be no optimal soln.

5.2

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & -1 & -2 \\ 0 & 2 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \overset{1}{l_{11}} & 0 \\ L_{21} & L_{22} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} u_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}}_{U}$$

$$1 = U_{11} \qquad [0 \quad 1] = U_{12} \qquad \begin{bmatrix} 3 \\ 0 \end{bmatrix} = u_{11} L_{21} \Rightarrow L_{21} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} = L_{21} U_{12} + L_{22} U_{22} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} [0 \quad 1] + L_{22} U_{22}$$

$$= \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} + L_{22} U_{22} \Rightarrow L_{22} U_{22} = \begin{bmatrix} -1 & -5 \\ 2 & -1 \end{bmatrix}$$

another LU decomposition

$$l'_{11} = 1 \qquad u'_{11} = -1 \qquad -5 = U'_{12}$$

$$2 = U'_{11} l'_{21} \Rightarrow l'_{21} = -2$$

$$-1 = l'_{21} U'_{12} + l'_{22} U'_{22}$$

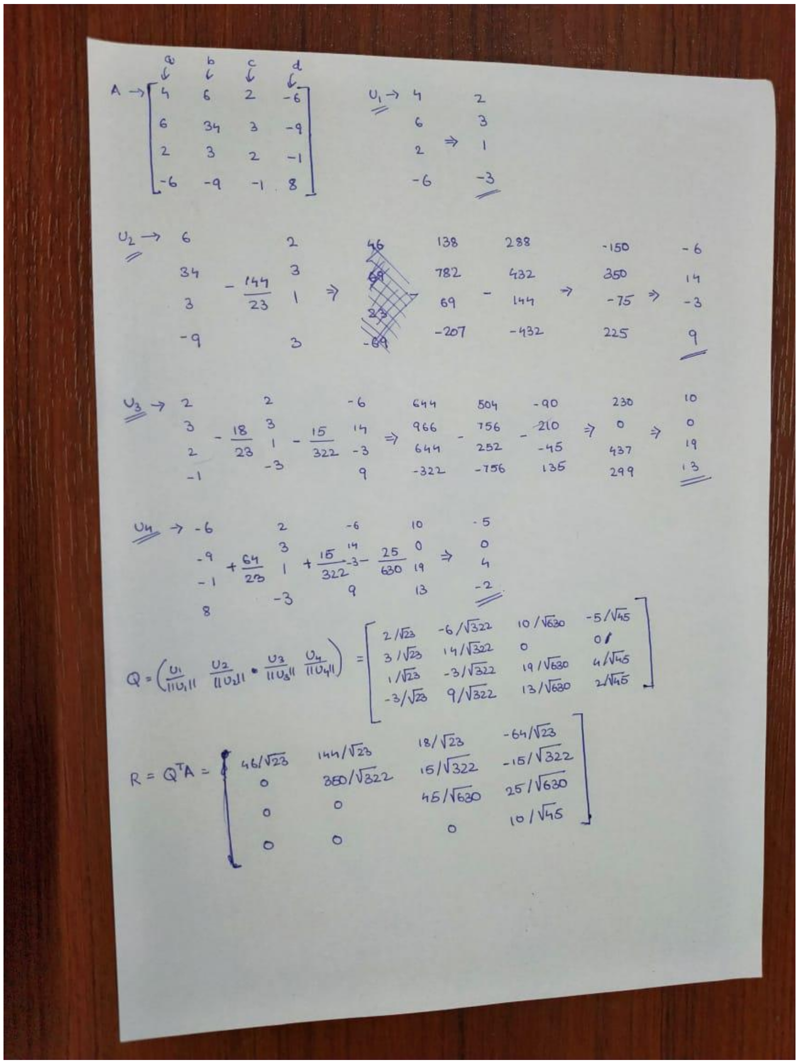
$$-1 = (-2)(-5) + l'_{22} U'_{22}$$

$$-11 = \underbrace{l'_{22}}_{1} U'_{22} \Rightarrow U'_{22} = -11$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -11 \end{bmatrix}$$

6.1



$$\begin{array}{c} \\ A \rightarrow \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 4 & 6 & 2 & -6 \\ 6 & 34 & 3 & -9 \\ 2 & 3 & 2 & -1 \\ -6 & -9 & -1 & 8 \end{bmatrix} \end{array} \qquad \underline{\underline{U_1}} \rightarrow \begin{matrix} 4 \\ 6 \\ 2 \\ -6 \end{matrix} \Rightarrow \underline{\underline{\begin{matrix} 2 \\ 3 \\ 1 \\ -3 \end{matrix}}}$$

$$\underline{\underline{U_2}} \rightarrow \begin{matrix} 6 \\ 34 \\ 3 \\ -9 \end{matrix} - \frac{144}{23} \begin{matrix} 2 \\ 3 \\ 1 \\ 3 \end{matrix} \Rightarrow \begin{matrix} \cancel{46} \\ \cancel{69} \\ \cancel{23} \\ \cancel{-69} \end{matrix} \quad \begin{matrix} 138 \\ 782 \\ 69 \\ -207 \end{matrix} - \begin{matrix} 288 \\ 432 \\ 144 \\ -432 \end{matrix} \rightarrow \begin{matrix} -150 \\ 350 \\ -75 \\ 225 \end{matrix} \Rightarrow \underline{\underline{\begin{matrix} -6 \\ 14 \\ -3 \\ 9 \end{matrix}}}$$

$$\underline{\underline{U_3}} \rightarrow \begin{matrix} 2 \\ 3 \\ 2 \\ -1 \end{matrix} - \frac{18}{23} \begin{matrix} 2 \\ 3 \\ 1 \\ -3 \end{matrix} - \frac{15}{322} \begin{matrix} -6 \\ 14 \\ -3 \\ 9 \end{matrix} \Rightarrow \begin{matrix} 644 \\ 966 \\ 644 \\ -322 \end{matrix} - \begin{matrix} 504 \\ 756 \\ 252 \\ -756 \end{matrix} - \begin{matrix} -90 \\ 210 \\ -45 \\ 135 \end{matrix} \Rightarrow \begin{matrix} 230 \\ 0 \\ 437 \\ 299 \end{matrix} \Rightarrow \underline{\underline{\begin{matrix} 10 \\ 0 \\ 19 \\ 13 \end{matrix}}}$$

$$\underline{\underline{U_4}} \rightarrow \begin{matrix} -6 \\ -9 \\ -1 \\ 8 \end{matrix} + \frac{64}{23} \begin{matrix} 2 \\ 3 \\ 1 \\ -3 \end{matrix} + \frac{15}{322} \begin{matrix} -6 \\ 14 \\ -3 \\ 9 \end{matrix} - \frac{25}{630} \begin{matrix} 10 \\ 0 \\ 19 \\ 13 \end{matrix} \Rightarrow \underline{\underline{\begin{matrix} -5 \\ 0 \\ 4 \\ -2 \end{matrix}}}$$

$$Q = \left(\frac{U_1}{\|U_1\|} \ \frac{U_2}{\|U_2\|} \cdot \frac{U_3}{\|U_3\|} \ \frac{U_4}{\|U_4\|} \right) = \begin{bmatrix} 2/\sqrt{23} & -6/\sqrt{322} & 10/\sqrt{630} & -5/\sqrt{45} \\ 3/\sqrt{23} & 14/\sqrt{322} & 0 & 0 \\ 1/\sqrt{23} & -3/\sqrt{322} & 19/\sqrt{630} & 4/\sqrt{45} \\ -3/\sqrt{23} & 9/\sqrt{322} & 13/\sqrt{630} & 2/\sqrt{45} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 46/\sqrt{23} & 144/\sqrt{23} & 18/\sqrt{23} & -64/\sqrt{23} \\ 0 & 350/\sqrt{322} & 15/\sqrt{322} & -15/\sqrt{322} \\ 0 & 0 & 45/\sqrt{630} & 25/\sqrt{630} \\ 0 & 0 & 0 & 10/\sqrt{45} \end{bmatrix}$$

9.3

min. $z = 6x_1 + 7x_2 + 3x_3$

st. $-5x_1 - 6x_2 - x_3 \geq 9$

$2x_1 + x_2 + 4x_3 \geq -3$

$-2x_1 - x_2 - 4x_3 \geq 3$

$-13x_1 + 8x_3 \geq 0$

primal problem.

$\downarrow$ dual.

max. $z = 9y_1 - 3y_2 + 3y_3$

St. $-5y_1 + 2y_2 - 2y_3 - 13y_4 \leq 6$

$-6y_1 + y_2 - y_3 \leq 4$

$-y_1 + 4y_2 - 4y_3 + 8y_4 \leq 3$

$y_i \geq 0$

dual problem