SIMPLEX METHOD-1

LP, SLACK VARIABLES AND BFS

earlier we had formulated LP as.

$$\min_{x} \; c^T x$$

$$\text{s.t.} \quad Ax \leq b$$

$$x \geq 0$$

where A is $m \times n$ matrix

$b \in \mathbb{R}^m$

$c \in \mathbb{R}^n$

we add auxiliary / slack variables $x_{n+1}, x_{n+2}, \ldots x_{n+m}$ to turn inequalities in constraints to equations:

$$\min_{x} \; c^T x$$

$$Ax = b$$

$$x \geq 0$$

this leads to:

where, A is a $m \times (m+n)$

$$Ax = b$$

$b \in \mathbb{R}^m$

- a basic solution to a system of m linear equations in n unknowns $(n \geq m)$ is obtained by setting $n-m$ variables to 0 and solving the resulting system to get the values of the other m variables.

- the variables set to 0 are called nonbasic variables; the variables obtained by solving the system are called basic variables.

- a basic solution that is feasible, is called basic feasible solution. they are the "vertices / corners" of the feasible regions.

BASIC AND NON-BASIC VARIABLES

simplex algorithm (vo):

1. start from any vertex as a feasible solution.

2. if a neighbouring vertex is better, solution is updated.

3. repeat step 2 until there is no better vertex.

from the set of n variables, we choose m basic variables as set B, $B \subseteq \{1, 2, \ldots n\}$ and set the rest to 0: $x_i = 0 \quad \forall \quad i \notin B$

$$[A_{B(1)}, A_{B(2)}, \dots A_{B(m)}] x_B = b$$

$$\bar{B} x_B = b$$

where,

$\bar{B} = [A_{B(1)}, A_{B(2)}, \dots A_{B(m)}]$ is a $m \times m$ matrix

$x_B \in \mathbb{R}^m$

$b \in \mathbb{R}^m$

this gives us a BFS:

$x [x_B = \bar{B}^{-1} b, \; x_i = 0, \; i \notin B$

the smart thing to do is:

given $B_1^{-1} b$, how do we efficiently find $B_2^{-1} b$ and so on.

BASIC FEASIBLE SOLUTION (BFS)

let $x_{new} = x + \theta d$

let j be the entering variable and l be the exiting variable

$d_j = 1$ and $d_i = 0$ for i not in the basic variable set.

$$A x_{new} = A(x + \theta d)$$

$$A x = b$$

$$\Rightarrow A d = 0$$

$$\sum_{i=1}^{m} d_{B(i)} A_{B(i)} + A_j = 0$$

$$\Rightarrow \bar{B} d_B + A_j = 0$$

$$\Rightarrow d_B = -\bar{B}^{-1} A_j$$

which j to pick? for a given variable j, difference in cost due to j^{th} variable being basic is

$$c_B^T d_B + c_j = c_j - c_B^T B^{-1} A_j$$

we do for each j and select:

$$\bar{c} = [c_1, c_2, \ldots, c_n]$$

where $\bar{c}_j = c_j - c_B^T B^{-1} A$

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SIMPLEX VI

algorithm:

1. Start with initial BFS

2. repeat:

- calculate $\bar{c}$, if all ≥ 0, then stop

- Select a j for $\bar{c}$ st. $\bar{c}_j < 0$

- $d_B = -B^{-1}A$

- find I and θ st. θ is increased ~~be~~ till basic variable becomes 0.

$$x_I + d_I \theta = 0$$

- $B = B - \{I\} + \{j\}$

3. Current BFS is optima.

note: $\theta^* = \min_{i=1,\dots m \mid d_B(i)<0} \left(-\frac{x_B(i)}{d_B(i)} \right)$

next: given B_1^{-1}, how do we find B_2^{-1} where B_2 is a new matrix whose column corresponding to I is replaced by that of j. i.e., $B_1^{-1} B_1 = I$. $B_1^{-1} B_2 = J$.

Prob: given J, use row transformations to convert to I.

EXAMPLES

(1) min $-x_1 - x_2$

st. $-x_1 + x_2 \leq 1$ $\quad x_1, x_2 \geq 0$

$x_1 \leq 3$

$x_2 \leq 2$

(2). min. $z = x_1 + x_2$

st. $x_1 + 5x_2 \leq 5$ $\quad x_1, x_2 \geq 0$

$2x_1 + x_2 \leq 4$

(3). min. $z = -10x_1 - 12x_2 - 12x_3$ $\quad x_1, x_2, x_3 \geq 0$

st. $x_1 + 2x_2 + 2x_3 \leq 20$

$2x_1 + x_2 + 2x_3 \leq 20$

$2x_1 + 2x_2 + x_3 \leq 20$

$$\underbrace{\begin{bmatrix} -1 & 1 & 1 \end{bmatrix}}_{\text{one } B} \quad$$

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

one B

basic soln. get 1
you can get other basic solns.

${}^5C_3 = 10$
5 feasible vertices.
rest are infeasible.

$$x_1 + 5x_2 = 5 \times 2$$
$$2x_4 + x_2 = 4$$

$$9x_2 = 6$$

$$x_2 = 2/3$$
$$x_1 = 5/3$$

$$\begin{bmatrix} 1 & 5 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$x_3 = 5$ $\quad$ $x_1 = 0$
$x_4 = 4$ $\quad$ $x_2 = 0$

$$\begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \Rightarrow \begin{matrix} 5/3 \\ 2/3 \end{matrix}$$

$$\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \Rightarrow \begin{matrix} 4 \\ -5 \end{matrix} \quad \times \text{ not feasible}$$

$$\left[\begin{array}{c} X_d \\ \hline 0 \end{array}\right] = \theta \left[\begin{array}{c} P_b \\ \hline 0 \end{array}\right] + (1-\theta) \left[\begin{array}{c} Q_b \\ \hline 0 \end{array}\right]$$

choose basic variable

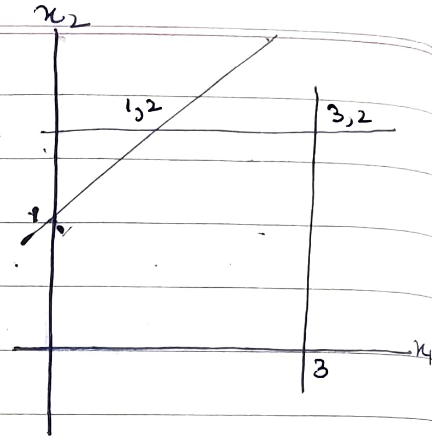
Solve $Bx = b$

$(n+m)\ C\ m$ times. (brute force?)

SIMPLEX ALGORITHM

go from one vertex to another as long as you get better solution.

add 1 variable to B, remove 1.



$x_3 = 1 + x_1 - x_2$

$x_4 = 3 - x_1$

$x_5 = 2 - x_2$

$x_1 \geq 3 \;,\; x_2 \geq 1$

bring x_2 to B.

$x_2 = 1 + x_1 - x_3$

$x_4 = 3 - x_1$

$x_5 = 2 - (1 + x_1 - x_3)$
$\quad = 1 - x_1 + x_3$

$B = \{x_2, x_4, x_5\}$

$z = -x_1 - (1 + x_1 - x_3)$
$\quad = -1 - 2x_1 + x_3$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \qquad z = -1$$

$x \leq 1$, bring x_1 to B

$x_1 = 1 + x_3 - x_5$

$x_2 = 2 - x_5$ $\qquad\qquad B = \{x_2, x_3, x_5\}$

$x_4 = 2 - x_3 + x_5$

$$z = -1 - 2(1 + x_3 - x_5) + x_3$$
$$= -3 - x_3 + 2x_5$$

bring x_3 to B

$x_3 = 2 - x_4 + x_5$

$x_1 = 3 - x_4$

$x_2 = 2 - x_5$

$$\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \\ \end{array} \begin{bmatrix} 3 \\ 2 \\ 2 \\ \\ \end{bmatrix}$$

$$z = -5 + \underbrace{x_4 + x_5}$$

$\underline{\underline{z = -5}}$.

no possibility of further improving objective.