SVD, LEAST SQUARES AND LEAST NORM

Singular value decomposition (SVD) is a very powerful and popular matrix factorization.

$$A = UDV^T$$

A : $m \times n$

U : $m \times n$ orthogonal.

D : $n \times n$ diagonal (D_{ii} the singular values)

V : $n \times n$ orthogonal.

$$U^TU = V^TV = VV^T = I$$

$$Ax = b$$

$b=0$: sys. of homogeneous eqns.

eigen vectors / values

simple

fast
- cholesky
- LU
- QR
- SVD : UDV^T

opt. problem
- least squares
- least norm

diagonal (often sorted)

$$\underset{m\times n}{A} = \underset{m\times n}{U}\ \underset{n\times n}{D}\ \underset{n\times n}{V^T}$$

orthogonal

$$UU^T = VV^T = V^TV = I$$

$A = LU$

(1) $Ax = b$ factorize A on LU.

$\downarrow$

$LUx = b$

(2a) $Lw = b$

(2b) $Ux = w$

$A = UDV^T$

(1) factorize A as UDV^T

(2a) compute $P = U^T b$

(2b) solve $Dw = P$

(2c) solve $V^T x = w$

$\Rightarrow x = Vw$

MORE ON SVD

relationship to eigen values and eigen vectors.

$A^T A V = V D^2$ $\quad$ $A A^T U = U D^2$

Date 14.2.2020
Page 3

finding transpose $A^T = VDU^T$

finding inverse $A^{-1} = VD^{-1}U^T$

$A^{-1} = (UDV^T)^{-1} = VD^{-1}U^T$

$A^T = (UDV^T)^T = VDU^T$

$A^TA = (UDV^T)^T UDV^T$

$= VD\cancel{U^T U}DV^T = VD^2V^T$

$A^TAV = VD^2$

$\quad \hookrightarrow$ diagonal

$= D^2V$

$A^TAV_1 = D_{11}^2 V_1$

V_1 is the eigen vector,

D_1^2 is the eigen values of A^TA

$A^TAV_i = D_{ii}^2 V_i$

V_i is the eigen vector,

D_{ii}^2 is the eigen value of A^TA

sorting D_{ii} and other matrices wont change A.

$A = UDV^T$

$A' = \begin{bmatrix} & \end{bmatrix}\begin{bmatrix} & \\ & 0 \end{bmatrix}\begin{bmatrix} & \end{bmatrix}$

$\min_{A'} \| A - A' \|$ rank $(A') = K$

A' is the nearest rank - $n-1$ matrix to A.

$A : A^{-1}$

$(A + \lambda I)$: regularization

D_{ii} = eigen values of A^TA, AA^T

LEAST SQUARE PROBLEM AND SOLUTION

given A and target vector b, least square error (MSE) problem is:

$$\min_x \| Ax - b \|$$

$$\min_x [Ax - b]^T [Ax - b]$$

$$\min_x (x^TA^TAx + b^Tb - 2x^TA^Tb)$$

$$2A^TAx - 2A^Tb = 0$$ (quadriatic differentiation)

$$A^T A x = A^T b$$

to obtain x, solve the above equation:

$$x = (A^T A)^{-1} A^T b.$$

LEAST NORM SOLUTION

minimize $\|x\|$

s.t. $Ax = b$.

soln: $x^* = A^T (A A^T)^{-1} b$

x^* satisfy the $Ax = b$

no other vector $x = x^* + (x - x^*)$ can be smaller.

$$\|x\|^2 = \|x^*\|^2 + \|(x - x^*)\|^2 + 2x^{*T}[x - x^*]$$

third term is zero

$$\|x\|^2 = \|x^*\|^2 + \|(x - x^*)\|^2$$

$$\|x\|^2 = \|x^*\|^2 + \text{positive term}$$

hence x^* is the smallest.

classmate

Date 14.2.2020

Page 6

EFFICIENT SOLUTIONS FOR LEAST SQUARES AND LEAST NORM

least squares

$$x^* = (A^T A)^{-1} A^T b$$

or solve $\quad A^T A x = A^T b$

least norm

$$x^* = A^T (A^T A)^{-1} b$$

how do we use:

- Cholesky
- LU
- QR
- SVD.

$$\underbrace{A}_{m \times n}\underbrace{x}_{n \times 1} = \underbrace{b}_{m \times 1}$$

$m > n$
(more rows)

LSE

MSE

$\min \|Ax-b\|$

$x = (A^TA)^{-1}A^Tb$

solve $\underbrace{A^TA}_{\downarrow \text{PSD}} x = A^Tb$

$m < n$
(more columns)

LN

$\min \|x\|$

st. $(Ax=b)$

$x = A^T(AA^T)^{-1}b$