FACTORIZATION OF MATRICES

$Ax = b$

$n \times n$

$x = A^{-1} b$ ✗ not attractive (more complex)

$A [x_1 \dots x_n] : [b_1 \dots b_n]$

if we have structure for A matrix it could be done efficiently.

I P D Tr.

PD matrix, Cholesky, LU.

$Ax = b$

$\downarrow$ ① factorization / decomposition, (costly)

$LUx = b$ SVD > cholesky

$Lw = b$ (2a) easy solution Δ.

$Ux = w$ (2b)

if this step reused extensively then ok.

$$A = \left[\begin{array}{c|c} \overset{1\times 1}{a_{11}} & \overset{1\times n-1}{A_{21}^T} \\ \hline \underset{(n-1)\times 1}{A_{21}} & \underset{(n-1)\times(n-1)}{A_{22}} \end{array}\right]_{n \times n} = \left[\begin{array}{c|c} l_{11} & 0 \\ \hline L_{21} & \underset{L\Delta}{L_{22}} \end{array}\right] \left[\begin{array}{c|c} l_{11} & L_{21}^T \\ \hline 0 & L_{22}^T \end{array}\right]$$

$a_{11} = l_{11} l_{11} \Rightarrow l_{11} = \sqrt{a_{11}}$ (1)

$A_{21} = L_{21} l_{11} \Rightarrow L_{21} = \frac{1}{l_{11}} A_{21}$ (2)

$A_{22} = L_{21} L_{21}^T + L_{22} L_{22}^T$

$\Rightarrow L_{22} L_{22}^T = A_{22} - L_{21} L_{21}^T$ (3)

(another cholesky decomposition) $((n-1) \times (n-1))$

$\therefore \cancel{T(n)} = \cancel{2} + \cancel{T(n-1)}$

$\Rightarrow \cancel{T(n)} =$ 1 scalar

+ remain scalar.vector

+ $(n-1) \times (n-1)$ cholesky.

$T(n) = n + T(n-1)$

$= n + (n-1) + \ldots$

$= \frac{n(n-1)}{2} = O(n^2)$

$$\begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}$$

$A = L U$

$a_{11} = l_{11} \cdot U_{11}$

assume $l_{11} = 1$ ① $\Rightarrow a_{11} = U_{11}$ ②

$A_{12} = U_{12} \times l_{11} \Rightarrow A_{12} = U_{12} \Rightarrow U_{12} = A_{12}$ ③

$A_{21} = L_{21} U_{11} \Rightarrow L_{21} = \dfrac{A_{21}}{U_{11}} = \dfrac{A_{21}}{a_{11}}$ ④

$A_{22} = L_{21} U_{12} + L_{22} U_{22}$

$\therefore L_{22} U_{22} = A_{22} - L_{21} U_{12}$

(another LU decomposition)

singular matrix should

PD matrix : diagonal elements > 0.

① $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 20 & 26 \\ 3 & 26 & 70 \end{bmatrix}$ (Cholesky).

$LL^T = A \quad \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} l_{11} & L_{21}^T \\ 0 & L_{22}^T \end{bmatrix}$

$l_{11} = \sqrt{a_{11}} = 1$

$L_{21} = \dfrac{1}{l_{11}} A_{21} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$L_{22}L_{22}^T = A_{22} - L_{21}L_{21}^T$$

$$= \begin{bmatrix} 20 & 26 \\ 26 & 70 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 26 \\ 26 & 70 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 20 \\ 20 & 61 \end{bmatrix}$$

$$l_{11} = \sqrt{a_{11}} = 4$$

$$L_{21} = \frac{1}{l_{11}} A_{21} = \frac{1}{4} \times 20 = 5$$

$$L_{22}L_{22}^T = A_{22} - L_{21}L_{21}^T$$

$$L_{22}^2 = 61 - 5^2 = 61 - 25$$

$$L_{22}^2 = 36 \Rightarrow L_{22} = 6.$$

$$\therefore \begin{bmatrix} 4 & 0 \\ 5 & 6 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$$

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$$LU = \begin{bmatrix} 6 & 3 & 1 \\ 2 & 4 & 3 \\ 9 & 5 & 2 \end{bmatrix}$$

Solve:

$$\begin{bmatrix} 6 & 3 & 1 \\ 2 & 4 & 3 \\ 9 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 17 \\ 29 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ 3/2 & 1/6 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 6 & 3 & 1 \\ 0 & 3 & 8/3 \\ 0 & 0 & 1/18 \end{bmatrix}$$

$Lw = b$

$$w = [18, 11, 1/6]^T$$

$Ux = w$

$$x = [2, 1, 3]^T.$$

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$$\begin{bmatrix} -a & \\ & b \end{bmatrix} AV = a^2 b^2$$

$AB = I$

$A b_i = e_i \quad i = 1, \dots, n.$ (1).

$QRx = b$

$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} [a \; b] = a^2 b^2$$

$Rx = Q^T b.$

a b

-a b

(2).
$$\begin{cases} 2x_1 + x_2 = 4 \\ 6x_1 + 3x_2 = 12 \end{cases}$$

(3). (a) . (b) . (a) .

PD: $v^T A v > 0$ & symmetric.

QR DECOMPOSITION

$A = QR$

$QQ^T = Q^T Q = I$

$A = UDV^T$ (SVD)

$Ax = b$

(costly)

$QRx = b$

~~numerical stability.~~

$Rx = Q^T b.$

Δ

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$$A = \underbrace{\left[\, \overset{n\times 1}{a_1} \,\middle|\, \overset{n\times(n-1)}{A_2} \,\right]}_{n\times n} \qquad Q = \underbrace{\left[\, \overset{n\times 1}{q_1} \,\middle|\, \overset{n\times(n-1)}{Q_2} \,\right]}_{n\times n} \qquad R = \underbrace{\begin{bmatrix} \overset{1\times 1}{r_{11}} & \overset{(n-1)\times 1}{R_{12}} \\ 0 & R_{22} \end{bmatrix}}_{n\times n}$$

R_{22}: $(n-1)\times(n-1)$

$QQ^T = I$ (given)

$a_1 = q_1 r_{11}$ $\qquad q_1 q_1^T = 1 \quad q_1^T Q = 0 \quad Q_2^T Q_2 = 1$

$A_2 = q_1 R_{12} + Q_2 R_{22}$

QR is possible where R is lower!

many possible factorizations.

$$[q_1 \; Q_2] \begin{bmatrix} q_1^T \\ Q_2^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow q_1 q_1^T = 1$

LEAST SQUARE PROBLEM

$$\min_x \|Ax - b\|$$

$$\min_x (Ax-b)^T (Ax-b)$$

$$\min_x (x^T A^T A x + b^T b - 2x^T A^T b)$$

$$2A^T A x - 2A^T b = 0$$

$$A^T A x = A^T b.$$