verizon class
230 + 50

THE ADDITION LANGUAGE

Syntax $\qquad$ exp ::= num | $\bar{+}$ exp exp

Semantics $\qquad$ num ::= $\bar{0}$ | $\bar{1}$ | ...

BNF grammar.

syntactic category.

literals

ex $\bar{2} + \bar{4}$

Q1) what are we talking about?
(in the addition language).

SEMANTIC DOMAINS

natural nos.

operations

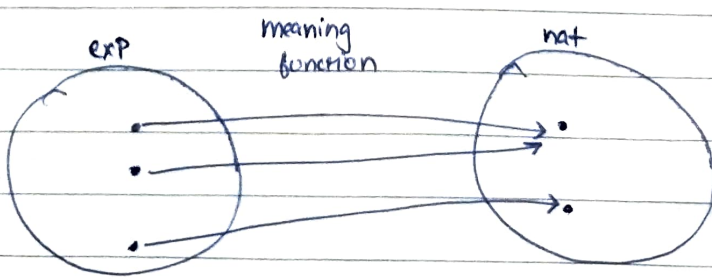
+ : Nat x Nat $\longrightarrow$ Nat

notion of composition already built into semantic domain.

now. we want a representation for that.

+(2, 3)

+ (7, +(2,3))



INDUCTIVE DEFNS.

antecedent

$$\frac{n \ \text{Nat}}{\bar{n} \ \text{Num}} \ \text{NAT}$$

consequent

if n is a Nat, then $\bar{n}$ is a Num

$$\frac{e \ \text{Num}}{e \ \text{Exp}} \ \text{NUM}$$

if e is a Num, then e is an Exp

antecedent

$$\frac{e_1 \ \text{Exp} \quad e_2 \ \text{Exp}}{+ \ e_1 \ e_2 \ \text{Exp}} \ \text{PLUS}$$

if e_1 is an Exp and e_2 is an Exp, then $+ \ e_1 \ e_2$ is an Exp.

~~NAT~~

rules give us a way to define a valid expression.
a valid expr. is one which allows me to derive an expr.

$+ \ \bar{3} \ \bar{4}$

1. 3 is a Nat
2. $\bar{3}$ is a Num from 1 using NAT
3. 4 is a Nat
4. $\bar{4}$ is a Num from 3 using NAT
5. $+ \ \bar{3} \ \bar{4}$ from 2, 4 using PLUS.

proof / deduction / derivation

$+ \ \bar{3}$ Exp ? X

→ this is not even an expression $\notin$ Exp.

$\vdash_{Exp}$ ______ deduce in the system Exp.
that + 3 4 is an Exp.

$\vdash_{Exp} + \ \bar{3} \ \bar{4} \quad Exp$

$\nvdash_{Exp} \bar{+} \ \bar{3} \quad Exp.$

↳ used to prove if programs are syntactically correct.

helps to have a semantic domain in mind
how to compose operations.

VALUATION JUDGEMENTS (VAL)

$$\underline{e \Rightarrow n} \qquad e: Exp \qquad n : Nat$$

① $$\frac{}{\bar{n} \Rightarrow n}\ NUM$$

② $$\frac{e_1 \Rightarrow n_1 \qquad e_2 \Rightarrow n_2}{\bar{+}\, e_1\, e_2 \Rightarrow n_1 + n_2}\ PLUS$$

or $\{I(\bar{+})(n_1, n_2)\}$

$(+ \bar{7} \ (+ \bar{2} \ \bar{3}))$

1. 7 is a Nat
2. $\bar{7}$ is a Num from ①, NAT
3. 2 is a Nat
4. $\bar{2}$ is a Num from ③, NAT
5. 3 is a Nat
6. $\bar{3}$ is a Num from ⑤, NAT
7. $\bar{2}$ is an Exp from ④, NUM
8. $\bar{3}$ is an Exp from ⑥, NUM
9. $+ \bar{2} \ \bar{3}$ is an Exp from ⑦, ⑧, PLUS
10. $\bar{7}$ is an Exp from ②, NUM
11. $(+ \bar{7} \ (+ \bar{2} \ \bar{3}))$ ⑩, ⑨, PLUS.

$\vdash_{Exp} \ (+ \ \bar{7} \ (+ \ \bar{2} \ \bar{3})) \ Exp$

commutative, associative

$|\,|$: Exp $\to$ Nat.

$|\bar{n}| = 1$

$|\bar{+} \ e_1 \ e_2| = 1 + |e_1| + |e_2|$

CONJECTURE

thm Uniqueness

$\forall e :$ Exp $\exists ! \ n \in$ Nat $: e \Rightarrow n$

a unique

$+ \ \bar{3} \ \bar{4} \Rightarrow 7$

to prove this claim.

1. $\bar{3} \Rightarrow 3$ from NUM ($\vdash_{VAL}$ NUM)
2. $\bar{4} \Rightarrow 4$ $\vdash_{VAL}$ NUM
3. $+(3, 4) \Rightarrow 7$ ARITHMETIC.

$\therefore + \bar{3} \ \bar{4} \Rightarrow 7$ 1, 2, 3 using PLUS ($\vdash_{VAL}$ PLUS)

proof $\forall e : Exp \ \exists! n : e \Rightarrow n.$

$:$ = element of

1. $\forall e : Exp \ \exists n \in Nat : e \Rightarrow n$
2. $e \Rightarrow n_1$ & $e \Rightarrow n_2$

IMPLIES $n_1 = n_2$

proof of (1) by induction on the structure of e.
(by induction on Exp)

base case:

1. $e \equiv \bar{n}$ assumption
2. $\bar{n} \Rightarrow n$ NUM rule

DONE

inductive case:

1. $e \equiv \bar{+}\ e_1\ e_2$ assumption.
2. $e_1 \Rightarrow n_1$ by IH on e_1

   for some n_1
3. $e_2 \Rightarrow n_2$ by IH on e_2

4. $\bar{+}\ e_1\ e_2 \Rightarrow +(n_1, n_2)$
5. $+(n_1, n_2)$ is well defined

   $+$ is total

   NAT X NAT

5. let $n = +(n_1, n_2)$
6. $e \Rightarrow n$ 3, 5, 1