classmate

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ABSTRACT REDUCTION SYSTEMS

(FORMAL METHODS)

(notes from baader & nipkon Ch1 & 2 on reserve in the library).

$A = \mathbb{R}$

$a \to b$ if $b = a^2$

$a \to b \to c \to d$.

$3+4 \to 7$ (reduction)

$2 * (3+4) \to 2 * 7 \to 14 \not\to$

$A = \mathbb{N}$

$a \to b$ if $a = b+1$

$5 \to 4 \to 3 \to 2 \to 1 \to 0 \not\to$

$a \to b$ iff $b = a+1$

$5 \to 6 \to 7 \to \ldots$

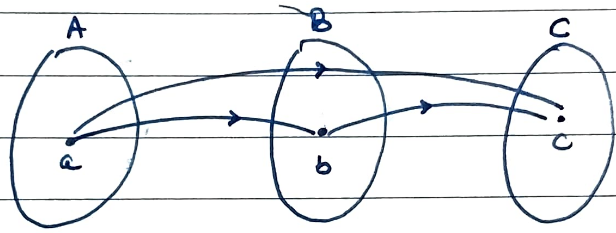
1. COMPOSITION OF RELATIONS

let $R \subseteq A \times B$ &

$S \subseteq B \times C$

the $R ; S \subseteq A \times C$

$R ; S = \{(a,c) \in A \times C \mid \exists\, b \in B : (a,b) \in R \;\&\; (b,c) \in S\}$

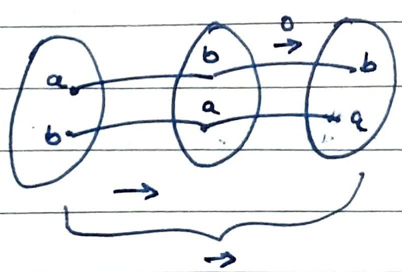


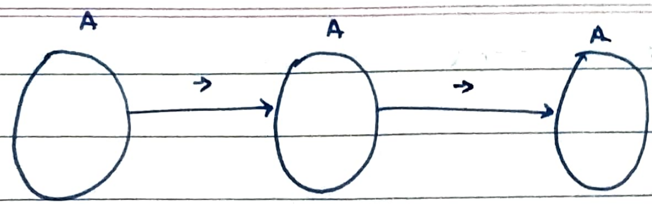
let $\rightarrow$ be a binary relation on A

[let $(A, \rightarrow)$ be an abstract reduction system].

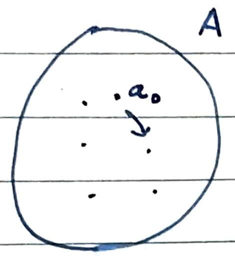
$\xrightarrow{0}$ defined as. $\{(a,a) \mid a \in A\}$ identity relation.
df.

$\xrightarrow{i+1}$ $=df.=$ $\rightarrow ; \xrightarrow{i}$ $i+1$ fold composition $i \geq 0$.





$\xrightarrow{i}$ i times.



i fold composition

$a \xrightarrow{i} b$ = there is a path of length i from a to b.

transitive closure

$\xrightarrow{+}$ =df= $\bigcup_{i \geq 0} \xrightarrow{i}$

$x \xrightarrow{+} y$ means that

a path of length i.

$x \xrightarrow{n} y$ for some $n \in \mathbb{N}$

reflexive transitive closure

$\xrightarrow{*}$ =df= $\xrightarrow{+} \cup \xrightarrow{0}$

$\xrightarrow{=}$ =df= $\longrightarrow \cup \xrightarrow{0}$ reflexive closure.

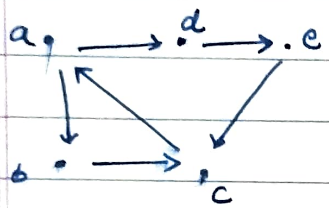
$\xrightarrow{-1}$ =df= $\{(a', a) \mid a \to a'\}$ inverse.

if $a \to b$ then by dfn. $b \xrightarrow{-1} a$ $\quad b \leftarrow a$

$a \leftrightarrow b$ =df= $\rightarrow \cup \xrightarrow{-1}$

$\xleftrightarrow{*}$ =df= $(\leftrightarrow)^*$ reflexive transitive closure of $\leftrightarrow$

$\xleftrightarrow{+}$ =df= $(\leftrightarrow)^+$ transitive closure



$A = \{a, b, c, d, e\}$

$\rightarrow = \{a \rightarrow d,\ c \rightarrow a,\ a \rightarrow b,\ d \rightarrow e,\ e \rightarrow c\}$

$\xrightarrow{0} = \{(a,a), (b,b), \ldots (e,e)\}$

$\xrightarrow{1} = \rightarrow$

$\xrightarrow{2} = \{a \xrightarrow{2} c,\ c \xrightarrow{2} b,\ a \xrightarrow{2} e,\ e \xrightarrow{2} a\}$

$\xrightarrow{+} = \{(x,y) \mid$ y is reachable from x in 1 or more steps $\}$

$\xrightarrow{-1}$ = reverse all edges.

$\rightarrow \cup \xrightarrow{-1} = \leftrightarrow$

1. a reduces to a' if $a \rightarrow a'$
   defined on the same set (ARS).

2.

$3+4 \rightarrow 7$

$5+2 \rightarrow 7$

$3+4 = 7$
$= 5+2$

$3+4 = 5+2$

$7 \xrightarrow{-1} 3+4$ equality is a symmetrical relation.

TERMINOLOGY

1. a reduces to a' =df= $a \rightarrow a'$

2. a is reducible if $\exists a' \in A : a \rightarrow a'$

   in the ARS $(\mathbb{N}, \rightarrow)$ where $a \rightarrow b$ iff $a = b+1$
   the element 0 is not reducible.
   all other elements are reducible.

discrete mathematics - all graphs

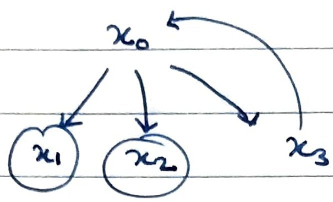
3. a simplifies to b iff

   $a \xrightarrow{*} b$

4. a is in normal form if a is irreducible.

   (model of computing)

5 b is a **normal form** of a if
$a \xrightarrow{*} b$ and b is a normal form.



$x_0 \xrightarrow{*} x_1$

$x_0 \xrightarrow{*} x_2$

$x_3 \xrightarrow{*} x_1$

$x_3 \xrightarrow{*} x_2$

$x_1 \xrightarrow{*} x_1$

$x_2 \xrightarrow{*} x_2$

x_1 is a NF x_1, x_0, x_3

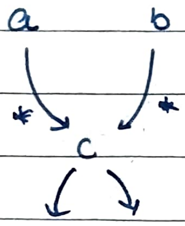
x_2 is a NF x_2, x_0, x_3

x_1 and x_2 NFs.

x_0 and x_3 are not NFs.

a and b are **joinable** if $\exists\, c \in A$ s.t.

$a \xrightarrow{*} c$ and $b \xrightarrow{*} c$



x_3 and x_0 are joinable (symmetric relation).
(meet)ing possible?

PROPERTIES OF ABSTRACT REDUCTION SYSTEMS

an ARS $(A, \to)$ is

1. <u>normalizing</u> iff every element of A simplifies to a normal form. i.e., every element <u>has</u> a normal form.

2. <u>terminating</u> iff there is no infinite sequence
$a_0 \to a_1 \to \dots$

$x_0 \to x_1 \to x_2 \to \dots$ (not normalizing)

$x_0 \to x_1$

	term.	non. term.
norm.		
not. normal.		

terminating ARS $\Rightarrow$ normal form.
if $(A, \to)$ is terminating then $(A, \to)$ is normalizing.
normalizing ARS $\not\Rightarrow$ terminating.