Linear Algebra

Problem set 2

1. * For the sake of contradiction, let us assume $\exists B, B'$ such that $BA = I$ and $B'A = I$
* Then corresponding to these left inverses of A, let $\exists C, C'$ such that $AC = I$ and $AC' = I$

(i) We prove that $B = C$ and $B' = C'$

$AC = I$	$AC' = I$
$B(AC) = B \cdot I$	$B' \cdot (AC') = B' \cdot I$
$(B \cdot A) \cdot C = B$	$(B'A) \cdot C' = B' \cdot I$
$I \cdot C = B$	$I \cdot C' = B'$
$C = B$	$C' = B'$

(ii) We now prove $B = B'$ and that our assumption was wrong

$BA = AB = I$

$B' \cdot (AB) = B' \cdot I$	$B'A = AB' = I$
$(B' \cdot A) \cdot B = B'$	$B \cdot (AB') = B \cdot I$
$I \cdot B = B'$	$(BA) \cdot B' = B$
$B = B'$	$I \cdot B' = B$
	$B' = B$

* Thus, a matrix A can ONLY have one unique left inverse

2. Consider the matrix $A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$, then the augmented matrix A' corresponding to $Ax = y$ is as follows –

$$A'_{m \times (n+1)} = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & y_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & y_2 \\ \vdots & & & & & \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} & y_m \end{array}\right]$$

We now discuss the 2 examples to find A^{-1}

i) Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Then $A' = \left[\begin{array}{cc|c} 2 & -1 & y_1 \\ 1 & 3 & y_2 \end{array}\right]$

R_1 is interchanged with R_2, $A' = \left[\begin{array}{cc|c} 1 & 3 & y_2 \\ 2 & -1 & y_1 \end{array}\right]$

$R_2 \leftarrow R_2 - 2R_1$, $A' = \left[\begin{array}{cc|c} 1 & 3 & y_2 \\ 0 & -7 & y_1 - 2y_2 \end{array}\right]$

$R_2 \leftarrow R_2/(-7)$, $A' = \left[\begin{array}{cc|c} 1 & 3 & y_2 \\ 0 & 1 & (y_1 - 2y_2)/(-7) \end{array}\right]$

$R_1 \leftarrow R_1 - 3R_2$, $A' = \left[\begin{array}{cc|c} 1 & 0 & (y_2 + 3y_1)/7 \\ 0 & 1 & (2y_2 - y_1)/7 \end{array}\right]$, meaning A is invertible

$AX = Y \Rightarrow X = A^{-1}Y$, meaning

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (y_2 + 3y_1)/7 \\ (2y_2 - y_1)/7 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

ii) Let us consider $A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$

Take $A' = \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & 0 & 1 \end{array}\right]$

We apply the following operations {in order}:

(a) $R_2 \leftarrow R_2 - \frac{R_1}{2}$
(b) $R_3 \leftarrow R_3 - \frac{R_1}{3}$
(c) $R_2 \leftarrow 12R_2$
(d) $R_1 \leftarrow R_1 - \frac{R_2}{2}$
(e) $R_3 \leftarrow 180R_3$
(f) $R_2 \leftarrow R_2 - R_3$
(g) $R_1 \leftarrow R_1 + \frac{1}{6}R_3$

To get $A' = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -36 & 30 \\ 0 & 1 & 0 & -36 & 192 & -180 \\ 0 & 0 & 1 & 30 & -180 & 180 \end{array}\right]$

Thus, $A^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$

3. The general procedure is as follows —

i) Consider the augmented matrix A',

$$A'_{m\times(n+1)} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & y_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & y_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & y_m \end{array}\right]$$

ii) Now, using row operations convert this matrix to a row reduced echelon matrix of the form $[I \mid Z]$.

iii) The solution is $X = Z$

Example: Take $A = \begin{bmatrix} 0 & 5 & -1 \\ 1 & 8 & -1 \\ 2 & 1 & 1 \end{bmatrix}$

$$A' = \begin{bmatrix} 0 & 5 & -1 & y_1 \\ 1 & 8 & -1 & y_2 \\ 2 & 1 & 1 & y_3 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$A' = \begin{bmatrix} 1 & 8 & -1 & y_2 \\ 0 & 5 & -1 & y_1 \\ 2 & 1 & 1 & y_3 \end{bmatrix}$$

$R_3 \leftarrow R_3 - 2R_1$

$$A' = \begin{bmatrix} 1 & 8 & -1 & y_2 \\ 0 & 5 & -1 & y_1 \\ 0 & -15 & 3 & y_3 - 2y_2 \end{bmatrix}$$

$R_2 \leftarrow R_2/5$

$$A' = \begin{bmatrix} 1 & 8 & -1 & y_2 \\ 0 & 1 & -1/5 & y_1/5 \\ 0 & -15 & 3 & y_3 - 2y_2 \end{bmatrix}$$

$R_1 \leftarrow R_1 - 8R_2$

$$A' = \begin{bmatrix} 1 & 0 & 3/5 & y_2 - 8y_1/5 \\ 0 & 1 & -1/5 & y_1/5 \\ 0 & -15 & 3 & y_3 - 2y_2 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 0 & 3/5 & y_2 - 8y_1/5 \\ 0 & 1 & -1/5 & y_1/5 \\ 0 & 0 & & \end{bmatrix}$$

Example: Take $A' = \left[\begin{array}{ccc|c} 0 & 5 & -1 & y_1 \\ 1 & 8 & -1 & y_2 \\ 2 & 1 & 1 & y_3 \end{array}\right]$

We apply the following operations in order —

(a) $R_1 \leftrightarrow R_2$

(b) $R_3 \leftarrow R_3 - 2R_1$

(c) $R_2 \leftarrow \frac{R_2}{5}$

(d) $R_1 \leftarrow R_1 - 8R_2$

(e) $R_3 \leftarrow R_3 + 15R_2$

$$\begin{pmatrix} 0 & 5 & -1 & y_1 \\ 1 & 8 & -1 & y_2 \\ 2 & 1 & 1 & y_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 8 & -1 & y_2 \\ 0 & 5 & -1 & y_1 \\ 2 & 1 & 1 & y_3 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 8 & -1 & y_2 \\ 0 & 5 & -1 & y_1 \\ 0 & -15 & 3 & y_3 - 2y_2 \end{pmatrix}$$

$$\downarrow$$

$$\cancel{1 \quad 0 \quad 3/5 \quad y_2 - 8y_1/5}$$

$$\begin{pmatrix} 1 & 8 & -1 & y_2 \\ 0 & 1 & -1/5 & y_1/5 \\ 0 & -15 & 3 & y_3 - 2y_2 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & 3/5 & y_2 - 8y_1/5 \\ 0 & 1 & -1/5 & y_1/5 \\ 0 & -15 & 3 & y_3 - 2y_2 \end{pmatrix}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3/5 & y_2 - 8y_1/5 \\ 0 & 1 & -1/5 & y_1/5 \\ 0 & 0 & 0 & y_3 - 2y_2 + 3y_1 \end{array}\right)$$

This implies the system is consistent iff $y_3 - 2y_2 + 3y_1 = 0$

$\cancel{\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_2 - \frac{8y_1}{5} \\ \frac{y_1}{5} \\ 0 \end{bmatrix}}$

$\cancel{\Rightarrow x_1 = \frac{5y_2 - 8y_1}{5}}$

$\cancel{x_2 = \frac{y_1}{5}}$

$\cancel{x_3 = 0}$

From the above RREF, we get —

(i) $x_1 + \frac{3}{5}x_3 = y_2 - \frac{8y_1}{5}$

(ii) $x_2 - \frac{x_3}{5} = \frac{y_1}{5}$

Taking $x_3 = c$, the solution set is —

$$\{x_1, x_2, x_3\} = \left\{ y_1 - \frac{8y_1}{5} - \frac{3c}{5}, \frac{y_1}{5} + \frac{c}{5}, c \right\}$$

4. a) Let us prove i) $\Rightarrow$ ii)

Given, matrix A is invertible

$$AX = Y$$
$$\Rightarrow X = A^{-1}Y$$
$\Rightarrow AX = Y$ has a solution for every matrix $Y_{n\times 1}$

b) Now let us prove ii) $\Rightarrow$ i)

* Given, $AX = Y$ has a solution for every matrix $Y_{n\times 1}$

  $\Rightarrow RX = Z$ has a solution for every Z
* It's then sufficient to show R is invertible, since $R \sim A$

  $\{ R = (E_1 E_2 \cdots E_s)A$ where E_i is elementary matrix $\forall\, i \in [1, s] \}$
* Take $Z_{n\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ {All 0's except last entry}

This $Z_{n\times 1}$ must also have a solution

$\Rightarrow R$ cannot have a zero row

$\Rightarrow R$ is invertible

$\Rightarrow A$ is invertible

5. If matrix A is invertible, then

~~$A E_1 E_2 E_3 \cdots E_s = I$~~

$I = E_1 E_2 E_3 \cdots E_s A$, where $E_1, E_2 \cdots E_s$ are elementary matrices

* Now take $A = A_1 A_2 A_3 A_4 \cdots A_k$

  $I = (E_1 E_2 E_3 \cdots E_s A_1)(A_2 A_3 \cdots A_k)$
* Since the product is I, $\left(\prod_{i=1}^{s} E_i\right)A_1$ has a right inverse

  $\Rightarrow \left(\prod_{i=1}^{s} E_i\right)A_1$ is invertible

  $\Rightarrow A_1$ is invertible, since $\prod_{i=1}^{s} E_i$ is invertible
* We can repeat this procedure for any j by rewriting product $A = (A_1 A_2 \cdots A_{j-1}) A_j (A_{j+1} \cdots A_k)$
* Thus, $\forall\, j \in [1, k]$, A_j is invertible. Hence proved!

6. * Given, A is an invertible matrix and $\exists\, E_i,\ i \in [1, s]$ such that

$$I = (E_1 E_2 E_3 \cdots E_s) A$$

* Right-multiply A^{-1}, we got –

$$I \cdot A^{-1} = \left(\prod_{i=1}^{s} E_i\right) A \cdot A^{-1}$$

$$\Rightarrow A^{-1} = \left(\prod_{i=1}^{s} E_i\right) I$$

$\Rightarrow$ The same sequence of operations when applied to I yields A^{-1}, hence proved!

7. * Take A^{-1} to be left inverse of A

$$\Rightarrow A^{-1} A = I$$

* Now consider $AX = 0$

$$\Rightarrow A^{-1} \cdot (AX) = A^{-1} \cdot 0$$

$$\Rightarrow (A^{-1} A) \cdot X = 0$$

$$\Rightarrow I \cdot X = 0$$

$$\Rightarrow X = 0$$

* Thus, $AX = 0$ has only trivial solutions. Hence proved!

8. * In a row-reduced echelon matrix, for all the non-zero rows, there exists a pivot element and this pivot element is 1

* The matrix is as below –

$$\begin{array}{c}
\begin{array}{ccccccc} & a_{1k_1} & & a_{2k_2} & & a_{3k_3} & \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & \cdots & 1 & \cdots & 0 & \cdots & & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & & \cdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots \\
 & & & & & \vdots & & & &
\end{bmatrix}
\begin{array}{l} \} u_1 \\ \} u_2 \\ \} u_3 \\ \end{array}
\end{array}$$

* Then, $\sum_{i=1}^{r} c_i u_i$ is of the form $\{c_1, c_2, c_3 \cdots c_r$, linear combinations of c_i's$\}$

* Since each c_i appears once individually $\{\forall\, i \in [1, r]\}$, and since $\sum_{i=1}^{r} c_i u_i = 0 \Rightarrow c_i = 0 \ \forall\, i \in [1, r]$

Thus, the given statement is true