Linear Algebra

Problem Set 3

1. ∘ Given, $H_n(\mathbb{C}) = \{ A \in \mathbb{C}^{n \times n} : A^H = A \}$.

∘ Let $A = \begin{Bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{Bmatrix}$, then $A^H = \begin{Bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & \overline{a_{n2}} \\ \vdots & & & \\ \overline{a_{1n}} & \overline{a_{2n}} & \cdots & \overline{a_{nn}} \end{Bmatrix}$

∘ By definition, $A^H = A \Rightarrow \overline{a_{ii}} = a_{ii} \quad \forall i \in [1, n]$
$\Rightarrow a_{ii} \in \mathbb{R} \quad \forall i \in [1, n]$

(a) Consider an arbitrary constant $k \in \mathbb{C}$, then kA may or may not belong to $H_n(\mathbb{C})$. {For instance, consider $a_{11} = 2$ and $k = \sqrt{2}i$, then $ka_{11} = 2\sqrt{2}i \notin \mathbb{R}$ and thus the matrix is no longer hermitian}.

(b) <i> Now considering an arbitrary constant $l \in \mathbb{R}$, then clearly $lA \in H_n(\mathbb{C})$ { $la_{ii} \in \mathbb{R} \; \forall i \in [1, n]$ by closure under the field $\mathbb{R}$ }

~~∘ Thus, $H_n(\mathbb{C})$ is a vector space over the field $\mathbb{R}$ but not over the field $\mathbb{C}$~~

<ii> Then consider $A \in H_n(\mathbb{C})$ and $B \in H_n(\mathbb{C})$, clearly $A + B \in H_n(\mathbb{C})$ { $a_{ii} + b_{ii} \in \mathbb{R}$ by closure under the field $\mathbb{R}$ }

<iii> The matrix $O = [0]_{ij} \; \forall i, j \in [1, n] \in H_n(\mathbb{C})$, and $O + A = A \; \forall A \in H_n(\mathbb{C})$

<iv> $\exists$ matrix $-A \in H_n(\mathbb{C})$ such that $A + -A = O$

<v> $A + B = B + A \; \forall A, B \in H_n(\mathbb{C})$

<vi> $(A + B) + C = A + (B + C) \; \forall A, B, C \in H_n(\mathbb{C})$

<vii> Taking $c_1, c_2 \in \mathbb{R}$; $(c_1 c_2)A = c_1(c_2 A)$ holds $\forall A \in H_n(\mathbb{C})$

<viii> $c(A + B) = cA + cB$; $(c_1 + c_2)A = c_1 A + c_2 A \; \forall c, c_1, c_2 \in \mathbb{R}$ and $\forall A, B \in H_n(\mathbb{C})$

∘ Thus, $H_n(\mathbb{C})$ is a vector space over the field $\mathbb{R}$ but not over the field $\mathbb{C}$

2. • For the sake of contradiction, assume neither statement holds, meaning –

<i> $\exists\, x \in W_1$ such that $x \notin W_2$

<ii> $\exists\, y \in W_2$ such that $y \notin W_1$

• Since $x \in W_1 \subseteq W_1 \cup W_2$ and $y \in W_2 \subseteq W_1 \cup W_2$, it implies $x, y \in W_1 \cup W_2$

• Then, since $W_1 \cup W_2$ is a subspace ; $x + y \in W_1 \cup W_2$ by closure under addition

$$x + y \in W_1 \cup W_2$$

<u>Case 1:</u> $x + y \in W_1$

• Then $\exists (-x)$ such that $x + (-x) = 0$ {Additive Inverse}

• $x + y + (-x) \in W_1$ via closure under addition

$\Rightarrow$ $y \in W_1$, a clear contradiction to the fact that $y \notin W_1$

<u>Case 2:</u> $x + y \in W_2$

• $\exists (-y) \in W_2$ such that $0 = y + (-y) \in W_2$ {Additive Inverse}

• $x + y + (-y) \in W_2$ via closure under addition

$\Rightarrow$ $x \in W_2$, a clear contradiction to the fact that $x \notin W_2$

• Thus, our initial assumption was wrong and hence $W_1 \cup W_2$ being a subspace implies $W_1 \subseteq W_2$ (or) $W_2 \subseteq W_1$

3. • Given, ~~$W = \bigcap_\alpha W_\alpha$~~ $W = \bigcap_a W_a$ is the intersection of subspaces $\{W_a\}_a$

• Now consider $\alpha, \beta \in W$ and c to be a scalar

$\alpha, \beta \in W$

$\Rightarrow \alpha, \beta \in W_a \;\forall a$ { Since $W = \bigcap_a W_a$ }

$\Rightarrow c\alpha + \beta \in W_a \;\forall a$ { Since W_a itself is a subspace }

$\Rightarrow c\alpha + \beta \in W$ { Since $W = \bigcap_a W_a$ }

• Hence proved $c\alpha + \beta \in W$

4. Given, F is a field and S is any non-empty set, and V is the set of all functions from S into F. Now consider —

(i) $(f+g)(s) = f(s) + g(s)$

(a) $f(s) + g(s) = g(s) + f(s)$ {commutativity holds in F}
$\Rightarrow (f+g)(s) = (g+f)(s)$, meaning functions $f+g$ and $g+f$ are identical

(b) $f(s) + \{g(s) + h(s)\} = \{f(s) + g(s)\} + h(s)$ {Associativity holds in F}
$\Rightarrow (f+(g+h))(s) = ((f+g)+h)(s)$, meaning functions $f+(g+h)$ and $(f+g)+h$ are identical

(c) $\exists$ zero function $\in V$ that assigns every value in S 0 from F.

(d) $\exists -f \in V \; \forall f \in V$ such that $(f+-f)(s)$
$= f(s) + (-f)(s) = f(s) + -f(s) = 0$

(ii) $(cf)(s) = cf(s)$

(a) consider scalars c, d ~~$\in F$~~ and $f(s) \in F$,
then $(cd) f(s) = c(d f(s))$ {Associativity in field F}
~~$\Rightarrow$ Associativity holds $\forall f \in V$~~

~~(b)~~ $\Rightarrow$ Also, $(c(df))(s) = c(df)(s)$
$= c((df)(s))$
$= (cd) f(s)$
$\Rightarrow (cd)f = c(df) \quad \forall f \in V$

(b) $\exists\, 1 \in F$ such that $(1 \cdot f)(s) = 1 \cdot (f(s)) = f(s)$
$\Rightarrow 1 \cdot f = f$

(c) $(c(f+g))(s) = c(f+g)(s)$
$= c\{f(s) + g(s)\}$
$= c f(s) + c g(s)$ {Distributivity in field F}

$(cf + cg)(s) = (cf)(s) + (cg)(s)$
$= cf(s) + cg(s)$
$\Rightarrow c(f+g) = cf + cg$

(d) $((c_1+c_2)f)(s) = (c_1+c_2)f(s) = c_1 f(s) + c_2 f(s)$

{Distributivity in field F}

$(c_1 f + c_2 f)(s) = (c_1 f)(s) + (c_2 f)(s)$

$= c_1 f(s) + c_2 f(s)$

$\Rightarrow (c_1+c_2) f = c_1 f + c_2 f \quad \forall\, c_1, c_2$ and $\forall\, f \in V$

{scalars}

5. • Given, $S = \{(x_1, x_2, x_3 \dots x_n) \in \mathbb{R}^n \mid x_1 = 1 + x_2\}$

• For S to be a subspace, it must obey the following –

<i> $\forall\, A, B \in S$, $A + B \in S$

<ii> $\forall\, A \in S$, $c \in F$ {field}, $cA \in S$

<iii> $\exists\, 0 \in S$ such that $0 + A = A \in S \ \forall\, A \in S$

<iv> $\forall\, A \in S$, $\exists (-A) \in S$ such that $A + (-A) = 0$

• We are interested in point <iii>, that is,

$(\underset{x_1}{0}, \underset{x_2}{0}, \dots \underset{x_n}{0}) \in S$

• clearly, $(\underset{x_1}{0}, \underset{x_2}{0}, \dots \underset{x_n}{0}) \notin S$ ~~since $x_1 = 0 \neq 1 + x_2 = 0$~~ since $x_1 \neq 1 + x_2$

• Thus, S is not a sub-space

6. • Given, $W = W_1 + W_2 + \dots W_K = \{v_1 + v_2 + v_3 + \dots v_K \mid v_i \in W_i\}$

where $W_i \ \forall\, i \in [1, K]$ is a subspace of V

• For $u_i, v_i \in W_i$ and $c \in F$, $cu_i + v_i \in W_i$ {since W_i is a subspace} {$\forall\, i \in [1, K]$}

<a> <i> $\underbrace{cu_1 + v_1}_{\in W_1} + \underbrace{(cu_2 + v_2}_{\in W_2} + \dots \underbrace{cu_K + v_K}_{\in W_K} \in W$

<ii> $\underbrace{c(u_1 + u_2 + \dots u_K)}_{\in W \text{ since } u_1 \in W_1,\ u_2 \in W_2 \dots u_K \in W_K} + \underbrace{(v_1 + v_2 + \dots v_K)}_{\in W \text{ since } v_1 \in W_1,\ v_2 \in W_2, \dots v_K \in W_K}$

$= (cu_1 + v_1) + (cu_2 + v_2) + \cdots (cu_k + v_k)$

- From <i> and <ii>, $c(\sum_{i=1}^{k} u_i) + \sum_{i=1}^{k} v_i \in W$, hence W is a subspace of V

<b> • Take any $w \in W$, $w = \underbrace{w_1}_{\in W_1} + \underbrace{w_2}_{\in W_2} + \cdots \underbrace{w_n}_{\in W_n}$

And also, $w_i \in W_i \subseteq W_1 \cup W_2 \cup \cdots W_n$

$w_i \in W_1 \cup W_2 \cup \cdots W_n \quad \forall i \in [1, n]$

This holds $\forall w \in W$, thus $W \subseteq \text{span}(W_1, W_2 \cdots W_n) = U$ {say}

- Consider $u \in U = \text{span}(W_1, W_2 \cdots W_n)$

$u = \sum_{i=1}^{n} a_i v_i \; ; \quad a_i \in \mathbb{R} \;\forall i \in [1, n]$

$v_i \in W_j$ ~~$\forall i \in [1,n]$ for in~~ for some $i, j \in [1, n]$

Take $u = \{\text{sum of terms from } W_1\} + \{\text{sum of terms from } W_2\} + \cdots \{\text{sum of terms from } W_n\}$

Thus, $u = w_1 + w_2 + \cdots w_n$, $w_i \in W_i \;\forall i \in [1, n]$

$\Rightarrow u \in W$

$\Rightarrow U \subseteq W$

- Thus, $U = W$ and hence $W = \text{span}(W_1 \cup W_2 \cup \cdots W_k)$

<b> • To prove, W is the subspace spanned by $W_1 \cup W_2 \cdots W_k$

- First, we show $W \subseteq \text{span}(W_1 \cup W_2 \cup \cdots W_k)$

Take any $\omega \in W$, $\omega = \omega_1 + \omega_2 + \omega_3 + \cdots \omega_k$

Now consider ω_1, $\omega_1 \in W_1 \subseteq W_1 \cup W_2 \cup \cdots W_k$

$\Rightarrow \omega_1 \in W_1 \cup W_2 \cup \cdots W_k$

$\Rightarrow \omega_1 \in \text{span}(W_1 \cup W_2 \cup \cdots W_k)$

{Why? Because $\text{span}(W_1 \cup W_2 \cup \cdots W_k) = c_1\omega_1 + c_2 x_2 + c_3 x_3 + \cdots$, and putting $c_i = 0$ {$i \geq 2$} and $c_1 = 1$ we get $\omega_1 \in \text{span}(W_1 \cup W_2 \cup \cdots W_k)$}

- Doing this for any such w_i, we see that $w_i \in \text{span}(W_1 \cup W_2 \cdots W_k)$ $\forall\, i \in [1, k]$

$\Rightarrow$ $w_1 + w_2 + w_3 + \cdots w_k \in \text{span}(W_1 \cup W_2 \cup \cdots W_k)$, since $\text{span}(W_1 \cup W_2 \cup \cdots W_k)$ is a subspace and closure under addition exists for subspaces

$\Rightarrow$ $w \in \text{span}(W_1 \cup W_2 \cup \cdots W_k)$

Since choice of w was arbitrary, this holds true for any such $w \in W$

$\Rightarrow$ $W \subseteq \text{span}(W_1 \cup W_2 \cup \cdots W_k)$

- Now we prove $\text{span}(W_1 \cup W_2 \cup \cdots W_k) \subseteq W$

Take any $v \in \text{span}(W_1 \cup W_2 \cup \cdots W_k)$

$\Rightarrow v = a_1 x_1 + a_2 x_2 + \cdots a_k x_k$

Here $x_i \{1 \le i \le k\} \in W_1 \cup W_2 \cup \cdots W_k$

$\Rightarrow x_i$ in general belongs to some W_j $\{1 \le j \le k\}$

{Note that i may or may not be equal to j in general}

Now grouping terms, $v = \underbrace{\{\text{sum of terms from } W_1\}}_{\in W_1} + \underbrace{\{\text{sum of terms from } W_2\}}_{\in W_2} + \cdots \underbrace{\{\text{sum of terms from } W_k\}}_{\in W_k}$

{Note, sum of terms from $W_i \in W_i$ since W_i is a subspace, and we make use of closure under addition}

$\Rightarrow v \in W$ {By definition of W}

Since choice of v was arbitrary, this holds true $\forall\, v \in \text{span}(W_1 \cup W_2 \cup \cdots W_k)$

$\Rightarrow \text{span}(W_1 \cup W_2 \cup \cdots W_k) \subseteq W$

- From the above statements, $\mathrm{span}(W_1 \cup W_2 \cup \cdots W_k) = W$

Hence proved!