Linear Algebra

Assignment 1

1. Two non-trivial subfields of complex numbers are -
   i) Real Numbers
   ii) Rational Numbers

We are given that the complex numbers form a field, so now consider

i) Real Numbers :

(a) Take two real numbers $a$ and $b \in \mathbb{R}$, then we can write them as $a + 0i \in \mathbb{C}$ and $b + 0i \in \mathbb{C}$ $\{$ where $i = \sqrt{-1}$ $\}$

$a + 0i + b + 0i = a+b+(0+0)i = a+b+0i \in \mathbb{R}$, thus ~~addition under~~ $\mathbb{R}$ ~~is closed~~ are closed under addition $\{$ since $a+b$ constitutes real part of complex numbers

(b) $a + 0i + b + 0i = a+b+(0+0)i = a+b+0i$

$b + 0i + a + 0i = \underbrace{a + 0i + b + 0i}_{\text{since } \mathbb{C} \text{ form a field, they must be commutative under addition}} = a+b+(0+0)i = a+b+0i$

$a + 0i + b + 0i = b + 0i + a + 0i$

$a + b = b + a$

$\mathbb{R}$ are commutative under addition

(c) Let $A = a + 0i$, $B = b + 0i$ and $C = c + 0i$, then $\{a, b, c \in \mathbb{R}\}$

$A + (B + C) = (A + B) + C$ $\{$ since $\mathbb{C}$ forms a field $\}$

$a + 0i + (b + 0i + c + 0i) = (a + 0i + b + 0i) + c + 0i$

$a + (b + c) = (a + b) + c \implies \mathbb{R}$ is associative under addition

(d) Take $D \in \mathbb{C}$ such that $D = 0 + 0i$

$A \in \mathbb{R}$ such that $a + 0i = A$, $a \in \mathbb{R}$, then

$A + D = A + 0 + 0i = \underbrace{a + 0i + 0 + 0i}_{\text{since } \mathbb{C} \text{ forms a field, it has identity element } 0+0i \text{ such that } b+0+0i = b \ \forall b \in \mathbb{C}} = a + 0i = a$

$D + A = 0 + 0i + A = 0 + 0i + a + 0i = a + 0i = a$

Thus, $\mathbb{R}$ has additive identity zero

(e) For $a + bi \in \mathbb{C}$, $\exists$ $c + di \in \mathbb{C}$ such that

$a + bi + c + di = 0 + 0i$

Taking $b = d = 0$, we get

$a + 0i + c + 0i = 0 + 0i$

$a + c + (0 + 0)i = 0 + 0i$

$a + c + 0i = 0 + 0i$

$a + c = 0 \implies c = -a$

Thus, $\forall a \in \mathbb{R}$ $\exists$ $c \in \mathbb{R}$ such that $a + c = 0$

f) $(a+bi)(c+di) \in \mathbb{C}$ since $\mathbb{C}$ forms a field

$ac + (ad+bc)i + bd(-1) \in \mathbb{C}$

Setting $b=d=0$, we get − $ac + (a\cdot 0 + 0\cdot c)i + 0\cdot(-1) \in \mathbb{C}$

$ac \in \mathbb{C}$

$ac + 0i \in \mathbb{C}$

Thus, $ac \in \mathbb{R}$ since it constitutes real part of complex number

g) Again take $A = a+0i$, $B = b+0i$ and $C = c+0i \in \mathbb{C}$ $\{a, b, c \in \mathbb{R}\}$

~~$A+B$~~ $(A\cdot B)\cdot C = A\cdot(B\cdot C)$ {since $\mathbb{C}$ forms a field}

$((a+0i)\times(b+0i))(c+0i) = (a+0i)\{(b+0i)(c+0i)\}$

$(ab)c = a(bc) \Rightarrow \mathbb{R}$ is associative under multiplication

h) From (g), consider $A$, $B$ and $C$

(1) $A(B+C) = A\cdot B + A\cdot C$ {since $\mathbb{C}$ is a field}

$(a+0i)(b+0i+c+0i) = (a+0i)(b+0i) + (a+0i)(c+0i)$

$a(b+c) = ab + ac$

(2) $(B+C)A = BA + CA$ {since $\mathbb{C}$ is a field}

$(b+0i+c+0i)(a+0i) = (b+0i)(a+0i) + (c+0i)(a+0i)$

$(b+c)a = ba + ca \Rightarrow \mathbb{R}$ obeys distributivity

i) $AB = BA$ {since $\mathbb{C}$ is a field}

$(a+0i)(b+0i) = (b+0i)(a+0i)$

$a\cdot b = b\cdot a \Rightarrow \mathbb{R}$ is commutative under multiplication

j) Considering all non-zero $\mathbb{R}$ numbers −

(1) They are closed under '×' from f)

(2) They are commutative under '×' from (i)

(3) They are associative under '×' from (h)

(4) $\exists\ 1+0i\ \forall\ a+bi \in \mathbb{C}$ such that

$(a+bi)(1+0i) = (1+0i)(a+bi) = a+bi$

Setting $b=0$, we get − $a\times 1 = 1\times a = a \Rightarrow 1 \in \mathbb{R}$ is multiplicative identity for $\mathbb{R}$

(5) $\exists\ c+di\ \forall\ a+bi \in \mathbb{C}$ such that

$(a+bi)(c+di) = 1 = 1+0i \Rightarrow ac+bd+(ad+bc)i$

Setting $b=d=0$, we get $ac = 1 \Rightarrow \exists\, c \in \mathbb{R}\ \forall\, a \in \mathbb{R}$ such that $ac = 1$

Thus, $\mathbb{R}$ forms a field and is thus a sub-field of $\mathbb{C}$!

(ii) Rational Numbers :

$\mathbb{Q}$ is a subset of $\mathbb{R}$, and since we've proved above $\mathbb{R}$ is a ~~subset~~ subfield of $\mathbb{C}$, it follows that $\mathbb{Q}$ is also a sub-field of $\mathbb{C}$!

2. We consider a general matrix $A_{m \times n}$ and the following operations -

i) Multiplying a row by a constant $c \{c \in \mathbb{R}$ and $c \neq 0\}$

$$e(A)_{ij} = \begin{cases} cA_{ij}, & \text{if } i = r \ \{r \in \mathbb{N}\} \\ A_{ij}, & \text{if } i \neq r \end{cases}$$

Now take $e'(A_{ij}) = \begin{cases} dA_{ij}, & \text{if } i = s \ \{s \in \mathbb{N},\ s \neq r,\ d \in \mathbb{R}\} \\ A_{ij}, & \text{if } i \neq s \end{cases}$

$$e'(e(A)_{ij}) = \begin{cases} cA_{ij}, & \text{if } i = r \\ A_{ij}, & \text{if } i \neq r, s \\ dA_{ij}, & \text{if } i = s \ \{r \neq s\} \end{cases} = B \neq A$$

$$e(e'(A_{ij})) = \begin{cases} dA_{ij}, & \text{if } i = s \\ A_{ij}, & i \neq r, s \\ cA_{ij}, & \text{if } i = r \end{cases} = B \neq A$$

(ii) Adding a multiple of one row to another row

Let $e$ and $e'$ both be defined as follows

$$e(A)_{ij} = e'(A_{ij}) = \begin{cases} A_{ij} + cA_{sj}, & \text{if } i = r, \{\text{where } c \in \mathbb{R},\ s \neq r\} \\ A_{ij}, & \text{if } i \neq r \end{cases}$$

$$e(e'(A_{ij})) = \begin{cases} A_{ij} + 2cA_{sj}, & \text{if } i = r \\ A_{ij}, & \text{if } i \neq r \end{cases} = e'(e(A)_{ij}) = B \neq A$$

(iii) Interchanging two rows

$$e(A)_{ij} = \begin{cases} A_{sj}, & \text{if } i = r \\ A_{ij}, & \text{if } i \neq r, s \\ A_{rj}, & \text{if } \cancel{i \neq s}\ i = s \end{cases}$$

$$\left\{ \begin{array}{l} \text{Here } r \neq s,\ u \neq v \\ r \neq u, v \quad s \neq u, v \end{array} \right\}$$

$$e'(A)_{ij} = \begin{cases} A_{uj}, & \text{if } i = v \\ A_{ij}, & \text{if } i \neq v, u \\ A_{vj}, & \text{if } i = u \end{cases}$$

$$e(e'(A)_{ij}) = e'(e(A)_{ij}) = \begin{cases} A_{sj}, & \text{if } i = r \\ A_{rj}, & \text{if } i = s \\ A_{uj}, & \text{if } i = v \\ A_{vj}, & \text{if } i = u \\ A_{ij}, & \text{if } i \neq r, s, u, v \end{cases} = B \neq A$$

Thus proved!

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3. We consider a matrix $A_{m \times n}$ as follows —

$$A_{m \times n} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn} \end{bmatrix}$$

* If a row is a zero row, do nothing if it is already appearing after all non-zero rows
* Otherwise, swap it with a non-zero row to send it to the bottom of the matrix, thus appearing after all non-zero rows
* ~~Now consider row 1, and say it's first non-zero entry is~~ $A_{1k}$. Then do — $R_1 \leftarrow R_1 / A_{1k}$ to make leading entry 1
* Then, do — $R_i \leftarrow R_i - A_{ik} R_1$ to make all entries in column '$k$' zero $\{i \in \mathbb{N}, i \neq 1\}$
* We repeat the process for all other rows 2 to m.
* In case leading entry for current row appears to the right of the leading entry in next row, swap the two rows
{ This ensures when we convert that entry to 1, in each row the leading entry 1 will appear to the right of the leading 1 in the previous row! }

* We've thus proved that every $m \times n$ matrix over the field $F$ is row-equivalent to a row-reduced echelon matrix!

4. Consider system I as given below

$$\begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 8 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 8 \\ -1 & 1 & 4 \end{bmatrix} \xrightarrow[\text{(ii) } R_3 \leftarrow R_3 + R_1]{\text{(i) } R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix}$$

$$\xrightarrow[\text{(ii) } R_3 \leftarrow R_3 - 3R_2]{\text{(i) } R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

* This is the rref of matrix A

$\Rightarrow$ $x_1 - x_3 = 0$

$x_2 + 3x_3 = 0$ is equivalent to system (I)

$\Rightarrow$ $\{t, -3t, t\}$ is general solution for system (I)

* Now take an arbitrary solution, say $\{1, -3, 1\}$ and try it in system (II)

$x_2 + 2x_3 = -3 + 2 = -1 \neq 0$

* Clearly, $\{1, -3, 1\}$ does not satisfy system (II) and thus, systems (I) and (II) are not equivalent!