LCB2023040

INDIAN INSTITUTE OF INFORMATION TECHNOLOGY LUCKNOW
END- SEMESTER EXAMINATION 2024
SUBJECT: PROBABILITY & STATISTICS

Course Code: PAS3300C
Time: 3 Hrs

Semester: III
Max. Marks: 75

Calculators are allowed. All questions are compulsory and carries 5 marks each.

1. Suppose we have a random sample of size $2n$ from a population denoted by X, and $E(X) = \mu$ and $Var(X) = \sigma^2$. Let

$$\bar{X_1} = \frac{1}{2n}\sum_{i=1}^{2n} X_i \text{ and } \bar{X_2} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

be two estimators of μ. Which is the better estimator of μ? Explain your choice.

2. Let the random variable X has the probability distribution

$$f(x) = \begin{cases} (\gamma+1)x^{\gamma}, & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n. Find the maximum likehood estimator of γ.

3. Consider the following frequency distribution.

x_i	-4	-3	-2	-1	0	1	2	3	4
f_i	60	120	180	200	240	190	160	90	30

Compute the sample coefficient of variation.

4. Consider the quantity $\sum_{i=1}^{n}(x_i - a)^2$, x_i denotes ith data. For what value of a is this quantity minimized?

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population, with mean μ and variance σ^2. Prove that the distribution of $\frac{(n-1)S^2}{\sigma^2}$ is χ^2_{n-1}, where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$.

6. Let $[X_1, X_2]$ be a continuous random vector with joint density function:

$$f(x_1, x_2) = \begin{cases} 4e^{-2(x_1+x_2)}, & x_1 > 0, x_2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find the distribution of $Y = \frac{X_1}{X_2}$, using $Z = X_1 + X_2$ as another random variable to find the Jacobian of the transformation.

7. Consider the probability distribution of the discrete random vector $[X_1, X_2]$, where X_1 represents the number of orders for aspirin in August at the neighborhood drugstore and X_2 represents the number of orders in September. The joint distribution is given in the form of a table:

$X_2 \downarrow X_1 \rightarrow$	51	52	53	54	55
51	0.06	0.05	0.05	0.01	0.01
52	0.07	0.05	0.01	0.01	0.01
53	0.05	0.10	0.10	0.05	0.05
54	0.05	0.02	0.01	0.01	0.03
55	0.05	0.06	0.05	0.01	0.03

i. Find the marginal distributions.

ii. Find the expected sales in September given that the sales in August were either 51 or 52.

8. Consider a situation in which the surface tension and acidity of a chemical product are measured. These variables are coded such that surface tension is measured on a scale $0 \leq X_1 \leq 2$, and acidity is measured on a scale $2 \leq X_2 \leq 4$. The pdf of $[X_1, X_2]$ is

$$f(x_1, x_2) = \begin{cases} k(6 - x_1 - x_2), & 0 \leq x_1 \leq 2, 2 \leq x_2 \leq 4, \\ 0 & \text{otherwise.} \end{cases}$$

   i. Find k.

   ii. Calculate $P(X_1 < 1, X_2 < 3)$ and $P(X_1 + X_2 \leq 4)$.

9. Let X be a Poisson random variable with parameter λ. Define the distribution.
   Show that $P(X \text{ is even }) = \frac{1}{2}(1 + e^{-2\lambda})$.

10. Let $X_1 \sim N(1.5, 0.0016)$ and $X_2 \sim N(1.48, 0.0009)$, consider a random variable $Y = X_1 - X_2$. Find mean and variance for Y and hence find $P(Y < 0)$. (Use the cdf values for normal distribution as $\Phi(-0.3) = 0.38209, \Phi(-0.4) = 0.34458, \Phi(-0.5) = 0.30854, \Phi(-0.6) = 0.27425$.)

11. Let X be a random variable with pdf:

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

   Find the density for $Y = 2X^2$ and $U = \ln X$.

12. Suppose X takes on values 5 and -5 with probabilities 0.5. Plot the quantity $P[|X - \mu| \geq k\sigma]$ as a function of k (for $k > 0$). On the same set of axes, plot the same probability determined by Chebyshev's inequality.

13. The random variable X is uniformly distributed over the interval $[0, 4]$. What is the probability that the roots of $y^2 + 4Xy + X + 1 = 0$ are real?

14. Is there an exponential density that satisfies the following condition?

$$P(X \leq 2) = \frac{2}{3}P(X \leq 3).$$

   If so, find the value of λ.

15. Two dice are tossed assuming each die is true. If we consider 2 events,

$$A = \{(d_1, d_2) : d_1 + d_2 = 4\},$$
$$B = \{(d_1, d_2) : d_2 \geq d_1\},$$

   where d_1 is the value of the up face of the first die and d_2 is the value of the up face of the second die. Find $P(A), P(B), P(A \cap B), P(B|A)$ and $P(A|B)$.

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