# Discrete Mathematics

Question Paper

February 19, 2025

## Contents

# 1 Class Test on 17 Feb 2025

**Max Marks: 10** **Max Time: 30 Min**

**Answer all the following questions. The use of mobile devices and calculators is prohibited during the class test. Each and every step of your calculation should be shown on the answer sheet with justification.**

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1. "The square of every real number is nonnegative." Write this in symbols, negate it using the symbols and then write the negation in words. 1.5

2. Let $R$ and $S$ be relations on sets $X$ and $Y$, respectively. Then there exists a natural relation $T$ on $X \times Y$ defined by $(x_1, y_1)T(x_2, y_2)$ iff $x_1 R x_2$ and $y_1 S y_2$. What properties of $R$ and $S$ are shared by $T$ w.r.t. reflexivity and antisymmetry? 1.5

3. Prove that the maximum number of regions formed by $n$ lines in a plane is $\frac{1}{2}(n^2 + n + 2)$ using mathematical induction. 3

4. Construct a bijection $g : \mathbb{N} \to A \setminus B$ with justification with the help of $f$, where $f : \mathbb{N} \to A$ is a bijection and $B = \{2, 17, 2025\}$. 4

## 1.1 Answer Key

1. "The square of every real number is nonnegative." Write this in symbols, negate it using the symbols and then write the negation in words. 1.5

   **Solution 1.1.** *In symbol, we can write the given sentence as*

$$\forall\, x \in \mathbb{R},\ x^2 \geq 0.$$

***OR***

$$\forall\, x \in \mathbb{R},\ (x^2 \geq 0).$$

*The negation of the above expression is*

$$\exists\, x \in \mathbb{R},\ x^2 \ngeq 0 \quad \Rightarrow \exists\, x \in \mathbb{R},\ x^2 < 0.$$

***OR***

$$\exists\, x \in \mathbb{R},\ (x^2 \ngeq 0) \quad \Rightarrow \exists\, x \in \mathbb{R},\ (x^2 < 0).$$

*The above expression can be written in words as:*

*There exists a real number whose square is not nonnegative.*

***OR***

*There exists a real number whose square is negative.*

2. Let $R$ and $S$ be relations on sets $X$ and $Y$, respectively. Then there exists a natural relation $T$ on $X \times Y$ defined by $(x_1, y_1)T(x_2, y_2)$ iff $x_1 R x_2$ and $y_1 S y_2$. What properties of $R$ and $S$ are shared by $T$ w.r.t. reflexivity and antisymmetry? 1.5

**Solution 1.2.** *(i)* ***W.r.t. reflexivity:*** *A relation $R$ on a set $X$ is reflexive if $xRx,\ \forall\ x \in X$.*

*First, we assume that $R$ and $S$ are both reflexive on $X$ and $Y$ respectively.*

*Now, we will check the reflexivity of $T$ on $X \times Y$.*

*We have $(x,y)T(x,y),\ \forall\ (x,y) \in X \times Y$ iff $xRx, \forall\ x \in X$ and $ySy,\ \forall\ y \in Y$. Since, according to our assumption $R$ and $S$ are both reflexive, therefore $T$ is reflexive.*

*(ii)* ***W.r.t antisymmetry:*** *A relation $R$ is antisymmetric if $aRb$ and $bRa$, then $a = b,\ \forall\ a, b \in X$.*

*In this case also, we first assume that both $R$ and $S$ are antisymmetric. Then exactly applying the above procedure, we can prove that $T$ is antisymmetric*

***Explicitly:*** *To prove $T$ is antisymmetric, we assume that $(x_1, y_1)T(x_2, y_2)$ and $(x_2, y_2)T(x_1, y_1)\ \Rightarrow x_1 R x_2,\ \ y_1 S y_2$ and $x_2 R x_1,\ \ y_2 S y_1$*

*$\Rightarrow x_1 = x_2$ and $y_1 = y_2\quad \because\ R$ and $S$ are antisymmetric.*

*$\Rightarrow (x_1, y_1) = (x_2, y_2)$*

*Thus, $T$ is antisymmetric, if $R$ and $S$ are both antisymmetric.*

3. Prove that the maximum number of regions formed by $n$ lines in a plane is $\frac{1}{2}(n^2+n+2)$ using mathematical induction. 3

**Solution 1.3.** ***Base step:*** *If $n = 1$, the plane is divided into $2$ regions and for $n = 1$, we have, $\frac{1}{2}(1^2 + 1 + 2) = 2$, so it is true for $n = 1$.*

***Induction hypothesis:*** *Now, we assume that the result is true for $n = k$; i.e., he maximum number of regions formed by $k$ lines in a plane is $\frac{1}{2}(k^2 + k + 2)$*

***Induction step:*** *Now, we have to check that result is true for $n = k + 1$.*

*From the induction hypothesis, we can see that we have maximum $\frac{1}{2}(k^2 + k + 2)$ regions using $k$ lines.*

*By add one more line, we see that it meets the existing lines at $k$ distinct points. These $k$ points subdivide the new line into $k + 1$ segments. Each segment divides one of the existing regions into two, so the number of regions is increased by $k + 1$.*

*Now, the total number of regions is*

$$\begin{aligned} &= \tfrac{1}{2}(k^2 + k + 2) + (k + 1) \\ &= \tfrac{1}{2}(k^2 + k + 2 + 2k + 2) \\ &= \tfrac{1}{2}[(k + 1)^2 + (k + 1) + 2] \end{aligned}$$

So, the result is true for $n = k + 1$.

Hence proved $\square$

4. Construct a bijection $g : \mathbb{N} \to A \setminus B$ with justification with the help of $f$, where $f : \mathbb{N} \to A$ is a bijection and $B = \{2, 17, 2025\}$ 4

**Solution 1.4.** *According to the given question, we have to construct **a** bijection $g : \mathbb{N} \to A \setminus B$ with the help of $f$, i.e., we do not have to construct **the** bijection $g : \mathbb{N} \to A \setminus B$ with the help of $f$.*

*It can be constructed in the following manner, though this approach is not necessarily the only way to achieve it:*

*Since it is given that $f : \mathbb{N} \to A$ is a bijection, we can assume that $f(x) = x$ is a bijection; therefore, we can construct the function $g : \mathbb{N} \to A \setminus B$ as follows:*

$$g(n) = \begin{cases} f(n) & \text{if } n < 2 \\ f(n+1) & \text{if } 2 \le n < 16 \\ f(n+2) & \text{if } 16 \le n < 2023 \\ f(n+3) & \text{if } n \ge 2023 \end{cases}$$

*Since $f : \mathbb{N} \to A$ is a bijection, so from the above definition $g$ is a injective function.*

*Now we can see $A \setminus B$ has exactly the same elements in $A$ except the elements $\{2, 17, 2025\}$. From the above definition, we can see that we removed the points $f(2), f(17)$, and $f(2025)$. Thus, $g$ is an onto function.*

*Hence, $g : \mathbb{N} \to A \setminus B$ is a bijective function.* $\square$