Discrete Mathematics

Question Paper

March 24, 2025

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1 Mid Semester Examination on 24 Feb 2025

Name: **R. No.:** ....

Indian Institute of Information Technology Lucknow

Mid Semester Examination

Regular/Back

Mathematics for CS I (MCS4300C)

BTech (IT, CS, CSAI & CSB) 4^{th} Semester

Date: 24 Feb 2025 **QPS:** Dhananjoy Dey

Max Marks: 30 **Max Time:** 2 Hours

Answer all of the following questions. The use of mobile phones and calculators is strictly prohibited in the examination hall. Each and every step of your calculation should be shown on the answer sheet with justification.

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1. What is the truth value of the following statement? Write your answer with justification. 2

$$\forall\, xP(x) \Rightarrow \exists\,!\, xP(x)$$

2. If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer. 2

3. Prove or disprove that if a and b are rational numbers, then a^b is also rational. 2

4. How many bytes contain either exactly four consecutive 0s or exactly four consecutive 1s? 4

5. Consider a function $f : \{0,1\}^{16} \to \{0,1\}^8$. What is the largest k such that in any set of 2000 inputs, there are atleast k inputs that f maps to the same value? 5

6. Prove that there is no surjective function from $\mathbb{N}$ onto its power set $\mathcal{P}(\mathbb{N})$. 5

7. Find $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1}$ in $GL(2, \mathbb{Z}_7)$. 4

8. (a) Find $\langle S_i \rangle$ using the following Cayley table of D_4, where each $S_i \subset D_4$ for $1 \leq i \leq 4$. Justify your answer.

   (i) $S_1 = \{\rho_2\}$, (ii) $S_2 = \{H, V\}$, (iii) $S_3 = \{\rho_2, \rho_3\}$ (iv) $S_4 = \{H, D\}$ 4

$\circ$	ρ_0	ρ_1	ρ_2	ρ_3	H	V	D	D'
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	H	V	D	D'
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	D'	D	H	V
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	V	H	D'	D
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	D	D'	V	H
H	H	D	V	D'	ρ_0	ρ_2	ρ_1	ρ_3
V	V	D'	H	D	ρ_2	ρ_0	ρ_3	ρ_1
D	D	V	D'	H	ρ_3	ρ_1	ρ_0	ρ_2
D'	D'	H	D	V	ρ_1	ρ_3	ρ_2	ρ_0

(b) Find all elements $g \in D_4$ such that $g^2 = \rho_0$. 2

1.1 Answer Key

1. What is the truth value of the following statement? Write your answer with justification. 2

$$\forall\, xP(x) \Rightarrow \exists\,!\, xP(x)$$

Solution 1.1. *We know that the conditional statement $p \Rightarrow q$ is false when p is true and q is false, and true otherwise.*

Now, to analyze the truth value of the given expression, we do the following:

- *If $\forall\, xP(x)$ is true, then $P(x)$ holds for every x. In this case, the uniqueness condition $\exists\,!\, xP(x)$ is not necessarily true, because $P(x)$ could hold for more than one x. Thus, the implication is false in this case.*
- *$\forall\, xP(x)$ is false, the implication is vacuously true.*

Thus, the statement $\forall\, xP(x) \Rightarrow \exists\,!\, xP(x)$ is not always true.

It is false when $\forall\, xP(x)$ is true, and true when $\forall\, xP(x)$ is false.

2. If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer. 2

Solution 1.2. *We know that if f and g are one-to-one function, then $f \circ g$ is one-to-one.*

Now, we assume that g is not one-to-one function. So, there exists x_1, x_2 where $x_1 \neq x_2 \Rightarrow g(x_1) = g(x_2) = y$.

Now we compute

$$\begin{aligned} f \circ g(x_1) &= f\left(g(x_1)\right) \\ &= f(y) \\ &= f\left(g(x_2)\right) \quad [\because\ y = g(x_2) = g(x_1)] \end{aligned}$$

This gives $f\left(g(x_1)\right) = f\left(g(x_2)\right)$ when $x_1 \neq x_2 \Rightarrow f \circ g$ is not one-to-one.

This is a contradiction. So, g must be one-to-one.

3. Prove or disprove that if a and b are rational numbers, then a^b is also rational. 2

Solution 1.3. *Consider $a = 2$ and $b = \frac{1}{2}$. So, both a and b are rational; however $a^b = \sqrt{2}$ is not rational.*

Thus,

$$a, b \in \mathbb{Q} \not\Rightarrow a^b \in \mathbb{Q}.$$

4. How many bytes contain either exactly four consecutive 0s or exactly four consecutive 1s? 4

Solution 1.4. *To solve this problem, we will compute explicitly by placing 0000 and 1111 blocks starting from 1st position to 5th position.*

(i) ***To count the number of bytes containing exactly four consecutive 0s:***

(a) ***In the 1st position:*** *If we place the 0000 block in the 1st position, so the 5th bit of it must be 1. So, we fixed the first 5 bits of a byte as 00001. Now, we can place any* b_i*'s from 6th to 8th position of those bytes, where* $b_i \in \{0,1\}$*. So, we have* $2^3 = 8$ *choices where each byte starts with the 0000 block.*

(b) ***In the 5th position:*** *Now, if we place the 0000 block in the 5th position, so the 4th bit of it must be 1. So, here we fix 10000 block from the 4th place. Now, we can write any* b_i*'s from 1st to 3rd position of those bytes. Again we have* $2^3 = 8$ *possibly bytes of this type.*

(c) ***In the 2nd position:*** *If we place the 0000 block in the 2nd position, so the 1st and 6th bit will be 1. In this case, we have to fix the block 100001 from the 1st position. Thus, we can take any* b_i*'s in the 7th and the 8th positions. So, we have* $2^2 = 4$ *possible bytes of this type.*

(d) ***In the 3rd & 4th positions:*** *Exactly, in the above logic, if we place the 0000 block in the 3rd and 4th positions, we have exactly 4 bytes in each position. So, in total we have 8 byte in these cases.*

Therefore, number of bytes containing exactly four consecutive 0s $= (8+8+4+8) = 28$.

(ii) ***To count the number of bytes containing exactly four consecutive 1s:***
Apply the method above, we can say that the number of bytes containing exactly four consecutive 1s $= 28$.

Now, we can see from the above counting process that the bytes 00001111 and 11110000 are counted twice.

Thus, the total number of required bytes:

$$= 28 + 28 - 2 = 54.$$

5. Consider a function $f : \{0,1\}^{16} \to \{0,1\}^8$. What is the largest k such that in any set of 2000 inputs, there are atleast k inputs that f maps to the same value? 5

Solution 1.5. *From the given* f*, we can see that the all possible output of* f *is 256.*

Now, according to the problem, we are considering any arbitrary set of 2000 inputs. These inputs will produce 2000 outputs. Since we have only 256 possible outputs, so we will definite have the collision for these 2000 outputs.

Applying pigeonhole principle, we have the minimum number of inputs that must map to the same output value is:

$$k = \left\lceil \frac{2000}{256} \right\rceil = \lceil 7.8125 \rceil = 8$$

Thus, we can say that there are at least 8 inputs that maps to the same value for any 2000 inputs.

Now, to verify that $k = 8$ is indeed the largest possible value for this case, we calculate the following:

First we see whether $k = 9$ is possible.

If $k = 9$, then at least one output value must have at least 9 inputs mapped to it.

However, it is possible to distribute the 2000 inputs such that no output value has more than 8 inputs mapped to it. Because $256 \times 8 = 2048 > 2000$.

So, we can distribute the 2000 given inputs across 256 outputs s/t no output value has more than 8 inputs mapped to it.

Thus, k cannot be equal to 9.

6. Prove that there is no surjective function from $\mathbb{N}$ onto its power set $\mathcal{P}(\mathbb{N})$. 5

Solution 1.6. *We will prove this by contradiction. First, we assume that g is any surjective function*

$$g : \mathbb{N} \xrightarrow{onto} \mathcal{P}(\mathbb{N})$$

Now, we construct a subset Y of $\mathbb{N}$ in the following way:

$$Y = \{n \in \mathbb{N} \ : \ n \notin g(n)\} \subseteq \mathbb{N}$$

Thus, $Y \in \mathcal{P}(\mathbb{N})$. Now, if we can prove that Y cannot belong to the range of g, this leads to the contradiction.

Now, according to our assumption, since g is surjective function, then there exists $n_0 \in \mathbb{N}$ s/t $g(n_0) = Y$.

Next we consider the following cases:

Case-1 *Let* $n_0 \in Y = g(n_0) \Rightarrow n_0 \in g(n_0)$.

Thus, $n_0 \notin Y$. *[from the definition of* Y*]*

Therefore, $n_0 \in Y \Rightarrow n_0 \notin Y$*; which is a contradiction.*

Now, we will consider the following case.

Case-2 $n_0 \notin Y = g(n_0) \Rightarrow n_0 \in Y$.

Thus, we have $n_0 \notin Y \Rightarrow n_0 \in Y$*; which is also a contradiction.*

Therefore, such an n_0 *cannot exist. This gives us that* g *is not a surjective function.*

Since, g *is any surjective function, this proves that there is no surjective function from* $\mathbb{N}$ *onto its power set* $\mathcal{P}(\mathbb{N})$.

7. Find $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1}$ in $GL(2, \mathbb{Z}_7)$. 4

Solution 1.7. *If* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*, then* $A^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The determinant of the given matrix $= (6 - 1) = 5$.

Now, we have to compute $5^{-1} \in \mathbb{Z}_7$*. We can see that* $5 \times 3 = 15 \equiv 1 \mod 7$.

So, we get $5^{-1} = 3$.

Thus,

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1} = 3 \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 6 \end{pmatrix}$$

8. (a) Find $\langle S_i \rangle$ using the following Cayley table of D_4, where each $S_i \subset D_4$ for $1 \leq i \leq 4$. Justify your answer.

(i) $S_1 = \{\rho_2\}$, (ii) $S_2 = \{H, V\}$, (iii) $S_3 = \{\rho_2, \rho_3\}$ (iv) $S_4 = \{H, D\}$ 4

$\circ$	ρ_0	ρ_1	ρ_2	ρ_3	H	V	D	D'
ρ_0	ρ_0	ρ_1	ρ_2	ρ_3	H	V	D	D'
ρ_1	ρ_1	ρ_2	ρ_3	ρ_0	D'	D	H	V
ρ_2	ρ_2	ρ_3	ρ_0	ρ_1	V	H	D'	D
ρ_3	ρ_3	ρ_0	ρ_1	ρ_2	D	D'	V	H
H	H	D	V	D'	ρ_0	ρ_2	ρ_1	ρ_3
V	V	D'	H	D	ρ_2	ρ_0	ρ_3	ρ_1
D	D	V	D'	H	ρ_3	ρ_1	ρ_0	ρ_2
D'	D'	H	D	V	ρ_1	ρ_3	ρ_2	ρ_0

Solution 1.8. *(i) From the table it is clear that* ρ_2 *is self inverse,* $\rho_2 \circ \rho_2 = \rho_0$*. Thus,*

$$\langle S_1 \rangle = \{\rho_2, \rho_0\}$$

(ii)

$$\langle S_2 \rangle = \{H, V, \rho_0, \rho_2\}$$

(iii)

$$\langle S_3 \rangle = \{\rho_2, \rho_3, \rho_0, \rho_1\}$$

(iv)

$$\langle S_4 \rangle = \{H, D, \rho_0, \rho_1, \rho_3, D', V, \rho_2, \} = D_4$$

(b) Find all elements $g \in D_4$ such that $g^2 = \rho_0$. 2

Solution 1.9.

$$\{\rho_0, \rho_2, H, V, D, D'\}$$