# Deodhar Diagram and Rational Dyck Path

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# Deodhar diagram and Rational Dyck Path

**Definition 1 (Deodhar Diagram)**

Let $a$ and $b$ be coprime. A Deodhar diagram is a filling $D$ of the boxes of an $a \times b$ rectangle with crossings and elbows satisfying the following condition:

1. Number the edges from bottom-left to top-right. The two numbers connected by one pipe should be the same.
2. (distinguished condition) For any elbow in $D$, the label of its bottom-left Path is less than the label of its top-right path.
3. $D$ contains $(a-1)(b-1)$ crossings, which is the maximal possible number of crossings with condition 1 and 2 being satisfied.

# Deodhar diagram and Rational Dyck Path

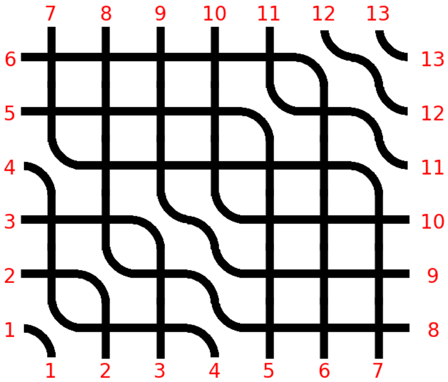


Figure 1: A deodhar diagram of type $(6, 7)$

## Definition 2 (Rational Dyck Path)

Let $a$ and $b$ be coprime. A $(a, b)$-Dyck path is a lattice path from $(0, 0)$ to $(a, b)$ which always remains below the main diagonal $y = \frac{a}{b}x$

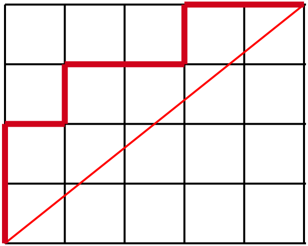

Figure 2: A $(5, 4)$-Dyck path

# Deodhar diagram and Rational Dyck Path

The number of the Deodhar diagram of size $(a, b)$ was firstly counted by Galashin and Lam[1].

## Theorem 3 (Galashin-Lam)

*The number of the Deodhar diagram of size* $(a, b)$ *is* $C_{a,b} = \frac{1}{a+b}\binom{a+b}{a}$.

## Theorem 4

*The number of* $(a, b)$*-Dyck path is* $C_{a,b} = \frac{1}{a+b}\binom{a+b}{a}$

[1] Pavel Galashin and Thomas Lam. *Positroids, knots, and q, t-Catalan numbers*. 2023. arXiv: 2012.09745 [math.CO]. URL: https://arxiv.org/abs/2012.09745.

# Our goal

**Conjecture 5 (Galashin-Lam)**

*There is a natural bijection between $(a, b)$-Dyck path and Deodhar diagram of size $(a, b)$.*

$$\{\text{Deodhar diagrams}\} \longleftrightarrow \{\text{rational Catalan objects}\}$$

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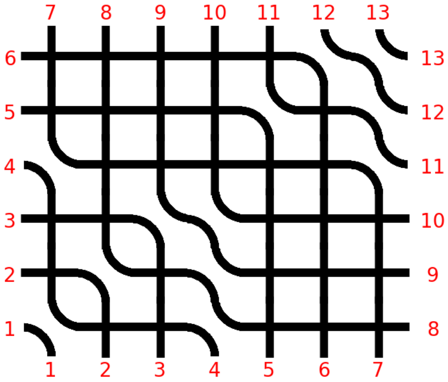
# $(n, n+1)$




1. There are two and only two elbows in each row.
2. The $n+1^{th}$ pipe turn twice, called the "Babel pipe".

# $(n, n+1)$

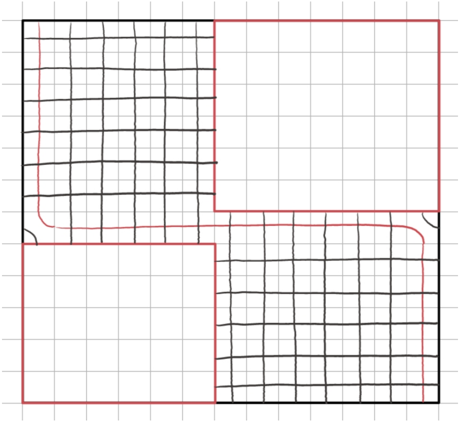


Figure 3: Recursion for $(n, n+1)$-Deodhar diagram

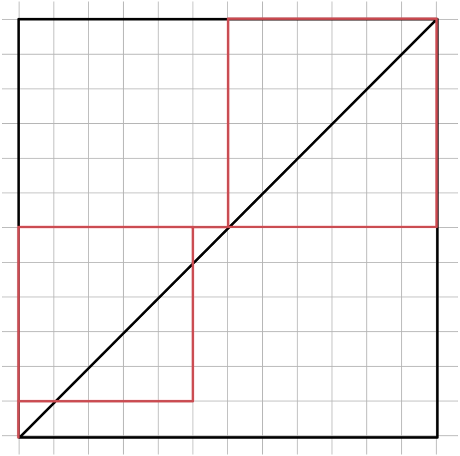


Figure 4: Recursion for $(n, n+1)$-Dyck path

We'll solve the $(n, kn+1)$ case through recursion. For (n,kn+1) Catalan number, we have

**Theorem 6**

$$C_{n,kn+1} = \sum_{i_0+i_1\cdots+i_k=n-1} \prod_{j=0}^{k} C_{i_j,ki_j+1}$$

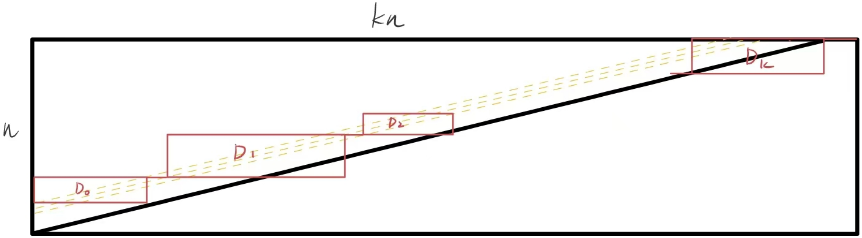



Figure 5: Recursion for $(n, kn+1)$-Dyck path

# $(n, kn+1)$ Deodhar diagram & Dyck path

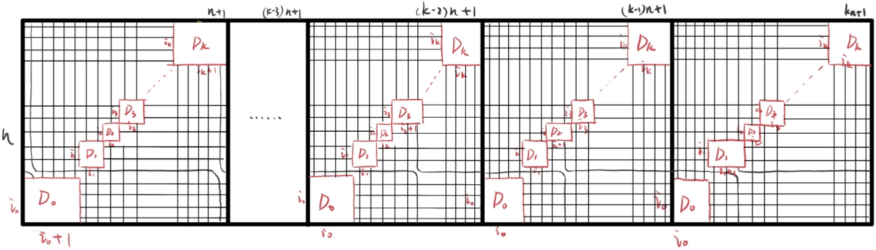
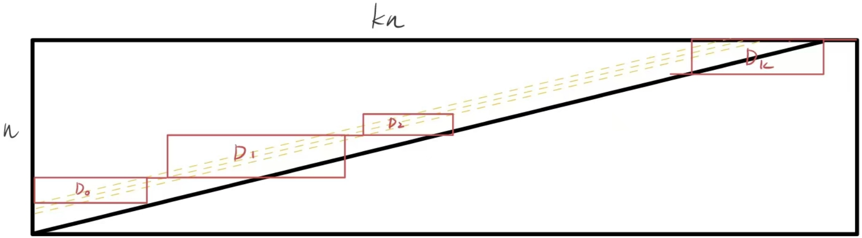


Figure 6: Recursion for $(n, kn+1)$-Deodhar diagram



# $(n, kn-1)$ Deodhar diagram & Dyck Path

**Corollary 7**

*There is a bijection between $n \times (kn-1)$ Deograms and $n \times (kn-1)$ Dyck paths.*

# $(n, kn-1)$-Deodhar diagram

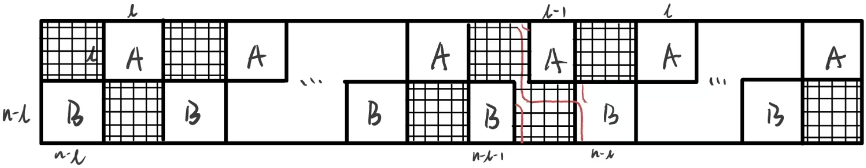



Figure 8: Recursion of $(n, kn-1)$ Deogram

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# Area of Deodhar Diagram – Crossing Tower

**Definition 8 (Area)**

Area is the number of complete tiles between the main diagonal and Dyck path.

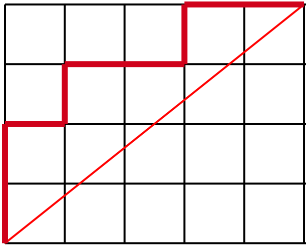

Figure 9: A $(5, 4)$-dyck path with area$= 2$

# Area of Deodhar Diagram – Crossing Tower

**Question 9 (Galashin-Lam)**

*What is "area" in the Deodhar diagram?*

**Definition 10 (Crossing Tower)**

A crossing tower of size $n$ is a list of numbers $(a_1, a_2, \ldots, a_n)$ such that $a_n = 0$ and $a_i \leq a_{i+1} + 1$ for $i = 1, 2, \ldots, n-1$

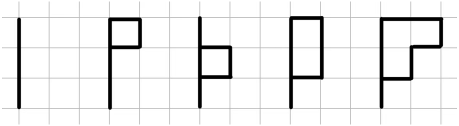

Figure 10: All crossing towers of size 3

**Theorem 11**

*The number of crossing towers of size $n$ is the Catalan number $C_n$*

# Area of Deodhar Diagram – Crossing Tower

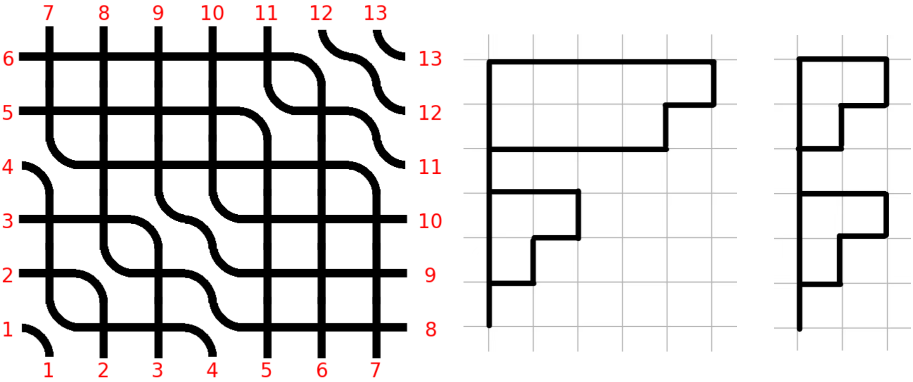


Figure 11: An example for bijections between deodhar diagram and crossing tower

# Area of Deodhar Diagram – Crossing Tower

## Definition 12 ($k$-Crossing Tower)

A $k$-crossing tower of size $n$ is a list of numbers $(a_1, a_2, \ldots, a_n)$ such that $a_n = 0$ and $a_i \leq a_{i+1} + k$ for $i = 1, 2, \ldots, n-1$

## Theorem 13

*The number of crossing towers of size $n$ is the rational Catalan number $C_{n,kn+1}$*

## Question 14

*Construct a bijection between $(n, kn+1)$-Deodhar diagram and $k$-crossing towers of size $n$.*

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# Deograms in the non relatively prime case

Let $\mathrm{Deo}^{\max}_{f_{k,n}}$ : denote the collection of $k \times n$ maximal identity Deograms. The primary result of this section is the following theorem:

**Theorem 15**

*If $gcd(a, b) = d$, $a = da_1$, $b = db_1$, then*

$$Deo^{\max}_{f_{a,b}} = \bigoplus_{i=1}^{d} Deo^{\max}_{f_{a_1,b_1}}$$

*More explictly, for each $r \in [d]$, rows and columns of an $a \times b$ Deogram corresponding to labels $i \equiv r \mod d$ forms an $a_1 \times b_1$ Deogram. These $d$ sub-Deograms determines uniquely the $a \times b$ Deogram and contains all elbows of it.*

# Deograms in the non relatively prime case

Combining with Galashin and Lam's result(Proposition 9.5 in [1]), this solves the enumeration problem of maximal identity Deograms for arbitary positive integer pairs:

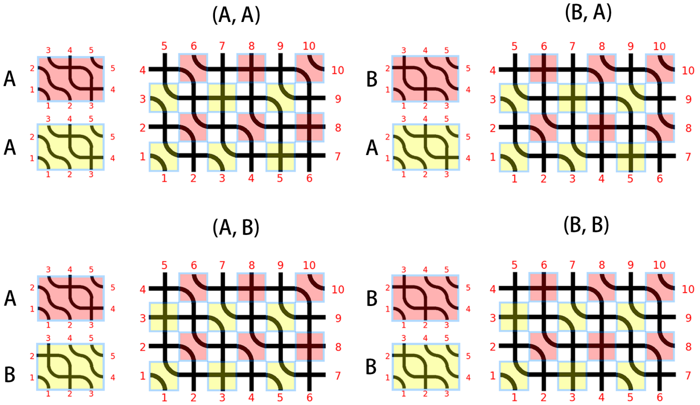
## Corollary 16

*If* $gcd(a, b) = 1$, $d \in \mathbb{N}$, *then*

$$\#Deo^{\max}_{f_{da,db}} = (\#Deo^{\max}_{f_{a,b}})^d = \left(\frac{1}{a+b}\binom{a+b}{a}\right)^d$$

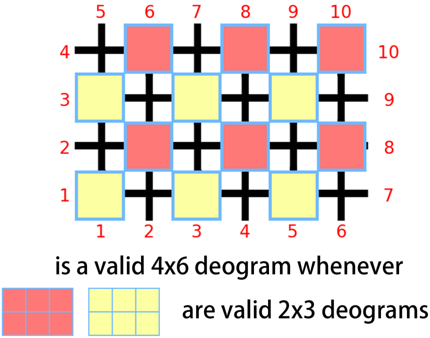
# Examples

There are two $2 \times 3$ maximal identity Deograms, denoted by A and B. Since gcd(4,6)=2, Each $4 \times 6$ Deogram is uniquely determined by a 2-tuple of $2 \times 3$ Deograms, as shown in the figure below. There are $2^2 = 4$ of them. Note that all the uncolored boxes are crossings.



# Proof idea

Given an $a \times b$ rectangle. When the stated $d$ regions are indeed $a_1 \times b_1$ Deograms, and all other boxes are filled with crossings, one can easily check that the whole diagram is indeed an $a \times b$ Deogram.




Therefore, it suffices to show that all $a \times b$ Deograms arise in this way. Namely, each box $(i, j)$ for which $d \nmid i - j$ must be filled with a crossing.

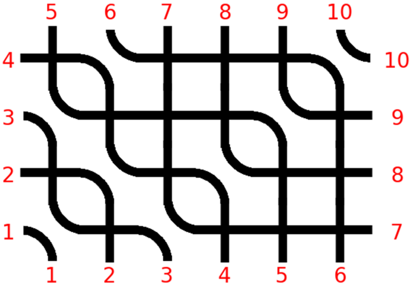
To this end we involve some graph theory. For each Deogram associate two graphs as follows:

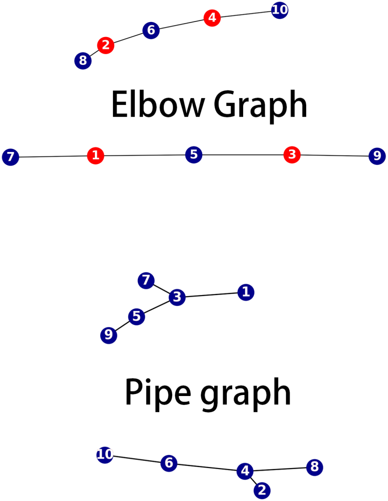
**Definition 17 (the Elbow Graph and the Pipe Graph)**

Given an $a \times b$ Deogram $\mathbb{D}$, define two graphs:
The *Elbow Graph* of $\mathbb{D}$ is a bipartite simple graph on vertices $\{1, 2, ..., a\} \sqcup \{a+1, a+2, ..., b\}$ where $i$ and $j$ are joined by an edge if there is an elbow placed in box (i,j). Here we are looking at the north-western labels.
The *Pipe Graph* of $\mathbb{D}$ is a multigraph on vertices $\{1, 2, ..., a+b\}$ where $i$ and $j$ are joined by $k$ edges iff there are $k$ elbows in $\mathbb{D}$ between pipe $i$ and pipe $j$.
Denote by $\Phi_E$ and $\Phi_P$ the maps from Deograms to their Elbow Graph and Pipe Graph, respectively.

# Examples

Below $\mathbb{D}$ is a $4 \times 6$ Deogram and $\Phi_E(\mathbb{D})$ and $\Phi_P(\mathbb{D})$ are drawn as examples.



# Structure of Elbow Graphs and Pipe Graphs

The structure of Elbow Graphs and Pipe Graphs are very straight-forward, as formulated by the following proposition:

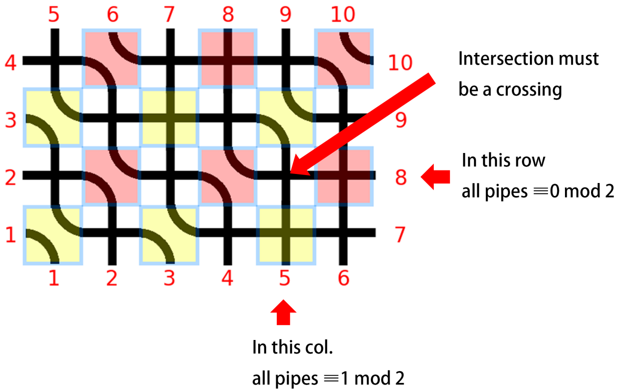
## Proposition 18

*Let $\mathbb{D} \in Deo^{\max}_{f_{a,b}}$, $gcd(a,b) = d$. Then both $\Phi_E(\mathbb{D})$ and $\Phi_P(\mathbb{D})$ are forests. In both case, each component is a tree on a residue class of $d$.*

Sketch of Proof: In both case, since there are $a + b - d$ edges, the conclusion follows from showing that the induced subgraph of each residue class is connected. For the Pipe Graph, notice that there always exists a path connecting $i$ and $i + a \ mod \ a + b$. For the Elbow graph, notice that the edges, viewed as transpositions and acted in lexicographical order, gives a permutation in $S_{a+b}$ each cycle of which is a residue class modulo $d$.

# Structure of Elbow Graphs and Pipe Graphs

The conclusion of Proposition 18 for the Elbow Graph immediately finishes the proof of Theorem 15.
The conclusion for the Pipe Graph gives an alternative proof of Theorem 15, as exemplified by the diagram below:



# A Noncrossing Conjecture

It is not difficult to show that:

## Proposition 19

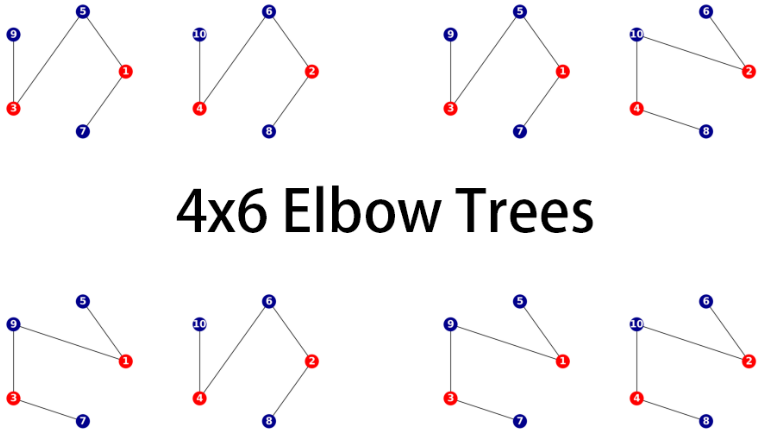
*$\Phi_E$ and $\Phi_P$ are injective.*

So in order to understand the combinatorics of Deograms, it is natural to ask what properties the image sets $\Phi_E(\mathrm{Deo}^{\max}_{f_{a,b}})$ and $\Phi_P(\mathrm{Deo}^{\max}_{f_{a,b}})$ must possess. We conclude this section with a uniform non-crossing conjecture for Elbow Graphs and Pipe Graphs:

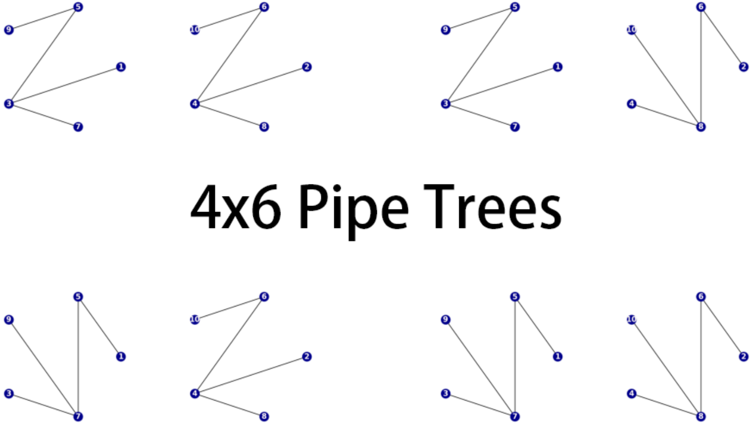
## Conjecture 20

*For each $r \in [d]$, arrange $r$, $r + a \bmod a + b$,..., $r + da \bmod a + b$ in counter-clockwise order on a circle, with no two circle intersecting each other. Draw chords corresponding to the edges of $\Phi_E(\mathbb{D})$(resp.$\Phi_P(\mathbb{D})$). Then no two chords cross each other in the interior.*

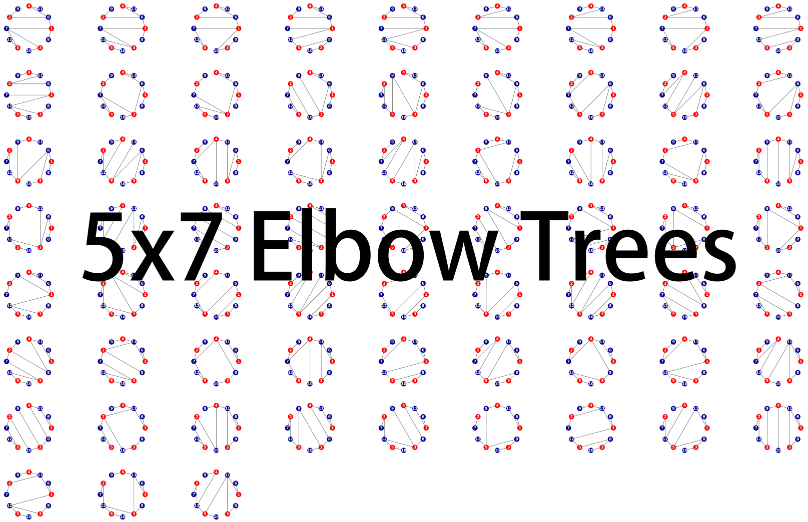
# A Noncrossing Conjecture



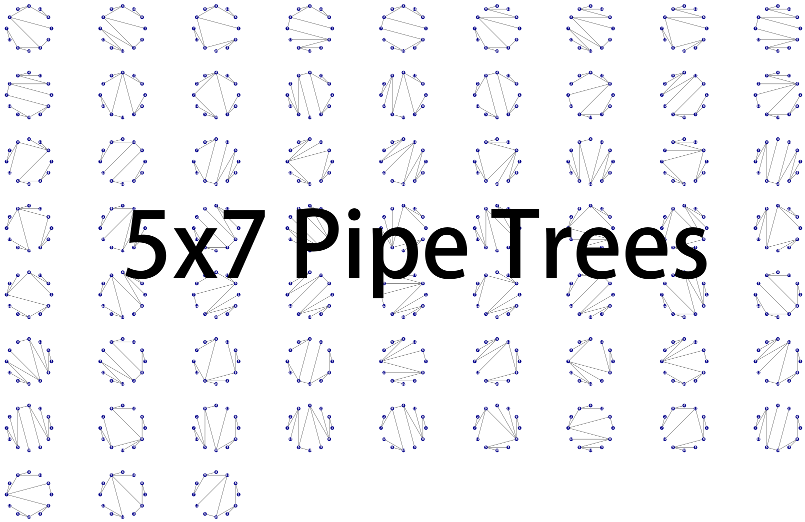
# A Noncrossing Conjecture



# The Noncrossing Conjecture



# A Noncrossing Conjecture



# References

[1] Pavel Galashin, Thomas Lam. *Positroids, knots, and q, t-Catalan numbers. arXiv:2012.09745v2, 2020..*