A BRIEF INTRODUCTION TO REPRESENTATIONS OF DYNKIN QUIVERS

AIWEI LIU

June 15, 2024

Abstract

Quiver representations are an important concept in algebraic representation theory. They provide numerous examples that integrate algebraic representation theory and have significant applications in various mathematical fields such as cluster algebras and algebraic geometry. The famous Gabriel's Theorem classifies finite type quivers and their indecomposable representations, highlighting the importance of Dynkin quivers.

This paper provides some basic definitions and crucial theorems about the representation of Dynkin quivers. Additionally, it discusses potential future research directions.

1 Quiver

Basicly speaking, a "quiver" is a directed graph. This concept is porposed by Peter Gabriel in his paper in 1972 that laid the foundation for quiver representation theory, since the term "graph" already includes too many related concepts.

Definition 1.1 (Quiver) *A quiver $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ is given by a set Q_0 of vertices and a set Q_1 of arrows. An arrow ρ starts at the vertex $s(\rho)$ and terminates at the vertex $t(\rho)$.*

In this paper, we assume that $Q_0 = [n]$ and $|Q_1|$ is finite.

2 Path Algebra and Quiver Representation

2.1 Path Algebra

Definition 2.1 (Path) *A (non-trival) path in Q is a sequence $\rho_m \cdots \rho_1 (m \geq 1)$ of arrows which satisfies $t(\rho_i) = s(\rho_{i+1})$ for $1 \leq i < m$.*

$$\bullet \xleftarrow{\rho_m} \bullet \quad \cdots \quad \bullet \xleftarrow{\rho_2} \bullet \xleftarrow{\rho_1} \bullet$$

We use e_i to denote the trival path for the vertex i, and $s(x), t(x)$ to denote the starting and terminating vertex of a path x.

Definition 2.2 (Path Algebra) *The path algebra kQ is the $k-algebra$ with basis all the possible paths in Q, and with the products of two paths x, y given by*

$$xy = \begin{cases} \textit{Composition of paths} & \textit{if } t(y) = s(x) \\ 0 & \textit{otherwise} \end{cases}$$

To prove that this multiplication is associative is trival. Note that the trival paths e_i should not be forgotten. As an example, if we consider a path x with $s(x) = 1$ and $t(x) = 2$, then $e_1 x = 0$ and $x e_1 = e_1$.

Here are some examples.

- If Q consists of one vertex and one loop, then $kQ \cong k[T]$. If Q has one vertex and r loops, then kQ is the free associative algebra on r letters.
- If there is at most one path between any two points, then kQ can be identified with the subalgebra

  $$\{C \in M_n(k) | c_{ij} = 0 \text{ if no path from } j \text{ to } i\}$$

  of $M_n(k)$. If Q is $1 \to 2 \to \cdots \to n$ then this is the lower triangular matrices.

2.2 Representations

Now we define the representations of quiver, which will be the main research object in this field.

Definition 2.3 (Representation of Quiver) *A representation V of Q is given by a vector space V_i for each $i \in Q_0$ and a linear map $V_\rho : V_{s(\rho)} \to V_{t(\rho)}$ for each $\rho \in Q_1$.*

The Representation category of Q is denoted as Rep(Q)

Definition 2.4 (Morphism) *A morphism $\theta : V \to W$ is given by linear maps $\theta_i : V_i \to W_i$ for each $i \in Q_0$ satisfying $W_\rho \circ \theta_{s(\rho)} = \theta_{t(\rho)} \circ V_\rho$ for each $\rho \in Q_1$.*

To sum up, we can use the following diagram to illustrate the definitions above.

$$
\begin{tikzcd}
V_{s(\rho)} \arrow[rrr, "V_\rho"] \arrow[dd, "\theta_{s(\rho)}"'] \arrow[dr, dashed, no head] & & & V_{t(\rho)} \arrow[dd, "\theta_{t(\rho)}"] \arrow[dl, dashed, no head] \\
& s(\rho) \bullet \arrow[r, dashed, "\rho"] \arrow[dl, dashed, no head] & \bullet\, t(\rho) \arrow[dr, dashed, no head] & \\
W_{s(\rho)} \arrow[rrr, "W_\rho"'] & & & W_{t(\rho)}
\end{tikzcd}
$$

A map $\theta : V \to W$ is a morphism if and only if the diagram above commutes.

2.3 $Rep(Q) \cong Mod_{kQ}$

One may notice that the representation of quiver is quite similar to module on path algebra – both of them are equipped with vector spaces, and the composition of arrows can be regarded as multiplication on modules. In fact, they are indeed congruent.

Theorem 2.1 *Given a quiver $Q = (Q_0, Q_1, s, t)$, then $Rep(Q) \cong Mod_{kQ}$.*

Proof

(1) $F : Rep(Q) \to Mod_{kQ}$

For $V \in Rep(Q)$, define $F(V) = \oplus_{i \in Q_0} V_i$. Define $\rho_m \cdots \rho_1 v = V_{\rho_m} \circ \cdots \circ V_{\rho_1}(v)$.

For $\theta \in Hom(V, W)$, $V, W \in Rep(Q)$, define $F(\theta) = \oplus_{i \in Q_0} \theta_i$. Namely, $F(\theta) : \oplus_{i \in Q_0} x_i \mapsto \oplus_{i \in Q_0} \theta_i(x_i)$.

(2) $G : Mod_{kQ} \to Rep(Q)$

For $X \in Mod_{kQ}$, define $G(X)_i = e_i X$ and $G(X)_\rho : G(X)_{s(\rho)} \to G(X)_{t(\rho)}$ as $x \mapsto \rho x$.

For $\phi \in Hom(X, Y)$, $X, Y \in Mod_{kQ}$, define $G(\phi)_i = e_i \phi$.

(3) It's easy to check that F and G are two functors. Now we'll prove that they are inverses.

For all $X \in Mod_{kQ}$, $F(G(X)) = \oplus_{i \in Q_0} G(X)_i = \oplus_{i \in Q_0} e_i X = X$

For all $V \in Rep(Q)$, $G(F(V))_i = e_i F(V) = i \oplus_{j \in Q_0} V_j = V_i$; $G(F(V))_\rho : V_{s(\rho)} \to V_{t(\rho)}, x \mapsto \rho x$.

Therefore, $F \circ G = id_{Mod_{kQ}}$ and $G \circ F = id_{Rep(Q)}$. $Rep(Q) \cong Mod_{kQ}$.

$\square$

From now on, we don't need to distinguish a module of path algebra and representation of quiver any more.

3 The Standard Resolution

In this section we'll give a standard resolution of left kQ module. We shall define "standard resolution" first.

Definition 3.1 (exact sequence) *A sequence*

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} A_n$$

of an specific algebra structure and its corresponding morphism is called "exact" if $im(f_i) = ker f_{i+1}$ *holds true for all* $1 \leq i < n$.

Definition 3.2 (Resolution of Modules) *Given a module M over a ring R, a resolution of M is an exact sequence of R-modules*

$$\cdots \xrightarrow{d_{n+1}} E_n \xrightarrow{d_n} \cdots \xrightarrow{d_3} E_2 \xrightarrow{d_2} E_1 \xrightarrow{d_1} E_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

We called a resolution of modules "projective" if each E_i is projective. The following theorem shows that we can provide a standard projective resolution for any kQ-module.

Theorem 3.1 (The Standard Resolution) *Let $A = kQ$, If M is a left A-module, there is an exact sequence*

$$0 \longrightarrow \bigoplus_{\rho \in Q_1} Ae_{t(\rho)} \otimes_k e_{s(\rho)} M \xrightarrow{f} \bigoplus_{i=1}^{n} Ae_i \otimes_k e_i M \xrightarrow{g} X \longrightarrow 0$$

where

$$\begin{aligned} g(a \otimes m) &= am && \text{for } a \in Ae_i, m \in e_i M \\ f(a \oplus m) &= a\rho \oplus m - a \oplus \rho m && \text{for } a \in Ae_{t(\rho)}, m \in e_{s(\rho)} M \end{aligned}$$

To prove this theorem, knowledge from some advanced algebra courses is required. Learning these knowledge is also one of my future research objectives; therefore, the proof will not be presented here.

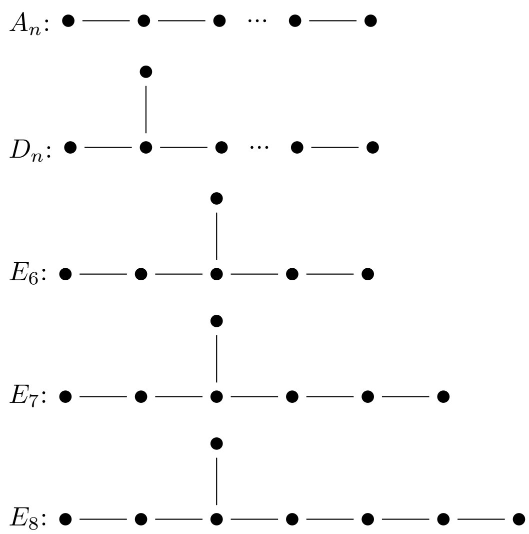
4 Gabriel's Theorem

In this section, we will introduce one of the most fundamental theorems in quiver representation – Gabriel's theorem. It classifies finite type quivers and their indecomposable representations, revealing the importance of Dynkin quivers.

4.1 Dynkin diagram

Basically, Dynkin diagrams are a specific class of finite undirected graphs that representative, and have been classified. They are used to describe and classify symmetric structures and Lie algebras in mathematics. The classification and specific shapes of these diagrams help mathematicians better understand and study these complex mathematical objects.

Understanding the rigorous definition and profound significance requires a more advanced mathematical background, such as Lie algebras, which is not the primary focus of this paper. Therefore, we will only introduce the specific types of Dynkin diagrams (ADE Dynkin diagrams) needed for the subsequent discussions: A_n, D_n, E_6, E_7, E_8.



4.2 Gabriel's Theorem

Theorem 4.1 *A connected quiver has only finitely many incomposable representations if and only if its underlying graph (when the directions of arrows are ignored) is one of the ADE Dynkin diagrams: A_n, D_n, E_6, E_7, E_8.*

Learning and summarizing the proof of this theorem is one of my future research objectives.

5 Research Objectives

My research goals primarily focus on quiver representation theory and related fields. By delving into the basic theory, reviewing and summarizing Gabriel's theorem, calculating minimal projective resolutions of Dynkin quivers, and exploring cutting-edge problems, I aim to contribute to the development of this field. Additionally, through systematic study and research, I seek to continuously improve my research capabilities, laying a solid foundation for my future academic career.

Quiver representation is an important concept in algebraic representation theory. It provides numerous instances that combine algebraic representation theory and has significant applications in areas such as bundle algebras and algebraic geometry. One of my future research goals is to deeply investigate the basic theory and important theorems of quiver representation theory, particularly the famous Gabriel's theorem and its classification of finite type quivers and their indecomposable representations. This will help me better understand the importance of Dynkin quivers and their critical role in representation theory.

To systematically understand Gabriel's theorem, I plan to meticulously review and summarize its proof. This will not only help reinforce my understanding of this important theorem but also provide a reference and assistance to other researchers. Through an in-depth study of Gabriel's theorem, I will explore its potential applications in other mathematical fields, further expanding its theoretical and practical impact. During this process, I will also focus on the forefront issues in quiver representation theory and stay updated with the latest research findings and trends.

Although quiver representations always have standard decompositions, these decompositions are often not minimal. Another of my research goals is to calculate the minimal projective resolutions of indecomposable modules in the finite-dimensional representation category of Dynkin quivers. This research direction is relatively rare in the existing literature; therefore, I hope to fill this gap and promote the development of this field. During the calculation process, I will integrate the latest research findings and methods, exploring cutting-edge issues in this area, and seeking new theoretical breakthroughs and applications.

Additionally, to achieve the above research goals, I will enhance my research capabilities and academic accumulation by auditing relevant courses, studying important literature, and participating in academic exchanges. Specifically, I will audit the graduate course “Foundations of Algebra”, study William Crawley-Boevey's “Lectures on Representations of Quivers”, and other related materials. These preparatory works will lay a solid foundation for my research and help me successfully complete the expected research tasks.

In summary, my research goals primarily focus on quiver representation theory and related fields. By deeply investigating the basic theory, reviewing and summarizing Gabriel's theorem, calculating minimal projective resolutions, and exploring cutting-edge problems, I aim to contribute to the development of this field. Simultaneously, through systematic study and research, I seek to continuously improve my research capabilities, laying a solid foundation for my future academic career.

References

[1] William Crawley-Boevey. Lectures on representations of quivers, 1992.

[2] Peter Gabriel. Unzerlegbare darstellungen i. *manuscripta mathematica*, 6:71–103, 1972.