A Brief Overview of Classical Ramsey Theory

AIWEI LIU

1 Introduction

We begin with an interesting problem.

Suppose there are 6 people, prove that there are either 3 people who know each other in pairs or 3 people who don't know each other in pairs.

With some basic knowledge of graph theory, we can use the terminologies to describe this problem. (Due to lenth constraints, the basic concepts in graph theory will not be restated.)

Theorem 0. *Any complete graph on* 6 *vertices,denoted as* K_6*, with its edges 2-colored must contain a complete graph on* 3 *vertices in the same color.*

Proof. Consider the six vertices labelled i as shown in Figure 1, where $i \in \langle 5 \rangle$. By the pigeonhole principle, there are at least three edges incident to vertex 0 that share the same color, say red for $\{0,1\}, \{0,2\}, \{0,4\}$. If there are no complete graph on 3 vertices, then $\{1,2\}, \{1,4\}, \{2,4\}$ is colored differently from red, say blue, then the complete graph with the vertices $1, 2, 3$ is in the same color.

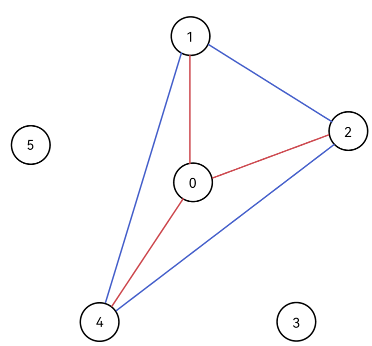

Figure 1

$\square$

Futhermore, we can make generalization as follows.

Definition 1 (Ramsey number)**.** *Let $r \geq 1, r \in \mathbb{N}$ and $q_i \geq r, q_i \in \mathbb{N}, i = 1, 2, \cdots, s$ be given. The minimal positive integer with the following property is called Ramsey number for r and $\{q_i\}$, denoted as $N(q_1, q_2, \cdots, q_s; r)$. Let S be a set with n elements. Suppose that all $\binom{n}{r}$ r-subsets of S are divided into s mutually exclusive families $T_1, T_2, \cdots, T_s$ ("colors"). Then if $n \geq N(q_1, q_2, \cdots, q_s; r)$ there is an i, $1 \leq i \leq s$, and some q_i-subsets of S for which every r-subset is in T_i.*

Notes: $\{T_i\}$ can be understanded as the coloring of vertices for $r = 1$ and of edges for $r = 2$. The case of $r > 2$ is a kind of generalization from "edges" to "hypergraphs".

Form now on, unless specified, all variables are assumed to be natural numbers except zero.

Theorem 1 (Ramsey theorem)**.** *Ramsey number is well-defined. (i.e. Ramsey number exists for any r and $\{q_i\}$.)*

Proof. The main idea is to adopt induction twice on r and $\sum_{i=1}^{s} p_i$ nestedly.

(a) By pigeonhole principle, the theorem is true for $r = 1$ since $N(q_1, q_2, \cdots, q_s; r) = \sum_{i=1}^{s} q_i - 1$.

(b) Trivally, for any r and $q_i \geq r$, if there exists an m_1 such that $q_i = r$ and $i \neq m$, then $N(q_1, q_2, \cdots, q_s; r) = q_m$.

(c) We proceed by induction on r.

Induction hypothesis: Ramsey number is well-defined for $r - 1$.

Then we proceed by induction on $\sum_{i=1}^{s} p_i$

Induction hypothesis: Ramsey number $N(q_1, \cdots, q_{i-1}, q_i - 1, q_{i+1}, \cdots, q_s; r)$ is well-defined for all $1 \leq i \leq s$.

Define $q_i' := N(q_1, \cdots, q_{i-1}, q_i - 1, q_{i+1}, \cdots, q_s; r)$. Let S be a set with n elements, where $n \geq N(q_1', \cdots, q_s'; r-1)+1$. As in the proof of Theorem 0, we pick an arbitary element a of S. For any r-coloring of S, there is a $(r-1)$-coloring of $S' := S \backslash \{a\}$ by giving $X \subset S'$ the same color with $X \cup \{a\}$. By induction and without the loss of generality, suppose that S' contains a subset A of size q_1' such that all its $(r-1)$-subsets are colored with T_1. Since A has $N(q_1 - 1, q_2, \cdots, q_s; r)$ elements, there are two possibilities. The first is that A has a subset of q_i elements with some $i \geq 2$ and all its r-subsets colored with T_i, in which case we are done. The second is that A has a subset A' of $q_1 - 1$ elements with all its r-subsets colored with T_1. Then the set $A' \cup \{a\}$ are also colored with T_1 because of the coloring rules of S'.

$\square$

The proof above also gives an upper bound of the Ramsey number. A useful special case occurs when we consider the case when $s = r = 2$. By adopting (a) in the proof above, we can achieve the following corollaries.

Corollary 1. $N(p,q;2) \leq N(p-1,q,2) + N(p,q-1;2)$.

Corollary 2. $N(p,q;2) \leq \binom{p+q-2}{p-1}$.

2 Bounds

The exact computation of Ramsey number experiences a significantly chanllenging for $s = r = 2$ when p and q increases slightly, not only because of the rapid increasing rate (exponential, as we will discuss later) but also because of the large possibilities of colorings for large n. Let alone the situation when $s, r > 2$. Therefore, a main problem in the Ramsey theory is to estimate it by finding the upper and lower bounds of Ramsey number.

Due to the (relatively) simplicity and the natural intuitive meaning in graph theory (edge coloring), the current mainstream research is based on $s = r = 2$, denoted as $R(p,q)$.

We focus on the case when $p = q$. By Corollary 2, we've already got an upper bound $\binom{2p-2}{p-1}$. To get a lower bound, we need to use a genius method in combinatorics called "probabilistic".

Consider a complete graph K_n and there are $2^{\binom{n}{2}}$ different colorings. Now fix a subgraph K_p and there are $2^{\binom{n}{2}-\binom{p}{2}+1}$ different colorings that K_p is in the same color. Since there are most $\binom{n}{p}2^{\binom{n}{2}-\binom{p}{2}+1}$ different colorings that some K_p is in the same color, once some n_0 let $\binom{n}{p}2^{\binom{n}{2}-\binom{p}{2}+1} < 2^{\binom{n}{2}}$ holds true, $R(p,p) > n_0$. We find that the inequality holds true when $n < 2^{\frac{p}{2}}$ (unless $n = 2$) by using the fact that $\binom{n}{p} < \frac{n^p}{p!}$. Therefore,

Theorem 2. $R(p,p) \geq 2^{\frac{p}{2}}$.

However, we can see that the process to obtain this lower bound involves many extremely loose bounding steps. There are also a method based on the above proof called alternating method that can give a tighter lower bound in the vast majority of cases. To understand alternating method, we need to look at the probabilistic method in a more "probability" way. Suppose $E(X_n)$ is the expectation of the number of subgraph K_p in the same color. Then $E(X_n) = \frac{\binom{n}{p}}{2^{\binom{p}{2}-1}}$ and we required $E(X_n) < 1$ in the proof above.

However, for $E(X_n) \geq 1$, we are not powerless as long as some other tricks are used. For $E(X_n) \leq k$, there must exist a coloring such that there are k K_ps in the same color. Therefore, we only need to delete at most k vertices to get a graph that with no K_p in the same color.

3 Some Open Problems

3.1 Exact value

As we've known, the computing of Ramsey numbers is quite hard. The only non-trival Ramsey numbers we known are the follows:

$R(3,3) = 6$, $R(3,4) = 9$, $R(3,5) = 14$, $R(3,6) = 18$, $R(3,7) = 23$, $R(3,8) = 28$, $R(3,9) = 36$, $R(4,4) = 18$, $R(4,5) = 25$.

For the others, all we've known now is some bounds. For example, $R(5,5)$ lies between 43 (Geoffrey Exoo (1989))[1] and 48 (Angeltveit and McKay (2017))[2].

3.2 Bounds

As we've mentioned before, $\binom{2p-2}{p-1}$ is an upper bound and $2^{\frac{p}{2}}$ is a lower bound. Notice that $\binom{2p}{p} \leq 2^{2p}$, we can get

$$\sqrt{2} \leq \sqrt[p]{R(p,p)} \leq 4.$$

It'll be nice if we can prove that the middle term converges for $p \to \infty$.

4 Conclusion

The field of Ramsey theory has a long-standing histroy, yet it remains far from thoroughly researched. It has many interesting application in other fields such as geometry (e.g. convex geometry) and computer science. In this paper we discuss the basic defination and some mainstream researching direction of the Ramsey number. The further research can be done in the open problems above. What's more, many works also remain to be done in other branches such as infinite Ramsey theorem (Ramsey number on $K_{\mathbb{N}}$).

References

[1] Geoffrey Exoo. A lower bound for $r(5,5)$. *J. Graph Theory*, 13:97–98, 1989.

[2] Vigleik Angeltveit and Brendan D. McKay. $r(5,5) \leq 48$. *Journal of Graph Theory*, 89:13 − 5, 2018.

[3] Jacobus Hendricus Van Lint and Richard Michael Wilson. *A course in combinatorics*. Cambridge university press, 2001.