A MORE GENERAL STATEMENT OF RIESZ REPRESENTATION THEOREM

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Abstract

Riesz representation theorem is one of the most important theorems in functional analysis. It proves the isomorphic relationship between Hilbert space and its continuous dual space, revealing the structure of continuous dual space – all continuous linear functionals are inner productions. In this paper, we will study this theorem from the perspective of advanced algebra rather than functional analysis, and make a more general statement of Riesz representation theorem.

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1 INTRODUCTION

Let $\mathbb{V}$ be a vector space and $\mathbb{V}'$ be its dual space. A reasonable way to study the structure of a vector space is to study its basis. First we need to define dual basis.

Definition 1.1 *Let $B = \{\beta_i\}$ be a basis of $\mathbb{V}$. We call a set of functional $\Phi = \{\phi_i\}$ the dual basis of B if:*

$$\phi_i(\beta_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

It's easy to prove that if $\mathbb{V}$ is finite dimensional, then Φ is a basis of $\mathbb{V}'$. However, it's not true in the case of infinite-dimensional $\mathbb{V}$.

Lemma 1.1 *if $\mathbb{V}$ is infinite dimensional, dual basis is not a basis of dual space.*

Proof Denote β_i as the elements of B and φ_i as the elements of Φ. (Noted that this does not suppose that B is a countable set.) Then there exist a linear functional $f \in \mathbb{V}' : \mathbb{V} \to \mathbb{F}$ $f(\beta_i) = 1$.

If Φ is a basis of $\mathbb{V}'$, then f can be represented as the finite sum of β_i, namely $f = \sum_{i=1}^{m} a_i \varphi_i$ for some m.
Therefore,

$$1 = f(\beta_j) = \sum_{i=1}^{m} a_i \varphi_i(\beta_j) = a_j.$$

Hence,

$$f = \sum_{i=1}^{m} \varphi_i,$$

$$1 = f(\beta_{m+1}) = \sum_{i=1}^{m} \varphi_i(\beta_{m+1}) = 0.$$

Contradiction. □

Since dual basis is not a basis, we doubt that dual space is isomorphic to the origin space. However, its doable to find a subspace of $\mathbb{V}'$, denoted as $\mathbb{V}^*$, that is isomorphic to $\mathbb{V}$. A new definition is needed in the first place.

Definition 1.2 *We call a function $G : \mathbb{V} \times \mathbb{V} \to \mathbb{F}$ "0-0 bilinear function" if:*

(i) G is bilinear.

(ii) $G(v, v) = 0$ if and only if $v = 0$.

2 SUBSPACE ISOMORPHIC TO $\mathbb{V}$

Now we can find $\mathbb{V}^*$. To make it clear, $span(\Phi)$ is absolutely a subspace isomorphic to $\mathbb{V}$. But we aimed at finding a more universal property of the needed subspace.

Theorem 2.1 *If there exists a "0-0 bilinear function" $G : \mathbb{V} \times \mathbb{V} \to \mathbb{F}$, $\forall f \in \mathbb{V}^*$, $\forall \alpha \in ker(f), \beta \in \mathbb{V}/ker(f)$, $G(\alpha, \beta) = 0$, then $\forall f \in \mathbb{V}^*$, $\exists! u$ such that $\forall v \in \mathbb{V}$, $f(v) = G(v, u)$.*

Proof First we will prove the existence.
Since $\mathbb{V}/ker(f) \cong im(f)$, $dim(\mathbb{V}/ker(f)) = dim(im(f)) = 1$. Therefore, $\exists \overline{v} \in \mathbb{V}/ker(f)$, $span\overline{v} \cong \mathbb{V}/ker(f)$.
$\forall v$, $\exists v_0 \in ker(f)$ such that $v = v_0 + m\overline{v}$
Therefore,

$$f(v) = f(v_0 + m\overline{v} = mf(\overline{v})),$$

$$G(v, \overline{v}) = G(v_0 + m\overline{v}, \overline{v}) = mG(\overline{v}, \overline{v}).$$

Hence,

$$f(v) = \frac{G(v, \overline{v})}{G(\overline{v}, \overline{v})} f(\overline{v}) = G(v, \frac{\overline{v}}{G(\overline{v}, \overline{v})} \overline{v}).$$

So there exist $u := \frac{\overline{v}}{G(\overline{v}, \overline{v})} \overline{v}$.
Then we will prove the uniqueness.

If $\exists u_1, u_2$ such that $f(v) = G(v, u_1) = G(v, u_2)$, then $G(v, u_1 - u_2) = 0$. Because of the randomcity of v, we can let v be $u_1 - u_2$. Then since $G(u_1 - u_2, u_1 - u_2) = 0$, $u_1 - u_2 = 0$. $\square$

Corollary 2.1 $\mathbb{V}^* \cong \mathbb{V}$.

Reisz representation theorem is constructed on Hilbert space. Consider the inner product on Hilbert space. The inner product is clearly a "0-0 bilinear function". And by changing $\mathbb{V}/ker(f)$ in **Theorem 2.1** into $ker(f)^{\perp}$, we can easily prove the following theorem.

Theorem 2.2 (Riesz representation theorem) *Let $\mathbb{H}$ be a Hilbert space. For every continuous linear functional $\varphi \in \mathbb{H}'$, there exists a unique vector $f_\varphi \in \mathbb{H}$, called the Riesz representation of φ, such that*

$$\varphi(x) = \langle x, f_\varphi \rangle$$

for all $x \in \mathbb{H}$.

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