A Trace Inequality of Positive Semi-definite Matrices

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1 Introduction

Positive semi-definite matrices (hereinafter referred to as PSD matrices) has some interesting properties in many discipline of mathematics. We know that the property of PSD matrices within the matrices space is analogous to the property of non-negative real numbers within the real number space, while there are many important inequalities of non-negative real numbers. Therefore, it is reasonable to infer that some of them holds true for PSD matrices. In this paper, we will establish the matrix version of one of the basic inequalities.

Theorem 1. *Let A, B be two PSD matrices. $\sqrt{M}$ denotes the unique matrix $\widetilde{M}$ such that $\widetilde{M}^2 = M$. Then*

$$tr(\sqrt{A}) + tr(\sqrt{B}) \geq tr(\sqrt{A+B})$$

.

Note: It has already been proved that all PSD matrices has unique square roots. This will no longer be proved in this paper.

2 Proof of The Main Result

To prove the theorem above, we need to prove a lemma first.

Lemma 1. *Let x, y be two n-dimensional vectors such that $x_1 \geq x_2 \cdots \geq x_n \geq 0$ and $y_1 \geq y_2 \cdots \geq y_n \geq 0$. If for all $1 \leq r \leq n$, $\sum_{i=1}^{r} x_i \leq \sum_{i=1}^{r} y_i$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then $\sum_{i=1}^{n} \sqrt{x_i} \geq \sum_{i=1}^{n} \sqrt{y_i}$.*

Proof. If $n = 1$, trival.

If $n = 2$, then $x_1 \leq y_1$ and $x_2 \geq y_2$. Since $f(x) = \sqrt{x}$ is concave, the inequality holds true.

If $n > 2$, by applying the case of $n = 2$ repeatedly, we have

$$
\begin{aligned}
&\sum_{i=1}^{n} \sqrt{x_i} \\
&\geq \sum_{i=1}^{n-2} \sqrt{x_i} + \sqrt{x_{n-1} + x_n - y_n} + \sqrt{y_n} \\
&\geq \sum_{i=1}^{n-3} \sqrt{x_i} + \sqrt{x_{n-2} + x_{n-1} + x_n - y_{n-1} - y_n} + \sqrt{y_{n-1}} + \sqrt{y_n} \\
&\geq \cdots \\
&\geq \sum_{i=1}^{r} \sqrt{x_i} + \sqrt{\sum_{i=r+1}^{n} x_i - \sum_{i=r+2}^{n} y_i} + \sqrt{y_{r+2}} + \sum_{i=r+3}^{n} \sqrt{y_i} \\
&\geq \sum_{i=1}^{n} \sqrt{y_i}
\end{aligned}
$$

□

Proof of Theorem 1. In the following proof, when we talk about the eigenvalues $\{\lambda_{k,M}\}$ of a matrix M, they are ranked decreasingly, namely $\lambda_{1,M} \geq \lambda_{2,M} \geq \cdots \geq \lambda_{n,M}$.

Performing the eigenvalue decomposition of A and B, we have

$$A = \sum_{k=1}^{n} \lambda_{k,A} p_k p_k^T,\ B = \sum_{k=1}^{n} \lambda_{k,B} q_k q_k^T.$$

Define $U := \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $W := \begin{bmatrix} A+B & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$tr(\sqrt{A}) + tr(\sqrt{B}) = \sum_{k=1}^{n} \sqrt{\lambda_{k,A}} + \sum_{k=1}^{n} \sqrt{\lambda_{k,B}} = \sum_{k=1}^{2n} \sqrt{\lambda_{k,U}};$$

$$tr(\sqrt{A+B}) = \sum_{k=1}^{n} \sqrt{\lambda_{k,A+B}} = \sum_{k=1}^{2n} \sqrt{\lambda_{k,W}}.$$

Additionally,

$$\sum_{k=1}^{2n} \lambda_{k,U} = tr(U) = tr(A) + tr(B) = tr(W) = \sum_{k=1}^{2n} \lambda_{k,W}$$

$$\sum_{k=1}^{r}\lambda_{k,U} \tag{1}$$

$$=\sum_{k=1}^{m}\lambda_{k,A}+\sum_{k=1}^{r-m}\lambda_{k,B} \tag{2}$$

$$=tr(\sum_{k=1}^{m}\lambda_{k,A}p_kp_k^T+\sum_{k=1}^{r-m}\lambda_{k,B}q_kq_k^T) \tag{3}$$

$$:=tr(X) \tag{4}$$

$$=tr(P_rX) \tag{5}$$

$$\leq tr(P_rX)+tr(P_r\widetilde{X}) \tag{6}$$

$$=tr(P_r(A+B)) \tag{7}$$

$$=tr(\begin{bmatrix} P_r & 0 \\ 0 & 0 \end{bmatrix} W) \tag{8}$$

$$\leq \sum_{k=1}^{r}\sigma_{k,W} \tag{9}$$

$$=\sum_{k=1}^{r}\lambda_{k,W} \tag{10}$$

Explanation of some equations or inequalities:

(4)=(5) Define a projection matrix P_r that project everything into the column space of X, then $X = P_rX$;

(5)≤(6) Define $\widetilde{X} := \sum_{k=m+1}^{n}\lambda_{k,A}p_kp_k^T + \sum_{k=r-m+1}^{n}\lambda_{k,B}q_kq_k^T$, then $\widetilde{X}$ is a PSD matrix. $tr(P_r\widetilde{X}) = tr(PrPr^T\widetilde{X}) = tr(Pr^T\widetilde{X}Pr) = tr(P_r^T\sqrt{\widetilde{X}}^T\sqrt{\widetilde{X}}P_r) = \sum[\sqrt{\widetilde{X}}P_r]_{ij}^2 \geq 0$;

(8)≤(9) The eigenvalues of P_r are r ones and n-r zeros. By Von Neumann trace inequality, the inequality holds true.

By applying Lemma 1,

$$tr(\sqrt{A})+tr(\sqrt{B})=\sum_{k=1}^{2n}\lambda_{k,U}\geq\sum_{k=1}^{2n}\lambda_{k,W}=tr(\sqrt{A+B}).$$

□

References

[1] Barry C. Arnold Albert W. Marshall, Ingram Olkin. Inequalities: Theory of majorization and its applications. *New York: Academic Press*, 1979.

[2] user8675309 (https://math.stackexchange.com/users/735806/user8675309). Inequality of square root inside trace norm. URL:https://math.stackexchange.com/q/3876040 (version: 2023-01-23).