Complexity analysis of polynomial algorithms

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Abstract

This work is about three different proofs of the same fact and a computational comparison between them, looking for the best one.

Let R be a real closed field and $n \geq 2$. We prove that: (1) for every finite subset F of R^n, the semialgebraic set $R^n \setminus F$ is a polynomial image of R^n; and (2) for any independent linear forms $l_1, \ldots, l_r$ of R^n, the semialgebraic set $\{l_1 > 0, \ldots, l_r > 0\} \subset R^n$ is a polynomial image of R^n.

The key proof here is that $Q = \{x > 0, y > 0\}$ is a polynomial image of $\mathbb{R}^2$. This assert is proved in three different ways: a first approach using real algebraic geometry; a second and shorter one, using the composition of 3 rather simple maps; and a third one that applies topology with no computer computations.

(...)

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Polynomial images of R^n

1.1 Introduction

Definition 1.1. Let R be a real closed field (see definition A.1) and $m, n \in \mathbb{N}_{>0}$. A map $f = (f_1, \ldots, f_n) : R^m \longrightarrow R^n$ is said to be polynomial if $f_i \in R[x_1, \ldots, x_m], i = 1, \ldots, n$.

A very famous theorem by Tarski and Seidenberg states:

Theorem (Tarski-Seidenberg)**.** The image of any polynomial map $f : R^m \longrightarrow R^n$ is a semialgebraic subset (see definition A.2) of R^n.

In this work we are studying sort of a converse of this statement. In an *Oberwolfach* week, J.M. Gamboa proposed to characterize the semialgebraic subsets of R^n that are polynomial images of R^m.

Notation. We need to mention to which topology we refer when we talk about closures, boundaries, etc. More specifically, the **exterior boundary** of a set S is $\delta S := \bar{S} \backslash S$, with $\bar{S}$ being the **closure** of S in R^n in the usual topology. $\bar{S}^{\text{zar}}$ is the closure of S with respect to the Zariski topology (see definition A.3). $A \subset R^n$ is **irreducible** if its Zariski closure $\bar{A}^{\text{zar}}$ is an irreducible algebraic set.

1.1.1 Neccesary conditions and examples

To begin working on this idea, we provide some neccesary conditions for a set $S \subset R^n$ to be polynomial image of R^m.

It is trivial that for $m = n = 1$ (so $f : R \to R$), the images of polynomial maps are either a set of one point or singletons (if the map is constant), or unbounded closed intervals (think of $f(x) = x^2$), or the whole R (think of $f(x) = x$).

In the general case, by theorem , .

$\mathcal{A}$ Auxiliar definitions and results

Definition A.1. A **real closed field** is a field R that has the 1^{st} order properties as the field of real numbers $\mathbb{R}$.

Definition A.2. A **semialgebraic set** is a subset $S \subset R^n$ (for some real closed field R) defined by a finite sequence of polynomial equations of the form:

$$
\begin{gathered}
P_1(x_1, \ldots, x_n) = 0 \\
\vdots \\
P_r(x_1, \ldots, x_n) = 0 \\
Q_1(x_1, \ldots, x_n) > 0 \\
\vdots \\
Q_l(x_1, \ldots, x_n) > 0
\end{gathered}
$$

. A **semialgebraic map** is a map that has semialgebraic graph. Moreover, the finite union, intersection and complement of semialgebraic sets is still a semialgebraic set.

Definition A.3 (Zariski topology)**.** It is a topology on algebraic varieties whose closed sets are the algebraic subsets of the variety. Its sets are defined as the set of solutions of a system of polynomial equations over a field R. In this topology, when we talk about the irreducibility of a element, we mean that it is not the union of two smaller sets that are closed under the Zariski topology.

Bibliography