Complexity Analysis of Polynomial Algorithms

Final Degree Project - Double Degree in Mathematics and Computer Science, Complutense University of Madrid (UCM).

Ignacio Iker Prado Rujas

Tutors: Juan R. Delgado, José F. Fernando & José M. Gamboa - Department of Algebra, Faculty of Mathematics.

Abstract

In this work we gather three different proofs appearing in [FG], [FGU] and [FU1] of the following result: let R be a real closed field and consider the open quadrant $\mathcal{Q} := \{x > 0, y > 0\} \subset R^2$. Then, there exists a polynomial map $f : R^2 \to R^2$ such that $f(R^2) = \mathcal{Q}$. In addition, we compare the computational complexity of the three maps f satisfying the equality $f(R^2) = \mathcal{Q}$ looking for the most efficient one, and we present 3 new candidates to solve this problem.

Introduction

Let R be a real closed field. A very famous theorem by Tarski and Seidenberg (1951) states the following:

Theorem 1. The image of every polynomial map $f : R^m \to R^n$ is a semialgebraic subset of R^n.

In this work we study a sort of converse of this statement proposed by J.M. Gamboa in the 1990 *Oberwolfach Reelle algebraische Geometrie* week:

'Inverse problem'. Characterize the semialgebraic subsets of R^n that are polynomial images of some R^m.

This problem has its origin in examples such as that $(0, +\infty)$ **is not** the image of any polynomial map $\mathbb{R} \to \mathbb{R}$, although there exists a certain polynomial map $\mathbb{R}^2 \to \mathbb{R}^2$ having $\mathcal{H} := \{y > 0\} \subset \mathbb{R}^2$ as its image.

Main results

The main results that we prove are:

Theorem 2. Let $n \geq 2$. For every finite set $F \subset R^n$, the semialgebraic set $R^n \setminus F$ is a polynomial image of R^n.

Theorem 3. Let $n \geq 2$. Given independent linear forms $h_1, \dots, h_r$ of R^n, the open semialgebraic set $\{h_1 > 0, \dots, h_r > 0\}$ is a polynomial image of R^n.

Theorem 4 (Open quadrant $\mathcal{Q}$ problem). The open quadrant $\mathcal{Q} := \{x > 0, y > 0\}$ is a polynomial image of R^2.

About the first proof

On the topic of finding a polynomial map $f : \mathbb{R}^2 \to \mathbb{R}^2$ verifying Theorem 4, a "major difficulty" is that **the closure of its image must contain the positive half-axes**. Aiming at this, we approach the positive half-axes with certain families of curves: $\alpha_\lambda, \beta_\mu : \mathbb{R} \to \mathbb{R}^2$. The map $g := (\mathcal{F}, \mathcal{G})$ defined as:

$$\boxed{\begin{aligned}\mathcal{F}(\mathtt{x},\mathtt{y}) &= (1 - \mathtt{x}^3\mathtt{y} + \mathtt{y} - \mathtt{x}\mathtt{y}^2)^2 + (\mathtt{x}^2\mathtt{y})^2,\\ \mathcal{G}(\mathtt{x},\mathtt{y}) &= (1 - \mathtt{x}\mathtt{y} + \mathtt{x} - \mathtt{x}^4\mathtt{y})^2 + (\mathtt{x}^2\mathtt{y})^2,\end{aligned}}$$

behaves well along those curves, namely:

$$\lim_{s\to 0} \mathcal{P}(\alpha_\lambda(s)) = (\lambda^2, 0) \text{ and } \lim_{s\to 0} \mathcal{P}(\beta_\mu(s)) = (0, \mu^2).$$

About the third proof

The map $\mathscr{F} := (\mathscr{F}_1, \mathscr{F}_2) : \mathbb{R}^2 \to \mathbb{R}^2$ defined as:

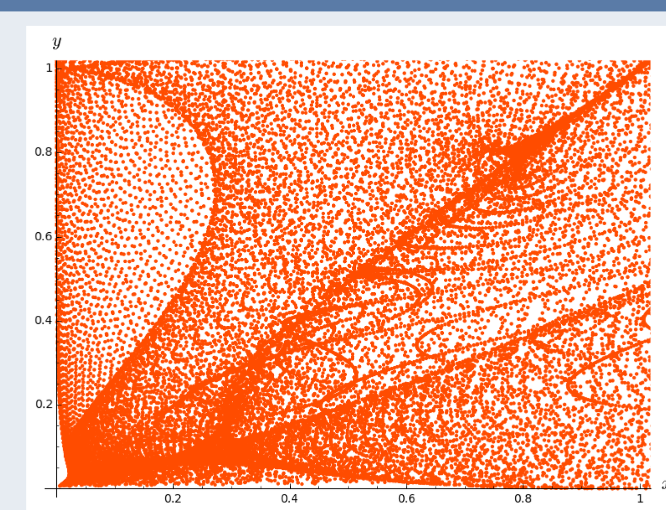
$$\boxed{\begin{aligned}\mathscr{F}_1(\mathtt{x},\mathtt{y}) &= (\mathtt{x}^2\mathtt{y}^4 + \mathtt{x}^4\mathtt{y}^2 - \mathtt{y}^2 - 1)^2 + \mathtt{x}^6\mathtt{y}^4,\\ \mathscr{F}_2(\mathtt{x},\mathtt{y}) &= (\mathtt{x}^6\mathtt{y}^2 + \mathtt{x}^2\mathtt{y}^2 - \mathtt{x}^2 - 1)^2 + \mathtt{x}^6\mathtt{y}^4,\end{aligned}}$$

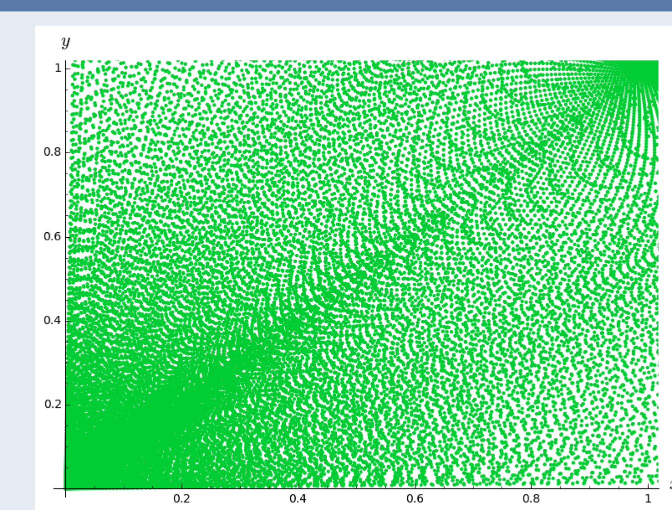
can be expressed as $\mathscr{F} = f_2 \circ f_1$, with $f_1(\mathtt{x},\mathtt{y}) = (\mathtt{x}^2, \mathtt{y}^2)$. This map verifies $f_1(\mathbb{R}^2) = \overline{\mathcal{Q}}$, so given $a, b > 0$, we must provide $x, y \geq 0$ such that:

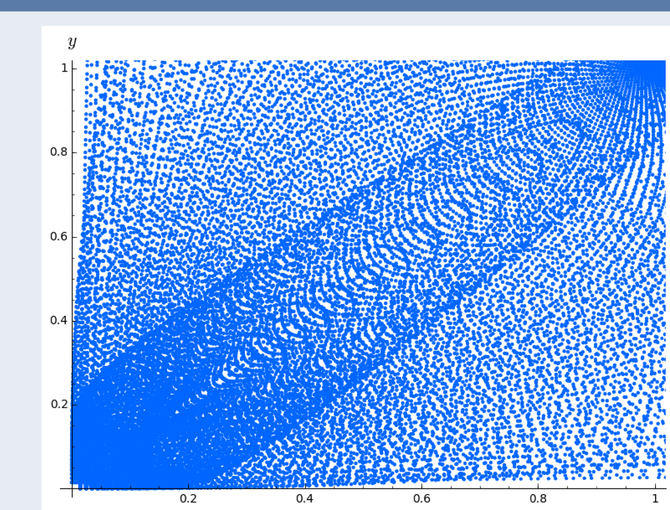
$$\begin{cases} f_{2,1}(x,y) = a, \\ f_{2,2}(x,y) = b. \end{cases}$$

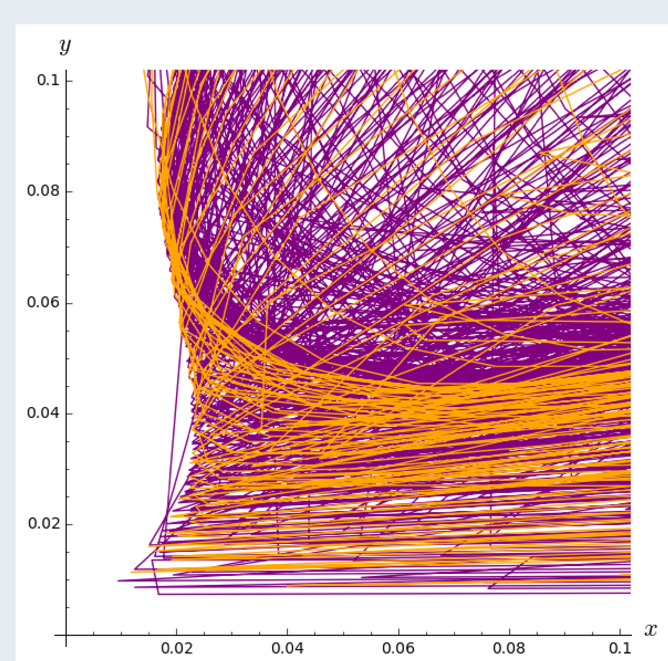
The proof is reduced to check that the boundaries of two certain topological subspaces of $\mathbb{R}^3$ homeomorphic to a closed disc meet $g(\overline{\mathcal{Q}})$, where $f_2 = h \circ g$.

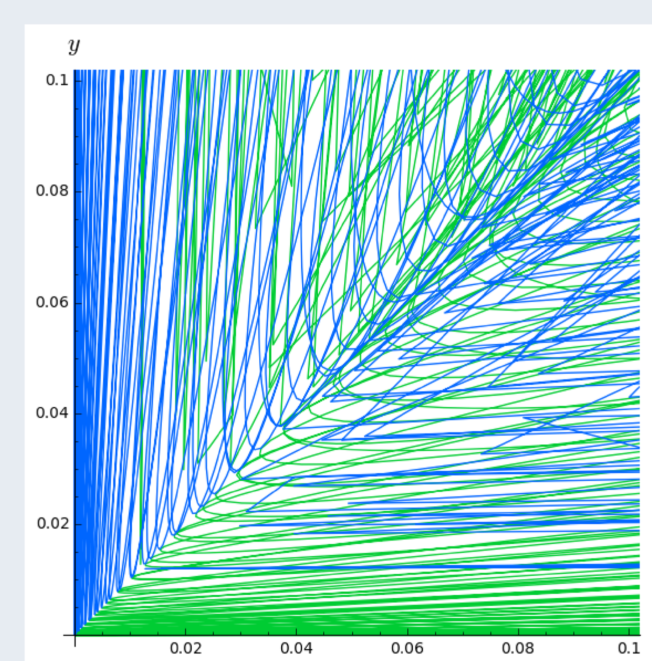
Some important graphs regarding the computational analysis

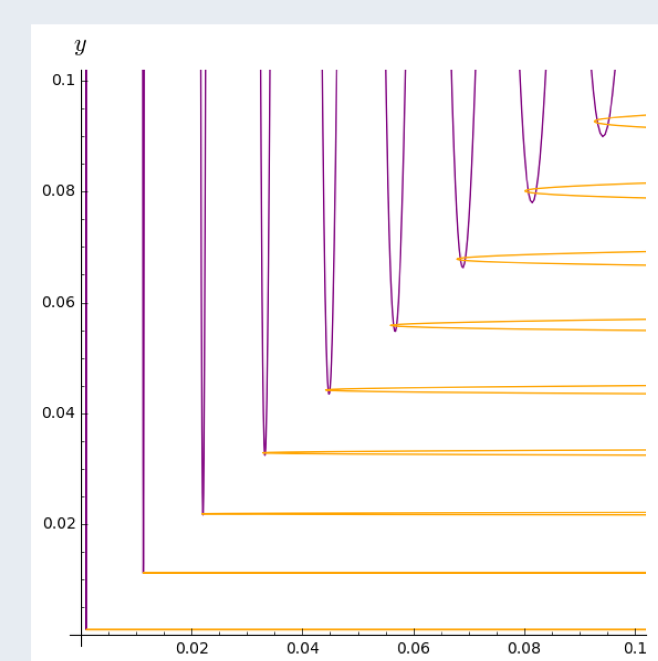

(a) $g(\mathscr{P}_6) \rightsquigarrow 1633.50$ s.


(b) $\mathcal{N}_1(\mathscr{P}_6) \rightsquigarrow 610.06$ s.


(c) $\mathcal{N}_3(\mathscr{P}_6) \rightsquigarrow 615.09$ s.


(d) $g(\alpha_\lambda)$ and $g(\beta_\mu)$ in $[0, 0.1]^2$.


(e) $\mathcal{N}_1(\sigma_\lambda)$ and $\mathcal{N}_1(\tau_\mu)$ in $[0, 0.1]^2$.


(f) $f_2(\gamma_\lambda)$ and $f_2(\delta_\mu)$ in $[0, 0.1]^2$.

About the second proof

Let $f := \mathcal{H} \circ \mathcal{G} \circ \mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$, where:

$$\boxed{\begin{aligned}\mathcal{F}(\mathtt{x},\mathtt{y}) &= \big((\mathtt{x}\mathtt{y} - 1)^2 + \mathtt{x}^2,\ (\mathtt{x}\mathtt{y} - 1)^2 + \mathtt{y}^2\big),\\ \mathcal{G}(\mathtt{x},\mathtt{y}) &= \big(\mathtt{x},\ \mathtt{y}(\mathtt{x}\mathtt{y} - 2)^2 + \mathtt{x}(\mathtt{x}\mathtt{y} - 1)^2\big),\\ \mathcal{H}(\mathtt{x},\mathtt{y}) &= \big(\mathtt{x}(\mathtt{x}\mathtt{y} - 2)^2 + \tfrac{1}{2}\mathtt{x}\mathtt{y}^2,\ \mathtt{y}\big).\end{aligned}}$$

In this case the proof is conducted by inspecting at the images of the aforementioned polynomials.

Complexity analysis

In this work we also present some new candidates whose images "should be" $\mathcal{Q}$: $\mathcal{N}_1, \mathcal{N}_2$ and $\mathcal{N}_3$. For the analysis, given a set of points we study how much time each map takes to compute the set of image points. We also examine which of the maps "covers more area of $\mathcal{Q}$" given the same set of points.

	Total degree	Total number of monomials	Non-escalar complexity
g	56	167	13
f	72	350	11
$\mathscr{F}$	28	22	11
$\mathcal{N}_1$	16	24	10
$\mathcal{N}_2$	16	26	8
$\mathcal{N}_3$	20	26	13

In the analysis we start with some grids of points contained in a square, like a set of 4 million points denoted as $\mathscr{P}_6$. We also use the families of curves that we mentioned before (and some others) in order to approach the positive half-axes.

Conclusions

Among the first 3 maps, the better one in terms of filling $\mathcal{Q}$ seems to be g, but if we look at computation time $\mathscr{F}$ is better. Although Theorem 4 hasn't been proved for the new maps, they outstrip g, f and $\mathscr{F}$ on the tests. It is remarkable that for instance for $\mathcal{N}_1$, even when looking at $[0, 0.1]^2$ the images of the curves stay really close to the positive half-axes (as we see on Figure 1e), whereas we don't see this behavior for g and f_2 (Figures 1d and 1f).

Some bibliography

[BCR] J. Bochnak, M. Coste, M.-F. Roy: Géométrie algébrique réelle. *Ergeb. Math.* **12**, Springer-Verlag, Berlin, Heidelberg, New York (1987).

[FG] J.F. Fernando, J.M. Gamboa: Polynomial images of $\mathbb{R}^n$. *J. Pure Appl. Algebra*, **179** (2003), 241–254.

[FGU] J.F. Fernando, J.M. Gamboa, C. Ueno: The open quadrant problem: A topological proof. A mathematical tribute to Professor José María Montesinos. Departamento de Geometría y Topología. Facultad de CC. Matemáticas UCM, (2016), 337–350.

[FU1] J.F. Fernando, C. Ueno: On the open quadrant as a polynomial image of $\mathbb{R}^2$ (revisited).

[U] C. Ueno: Imágenes polinómicas y regulares de espacios euclídeos. Ph.D. Thesis UCM, (2012).

Further reading

Full text and code can be found at `https://github.com/iipr/Final-Degree-Project`.