Complexity Analysis of Polynomial Algorithms

Ignacio Iker Prado Rujas
Complutense University of Madrid (UCM)
Double Degree in Mathematics and Computer Science

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Tutors:
Juan R. Delgado, José F. Fernando & José M. Gamboa
Algebra Department

Abstract

This work is about different proofs of the same fact and a computational comparison between them, looking for the most efficient one.

Let R be a real closed field and $n \geq 2$. Then:

(1) For every finite subset F of R^n, the semialgebraic set $R^n \setminus F$ is a polynomial image of R^n.

(2) Given independent linear forms $h_1, \ldots, h_r$ of R^n, the semialgebraic set $\{h_1 > 0, \ldots, h_r > 0\} \subset R^n$ is a polynomial image of R^n.

The key result here is that $\mathcal{Q} := \{x > 0, y > 0\}$ is a polynomial image of R^2. This assert is proved in three different ways: a first approach using some basic results in real algebraic geometry (such as Sturm's theorem and the Curve Selection Lemma) and the aid of a computer; a second and shorter one, using the composition of three rather simple polynomial maps; and a third one that applies arguments of algebraic topology, without the aid of computer computations.

(...)

Resumen

Este trabajo contiene distintas pruebas del mismo hecho y pretende hacer una comparacin computacional entre ellas, tratando de encontrar la más eficiente.

Sea R un cuerpo real cerrado y $n \geq 2$. Entonces:

(1) Para cada subconjunto finito F de R^n, el conjunto semialgebraico $R^n \setminus F$ es imagen polinomial de R^n.

(2) Dadas unas formas linealmente independientes $h_1, \ldots, h_r$ de R^n, el conjunto semialgebraico $\{h_1 > 0, \ldots, h_r > 0\} \subset R^n$ es imagen polinomial de R^n.

El resultado clave aquí es que $\mathcal{Q} := \{x > 0, y > 0\}$ es imagen polinomial de R^2. Esta afirmación es probada de tres maneras diferentes: una primera usando algunos resultados básicos de la geometría algebraica real (como son el teorema de Sturm y el Lema de Selección de Curvas) y la ayuda de un odenador; una segunda forma más corta, usando la composición de tres aplicaciones polinomiales bastante sencillas; y una tercera y última que comprende argumentos de la rama de la topología algebraica, sin la necesidad de realizar computaciones en un ordenador.

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Introduction to polynomial images of R^n

1.1 Introduction

Definition 1.1. Let R be a real closed field and $m, n \in \mathbb{N}_{>0}$. A map $f := (f_1, \ldots, f_n) : R^m \to R^n$ is said to be *polynomial* if $f_i \in R[x_1, \ldots, x_m]$ for $i = 1, \ldots, n$.

A very famous theorem by Tarski and Seidenberg states the following:

Theorem 1.2 (Tarski-Seidenberg)

The image of every polynomial map $f : R^m \longrightarrow R^n$ is a semialgebraic subset of R^n.

In this work we study a sort of converse of this statement. In the 1990 *Oberwolfach Reelle algebraische Geometrie* week, J.M. Gamboa [G] proposed the following

Problem. Characterize the semialgebraic subsets of R^n that are polynomial images of some R^m.

Particularly interesting seems to be the study of those open semialgebraic subsets of R^n that are polynomial images of R^n, because this is related with the real jacobian conjecture.

Notation. We need to mention to which topology we refer to when we talk about closures, boundaries, etc. More specifically, the *exterior boundary* of a subset $S \subset R^n$ is $\delta S := \overline{S} \setminus S$, with $\overline{S}$ being the *closure* of S in the usual topology of R^n. In addition we will denote by $\overline{S}^{\text{zar}}$ the closure of S with respect to the Zariski topology of R^n. We will say that a semialgebraic subset $A \subset R^n$ is *Zariski-irreducible* if its Zariski closure $\overline{A}^{\text{zar}}$ is an irreducible algebraic set.

1.1.1 Necessary conditions and examples

To begin working on this idea, we provide some necessary conditions for a set $S \subset R^n$ to be a polynomial image of R^m.

If $m = n = 1$, that is, for a polynomial function $f : R \to R$, its image $f(R)$ is either a *singleton*, that is, a set with a unique point if the function f is constant, or an unbounded closed interval, for example if $f(x) = x^2$, or $f(R) = R$ if f is a polynomial of odd degree.

In the general case, by Tarski-Seidenberg, S must be a semialgebraic set and, as R^n is semialgebraically connected, S is semialgebraically connected too. Even more, by the identity principle for polynomials, S is Zariski-irreducible and pure dimensional.

But to be a polynomial image of some R^m is a restrictive condition, and there are more constrains than those quoted above.

Definition 1.3. A polynomial map $f : R^m \to R^n$ is said to be *semialgebraically proper at a point* $p \in R^n$ if there exists an open neighbourhood K of p such that the restriction $f|_{f^{-1}(K)} : f^{-1}(K) \to K$ is a semialgebraically proper map.

Definition 1.4. A *parametric semiline* of R^n is the image of R under a non-constant polynomial map $R \to R^n$.

It is clear that every parametric semiline is semialgebraically closed, since every polynomial map from R to R^n is semialgebraically proper. Let $\mathscr{S}_f$ denote the set of points $p \in R^n$ at which f is **not** semialgebraically proper.

Theorem 1.5 (Jelonek)

Let $f : R^2 \to R^2$ be a dominant polynomial map. Then $\mathscr{S}_f$ is a finite union of parametric semilines.

With these ideas in mind, we present in the following proposition some obstructions for a semialgebraic set to be a polynomial image of R^n.

Proposition 1.6. *Let $f : R^m \to R^n$ be a polynomial map and $S := f(R^m)$.*

(1) $\delta S \subset \mathscr{S}_f$.

Proof. Suppose $p \in \delta S \setminus \mathscr{S}_f$. Since $p \notin \mathscr{S}_f$, there exists an open neighbourhood K of p such that the restriction $f|_{f^{-1}(K)} : f^{-1}(K) \to K$ of f is proper. Thus its image $K \cap S$ is a closed subset of K. Hence, $p \in K \cap \overline{S} = K \cap (\overline{K \cap S}) = K \cap S$, which yields in a contradiction.

(2) *Let $m = n = 2$ and Γ be a 1-dimensional irreducible component of $\overline{\delta S}^{zar}$. Then Γ is the Zariski closure of a parametric semiline of R^2.*

Proof. As f is a dominant map, it follows from Theorem 1.5 that $\mathscr{S}_f$ is a finite union of parametric semilines, say $M_1, \ldots, M_s$ in R^2. Then, using (1) we get: $\Gamma \subset \overline{\delta S}^{\text{zar}} \subset \overline{\mathscr{S}_f}^{\text{zar}} = \bigcup_{i=1}^s \overline{M_i}^{\text{zar}}$. Lastly, using that both Γ and the $\overline{M_i}^{\text{zar}}$'s are irreducible, $\Gamma = \overline{M_i}^{\text{zar}}$ for some $i = 1, \ldots, s$.

(3) *Let $p : R^n \to R$ be a polynomial map which is non-constant on S. Then $p(S)$ is unbounded.*

Proof. For each $a \in R^m$ let $\varphi_a : R \to R$ defined as $\varphi_a(t) := p(f(ta))$. Then $p(S)$ would contain the image $\varphi_a(R)$ for all $a \in R^m$. Suppose now that $\varphi_a(R)$ is bounded. Then $\varphi_a(R)$ would be a point r_a, and given two points $a, b \in R^m$ we would have

$$\varphi_a(1) = p(f(ta)) = r_a = \varphi_a(0) = \varphi_b(0) = r_b = p(f(tb)) = \varphi_b(1).$$

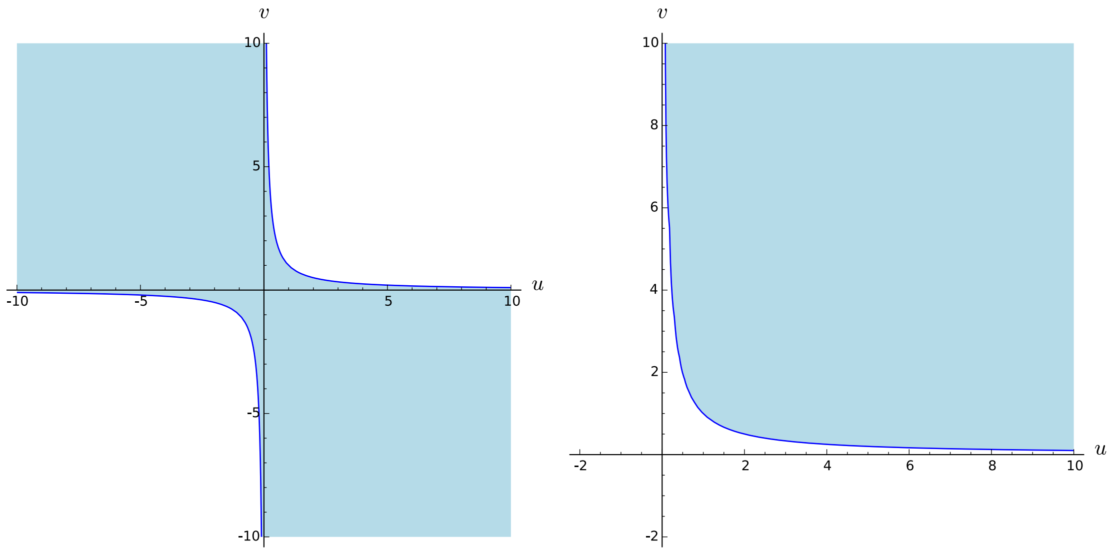
Then p would be constant on S, which is a contradiction.

Corollary 1.7. *Because of* (3) *in Proposition 1.6, all linear projections of S are either a point or unbounded. Thus, S is also unbounded or a point.*

Examples 1.8.

(i) The exterior of the closed unit disc $S := \{u^2 + v^2 > 1\}$ **is not** a polynomial image of R^2. This is so because the only irreducible component of $\overline{\delta S}^{\text{zar}}$ is $\{u^2 + v^2 = 1\}$ and this set is not a parametric semiline because it is bounded.

(ii) Let $S_1 := \{uv < 1\}$ and $S_2 := \{uv > 1, u > 0\}$ (see fig. 1.1). They both **are not** polynomial images of R^2 since the Zariski closure of their exterior boundaries $\overline{\delta S_1}^{\text{zar}} = \overline{\delta S_2}^{\text{zar}}$ is the hyperbola $\{uv = 1\}$, which is not a parametric semiline.



(a) $S_1 := \{uv < 1\}$. (b) $S_2 := \{uv > 1, u > 0\}$.

Figure 1.1: Plots of the regions defined in Example 1.8 (ii).

(iii) Let $S := R^2 \setminus \{(0,0)\}$ be the punctured plane. Then S is the image of the polynomial map $(x,y) \mapsto (xy - 1, (xy - 1)x^2 - y)$.

(iv) Let $\mathcal{H} := \{v > 0\}$ be the open upper half-plane. Then $\mathcal{H}$ is the image of $(x,y) \mapsto (y(xy-1), (xy-1)^2 + x^2)$. This implies that every open half-plane is a polynomial image of R^2. This is probably the simplest polynomial map whose image is $\mathcal{H}$.

1.1.2 Statement of the main results

The main results of this chapter are generalizations of examples (iii) and (iv) from Examples 1.8.

Theorem 1.9

Let $n \geq 2$. For every finite set $F \subset R^n$, the semialgebraic set $R^n \setminus F \subset R^n$ is a polynomial image of R^n.

Theorem 1.10

Let $n \geq 2$. For any independent linear forms $h_1, \dots, h_r$ of R^n, the open semialgebraic set $\{h_1 > 0, \dots, h_r > 0\}$ is a polynomial image of R^n.

Before the paper [FG] was published, the exterior boundary of all open sets that were known to be polynomial images of R^2 was Zariski-irreducible, and all of them were deformations of $\mathcal{H}$. J.M. Gamboa outlined the problem of finding if $\mathcal{Q} = \{x > 0, y > 0\}$ is a polynomial image of R^2 or not, since its exterior boundary is Zariski-reducible. The solution of this problem is a key particular case of the content of Theorem 1.10. Before that, the closest known approach to look for a solution of this problem was the transformation

$$\psi : R^2 \to R^2,\ (x, y) \mapsto (x^4y^2, x^2y^4)$$

whose image is $\mathcal{Q} \cup \{(0, 0)\}$. The answer to the first intriguing problem in this field was given in the following theorem:

Theorem 1.11

The first open quadrant $\mathcal{Q}$ is a polynomial image of R^2.

Remark 1.12. The first proof of Theorem 1.11 consists of two parts:

- ⋆ Choosing a "good" candidate to be a polynomial map whose image is close enough to $\mathcal{Q}$, and giving the reasons behind this choice (see section 2.1).
- ⋆ Checking that the image of the map is $\mathcal{Q}$ indeed. After some arguments, this can be reduced to prove the non-existence of real roots of certain polynomials in one variable on certain intervals, and to compare some rational functions on those intervals. In order to do this, we use symbolic computations with tools like Sage and Maple. Because of the high degree of the involved polynomials, the actual checking of the non-existence of roots is done with a Maple package that performs Sturm algorithm and a Python programme that implements Laguerre's method.

Let us see how Theorem 1.10 follows from Theorem 1.11.

[Proof of theorem 1.10] After a linear change of coordinates we can suppose that $h_1 := x_1, \dots, h_r := x_r$, so we only have to prove that for every pair of positive integers $r \le n$ the semialgebraic set $\{x_1 > 0, \dots, x_r > 0\} \subset R^n$ is a polynomial image of R^n. This is reduced to prove the following two steps:

- ⋆ $\mathcal{H} := \{x_1 > 0\}$ and $\mathcal{Q} := \{x_1 > 0,\ x_2 > 0\} \subset R^2$ are polynomial images of R^2, which is true, respectively, by Example 1.8 (iv) and Theorem 1.11.
- ⋆ This implies that $\mathcal{O} := \{x_1 > 0,\ x_2 > 0,\ x_3 > 0\} \subset R^3$ is a polynomial image of R^3. Indeed, let $H_1, H_2 : R^2 \to R^2$ be polynomial maps whose respective images are $\mathcal{H}$ and $\mathcal{Q}$. Let us define:

  $$(H_1, \mathrm{id}_R) : R^3 = R^2 \times R \longrightarrow R^3 = R^2 \times R$$
  $$(\mathrm{id}_R, H_2) : R^3 = R \times R^2 \longrightarrow R^3 = R \times R^2.$$

  Then, $\mathcal{O}$ is the image of the polynomial map

  $$H := (\mathrm{id}_R, H_2) \circ (H_1, \mathrm{id}_R) : R^3 \to R^3$$

Once the case $n = 3$ is solved the case of arbitrary n follows straightforwardly. □

The original proofs of Theorems 1.9 and 1.11 are written for $R := \mathbb{R}$. As for both theorems explicit polynomial maps are given, the results can be extended to arbitrary real closed field R by the Transfer Principle.

1.2 Complementary set of a finite set

We proceed to prove theorem 1.9:

[Proof of theorem 1.9] Let $F := \{p_1, \dots, p_k\}$. Let us see that it suffices to prove the result for points of the form $p_j := (a_j, \vec{0}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. After a linear change of coordinates we can assume that the first coordinates of the given points are pairwise distinct.

In other words, we denote $p_j := (a_{1j}, \dots, a_{nj})$ and we may suppose that $a_{1j} \neq a_{1\ell}$ when $j \neq \ell$. Then, there exists a polynomial $P_1 \in \mathbb{R}[T]$ such that $P_1(a_{1j}) = a_{nj}$, with $j = 1, \dots, n$, and denoting $x' := (x_1, \dots, x_{n-1})$, we define the polynomial map

$$h_1 : \mathbb{R}^n \to \mathbb{R}^n,\ (x', x_n) \mapsto (x', x_n + P_1(x_1)).$$

Indeed h_1 is bijective. Note first that every point of $\mathbb{R}^n$ has a preimage in $\mathbb{R}^n$, namely if $x := (x_1, \dots, x_n)$, then $z := (x', x_n - P_1(x_1))$ satisfies $h_1(z) = x$ and h_1 is onto. As for being injective, let $x, y \in \mathbb{R}^n$ such that

$$h_1(x) = (x_1, \dots, x_n + P_1(x_1)) = (y_1, \dots, y_n + P_1(y_1)) = h_1(y).$$

Then $x_i = y_i$ for $i = 1, \dots, n-1$. Also $x_n + P_1(x_1) = y_n + P_1(y_1)$ and $P_1(x_1) = P_1(y_1)$ because $x_1 = y_1$. Therefore $x_n = y_n$ and $x = y$.

Now, for $p'_j := (a_{1j}, \dots, a_{(n-1)j}, 0)$ we have $h_1(p'_j) = p_j$. Analogously, there exists $P_2 \in \mathbb{R}[T]$ such that $P_2(a_{1j}) = a_{(n-1)j}$, and define the polynomial bijection

$$h_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^n,\ (x'', x_{n-1}, x_n) \mapsto (x'', x_{n-1} + P_2(x_1), x_n),$$

where $x'' = (x_1, \dots, x_{n-2})$. Then $h_2(p''_j) = p'_j$ for $p''_j = (a_{1j}, \dots, a_{(n-2)j}, 0, 0)$. In this way the polynomial bijection $h_1 \circ h_2 : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$(h_1 \circ h_2)(p''_j) = h_1(h_2(p''_j)) = h_1(p'_j) = p_j,$$

and we can inductively construct a polynomial bijection $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(q_j) = p_j$ for $q_j := (a_{1j}, \vec{0}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Now let $G := \{q_1, \dots, q_k\}$ and suppose that there exists a polynomial map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g(\mathbb{R}^n) = \mathbb{R}^n \setminus G$. Then $(h \circ g)(\mathbb{R}^n) = \mathbb{R}^n \setminus F$, which concludes the first part of the proof. Thus in what follows we suppose that $p_j = (a_j, \vec{0})$.

We claim that the image of the polynomial map $f := (f_1, \dots, f_n)$:

$$f(x) = \big(x_1x_2 - r + a_1,\ x_1^4\rho(x) + x_1^2\sigma(x) + x_2,\ x_3, \dots,\ x_n\big)$$

is $\mathbb{R}^n \setminus F$, where r is an integer such that $r \neq a_1 - a_j$ for $j = 1, \dots, k$,

$$\sigma(x) := \sum_{j=3}^{n} x_j^2 \text{ and } \rho(x) := \prod_{j=1}^{k}(x_1x_2 - r + a_1 - a_j).$$

First, suppose that there exists $b := (b_1, \dots, b_n) \in R^n$ such that $f(b) = p_\ell$ for some $\ell = 1, \dots, k$. Then $f_1(b) = b_1b_2 - r + a_1 = a_\ell$. Thus the ℓ^{th}-factor of the polynomial ρ evaluated at $x := b$ is

$$b_1b_2 - r + a_1 - a_\ell = a_\ell - a_\ell = 0,$$

and $\rho(b) = 0$. In addition, the equality $f(b) = p_\ell = (a_\ell, \vec{0})$ implies that $f_i(b) = 0$ for $i = 2, \dots, n$. In particular, since $f_i \equiv \mathrm{id}$ for $i = 3, \dots, n$ we get $b_i = 0$ for $i = 3, \dots, n$.

Hence $\sigma(b) = 0$. As $\sigma(b) = \rho(b) = 0$ we get $b_2 = f_2(b) = 0$, so $b_2 = 0$ and $a_\ell = f_1(b) = a_1 - r$, that is, $r = a_1 - a_\ell$, which is a contradiction. So $\operatorname{im}(f) \subset \mathbb{R}^n \setminus F$. Conversely, let $u := (u_1, \dots, u_n) \in \mathbb{R}^n \setminus F$. We must prove that the system of polynomial equations:

$$\begin{cases} f_1(x) &= x_1x_2 - r + a_1 = u_1 \\ f_2(x) &= x_1^4\rho(x) + x_1^2\sigma(x) + x_2 = u_2 \\ f_j(x) &= x_j = u_j, \quad j \geq 3 \end{cases}$$

has a solution.

(i) If $u_1 = a_1 - r$ then $f(0, u_2, \dots, u_n) = u$.

(ii) If $u_1 \neq a_1 - r$ we use the first equation to substitute

$$x_2 = \frac{u_1 - a_1 + r}{x_1} \qquad \text{and} \qquad x_j = u_j \qquad \text{for } j \geq 3.$$

Now, we expand $f_2(x)$:

$$x_1^4\rho(x) + x_1^2\sigma(x) - u_2 = -x_2 = -\frac{u_1 - a_1 + r}{x_1}$$

and multiplying by x_1 we get

$$x_1^5\rho(x) + x_1^3\sigma(x) - u_2x_1 + (u_1 - a_1 + r) = 0.$$

Then, $\rho(x) = \prod_{j=1}^k (u_1 - a_j)$ and $\sigma(x) = \sigma(u)$. Now it is clear that x_1 must be a nonzero root of the polynomial:

$$Q(T) = \Big(\prod_{j=1}^k (u_1 - a_j)\Big)T^5 + \sigma(u)T^3 - u_2T + (r - a_1 + u_1),$$

which has odd degree, and so a real root, unless

$$\prod_{j=1}^k (u_1 - a_j) = \sigma(u) = u_2 = 0.$$

If this were the case, then $u_1 = a_j$ for some $j = 1, \dots, k$ and $u_2 = u_3 = \cdots = u_k = 0$. This is not possible because $u \notin F$. Thus, $Q(T)$ has a real root b_1, and in fact $b_1 \neq 0$ because $Q(0) = r - a_1 + u_1 \neq 0$. Finally,

$$f\left(b_1,\ \frac{u_1 - a_1 + r}{b_1},\ u_3,\ \dots,\ u_n\right) = u,$$

as required.

$\square$

2

The open quadrant $\mathcal{Q}$ problem: First proof

2.1 Reasons behind the choice

It is remarkable that even though the open interval $I := (0, +\infty)$ is the image of $\mathbb{R}^2$ under the polynomial map $h : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto (xy - 1)^2 + x^2$, see fig. 2.1, the interval I is not a polynomial image of $\mathbb{R}$ because polynomial images of the real line are closed subsets of $\mathbb{R}$.

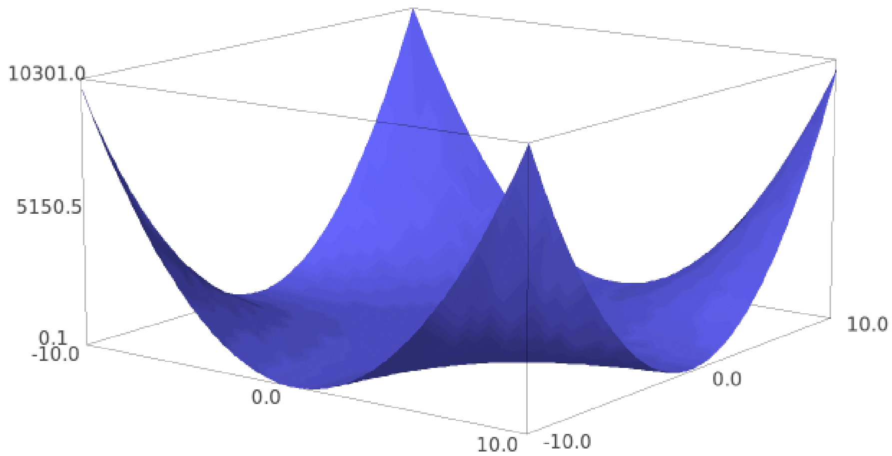


Figure 2.1: $h(x, y) = (xy - 1)^2 + x^2$.

However, although $h(\mathbb{R}^2) = I$, the polynomial h does not help to obtain $\mathcal{Q}$ at all:

Remark 2.1. There is no polynomial map $f := (P_1, P_2) : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $f(\mathbb{R}^2) = \mathcal{Q}$ and $P_1(x, y) = (xy - 1)^2 + x^2$.

The proof of this remark relies on a suitable use of the Curve Selection Lemma ([ABR, VIII.2.6]) to approach a point $(\lambda^2, 0) \in \overline{\mathcal{Q}}$ with $\lambda > 0$, to get a contradiction.

On the topic of finding a polynomial map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $\Phi(\mathbb{R}^2) = \mathcal{Q}$, "a major difficulty" is the following:

$$\boxed{\textit{The closure of its image must contain the positive half-axes.}} \qquad (\clubsuit)$$

Remark 2.2. Using Theorem 1.9, we just need to find a polynomial map

$$\mathcal{P} = (\mathcal{F}, \mathcal{G}) : \mathbb{R}^2 \to \mathbb{R}^2$$

such that $\mathcal{P}(\mathbb{R}^2)$ is the disjoint union of $\mathcal{Q}$ and a set with finite preimage, say $\mathcal{P}(\mathbb{R}^2) = \mathcal{Q} \sqcup F$ with $\mathcal{P}^{-1}(F)$ a finite set. By Theorem 1.9 there exists a polynomial map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(\mathbb{R}^2) = \mathbb{R}^2 \setminus \mathcal{P}^{-1}(F)$ and the polynomial map $\Phi := \mathcal{P} \circ \varphi$ satisfies

$$\Phi(\mathbb{R}^2) = \mathcal{P}(\varphi(\mathbb{R}^2)) = \mathcal{P}(\mathbb{R}^2 \setminus \mathcal{P}^{-1}(F)) = \mathcal{P}(\mathbb{R}^2) \setminus F = (\mathcal{Q} \sqcup F) \setminus F = \mathcal{Q}.$$

We are going to define a map $\mathcal{P} := (\mathcal{F}, \mathcal{G})$ that accomplish this task, with the set F being $F := \{(-1, 0), (0, -1)\}$. If we were able to find such map $\mathcal{P}$, then condition ($\clubsuit$) will immediately be satisfied.

Suppose for a while that such a map $\mathcal{P}$ exists. Then, for every $\lambda, \mu \geq 0$ there will exist Nash half-branch curve germs $\alpha_\lambda(s), \beta_\mu(s)$ which cannot be extended to 0 and such that:

$$\lim_{s\to 0} P(\alpha_\lambda(s)) = (\lambda^2, 0) \qquad \text{and} \qquad \lim_{s\to 0} P(\beta_\mu(s)) = (0, \mu^2).$$

We can try parameterizations of the form:

$$\alpha_\lambda(s) := \left(s^{n_\lambda}, \frac{a_{\lambda 0} + a_{\lambda 1}s + \cdots}{s^{m_\lambda}}\right) \qquad \text{and} \qquad \beta_\mu(s) := \left(\frac{b_{\mu 0} + b_{\mu 1}s + \cdots}{s^{\ell_\mu}}, s^{k_\mu}\right).$$

Then $a_{\lambda 0}, b_{\mu 0}$ must be constants (except maybe for finitely many values of λ and μ). This leads us to choose curves of the type:

$$\alpha_\lambda(s) := \left(s^{n_\lambda}, \frac{1 + a_{\lambda 1}s + \cdots}{s^{m_\lambda}}\right) \qquad \text{and} \qquad \beta_\mu(s) := \left(\frac{1 + b_{\mu 1}s + \cdots}{s^{\ell_\mu}}, s^{k_\mu}\right),$$

and among them we make the simplest choice:

$$\alpha_\lambda(s) := \left(s, \frac{1 + a_\lambda s}{s}\right) \qquad \text{and} \qquad \beta_\mu(s) := \left(\frac{1 + b_\mu s}{s}, s^3\right).$$

The following pair of polynomials:

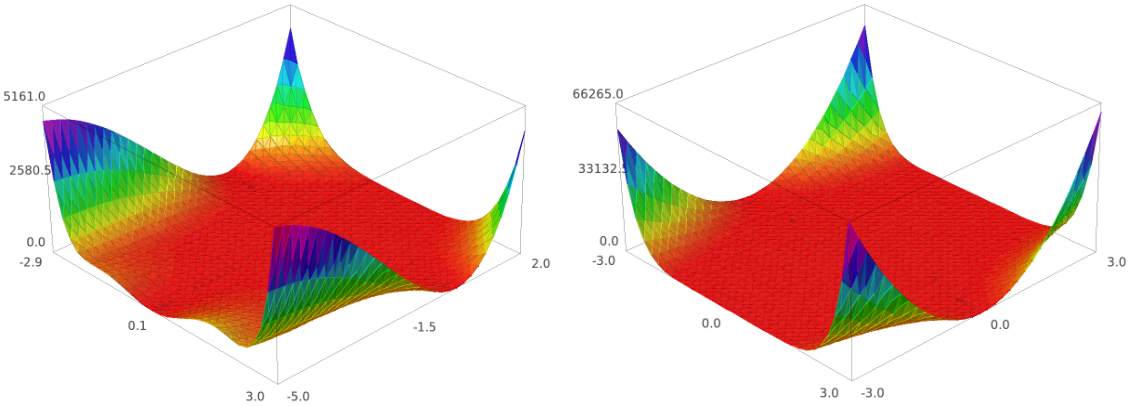
$$\boxed{\begin{aligned} \mathcal{F}(x, y) &:= (1 - x^3y + y - xy^2)^2 + (x^2y)^2 &&= \mathcal{F}_1^2 + \mathcal{F}_2^2 \\ \mathcal{G}(x, y) &:= (1 - xy + x - x^4y)^2 + (x^2y)^2 &&= \mathcal{G}_1^2 + \mathcal{G}_2^2 \end{aligned}}$$

enjoy a nice behavior along these curves, as figure 2.2 show. Next, notice that

(a) · $\mathcal{F}_1 \circ \alpha_\lambda = 1 - a_\lambda - a_\lambda^2 s - s^2 - a_\lambda s^3 \in \mathbb{R}[s, a_\lambda]$, $\mathcal{F}_1 \circ \alpha_\lambda(0) = 1 - a_\lambda$.
· $\mathcal{F}_1 \circ \beta_\mu = -3\,b_\mu s - 3\,b_\mu^2 s^2 - \big(b_\mu^3 - 1\big)s^3 - s^5 - b_\mu s^6 \in \mathbb{R}[s, b_\mu]$, $\mathcal{F}_1 \circ \beta_\mu(0) = 0$.

(b) · $\mathcal{G}_1 \circ \alpha_\lambda = (1 - a_\lambda)s - s^3 - a_\lambda s^4 \in \mathbb{R}[s, a_\lambda]$, $\mathcal{G}_1 \circ \alpha_\lambda(0) = 0$.
· $\mathcal{G}_1 \circ \beta_\mu = 1 - 3\,b_\mu - 6\,b_\mu^2 s - \big(4\,b_\mu^3 + 1\big)s^2 - \big(b_\mu^4 + b_\mu\big)s^3 \in \mathbb{R}[s, b_\mu]$, and $\mathcal{G}_1 \circ \beta_\mu(0) = 1 - 3b_\mu$.

(c) · $\mathcal{F}_2 \circ \alpha_\lambda = s + a_\lambda s^2 = \mathcal{G}_2 \circ \alpha_\lambda \in \mathbb{R}[s, a_\lambda]$.
· $\mathcal{F}_2 \circ \beta_\mu = s + 2\,b_\mu s^2 + b_\mu^2 s^3 = \mathcal{G}_2 \circ \beta_\mu \in \mathbb{R}[s, b_\mu]$.
· $\mathcal{F}_2 \circ \alpha_\lambda(0) = \mathcal{G}_2 \circ \alpha_\lambda(0) = \mathcal{F}_2 \circ \beta_\mu(0) = \mathcal{G}_2 \circ \beta_\mu(0) = 0$.



(a) $\mathcal{F}(x,y) := (1 - x^3y + y - xy^2)^2 + (x^2y)^2$. (b) $\mathcal{G}(x,y) = (1 - xy + x - x^4y)^2 + (x^2y)^2$.

Figure 2.2: Plots of the polynomials $\mathcal{F}, \mathcal{G} : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

All of these map compositions were computed by Sage. Thus, we get the following properties:

(i) The polynomials $\mathcal{F}, \mathcal{G}$ are non-negative in $\mathbb{R}^2$.

(ii) $\cdot\ \mathcal{F}^{-1}(0) = \mathcal{F}_1^{-1}(0) \cap \mathcal{F}_2^{-1}(0) = \{(0,-1)\} \xmapsto{\mathcal{P}} \{(0,1)\}$.

$\cdot\ \mathcal{G}^{-1}(0) = \mathcal{G}_1^{-1}(0) \cap \mathcal{G}_2^{-1}(0) \ = \{(-1,0)\} \xmapsto{\mathcal{P}} \{(1,0)\}$.

(iii) $\cdot\ \mathcal{P} \circ \alpha_\lambda = (\mathcal{F} \circ \alpha_\lambda, \mathcal{G} \circ \alpha_\lambda) =$

$$\begin{aligned}
&\big(\, a_\lambda^2 - 2\,a_\lambda + 1 + 2\,(a_\lambda^3 - a_\lambda^2)s + (a_\lambda^4 + 2\,a_\lambda - 1)s^2 + \, a_\lambda^2 s^3 + \\
&\quad (2\,a_\lambda^3 + a_\lambda^2 + 1)s^4 + 2\,a_\lambda s^5 + a_\lambda^2 s^6, \\
&\ (a_\lambda^2 - 2\ a_\lambda + 2)s^2 + 2\,a_\lambda s^3 + (a_\lambda^2 + 2\,a_\lambda - 2)s^4 + \\
&\quad 2\,(a_\lambda^2 - a_\lambda)s^5 + s^6 + 2\,a_\lambda s^7 + a_\lambda^2 s^8 \,\big)\,.
\end{aligned}$$

$\cdot\ \mathcal{P} \circ \beta_\mu = (\mathcal{F} \circ \beta_\mu, \mathcal{G} \circ \beta_\mu) =$

$$\begin{aligned}
&\big(\,(9\,b_\mu^2 + 1)s^2 + 2\,(9\,b_\mu^3 + 2\,b_\mu)s^3 + 3\,(5\,b_\mu^4 + 2\,b_\mu^2 - 2\,b_\mu)s^4 + \\
&\quad 2\,(3\,b_\mu^5 + 2\,b_\mu^3 - 3\,b_\mu^2)s^5 + (b_\mu^6 + b_\mu^4 - 2\,b_\mu^3 + 6\,b_\mu + 1)s^6 + \\
&\quad 12\,b_\mu^2 s^7 + 2\,(4\,b_\mu^3 - 1)s^8 + 2\,(b_\mu^4 - b_\mu)s^9 + s^{10} + 2\,b_\mu s^{11} + b_\mu^2 s^{12}, \\
&\ 9\,b_\mu^2 - 6\,b_\mu + 1 + 12\,(3\,b_\mu^3 - b_\mu^2)s + (60\,b_\mu^4 - 8\,b_\mu^3 + 6\,b_\mu - 1)s^2 + \\
&\quad 2\,(27\,b_\mu^5 - b_\mu^4 + 9\,b_\mu^2 + b_\mu)s^3 + (28\,b_\mu^6 + 20\,b_\mu^3 + 6\,b_\mu^2 + 1)s^4 + \\
&\quad 2\,(4\,b_\mu^7 + 5\,b_\mu^4 + 2\,b_\mu^3 + b_\mu)s^5 + (b_\mu^8 + 2\,b_\mu^5 + b_\mu^4 + b_\mu^2)s^6 \,\big)\,.
\end{aligned}$$

The polynomials $\mathcal{F} \circ \alpha_\lambda,\ \mathcal{G} \circ \alpha_\lambda \in \mathbb{R}[s, a_\lambda]$ and $\mathcal{F} \circ \beta_\mu,\ \mathcal{G} \circ \beta_\mu \in \mathbb{R}[s, b_\mu]$ were computed with Sage. As we anticipated before, by condition (ii) the set $F = \{(-1,0),(0,-1)\}$.

2.2 The first proof

[Proof of theorem 1.11] We are going to prove that $\mathcal{Q} \subset \mathcal{P}(\mathbb{R}^2)$. To do this it is enough to fix $v > 0$ and see that the image under $\mathcal{F}$ of the curve $\{\mathcal{G} = v\}$ contains the open half-line

$(0, +\infty)$.

$\boxed{\textbf{Step 1}}$ *Parametrization of the curve* $\{\mathcal{G} - v = 0\}$.

We start by solving the equation $\mathcal{G} - v = 0$, that is,

$$(1 - xy + x - x^4 y)^2 + (x^2 y)^2 - v = 0$$

As it has degree 2 with respect to y, we can compute its roots $y^+(x, v)$ and $y^-(x, v)$ given by:

$$y^+(x,v) := \frac{1 + x + x^3 + x^4 + \sqrt{\Delta(x,v)}}{x(x^2 + (x^3+1)^2)}$$
$$y^-(x,v) := \frac{1 + x + x^3 + x^4 - \sqrt{\Delta(x,v)}}{x(x^2 + (x^3+1)^2)}$$

where $\Delta(x, v) = \Delta_v(x) := v(x^2 + (x^3 + 1)^2) - x^2(x + 1)^2$, $\deg_x(\Delta) = 6$. We can see on figure 2.3 how y^+ and y^- look like for instance for $v := 0.8$. As we can see on figure 2.4, for $v := 1$ there are no singularities on y^- because $\lim_{x \to 0} y^-(x, 1) = 1$. This observation is used later, in Step 2.

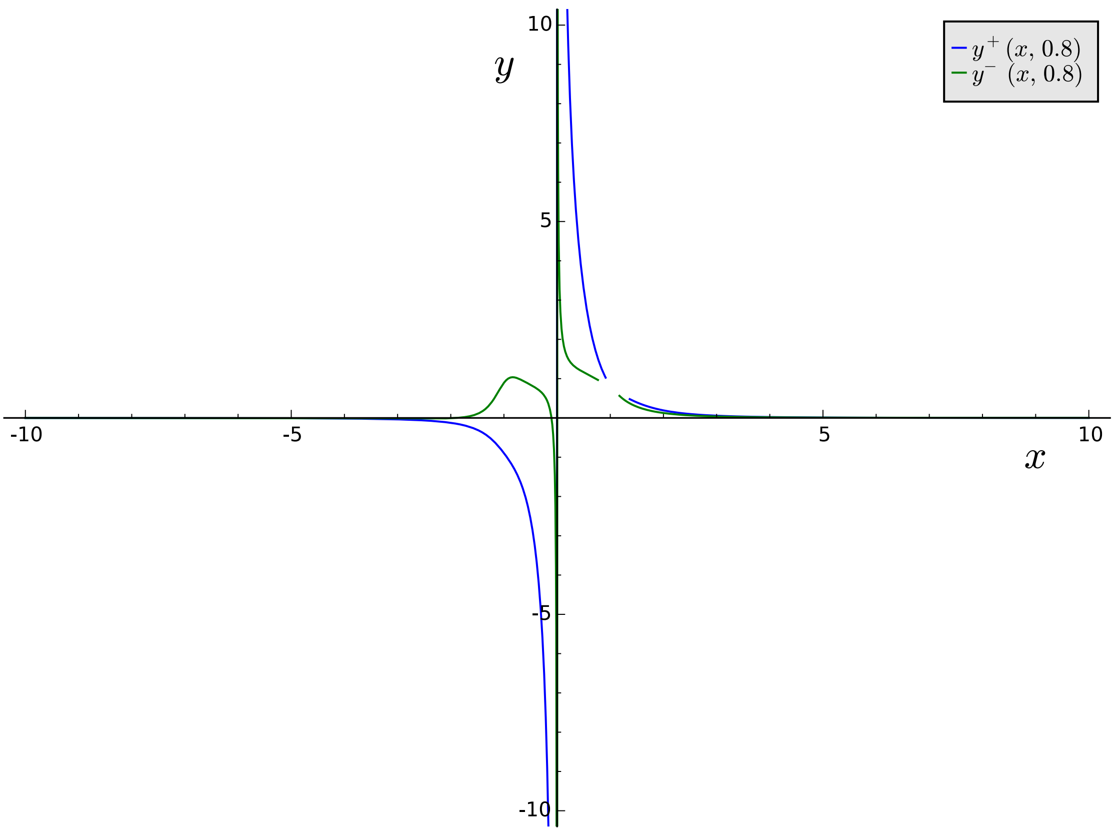


Figure 2.3: $y^+(x, v)$ and $y^-(x, v)$ for $v := 0.8$.

The common domain of these two functions is the set

$$D_v := \{x \in R : \Delta(x, v) \geq 0, x \neq 0\}.$$

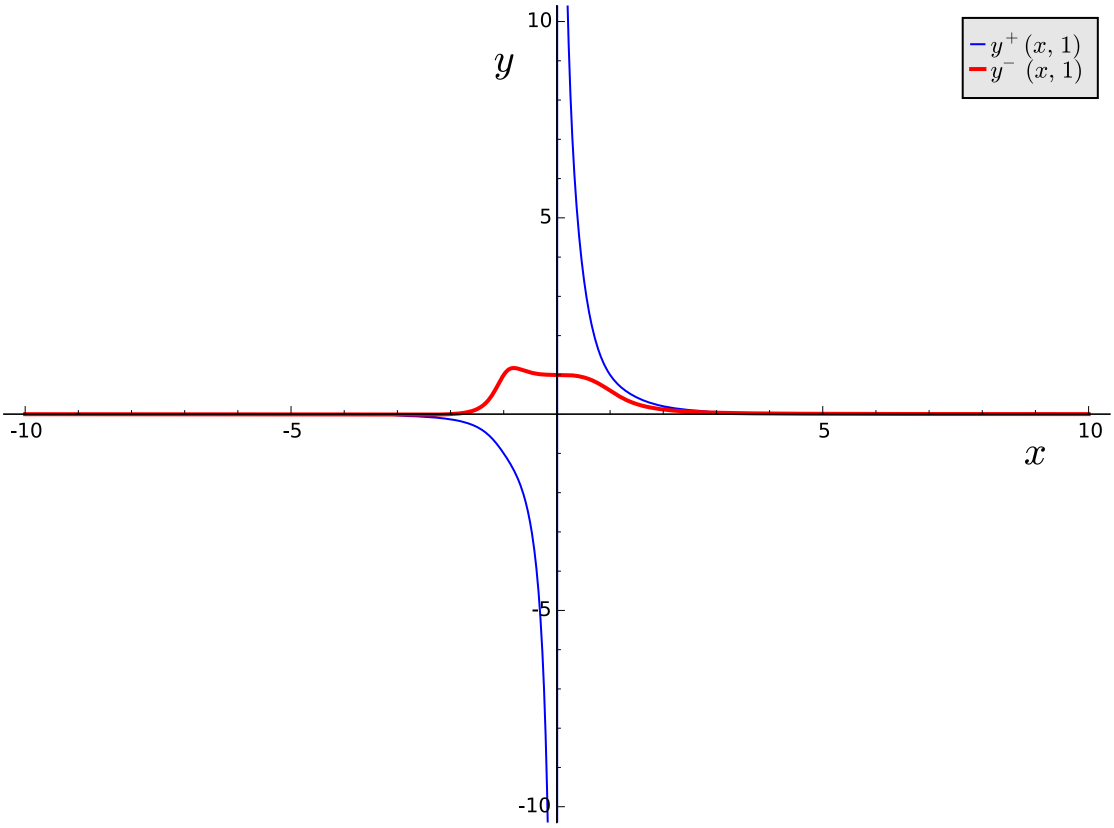


Figure 2.4: $y^+(x,v)$ and $y^-(x,v)$ for $v := 1$.

Notice that the only real root of the denominator is $x_0 := 0$[1] Let

$$\gamma_v^+ : D_v \to \mathbb{R},\ x \mapsto \mathcal{F}(x, y^+(x,v)) \quad \text{and} \quad \gamma_v^- : D_v \to \mathbb{R},\ x \mapsto \mathcal{F}(x, y^-(x,v))$$

Notice that $\mathcal{F}(\{\mathcal{G} = v\}) = \operatorname{im}(\gamma_v^+) \cup \operatorname{im}(\gamma_v^-)$, so our aim is to prove that $(0,+\infty) \subset \operatorname{im}(\gamma_v^+) \cup \operatorname{im}(\gamma_v^-)$.

Step 2 *Main properties of γ_v^+ and γ_v^-.*

In this section we are going to prove that:

$$\text{(i)} \lim_{x\to\pm\infty} \gamma_v^+(x) = \lim_{x\to\pm\infty} \gamma_v^-(x) = 0.$$

$$\text{(ii)} \lim_{x\to 0} \gamma_v^+(x) = +\infty, \quad \lim_{x\to 0} \gamma_v^-(x) = \begin{cases} +\infty & \text{for } v \neq 1 \\ 4 & \text{for } v = 1 \end{cases}$$

Using Sage we can symbolically check how γ_v^+ and γ_v^- look like, getting polynomials $A_1, A_2, B_1, B_2 \in \mathbb{R}[x,v]$ and $C \in \mathbb{R}[x]$ such that:

$$\text{(a) } \gamma_v^+(x) = \frac{A_1(x,v) + B_1(x,v)\sqrt{\Delta(x,v)}}{C(x)}, \quad \gamma_v^-(x) = \frac{A_2(x,v) + B_2(x,v)\sqrt{\Delta(x,v)}}{C(x)}$$

[1] We checked with Laguerre's method, implemented with Python 2.7, that the polynomial $x^7+2x^4+x^3+x$ has 6 complex roots.

$$\text{(b)} \quad \begin{aligned} A_1(x,v) &= A_2(x,v)\,, & \deg_x(A_1) = \deg_x(A_2) = 24 \\ B_1(x,v) &= -B_2(x,v)\,, & \deg_x(B_1) = \deg_x(B_2) = 21 \\ C(x) &= x^2(x^2+(x^3+1)^2)^4\,, & \deg_x(C) = 26\,. \end{aligned}$$

We proceed to study γ_v^+ and γ_v^- at the origin. Since Δ has even degree and positive leading coefficient with respect to x, it is positive for $|x|$ large enough, so (i) holds.

Now, for $x = 0$, we get $\Delta(0,v) = v > 0$, so $0 \in \overline{D_v}$. Also:

- ⋆ $A_1(0,v) + B_1(0,v)\sqrt{\Delta(0,v)} = v(1+\sqrt{v})^2 > 0$.
- ⋆ $A_2(0,v) + B_2(0,v)\sqrt{\Delta(0,v)} = v(1-\sqrt{v})^2 \geq 0$, and equality holds if and only if $v = 1$.

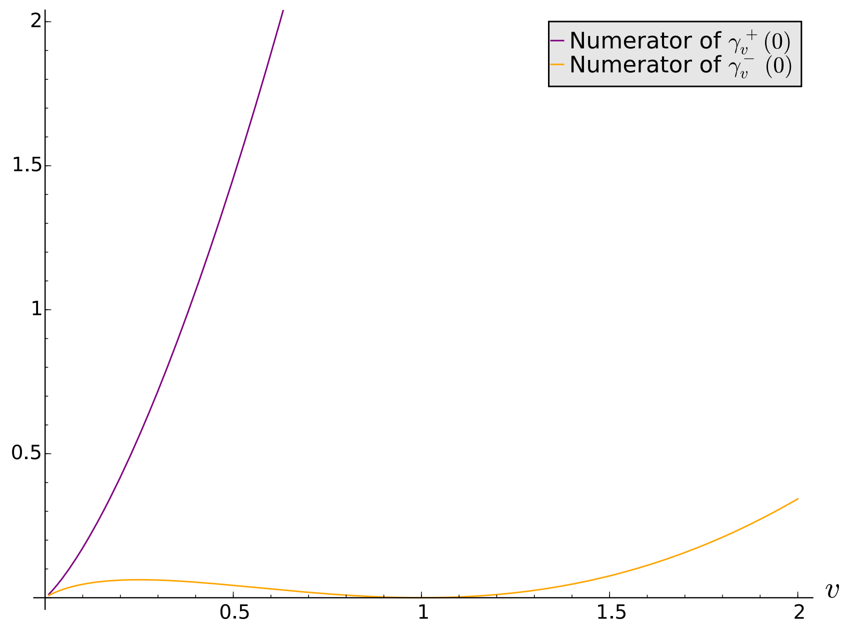


Figure 2.5: Numerators of γ_v^+ and γ_v^- for $x := 0$.

Thus, (ii) holds (we also checked it with Sage). The result for $v := 1$ in (ii) is not relevant here (see figure 2.6).

Step 3 *When* $v \geq 0.28^2$ *we have* $(0,+\infty) \subset \mathrm{im}(\gamma_v^+)$.

In order to see whether $(0,+\infty) \subset \mathrm{im}(\gamma_v^+) \cup \mathrm{im}(\gamma_v^-)$ or not we are now going to study the domain D_v. To that end we need to study when $\Delta(x,v) = 0$, so it seems convenient to define:

$$v(x) := \frac{x^2(x+1)^2}{x^2+(x^3+1)^2},$$

whose graph can be seen in figure 2.7. If $x \in (-\infty, 0)$ we checked using Laguerre's method that the polynomial

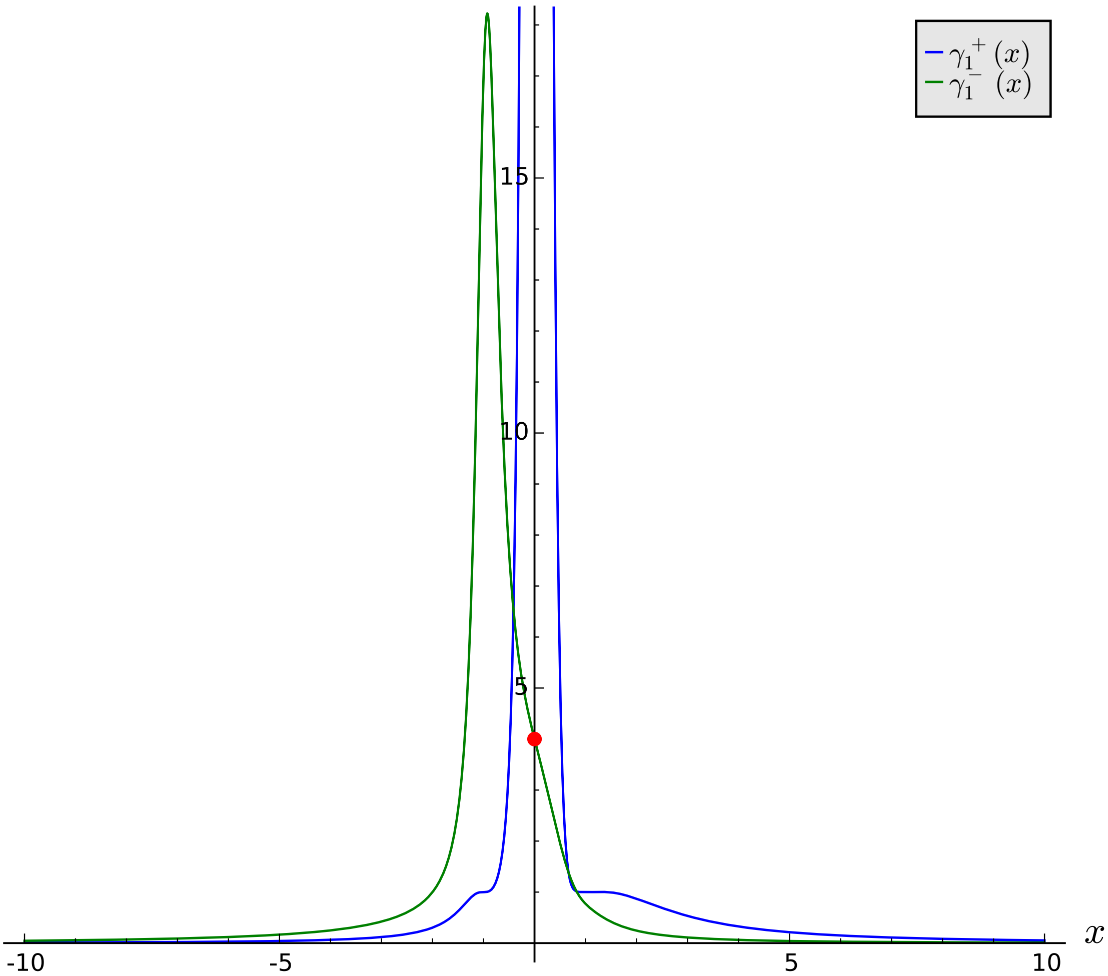


Figure 2.6: Notice the value of $\gamma_1^-(x)$ at $x = 0$.

$$\Delta(x, 0.28^2) = 0.0784\, x^6 - x^4 - 1.8432\, x^3 - 0.9216\, x^2 + 0.0784$$

has 4 complex roots and 2 real ones[2], with the real ones being $\delta_0 \approx 0.236$ and $\delta_1 \approx 4.336$. Thus $\Delta(x, v)$ has no negative roots when $v \geq 0.28^2$ and, in addition, it is positive. Therefore $(-\infty, 0) \subset D_v$. But then, as γ_v^+ is continuous and recalling the limits computed in Step 2, we get

$$(0, +\infty) \subset \mathrm{im}(\gamma_v^+) \subset \mathrm{im}(\gamma_v^+) \cup \mathrm{im}(\gamma_v^-).$$

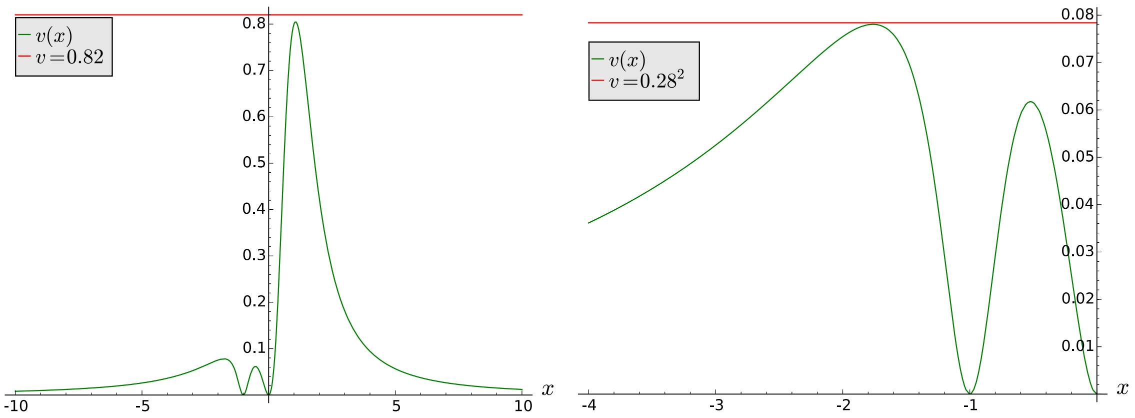
$\boxed{\textbf{Step 4}}$ *When* $0 < v < 0.28^2$ *we have* $(0, +\infty) \subset \mathrm{im}(\gamma_v^-)$.

To prove that for $0 < v < 0.28^2$ the inclusion $(0, +\infty) \subset \mathrm{im}(\gamma_v^-)$ holds it is enough to prove the existence of $N_v, \delta_v \in \mathbb{R}$ satisfying

$$\boxed{N_v < \delta_v, \quad (-\infty, N_v] \cup [\delta_v, +\infty) \subset D_v \text{ and } \gamma_v^-(N_v) > \gamma_v^+(\delta_v)} \qquad (\spadesuit)$$

See figure 2.8 to get an idea of what we are saying here.

[2] The value $v_0 := 0.28^2$ comes from a careful observation of the plot from figure 2.7b.



(a) Plot of $v(x)$.

(b) Detail of the function when $x \in (-4, 0)$.

Figure 2.7: Plot of the univariate function $v(x)$.

To prove the existence of such N_v and δ_v, we must compute the roots of $\Delta_v(x)$ in an algebraic closure of the field of rational functions $\mathbb{R}(v)$. Such an algebraic closure is the field of Puiseux series $\mathbb{C}\left(\{v^*\}\right)$, see A.9. These roots are power series in $\mathbb{C}\left(\{w^*\}\right)$ with $w := \sqrt{v}$, and we consider the largest and the smallest negative roots $\eta_v, \xi_v \in \mathbb{R}\left(\{v^*\}\right)$ of Δ_v with respect to the unique ordering in $\mathbb{R}\left(\{v^*\}\right)$ that makes v positive and infinitesimal with respect to $\mathbb{R}$. These roots are:

$$\begin{cases} \eta_v := -\dfrac{1}{w} + 1 + w + w^2 + \dfrac{5}{2}w^3 + \cdots \\ \xi_v := -w - w^2 - \dfrac{5}{2}w^3 - 6w^4 + \cdots \end{cases}$$

Notice that, by the definition of the ordering in $\mathbb{R}\left(\{v^*\}\right)$, the first coefficient of a series is the "most meaningful orderwise". In particular $\eta_v < \xi_v$. To perform calculations, we handle suitable truncations of the involved series. Here the word suitable means "as short as possible but order preserving"; in other words, we look for N_v and δ_v with $N_v < \eta_v < \xi_v < \delta_v$, and in fact we choose

$$\begin{cases} N_v := -\dfrac{1}{w} + 1 + w + w^2 = \eta_v - \left(\dfrac{5}{2}w^3 + \cdots\right) < \eta_v \\ \delta_v := -w - w^2 - \dfrac{5}{2}w^3 = \xi_v - \left(-6w^4 + \cdots\right) > \xi_v \end{cases}$$

We checked with Sage that $-\infty < N_v < \delta_v < 0$ for $v \in (0, 0.28^2)$ that is, for $w \in (0, 0.28)$, see fig. 2.9a. Now we can focus on proving (♠). Since $\Delta(N_{w^2}, w^2)$ and $\Delta(\delta_{w^2}, w^2)$ are positive (see fig. 2.9b) for $w \in (0, 0.28)$, we get that $N_v, \delta_v \in D_v$. For the first part, let

$$D := \bigcup_{v>0} D_v = \bigcup_{v>0} \{x \in \mathbb{R} : \Delta(x, v) \geq 0,\ x \neq 0\},$$

whose boundary is the union of the axis $\{x = 0\}$ and the curve given by the equation

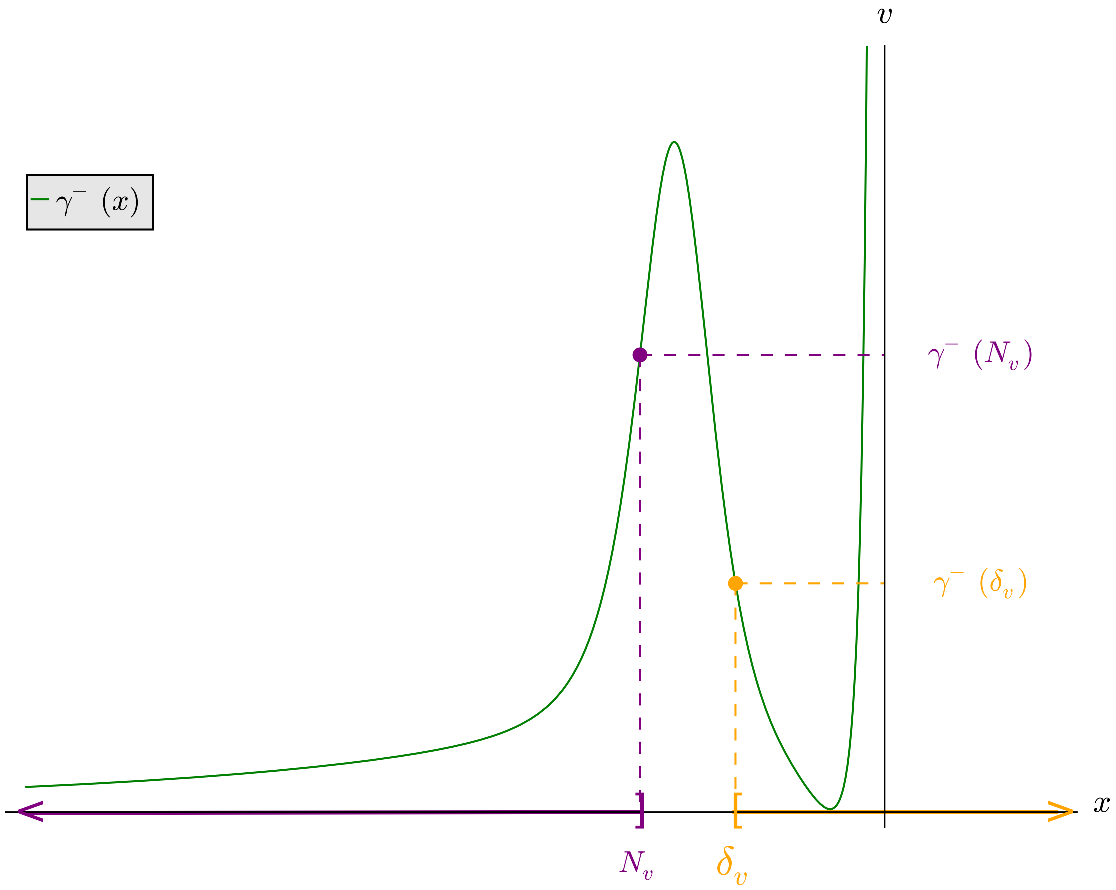


Figure 2.8: Idea of what we are saying with (♠). Here $v := 0.1$.

$\Delta(x, v) = 0$, that is, the graph of the regular function

$$v(x) = \frac{x^2(x+1)^2}{x^2 + (x^3+1)^2}.$$

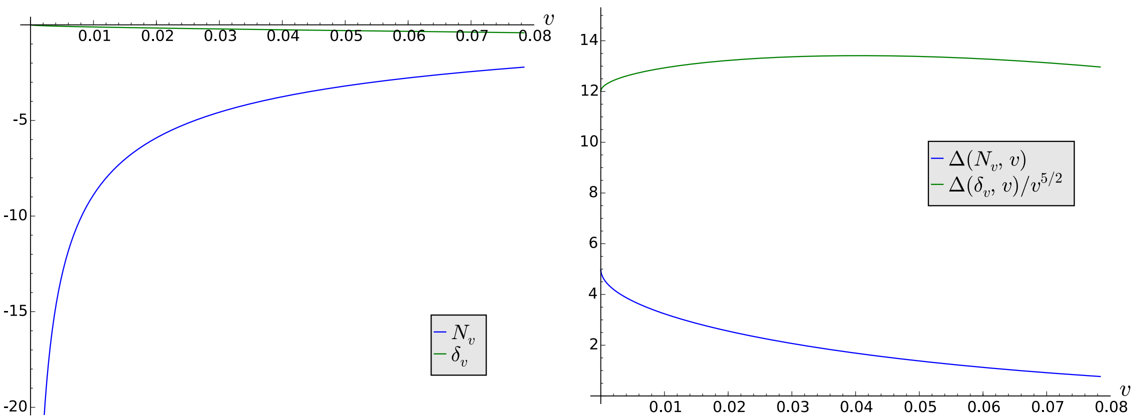
This graph is above the axis $\{v = 0\}$. Then, $(-\infty, N_v]$ and $[\delta_v, 0)$ are contained in the interior of D_v for $v \in (0, 0.28^2)$, because the curves

$$\{(\delta_v, v) : 0 < v < 0.28^2\} \quad \text{and} \quad \{(N_v, v) : 0 < v < 0.28^2\}$$

are contained in D, they are graphs above the vertical axis $\{x = 0\}$, and $\delta_v < \xi_v$ and $N_v < \eta_v$ as we saw before. Look at figure 2.10.

So the only thing left to do is checking that $\gamma_v^-(N_v) > \gamma_v^-(\delta_v)$. Recall that

$$\gamma_v^-(x) = \frac{A_2(x, v) + B_2(x, v)\sqrt{\Delta(x, v)}}{C(x)},$$



(a) $-\infty < N_v < \delta_v < 0$. (b) $\Delta(N_v, v), \Delta(\delta_v, v) > 0$.

Figure 2.9: Plots of $N_v, \delta_v < 0$ and $\Delta(N_v, v),\ \Delta(\delta_v, v) > 0$ for $v \in (0, 0.28^2)$.

with $\deg_x(A_2) = 24, \deg_x(B_2) = 21, \deg_x(\Delta) = 6$ and $\deg_x(C) = 26$. Consider:

- $f_1(w) = A_2(N_{w^2}, w^2) \cdot w^{24}$
- $g_1(w) = B_2(N_{w^2}, w^2) \cdot w^{21}$
- $q_1(w) = \Delta(N_{w^2}, w^2)$
- $h_1(w) = C(N_{w^2}) \cdot w^{26}$
- $f_2(w) = A_2(\delta_{w^2}, w^2)$
- $g_2(w) = B_2(\delta_{w^2}, w^2)$
- $q_2(w) = \Delta(\delta_{w^2}, w^2)$
- $h_2(w) = C(\delta_{w^2})$.

Thus, we need to prove that for $w \in (0, 0.28)$:

$$\frac{f_1 \cdot (w^{24})^{-1} + g_1 \cdot (w^{21})^{-1}\sqrt{q_1}}{h_1 \cdot (w^{26})^{-1}} > \frac{f_2 + g_2\sqrt{q_2}}{h_2} \iff$$

$$\frac{w^2 h_2 f_1 - f_2 h_1}{h_1 h_2} + \frac{w^5 g_1 \sqrt{q_1}}{h_1} - \frac{g_2\sqrt{q_2}}{h_2} > 0,$$

and we are going to prove that

$$\Lambda_1 := \frac{w^2 h_2 f_1 - f_2 h_1}{h_1 h_2}, \quad \Lambda_2 := \frac{w^5 g_1\sqrt{q_1}}{h_1} \text{ and } \Lambda_3 := -\frac{g_2\sqrt{q_2}}{h_2}$$

are positive in the given interval, which only contains positive values. Since q_1, q_2 are positive, we can clear away w^5 and $\sqrt{q_1}$ from Λ_2, $\sqrt{q_2}$ from Λ_3. Furthermore, $C(x) = x^2(x^2 + (x^3+1)^2)^4 > 0$, so we can also remove h_1 and h_2 from $\Lambda_1, \Lambda_2, \Lambda_3$. Thus it suffices to see that

$$L := \frac{w^2 h_2 f_1 - f_2 h_1}{w^4}, \quad g_1, \quad K := -\frac{g_2}{w^3}$$

are positive for $w \in (0, 0.28)$.

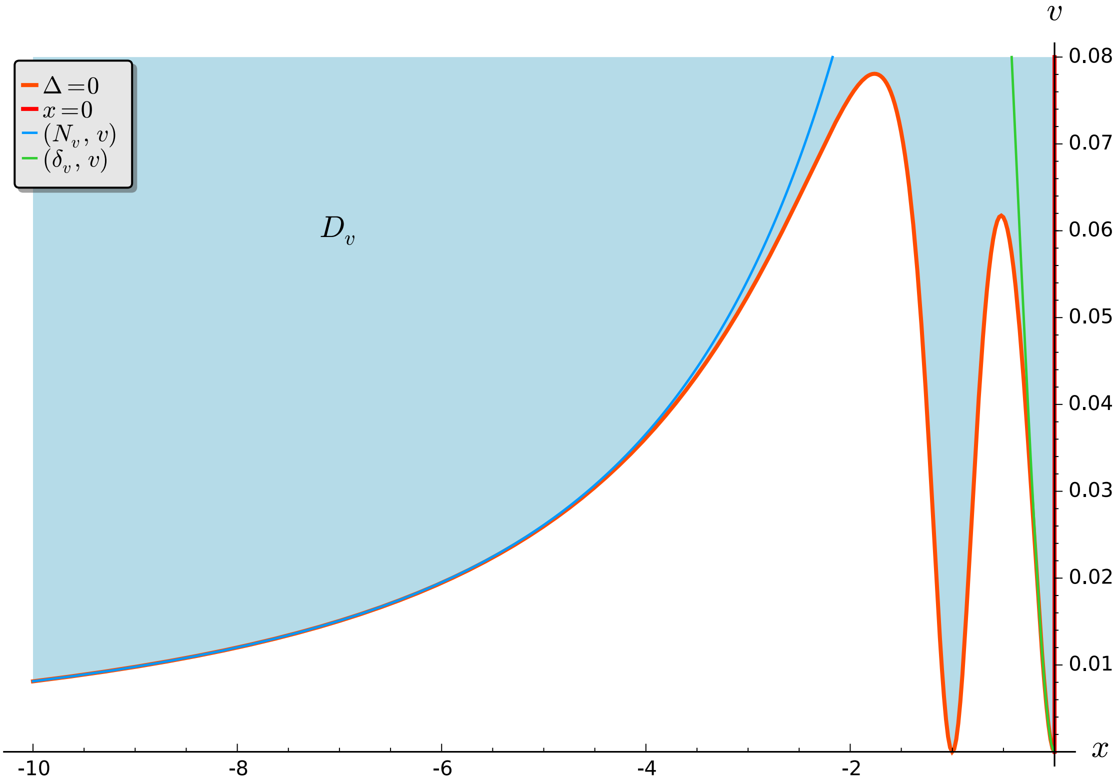
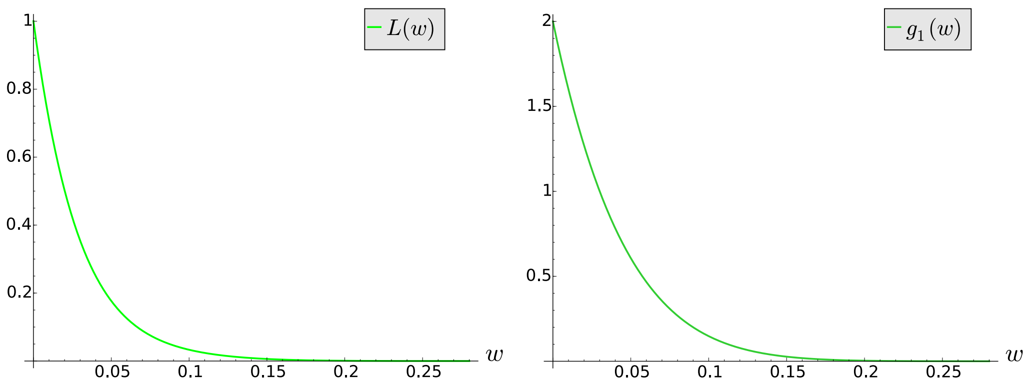
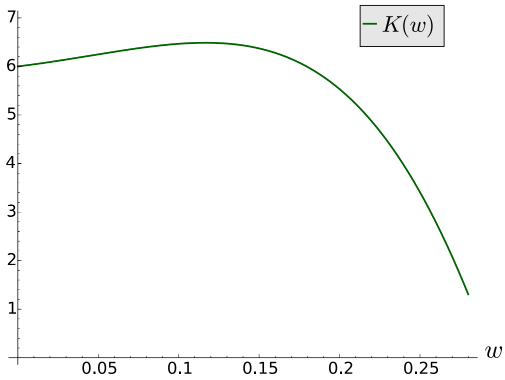


Figure 2.10: Plot of $\{(N_v, v)\}, \{(\delta_v, v)\} \subset D_v$, for $0 < v < 0.28^2$.

As we see in the figures they indeed are. This has been also checked with Sturm algorithm and numerically with Sage. Thus, (♠) holds and the result is proved.




K(w)
w
7
6
5
4
3
2
1
0.05
0.1
0.15
0.2
0.25

□

3

A short proof for the open quadrant $\mathcal{Q}$ problem

3.1 A new approach

In the previous chapter, we proved that $\mathcal{Q}$ is a polynomial image of $\mathbb{R}^2$ by the polynomial map $\mathcal{P} = (\mathcal{F}, \mathcal{G})$. This fact alongside theorem 1.9 were key for proving theorem 1.10. To achieve the result 1.11, we needed the aid of a computer in order to check that certain polynomials didn't have roots on particular intervals or that they were positive on them. Although this procedure is legit, the authors kept on working on the problem of characterizing which semialgebraic subsets $\mathcal{S} \subset \mathbb{R}^m$ are polynomial images of $\mathbb{R}^n$ and wrote a new paper ([FU]) with a much simpler proof.

In this second chapter we present a new approach to the open quadrant problem. We will show that $\mathcal{Q}$ is the image of the composition of three simple polynomial maps: $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. The proof of theorem 1.11 will be conducted by inspecting at the images of the aforementioned polynomials, albeit the lack of precise tools to determine the image of a polynomial map.

3.1.1 The new polynomial maps

In this subsection we introduce the new polynomial maps that meet the requirements to prove theorem 1.11 in a different fashion. To be more precise, we define

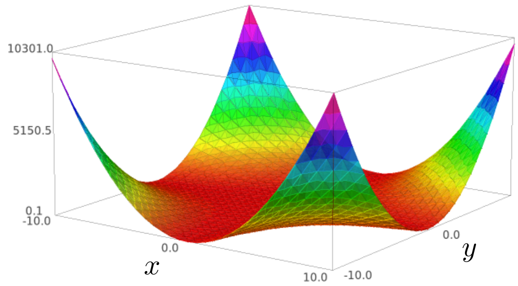
$$f := \mathcal{H} \circ \mathcal{G} \circ \mathcal{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

where

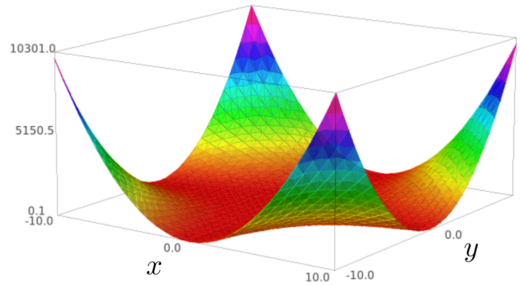
$$\boxed{\begin{array}{l} \mathcal{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ (x,y) \longmapsto ((xy-1)^2 + x^2,\ (xy-1)^2 + y^2), \\ \mathcal{G} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ (x,y) \longmapsto (x,\ y(xy-2)^2 + x(xy-1)^2), \\ \mathcal{H} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ (x,y) \longmapsto (x(xy-2)^2 + \frac{1}{2}xy^2,\ y). \end{array}}$$

We can see how these maps look like in figure 3.1, and appreciate the symmetry between the x and y variables while the transformation is performed.

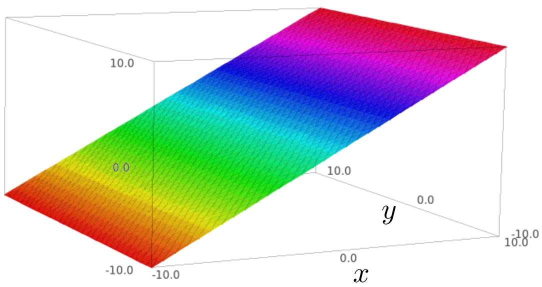
In the next section we proceed to develop the proof, which is split into three different lemmas.



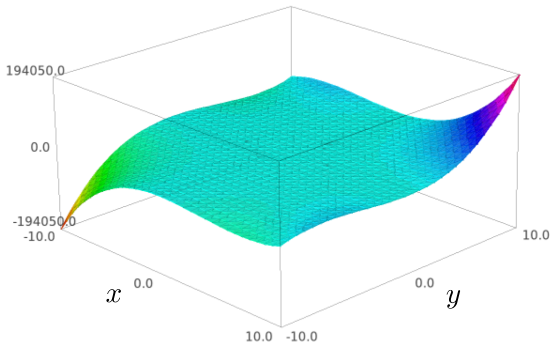
(a) $\mathcal{F}_1(x, y) = (xy - 1)^2 + x^2$.



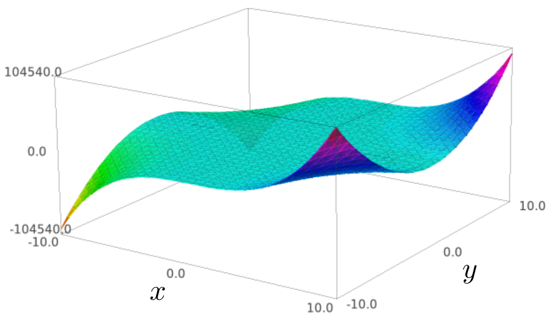
(b) $\mathcal{F}_2(x, y) = (xy - 1)^2 + y^2$.



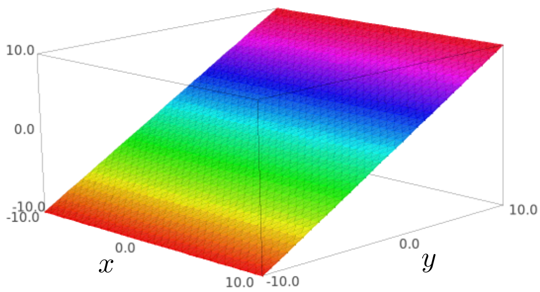
(c) $\mathcal{G}_1(x, y) = x$.



(d) $\mathcal{G}_2(x, y) = y(xy - 2)^2 + x(xy - 1)^2$.



(e) $\mathcal{H}_1(x, y) = x(xy - 2)^2 + \frac{1}{2}xy^2$.



(f) $\mathcal{H}_2(x, y) = y$.

Figure 3.1: The polynomial maps from the second proof: $\mathcal{H} \circ \mathcal{G} \circ \mathcal{F}(\mathbb{R}^2) = \mathcal{Q}$.

3.2 The new proof

As we have anticipated before, $\mathcal{Q}$ is the image of the composition $\mathcal{H} \circ \mathcal{G} \circ \mathcal{F}$, but before proving theorem 1.11 we need three auxiliar lemmas in order to shed some light on what properties the images of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ have.

3.2.1 The first lemma

Lemma 3.1. Let $\mathcal{A} := \{xy \geq 1\} \cap \mathcal{Q}$. Then the image of

$$\mathcal{F} := (\mathcal{F}_1, \mathcal{F}_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, (x, y) \longmapsto ((xy-1)^2 + x^2,\ (xy-1)^2 + y^2)$$

satisfies that $\mathcal{A} \subset \mathcal{F}(\mathbb{R}^2) \subset \mathcal{Q}$.

Proof. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are clearly positive on $\mathbb{R}^2$, the inclusion $\mathcal{F}(\mathbb{R}^2) \subset \mathcal{Q}$ is trivial. To prove the other inclusion, we show that for $(a, b) \in \mathcal{A}$ the system of polynomial equations

$$\begin{cases} (xy-1)^2 + x^2 = a \\ (xy-1)^2 + y^2 = b \end{cases} \tag{3.1}$$

has a solution $(x_0, y_0) \in \mathbb{R}^2$. Set $z := xy - 1$ in order to rewrite the system 3.1 in terms of x and z. Since $y = \frac{z+1}{x}$ we get:

$$\begin{cases} z^2 + x^2 & = a \\ z^2 + \dfrac{(z+1)^2}{x^2} & = b. \end{cases} \tag{3.2}$$

By eliminating x on the system 3.2 we deduce that z must be a root of the polynomial

$$P(z) := z^4 - (a+b+1)z^2 - 2z + (ab-1) = 0.$$

Taking into account that $(a, b) \in \mathcal{A}$ satisfy that $a, b > 0$ and $ab \geq 1$, we notice that P is a monic polynomial of even degree satisfying

$$P(0) = ab - 1 \geq 0 \quad \text{and} \quad P(\sqrt{a}) = -2\sqrt{a} - a - 1 < 0.$$

Thus, P has a real root z_0 such that $0 \leq z_0 < \sqrt{a}$, so we set:

$$x_0 := \sqrt{a - z_0^2} \quad \text{and} \quad y_0 := \frac{z_0 + 1}{x_0}.$$

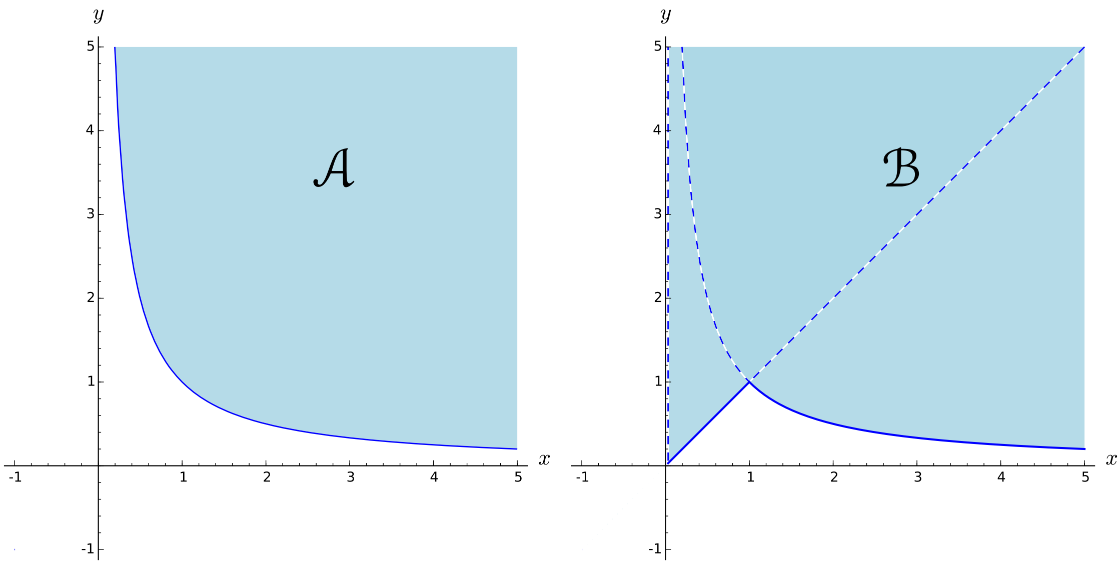
Then, we have $F(x_0, y_0) = (a, b)$ and $\mathcal{A} \subset \mathcal{F}(\mathbb{R}^2)$, as required. □

3.2.2 The second lemma

Lemma 3.2. Let $\mathcal{B} := \mathcal{A} \cup \{y \geq x > 0\}$. Then the image of

$$\mathcal{G} := (\mathcal{G}_1, \mathcal{G}_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, (x, y) \longmapsto (x,\ y(xy-2)^2 + x(xy-1)^2)$$

satisfies that $\mathcal{B} \subset \mathcal{G}(\mathcal{A}) \subset \mathcal{G}(\mathcal{Q}) \subset \mathcal{Q}$.



(a) $\mathcal{A} = \{xy \geq 1\} \cap \mathcal{Q}$.

(b) $\mathcal{B} = \mathcal{A} \cup \{y \geq x > 0\}$.

Figure 3.2: Relevant sets for lemmas 3.1, 3.2 and 3.3.

Proof. The inclusion $\mathcal{G}(\mathcal{A}) \subset \mathcal{G}(\mathcal{Q})$ is trivial since $\mathcal{A} \subset \mathcal{Q}$. The last inclusion $\mathcal{G}(\mathcal{Q}) \subset \mathcal{Q}$ also holds because $\mathcal{G}$ is strictly positive on $\mathcal{Q}$: if $x_0, y_0 > 0$ then $\mathcal{G}_1(x_0, y_0) = x_0 > 0$ and $\mathcal{G}_2(x_0, y_0) = y_0(x_0y_0 - 2)^2 + x_0(x_0y_0 - 1)^2 > 0$. Notice that $\mathcal{G}_2(x_0, y_0) \neq 0$ because if not there would exists a solution to the system of equations

$$\begin{cases} xy - 1 = 0 \\ xy - 2 = 0, \end{cases}$$

which is not possible.

Now we can focus on proving $\mathcal{B} \subset \mathcal{G}(\mathcal{A})$. First of all, notice that we can express the set $\mathcal{B}$ in the following way:

$$\mathcal{B} = \bigsqcup_{x>0} (\{x\} \times [y_x, +\infty)) := \bigsqcup_{x>0} (\{x\} \times \mathcal{B}_x),$$

where $y_x := \min\{x, 1/x\}$. Intuitively, we are "slicing" the set $\mathcal{B}$ vertically, depending on x. Now look at the definition of $\mathcal{G}_2$ and consider for each $x > 0$ the polynomial map ϕ_x dependent on the variable $\mathtt{y}$:

$$\phi_x(\mathtt{y}) := \mathtt{y}(x\mathtt{y} - 2)^2 + x(x\mathtt{y} - 1)^2 = x^2\mathtt{y}^3 + (x^3 - 4x)\mathtt{y}^2 + (4 - 2x^2)\mathtt{y} + x.$$

The polynomials $\phi_x(\mathtt{y})$ have odd degree and positive leading coefficient since $x > 0$. Now notice that if we fix $x_0 > 0$ then we get the following sequence of inclusions:

$$\begin{array}{ccc} \phi_x([1/x_0, +\infty)) & \supset^1 & \phi_x([2/x_0, +\infty)) \\ \cup^2 & & \cup^3 \\ [1/x_0, +\infty) & & [x_0, +\infty) \end{array}$$

Inclusion [1] is followed from the fact that $[1/x_0, +\infty) \supset [2/x_0, +\infty)$ (notice that for $x > 0$ the graph of the map $2/x$ is "above" the one of $1/x$). Inclusion [2] and [3] are followed from the fact that $\phi_{x_0}(\mathtt{y})$ has positive leading coefficient and the computation of the images of $1/x_0$ and $2/x_0$ through $\phi_{x_0}(\mathtt{y})$:

$$\phi_x\left(\frac{1}{x_0}\right) = \frac{1}{x_0} + \left((x_0^3 - 4x_0)\frac{1}{x_0^2} + (4 - 2x_0^2)\frac{1}{x_0} + x_0\right) = \frac{1}{x_0}$$
$$\phi_x\left(\frac{2}{x_0}\right) = \left(\frac{8}{x_0} + (x_0^3 - 4x_0)\frac{4}{x_0^2} + (4 - 2x_0^2)\frac{2}{x_0}\right) + x_0 = x_0.$$

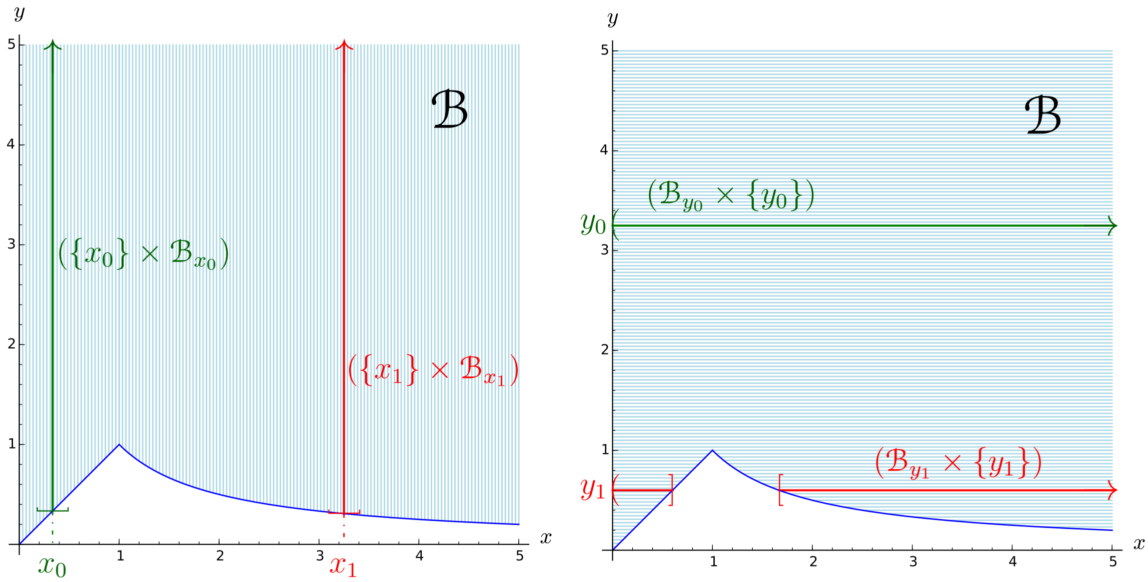
Now it is clear that:

$$\mathcal{B}_x = [y_x, +\infty) \subset \phi_x([1/x, +\infty)).$$

Then we can prove the desired inclusion in this way:

$$\mathcal{B} = \bigsqcup_{x>0} (\{x\} \times \mathcal{B}_x) \subset \bigsqcup_{x>0} (\{x\} \times \phi_x([1/x, +\infty))) = \bigsqcup_{x>0} \mathcal{G}(\{x\} \times [1/x, +\infty)) = \mathcal{G}(\mathcal{A}),$$

which concludes the proof. $\square$



(a) $\mathcal{B} = \bigsqcup_{x>0}(\{x\} \times \mathcal{B}_x)$. (b) $\mathcal{B} = \bigsqcup_{y>0}(\mathcal{B}_y \times \{y\})$.

Figure 3.3: Idea of how we represent the set $\mathcal{B}$ in lemmas 3.2 and 3.3.

3.2.3 The third lemma

Lemma 3.3. The polynomial map

$$\mathcal{H} := (\mathcal{H}_1, \mathcal{H}_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, (x, y) \longmapsto (x(xy - 2)^2 + \tfrac{1}{2}xy^2,\ y)$$

satisfies $\mathcal{H}(\mathcal{B}) = \mathcal{H}(\mathcal{Q}) = \mathcal{Q}$.

Proof. The inclusion $\mathcal{H}(\mathcal{B}) \subset \mathcal{H}(\mathcal{Q})$ is trivial since $\mathcal{B} \subset \mathcal{Q}$. As for $\mathcal{H}(\mathcal{Q}) \subset \mathcal{Q}$, since $\mathcal{H}_1$ and $\mathcal{H}_2$ are strictly positive on $\mathcal{Q}$ the inclusion holds.

We procced now to prove that $\mathcal{Q} \subset \mathcal{H}(\mathcal{B})$ and therefore get $\mathcal{Q} \subset \mathcal{H}(\mathcal{B}) \subset \mathcal{H}(\mathcal{Q}) \subset \mathcal{Q}$, which implies that $\mathcal{H}(\mathcal{B}) = \mathcal{H}(\mathcal{Q}) = \mathcal{Q}$.

Firstly, notice that the set $\mathcal{B}$ can be expressed as follows:

$$\mathcal{B} = \bigsqcup_{y>0} (\mathcal{B}_y \times \{y\}), \qquad \text{where } \mathcal{B}_y := \begin{cases} (0, +\infty) & \text{if } y \geq 1, \\ (0, y] \cup [1/y, +\infty) & \text{if } 0 < y < 1. \end{cases}$$

Intuitively, we are "slicing" the set $\mathcal{B}$ horizontally, depending on y. Now, looking at the definition of $\mathcal{H}_1$ we can define for each $y > 0$ the polynomial ψ_y dependent on the variable $\mathtt{x}$:

$$\psi_y(\mathtt{x}) := \mathtt{x}(\mathtt{x}y - 2)^2 + \tfrac{1}{2}\mathtt{x}y^2 = y^2\mathtt{x}^3 - 4y\mathtt{x}^2 + (4 + \tfrac{1}{2}y^2)\mathtt{x}.$$

Notice first that $\psi_y(\mathtt{x})$ has odd degree, positive leading coefficient and the following properties:

(i) $\lim_{x\to+\infty} \psi_y(x) = +\infty$.

(ii) $\psi_y(0) = 0$.

(iii) $\psi_y(y) = y((y^2 - 2)^2 + \frac{1}{2}y^2) > y$ for $0 < y < 1$.

(iv) $\psi_y\left(\frac{2}{y}\right) = y$.

Property (iii) holds because $(y^2 - 2)^2 + \frac{1}{2}y^2 > 1$ when $0 < y < 1$. Next, we are going to show that:

$$\psi_y(\mathcal{B}_y) = \begin{cases} \psi_y((0, +\infty)) =^1 (0, +\infty) & \text{if } y \geq 1, \\ \psi_y\big((0, y] \cup \big[\frac{1}{y}, +\infty\big)\big) \supset^2 (0, \psi_y(y)] \cup \big[\psi_y\big(\frac{2}{y}\big), +\infty\big) =^3 (0, +\infty) & \text{if } 0 < y < 1. \end{cases}$$

Equality ^{1} holds because of (i), (ii) and the fact that ψ_y is strictly positive on $(0, +\infty)$. Inclusion ^{2} is followed from the fact that $[1/y_0, +\infty) \supset [2/y_0, +\infty)$ for y_0 fixed. Finally, inclusion ^{3} holds because of (iii) and (iv): $\psi_y(y) > y = \psi_y\big(\frac{2}{y}\big)$.

We are now in conditions of writting:

$$\mathcal{Q} = \bigsqcup_{y>0} \big((0, +\infty) \times \{y\}\big) \subset \bigsqcup_{y>0} \big(\psi_y(\mathcal{B}_y) \times \{y\}\big) = \bigsqcup_{y>0} \mathcal{H}\big(\mathcal{B}_y \times \{y\}\big) = \mathcal{H}(\mathcal{B}_y),$$

which concludes the proof. □

3.2.4 The second proof

To wrap up this chapter, we can now write the proof of theorem 1.11 relying on lemmas 3.1, 3.2 and 3.3:

[Proof of theorem 1.11] Appliying the lemmas, we deduce that

$$\mathcal{Q} \overset{3.3}{=} \mathcal{H}(\mathcal{B}) \overset{3.2}{\subset} (\mathcal{H} \circ \mathcal{G})(\mathcal{A}) \overset{3.1}{\subset} (\mathcal{H} \circ \mathcal{G} \circ \mathcal{F})(\mathbb{R}^2) \overset{3.1}{\subset} (\mathcal{H} \circ \mathcal{G})(\mathcal{Q}) \overset{3.2}{\subset} \mathcal{H}(\mathcal{Q}) \overset{3.3}{=} \mathcal{Q},$$

which means

$$(\mathcal{H} \circ \mathcal{G} \circ \mathcal{F})(\mathbb{R}^2) = \mathcal{Q}$$

as required. □

$\mathcal{A}$

Auxiliary definitions and results

A.1 Real algebraic geometry basics

In this section we give a basic insight into the required concepts, definitions and results from real algebraic geometry. They are mostly used on the first chapter.

We begin with some definitions like *real closed field* and *semialgebraic set*. The *Transfer Principle* allows us to move our results from $\mathbb{R}$ to an arbitrary real closed field R. We also give an overview of the *Zariski topology*, in order to talk about concepts like closure or *reducibility* in this topology. Finally, after defining other concepts like *proper maps* and *germ Nash half-branch curves*, we finish the section with *Puiseux series*, which guarantees us a solution of $\Delta_v(x)$ in the first proof of theorem 1.11.

Definition A.1. A *real closed field* is an ordered field R such that $R(\sqrt{-1})$ is an algebraically closed field. There exist many characterizations of real closed fields. A very enlightening is the following one: R is a real closed field if it is an ordered field that shares with the field $\mathbb{R}$ of real numbers its properties of the first-order language of ordered fields.

Indeed, Tarski-Seidenberg Theorem admits a useful formulation in model theory that explains accurately our last sentence.

Definitions A.2. (i) Let R be a real closed field. A subset $S \subset R^n$ is *semialgebraic* if it is defined as a finite union of sets defined by a conjunction of polynomial equalities and inequalities:

$$\begin{cases} P_1(x_1, \dots, x_n) = 0 \\ \vdots \\ P_r(x_1, \dots, x_n) = 0 \\ Q_1(x_1, \dots, x_n) > 0 \\ \vdots \\ Q_\ell(x_1, \dots, x_n) > 0 \end{cases}$$

It is easily seen that finite unions and intersections of semialgebraic sets are semialgebraic sets too. In addition, the complementary set of a semialgebraic set is also a semialgebraic set. The easy proof of these facts can be studied in [BCR], Ch.2.

(ii) A map $f : S \to T$ between semialgebraic subsets $S \subset R^n$ and $T \subset R^m$ is *semialgebraic* if its graph is a semialgebraic subset of R^{m+n}.

(iii) A *semialgebraic homeomorphism* between two semialgebraic subsets $S \subset R^n$ and $T \subset R^m$ is a continuous and bijective semialgebraic map $f : S \to T$. It is easily seen that in such a case its inverse $f^{-1} : T \to S$ is also semialgebraic.

Theorem A.3 (Transfer Principle)

Let $\mathcal{L}(R)$ be the first-order language of ordered fields with parameters in the real closed field R and let Φ be a formula of $\mathcal{L}(R)$. Then, there exists a quantifier-free formula Ψ of $\mathcal{L}(R)$ with the same free variables $x_1, \dots, x_n$ as Φ such that, for every real closed field extension K of R and every $x \in K^n$, the sentence $\Phi(x)$ holds true if and only if $\Psi(x)$ holds true.

Definition A.4 (Zariski topology)**.** (i) Let K be a field. A subset $X \subset K^n$ is *algebraic* if it is the set of common zeros of a finite family of polynomials $f_1, \dots, f_m \in K[x_1, \dots, x_n]$.

(ii) The *Zariski topology* of an algebraic set $X \subset K^n$ is the topology whose closed sets are the algebraic subsets of K^n contained in X. It is indeed a topology, that is, the arbitrary intersection of algebraic sets is also an algebraic set as an straightforward consequence of Hilbert's basis theorem.

(iii) An algebraic subset $X \subset K^n$ is said to be *reducible* if there exist algebraic subsets $Y \subsetneq X$ and $Z \subsetneq X$ such that $X = Y \cup Z$. If X is not reducible it is said to be *irreducible.*

Definition A.5. (i) Let $S \subset \mathbb{R}^n$ be a semialgebraic set and $p \in S$. The *local dimension of S at p,* denoted $\dim(S_p)$, is the largest non-negative integer d such that for every open ball B centered at p the intersection $S \cap B$ contains a semialgebraic subset semialgebraically homeomorphic to the cube $[0,1]^d$.

It is said that S is *pure dimensional* if $\dim(S_p) = \dim(S_q)$ for every pair of points $p, q \in S$.

Definitions A.6. (i) A continuous map $f : X \to Y$ between topological spaces X and Y is said to be *proper* if $f^{-1}(K)$ is a compact subspace of X for every compact subspace K of Y.

(ii) A semialgebraic map $f : S \to T$ between semialgebraic sets $S \subset R^n$ and $T \subset R^m$ is said to be *semialgebraically proper* if $f^{-1}(K)$ is bounded and closed in S for every bounded and closed in T subset K of T.

Definition A.7. A polynomial map $f : X \to Y$ between algebraic sets X and Y is said to be *dominant* if its image $f(X)$ is a dense subset of Y in its Zariski topology.

Definitions A.8. (i) A function $f : U \to \mathbb{R}$ defined in an open semialgebraic subset $U \subset \mathbb{R}^n$ is a *Nash function* if it is analytic and semialgebraic. A map $f := (f_1, \dots, f_m) : U \to \mathbb{R}^m$ is a *Nash map* if each coordinate $f_i : U \to \mathbb{R}$ is a Nash function.

(ii) Let $\Gamma \subset \mathbb{R}^n$ be an algebraic curve and $p \in \Gamma$. For every small enough $\varepsilon > 0$ the intersection $B_\varepsilon \cap \Gamma \setminus \{a\}$, where $B_\varepsilon \subset \mathbb{R}^n$ is the open ball of radius ε centered at the point p, has finitely many connected components $C_1, \dots, C_k$. Each C_i is semialgebraically homeomorphic to the interval $(0,1]$ and in fact for $i = 1, \dots, k$ there exists a Nash homeomorphism $f_i : [0,1] \to C_i \cup \{p\}$, with $f_i(0) = p$.

Indeed, this result is a particular case of the local conic structure theorem of semialgebraic sets.

Observe that, by its very definition, it makes sense to define the germs $C_{i,p}$ of C_i at the point p for $i = 1, \dots, k$ as they are independent of the radius ε. These germs $C_{i,p}$ are called the *germ Nash half-banches* of the curve Γ centered at p. For more details see [BCR, IX.5.2].

Definition A.9. It is proved in [W, pp. 98-102] that given an algebraically closed field K and an indeterminate $\mathtt{t}$ over K, the field $K(\{\mathtt{t}^*\})$ of *Puiseux series* with coefficients in K is algebraically closed. As it is an algebraic extension of the field $K(\mathtt{t})$ of rational functions over K, the field $K(\{\mathtt{t}^*\})$ is an algebraic closure of the field $F(\mathtt{t})$ for every subfield F of K such that the field extension $K|F$ is algebraic. In particular, for $F := \mathbb{R}$ it follows that $\mathbb{C}(\{\mathtt{t}^*\})$ is an algebraic closure of $\mathbb{R}(\mathtt{t})$.

A.2 Root finding algorithms

In the first proof of theorem 1.11 many times we require to check wether a given polynomial has any root on a certain interval. Some other times we want to check that a polynomial is positive or negative on an interval, which is done by checking that it does not have a root in the interval and evaluating on a value that belongs to the interval to check the positiveness/negativeness.

Sturm's method is used to obtain the number of different real roots of a polynomial in a given interval. Note that it is an algebraic method that doesn't rely on approximations.

Definitions A.10. (i) Let R be a real closed field and $f, g \in \mathbb{R}[x]$ be polynomials. A *Sturm sequence* (or *Sturm chain*) is a finite sequence of polynomials $(f_0, f_1, \dots, f_k)$ where

$$\begin{aligned} &f_0 := f \\ &f_1 := f'g \\ &\quad \dots \\ &f_i := f_{i-1}q_i - f_{i-2}, \text{ with } q_i \in R[x] \text{ and } \deg(f_i) < \deg(f_{i-1}) \text{ for } i = 2, \dots, k. \end{aligned}$$

Then, by Euclid's Algorithm, there is an integer k verifying $f_k = \gcd(f, f'g)$.

(ii) Given a sequence $(a_0, a_1, \dots, a_k)$ of elements of R with $a_0 \neq 0$, we define the *number of sign changes in the sequence* $(a_0, \dots, a_k)$ as follows: count one sign change every time $a_i a_\ell < 0$, with

$$\begin{aligned} &\ell = i + 1, \text{ or} \\ &\ell > i + 1 \text{ and } a_j = 0 \text{ for every } j \text{ verifying } i < j < \ell. \end{aligned}$$

(iii) If $a \in R$ is not a root of f and $(f_0, \dots, f_k)$ is the Sturm sequence of f and g, we define $v(f, g, a)$ to be the number of sign changes in $(f_0(a), \dots, f_k(a))$.

Proposition A.11 (Sturm's Theorem)**.** *Let R be a real closed field and $f \in R[x]$. Let $a, b \in R$ be such that $a < b$ and neither a nor b are roots of f. Then the number of roots of f in the interval (a, b) is equal to $v(f, 1, a) - v(f, 1, b)$.*

The proof of this proposition can be studied in [BCR, 1.2.10].

Laguerre's method differs from Sturm's method in that it is a numerical algorithm, rather than algebraic. It is remarkable that it converges to a root, with very few exceptions, from any starting value.

Proposition A.12 (Laguerre's method)**.** *Let $f \in \mathbb{R}[x]$ with degree n such that*

$$f(x) = (x-r)(x-q)^{n-1}, \tag{A.1}$$

for $r, q \in \mathbb{C}$ unknown. Let

$$g(x) := \frac{f'(x)}{f(x)} = \frac{1}{x-r} + \frac{n-1}{x-q} \quad \textit{and} \quad h(x) := g^2(x) - \frac{f''(x)}{f(x)} = \frac{1}{(x-r)^2} + \frac{n-1}{(x-q)^2}.$$

If we solve for $x-q$ in $g(x)$ and substitute the result on $h(x)$, we get a quadratic equation for $x-r$, and its solution is named Laguerre's formula:

$$x - r = n\left(g(x) \pm \sqrt{(n-1)(nh(x) - g^2(x))}\right)^{-1}. \tag{A.2}$$

Then, the equation given by A.2 defines a numerical method by choosing on each step the sign that results in the larger magnitude of the denominator and taking the new root to be:

$$x_{k+1} = x_k - n\left(g(x_k) \pm \sqrt{(n-1)(nh(x_k) - g^2(x_k))}\right)^{-1}.$$

Furthermore, this iterative formula works with any polynomial, not just with the ones described by equation A.1.

B
Sage code

C

Python code

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