Complexity Analysis of Polynomial Algorithms

Final Degree Project

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Abstract

This work is about different proofs of the same fact and a computational comparison between them, looking for the most efficient one.

Let R be a real closed field and $n \geq 2$. Then:

(1) For every finite subset F of R^n, the semialgebraic set $R^n \setminus F$ is a polynomial image of R^n.

(2) Given linearly independent linear forms $h_1, \ldots, h_r$ of R^n, the semialgebraic set $\{h_1 > 0, \ldots, h_r > 0\} \subset R^n$ is a polynomial image of R^n.

The key result here is that $\mathcal{Q} := \{x > 0, y > 0\} \subset R^2$ is a polynomial image of R^2. This assert is proved in three different ways: a first approach using some basic results in real algebraic geometry (such as Sturm's theorem and the Curve Selection Lemma) and the aid of a computer; a second and shorter one, using the composition of three rather simple polynomial maps; and a third one that applies arguments of algebraic topology, without the aid of computer computations.

(...)

Keywords. Polynomial map, polynomial image, semialgebraic set, open quadrant, complexity.

Resumen

Este trabajo contiene distintas pruebas del mismo hecho y pretende hacer una comparación computacional entre ellas, tratando de encontrar la más eficiente.

Sean R un cuerpo real cerrado y $n \geq 2$. Entonces:

(1) Para cada subconjunto finito F de R^n, el conjunto semialgebraico $R^n \setminus F$ es imagen polinómica de R^n.

(2) Dadas formas linealmente independientes $h_1, \ldots, h_r$ de R^n, el conjunto semialgebraico $\{h_1 > 0, \ldots, h_r > 0\} \subset R^n$ es imagen polinómica de R^n.

El resultado clave aquí es que $\mathcal{Q} := \{x > 0, y > 0\} \subset R^2$ es imagen polinómica de R^2. Esta afirmación es probada de tres maneras diferentes: una primera usando algunos resultados básicos de geometría algebraica real (como son el teorema de Sturm y el Lema de Selección de Curvas) y la ayuda de un ordenador; una segunda forma más breve, empleando la composición de tres aplicaciones polinomiales bastante sencillas; y una tercera y última en la que se utilizan argumentos de topología algebraica, que evitan efectuar cálculos en un ordenador.

(...)

Palabras clave. Aplicación polinomial, imagen polinomial, conjunto semialgebraico, quadrante abierto, complejidad.

Contents

Introduction to polynomial images of R^n

1.1 Statement of the problem and main results

Definition 1.1. Given a real closed field R and positive integers m, n it is said that $f := (f_1, \dots, f_n) : R^m \to R^n$ is a *polynomial map* if $f_i \in R[\mathtt{x}_1, \dots, \mathtt{x}_m]$ for $i = 1, \dots, n$.

An important example of real closed field (and the only one with the Archimedean property) if the field $\mathbb{R}$ of real numbers. There are several concepts, like the one of real closed field, that appear in the Introduction but that we do not recall until Appendix A in order to lighten the presentation and soften the reading.

A very famous theorem by Tarski and Seidenberg states the following:

Theorem 1.2 (Tarski-Seidenberg)

> The image of every polynomial map $f : R^m \longrightarrow R^n$ is a semialgebraic subset of R^n.

In this work we study a sort of converse of this statement. In the 1990 *Oberwolfach Reelle algebraische Geometrie* week, J.M. Gamboa [G] proposed the following

Problem. Characterize the semialgebraic subsets of R^n that are polynomial images of some R^m.

Particularly interesting seems to be the study of those open semialgebraic subsets of R^n that are polynomial images of R^n, because this is related with the real jacobian conjecture.

Notation. We need to mention to which topology we refer to when we talk about closures, boundaries, etc. More specifically, the *exterior boundary* of a subset $S \subset R^n$ is $\delta S := \overline{S} \setminus S$, with $\overline{S}$ being the *closure* of S in the usual topology of R^n. In addition we will denote by $\overline{S}^{\mathrm{zar}}$ the closure of S with respect to the Zariski topology of R^n. We will say that a semialgebraic subset $A \subset R^n$ is *Zariski-irreducible* if its Zariski closure $\overline{A}^{\mathrm{zar}}$ is an irreducible algebraic set.

1.1.1 Necessary conditions and examples

To begin working on this idea, we provide some necessary conditions for a set $S \subset R^n$ to be a polynomial image of R^m.

If $m = n = 1$, that is, for a polynomial function $f : R \to R$, its image $f(R)$ is either a *singleton,* that is, a set with a unique point if the function f is constant, or an unbounded

closed interval. For example if $f(\mathbf{x}) = \mathbf{x}^2$ we have $f(R) = [0, +\infty)$, and $f(R) = R$ if f is a polynomial of odd degree.

In the general case, by Tarski-Seidenberg, S must be a semialgebraic set and, as R^n is semialgebraically connected, S is semialgebraically connected too. In addition, by the identity principle for polynomials, S is Zariski-irreducible and pure dimensional.

But to be a polynomial image of some R^m is a restrictive condition, and there are more constrains than those quoted above.

Definition 1.3. A polynomial map $f : R^m \to R^n$ is said to be *semialgebraically proper at a point* $p \in R^n$ if there exists an open neighbourhood K of p such that the restriction $f|_{f^{-1}(K)} : f^{-1}(K) \to K$ is a semialgebraically proper map.

Definition 1.4. A *parametric semiline* of R^n is the image of R under a non-constant polynomial map $R \to R^n$.

It is clear that every parametric semiline is semialgebraically closed, since every polynomial map from R to R^n is semialgebraically proper. Let $\mathscr{S}_f$ denote the set of points $p \in R^n$ at which f is **not** semialgebraically proper.

Theorem 1.5 (Jelonek)

> Let $f : R^2 \to R^2$ be a dominant polynomial map. Then $\mathscr{S}_f$ is a finite union of parametric semilines.

With these ideas in mind, we present in the following proposition some obstructions for a semialgebraic set to be a polynomial image of R^n.

Proposition 1.6. *Let* $f : R^m \to R^n$ *be a polynomial map and* $S := f(R^m)$.

(1) $\delta S \subset \mathscr{S}_f$.

Proof. Suppose $p \in \delta S \setminus \mathscr{S}_f$. Since $p \notin \mathscr{S}_f$, there exists an open neighbourhood K of p such that the restriction $f|_{f^{-1}(K)} : f^{-1}(K) \to K$ of f is proper. Thus its image $K \cap S$ is a closed subset of K. Hence, $p \in K \cap \overline{S} = K \cap (\overline{K \cap S}) = K \cap S$, which yields in a contradiction.

(2) *Let* $m = n = 2$ *and* Γ *be a 1-dimensional irreducible component of* $\overline{\delta S}^{zar}$. *Then* Γ *is the Zariski closure of a parametric semiline of* R^2.

Proof. As f is a dominant map, $\mathscr{S}_f$ is, by Theorem 1.5, a finite union of parametric semilines, say $M_1, \ldots, M_s$ in R^2. Using (1) we get: $\Gamma \subset \overline{\delta S}^{\text{zar}} \subset \overline{\mathscr{S}_f}^{\text{zar}} = \bigcup_{i=1}^{s} \overline{M_i}^{\text{zar}}$. Lastly, using that both Γ and the $\overline{M_i}^{\text{zar}}$'s are irreducible, $\Gamma = \overline{M_i}^{\text{zar}}$ for some index $i = 1, \ldots, s$.

(3) *Let* $p : R^n \to R$ *be a polynomial map which is non-constant on* S. *Then* $p(S)$ *is unbounded.*

Proof. For each $a \in R^m$ let $\varphi_a : R \to R$ defined as $\varphi_a(t) := p(f(ta))$. Then $p(S)$ would contain the image $\varphi_a(R)$ for all $a \in R^m$. Suppose now that $\varphi_a(R)$ is bounded. Then $\varphi_a(R)$ would be a point r_a, and given two points $a, b \in R^m$ we would have

$$\varphi_a(1) = p(f(a)) = r_a = \varphi_a(0) = \varphi_b(0) = r_b = p(f(b)) = \varphi_b(1).$$

Then p would be constant on S, which is a contradiction.

Corollary 1.7. *Because of statement* (3) *in Proposition* 1.6, *all linear projections of a polynomial image $S = f(R^m)$ are either a point or unbounded. Thus, S is also unbounded or a point.*

Examples 1.8.

(i) The exterior of the closed unit disc $S := \{u^2 + v^2 > 1\} \subset R^2$ **is not** a polynomial image of R^2. This is so because the only irreducible component of $\overline{\delta S}^{\mathrm{zar}}$ is the circle $\{u^2 + v^2 = 1\} \subset R^2$ and this set is not a parametric semiline because it is bounded.

(ii) Let $S_1 := \{uv < 1\} \subset R^2$ and $S_2 := \{uv > 1, u > 0\} \subset R^2$ (see Figure 1.1). They both **are not** polynomial images of R^2 since the Zariski closure of their exterior boundaries $\overline{\delta S_1}^{\mathrm{zar}} = \overline{\delta S_2}^{\mathrm{zar}}$ is the hyperbola $\{uv = 1\} \subset R^2$, which is not a parametric semiline.

(a) $S_1 := \{uv < 1\}$.   (b) $S_2 := \{uv > 1, u > 0\}$.
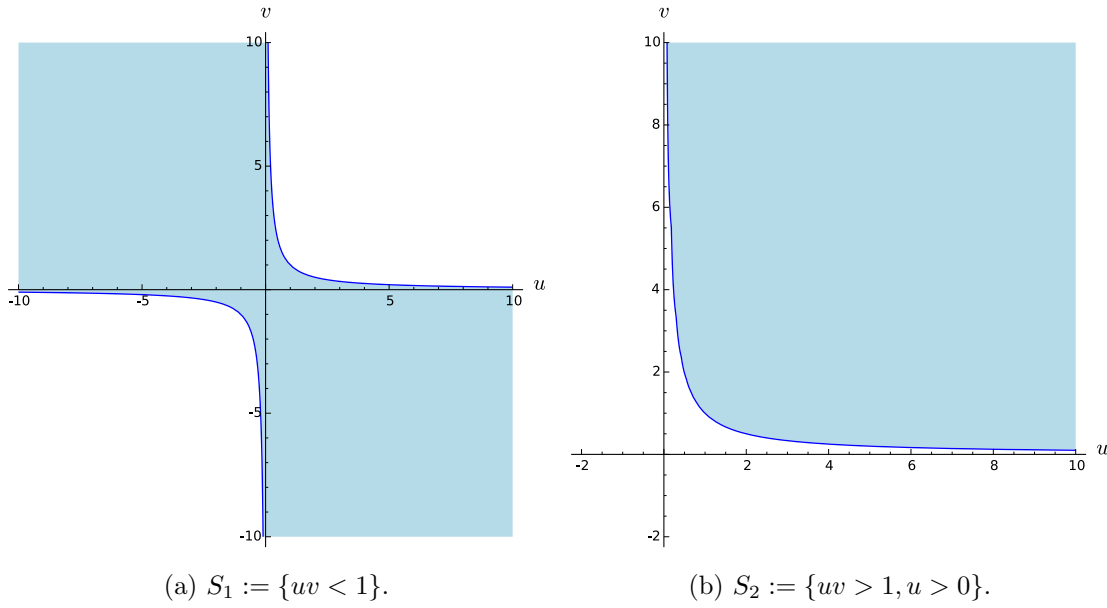
Figure 1.1: Plots of the regions defined in Example 1.8 (ii).

(iii) Let $S := R^2 \setminus \{(0,0)\}$ be the punctured plane. Then S is the image of the polynomial map $(x, y) \mapsto (xy - 1, (xy - 1)x^2 - y)$.

(iv) The open upper half-plane $\mathcal{H} := \{v > 0\} \subset R^2$ is the image of the polynomial map

$$R^2 \to R^2, \ (x, y) \mapsto (y(xy - 1), (xy - 1)^2 + x^2),$$

This implies that every open half-plane is a polynomial image of R^2. This is probably the simplest polynomial map whose image is $\mathcal{H}$.

1.1.2 Statement of the main results

The main results of this chapter are generalizations of items (iii) and (iv) from Examples 1.8.

Theorem 1.9

Let $n \geq 2$. For every finite set $F \subset R^n$, the semialgebraic set $R^n \setminus F$ is a polynomial image of R^n.

Theorem 1.10

Let $n \geq 2$. Given independent linear forms $h_1, \ldots, h_r$ of R^n, the open semialgebraic set $\{h_1 > 0, \ldots, h_r > 0\}$ is a polynomial image of R^n.

Before the paper [FG] was published, the exterior boundary of all open sets that were known to be polynomial images of R^2 was Zariski-irreducible, and all of them were deformations of $\mathcal{H}$. J.M. Gamboa outlined the problem of finding if the open quadrant $\mathcal{Q} := \{x > 0, y > 0\} \subset R^2$ is a polynomial image of R^2 or not, since its exterior boundary is Zariski-reducible. The solution of this problem is a key particular case of the content of Theorem 1.10. Before that, the closest known approach to look for a solution of this problem was the transformation

$$\psi : R^2 \to R^2, \, (x, y) \mapsto (x^4 y^2, x^2 y^4)$$

whose image is $\mathcal{Q} \cup \{(0, 0)\}$. The answer to the first intriguing problem in this field was given in the following theorem:

Theorem 1.11

The first open quadrant $\mathcal{Q}$ is a polynomial image of R^2.

Remark 1.12. The first proof of Theorem 1.11 consists of two parts:

* Choosing a "good" candidate to be a polynomial map whose image is close enough to $\mathcal{Q}$, and giving the reasons behind this choice (see section 2.1).

* Checking that the image of the map is $\mathcal{Q}$ indeed. After some arguments, this can be reduced to prove the non-existence of real roots of certain polynomials in one variable on certain intervals, and to compare some rational functions on those intervals. In order to do this, we use symbolic computations with tools like Sage and Maple. Because of the high degree of the involved polynomials, the actual checking of the non-existence of roots is done with a Maple package that performs Sturm algorithm and a Python programme that implements Laguerre's method.

Let us see how Theorem 1.10 follows from Theorem 1.11.

[Proof of Theorem 1.10] After a linear change of coordinates we can suppose that $h_1 := \mathtt{x}_1, \ldots, h_r := \mathtt{x}_r$, so we only have to prove that for every pair of positive integers $r \leq n$ the semialgebraic set $\{x_1 > 0, \ldots, x_r > 0\} \subset R^n$ is a polynomial image of R^n. This is reduced to prove the following two steps:

* $\mathcal{H} := \{x_1 > 0\} \subset R^2$ and $\mathcal{Q} := \{x_1 > 0, x_2 > 0\} \subset R^2$ are polynomial images of R^2, which is true, respectively, by Example 1.8 (iv) and Theorem 1.11.

* This implies that $\mathcal{O} := \{x_1 > 0, x_2 > 0, x_3 > 0\} \subset R^3$ is a polynomial image of R^3. Indeed, let $f, g : R^2 \to R^2$ be polynomial maps whose respective images are $\mathcal{H}$ and

$\mathcal{Q}$. Let us define:

$$(f, \mathrm{id}_R) : R^3 = R^2 \times R \longrightarrow R^3 = R^2 \times R$$
$$(x_1, x_2, x_3) \longmapsto (f_1(x_1, x_2), f_2(x_1, x_2), x_3)$$
$$(\mathrm{id}_R, g) : R^3 = R \times R^2 \longrightarrow R^3 = R \times R^2$$
$$(x_1, x_2, x_3) \longmapsto (x_1, g_1(x_2, x_3), g_2(x_2, x_3)).$$

Then, $\mathcal{O}$ is the image of the polynomial map

$$H := (\mathrm{id}_R, g) \circ (f, \mathrm{id}_R) : R^3 \longrightarrow R^3$$
$$(x_1, x_2, x_3) \longmapsto (f_1(x_1, x_2), g_1(f_2(x_1, x_2), x_3), g_2(f_2(x_1, x_2), x_3)).$$

Once the case $n = 3$ is solved the case of arbitrary n follows straightforwardly, by taking $g \times \overset{k}{\cdots} \times g : R^n \longrightarrow R^n$ when $n = 2k$ and $H \times g \times \overset{k-1}{\cdots} \times g : R^n \longrightarrow R^n$ when $n = 2k - 1$. $\quad\square$

The original proofs of Theorems 1.9 and 1.11 are written for $R := \mathbb{R}$. As for both theorems explicit polynomial maps are given, the results can be extended to arbitrary real closed field R by the Transfer Principle.

1.2 Complementary set of a finite set

We proceed to prove Theorem 1.9:

[Proof of Theorem 1.9] Let $F := \{p_1, \ldots, p_k\}$. Let us see that it suffices to prove the result for points of the form $p_j := (a_j, \vec{0}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. After a linear change of coordinates we can assume that the first coordinates of the given points are pairwise distinct.

In other words, we denote $p_j := (a_{1j}, \ldots, a_{nj})$ and we may suppose that $a_{1j} \neq a_{1\ell}$ when $j \neq \ell$. Then, there exists a polynomial $P_1 \in \mathbb{R}[\mathsf{t}]$ such that $P_1(a_{1j}) = a_{nj}$, with $j = 1, \ldots, n$, and denoting $x' := (x_1, \ldots, x_{n-1})$, we define the polynomial map

$$h_1 : \mathbb{R}^n \to \mathbb{R}^n, \ (x', x_n) \mapsto (x', x_n + P_1(x_1)).$$

Indeed h_1 is bijective. Note first that every point of $\mathbb{R}^n$ has a preimage in $\mathbb{R}^n$, namely if $x := (x_1, \ldots, x_n)$, then $z := (x', x_n - P_1(x_1))$ satisfies $h_1(z) = x$ and h_1 is onto. As for being injective, let $x, y \in \mathbb{R}^n$ such that

$$h_1(x) = (x_1, \ldots, x_n + P_1(x_1)) = (y_1, \ldots, y_n + P_1(y_1)) = h_1(y).$$

Then $x_i = y_i$ for $i = 1, \ldots, n - 1$. Also $x_n + P_1(x_1) = y_n + P_1(y_1)$ and $P_1(x_1) = P_1(y_1)$ because $x_1 = y_1$. Therefore $x_n = y_n$ and $x = y$.

Now, for $p'_j := (a_{1j}, \ldots, a_{(n-1)j}, 0)$ we have $h_1(p'_j) = p_j$. Analogously, there exists $P_2 \in \mathbb{R}[\mathsf{t}]$ such that $P_2(a_{1j}) = a_{(n-1)j}$, and define the polynomial bijection

$$h_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \ (x'', x_{n-1}, x_n) \mapsto (x'', x_{n-1} + P_2(x_1), x_n),$$

where $x'' := (x_1, \ldots, x_{n-2})$. Then $h_2(p''_j) = p'_j$ for $p''_j = (a_{1j}, \ldots, a_{(n-2)j}, 0, 0)$. In this way the polynomial bijection $h_1 \circ h_2 : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$(h_1 \circ h_2)(p''_j) = h_1(h_2(p''_j)) = h_1(p'_j) = p_j,$$

and we can inductively construct a polynomial bijection $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(q_j) = p_j$ for $q_j := (a_{1j}, \vec{0}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Now let $G := \{q_1, \ldots, q_k\}$ and suppose that there exists a polynomial map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $g(\mathbb{R}^n) = \mathbb{R}^n \setminus G$. Then $(h \circ g)(\mathbb{R}^n) = \mathbb{R}^n \setminus F$, which concludes the first part of the proof. Thus in what follows we suppose that $p_j := (a_j, \vec{0})$.

We claim that the image of the polynomial map $f := (f_1, \ldots, f_n)$ defined as:

$$f(\mathbf{x}) := \big(\mathbf{x}_1 \mathbf{x}_2 - r + a_1, \; \mathbf{x}_1^4 \rho(\mathbf{x}) + \mathbf{x}_1^2 \sigma(\mathbf{x}) + \mathbf{x}_2, \; \mathbf{x}_3, \ldots, \; \mathbf{x}_n\big) \quad \text{for } \mathbf{x} := (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbb{R}^n$$

is $\mathbb{R}^n \setminus F$, where r is an integer such that $r \neq a_1 - a_j$ for $j = 1, \ldots, k$,

$$\sigma(\mathbf{x}) := \sum_{j=3}^{n} \mathbf{x}_j^2 \quad \text{and} \quad \rho(\mathbf{x}) := \prod_{j=1}^{k} (\mathbf{x}_1 \mathbf{x}_2 - r + a_1 - a_j).$$

First, suppose that there exists $b := (b_1, \ldots, b_n) \in R^n$ with $f(b) = p_\ell$ for some $\ell = 1, \ldots, k$. Then $f_1(b) = b_1 b_2 - r + a_1 = a_\ell$. Thus the ℓth-factor of the polynomial ρ evaluated at $x := b$ is

$$b_1 b_2 - r + a_1 - a_\ell = a_\ell - a_\ell = 0,$$

and $\rho(b) = 0$. In addition, the equality $f(b) = p_\ell = (a_\ell, \vec{0})$ implies that $f_i(b) = 0$ for $i = 2, \ldots, n$. In particular, as $f_i \equiv \mathrm{id}$ for $i = 3, \ldots, n$ we get $b_i = 0$ for $i = 3, \ldots, n$. Consequently $\sigma(b) = 0$. Since $\sigma(b) = \rho(b) = 0$ we get $b_2 = f_2(b) = 0$, so $b_2 = 0$ and $a_\ell = f_1(b) = a_1 - r$, that is, $r = a_1 - a_\ell$, which is a contradiction. So $\mathrm{im}(f) \subset \mathbb{R}^n \setminus F$. Conversely, let $u := (u_1, \ldots, u_n) \in \mathbb{R}^n \setminus F$. We must prove that the system of polynomial equations:

$$\begin{cases} f_1(\mathbf{x}) = \mathbf{x}_1 \mathbf{x}_2 - r + a_1 = u_1 \\ f_2(\mathbf{x}) = \mathbf{x}_1^4 \rho(\mathbf{x}) + \mathbf{x}_1^2 \sigma(\mathbf{x}) + \mathbf{x}_2 = u_2 \\ f_j(\mathbf{x}) = \mathbf{x}_j = u_j, \quad j \geq 3 \end{cases}$$

has a solution.

(i) If $u_1 = a_1 - r$ then $f(0, u_2, \ldots, u_n) = u$.

(ii) If $u_1 \neq a_1 - r$ we use the first equation to substitute

$$\mathbf{x}_2 = \frac{u_1 - a_1 + r}{\mathbf{x}_1} \quad \text{and} \quad \mathbf{x}_j = u_j \quad \text{for } j \geq 3.$$

Next, we expand $f_2(\mathbf{x})$:

$$\mathbf{x}_1^4 \rho(\mathbf{x}) + x_1^2 \sigma(\mathbf{x}) - u_2 = -\mathbf{x}_2 = -\frac{u_1 - a_1 + r}{\mathbf{x}_1},$$

and multiplying by $\mathbf{x}_1$ we get

$$\mathbf{x}_1^5 \rho(\mathbf{x}) + \mathbf{x}_1^3 \sigma(\mathbf{x}) - u_2 \mathbf{x}_1 + (u_1 - a_1 + r) = 0. \tag{1.1}$$

Then, $\rho(\mathbf{x}) = \prod_{j=1}^{k}(u_1 - a_j)$ and $\sigma(\mathbf{x}) = \sigma(u)$. Now it is clear that each solution x_1 of equation (1.1) must be a nonzero root of the polynomial:

$$Q(\mathbf{t}) = \left(\prod_{j=1}^{k} (u_1 - a_j) \right) \mathbf{t}^5 + \sigma(u) \mathbf{t}^3 - u_2 \mathbf{t} + (r - a_1 + u_1),$$

which has odd degree, and so it has a real root, unless

$$\prod_{j=1}^{k}(u_1 - a_j) = \sigma(u) = u_2 = 0.$$

If this were the case, then $u_1 = a_j$ for some $j = 1, \ldots, k$ and $u_2 = u_3 = \cdots = u_k = 0$. This is not possible because $u \notin F$. Thus, $Q(\mathtt{t})$ has a real root, say b_1, and in fact $b_1 \neq 0$ because $Q(0) = r - a_1 + u_1 \neq 0$. Finally,

$$f\left(b_1, \ \frac{u_1 - a_1 + r}{b_1}, \ u_3, \ \ldots, \ u_n\right) = u,$$

as required.

$\square$

The open quadrant $\mathcal{Q}$ problem: First proof

2.1 How to find a "potential" map to solve the problem

It is remarkable that though the open interval $I := (0, +\infty)$ is the image of $\mathbb{R}^2$ under the polynomial map $h : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto (xy - 1)^2 + x^2$, see Figure 2.1, the interval I is not a polynomial image of $\mathbb{R}$ because polynomial images of the real line are closed subsets of $\mathbb{R}$.
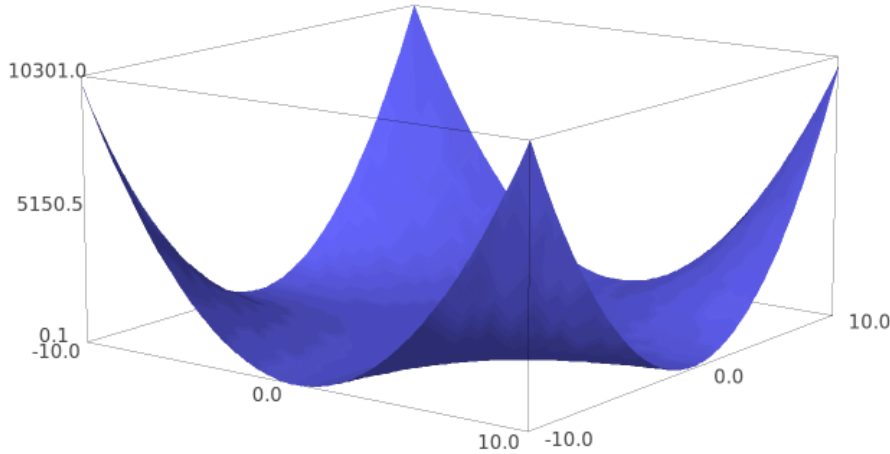

Figure 2.1: $h(\mathbf{x}, \mathbf{y}) = (\mathbf{x}\mathbf{y} - 1)^2 + \mathbf{x}^2$.

However, although $h(\mathbb{R}^2) = I$, the polynomial h does not help to obtain $\mathcal{Q}$ at all:

Remark 2.1. There is no polynomial map $f := (P_1, P_2) : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $f(\mathbb{R}^2) = \mathcal{Q}$ and $P_1(\mathbf{x}, \mathbf{y}) = (\mathbf{x}\mathbf{y} - 1)^2 + \mathbf{x}^2$.

The proof of this remark relies on a suitable use of the Curve Selection Lemma ([ABR, VIII.2.6]) to approach a point $(\lambda^2, 0) \in \overline{\mathcal{Q}}$ with $\lambda > 0$, to get a contradiction.

On the topic of finding a polynomial map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $\Phi(\mathbb{R}^2) = \mathcal{Q}$, "a major difficulty" is the following:

$$\boxed{\textit{The closure of its image must contain the positive half-axes.}} \qquad (\clubsuit)$$

Remark 2.2. Using Theorem 1.9, we just need to find a polynomial map

$$\mathcal{P} = (\mathcal{F}, \mathcal{G}) : \mathbb{R}^2 \to \mathbb{R}^2$$

such that $\mathcal{P}(\mathbb{R}^2)$ is the disjoint union of $\mathcal{Q}$ and a set with finite preimage, say $\mathcal{P}(\mathbb{R}^2) = \mathcal{Q} \sqcup F$ with $\mathcal{P}^{-1}(F)$ a finite set. Suppose for a while this is proved. Then, by Theorem 1.9, there exists a polynomial map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(\mathbb{R}^2) = \mathbb{R}^2 \setminus \mathcal{P}^{-1}(F)$ and the polynomial map $\Phi := \mathcal{P} \circ \varphi$ satisfies

$$\Phi(\mathbb{R}^2) = \mathcal{P}(\varphi(\mathbb{R}^2)) = \mathcal{P}(\mathbb{R}^2 \setminus \mathcal{P}^{-1}(F)) = \mathcal{P}(\mathbb{R}^2) \setminus F = (\mathcal{Q} \sqcup F) \setminus F = \mathcal{Q}.$$

We are going to define a map $\mathcal{P} := (\mathcal{F}, \mathcal{G})$ that accomplish this task, with the set F being $F := \{(-1, 0), (0, -1)\}$. If we were able to find such map $\mathcal{P}$, then condition (♣) will immediately be satisfied.

Suppose for a while that such a map $\mathcal{P}$ exists. Then, for every $\lambda, \mu \geq 0$ there will exist Nash half-branch curve germs $\alpha_\lambda(s), \beta_\mu(s)$ which cannot be extended to 0 and such that:

$$\lim_{s \to 0} P(\alpha_\lambda(s)) = (\lambda^2, 0) \qquad \text{and} \qquad \lim_{s \to 0} P(\beta_\mu(s)) = (0, \mu^2).$$

We can try parameterizations of the form:

$$\alpha_\lambda(s) := \left(s^{n_\lambda}, \frac{a_{\lambda 0} + a_{\lambda 1} s + \cdots}{s^{m_\lambda}} \right) \qquad \text{and} \qquad \beta_\mu(s) := \left(\frac{b_{\mu 0} + b_{\mu 1} s + \cdots}{s^{\ell_\mu}}, s^{k_\mu} \right).$$

Then $a_{\lambda 0}, b_{\mu 0}$ must be constants (except maybe for finitely many values of λ and μ). This leads us to choose curves of the type:

$$\alpha_\lambda(s) := \left(s^{n_\lambda}, \frac{1 + a_{\lambda 1} s + \cdots}{s^{m_\lambda}} \right) \qquad \text{and} \qquad \beta_\mu(s) := \left(\frac{1 + b_{\mu 1} s + \cdots}{s^{\ell_\mu}}, s^{k_\mu} \right),$$

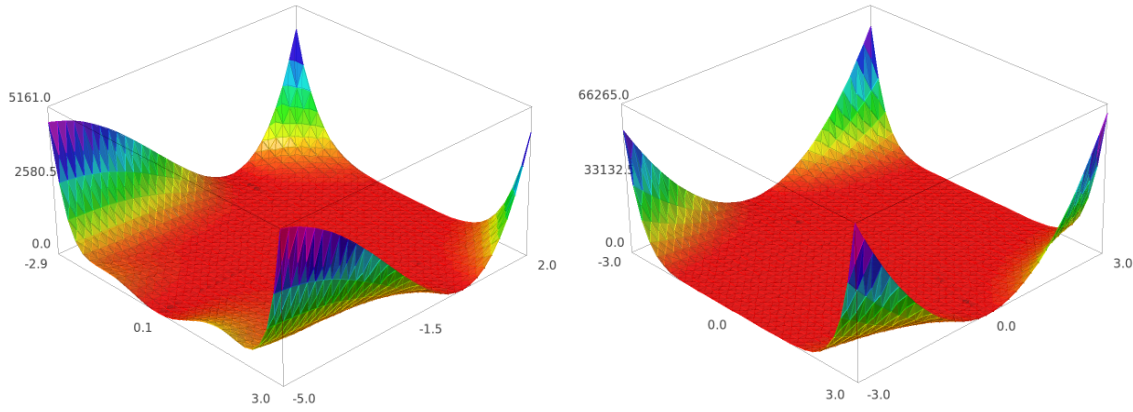and among them we make the simplest choice:

$$\alpha_\lambda(s) := \left(s, \frac{1 + a_\lambda s}{s} \right) \qquad \text{and} \qquad \beta_\mu(s) := \left(\frac{1 + b_\mu s}{s}, s^3 \right).$$

The following pair of polynomials:

$$\boxed{\begin{array}{lll} \mathcal{F}(\mathbf{x}, \mathbf{y}) := (1 - \mathbf{x}^3\mathbf{y} + \mathbf{y} - \mathbf{xy}^2)^2 + (\mathbf{x}^2\mathbf{y})^2 & = \mathcal{F}_1^2 + \mathcal{F}_2^2 \\ \mathcal{G}(\mathbf{x}, \mathbf{y}) := (1 - \mathbf{xy} + \mathbf{x} - \mathbf{x}^4\mathbf{y})^2 + (\mathbf{x}^2\mathbf{y})^2 & = \mathcal{G}_1^2 + \mathcal{G}_2^2 \end{array}}$$

enjoy a nice behavior along these curves, as figure 2.2 show. Next, notice that

(a) · $\mathcal{F}_1 \circ \alpha_\lambda = 1 - a_\lambda - a_\lambda^2 s - s^2 - a_\lambda s^3 \in \mathbb{R}[s, a_\lambda]$, $\qquad\qquad$ $\mathcal{F}_1 \circ \alpha_\lambda(0) = 1 - a_\lambda$.
   · $\mathcal{F}_1 \circ \beta_\mu = -3 b_\mu s - 3 b_\mu^2 s^2 - (b_\mu^3 - 1) s^3 - s^5 - b_\mu s^6 \in \mathbb{R}[s, b_\mu]$, $\mathcal{F}_1 \circ \beta_\mu(0) = 0$.

(b) · $\mathcal{G}_1 \circ \alpha_\lambda = (1 - a_\lambda) s - s^3 - a_\lambda s^4 \in \mathbb{R}[s, a_\lambda]$, $\qquad\qquad\qquad$ $\mathcal{G}_1 \circ \alpha_\lambda(0) = 0$.
   · $\mathcal{G}_1 \circ \beta_\mu = 1 - 3 b_\mu - 6 b_\mu^2 s - (4 b_\mu^3 + 1) s^2 - (b_\mu^4 + b_\mu) s^3 \in \mathbb{R}[s, b_\mu]$, $\mathcal{G}_1 \circ \beta_\mu(0) = 1 - 3b_\mu$.

(c) · $\mathcal{F}_2 \circ \alpha_\lambda = s + a_\lambda s^2 = \mathcal{G}_2 \circ \alpha_\lambda \in \mathbb{R}[s, a_\lambda]$.
   · $\mathcal{F}_2 \circ \beta_\mu = s + 2 b_\mu s^2 + b_\mu^2 s^3 = \mathcal{G}_2 \circ \beta_\mu \in \mathbb{R}[s, b_\mu]$.
   · $\mathcal{F}_2 \circ \alpha_\lambda(0) = \mathcal{G}_2 \circ \alpha_\lambda(0) = \mathcal{F}_2 \circ \beta_\mu(0) = \mathcal{G}_2 \circ \beta_\mu(0) = 0$.

(a) $\mathcal{F}(x, y) := (1 - x^3 y + y - xy^2)^2 + (x^2 y)^2.$ (b) $\mathcal{G}(x, y) = (1 - xy + x - x^4 y)^2 + (x^2 y)^2.$

Figure 2.2: Plots of the polynomials $\mathcal{F}, \mathcal{G} : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

All of these map compositions were computed by Sage. Thus, we get the following properties:

(i) The polynomials $\mathcal{F}, \mathcal{G}$ are non-negative in $\mathbb{R}^2$.

(ii) $\cdot \, \mathcal{F}^{-1}(0) = \mathcal{F}_1^{-1}(0) \cap \mathcal{F}_2^{-1}(0) = \{(0, -1)\} \xmapsto{\mathcal{P}} \{(0, 1)\}.$

$\cdot \, \mathcal{G}^{-1}(0) = \mathcal{G}_1^{-1}(0) \cap \mathcal{G}_2^{-1}(0) \, = \{(-1, 0)\} \xmapsto{\mathcal{P}} \{(1, 0)\}.$

(iii) $\cdot \, \mathcal{P} \circ \alpha_\lambda = (\mathcal{F} \circ \alpha_\lambda, \mathcal{G} \circ \alpha_\lambda) =$

$(\, a_\lambda^2 - 2\, a_\lambda + 1 + 2\, (a_\lambda^3 - a_\lambda^2)s + (a_\lambda^4 + 2\, a_\lambda - 1)s^2 + a_\lambda^2 s^3 +$
$(2\, a_\lambda^3 + a_\lambda^2 + 1)s^4 + 2\, a_\lambda s^5 + a_\lambda^2 s^6,$
$(a_\lambda^2 - 2\, a_\lambda + 2)s^2 + 2\, a_\lambda s^3 + (a_\lambda^2 + 2\, a_\lambda - 2)s^4 +$
$2\, (a_\lambda^2 - a_\lambda)s^5 + s^6 + 2\, a_\lambda s^7 + a_\lambda^2 s^8 \,).$

$\cdot \, \mathcal{P} \circ \beta_\mu = (\mathcal{F} \circ \beta_\mu, \mathcal{G} \circ \beta_\mu) =$

$(\, (9\, b_\mu^2 + 1)s^2 + 2\, (9\, b_\mu^3 + 2\, b_\mu)s^3 + 3\, (5\, b_\mu^4 + 2\, b_\mu^2 - 2\, b_\mu)s^4 +$
$2\, (3\, b_\mu^5 + 2\, b_\mu^3 - 3\, b_\mu^2)s^5 + (b_\mu^6 + b_\mu^4 - 2\, b_\mu^3 + 6\, b_\mu + 1)s^6 +$
$12\, b_\mu^2 s^7 + 2\, (4\, b_\mu^3 - 1)s^8 + 2\, (b_\mu^4 - b_\mu)s^9 + s^{10} + 2\, b_\mu s^{11} + b_\mu^2 s^{12},$
$9\, b_\mu^2 - 6\, b_\mu + 1 + 12\, (3\, b_\mu^3 - b_\mu^2)s + (60\, b_\mu^4 - 8\, b_\mu^3 + 6\, b_\mu - 1)s^2 +$
$2\, (27\, b_\mu^5 - b_\mu^4 + 9\, b_\mu^2 + b_\mu)s^3 + (28\, b_\mu^6 + 20\, b_\mu^3 + 6\, b_\mu^2 + 1)s^4 +$
$2\, (4\, b_\mu^7 + 5\, b_\mu^4 + 2\, b_\mu^3 + b_\mu)s^5 + (b_\mu^8 + 2\, b_\mu^5 + b_\mu^4 + b_\mu^2)s^6 \,).$

The polynomials $\mathcal{F} \circ \alpha_\lambda, \ \mathcal{G} \circ \alpha_\lambda \in \mathbb{R}[s, a_\lambda]$ and $\mathcal{F} \circ \beta_\mu, \ \mathcal{G} \circ \beta_\mu \in \mathbb{R}[s, b_\mu]$ were computed with Sage. As we anticipated before, by condition (ii) the set $F = \{(-1, 0), (0, -1)\}$.

2.2 The first proof for the open quadrant problem

[Proof of Theorem 1.11] We are going to prove that $\mathcal{Q} \subset \mathcal{P}(\mathbb{R}^2)$. To do this it is enough to fix $v > 0$ and to see that the image under $\mathcal{F}$ of the curve $\{\mathcal{G} = v\}$ contains the open

half-line $(0, +\infty)$.

$\boxed{\textbf{Step 1}}$ *Parametrization of the curve* $\{\mathcal{G} - v = 0\}$.

We start by solving the equation $\mathcal{G} - v = 0$, that is,

$$(1 - \mathbf{xy} + \mathbf{x} - \mathbf{x}^4\mathbf{y})^2 + (\mathbf{x}^2\mathbf{y})^2 - v = 0$$

As it has degree 2 with respect to y, we can compute its roots $y^+(\mathbf{x}, v)$ and $y^-(\mathbf{x}, v)$ given by:

$$y^+(\mathbf{x}, v) := \frac{1 + \mathbf{x} + \mathbf{x}^3 + \mathbf{x}^4 + \sqrt{\Delta(\mathbf{x}, v)}}{\mathbf{x}(\mathbf{x}^2 + (\mathbf{x}^3 + 1)^2)}$$

$$y^-(\mathbf{x}, v) := \frac{1 + \mathbf{x} + \mathbf{x}^3 + \mathbf{x}^4 - \sqrt{\Delta(\mathbf{x}, v)}}{\mathbf{x}(\mathbf{x}^2 + (\mathbf{x}^3 + 1)^2)}$$

where $\Delta(\mathbf{x}, v) = \Delta_v(\mathbf{x}) := v(\mathbf{x}^2 + (\mathbf{x}^3 + 1)^2) - \mathbf{x}^2(\mathbf{x} + 1)^2$, $\deg_{\mathbf{x}}(\Delta) = 6$. We can see on figure 2.3 how y^+ and y^- look like for instance for $v := 0.8$. As we can see on figure 2.4, for $v := 1$ there are no singularities on y^- because $\lim_{x \to 0} y^-(x, 1) = 1$. This observation is used later, in Step 2.
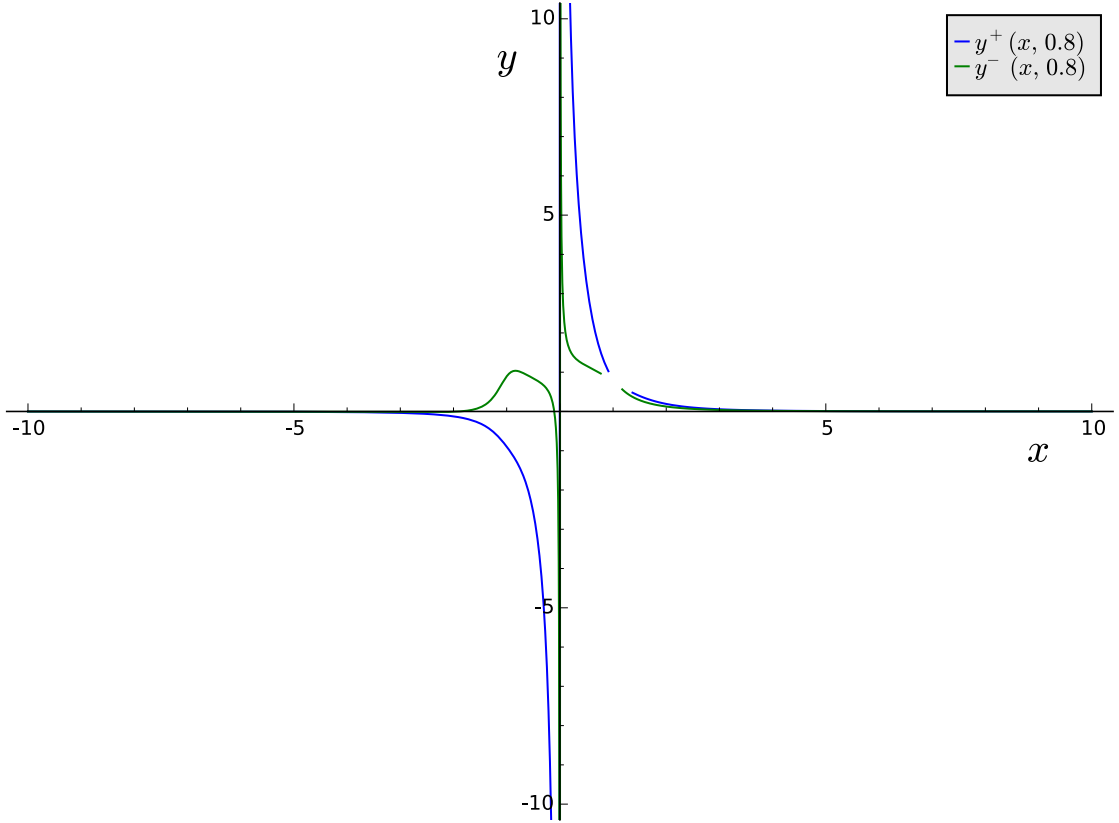

Figure 2.3: $y^+(x, v)$ and $y^-(x, v)$ for $v := 0.8$.

The common domain of these two functions is the set
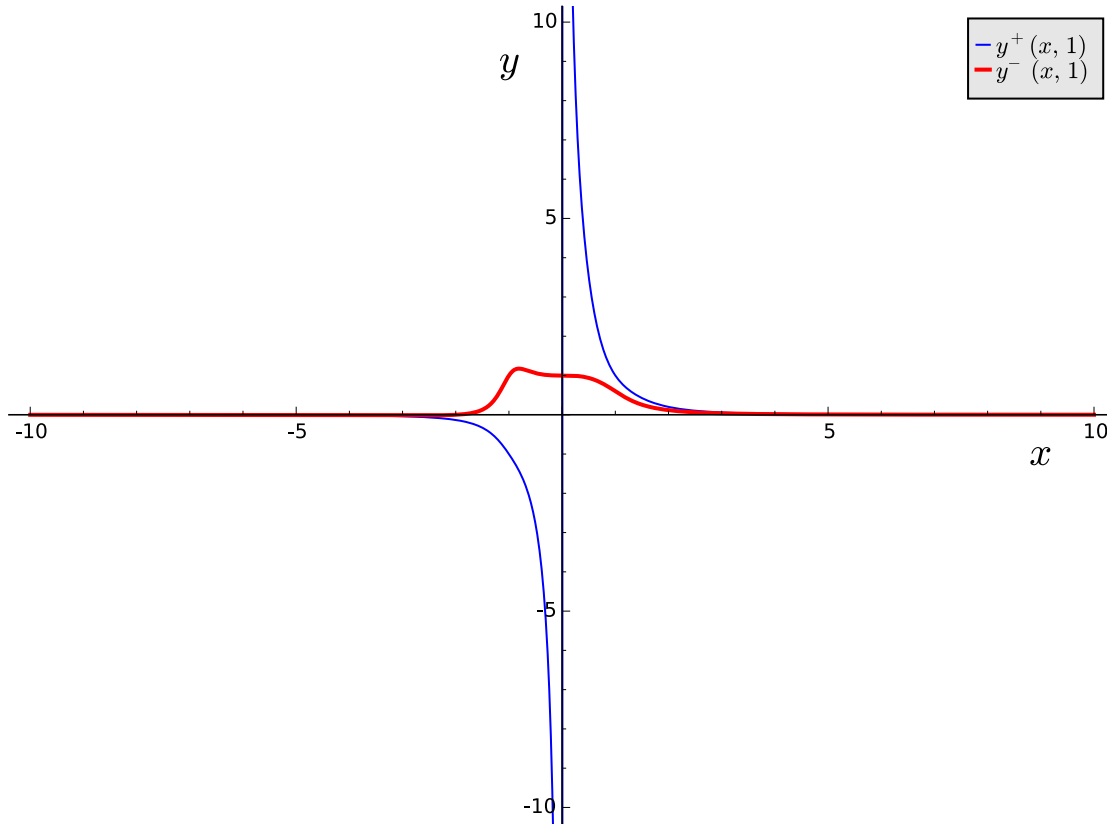
$$D_v := \{x \in R : \Delta(x, v) \geq 0, \, x \neq 0\}.$$

Figure 2.4: $y^+(x,v)$ and $y^-(x,v)$ for $v := 1$.

Notice that the only real root of the denominator is $x_0 := 0$.[1] Let

$$\gamma_v^+ : D_v \to \mathbb{R}, \; x \mapsto \mathcal{F}(x, y^+(x,v)) \quad \text{and} \quad \gamma_v^- : D_v \to \mathbb{R}, \; x \mapsto \mathcal{F}(x, y^-(x,v))$$

Note that $\mathcal{F}(\{\mathcal{G} = v\}) = \operatorname{im}(\gamma_v^+) \cup \operatorname{im}(\gamma_v^-)$, so all reduce to prove the inclusion

$$(0, +\infty) \subset \operatorname{im}(\gamma_v^+) \cup \operatorname{im}(\gamma_v^-).$$

$\boxed{\textbf{Step 2}}$ *Main properties of γ_v^+ and γ_v^-.*

In this section we are going to prove that:

$$(i) \lim_{x \to \pm\infty} \gamma_v^+(x) = \lim_{x \to \pm\infty} \gamma_v^-(x) = 0.$$

$$(ii) \lim_{x \to 0} \gamma_v^+(x) = +\infty, \quad \lim_{x \to 0} \gamma_v^-(x) = \begin{cases} +\infty & \text{for } v \neq 1 \\ 4 & \text{for } v = 1 \end{cases}$$

Using Sage we can symbolically check how γ_v^+ and γ_v^- look like, getting polynomials $A_1, A_2, B_1, B_2 \in \mathbb{R}[x,v]$ and $C \in \mathbb{R}[x]$ such that:

$$(a) \; \gamma_v^+(x) = \frac{A_1(x,v) + B_1(x,v)\sqrt{\Delta(x,v)}}{C(x)}, \quad \gamma_v^-(x) = \frac{A_2(x,v) + B_2(x,v)\sqrt{\Delta(x,v)}}{C(x)},$$

---

[1] We checked with Laguerre's method, implemented with Python 2.7, that the polynomial $x^7 + 2x^4 + x^3 + x$ has 6 complex roots.

$$A_1(x,v) = A_2(x,v), \qquad \deg_x(A_1) = \deg_x(A_2) = 24$$
$$\text{(b)} \quad B_1(x,v) = -B_2(x,v), \qquad \deg_x(B_1) = \deg_x(B_2) = 21$$
$$C(x) = x^2(x^2 + (x^3 + 1)^2)^4, \qquad \deg_x(C) = 26.$$

We proceed to study γ_v^+ and γ_v^- at the origin. Since Δ has even degree and positive leading coefficient with respect to x, it is positive for $|x|$ large enough, so (i) holds.

Now, for $x = 0$, we get $\Delta(0,v) = v > 0$, thus $0 \in \overline{D_v}$. Also:

★ $A_1(0,v) + B_1(0,v)\sqrt{\Delta(0,v)} = v(1 + \sqrt{v})^2 > 0$.

★ $A_2(0,v) + B_2(0,v)\sqrt{\Delta(0,v)} = v(1 - \sqrt{v})^2 \geq 0$, and equality holds if and only if $v = 1$.

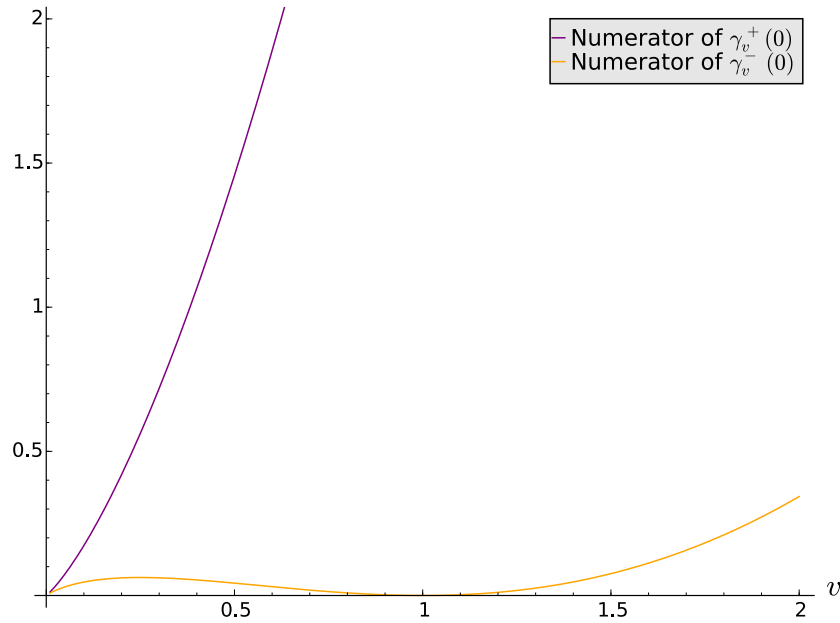Figure 2.5: Numerators of γ_v^+ and γ_v^- for $x := 0$.

Thus, (ii) holds (we also checked it with Sage). The result for $v := 1$ in (ii) is not relevant here (see figure 2.6).

$\boxed{\textbf{Step 3}}$ *When $v \geq 0.28^2$ we have $(0, +\infty) \subset \text{im}(\gamma_v^+)$.*

In order to see whether $(0, +\infty) \subset \text{im}(\gamma_v^+) \cup \text{im}(\gamma_v^-)$ or not we are now going to study the domain D_v. To that end we need to study when $\Delta(x,v) = 0$, so it seems convenient to define:

$$v(x) := \frac{x^2(x+1)^2}{x^2 + (x^3+1)^2},$$

whose graph can be seen in figure 2.7. If $x \in (-\infty, 0)$ we checked using Laguerre's method that the polynomial
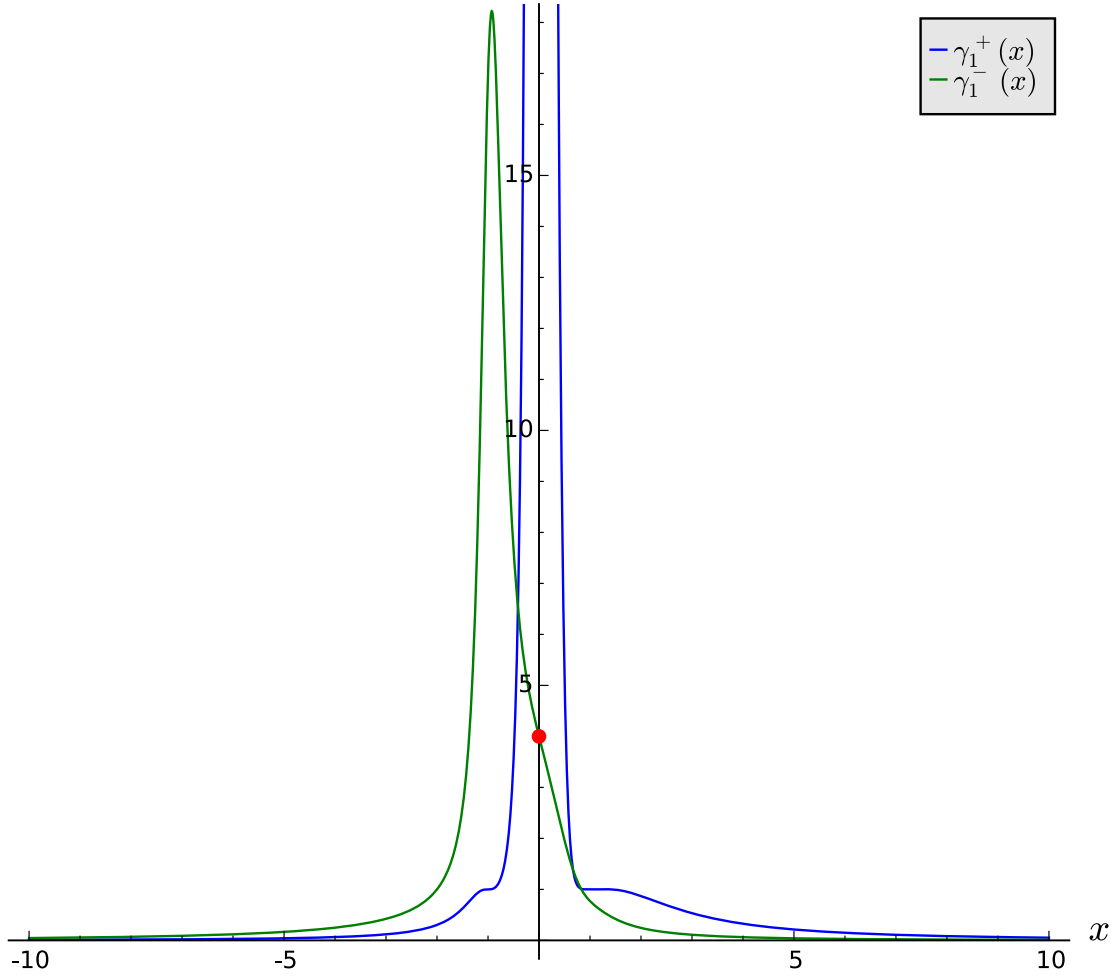
Figure 2.6: Notice the value of $\gamma_1^-(x)$ at $x = 0$.

$$\Delta(x, 0.28^2) = 0.0784\,x^6 - x^4 - 1.8432\,x^3 - 0.9216\,x^2 + 0.0784$$

has 4 complex roots and 2 real ones[2]. These last are $\delta_0 \approx 0.236$ and $\delta_1 \approx 4.336$. Thus $\Delta(\mathbf{x}, v)$ has no negative roots for $v \geq 0.28^2$ and, in addition, it only attains positive values. Therefore $(-\infty, 0) \subset D_v$. But then, as γ_v^+ is continuous and recalling the limits computed in Step 2, we get

$$(0, +\infty) \subset \mathrm{im}(\gamma_v^+) \subset \mathrm{im}(\gamma_v^+) \cup \mathrm{im}(\gamma_v^-).$$

Step 4 *When $0 < v < 0.28^2$ we have $(0, +\infty) \subset \mathrm{im}(\gamma_v^-)$.*

To prove that for $0 < v < 0.28^2$ the inclusion $(0, +\infty) \subset \mathrm{im}(\gamma_v^-)$ holds it is enough to prove the existence of $N_v, \delta_v \in \mathbb{R}$ satisfying

$$\boxed{N_v < \delta_v, \quad (-\infty, N_v] \cup [\delta_v, +\infty) \subset D_v \text{ and } \gamma_v^-(N_v) > \gamma_v^+(\delta_v)} \qquad (\spadesuit)$$

See figure 2.8 to get an idea of what we are saying here.

---

[2]The value $v_0 := 0.28^2$ comes from a careful observation of the plot from figure 2.7b.

(a) Plot of $v(x)$.

(b) Detail of the function when $x \in (-4, 0)$.
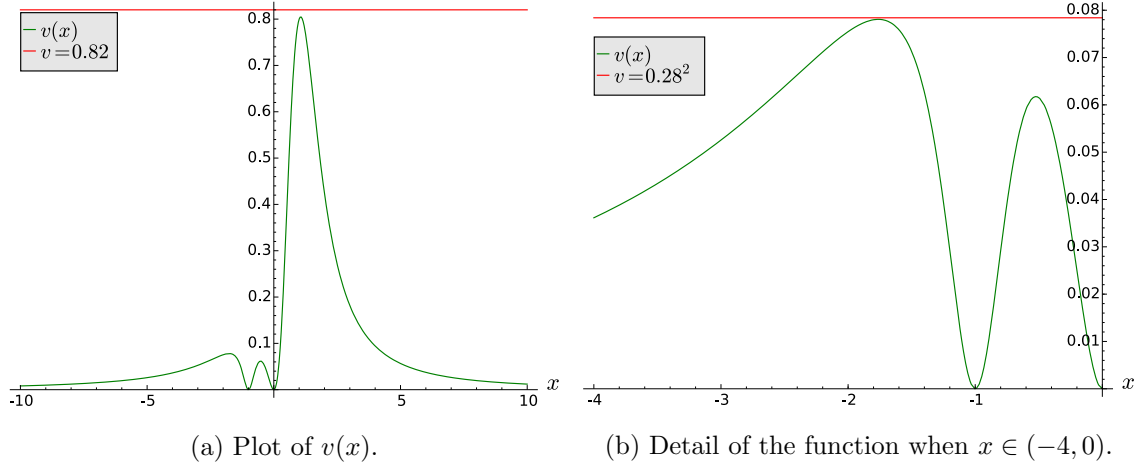
Figure 2.7: Plot of the univariate function $v(x)$.

To prove the existence of such N_v and δ_v, we must compute the roots of $\Delta_v(\mathbf{x})$ in an algebraic closure of the field of rational functions $\mathbb{R}(v)$. Such an algebraic closure is the field of Puiseux series $\mathbb{C}(\{v^*\})$, see A.9. These roots are power series in $\mathbb{C}(\{w^*\})$ with $w := \sqrt{v}$, and we consider the largest and the smallest negative roots $\eta_v, \xi_v \in \mathbb{R}(\{v^*\})$ of Δ_v with respect to the unique ordering in $\mathbb{R}(\{v^*\})$ that makes v positive and infinitesimal with respect to $\mathbb{R}$. These roots are:

$$\begin{cases} \eta_v := -\dfrac{1}{w} + 1 + w + w^2 + \dfrac{5}{2}w^3 + \cdots \\ \xi_v := -w - w^2 - \dfrac{5}{2}w^3 - 6w^4 + \cdots \end{cases}$$

Notice that, by the definition of the ordering in $\mathbb{R}(\{v^*\})$, the first coefficient of a series is the "most meaningful orderwise". In particular $\eta_v < \xi_v$. To perform calculations, we handle suitable truncations of the involved series. Here the word suitable means "as short as possible but order preserving"; in other words, we look for N_v and δ_v with $N_v < \eta_v < \xi_v < \delta_v$, and in fact we choose

$$\begin{cases} N_v := -\dfrac{1}{w} + 1 + w + w^2 = \eta_v - \left(\dfrac{5}{2}w^3 + \cdots\right) < \eta_v \\ \delta_v := -w - w^2 - \dfrac{5}{2}w^3 = \xi_v - (-6w^4 + \cdots) > \xi_v \end{cases}$$

We checked with Sage that $-\infty < N_v < \delta_v < 0$ for $v \in (0, 0.28^2)$ that is, for $w \in (0, 0.28)$, see Figure 2.9a. Now we can focus on proving ($\spadesuit$). Since $\Delta(N_{w^2}, w^2)$ and $\Delta(\delta_{w^2}, w^2)$ are positive (see Figure 2.9b) for $w \in (0, 0.28)$, we get that $N_v, \delta_v \in D_v$. For the first part, let

$$D := \bigcup_{v>0} D_v = \bigcup_{v>0} \{x \in \mathbb{R} : \Delta(x, v) \geq 0, \ x \neq 0\},$$

whose boundary is the union of the axis $\{x = 0\} \subset \mathbb{R}^2$ and the curve given by the equation

$-\gamma^-(x)$

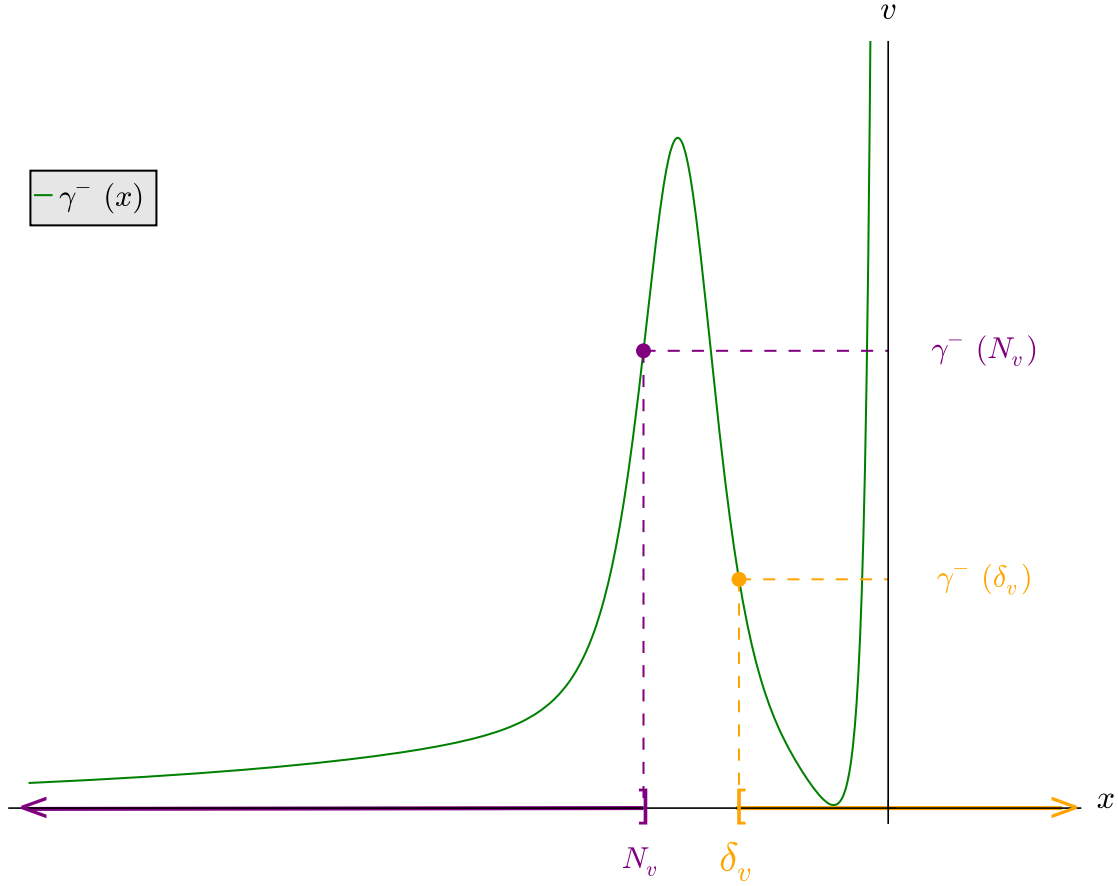$\gamma^-(N_v)$

$\gamma^-(\delta_v)$

N_v δ_v

Figure 2.8: Idea of what we are saying with ($\spadesuit$). Here $v := 0.1$.

$\Delta(x, v) = 0$, that is, the graph of the regular function

$$v(x) = \frac{x^2(x+1)^2}{x^2 + (x^3 + 1)^2}.$$

This graph is above the axis $\{v = 0\} \subset R^2$. Then, $(-\infty, N_v]$ and $[\delta_v, 0)$ are contained in the interior of D_v for $v \in (0, 0.28^2)$, because the curves

$$\{(\delta_v, v) : 0 < v < 0.28^2\} \text{ and } \{(N_v, v) : 0 < v < 0.28^2\}$$

are contained in D, they are graphs above the vertical axis $\{x = 0\} \subset R^2$, and $\delta_v < \xi_v$ and $N_v < \eta_v$ as we saw before. Look at figure 2.10.

So the only thing left to do is checking that $\gamma_v^-(N_v) > \gamma_v^-(\delta_v)$. Recall that

$$\gamma_v^-(\mathbf{x}) = \frac{A_2(\mathbf{x}, v) + B_2(\mathbf{x}, v)\sqrt{\Delta(\mathbf{x}, v)}}{C(\mathbf{x})},$$

(a) $-\infty < N_v < \delta_v < 0$.
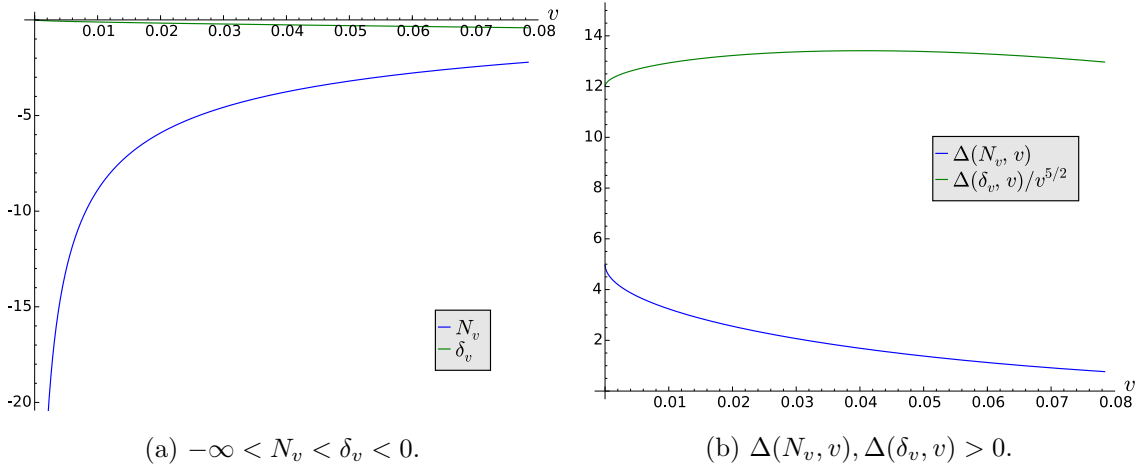
(b) $\Delta(N_v, v), \Delta(\delta_v, v) > 0$.

Figure 2.9: Plots of $N_v, \delta_v < 0$ and $\Delta(N_v, v),\ \Delta(\delta_v, v) > 0$ for $v \in (0, 0.28^2)$.

with $\deg_{\mathbf{x}}(A_2) = 24,\ \deg_{\mathbf{x}}(B_2) = 21,\ \deg_{\mathbf{x}}(\Delta) = 6\ $ and $\deg_{\mathbf{x}}(C) = 26$. Consider:

$$\cdot\ f_1(w) = A_2(N_{w^2}, w^2) \cdot w^{24} \qquad \cdot\ f_2(w) = A_2(\delta_{w^2}, w^2)$$

$$\cdot\ g_1(w) = B_2(N_{w^2}, w^2) \cdot w^{21} \qquad \cdot\ g_2(w) = B_2(\delta_{w^2}, w^2)$$

$$\cdot\ q_1(w) = \Delta(N_{w^2}, w^2) \qquad\qquad \cdot\ q_2(w) = \Delta(\delta_{w^2}, w^2)$$

$$\cdot\ h_1(w) = C(N_{w^2}) \cdot w^{26} \qquad\quad\ \cdot\ h_2(w) = C(\delta_{w^2}).$$

Thus, we need to prove that for $w \in (0, 0.28)$:

$$\frac{f_1 \cdot (w^{24})^{-1} + g_1 \cdot (w^{21})^{-1}\sqrt{q_1}}{h_1 \cdot (w^{26})^{-1}} > \frac{f_2 + g_2\sqrt{q_2}}{h_2} \iff$$

$$\frac{w^2 h_2 f_1 - f_2 h_1}{h_1 h_2} + \frac{w^5 g_1 \sqrt{q_1}}{h_1} - \frac{g_2 \sqrt{q_2}}{h_2} > 0,$$

and we are going to prove that

$$\Lambda_1 := \frac{w^2 h_2 f_1 - f_2 h_1}{h_1 h_2}, \quad \Lambda_2 := \frac{w^5 g_1 \sqrt{q_1}}{h_1} \text{ and } \Lambda_3 := -\frac{g_2 \sqrt{q_2}}{h_2}$$

are positive in the given interval, which only contains positive values. As q_1, q_2 are positive, we can clear away w^5 and $\sqrt{q_1}$ from Λ_2, and $\sqrt{q_2}$ from Λ_3. Furthermore, the polynomial $C(\mathbf{x}) = \mathbf{x}^2(\mathbf{x}^2 + (\mathbf{x}^3 + 1)^2)^4 > 0$, so we can also remove h_1 and h_2 from $\Lambda_1, \Lambda_2, \Lambda_3$. Thus it suffices to see that

$$L := \frac{w^2 h_2 f_1 - f_2 h_1}{w^4}, \quad g_1, \quad K := -\frac{g_2}{w^3}$$
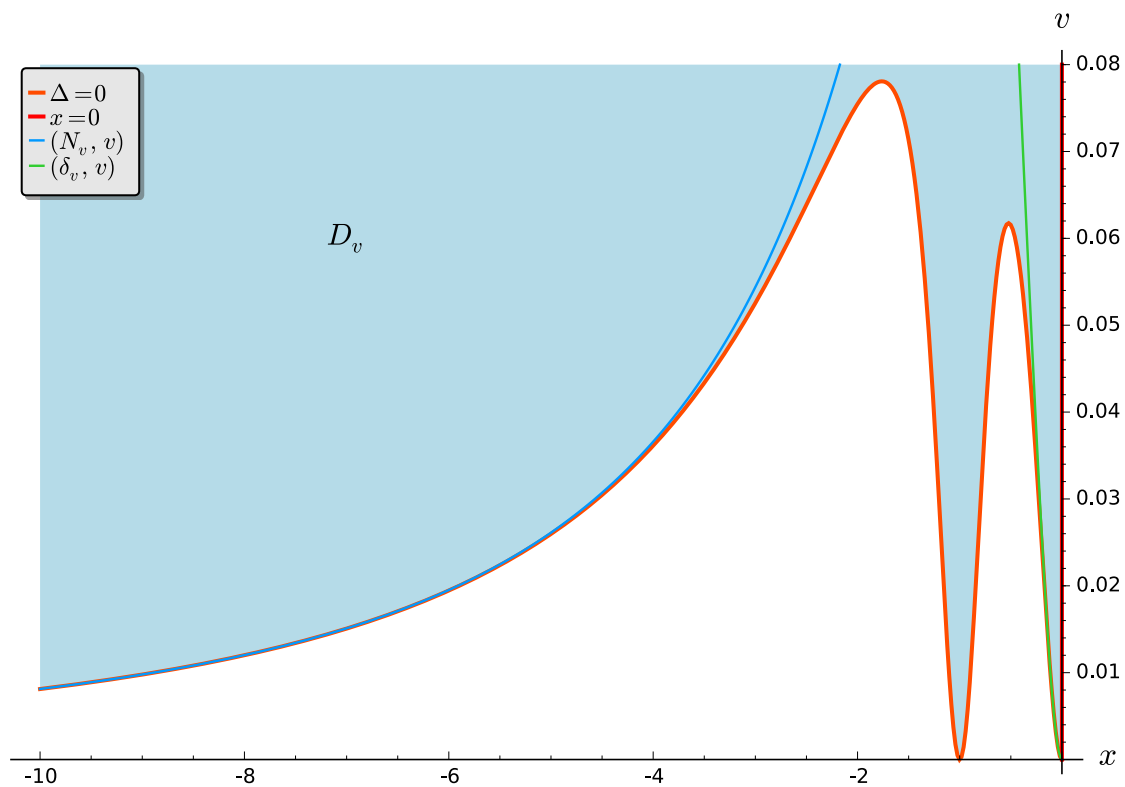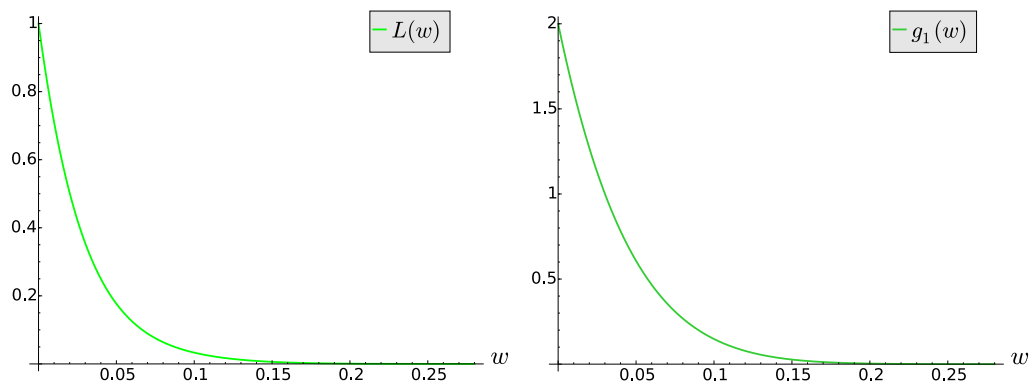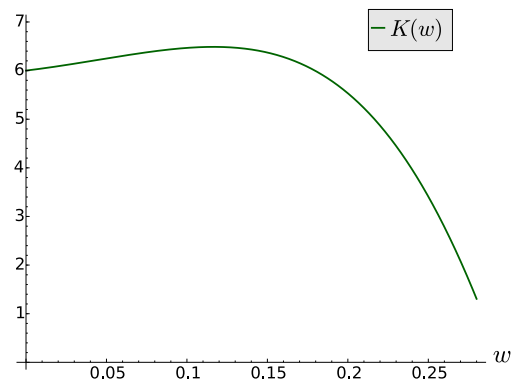
are positive for $w \in (0, 0.28)$.

Figure 2.10: Plot of $\{(N_v, v)\}, \{(\delta_v, v)\} \subset D_v$, for $0 < v < 0.28^2$.

As we see in the figures they indeed are. This has been also checked with Sturm algorithm and numerically with Sage. Thus, ($\spadesuit$) holds and the result is proved.

A short proof for the open quadrant $\mathcal{Q}$ problem

3.1 A new approach

In the previous chapter, we proved that $\mathcal{Q}$ is the image of $\mathbb{R}^2$ under the polynomial map $\mathcal{P} := (\mathcal{F}, \mathcal{G})$. This fact alongside Theorem 1.9 were key for proving Theorem 1.10. To achieve Theorem 1.11, we needed the aid of a computer in order to check that certain polynomials do not have roots on particular intervals or that they are positive on them. Although legit, this procedure is controversial, and the authors kept on working on the problem of characterizing which semialgebraic subsets $\mathcal{S} \subset \mathbb{R}^m$ are polynomial images of $\mathbb{R}^n$ and wrote a new paper ([FU]) with a much simpler proof.

In this third chapter we present a new approach to the open quadrant problem. We will show that $\mathcal{Q}$ is the image of the composition of three simple polynomial maps: $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. The proof of Theorem 1.11 will be conducted by inspecting at the images of the aforementioned polynomials, albeit the lack of precise tools to determine the image of a polynomial map.

3.1.1 The new polynomial maps

In this subsection we introduce new polynomial maps that satisfy the requirements to prove Theorem 1.11 in a different way. To be more precise, we define

$$f := \mathcal{H} \circ \mathcal{G} \circ \mathcal{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

where

$$
\boxed{
\begin{aligned}
&\mathcal{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ (x, y) \longmapsto ((xy - 1)^2 + x^2, \ (xy - 1)^2 + y^2), \\
&\mathcal{G} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ (x, y) \longmapsto (x, \ y(xy - 2)^2 + x(xy - 1)^2), \\
&\mathcal{H} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ (x, y) \longmapsto (x(xy - 2)^2 + \tfrac{1}{2}xy^2, \ y).
\end{aligned}
}
$$

We can see how the graphs of the components of these maps look like in figure 3.1, and appreciate the symmetry between the x and y variables while the transformation is performed.

In the next section we proceed to develop the proof, which splits into three lemmas.
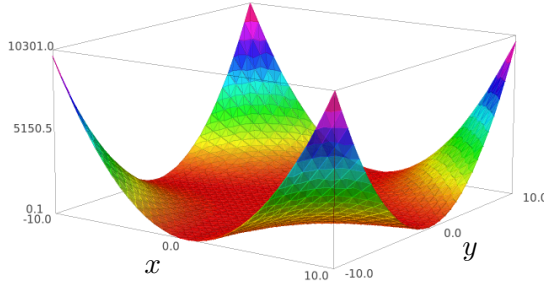
(a) $\mathcal{F}_1(\mathbf{x}, \mathbf{y}) = (\mathbf{x}\mathbf{y} - 1)^2 + \mathbf{x}^2.$
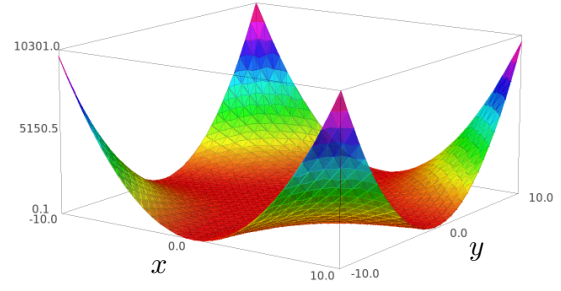
(b) $\mathcal{F}_2(\mathbf{x}, \mathbf{y}) = (\mathbf{x}\mathbf{y} - 1)^2 + \mathbf{y}^2.$
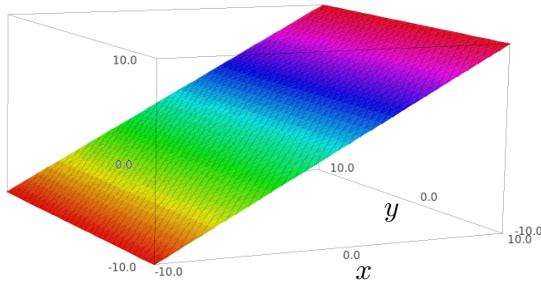
(c) $\mathcal{G}_1(\mathbf{x}, \mathbf{y}) = \mathbf{x}.$
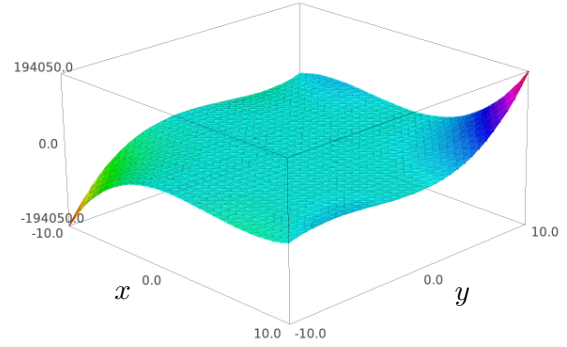
(d) $\mathcal{G}_2(\mathbf{x}, \mathbf{y}) = \mathbf{y}(\mathbf{x}\mathbf{y} - 2)^2 + \mathbf{x}(\mathbf{x}\mathbf{y} - 1)^2.$
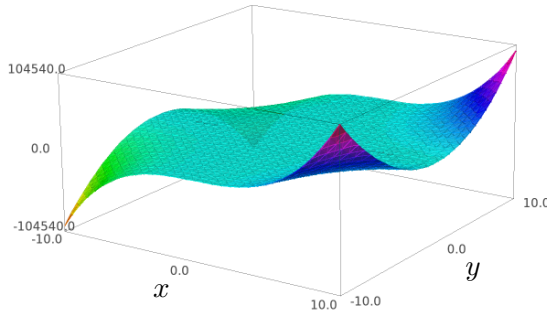
(e) $\mathcal{H}_1(\mathbf{x}, \mathbf{y}) = \mathbf{x}(\mathbf{x}\mathbf{y} - 2)^2 + \frac{1}{2}\mathbf{x}\mathbf{y}^2.$
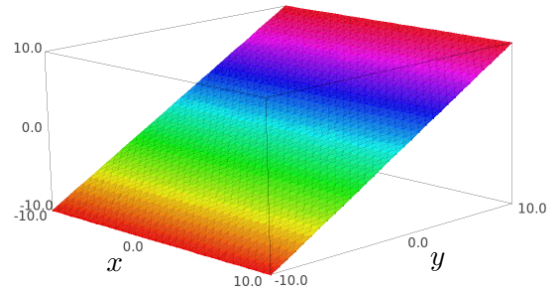
(f) $\mathcal{H}_2(\mathbf{x}, \mathbf{y}) = \mathbf{y}.$

Figure 3.1: The polynomial maps from the second proof: $(\mathcal{H} \circ \mathcal{G} \circ \mathcal{F})(\mathbb{R}^2) = \mathcal{Q}.$

3.2 The new proof for the open quadrant problem

As we have anticipated before, $\mathcal{Q}$ is the image of the composition $\mathcal{H} \circ \mathcal{G} \circ \mathcal{F}$, but before proving this fact we need three auxiliar lemmas in order to shed some light on the properties that the images of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ enjoy.

3.2.1 The first lemma

Lemma 3.1. Let $\mathcal{A} := \{xy \geq 1\} \cap \mathcal{Q}$. Then the image of the map

$$\mathcal{F} := (\mathcal{F}_1, \mathcal{F}_2) : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto ((xy - 1)^2 + x^2, (xy - 1)^2 + y^2)$$

satisfies that $\mathcal{A} \subset \mathcal{F}(\mathbb{R}^2) \subset \mathcal{Q}$.

Proof. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are positive on $\mathbb{R}^2$, the inclusion $\mathcal{F}(\mathbb{R}^2) \subset \mathcal{Q}$ is evident. To prove the other inclusion, we must show that for $(a, b) \in \mathcal{A}$ the system of polynomial equations

$$\begin{cases} (\mathbf{xy} - 1)^2 + \mathbf{x}^2 = a \\ (\mathbf{xy} - 1)^2 + \mathbf{y}^2 = b \end{cases} \tag{3.1}$$

has a solution $(x_0, y_0) \in \mathbb{R}^2$. Set $\mathbf{z} := \mathbf{xy} - 1$ in order to rewrite the system 3.1 in terms of $\mathbf{x}$ and $\mathbf{z}$. Since $\mathbf{y} = \frac{\mathbf{z}+1}{\mathbf{x}}$ we get:

$$\begin{cases} \mathbf{z}^2 + \mathbf{x}^2 = a \\ \mathbf{z}^2 + \dfrac{(\mathbf{z}+1)^2}{\mathbf{x}^2} = b. \end{cases} \tag{3.2}$$

By eliminating $\mathbf{x}$ on the system 3.2 we deduce that $\mathbf{z}$ must be a root of the polynomial

$$P(\mathbf{z}) := \mathbf{z}^4 - (a + b + 1)\mathbf{z}^2 - 2\mathbf{z} + (ab - 1) = 0.$$

Taking into account that $(a, b) \in \mathcal{A}$ satisfy that $a, b > 0$ and $ab \geq 1$, we notice that P is a monic polynomial of even degree satisfying

$$P(0) = ab - 1 \geq 0 \quad \text{and} \quad P(\sqrt{a}) = -2\sqrt{a} - a - 1 < 0.$$

Thus, P has a real root z_0 such that $0 \leq z_0 < \sqrt{a}$, so we set:

$$x_0 := \sqrt{a - z_0^2} \quad \text{and} \quad y_0 := \frac{z_0 + 1}{x_0}.$$

Then $F(x_0, y_0) = (a, b)$ and $\mathcal{A} \subset \mathcal{F}(\mathbb{R}^2)$, as required. $\qquad \square$

3.2.2 The second lemma

Lemma 3.2. Let $\mathcal{B} := \mathcal{A} \cup \{y \geq x > 0\}$. Then, the map

$$\mathcal{G} := (\mathcal{G}_1, \mathcal{G}_2) : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x, \ y(xy - 2)^2 + x(xy - 1)^2)$$

satisfies that $\mathcal{B} \subset \mathcal{G}(\mathcal{A}) \subset \mathcal{G}(\mathcal{Q}) \subset \mathcal{Q}$.

(a) $\mathcal{A} = \{xy \geq 1\} \cap \mathcal{Q}$.
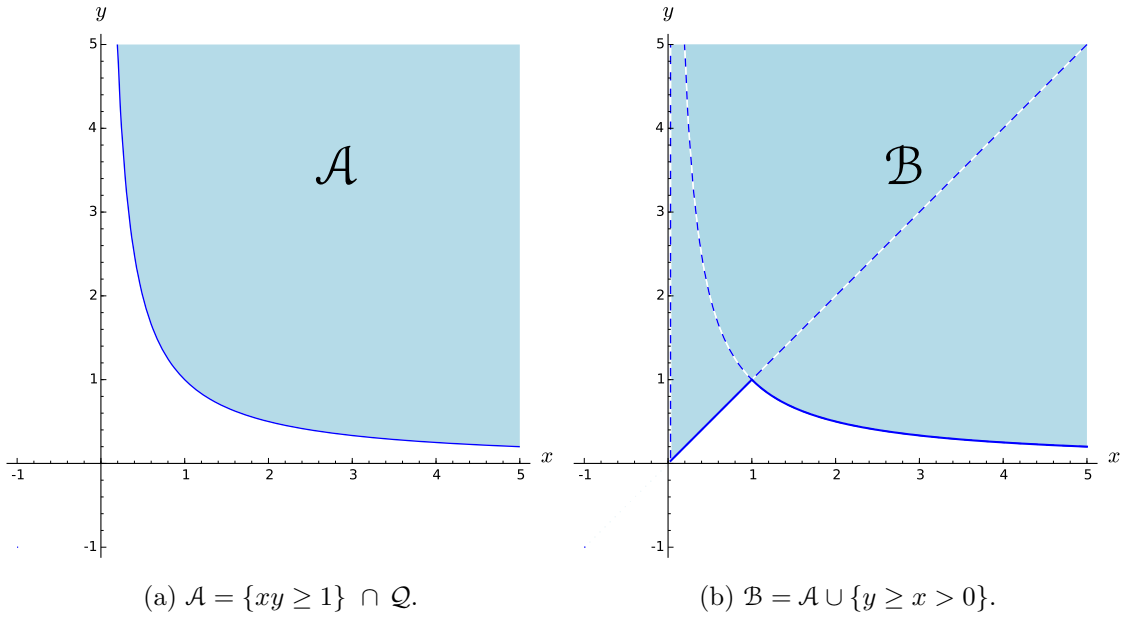
(b) $\mathcal{B} = \mathcal{A} \cup \{y \geq x > 0\}$.

Figure 3.2: Relevant sets for Lemmas 3.1, 3.2 and 3.3.

Proof. The inclusion $\mathcal{G}(\mathcal{A}) \subset \mathcal{G}(\mathcal{Q})$ is trivial because $\mathcal{A} \subset \mathcal{Q}$. The last inclusion $\mathcal{G}(\mathcal{Q}) \subset \mathcal{Q}$ also holds since $\mathcal{G}_1$ and $\mathcal{G}_2$ are strictly positive on $\mathcal{Q}$: if $x_0, y_0 > 0$ then $\mathcal{G}_1(x_0, y_0) = x_0 > 0$ and $\mathcal{G}_2(x_0, y_0) = y_0(x_0y_0 - 2)^2 + x_0(x_0y_0 - 1)^2 > 0$. Notice that $\mathcal{G}_2(x_0, y_0) \neq 0$. Otherwise there would exists a solution of the system of equations

$$\begin{cases} xy - 1 = 0 \\ xy - 2 = 0, \end{cases}$$

which is not possible.

Now we can focus on proving $\mathcal{B} \subset \mathcal{G}(\mathcal{A})$. First of all, notice that we can express the set $\mathcal{B}$ as a union of closed half-lines. Precisely, for each $x > 0$ let $\mathcal{B}_x := [y_x, +\infty)$, where $y_x := \min\{x, 1/x\}$ and notice that

$$\mathcal{B} = \bigsqcup_{x>0} (\{x\} \times \mathcal{B}_x).$$

That is, we are "slicing" the set $\mathcal{B}$ vertically, depending on x. Now look at the definition of the polynomial $\mathcal{G}_2$ and consider for each $x > 0$ the polynomial $\phi_x \in \mathbb{R}[\mathrm{y}]$ (dependent on the variable y):

$$\phi_x(\mathrm{y}) := \mathrm{y}(x\mathrm{y} - 2)^2 + x(x\mathrm{y} - 1)^2 = x^2\mathrm{y}^3 + (x^3 - 4x)\mathrm{y}^2 + (4 - 2x^2)\mathrm{y} + x.$$

The polynomials $\phi_x(\mathrm{y})$ have odd degree and positive leading coefficient since $x > 0$. Now notice that if we fix $x_0 > 0$ then we get the following sequence of inclusions:

$$\phi_{x_0}([1/x_0, +\infty)) \supset^1 \phi_{x_0}([2/x_0, +\infty))$$
$$\cup^2 \qquad\qquad\qquad \cup^3$$
$$[1/x_0, +\infty) \qquad\qquad [x_0, +\infty)$$

Inclusion [1] follows from the fact that $[1/x_0, +\infty) \supset [2/x_0, +\infty)$ (notice that for $x > 0$ the graph of the map $2/x$ is "above" the one of $1/x$). Inclusion [2] and [3] follow from the fact that $\phi_{x_0}(\mathbf{y})$ has positive leading coefficient and the computation of the images of $1/x_0$ and $2/x_0$ through $\phi_{x_0}(\mathbf{y})$:

$$\phi_{x_0}\left(\frac{1}{x_0}\right) = \frac{1}{x_0} + \left((x_0^3 - 4x_0)\frac{1}{x_0^2} + (4 - 2x_0^2)\frac{1}{x_0} + x_0\right) = \frac{1}{x_0}$$

$$\phi_{x_0}\left(\frac{2}{x_0}\right) = \left(\frac{8}{x_0} + (x_0^3 - 4x_0)\frac{4}{x_0^2} + (4 - 2x_0^2)\frac{2}{x_0}\right) + x_0 = x_0.$$

Now it is clear that:

$$\mathcal{B}_x = [y_x, +\infty) \subset \phi_x([1/x, +\infty)).$$

Then we can prove the desired inclusion in this way:

$$\mathcal{B} = \bigsqcup_{x>0}\left(\{x\} \times \mathcal{B}_x\right) \subset \bigsqcup_{x>0}\left(\{x\} \times \phi_x([1/x, +\infty))\right) = \bigsqcup_{x>0}\mathcal{G}\left(\{x\} \times [1/x, +\infty)\right) = \mathcal{G}(\mathcal{A}),$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

(a) $\mathcal{B} = \bigsqcup_{x>0}(\{x\} \times \mathcal{B}_x)$.

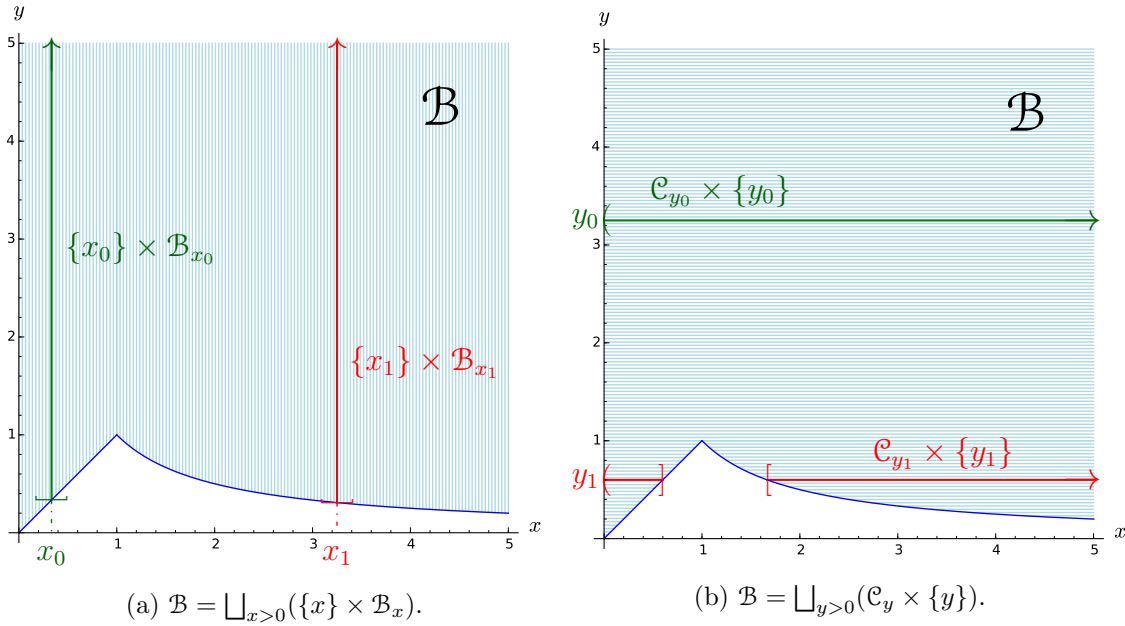(b) $\mathcal{B} = \bigsqcup_{y>0}(\mathcal{C}_y \times \{y\})$.

Figure 3.3: Idea of how we represent the set $\mathcal{B}$ in Lemmas 3.2 and 3.3.

3.2.3 The third lemma

Lemma 3.3. The polynomial map

$$\mathcal{H} := (\mathcal{H}_1, \mathcal{H}_2) : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x(xy - 2)^2 + \tfrac{1}{2}xy^2,\ y)$$

satisfies $\mathcal{H}(\mathcal{B}) = \mathcal{H}(\mathcal{Q}) = \mathcal{Q}$.

Proof. The inclusion $\mathcal{H}(\mathcal{B}) \subset \mathcal{H}(\mathcal{Q})$ is trivial because $\mathcal{B} \subset \mathcal{Q}$. As for $\mathcal{H}(\mathcal{Q}) \subset \mathcal{Q}$, since $\mathcal{H}_1$ and $\mathcal{H}_2$ are strictly positive on $\mathcal{Q}$ the inclusion holds.

We proceed now to prove that $\mathcal{Q} \subset \mathcal{H}(\mathcal{B})$ and therefore get $\mathcal{Q} \subset \mathcal{H}(\mathcal{B}) \subset \mathcal{H}(\mathcal{Q}) \subset \mathcal{Q}$, which implies that $\mathcal{H}(\mathcal{B}) = \mathcal{H}(\mathcal{Q}) = \mathcal{Q}$.

Firstly, notice that the set $\mathcal{B}$ can be expressed as follows:

$$\mathcal{B} = \bigsqcup_{y>0} (\mathcal{C}_y \times \{y\}), \quad \text{where } \mathcal{C}_y := \begin{cases} (0, +\infty) & \text{if } y \geq 1, \\ (0, y] \cup [1/y, +\infty) & \text{if } 0 < y < 1. \end{cases}$$

Intuitively, we are "slicing" the set $\mathcal{B}$ horizontally, depending on y. Looking at the definition of $\mathcal{H}_1$ we define for each $y > 0$ the polynomial ψ_y dependent on the variable $\mathtt{x}$:

$$\psi_y(\mathtt{x}) := \mathtt{x}(y\mathtt{x} - 2)^2 + \tfrac{1}{2}y^2\mathtt{x} = y^2\mathtt{x}^3 - 4y\mathtt{x}^2 + (4 + \tfrac{1}{2}y^2)\mathtt{x}.$$

Notice first that $\psi_y(\mathtt{x})$ has odd degree, positive leading coefficient and the following properties:

(i) $\lim_{x \to +\infty} \psi_y(x) = +\infty$.   (iii) $\psi_y(y) = y((y^2 - 2)^2 + \tfrac{1}{2}y^2) > y$ for $0 < y < 1$.

(ii) $\psi_y(0) = 0$.   (iv) $\psi_y\left(\tfrac{2}{y}\right) = y$.

Property (iii) holds because $(y^2 - 2)^2 + \tfrac{1}{2}y^2 > 1$ when $0 < y < 1$. Next, we are going to show that:

$$\psi_y(\mathcal{C}_y) = \begin{cases} \psi_y((0, +\infty)) =^1 (0, +\infty) & \text{if } y \geq 1, \\ \psi_y\left((0, y] \cup [\tfrac{1}{y}, +\infty)\right) \supset^2 (0, \psi_y(y)] \cup [\psi_y(\tfrac{2}{y}), +\infty) =^3 (0, +\infty) & \text{if } 0 < y < 1. \end{cases}$$

Equality ^{1} holds because of (i), (ii) and the fact that ψ_y is strictly positive on $(0, +\infty)$. Inclusion ^{2} follows from the fact that $[1/y_0, +\infty) \supset [2/y_0, +\infty)$ for each fixed y_0. Finally, equality ^{3} holds because of (iii) and (iv): $\psi_y(y) > y = \psi_y(\tfrac{2}{y})$.

Then we can write:

$$\mathcal{Q} = \bigsqcup_{y>0} \left((0, +\infty) \times \{y\}\right) \subset \bigsqcup_{y>0} \left(\psi_y(\mathcal{C}_y) \times \{y\}\right) = \bigsqcup_{y>0} \mathcal{H}(\mathcal{C}_y \times \{y\}) = \mathcal{H}(\mathcal{C}_y),$$

which concludes the proof. $\qquad\square$

3.2.4 The second proof for the open quadrant problem

To wrap up this chapter, we can now write the proof of Theorem 1.11 relying on Lemmas 3.1, 3.2 and 3.3:

[Proof of Theorem 1.11] Applying the lemmas, we deduce that

$$\mathcal{Q} \overset{3.3}{=} \mathcal{H}(\mathcal{B}) \overset{3.2}{\subset} (\mathcal{H} \circ \mathcal{G})(\mathcal{A}) \overset{3.1}{\subset} (\mathcal{H} \circ \mathcal{G} \circ \mathcal{F})(\mathbb{R}^2) \overset{3.1}{\subset} (\mathcal{H} \circ \mathcal{G})(\mathcal{Q}) \overset{3.2}{\subset} \mathcal{H}(\mathcal{Q}) \overset{3.3}{=} \mathcal{Q},$$

which means

$$(\mathcal{H} \circ \mathcal{G} \circ \mathcal{F})(\mathbb{R}^2) = \mathcal{Q},$$

as required. $\qquad\square$

A topological proof for the open quadrant $\mathcal{Q}$ problem

4.1 A topological perspective

Not happy with having proved that $\mathcal{Q}$ is the image of $\mathbb{R}^2$ under two different polynomial maps using diverse techniques, the authors developed a whole new proof that relies on algebraic topology for a new polynomial map $\mathscr{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ on the paper [FGU]. The genesis of this work was their collaboration with C. Ueno while he was writing his Ph.D. Thesis [U].

In this chapter we will build up the topological machinery that will allow us to prove theorem 1.11 in a very geometrical way, and completely avoiding the aid of computer calculations.

4.1.1 The new polynomial map

Without further ado, the polynomial map we are going to be working with is $\mathscr{F}$ and it is defined in the following way:

$$\mathscr{F}(\mathbf{x}, \mathbf{y}) := \left((\mathbf{x}^2\mathbf{y}^4 + \mathbf{x}^4\mathbf{y}^2 - \mathbf{y}^2 - 1)^2 + \mathbf{x}^6\mathbf{y}^4, \ (\mathbf{x}^6\mathbf{y}^2 + \mathbf{x}^2\mathbf{y}^2 - \mathbf{x}^2 - 1)^2 + \mathbf{x}^6\mathbf{y}^4\right).$$

As we anticipated before, in the next section we will give some topological definitions and proof some facts about a family of topological spaces homeomorphic[1] to a closed disc in $\mathbb{R}^3$, and paths that go through them.

Afterwards, by decomposing $\mathscr{F}$ into the composition $\mathscr{F} = f_2 \circ f_1$ with $f_2 = h \circ g$, it will be clear that we can reduce the proof to check that the boundaries of two certain topological spaces (homeomorphic to a closed disc) have non-empty intersection with the image $g(\overline{\mathcal{Q}})$, where $\overline{\mathcal{Q}} = \{x \geq 0, y \geq 0\}$ is the closure of the open quadrant $\mathcal{Q}$.

4.2 Topological tools

4.2.1 Some definitions

We begin by defining the key topological varieties we are going to be working with:

---

[1]Meaning there exists a continuos map that has a continuous inverse.

Definitions 4.1. (i) Let $\mathbb{D}_A = \{(x, y) : x^2 + y^2 \leq A^2\} \subset \mathbb{R}^2$ be the closed disc of center the origin and radius $A > 0$. A *warped disc* is a subset

$$\mathcal{D}_{A,\xi} := \{(x, y, z) : x^2 + y^2 \leq A^2, \ z = \xi(x, y)\} \subset \mathbb{R}^3$$

where $\xi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a continuous function. Let ζ be the homeomorphism

$$\zeta : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \ (x, y, z) \longmapsto \big(x, \ y, \ z - \xi(x, y)\big).$$

Then, it is clear that $\zeta(\mathcal{D}_{A,\xi}) = \mathbb{D}_A \times \{0\}$ is the projection or "flattening" of the warped disc $\mathcal{D}_{A,\xi}$ onto the $x - y$ plane. We will also call a warped disc the image of $\mathcal{D}_{A,\xi}$ under a permutation of the variables of $\mathbb{R}^3$:

$$\mathcal{D}_{A,\xi} := \{(x, y, z) : y^2 + z^2 \leq A^2, \ x = \xi(y, z)\} \ \text{ or}$$
$$\mathcal{D}_{A,\xi} := \{(x, y, z) : x^2 + z^2 \leq A^2, \ y = \xi(x, z)\},$$

with ξ defined as above.

(ii) For each $\varepsilon > 0$ let us define an open neighborhood of $\mathbb{D}_A$ as follows:

$$\mathbb{D}_A(\varepsilon) := \{(x, y) : x^2 + y^2 < (A + \varepsilon)^2\} \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^3.$$

Clearly, in this way we obtain an open neighborhood of $\mathcal{D}_{A,\xi}$ by defining:

$$\mathcal{D}_{A,\xi}(\varepsilon) := \zeta^{-1}\big(\mathbb{D}_A(\varepsilon)\big) \subset \mathbb{R}^3,$$

which is guaranteed to be well-defined because ζ is an homeomorphism.

Next, we define the topological concepts of *path* and *path concatenation*, and we explain what *meeting transversally once a warped disc* means, which is just a formalization of our intuition. Finally, we will define the *fundamental group* of a topological space in the usual way.

Definitions 4.2. (i) Let X be a topological space. A *path* in X is a continuos map $\alpha : [a, b] \to X$, where $[a, b] \subset \mathbb{R}$ is a closed interval. If α verifies that $\alpha(a) = \alpha(b) = x_0 \in \mathbb{R}^3$, then we say that α is a *loop* with base point x_0. When this is the case, we can consider α as a a map whose domain is the quotient space $[a, b]/\sim$ obtained by identifying the end points of $[a, b]$, which is homeomorphic to $\mathbb{S}^1 := \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$.

(ii) Given two paths $\alpha : [a, b] \to X$ and $\beta : [c, d] \to X$ verifying $\alpha(b) = \beta(c)$, we define the *composition (or product) of paths* in the following way:

$$\alpha * \beta : [a, b + (d - c)] \longrightarrow X, \ t \longmapsto \begin{cases} \alpha(t) & \text{if } t \in [a, b], \\ \beta(t + c - b) & \text{if } t \in [b, b + (d - c)]. \end{cases}$$

The composition of paths verifies the associative property. Also, there exists an identity element e such that for every path α we have $\alpha * e = e * \alpha = \alpha$. Lastly, for every path α there exists an inverse element α^{-1} such that $\alpha * \alpha^{-1} = \alpha^{-1} * \alpha = e$.

(iii) A path $\alpha : [a, b] \to \mathbb{R}^3$ *meets transversally once* the warped disc $\mathcal{D}_{A,\xi}$ if there exist $s_0 \in (a, b)$ and $\varepsilon > 0$ such that

$$J := \alpha^{-1}(\mathcal{D}_{A,\xi}(\varepsilon)) = (s_0 - \varepsilon, s_0 + \varepsilon)$$

is an open subinterval of $[a, b]$ and

$$(\zeta \circ \alpha)|_J(t) = (0, 0, t - s_0),$$

where obviously $(\zeta \circ \alpha)|_J(t) \subset \mathbb{D}_A(\varepsilon)$ because $\alpha|_J(t) \subset \zeta^{-1}(\mathbb{D}_A(\varepsilon)) = \mathcal{D}_{A,\xi}(\varepsilon)$.

(iv) Let C be a topological space homeomorphic to a closed disc and let $\phi : C \longrightarrow \mathbb{R}^3$ be a continuous map. The restriction $\partial\phi := \phi|_{\partial C}$ is called the *boundary map* of ϕ. We say that the boundary map $\partial\phi$ *meets transversally once a warped disc* $\mathcal{D}_{A,\xi} \subset \mathbb{R}^3$ if there exists a parameterization $\beta : [a, b] \to \partial C \cong \mathbb{S}^1$ such that the path $\alpha := \phi \circ \beta : [a, b] \to \mathbb{R}^3$ meets transversally once the warped disc $\mathcal{D}_{A,\xi}$.

(iv) Let $I := [0, 1] \subset \mathbb{R}$, $s \in I$ and X, Y be two path-connected topological spaces. We say that a *homotopy* is a continuous map $H : X \times I \longrightarrow Y$. We also say that the set $\{H_s : X \longrightarrow Y, \; x \longmapsto H(x, s)$ where the variable s is fixed$\}$ is a family of continuous maps. If $f = H_0 : X \longrightarrow Y$ and $g = H_1 : X \longrightarrow Y$ are homotopic, we denote it with $f \overset{H}{\simeq} g$, and if g is constant we say that f is *null-homotopic*.

(vi) Given a path-connected topological space X and a point $x_0 \in X$ we define the *fundamental group of X at the base point x_0* as the quotient space modulo the homotopy relation, and we denote it with $\pi_1(X, x_0)$. If α is a loop with base point x_0, then it represents an element of $\pi_1(X, x_0)$ denoted with $[\alpha]$.

Remark 4.3. If the path $\alpha : [a, b] \to \mathbb{R}^3$ meets transversally once the warped disc $\mathcal{D}_{A,\xi}$, then $\alpha([a, b]) \cap \partial\mathcal{D}_{A,\xi} = \varnothing$. This will be helpful in Proposition 4.7.

4.2.2 Useful results

We proceed now to prove some facts that later on will be helpful for the proof of Theorem 1.11. Firstly, we prove in Lemma 4.4 that if a loop meets transversaly once a warped disc, then the topological space $\mathbb{R}^3$ minus the border of the warped disc has $\mathbb{Z}$ as fundamental group. In order to do this, Figure 4.1 will be very helpful for visualizing the idea of the proof.

Lemma 4.4. Let $\mathcal{D}_{A,\xi} \subset \mathbb{R}^3$ be a warped disc and let $X := \mathbb{R}^3 \setminus \partial\mathcal{D}_{A,\xi}$. Let $\alpha : [a, b] \to X$ be a loop with base point $x_0 \in X$ that meets transversally once $\mathcal{D}_{A,\xi}$. Then $[\alpha]$ is a generator of $\pi_1(X, x_0) \cong \mathbb{Z}$.

Proof. Using the notation introduced in Definitions 4.2, let $s_0 \in (a, b)$ and $\varepsilon > 0$ be such that

$$J := \alpha^{-1}(\mathcal{D}_{A,\xi}(\varepsilon)) = (s_0 - \varepsilon, s_0 + \varepsilon) \subset [a, b]$$

and $(\zeta \circ \alpha)|_J(t) = (0, 0, t - s_0)$. We can assume $s_0 = 0$ after a reparameterization of α, and so $J = (-\varepsilon, \varepsilon)$ and $(\zeta \circ \alpha)|_J(t) = (0, 0, t)$. Note that we can write the interval $[a, b]$ like $[a, -\varepsilon] \cup (-\varepsilon, \varepsilon) \cup [\varepsilon, b]$.

Since ζ is a homeomorphism of $\mathbb{R}^3$, instead of working in the warped disc we will prove the statement in $\mathbb{D}_A$ for $\beta := \zeta \circ \alpha$, $Y := \mathbb{R}^3 \setminus \partial\mathbb{D}_A$ and the base point $y_0 := \beta(-\varepsilon) = (0, 0, -\varepsilon)$. Consider the path $\gamma : [0, 1] \to \mathbb{R}^3$ (see Figure 4.1) defined in the following way:

$$\gamma(t) := \begin{cases} (3(A + \varepsilon)t, \, 0, \, \varepsilon) & \text{if } 0 \leq t \leq \frac{1}{3}, \\ (A + \varepsilon, \, 0, \, \varepsilon - (t - \frac{1}{3})6\varepsilon) & \text{if } \frac{1}{3} < t \leq \frac{2}{3}, \\ (A + \varepsilon - 3(A + \varepsilon)(t - \frac{2}{3}), \, 0, \, -\varepsilon) & \text{if } \frac{2}{3} < t \leq 1. \end{cases}$$

Write $\beta_0 := \beta|_J = \beta|_{(-\varepsilon,\varepsilon)}$ and $\beta_1 := \beta|_{[\varepsilon,b]} * \beta|_{[a,-\varepsilon]}$. Using the properties of path composition we get:

$$[\beta] = [\beta_0 * \beta_1] = [\beta_0 * (\gamma * \gamma^{-1}) * \beta_1] = [\beta_0 * \gamma] \cdot [\gamma^{-1} * \beta_1].$$

We claim that $[\beta_0 * \gamma] = g$ and $[\gamma^{-1} * \beta_1] = e$, where g and e are respectively a generator of $\pi_1(Y, y_0) \cong \mathbb{Z}$ and the identity element.
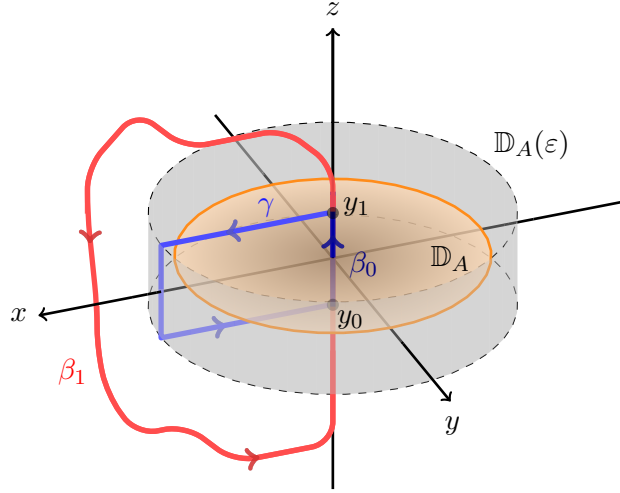

Figure 4.1: The loop $\beta = \beta_0 * \beta_1$ meets transversally once the disk $\mathbb{D}_A$. Here $y_0 := \beta(-\varepsilon)$ and $y_1 := \beta(\varepsilon)$, so the path β_0 goes from y_0 to y_1 (in purple), γ from y_1 to y_0 (in blue) and β_1 from y_1 to y_0 (in red).

Firstly, the loop $\gamma^{-1} * \beta_1$ with base point y_0 is contained in $\mathbb{R}^3 \setminus \mathbb{D}_A$, which is a simply connected space. Consequently, the loop $\gamma^{-1} * \beta_1$ is null-homotopic and $[\gamma^{-1} * \beta_1] = e \in \pi_1(Y, y_0)$.

On the other hand, the class $[\beta_0 * \gamma]$ generates $\pi_1(Y, y_0)$. Indeed, Y has as deformation retract the set $Z := \partial \mathbb{D}_A(\varepsilon) \cup I_\varepsilon$ where $\partial \mathbb{D}_A(\varepsilon)$ is the boundary of $\mathbb{D}_A(\varepsilon)$ and $I_\varepsilon := \{(0,0,z) : -\varepsilon \leq z \leq \varepsilon\}$. Finally, because the set Z is homeomorphic to a sphere union one of its diameters we can use Lemma 4.5 and obtain that $[\beta_0 * \gamma]$ is a generator of $\pi_1(Z, y_0) \cong \pi_1(Y, y_0) \cong \mathbb{Z}$, as required. $\qquad\square$

Lemma 4.5. Let $S = \{(x,y,z) : x^2 + y^2 + z^2 = A^2\} \subset \mathbb{R}^3$ be the sphere of center the origin and radius $A > 0$, let D be one of its diameters and let $M := S \cup D$. If $x_0 \in M$ then $\pi_1(M, x_0) \cong \mathbb{Z}$.

Proof. Let $x_0 \in S \setminus D$ and let $U := M \setminus \{x_0\}$. Also let V be a neighbourhood of x_0 homeomorphic to a closed disc and small enough as for to having empty intersection with the diameter D. It is clear that $M \cong U \cup V$. Consider φ the group homeomorphism

$$\varphi : \pi_1(U \cap V, x_0) \longrightarrow \pi_1(U, x_0)$$

induced by the inclusion map $\mathbf{j} : (U \cap V) \hookrightarrow U$. Because V is a simply connected space, we deduce from the Theorem of Seifert and Van Kampen [M, Chapter IV: §4, page 95]

that

$$\pi_1(M, x_0) = \pi_1(U, x_0) \Big/ \operatorname{im}(\varphi).$$

Next, notice that U is homeomorphic to a closed disc joined with an arc that connects two points of the disc and that doesn't touch its border. Thus, U has $\mathbb{S}^1$ as deformation retract and $\pi_1(U, x_0) \cong \mathbb{Z}$. What is more, since $U \cap V \cong V \setminus \{x_0\}$ the set $U \cap V$ is homeomorphic to a disc minus a point, so $\pi_1(U \cap V, x_0) \cong \pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z}$, and notice that a generator of this fundamental group is contained in $S \setminus \{x_0\} \subset U$. But then, since $S \setminus \{x_0\}$ is contractible, we obtain that $\varphi(\pi_1(U \cap V, x_0)) = 0$ and then $\pi_1(M, x_0) = \pi_1(U, x_0)$ is generated by the class of a loop α like in Figure 4.2. Thus, $\pi_1(M, x_0) = \mathbb{Z}$ as required. □
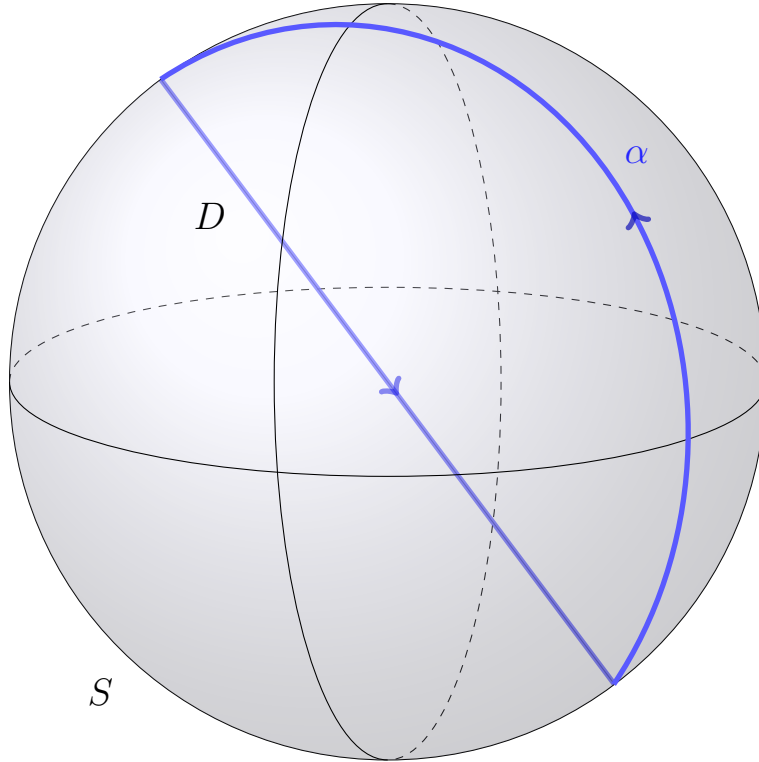

Figure 4.2: The set $M := S \cup D$ and a generator α of $\pi_1(M, x_0)$ from Lemma 4.5.

Lemma 4.6. Let $z_0 \in \partial C$ where C is a space homeomorphic to a closed disc. Let $\phi : C \longrightarrow X$ be a continuous map and consider $\beta : [a, b] \longrightarrow \partial C$ a parameterization starting and ending at z_0. Then $[\phi \circ \beta]$ is the identity element of $\pi_1\big(X, \phi(z_0)\big)$.

Proof. Let $\psi : C \longrightarrow \{x^2 + y^2 \leq 1\}$ be a homeomorphism. Let H be the continuous map

$$H : [0, 1] \times [a, b] \longrightarrow X, \ (\rho, t) \longmapsto \big(\phi \circ \psi^{-1}\big)\big(\rho \cdot (\psi \circ \beta)(t) + (1 - \rho) \cdot \psi(z_0)\big).$$

Notice that $H(0, t) = \phi(z_0)$ and that $H(1, t) = \phi\big(\beta(t)\big)$. Thus, it is clear that H is a homotopy map between $\phi \circ \beta$ and the constant path, as required. □

Proposition 4.7. *Let C be a topological space homeomorphic to a closed disc and $\phi : C \longrightarrow \mathbb{R}^3$ a continuous map. If the boundary map of ϕ (namely $\partial\phi$) meets transversally once a warped disc $\mathcal{D} \subset \mathbb{R}^3$, then $\partial\mathcal{D} \cap \phi(\overset{\circ}{C}) \neq \varnothing$.*

Proof. Suppose instead that $\partial\mathcal{D} \cap \phi(\overset{\circ}{C}) = \varnothing$. Since $\partial\phi$ meets transversally once $\mathcal{D}$, by Remark 4.3 the image of ∂C under the map ϕ does not intersect $\partial\mathcal{D}$. This alongside our initial asumption gives us that $\phi(C) \subset X := \mathbb{R}^3 \setminus \partial\mathcal{D}$.

Let $\beta : [a, b] \to \partial C$ be a parameterization starting and ending at $z_0 \in \partial C$ such that $\phi \circ \beta$ meets transversally once $\mathcal{D}$. By Lemma 4.6 the class $[\phi \circ \beta]$ is the identity element of $\pi_1(X, \phi(z_0))$. However, by Lemma 4.4 the class $[\phi \circ \beta]$ is a generator of $\pi_1(X, \phi(z_0)) \cong \mathbb{Z}$, which leads to a contradiction. Consequently, $\partial\mathcal{D} \cap \phi(\overset{\circ}{C}) \neq \varnothing$, as we wanted to. $\qquad \square$

4.3 The topological proof for the open quadrant problem

4.3.1 Decomposing $\mathscr{F}$

The authors observed in [FGU, Section 3] that the map $\mathscr{F}$ can be decomposed as $\mathscr{F} = f_2 \circ f_1$, where the polynomial maps f_1, f_2 are defined as follows:

$$f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x, y) \longmapsto (x^2, y^2),$$
$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x, y) \longmapsto ((xy^2 + x^2y - y - 1)^2 + x^3y^2, (x^3y + xy - x - 1)^2 + x^3y^2).$$

Since $f_1(\mathbb{R}^2) = \overline{\mathcal{Q}}$ it is then clear that we just need to prove that

$$f_2(\overline{\mathcal{Q}}) = \mathcal{Q}. \tag{4.1}$$

Note that both components of f_2 are strictly positive on $\overline{\mathcal{Q}}$ so the inclusion $f_2(\overline{\mathcal{Q}}) \subset \mathcal{Q}$ holds. We need to prove the converse inclusion:

$$\mathcal{Q} \subset f_2(\overline{\mathcal{Q}}). \tag{4.2}$$

A careful observation of f_2 leads us to decompose it into $f_2 = h \circ g$, where the continuous semialgebraic maps g, h are defined in the following way:

$$g : \overline{\mathcal{Q}} \longrightarrow \mathbb{R}^3, \quad (x, y) \longmapsto (xy^2 + x^2y - y - 1, x^{3/2}y, x^3y + xy - x - 1),$$
$$h : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad (x, y, z) \longmapsto (x^2 + y^2, y^2 + z^2).$$

In this way we can reduce the proof to show that for each pair of real numbers $(A^2, B^2) \in \mathcal{Q}$ there exists $(x_0, y_0) \in \overline{\mathcal{Q}}$ such that $(h \circ g)(x_0, y_0) = (A^2, B^2)$. This is equivalent to check that

$$\boxed{\forall (A^2, B^2) \in \mathcal{Q} : \ h^{-1}(\{(A^2, B^2)\}) \bigcap g(\overline{\mathcal{Q}}) \neq \varnothing.}$$

Let $\mathcal{S} := g(\overline{\mathcal{Q}})$ and fix values $B \geq A > 0$. By the definition of h we know that

$$h^{-1}(\{(A^2, B^2)\}) = \{(x, y) : x^2 + y^2 = A^2, \ y^2 + z^2 = B^2\} \text{ and}$$
$$h^{-1}(\{(B^2, A^2)\}) = \{(x, y) : y^2 + z^2 = A^2, \ x^2 + y^2 = B^2\},$$

which both of them are the intersection of two perpendicular cylinders. Besides, those sets contain respectively the boundaries of the warped discs

$$\mathcal{D}_1 := \{(x, y, z) : x^2 + y^2 \leq A^2,\ z = \xi_1(x, y)\} \text{ and} \tag{4.3}$$

$$\mathcal{D}_2 := \{(x, y, z) : y^2 + z^2 \leq A^2,\ x = \xi_2(y, z)\}, \tag{4.4}$$

for the continuous and semialgebraic functions

$$\xi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (x, y) \longmapsto \sqrt{B^2 - \min\{y^2, B^2\}}, \tag{4.5}$$

$$\xi_2 : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (y, z) \longmapsto \sqrt{B^2 - \min\{y^2, B^2\}}, \tag{4.6}$$

see Figure 4.3.

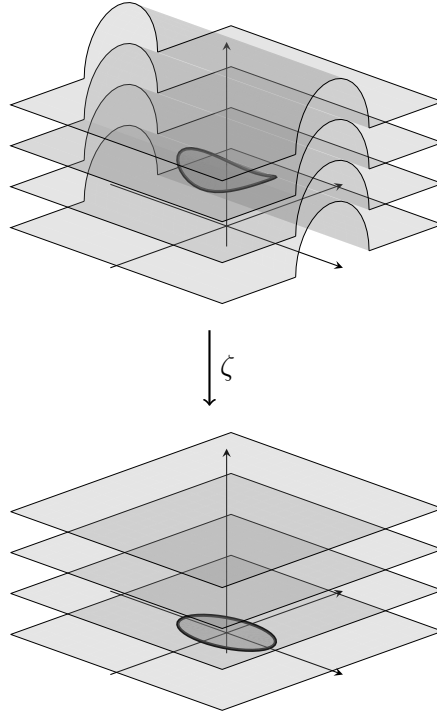Figure 4.3: The homeomorphism ζ for $\xi_1(x, y) := \sqrt{B^2 - \min(y^2, B^2)}$ acting on $\mathbb{R}^3$. Notice how it "flattens" the warped disc $\mathcal{D}_1$.

Now it is clear that our aim is to prove the following fact:

For fixed values $B \geq A > 0$ the intersections $\partial\mathcal{D}_1 \cap \mathcal{S}$ and $\partial\mathcal{D}_2 \cap \mathcal{S}$ are non-empty. $(\Diamond)$

4.3.2 Proof of ($\Diamond$)

In order to proof ($\Diamond$), we first need to introduce a map ϕ and show some of its properties. Let $\phi := (\phi_1, \phi_2, \phi_3) : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, whose components are defined as

$$\phi_1(\rho, \theta) := \cos\theta\sin\theta(\cos\theta - \sin\theta)^2 + \rho(2\cos^4\theta\sin\theta + \cos\theta\sin^4\theta + \cos^5\theta) + \rho^2\cos^5\theta\sin\theta,$$

$$\phi_2(\rho, \theta) := \sqrt{\cos\theta\sin\theta}(\cos\theta + \sin\theta + \rho\cos\theta\sin\theta),$$

$$\phi_3(\rho, \theta) := \rho\sin\theta.$$

Let also $\mathcal{R} := [0, +\infty) \times \left(0, \frac{\pi}{2}\right)$ and $\overline{\mathcal{R}} = [0, +\infty) \times \left[0, \frac{\pi}{2}\right]$ be its closure. We procced now to prove some of the properties that ϕ enjoys, where the sets $\mathcal{R}$ and $\overline{\mathcal{R}}$ will be the domain of the map ϕ.

Lemma 4.8. With the notation introduced above, $\phi(\mathcal{R}) \subset \mathcal{S}$.

Proof. Let ψ be the analytic map

$$\psi := (\psi_1, \psi_2) : \mathcal{R} \longrightarrow \mathcal{Q}, \quad (\rho, \theta) \longmapsto \left(\frac{\sin\theta}{\cos\theta}, \frac{(\cos\theta + \sin\theta + \rho\cos\theta\sin\theta)\cos^2\theta}{\sin\theta}\right).$$

Note that $g \circ \psi = \phi|_{\mathcal{R}}$. If $\theta \in \left(0, \frac{\pi}{2}\right)$ then $\cos\theta, \sin\theta \in (0, 1)$, so ψ is well-defined. This alongside $\rho \in [0, +\infty)$ gives us that $\psi_1, \psi_2 > 0$, so $\psi(\mathcal{R}) \subset \mathcal{Q} \subset \overline{\mathcal{Q}}$. Thus,

$$\phi|_{\mathcal{R}} = g(\psi(\mathcal{R})) \subset g(\overline{\mathcal{Q}}) = \mathcal{S}.$$

$\square$

Lemma 4.9. For all $(\rho, \theta) \in \overline{\mathcal{R}}$, the inequality $\phi_1^2(\rho, \theta) + \phi_3^2(\rho, \theta) \geq \frac{\rho^2}{4}$ holds. Consequently,

$$\forall (\rho, \theta) \in \overline{\mathcal{R}}: \quad \text{dist}(\phi(\rho, \theta), \mathbf{0}) \geq \frac{\rho}{2}. \tag{4.7}$$

Proof. Firstly, if the inequality $\phi_1^2(\rho, \theta) + \phi_3^2(\rho, \theta) \geq \frac{\rho^2}{4}$ holds, we get that

$$\text{dist}(\phi(\rho, \theta), \mathbf{0}) = \sqrt{\phi_1^2(\rho, \theta) + \phi_3^2(\rho, \theta) + \phi_2^2(\rho, \theta)} \geq \sqrt{\frac{\rho^2}{4} + \phi_2^2(\rho, \theta)} \geq \sqrt{\frac{\rho^2}{4}} \geq \frac{\rho}{2},$$

so 4.7 holds. Now, for the first part, if $(\rho, \theta) \in \overline{\mathcal{R}}$ then $\cos\theta, \sin\theta \in [0, 1]$ and $\rho \geq 0$, so we have:

$$\phi_1(\rho, \theta) \geq \rho(\cos\theta\sin^4\theta + \cos^5\theta) = \rho\cos\theta(\cos^4\theta + \sin^4\theta)$$

$$=^1 \rho\cos\theta(1 - 2\cos^2\theta\sin^2\theta) =^2 \rho\cos\theta\left(1 - \frac{\sin^2(2\theta)}{2}\right) \geq^3 \frac{\rho}{2}\cos\theta.$$

In the equality [1] we have used that $1^2 = (\cos\theta^2 + \sin\theta^2)^2 = \cos\theta^4 + \sin\theta^4 + 2\cos\theta^2\sin\theta^2$. Equality [2] holds because $2\cos\theta\sin\theta = \sin(2\theta)$. Also, since $\theta \in [0, \frac{\pi}{2}]$ then $2\theta \in [0, \pi]$, so $\sin(2\theta) \in [0, 1]$ and because of this $\sin^2(2\theta) \in [0, 1]$ and $\frac{\sin(2\theta)}{2} \in [0, \frac{1}{2}]$. Thus, inequality [3] holds.

In addition, $\phi_3(\rho, \theta) = \rho\sin\theta \geq \frac{\rho}{2}\sin\theta$, so we obtain the desired inequality:

$$\phi_1^2(\rho, \theta) + \phi_3^2(\rho, \theta) \geq \frac{\rho^2}{4}\cos^2\theta + \frac{\rho^2}{4}\sin^2\theta = \frac{\rho^2}{4}.$$

$\square$

Construction 4.10. Thanks to the periodicity of the sine and the cosine:

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \text{ and } \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \ \forall\theta \in \mathbb{R}.$$

Thus, we observe that $\phi(0,\theta) = \phi\left(0, \frac{\pi}{2} - \theta\right)$ when $\theta \in [0, \frac{\pi}{2}]$. This induces us to make an identification described as follows. Fix $M > 0$ and consider the rectangle

$$\overline{\mathcal{R}}_M := [0, M] \times [0, \tfrac{\pi}{2}] \subset \overline{\mathcal{R}}.$$

Let $\phi_M = \phi|_{\overline{\mathcal{R}}_M}$. Now, identify the points $(0,\theta)$ and $(0, \frac{\pi}{2} - \theta)$ for $\theta \in [0, \frac{\pi}{2}]$ (see Figure 4.4) and endow the quotient space $\widetilde{\overline{\mathcal{R}}}_M$ with the quotient topology. Observe that, as a topological manifold with boundary, the interior of $\widetilde{\overline{\mathcal{R}}}_M$ is the quotient space $\tilde{\mathcal{R}}_M$ obtained identifying the points $(0,\theta)$ and $(0, \frac{\pi}{2} - \theta)$ of $\mathcal{R}_M := [0, M) \times (0, \frac{\pi}{2})$, where $\theta \in (0, \frac{\pi}{2})$.

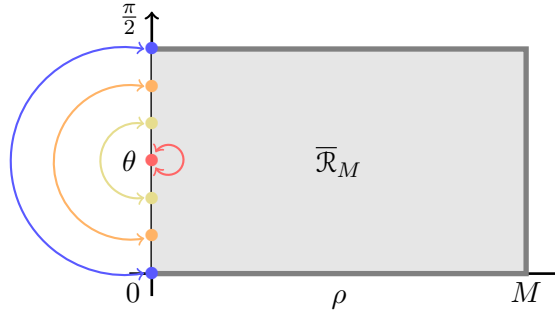
Figure 4.4: Idea of the identification in Construction 4.10.

Consider the canonical projection $\pi_M : \overline{\mathcal{R}}_M \longrightarrow \widetilde{\overline{\mathcal{R}}}_M$, which is continuous. Since ϕ_M is compatible with π_M, there exists a continuous map $\tilde{\phi}_M : \widetilde{\overline{\mathcal{R}}}_M \longrightarrow \mathbb{R}^3$ such that the following diagram is commutative. In addition, since $\mathcal{R}_M \subset \mathcal{R}$, Lemma 4.8 gives us that $\tilde{\phi}_M(\tilde{\mathcal{R}}_M) = \phi(\mathcal{R}_M) \subset \phi(\mathcal{R}) \subset \mathcal{S}$.
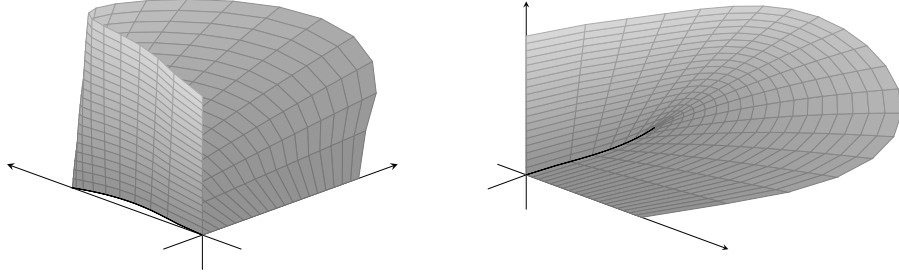
$$
\begin{array}{ccc}
\mathcal{R}_M & \hookrightarrow & \overline{\mathcal{R}}_M \\
{\scriptstyle \pi_M|_{\mathcal{R}_M}}\downarrow & {\scriptstyle \pi_M}\downarrow & \searrow {\scriptstyle \phi_M} \\
\tilde{\mathcal{R}}_M & \hookrightarrow & \widetilde{\overline{\mathcal{R}}}_M \xrightarrow[\tilde{\phi}_M]{} \mathbb{R}^3
\end{array}
$$

Lemma 4.11. The set $\widetilde{\overline{\mathcal{R}}}_M$ is homeomorphic to a disc and its boundary is the set

$$\pi_M\left(\{\rho = M\} \cup \{\theta = 0\} \cup \{\theta = \tfrac{\pi}{2}\}\right).$$

Proof. Identify $\mathbb{R}^2$ with $\mathbb{C}$ (interchanging the order of the variables $(\rho, \theta) \rightsquigarrow (\theta, \rho)$) and consider the continuous map defined as

$$\mu : \mathbb{C} \longrightarrow \mathbb{C}, \ z := \theta + \mathbf{i}\rho \longmapsto w := u + \mathbf{i}v = \left(\tfrac{4}{\pi}z - 1\right)^2,$$

Figure 4.5: Left and right views of $\phi_M(\mathcal{R}_M) \subset \mathcal{S}$.

where $\mathbf{i}$ is the imaginary unit $\sqrt{-1}$. If we restrict ourselves to the open upper half plane, we obtain the homeomorphism $\mu|_{\{\rho > 0\}} : \{\rho > 0\} \longrightarrow \mathbb{C} \setminus ([0, +\infty) \times \{0\})$. Now observe that the image of $\overline{\mathcal{R}}_M \setminus \{\rho = 0\}$ is

$$\mathcal{T}_M := \left\{ (u, v) \in \mathbb{R}^2 : \left(\tfrac{\pi v}{8M}\right)^2 - \left(\tfrac{4M}{\pi}\right)^2 \leq u \leq 1 - \left(\tfrac{v}{2}\right)^2 \right\} \setminus \left\{ (u, 0) : u \in [0, 1] \right\}.$$

The closure $\overline{\mathcal{T}}_M$ of $\mathcal{T}_M$ is a compact convex set because it is a closed bounded intersection of two convex sets, see Figure 4.6. Because of [B, Corolary 11.3.4], $\overline{\mathcal{T}}_M$ is homeomorphic to a closed disc. On top of that,

$$\mu|_{\{\rho = 0\}} : \{\rho = 0\} \longrightarrow \left\{ (u, 0) : u \in [0, +\infty) \right\}, \ \theta \longmapsto \left(\tfrac{4}{\pi}\theta - 1\right)^2$$

transforms the segment $[0, \tfrac{\pi}{2}] \times \{0\}$ onto the interval $[0, 1] \times \{0\}$. Looking at the definition of $\mu|_{\{\rho = 0\}}$ we see that the preimage of $t_0 \in [0, 1]$ is composed by two points, namely:

$$\left\{ \theta_1 := \tfrac{\pi}{4}\left(1 + \sqrt{t_0}\right), \theta_2 := \tfrac{\pi}{4}\left(1 - \sqrt{t_0}\right) \right\},$$

which makes sense considering the identification described in Construction 4.10.

Following Construction 4.10 and since $\theta_1 = \tfrac{\pi}{2} - \theta_2$, the map $\lambda := \mu|_{\overline{\mathcal{R}}_M} : \overline{\mathcal{R}}_M \longrightarrow \overline{\mathcal{T}}_M$ factors through $\widetilde{\overline{\mathcal{R}}}_M$ and there exists a continuous map $\tilde{\lambda} : \widetilde{\overline{\mathcal{R}}}_M \longrightarrow \overline{\mathcal{T}}_M$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{R}_M & \hookrightarrow & \overline{\mathcal{R}}_M \\
{\scriptstyle \pi_M|_{\mathcal{R}_M}} \downarrow & {\scriptstyle \pi_M} \downarrow & \searrow^{\lambda} \\
\widetilde{\mathcal{R}}_M & \hookrightarrow & \widetilde{\overline{\mathcal{R}}}_M \xrightarrow[\tilde{\lambda}]{} \mathcal{T}_M
\end{array}
$$

The map $\tilde{\lambda}$ is continuous and bijective and it is a homeomorphism since it maps the compact set $\widetilde{\overline{\mathcal{R}}}_M$ onto the Hausdorff space[2] $\overline{\mathcal{T}}_M$. Thus, $\widetilde{\overline{\mathcal{R}}}_M$ is homeomorphic to a disc and its boundary is

$$\pi_M\left(\{\rho = M\} \cup \{\theta = 0\} \cup \{\theta = \tfrac{\pi}{2}\}\right),$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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[2]Meaning every pair of distinct points have disjoint neigbourhoods.
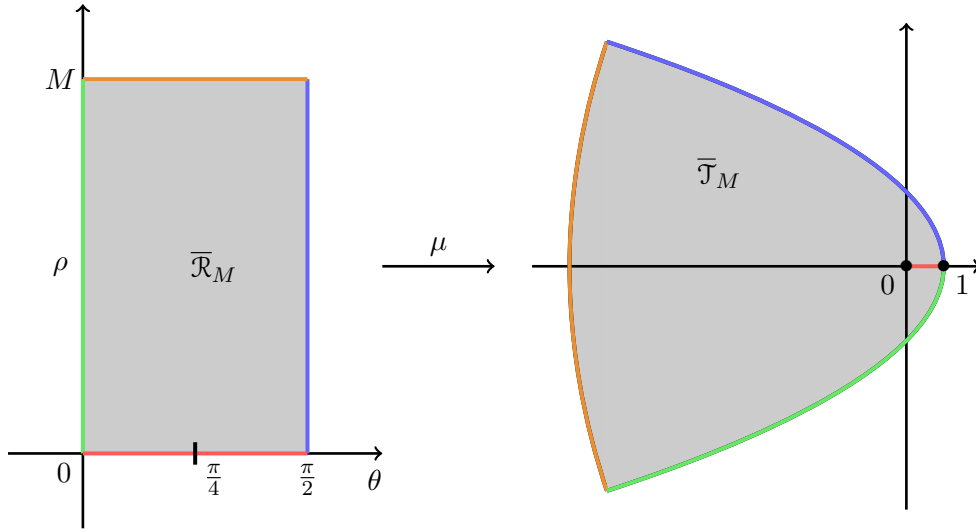
Figure 4.6: Behavior of the map $\mu : \overline{\mathcal{R}}_M \longrightarrow \overline{\mathcal{T}}_M$.

We are going to prove now the main result of this section, which will be the key for the proof of Theorem 1.11.

Lemma 4.12. Fix $B \geq A > 0$ and consider the warped discs $\mathcal{D}_1$ and $\mathcal{D}_2$ introduced in (4.3) and (4.4). Then there exists $M > 0$ such that the boundary map $\partial \tilde{\phi}_M : \partial \tilde{\overline{\mathcal{R}}}_M \longrightarrow \mathbb{R}^3$ meets transversally once both discs $\mathcal{D}_1$ and $\mathcal{D}_2$.

Proof. Firstly, since $\mathcal{D}_1$ and $\mathcal{D}_2$ are bounded, there exists $M_0 > 0$ verifying

$$\mathcal{D}_1 \cup \mathcal{D}_2 \subset \{(x, y, z) : \|(x, y, z)\| < M_0\}.$$

Take $M := 4M_0$ and consider the set $\tilde{\overline{\mathcal{R}}}_M$ and the continuous map $\tilde{\phi}_M$ introduced in Construction 4.10.

We are going to prove that the boundary map $\partial \tilde{\phi}_M : \partial \tilde{\overline{\mathcal{R}}}_M \to \mathbb{R}^3$ meets transversally once $\mathcal{D}_1$.

Consider the parameterization of $\partial \tilde{\overline{\mathcal{R}}}_M$ given by

$$\beta_1(t) := \begin{cases} \pi_M\left(t, \frac{\pi}{2}\right) & \text{if } 0 \leq t \leq M, \\ \pi_M\left(M, M + \frac{\pi}{2} - t\right) & \text{if } M < t \leq M + \frac{\pi}{2}, \\ \pi_M\left(2M + \frac{\pi}{2} - t, 0\right) & \text{if } M + \frac{\pi}{2} < t \leq 2M + \frac{\pi}{2}. \end{cases}$$

Let $\alpha_1(t) := \tilde{\phi}_M \circ \beta_1(t)$ defined as:

$$\alpha_1(t) = \begin{cases} \phi\left(t, \frac{\pi}{2}\right) & \text{if } 0 \leq t \leq M, \\ \phi\left(M, M + \frac{\pi}{2} - t\right) & \text{if } M < t \leq M + \frac{\pi}{2}, \\ \phi\left(2M + \frac{\pi}{2} - t, 0\right) & \text{if } M + \frac{\pi}{2} < t \leq 2M + \frac{\pi}{2}. \end{cases}$$

Choose ε such that $0 < \varepsilon < \min\{B, M_0 - B\}$ and consider the homeomorphism

$$\zeta_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \ (x, y, z) \longmapsto (x, y, z - \xi_1(x, y)),$$
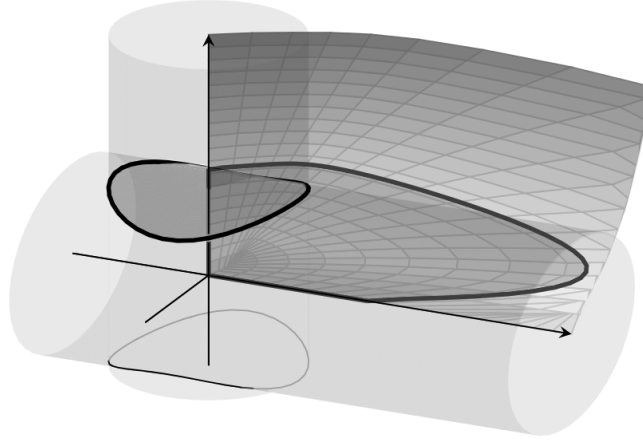
Figure 4.7: The boundary map $\partial\tilde{\phi}_M : \partial\tilde{\tilde{\mathcal{R}}}_M \to \mathbb{R}^3$ meets transversally once $\mathcal{D}_1$.
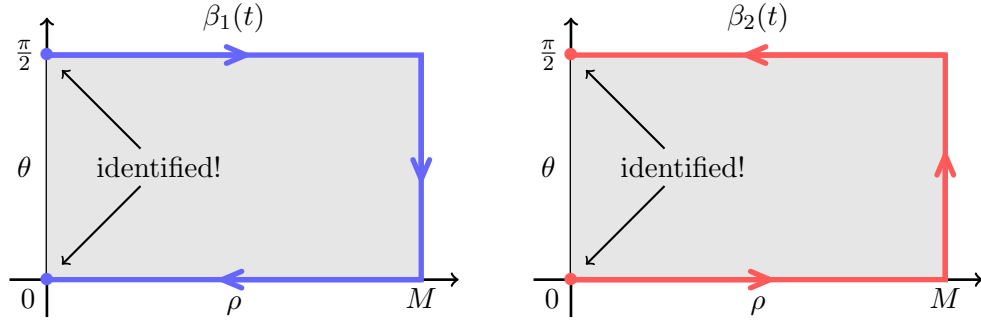

Figure 4.8: Behavior of the paths β_1 and β_2, surrounding $\tilde{\tilde{\mathcal{R}}}_M$.

where ξ_1 is the continuous semialgebraic function introduced in (4.5). If we denote $\mathcal{D}_1(\varepsilon) := \zeta_1^{-1}\big(\mathbb{D}_A(\varepsilon)\big)$, we can do as we have done in other proofs and rely on $\mathbb{D}_A(\varepsilon)$ to conclude this part of the proof, so is enough to check:

$$\alpha_1^{-1}(\mathcal{D}_1(\varepsilon)) = (B - \varepsilon, B + \varepsilon).$$

If $t_0 \in [0, 2M + \frac{\pi}{2}]$, pick $p_0 := \alpha_1(t_0) \in \text{im}(\alpha_1)$. By the definition of α_1, we distinguish three cases:

(i) If $0 \leq t_0 \leq M$, then

$$\zeta_1(p_0) = \zeta_1(\alpha(t_0)) = \zeta_1(\phi(t_0, 0)) = \zeta_1(\phi(t_0, \tfrac{\pi}{2})) = (0, 0, t_0 - B),$$

where we have used the definition of ϕ given in the beginning of this subsection. Thus, $\zeta_1(p_0) \in \mathbb{D}_A(\varepsilon)$ if and only if $-B < -\varepsilon < t_0 - B < \varepsilon < M - B$, or $0 < B - \varepsilon < t_0 < B + \varepsilon < M$.

(ii) If $M < t_0 \leq M + \frac{\pi}{2}$, we have by (4.7) that

$$\text{dist}(p_0, \mathbf{0}) = \text{dist}\big(\phi\big(M, M + \tfrac{\pi}{2} - t\big), \mathbf{0}\big) \geq \tfrac{M}{2} = 2M_0 > \sqrt{2}M_0 > \text{dist}(q, \mathbf{0}) \,\forall q \in \mathcal{D}_1(\varepsilon).$$

Therefore $p_0 \notin \mathcal{D}_1(\varepsilon)$.

(iii) If $M + \frac{\pi}{2} < t_0 \leq 2M + \frac{\pi}{2}$, then

$$p_0 = \alpha_1(t_0) = \phi\left(2M + \tfrac{\pi}{2} - t_0, 0\right) = \left(2M + \tfrac{\pi}{2} - t_0, 0, 0\right),$$

so $\zeta_1(p_0) = (2M + \frac{\pi}{2} - t_0, 0, -B)$. By the definition of ε, we know that $\varepsilon < B$ and thus

$$\zeta_1(p_0) \notin \mathbb{D}_A(\varepsilon) \Rightarrow p_0 \notin \mathcal{D}_1(\varepsilon).$$

We conclude $\alpha_1^{-1}\left(\mathcal{D}_1(\varepsilon)\right) = \left(B - \varepsilon, B + \varepsilon\right)$, so α_1 meets transversally once $\mathcal{D}_1$.

Analogously one shows that the boundary map $\partial\tilde{\phi}_M : \partial\tilde{\overline{\mathcal{R}}}_M \longrightarrow \mathbb{R}^3$ meets transversally once $\mathcal{D}_2$.
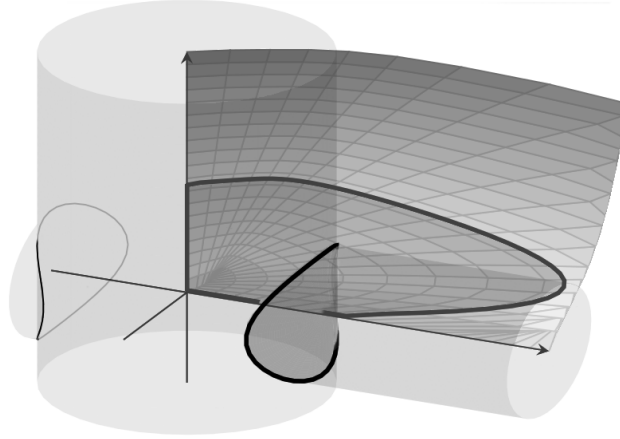

Figure 4.9: The boundary map $\partial\tilde{\phi}_M : \partial\tilde{\overline{\mathcal{R}}}_M \to \mathbb{R}^3$ meets transversally once $\mathcal{D}_2$.

Consider in this case the parameterization of $\partial\tilde{\overline{\mathcal{R}}}_M$ given by

$$\beta_2(t) := \begin{cases} \pi_M\left(t, 0\right) & \text{if } 0 \leq t \leq M, \\ \pi_M\left(M, t - M\right) & \text{if } M < t \leq M + \frac{\pi}{2}, \\ \pi_M\left(2M + \frac{\pi}{2} - t, \frac{\pi}{2}\right) & \text{if } M + \frac{\pi}{2} < t \leq 2M + \frac{\pi}{2}. \end{cases}$$

(see Figure 4.8). We can then define $\alpha_2(t) := \tilde{\phi}_M \circ \beta_2(t)$ as:

$$\alpha_2(t) = \begin{cases} \phi\left(t, 0\right) & \text{if } 0 \leq t \leq M, \\ \phi\left(M, t - M\right) & \text{if } M < t \leq M + \frac{\pi}{2}, \\ \phi\left(2M + \frac{\pi}{2} - t, \frac{\pi}{2}\right) & \text{if } M + \frac{\pi}{2} < t \leq 2M + \frac{\pi}{2}. \end{cases}$$

Proceed as above keeping the same values for A and ε and using in this case the homeomorphism

$$\zeta_2 : \mathbb{R}^3 \to \mathbb{R}^3, \ (x, y, z) \longmapsto (z, y, x - \xi_2(z, y)),$$

where ξ_2 is the continuous semialgebraic function introduced in (4.6), to prove that α_2 meets transversally once the warped disk $\mathcal{D}_2$.

This concludes the proof. $\qquad\square$

Now we are in conditions of proving ($\lozenge$) and thus conclude the topological proof.

[Proof of Theorem 1.10] Firstly, by Lemma 4.11 $\overline{\mathcal{R}}_M$ is homeomorphic to a closed disc. By Proposition 4.7 applied to the continuous map $\phi_M : \overline{\mathcal{R}}_M \to \mathbb{R}^3$ and Lemma 4.12, we deduce that the boundaries of both warped discs $\mathcal{D}_1$ and $\mathcal{D}_2$ meet $\phi_M(\mathcal{R}_M) \subset \mathcal{S}$. Thus, ($\lozenge$) holds, which concludes the proof. $\qquad\square$

A

Auxiliary definitions and results

A.1 Real algebraic geometry basics

In this section we give a basic insight into the required concepts, definitions and results from real algebraic geometry. They are mostly used on the first chapter.

We begin with some definitions like *real closed field* and *semialgebraic set*. The *Transfer Principle* allows us to move our results from $\mathbb{R}$ to an arbitrary real closed field R. We also give an overview of the *Zariski topology*, in order to talk about concepts like closure or *reducibility* in this topology. Finally, after defining other notions as *proper maps* and *germ Nash half-branch curves*, we finish the section recalling *Puiseux series*, since this is the nature of the solutions of the equation $\Delta_v(x) = 0$ in the first proof of Theorem 1.11.

Definition A.1. A *real closed field* is an ordered field R such that $R(\sqrt{-1})$ is an algebraically closed field. There exist many characterizations of real closed fields. A very enlightening one is the following: R is a real closed field if it is an ordered field that shares with the field $\mathbb{R}$ of real numbers its properties of the first-order language of ordered fields.

Indeed, Tarski-Seidenberg Theorem admits a useful formulation in model theory that explains accurately our last sentence.

Definitions A.2. (i) Let R be a real closed field. A subset $S \subset R^n$ is *semialgebraic* if it is defined as a finite union of sets defined by a conjunction of polynomial equalities and inequalities:

$$\begin{cases} P_1(\mathtt{x}_1, \ldots, \mathtt{x}_n) = 0 \\ \qquad \vdots \\ P_r(\mathtt{x}_1, \ldots, \mathtt{x}_n) = 0 \\ Q_1(\mathtt{x}_1, \ldots, \mathtt{x}_n) > 0 \\ \qquad \vdots \\ Q_\ell(\mathtt{x}_1, \ldots, \mathtt{x}_n) > 0 \end{cases}$$

It is easily seen that finite unions and intersections of semialgebraic sets are semialgebraic sets too. In addition, the complementary set of a semialgebraic set is also a semialgebraic set. The easy proof of these facts can be studied in [BCR], Ch.2.

(ii) A map $f : S \to T$ between semialgebraic subsets $S \subset R^n$ and $T \subset R^m$ is *semialgebraic* if its graph is a semialgebraic subset of R^{m+n}.

(iii) A *semialgebraic homeomorphism* between two semialgebraic subsets $S \subset R^n$ and $T \subset R^m$ is a continuous and bijective semialgebraic map $f : S \to T$. It is easily seen that in such a case its inverse $f^{-1} : T \to S$ is also semialgebraic.

Theorem A.3 (Transfer Principle)

Let $\mathcal{L}(R)$ be the first-order language of ordered fields with parameters in the real closed field R and let Φ be a formula of $\mathcal{L}(R)$. Then, there exists a quantifier-free formula Ψ of $\mathcal{L}(R)$ with the same free variables $\mathtt{x}_1, \ldots, \mathtt{x}_n$ as Φ such that, for every real closed field extension K of R and every $x \in K^n$, the sentence $\Phi(x)$ holds true if and only if $\Psi(x)$ holds true.

Definition A.4 (Zariski topology). (i) Let K be a field. A subset $X \subset K^n$ is *algebraic* if it is the set of common zeros of a finite family of polynomials $f_1, \ldots, f_m \in K[\mathtt{x}_1, \ldots, \mathtt{x}_n]$.

(ii) The *Zariski topology* of an algebraic set $X \subset K^n$ is the topology whose closed sets are the algebraic subsets of K^n contained in X. It is indeed a topology, that is, the arbitrary intersection of algebraic sets is also an algebraic set as an straightforward consequence of Hilbert's basis theorem.

(iii) An algebraic subset $X \subset K^n$ is said to be *reducible* if there exist algebraic subsets $Y \subsetneq X$ and $Z \subsetneq X$ such that $X = Y \cup Z$. If X is not reducible it is said to be *irreducible*.

Definition A.5. (i) Let $S \subset \mathbb{R}^n$ be a semialgebraic set and $p \in S$. The *local dimension of S at p*, denoted $\dim(S_p)$, is the largest non-negative integer d such that for every open ball B centered at p the intersection $S \cap B$ contains a semialgebraic subset semialgebraically homeomorphic to the cube $[0,1]^d$.

It is said that S is *pure dimensional* if $\dim(S_p) = \dim(S_q)$ for every pair of points $p, q \in S$.

Definitions A.6. (i) A continuous map $f : X \to Y$ between topological spaces X and Y is said to be *proper* if $f^{-1}(K)$ is a compact subspace of X for every compact subspace K of Y.

(ii) A semialgebraic map $f : S \to T$ between semialgebraic sets $S \subset R^n$ and $T \subset R^m$ is said to be *semialgebraically proper* if $f^{-1}(K)$ is bounded and closed in S for every bounded and closed in T subset K of T.

Definition A.7. A polynomial map $f : X \to Y$ between algebraic sets X and Y is said to be *dominant* if its image $f(X)$ is a dense subset of Y in its Zariski topology.

Definitions A.8. (i) A function $f : U \to \mathbb{R}$ defined in an open semialgebraic subset $U \subset \mathbb{R}^n$ is a *Nash function* if it is analytic and semialgebraic.

(ii) A map $f := (f_1, \ldots, f_m) : U \to \mathbb{R}^m$ is a *Nash map* if each coordinate $f_i : U \to \mathbb{R}$ is a Nash function.

(iii) Let $\Gamma \subset \mathbb{R}^n$ be an algebraic curve and $p \in \Gamma$. For every small enough $\varepsilon > 0$ the intersection $B_\varepsilon \cap \Gamma \setminus \{a\}$, where $B_\varepsilon \subset \mathbb{R}^n$ is the open ball of radius ε centered at the point p, has finitely many connected components $C_1, \ldots, C_k$. Each C_i is semialgebraically homeomorphic to the interval $(0, 1]$ and in fact for $i = 1, \ldots, k$ there exists a Nash homeomorphism $f_i : [0, 1] \to C_i \cup \{p\}$, with $f_i(0) = p$.

Indeed, this result is a particular case of the local conic structure theorem of semialgebraic sets, [BCR, IX.3.6].

Observe that, by its very definition, it makes sense to define the germs $C_{i,p}$ of C_i at the point p for $i = 1, \ldots, k$ as they are independent of the radius ε. These germs $C_{i,p}$ are called the *germ Nash half-banches* of the curve Γ centered at p. For more details see [BCR, IX.5.2].

Definition A.9. It is proved in [W, pp. 98-102] that given an algebraically closed field K and an indeterminate $\mathtt{t}$ over K, the field $K(\{\mathtt{t}^*\})$ of *Puiseux series* with coefficients in K is algebraically closed. As it is an algebraic extension of the field $K(\mathtt{t})$ of rational functions over K, the field $K(\{\mathtt{t}^*\})$ is an algebraic closure of the field $F(\mathtt{t})$ for every subfield F of K such that the field extension $K|F$ is algebraic. In particular, for $F := \mathbb{R}$ it follows that $\mathbb{C}(\{\mathtt{t}^*\})$ is an algebraic closure of $\mathbb{R}(\mathtt{t})$.

A.2   Root finding algorithms

The first proof of Theorem 1.11 requires frequently to check whether a given polynomial f has some root on a certain interval I. Sometimes we want to check that f is positive or negative on I. To that end it suffices to evaluate it at an arbitrary point $a \in I$ and to check that f has no root on I.

Sturm's method is used to compute the number of different real roots of a polynomial in a given interval. Note that it is an algebraic method that doesn't rely on approximations.

Definitions A.10. (i) Let R be a real closed field and let $f \in \mathbb{R}[\mathtt{t}]$. The *Sturm sequence* (or *Sturm chain*) of f is the finite sequence of polynomials $(f_0, f_1, \ldots, f_k)$ defined as follows:

$$f_0 := f$$
$$f_1 := f'$$
$$\cdots$$
$$f_i := f_{i-1}q_i - f_{i-2}, \text{ with } q_i \in R[\mathtt{x}] \text{ and } \deg(f_i) < \deg(f_{i-1}) \text{ for } i = 2 \ldots, k.$$

Then, by Euclid's Algorithm, there is an integer k satisfying $f_k = \gcd(f, f')$.

(ii) Given a sequence $(a_0, a_1, \ldots, a_k)$ of elements of R with $a_0 \neq 0$, we define the *number of sign changes in the sequence* $(a_0, \ldots, a_k)$ as follows: count one sign change every time $a_i a_\ell < 0$, with

$$\ell = i + 1, \text{ or}$$
$$\ell > i + 1 \text{ and } a_j = 0 \text{ for every } j \text{ satisfying } i < j < \ell.$$

(iii) If $a \in R$ is not a root of f and $(f_0, \ldots, f_k)$ is the Sturm sequence of f, we define $v(f; a)$ to be the number of sign changes in $(f_0(a), \ldots, f_k(a))$.

Proposition A.11 (Sturm's Theorem)**.** *Let R be a real closed field and $f \in R[\mathtt{t}]$. Let $a, b \in R$ be such that $a < b$ and neither a nor b are roots of f. Then the number of roots of f in the interval (a, b) equals $v(f; a) - v(f; b)$.*

The proof of this proposition can be studied in [BCR, 1.2.10].

Laguerre's method differs from Sturm's method in that it is a numerical algorithm, rather than algebraic, to decide if a polynomial with real coefficients has a real root. It is remarkable that it converges to a root, with very few exceptions, from any initial value.

Proposition A.12 (Laguerre's method). *Let $f \in \mathbb{R}[\mathbf{x}]$ be a polynomial of degree n and write it as*

$$f(\mathbf{x}) = (\mathbf{x} - r)(\mathbf{x} - q)^{n-1}, \tag{A.1}$$

where $r, q \in \mathbb{C}$ represent unknowns. Let

$$g(\mathbf{x}) := \frac{f'(\mathbf{x})}{f(\mathbf{x})} = \frac{1}{\mathbf{x} - r} + \frac{n-1}{\mathbf{x} - q} \quad and \quad h(\mathbf{x}) := g^2(\mathbf{x}) - \frac{f''(\mathbf{x})}{f(\mathbf{x})} = \frac{1}{(\mathbf{x} - r)^2} + \frac{n-1}{(\mathbf{x} - q)^2}.$$

If we solve for $\mathbf{x} - q$ in $g(\mathbf{x})$ and substitute the result on $h(\mathbf{x})$, we get a quadratic equation for $\mathbf{x} - r$, whose solution is named Laguerre's formula*:*

$$\mathbf{x} - r = n \left(g(\mathbf{x}) \pm \sqrt{(n-1)(nh(\mathbf{x}) - g^2(\mathbf{x}))} \right)^{-1}. \tag{A.2}$$

Equation A.2 *defines a numerical method by choosing on each step the sign that results in the larger magnitude of the denominator and taking the new root to be:*

$$x_{k+1} = x_k - n \left(g(x_k) \pm \sqrt{(n-1)(nh(x_k) - g^2(x_k))} \right)^{-1}.$$

Furthermore, this iterative formula works with any polynomial, not just with the ones described by equation A.1*.*

El algoritmo, tal y como se describe, parece fuertemente ligado al hecho de que f tenga una raíz, tal vez compleja, de multiplicidad $n - 1$. Por eso no entiendo qué significa que la fórmula funciona para polinomios cualesquiera!!!

B

Sage code

C

Python code

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