Complexity Analysis of Polynomial Algorithms
Final Degree Project

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The problem and its origin

· Let R be a real closed field.

Theorem (Tarski-Seidenberg, 1951)

The image of every polynomial map $f : R^m \to R^n$ is a semialgebraic subset of R^n.

'Inverse problem' (Proposed by J.M. Gamboa, 1990)

Characterize the semialgebraic subsets of R^n that are polynomial images of some R^m.

Some introductory examples

(i) $[0, +\infty)$ is the image of $\mathbb{R}$ under $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$.

(ii) $(0, +\infty)$ **is not** the image of any polynomial map $\mathbb{R} \to \mathbb{R}$.

(iii) $(0, +\infty)$ is the image of $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto (xy - 1)^2 + y^2$.

(iv) $S := \{x^2 + y^2 > 1\} \subset R^2$ **is not** a polynomial image of R^2.

(v) $\mathcal{H} := \{y > 0\} \subset \mathbb{R}^2$ is the image of the polynomial map

$$\mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto \big(y(xy - 1), (xy - 1)^2 + x^2\big).$$

(vi) $S_1 := \{xy < 1\} \subset R^2$ and $S_2 := \{xy > 1, x > 0\} \subset R^2$ **are not** polynomial images of R^2.

(vii) $S_3 := R^2 \setminus \{(0, 0)\}$ is the image of the polynomial map

$$\mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \mapsto \big(xy - 1, (xy - 1)x^2 - y\big).$$

Statement of the main results

Theorem 1.9

Let $n \geq 2$. For every finite set $F \subset R^n$, the semialgebraic set $R^n \setminus F$ is a polynomial image of R^n.

Theorem 1.10

Let $n \geq 2$. Given independent linear forms $h_1, \ldots, h_r$ of R^n, the open semialgebraic set $\{h_1 > 0, \ldots, h_r > 0\}$ is a polynomial image of R^n.

Theorem 1.11

The open quadrant $\mathcal{Q} := \{x > 0, y > 0\}$ is a polynomial image of R^2.

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The first proof: $g := \mathcal{P} \circ Q \circ H$

$\star$ The map $H : \mathbb{R}^2 \to \mathbb{R}^2$ transforms $\mathbb{R}^2$ in $\mathbb{R}^2 \setminus \{(0,0),(-1,0)\}$:

$$H(\mathtt{x},\mathtt{y}) := \big(\mathtt{xy}+1,\ \mathtt{x}^2(\mathtt{xy}+1)(\mathtt{xy}+2)+\mathtt{y}\big).$$

$\star$ The map $Q : \mathbb{R}^2 \to \mathbb{R}^2$ maps $(0,0) \mapsto (0,-1)$ and $(-1,0) \mapsto (-1,0)$:

$$Q(\mathtt{x},\mathtt{y}) = (\mathtt{x},\mathtt{y}-\mathtt{x}-1).$$

$\star$ The map $\mathcal{P} := (\mathcal{F},\mathcal{G})$ transforms $\mathbb{R}^2$ into $\mathcal{Q} \sqcup \{(1,0),(0,1)\}$ and $\mathcal{P}^{-1}\big(\{(1,0),(0,1)\}\big) = \{(0,-1),(-1,0)\}$:

$$\boxed{\begin{aligned}\mathcal{F}(\mathtt{x},\mathtt{y}) &:= (1-\mathtt{x}^3\mathtt{y}+\mathtt{y}-\mathtt{xy}^2)^2+(\mathtt{x}^2\mathtt{y})^2,\\ \mathcal{G}(\mathtt{x},\mathtt{y}) &:= (1-\mathtt{xy}+\mathtt{x}-\mathtt{x}^4\mathtt{y})^2+(\mathtt{x}^2\mathtt{y})^2.\end{aligned}}$$

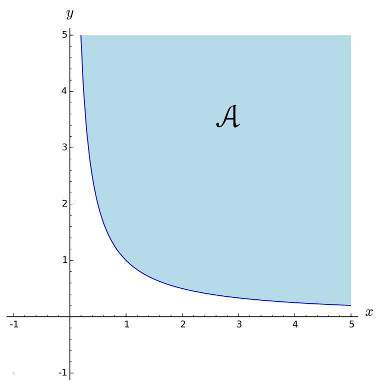
$\star$ Then $g\big(\mathbb{R}^2\big) = \mathcal{Q}$, where:

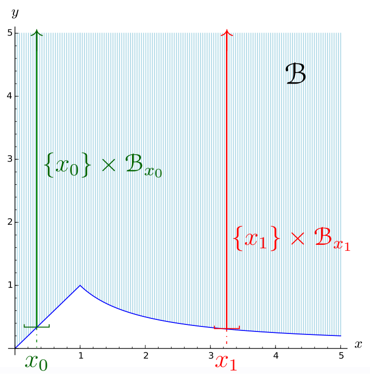
$$g := \mathcal{P} \circ Q \circ H : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

The short proof: $f := \mathcal{H} \circ \mathcal{G} \circ \mathcal{F}$

$\star$ The map $f := \mathcal{H} \circ \mathcal{G} \circ \mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2$ verifies $f\left(\mathbb{R}^2\right) = \mathcal{Q}$, where:

$$\boxed{\begin{aligned} &\mathcal{F} : \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto \left((xy-1)^2 + x^2,\ (xy-1)^2 + y^2\right),\\ &\mathcal{G} : \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto \left(x,\ y(xy-2)^2 + x(xy-1)^2\right),\\ &\mathcal{H} : \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto \left(x(xy-2)^2 + \tfrac{1}{2}xy^2,\ y\right).\end{aligned}}$$





The short proof

Lemma 3.1

The polynomial map $\mathcal{F}$ verifies that $\mathcal{A} \subset \mathcal{F}(\mathbb{R}^2) \subset \mathcal{Q}$.

Lemma 3.2

The polynomial map $\mathcal{G}$ verifies that $\mathcal{B} \subset \mathcal{G}(\mathcal{A}) \subset \mathcal{G}(\mathcal{Q}) \subset \mathcal{Q}$.

Lemma 3.3

The polynomial map $\mathcal{H}$ verifies that $\mathcal{H}(\mathcal{B}) = \mathcal{H}(\mathcal{Q}) = \mathcal{Q}$.

$$\mathcal{Q} \overset{3.3}{=} \mathcal{H}(\mathcal{B}) \overset{3.2}{\subset} (\mathcal{H}\circ\mathcal{G})(\mathcal{A}) \overset{3.1}{\subset} (\mathcal{H}\circ\mathcal{G}\circ\mathcal{F})(\mathbb{R}^2) \overset{3.1}{\subset} (\mathcal{H}\circ\mathcal{G})(\mathcal{Q}) \overset{3.2}{\subset} \mathcal{H}(\mathcal{Q}) \overset{3.3}{=} \mathcal{Q}.$$

The topological proof: $\mathscr{F} = f_2 \circ f_1$

$\star$ The map $f_1(\mathtt{x},\mathtt{y}) := \big(\mathtt{x}^2,\, \mathtt{y}^2\big)$ verifies $f_1\big(\mathbb{R}^2\big) = \overline{\mathcal{Q}} := \{x \geq 0, y \geq 0\}$.

$\star$ The map $f_2 := h \circ g$, where $g : \overline{\mathcal{Q}} \to \mathbb{R}^3,\ h : \mathbb{R}^3 \to \mathbb{R}^2$ and:

$$\begin{aligned} g(\mathtt{x},\mathtt{y}) &:= \big(\mathtt{x}\mathtt{y}^2 + \mathtt{x}^2\mathtt{y} - \mathtt{y} - 1,\, \mathtt{x}^{3/2}\mathtt{y},\, \mathtt{x}^3\mathtt{y} + \mathtt{x}\mathtt{y} - \mathtt{x} - 1\big), \\ h(\mathtt{x},\mathtt{y},\mathtt{z}) &:= \big(\mathtt{x}^2 + \mathtt{y}^2,\, \mathtt{y}^2 + \mathtt{z}^2\big), \end{aligned}$$

verifies that $f_2(\overline{\mathcal{Q}}) = \mathcal{Q}$.

$\star$ The map $\mathscr{F} := (\mathscr{F}_1, \mathscr{F}_2) : \mathbb{R}^2 \to \mathbb{R}^2$ defined as:

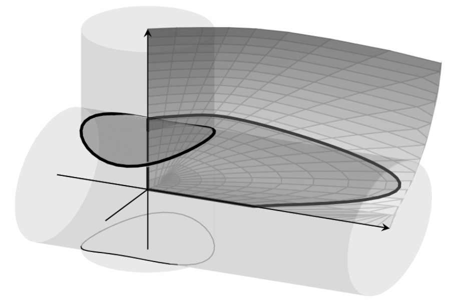
$$\boxed{\begin{aligned} \mathscr{F}_1(\mathtt{x},\mathtt{y}) &:= (\mathtt{x}^2\mathtt{y}^4 + \mathtt{x}^4\mathtt{y}^2 - \mathtt{y}^2 - 1)^2 + \mathtt{x}^6\mathtt{y}^4, \\ \mathscr{F}_2(\mathtt{x},\mathtt{y}) &:= (\mathtt{x}^6\mathtt{y}^2 + \mathtt{x}^2\mathtt{y}^2 - \mathtt{x}^2 - 1)^2 + \mathtt{x}^6\mathtt{y}^4. \end{aligned}}$$

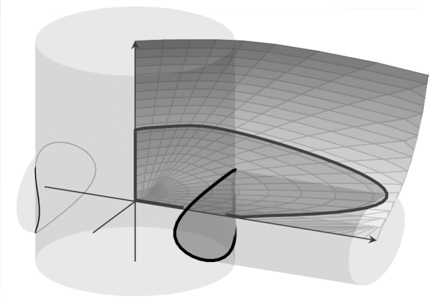
Note that $\mathscr{F} = f_2 \circ f_1$, so we have that $\mathscr{F}\big(\mathbb{R}^2\big) = \mathcal{Q}$.

The topological proof: The key step

The boundary map $\partial\tilde{\phi}_M : \partial\tilde{\overline{\mathcal{R}}}_M \to \mathbb{R}^3$ meets transversally once $\mathcal{D}_1$.

The boundary map $\partial\tilde{\phi}_M : \partial\tilde{\overline{\mathcal{R}}}_M \to \mathbb{R}^3$ meets transversally once $\mathcal{D}_2$.





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Some new candidates

$\star$ New polynomial maps $\mathbb{R}^2 \to \mathbb{R}^2$ whose images "should be" $\mathcal{Q}$:

$$\cdot\ \mathcal{N}_1(\mathtt{x},\mathtt{y}) := \big(\mathtt{x}^4\mathtt{y}^4 + (\mathtt{x}^2\mathtt{y} + \mathtt{x}\mathtt{y}^2 - 1)^2(\mathtt{y}^2+1),\\ \mathtt{x}^4\mathtt{y}^4 + (\mathtt{x}^2\mathtt{y} + \mathtt{x}\mathtt{y}^2 - 1)^2(\mathtt{x}^2+1)\big).$$

$$\cdot\ \mathcal{N}_2(\mathtt{x},\mathtt{y}) := \big(\mathtt{x}^2\mathtt{y}^2 + (\mathtt{x}^2\mathtt{y} + \mathtt{x}\mathtt{y}^2 - 1)^2(\mathtt{y}^2+1),\\ \mathtt{x}^2\mathtt{y}^2 + (\mathtt{x}^2\mathtt{y} + \mathtt{x}\mathtt{y}^2 - 1)^2(\mathtt{x}^2+1)\big).$$

$$\cdot\ \mathcal{N}_3(\mathtt{x},\mathtt{y}) := \big(\mathtt{x}^6\mathtt{y}^4 + (\mathtt{x}^2\mathtt{y} + \mathtt{x}\mathtt{y}^2 - 1)^2(\mathtt{y}^2+1),\\ \mathtt{x}^4\mathtt{y}^6 + (\mathtt{x}^2\mathtt{y} + \mathtt{x}\mathtt{y}^2 - 1)^2(\mathtt{x}^2+1)\big).$$

Measurements of the complexity

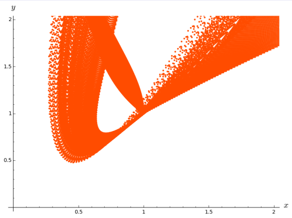
⋆ **Optimal algebraic structure:** Minimum total degree and sparseness (minimum number of monomials).

⋆ **Optimal multiplicative complexity:** Minimum number of non-scalar products.

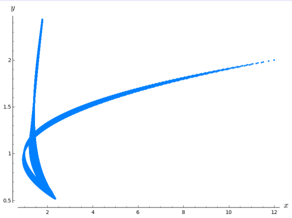
	Total degree	Total number of monomials	Non-escalar complexity
$g = \mathcal{P} \circ Q \circ H$	56	167	13
$f = \mathcal{H} \circ \mathcal{G} \circ \mathcal{F}$	72	350	11
$\mathscr{F} = f_2 \circ f_1$	28	22	11
$\mathcal{N}_1$	16	24	10
$\mathcal{N}_2$	16	26	8
$\mathcal{N}_3$	20	26	13

Uniformly distributed points contained in a square

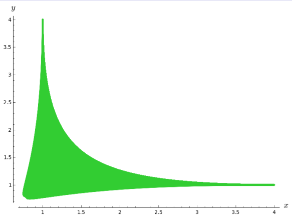
	$g = \mathcal{P} \circ Q \circ H$	$f = \mathcal{H} \circ \mathcal{G} \circ \mathcal{F}$	$\mathscr{F}$
$\mathscr{P}_1 \rightsquigarrow [-10, 10]^2$	0.19 s	0.17 s	0.07 s
$\mathscr{P}_2 \rightsquigarrow [-100, 100]^2$	17.38 s	16.23 s	6.88 s
$\mathscr{P}_3 \rightsquigarrow [0, 1]^2$	410.73 s	389.74 s	155.09 s
$\mathscr{P}_4 \rightsquigarrow [-1, 1]^2$	408.88 s	392.34 s	155.29 s
$\mathscr{P}_5 \rightsquigarrow [-10, 10]^2$	17.10 s	15.45 s	7.00 s
$\mathscr{P}_6 \rightsquigarrow [-10, 10]^2$	1633.50 s	1640.01 s	625.11 s



(a) $g(\mathscr{P}_4)$.

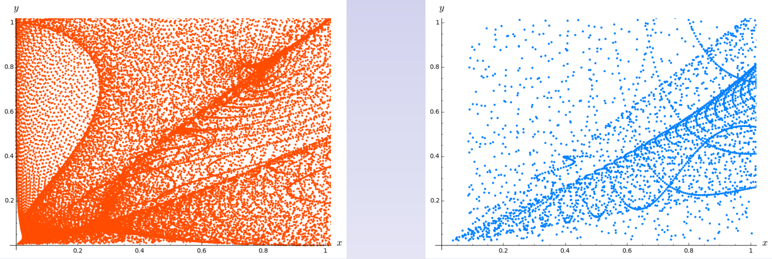


(b) $f(\mathscr{P}_3)$.



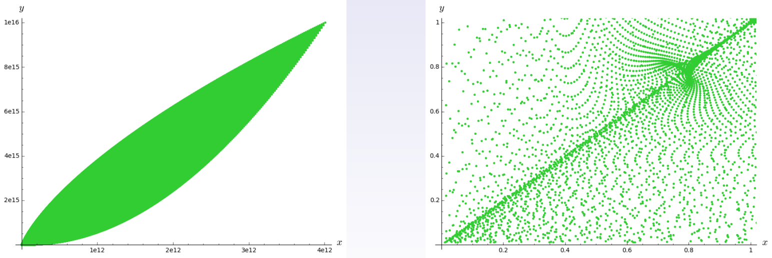
(c) $\mathscr{F}(\mathscr{P}_3)$.

Grid of $4 \cdot 10^6$ points in $[-10, 10] \times [-10, 10]$



(a) $g(\mathscr{P}_6) \rightsquigarrow 1633.50$ s.

(b) $f(\mathscr{P}_6) \rightsquigarrow 1640.01$ s.

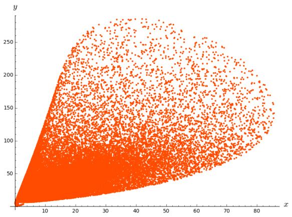


(c) $\mathscr{F}(\mathscr{P}_6) \rightsquigarrow 625.11$ s.

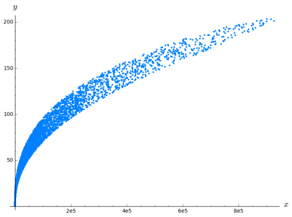
(d) $\mathscr{F}(\mathscr{P}_6) \rightsquigarrow 625.11$ s.

Randomly distributed points contained in a disc

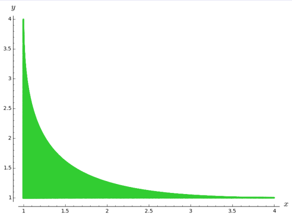
	$g = \mathcal{P} \circ Q \circ H$	$f = \mathcal{H} \circ \mathcal{G} \circ \mathcal{F}$	$\mathscr{F}$
$\mathbb{D}_1$	131.97 s	99.56 s	32.02 s
$\mathbb{D}_{100}$	131.62 s	100.84 s	32.36 s



(a) $g(\mathbb{D}_1)$.

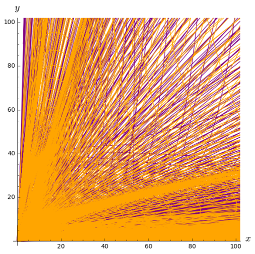


(b) $f(\mathbb{D}_1)$.

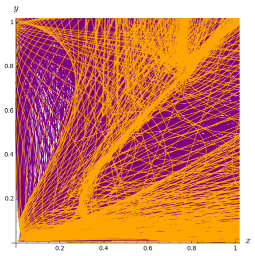


(c) $\mathscr{F}(\mathbb{D}_1)$.

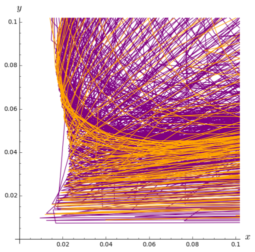
Using families of curves for g and f_2



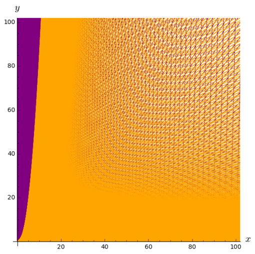
(a) g in $[0, 100]^2$.



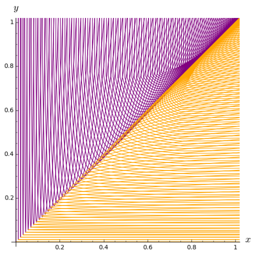
(b) g in $[0, 1]^2$.



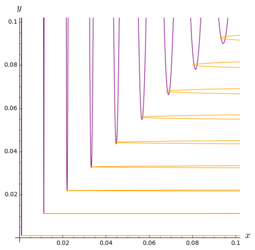
(c) g in $[0, 0.1]^2$.



(d) f_2 in $[0, 100]^2$.

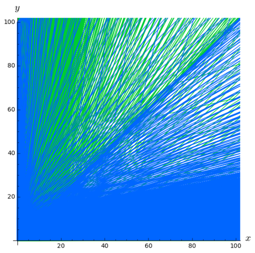


(e) f_2 in $[0, 1]^2$.

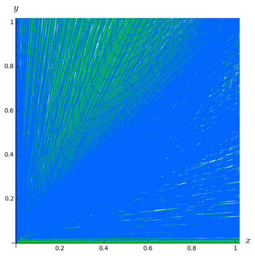


(f) f_2 in $[0, 0.1]^2$.

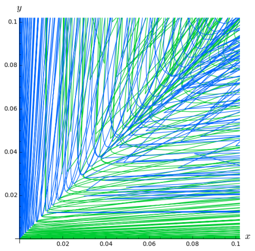
Using families of curves for $\mathcal{N}_1$ and $\mathcal{N}_2$



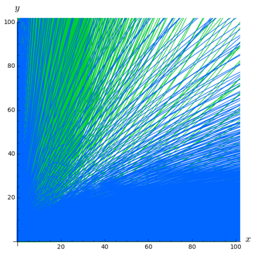
(a) $\mathcal{N}_1$ in $[0, 100]^2$.



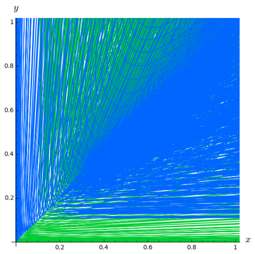
(b) $\mathcal{N}_1$ in $[0, 1]^2$.



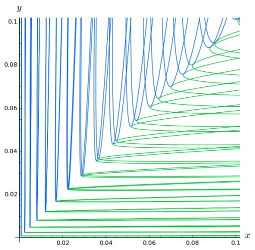
(c) $\mathcal{N}_1$ in $[0, 0.1]^2$.



(d) $\mathcal{N}_2$ in $[0, 100]^2$.



(e) $\mathcal{N}_2$ in $[0, 1]^2$.



(f) $\mathcal{N}_2$ in $[0, 0.1]^2$.

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The first proof: $g := \mathcal{P} \circ Q \circ H$

$\star$ $\mathcal{Q} \subset \mathcal{P}(\mathbb{R}^2) \rightsquigarrow$ Fix $v > 0$ and proof that $\mathcal{F}\big(\{\mathcal{G} = v\}\big) \supset (0, +\infty)$:

$\cdot$ **Step 1:** Parametrization of the curve $\{\mathcal{G} - v = 0\}$. Define $\gamma_v^+(\mathtt{x}, v) := \mathcal{F}(\mathtt{x}, y^+(\mathtt{x}, v))$ and $\gamma_v^-(\mathtt{x}, v) := \mathcal{F}(\mathtt{x}, y^-(\mathtt{x}, v)) \rightsquigarrow$

$$(0, +\infty) \overset{?}{\subset} \operatorname{im}(\gamma_v^+) \cup \operatorname{im}(\gamma_v^-).$$

$\cdot$ **Step 2:** Main properties of γ_v^+ and γ_v^-. Except for $\gamma_1^-(x)$, we have:

$$\lim_{x \to \pm\infty} \gamma_v^+(x) = \lim_{x \to \pm\infty} \gamma_v^-(x) = 0, \quad \lim_{x \to 0} \gamma_v^+(x) = \lim_{x \to 0} \gamma_v^-(x) = +\infty.$$

$\cdot$ **Step 3:** When $v \geq 0.28^2$ we have $(0, +\infty) \subset \operatorname{im}(\gamma_v^+)$.

$\cdot$ **Step 4:** When $0 < v < 0.28^2$ we have $(0, +\infty) \subset \operatorname{im}(\gamma_v^-)$.

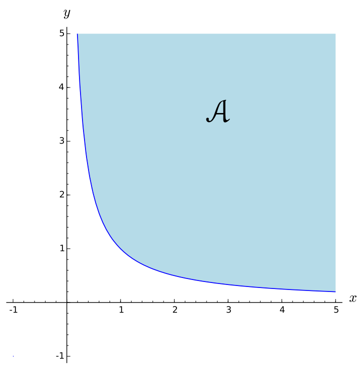
The short proof: The first lemma

Lemma 3.1

Let $\mathcal{A} := \{xy \geq 1\} \cap \mathcal{Q}$. *Then the image of the map*

$$\mathcal{F} := (\mathcal{F}_1, \mathcal{F}_2) : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto ((xy-1)^2 + x^2, (xy-1)^2 + y^2)$$

satisfies that $\mathcal{A} \subset \mathcal{F}(\mathbb{R}^2) \subset \mathcal{Q}$.



The short proof: The second lemma

Lemma 3.2

Let $\mathcal{B} := \mathcal{A} \cup \{y \geq x > 0\}$. *Then, the map*

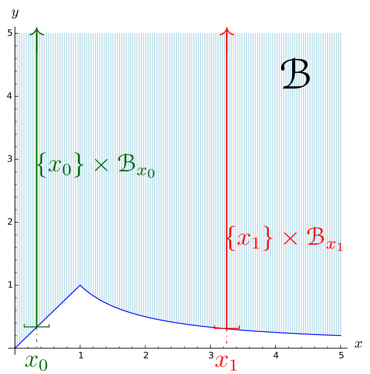
$$\mathcal{G} := (\mathcal{G}_1, \mathcal{G}_2) : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x,\ y(xy-2)^2 + x(xy-1)^2)$$

satisfies that $\mathcal{B} \subset \mathcal{G}(\mathcal{A}) \subset \mathcal{G}(\mathcal{Q}) \subset \mathcal{Q}$.

$$y_x := \min\{x, 1/x\},$$

$$\mathcal{B}_x := [y_x, +\infty),$$

$$\mathcal{B} = \bigsqcup_{x>0} (\{x\} \times \mathcal{B}_x).$$



The short proof: The third lemma

Lemma 3.3

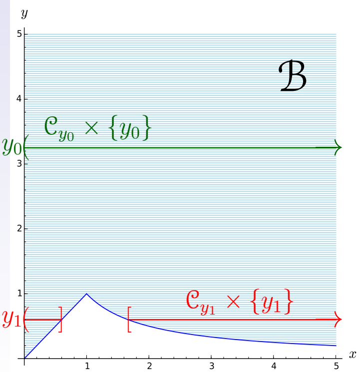
The polynomial map

$$\mathcal{H} := (\mathcal{H}_1, \mathcal{H}_2) : \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto \left(x(xy-2)^2 + \tfrac{1}{2}xy^2,\ y\right)$$

satisfies $\mathcal{H}(\mathcal{B}) = \mathcal{H}(\mathcal{Q}) = \mathcal{Q}$.

$$\mathcal{C}_y := \begin{cases} (0, +\infty) & \text{if } y \geq 1, \\ (0, y] \cup [1/y, +\infty) & \text{if } 0 < y < 1. \end{cases}$$

$$\mathcal{B} = \bigsqcup_{y>0} (\mathcal{C}_y \times \{y\}).$$



The short proof

Lemma 3.1

The polynomial map $\mathcal{F}$ verifies that $\mathcal{A} \subset \mathcal{F}(\mathbb{R}^2) \subset \mathcal{Q}$.

Lemma 3.2

The polynomial map $\mathcal{G}$ verifies that $\mathcal{B} \subset \mathcal{G}(\mathcal{A}) \subset \mathcal{G}(\mathcal{Q}) \subset \mathcal{Q}$.

Lemma 3.3

The polynomial map $\mathcal{H}$ verifies that $\mathcal{H}(\mathcal{B}) = \mathcal{H}(\mathcal{Q}) = \mathcal{Q}$.

$$\mathcal{Q} \overset{3.3}{=} \mathcal{H}(\mathcal{B}) \overset{3.2}{\subset} (\mathcal{H} \circ \mathcal{G})(\mathcal{A}) \overset{3.1}{\subset} (\mathcal{H} \circ \mathcal{G} \circ \mathcal{F})(\mathbb{R}^2) \overset{3.1}{\subset} (\mathcal{H} \circ \mathcal{G})(\mathcal{Q}) \overset{3.2}{\subset} \mathcal{H}(\mathcal{Q}) \overset{3.3}{=} \mathcal{Q}.$$

The topological proof: $\mathscr{F} = f_2 \circ f_1$

- **Step 1:** Factor $\mathscr{F} = f_2 \circ f_1$, with:

$$f_1(\mathtt{x},\mathtt{y}) := \left(\mathtt{x}^2,\, \mathtt{y}^2\right) \rightsquigarrow f_1\left(\mathbb{R}^2\right) = \overline{\mathcal{Q}} := \{x \geq 0, y \geq 0\},$$
$$f_2(\mathtt{x},\mathtt{y}) := \left((\mathtt{x}\mathtt{y}^2 + \mathtt{x}^2\mathtt{y} - \mathtt{y} - 1)^2 + \mathtt{x}^3\mathtt{y}^2,\, (\mathtt{x}^3\mathtt{y} + \mathtt{x}\mathtt{y} - \mathtt{x} - 1)^2 + \mathtt{x}^3\mathtt{y}^2\right).$$

- **Step 2:** Factor $f_2 = h \circ g$, where $g : \overline{\mathcal{Q}} \to \mathbb{R}^3,\ h : \mathbb{R}^3 \to \mathbb{R}^2$ and:

$$g(\mathtt{x},\mathtt{y}) := \left(\mathtt{x}\mathtt{y}^2 + \mathtt{x}^2\mathtt{y} - \mathtt{y} - 1,\, \mathtt{x}^{3/2}\mathtt{y},\, \mathtt{x}^3\mathtt{y} + \mathtt{x}\mathtt{y} - \mathtt{x} - 1\right),$$
$$h(\mathtt{x},\mathtt{y},\mathtt{z}) := \left(\mathtt{x}^2 + \mathtt{y}^2,\, \mathtt{y}^2 + \mathtt{z}^2\right).$$

- **Step 3:** Let $\mathcal{S} := g\left(\overline{\mathcal{Q}}\right)$. Then:

$$\forall\ (A^2, B^2) \in \mathcal{Q} :\ h^{-1}\left(\left\{(A^2, B^2)\right\}\right) \bigcap \mathcal{S} \neq \varnothing.$$

- **Step 4:** For fixed values $B \geq A > 0$: $\partial\mathcal{D}_1 \cap \mathcal{S} \neq \varnothing \neq \partial\mathcal{D}_2 \cap \mathcal{S}$.

The topological proof: **Step 4**

The boundary map $\partial\tilde{\phi}_M : \partial\tilde{\bar{\mathcal{R}}}_M \to \mathbb{R}^3$ meets transversally once $\mathcal{D}_1$.

The boundary map $\partial\tilde{\phi}_M : \partial\tilde{\bar{\mathcal{R}}}_M \to \mathbb{R}^3$ meets transversally once $\mathcal{D}_2$.