ASSIGNMENT 1

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1. $15! \times 14$ ways

2. $8! \binom{9}{6} 6!$ ways.

3. a. $4 \binom{13}{5}$ ways

b. $(1)(48) = 48$ ways

c. $(13)(48) = 624$ ways

d. $$\binom{4}{3}\binom{4}{2} = \left(\frac{4!}{3!\,(4-3)!}\right)\left(\frac{4!}{2!\,(4-2)!}\right)$$

$$= \left(\frac{4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot\cancel{1}\cdot 1}\right)\left(\frac{\overset{2}{\cancel{4}}\cdot 3\cdot\cancel{2}\cdot\cancel{1}}{\cancel{2}\cdot\cancel{1}\cdot\cancel{2}\cdot 1}\right)$$

$= 4 \cdot 6 = 24$ ways.

e. $\binom{4}{3}\binom{12}{1}\binom{4}{2} = (4)(12)(6) = 288$ ways.

f. $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = (13)(4)(12)(6) = 3744$ ways.

g. $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1} = 13 \cdot 4 \cdot 66 \cdot 4 \cdot 4$
$= 54912$ ways.

$$\frac{12!}{2!\,10!} = \frac{\overset{6}{\cancel{12}}\cdot 11\cdot \cancel{10}\cdot \cancel{9}\cdot \cancel{8}\cdot \cancel{7}\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{2}\cdot 1\cdot \cancel{10!}} = 66$$

④ a. $\binom{10}{4}\binom{6}{3}\binom{3}{3} = (210)(20)(1) = 4200$

$$\frac{10\cdot \overset{3}{\cancel{9}}\cdot \overset{2}{\cancel{8}}\cdot 7\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}\ \ \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = 210$$

2.1

$$\frac{\cancel{6}\cdot 5\cdot 4\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{3!}\ \cancel{3}\cdot \cancel{2}\cdot 1} = 20$$

b. $\binom{10}{8}2^2 + \binom{10}{9}2^1 + 1$

$$\underset{45}{\binom{10}{8}} \qquad \underset{10}{\binom{10}{9}}$$

$$\frac{\overset{5}{\cancel{10}}\cdot 9\cdot \cancel{8}\cdot \cancel{7}\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{\cancel{8!}\ \cancel{2}\cdot 1} = 45$$

$$= (45)(4) + (10)(2) + 1 = 180 + 21 = 201$$

c. $\binom{10}{4}\binom{6}{6} + \binom{10}{2}\binom{8}{1}\binom{7}{7} + \binom{10}{2}\binom{8}{8}$

$$\binom{6}{6} = 1 \qquad \binom{8}{1} = 8 \qquad \binom{7}{7} = 1 \qquad \binom{8}{8} = 1$$

$$= (210)(1) + (45)(8)(1) + (45)(1)$$

$$= 210 + 360 + 45 = 615$$

$$\frac{10 \cdot \overset{3}{\cancel{9}} \cdot \overset{2}{\cancel{8}} \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4!}}{\cancel{4!}\ \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 210$$

$$\frac{\overset{5}{\cancel{10}} \cdot 9 \cdot \cancel{8!}}{\cancel{2}!\ \cancel{8!}} = 45$$

$$\begin{array}{r} 210 \\ 360 \\ 45 \\ \hline 615 \end{array}$$

5. a. $$\frac{10!}{2!2!2!2!2!} = 113400$$

$$\frac{10 \cdot 9 \cdot \overset{2}{\cancel{8}} \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4} \cdot 3 \cdot \cancel{2!}}{\cancel{2} \cdot 1 \cdot \cancel{2} \cdot 1 \cdot \cancel{2} \cdot 1 \cdot \cancel{2} \cdot 1 \cdot \cancel{2 \cdot 1}} = 113400$$

b. $$\frac{12!}{2!2!2!2!4!}\left[\underset{4}{\underset{\parallel}{(2)^2}}\ \underset{1}{\underset{\parallel}{(-1)^2}}\ \underset{9}{\underset{\parallel}{(3)^2}}\ \underset{1}{\underset{\parallel}{(1)^2}}\ \underset{16}{\underset{\parallel}{(-2)^4}}\right]$$

$$\parallel$$

$$576$$

$$\frac{12 \cdot 11 \cdot \overset{5}{\cancel{10}} \cdot 9 \cdot \cancel{8} \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{\cancel{2!}\,\cancel{2!}\,\cancel{2!}\,\cancel{2!}\,\cancel{4!} \cdot 1} = 1247400$$

$$= (1247400)(576) = 718502400$$

c. $$\frac{12!}{2!2!2!2!4!}\left[\overset{1}{\overset{\parallel}{(1)^2}}\ \overset{4}{\overset{\parallel}{(-2)^2}}\ \overset{1}{\overset{\parallel}{(1)^2}}\ \overset{25}{\overset{\parallel}{(5)^2}} \cdot \overset{81}{\overset{\parallel}{(3)^4}}\right] \overset{8100}{\parallel}$$

$$= (1247400)(8100) = 10103940000$$

6 a. $$\sum_{i=0}^{n} \frac{1}{i!(n-i)!}$$

$$= \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i}$$

$$= \frac{2^n}{n!}$$

b. $$\sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!}$$

$$= \frac{1}{n!} \underbrace{\sum_{i=0}^{n} \binom{n}{i} (-1)^i}_{=0}$$

$$= 0$$

7 a. $x_1 + x_2 + x_3 + x_4 + x_5 < 40$

$\therefore x_1 + x_2 + x_3 + x_4 + x_5 \leq 39$

$x_1 + x_2 + x_3 + x_4 + x_5 + \underbrace{M}_{\geq 0.} = 39$

$\therefore \binom{n+r-1}{r-1}$ where $n = 39,\ r = 6$

$$= \binom{39+6-1}{6-1} = \binom{44}{5}$$

b. $x_1 - 3 + x_2 - 3 + x_3 - 3 + x_4 - 3 + x_5 - 3 < 40$

$x_1 + x_2 + x_3 + x_4 + x_5 - 15 \leq 39$

$x_1 + x_2 + x_3 + x_4 + x_5 \leq 54$

$x_1 + x_2 + x_3 + x_4 + x_5 + m = 54.$

$\therefore \binom{n+r-1}{r-1}$ wher $n = 54, r = 6$

$= \binom{54+6-1}{6-1} = \binom{59}{5}$

8. $a_1 + a_2 + a_3 + \dots + a_{19} + 19 = n,$

where $x_1 = a_1 + 1, \dots x_{19} = a_{19} + 1$

$\therefore a_1 + a_2 + a_3 + \dots a_{19} = \underbrace{n - 19}_{= n}$

$b_1 + b_2 + b_3 + \dots + b_{64} + 64 = n$

where $y_1 = b + 1, \dots y_{64} = b_{64} + 1$

$\therefore b_1 + b_2 + b_3 + \dots + b_{64} = \underbrace{n - 64}_{= n}$

$\binom{n+r-1}{n} \quad \binom{n-1}{k} = \binom{n}{n-k}$

for a.

$$\binom{n-19+19-1}{n-19} = \binom{n-1}{n-19}$$

for b.

$$\binom{n-64+64-1}{n-64} = \binom{n-1}{n-64}$$

$$= \binom{n-1}{\cancel{n}-1-\cancel{n}+64} = \binom{n-1}{63}$$

$$\binom{n-1}{n-19} \neq \binom{n-1}{63}$$

$$\therefore \quad n-19 = 63$$

$$= n = 63+19$$

$$n = 82$$

⑨ a. $\binom{n}{k} = \binom{6+3-1}{3-1} = \binom{8}{2} = 28$ $\quad \frac{\overset{4}{\cancel{8}} \cdot 7 \cdot \cancel{6!}}{\cancel{2!}\,\cancel{6!}} = 28$

$\binom{n}{k} = \binom{31+4-1}{4-1} = \binom{34}{3}$ $\quad \frac{34 \cdot \overset{11}{\cancel{33}} \cdot 32 \cdot \cancel{31!}}{\cancel{3!}\,\cancel{31!} \cdot 2} = 5984$

$= 5984$

b. $\binom{n}{k} = \binom{31+3-1}{3-1} = \binom{33}{2} = 528$ $\quad \frac{33 \cdot 32 \cdot \cancel{31!}}{2!\,\cancel{31!}} = 528$

$\therefore \binom{34}{3} - \binom{33}{2} = 5984 - 528$

$= 5456.$

⑩ for $n \geq 0$, let S be a set with $|S| = 2^n$ for $n \geq 0$
$\therefore$ let $|S| = 2^n$

because S_1 and S_2 be 2 sets where $|S_1| = m$, $|S_2| = r$, for $m, r \in \mathbb{Z}^+$, if we divide S into 2 subsets S_1 & S_2, each of the subsets would be 2^{n-1}

to sort each half: $T(n-1)$ and $\therefore 2[T(n-1)]$

goal: recursively sort each subset then merge by making no more than $m+r-1$ comparisons.

$2^{n-1} + 2^{n-1} - 1 = 2^n - 1$ $\quad = \quad$ $m + r - 1 = 2^{n-1}$

$T(n) = 2T(n-1) + (2^{n-1})$

Suppose for all $n \geq 0$, $T(n) \leq n(2^n)$

Base case : $n = 0$ $\therefore T(0) = 0$

Inductive step: let $T(n-1) \leq (n-1)(2^{n-1})$

induction hypothesis

$$T(n) = 2T(n-1) + 2^n - 1$$

$$\leq 2((n-1)2^{n-1}) + 2^n - 1$$

$$\leq (n-1)2^n + 2^n - 1$$

(by induction hypothesis)

$$= n(2^n - 1)$$

$$\therefore T(n) \leq n \cdot 2^n$$

$\therefore$ It was proven through induction that the number of comparisons needed to place the elements of S in ascending order is bounded above by $n \cdot 2^n$.

11. Proof by induction.

Base case : $n = 2$

Inductive step: if for $k \geq 2$,

induction hypothesis

$$|x_1 + x_2 + \cdots + x_k| \leq |x_1| + \cdots + |x_k|$$

let $x_1 + x_2 + \ldots + x_k,\ x_{k+1} \in \mathbb{R}$

$$|x_1 + x_2 + \ldots + x_k + x_{k+1}|$$

$$= |X + Y| \leq |X| + |Y|$$

(by induction hypothesis)

$$\leq (|x_1| + |x_2| + \ldots + |x_k|) + |x_{k+1}|$$

Through induction, because it holds for $k+1$, then it holds for $n \geq 2$.

(12) Proof by induction.

Base case: $n = 0$ $\quad \sum_{i=0}^{0} F_i = F_0 = F_{2} - 1$

9.1

Inductive step: if for $k \geq 0$,

induction hypothesis

$$\sum_{i=0}^{n} F_i = F_{k+2} - 1$$

$$\sum_{i=0}^{k+1} F_i = \left[\sum_{i=0}^{k} F_i\right] + F_{k+1}$$

$$= (F_{k+2} - 1) + F_{k+1}$$

(by induction hypothesis)

$$= F_{k+2} + F_{k+1} - 1$$

$$= F_{k+3} - 1$$

Through induction, because it holds for $k+1$, it holds for all $k \geq 0$ $\therefore$ for all $n \geq 0$.

(13) a.

$k=0$	321	0 ascent
$k=1$	132	1 ascent
	213	1 ascent
	231	1 ascent
	312	1 ascent
$k=2$	123	2 ascent

$\therefore$ 1 permutation for $k=0$
4 permutations for $k=1$
1 permutation for $k=2$

D.

$k=0$	2431	0 ascent	5 permutations for $k=0$
	3241	0 ascent	
	3421	0 ascent	
	4231	0 ascent	
	4321	0 ascent	
$k=1$	1342	1 ascent	11 permutations for $k=1$
	1432	1 ascent	
	2143	1 ascent	
	2314	1 ascent	
	2413	1 ascent	
	3142	1 ascent	
	3214	1 ascent	
	3412	1 ascent	
	4132	1 ascent	
	4213	1 ascent	
	4312	1 ascent	
$k=2$	1243	2 ascent	7 permutations for $k=2$
	1324	2 ascent	
	1423	2 ascent	
	2134	2 ascent	
	2341	2 ascent	
	3124	2 ascent	
	4123	2 ascent	
$k=3$	1234	3 ascent	1 permutation for $k=3$

c. $\{1,2,3,4,5,6,7\}$ $m=7$

ascents + descents $= m-1$

4 + descents $= 7-1 = 6$

descents $= 6-4 = 2$

$\therefore$ 2 descends.

d. ascents + descents $= m-1$

K + descents $= m-1$

descents $= m-1-K$

$\therefore$ $m-1-K$ descents.

e. start.

9 1 2 4 3 6 5 8 7

it's bigger $\therefore$ No new ascent.

$\therefore$ this position gives 4 ascents

end.

1 2 4 3 6 5 8 7 9

1 new ascent $\therefore$ this position gives 5 ascents

in between

ORIGINAL 1 2 4 3 6 5 8 7 4 ascents

ADDING "9"

1 9 2 4 3 6 5 8 7 4 ascents

1 2 9 4 3 6 5 8 7 4 ascents

1 2 4 9 3 6 5 8 7 5 ascents

1 2 4 3 9 6 5 8 7 4 ascents

1 2 4 3 6 9 5 8 7 5 ascents

1 2 4 3 6 5 9 8 7   4 ascents

↑ ↑ ↑ ↑

1 2 4 3 6 5 8 9 7   5 ascents

↑ ↑ ↑ ↑ ↑

i) 4 ascents : 5 positions

ii) 5 ascents : 4 positions

f. $\pi_{m,k}$ = number of permutations

if insert m in a position of ascent $(k+1)$
and it doesn't change the number of ascent:

$(k+1)\pi_{m-1,k}$

if insert m in a position of descent
$(m-1)-k$ and it gives a new ascent:

$(m-k)\pi_{m-1,k-1}$

$\therefore \pi_{m,k} = (k+1)\pi_{m-1,k} + (m-k)\pi_{m-1,k-1}$

(14) if n is even $\rightarrow$ $n = 2K$

26 letters $\rightarrow$ $26^{K} = 26^{\frac{n}{2}}$ palindromes

if n is odd $\rightarrow$ $n = 2K+1$

26 letters $\rightarrow$ $26^{K+1} \rightarrow 26^{\frac{n+1}{2}}$ palindromes

$\therefore$ we can combine and the closest is $26^{\lceil \frac{n}{2} \rceil}$

$\therefore$ the number of palindromes of length n over the English alphabet is $26^{\lceil \frac{n}{2} \rceil}$

Index of comments

1.1 Additional explanation needed

2.1 Needs more explanation in parts (b) and (c)

9.1 Show how base case holds