Euler Bernoulli Beam Theories

Gaurav Srivastava

September 26, 2023

Introduction

Beams are fundamental structural elements widely used in various engineering applications, such as bridges, buildings, and mechanical systems. Understanding their behavior under loading conditions is crucial for designing safe and efficient structures. The Euler beam theory, also known as the Euler-Bernoulli beam theory, provides a simplified yet valuable framework for analyzing the deformation and bending of slender beams subjected to small deformations.

The fundamental concept in the Euler beam theory is that a beam can resist bending due to its ability to carry moments resulting from external loads. The theory assumes that beams deform primarily by bending, neglecting the effects of shear deformation and axial extension or compression. This assumption is valid for beams that have relatively small cross-sectional dimensions compared to their lengths.

By understanding the principles of the Euler beam theory, engineers can analyze and design beams to meet specific requirements, ensuring structural integrity and performance. The theory serves as a useful tool for designing various types of beams, including simply supported beams, cantilever beams, continuous beams, and more complex beam configurations.

In summary, the Euler beam theory provides a simplified yet effective approach for analyzing the behavior of slender beams subjected to small deformations. By considering the deflection, bending moment, and internal stresses, engineers can gain insights into the response of beams under various loading conditions, aiding in the design and optimization of beam structures.

1 Euler Bernoulli linear beam theory

The Euler-Bernoulli linear beam theory, often referred to as simply the Euler beam theory is a fundamental theoretical framework for analyzing the deformation and bending behavior of slender beams under small deformations. This theory provides a simplified mathematical description of the behavior of beams, allowing engineers to analyze and design structures efficiently.

This theory is based on a set of assumptions that enable the derivation of governing equations for beam deflection, curvature, and the distribution of internal forces. This equation is a fourth-order differential equation that relates the bending moment, the flexural rigidity of the beam, and the curvature of the beam. The governing equations of the Euler beam theory are derived from the equilibrium and compatibility equations for beam deformation. The governing equation is given below:-

$$\frac{\partial^2}{\partial x^2}(EI\,\frac{\partial^2 w}{\partial x^2}) = f(x)$$

Here:-

w is Deflection

E is Young Modulus

I is moment of Inertia

The main assumptions of the Euler Beam theory are given here:-

1. The deformation is small
2. Effects of Poisson's ratio are ignored
3. The beam is made of a linear elastic isotropic material
4. Plane sections remain plane

1.1 Derivation of Curvature

According to the Euler-Bernoulli linear beam theory, the curvature of the beam is related to the applied bending moment and the flexural rigidity of the shaft.

Let's consider a beam that is initially straight and undeflected along the x-axis. When we apply a load then it is deflected. The beam deflection at any point along its length is denoted by $\delta(x)$, where x is the axial coordinate along the beam.

The curvature of the beam at a given point is defined as the rate of change of the slope of the deflection curve with respect to the axial coordinate x. And the slope of the deflection curve, dδ/dx, represents the angle between the tangent to the deflection curve and the horizontal x-axis. So we can also say that the curvature is defined as the rate of change of the slope.

Here

$$tan\theta = \lim_{\Delta x \to 0} \frac{d\delta}{dx} \text{ .... 1.1.1}$$

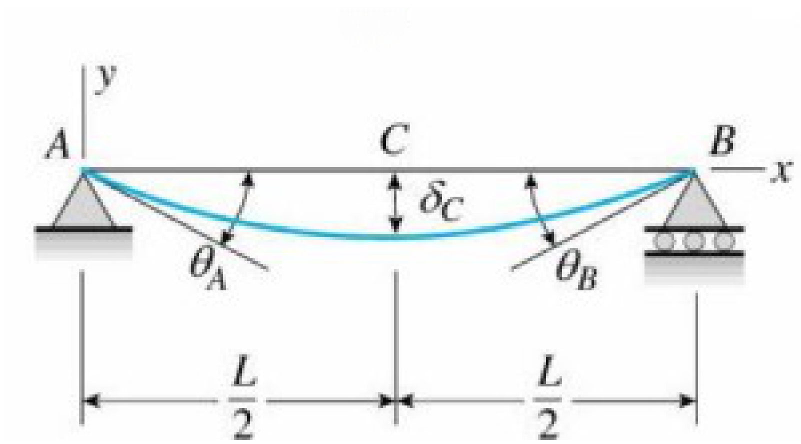

Figure 1:

if θ is small then we can also write

$$\theta = \frac{d\delta}{dx} \quad .... \ 1.1.2$$

Here

$$\theta = angle\ between\ deflection\ curve\ and\ horizontal\ x - axis$$

Now

$$Curvature\ K(Kappa) = \frac{d\theta}{dx} = \frac{d^2\delta}{dx^2} \quad .... \ 1.1.3$$

1.2 Derivation of Moment(M) and Curvature(K) relation

we can assume a beam that is bent as shown below:

Here

$$ab = (\rho - y)\Delta\phi$$

$$a_1 b_1 = \rho\Delta\phi$$

Here
ρ is the radius of curvature

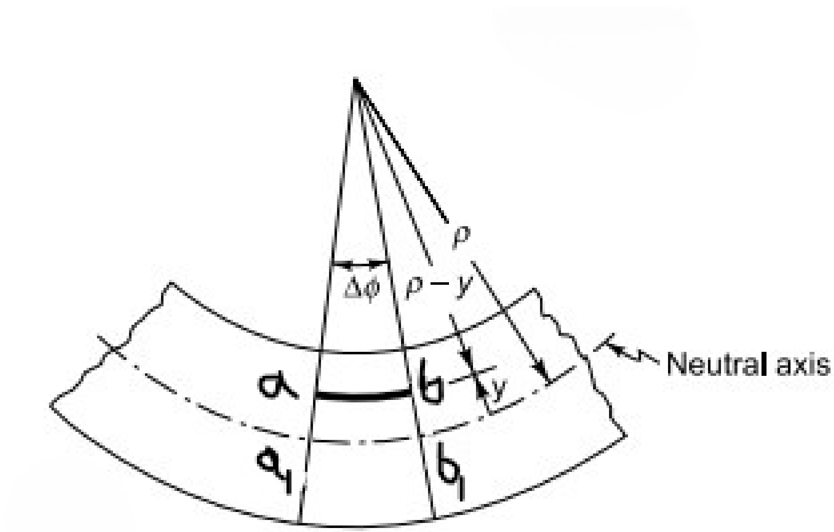

Figure 2:

$$\rho = \frac{1}{Kappa} \text{ .... 1.2.1}$$

Now

$$Strain\ (\epsilon_x) = \frac{ab - a_1 b_1}{a_1 b_1} = \frac{(\rho - y)\Delta\phi - \rho\Delta\phi}{\rho\Delta\phi}$$

$$\epsilon_x = \frac{-y}{\rho}$$

and
we also know that:-

$$\sigma_x = \epsilon_x E$$

Here

$E = Elastic\ Modulus$

Then

$$\sigma_x = \frac{-y}{\rho}E \text{ .... 1.2.2}$$

Now We can need to take a small area of a beam.

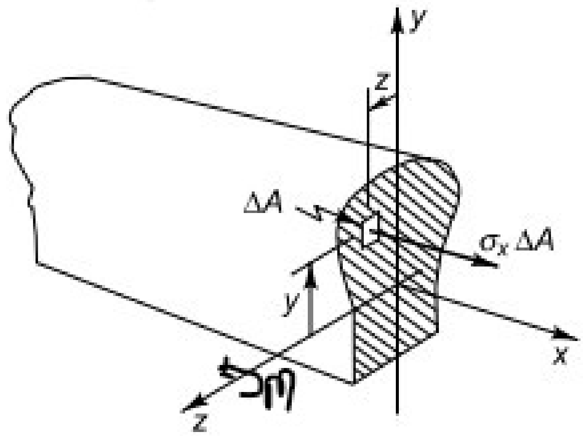

Figure 3:

Now
Take equilibrium

$$\sum F_x = 0$$

$$\int_A \sigma_x dA = \int_A \frac{-Ey}{\rho} dA = 0$$

$$= \frac{-E}{\rho} \int_A y dA = 0$$

$$\sum M_y = 0$$

$$\int_A z\sigma_x \, dA = \int_A \frac{-zEy}{\rho} dA = 0$$

$$= \frac{-E}{\rho} \int_A zy \, dA = 0$$

$$\sum M_z = M$$

$$\int_A y \, \sigma_x dA = \int_A yE \, \frac{y}{\rho} \, dA = M$$

$$\boxed{\frac{E}{\rho}\int_A y^2 dA = M \text{ .... } 1.2.3}$$

Here, We know that

$$\int_A y^2 dA = I_{zz}$$

$$I_{zz} = Moment\ of\ inertia$$

Then

$\frac{E}{\rho} I_{zz} = M$

or we can also write as:

$$\boxed{\frac{M}{EI_{zz}} = \frac{1}{\rho} \text{ .... } 1.2.4}$$

From Equation 1.1.1

$$\tan\theta = \lim_{\Delta x \to 0} \frac{d\delta}{dx}$$

Now, differentiate with respect to the s

$$\frac{d\theta}{ds}\ \sec^2\theta = \frac{d^2\delta}{dx^2}\ \frac{dx}{ds}$$

From the figure 1

$$\frac{dx}{ds} = \cos\theta$$

then, We can write;

$$\frac{d\theta}{ds} = \frac{d^2\delta}{dx^2}\ \cos^3\theta$$

if θ is small

Then

$\cos\theta = 1$

so, The equation is

$$\frac{d\theta}{ds} = \frac{d^2\delta}{dx^2}$$

Now, From Equation 1.1.3 and 1.2.1

$$\frac{d\theta}{ds} = \frac{1}{\rho}$$

So, we can write

$$\frac{d^2\delta}{dx^2} = \frac{1}{\rho}$$

From equation 1.2.4, We can write the above equation is

$$\frac{d^2\delta}{dx^2} = \frac{M}{E\ I_{zz}}$$

&

$$\boxed{Kappa\ (K) = \frac{d^2\delta}{dx^2}}$$

So, here is the relation between K and M

$$\boxed{k = \frac{M}{E\ I_{zz}}}$$

1.3 Derivation of $\boxed{\sigma_x = \frac{My}{I}}$

From the Equation 1.2.2

$$\sigma_x = \frac{-y}{\rho}\ E$$

From the equation 1.2.4

$$\frac{1}{\rho} = \frac{M}{EI_{zz}}$$

Now compare the both equations
We can find out

$$\sigma_x = -y\ E\frac{M}{EI_{zz}}$$

Now, the equation is

$$\boxed{\sigma_x = -y\ \frac{M}{I_{zz}}}$$

1.4 Analytical solution of a simply supported beam

Let's assume a simply supported beam on which applied a uniform distributed load. And the length of the beam is L and applied uniform load is q_0 in the downward direction.

From the Euler Bernoulli theory

$$\frac{\partial^2}{\partial x^2}(EI\ \frac{\partial^2 w}{\partial x^2}) = -q_0$$

or

$$EI\ \frac{\partial^4 \delta}{\partial x^4} = -q_0$$

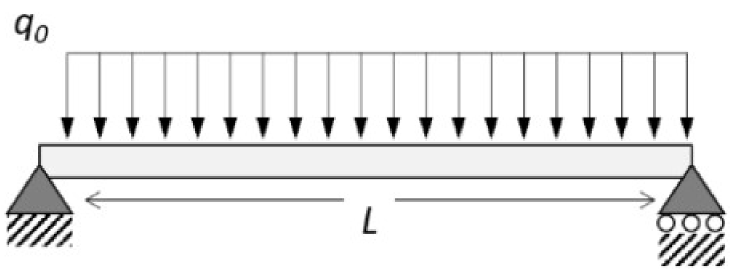

Figure 4:

Now integrate this equation

$$EI\ \frac{\partial^3 \delta}{\partial x^3} = -q_0\ x + c_1$$

$$EI\ \frac{\partial^2 \delta}{\partial x^2} = \frac{-q_0 x^2}{2} + c_1 x + c_2$$

$$EI\ \frac{\partial \delta}{\partial x} = \frac{-q_0 x^3}{6} + \frac{c_1 x^2}{2} + c_2 x$$

$$EI\ \delta_x = -\frac{q_0 x^4}{24} + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 \text{ .... 1.4.1}$$

From the figure 4

$$\delta(x = 0) = 0$$

Then

$$c_4 = 0$$

and

$$\frac{\partial^2 \delta}{\partial x^2}(x = 0) = 0$$

Then

$$c_2 = 0$$

and

$$\frac{\partial^2 \delta}{\partial x^2}(x = L) = 0$$

Then

$$c_1 = \frac{q_0 L}{2}$$

and

$$\delta(x = L) = 0$$

Then

$$c_3 = -\frac{q_0 L^3}{24}$$

Now, Put the value of c_1 c_2 c_3 and c_4 in the equation

Then we find out the value of δ as a function of x

$$\delta_x = \frac{1}{EI}\left(\frac{-q_0 x^4}{24} + \frac{q_0 x^3 L}{12} - \frac{q_0 L^3 x}{24}\right) \text{ .... 1.4.2}$$

or

$$\delta_x = \frac{-q_0 x}{24\ EI}(x^3 - 2\ x^2 L + L^3) \text{ .... 1.4.3}$$

Now, from the figure 1 we can see the maximum deflection is δ_c at L/2

$$\boxed{\delta_c = -\frac{5\ q_0 L^4}{384\ EI}}$$

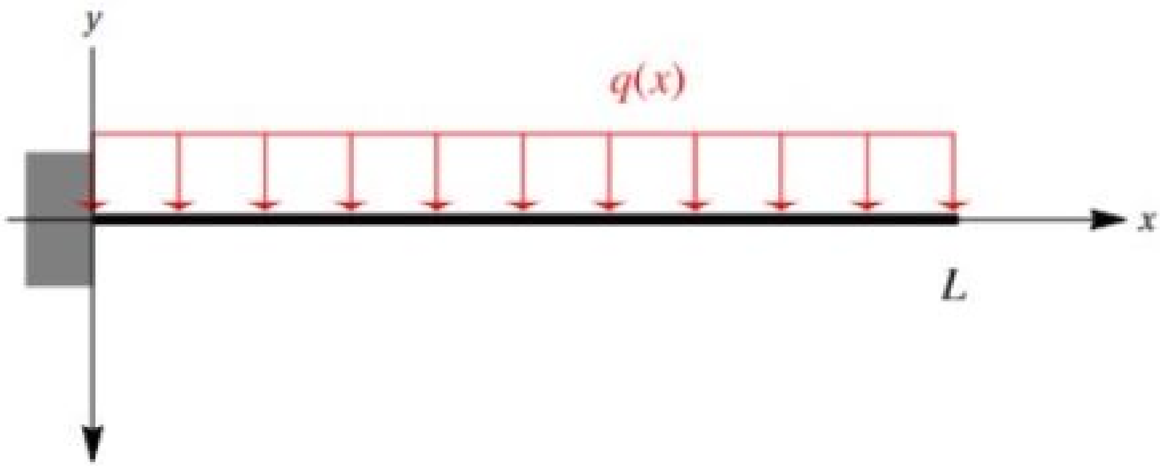

Figure 5:

1.5 Analytical solution of a Cantilever beam

Let's assume a cantilever in the figure 5 with the length. A uniform load q_x is applied on it in the downward direction.

Here, we can take

$$q_x = -q_0$$

From the equation 1.4.1

$$EI\ \delta_x = -\frac{q_0 x^4}{24} + \frac{c_1 x^3}{6} + \frac{c_2 x^2}{2} + c_3 x + c_4 \text{ .... } 1.5.1$$

We can see in the figure that

$$\delta(x = 0) = 0$$

Then

$$c_4 = 0$$

and

$$\frac{\partial \delta}{\partial x}(x = 0) = 0$$

Then

$$c_3 = 0$$

and

$$\frac{\partial^3 \delta}{\partial x^3}(x = L) = 0$$

Then

$$c_2 = -\frac{q_0 L^2}{2}$$

and

$$\frac{\partial^2 \delta}{\partial x^2}(x = L) = 0$$

Then

$$c_1 = q_0 L$$

Now, Put the value of c_1 c_2; c_3 and c_4 in the equation

Then we find out the value of δ as a function of x

$$\delta_x = \frac{1}{EI}\left(-\frac{q_0 x^4}{24} + q_0 L \frac{x^3}{6} - \frac{q_0 L^2 x^2}{4}\right)$$

or

$$\boxed{\delta_x = -\frac{q_0 x^2}{24\ EI}(x^2 + 6L^2 - 4Lx)}$$

2 Euler Bernoulli Non linear beam theory

The non-linear beam theory, also known as geometrically non-linear beam theory or large displacement theory, is a mathematical framework used to analyze the behavior of beams subjected to large deformations and displacements.

The non-linear beam theory is typically formulated using the Euler-Bernoulli beam theory, which describes the beam's behavior based on deflection, rotation, and internal bending and shear forces. However, in the non-linear theory, additional terms and equations are introduced to account for the large displacements and rotations.

The Non-linear beam theory takes some assumptions given below:-

1. Large Displacement: The theory allows for significant deflections and rotations of the beam, where the displacement of a point in the deformed configuration is not necessarily small compared to its original position.

2. Small Strains: Although the displacements can be large, the theory assumes that the strains within the beam material are still small. This assumption simplifies the analysis since linear elastic material behavior can still be assumed.

3. Plane sections remain plane: The theory assumes that any cross-section of the beam, which is initially planar, remains planar after deformation. This assumption allows for the separation of bending and extensional deformations.

In summary, the non-linear beam theory extends the traditional linear beam theory to account for large displacements and rotations. By considering the non-linear behavior of materials and the deformed geometry of the beam, this theory provides a more accurate representation of beam structures under significant loads or deformations.

The governing equation for the linear beam theory is derived based on the assumptions of small deformations and displacements. It describes the behavior of beams under small loads or displacements and is known as the Euler-Bernoulli beam equation.

The non-linear beam theory is typically formulated using the Euler-Bernoulli beam theory, which describes the beam's behavior based on its deflection, rotation, and internal bending and shear forces. However, in the non-linear theory, additional terms and equations are introduced to account for the large displacements and rotations.

The governing equations of the non-linear beam theory are derived by considering the equilibrium of small beam segments. These equations involve the bending moment, shear force, axial force, and the beam's geometry. The constitutive relationship, which relates the stresses and strains within the beam material, is typically described using a non-linear stress-strain curve that accounts for material plasticity or other non-linear behaviors. The governing equation is given below:-

$$\frac{d^2w}{dz^2} = \frac{1}{EI}\left(-M_2 + \frac{M_1 + M_2}{L}z + Pw\right); \quad P > 0$$

Here:-
w is the deflection
E is Young Modulus
I is moment of Inertia
P is the axial Force

$$P = EA\left[\frac{u}{L} + b_1(\phi_1 + \phi_2)^2 + b_2(\phi_1 - \phi_2)^2\right]$$

$$b_1 = b_2 = BowingFunction$$

$$\phi_1 = \phi_2 = angles$$

u is the axial displacement due to axial force
L is the length of the beam

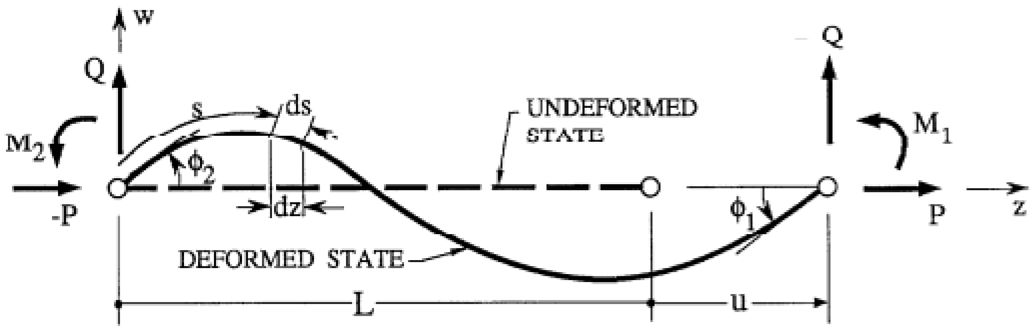

Figure 6:

$$M_1 \text{ and } M_2 \text{ is the moment}$$

$$M_1 = \frac{EI}{L}(c_1\phi_1 + c_2\phi_2)$$
$$M_2 = \frac{EI}{L}(c_2\phi_1 + c_1\phi_2)$$

$$c_1 \text{ and } c_2 \text{ is the Stability Fucntion}$$

2.1 Derivation of the Governing Equation

Let's assume a beam with length L and cross-section area A. We applied an axial force and the moments at the end of the beam. After this, the beam deflected as shown in the figure 6 at the angles ϕ_1 and ϕ_2. Due to axial force, the length of the beam increases by u.

The free-body diagram of the beam is shown in the figure 7. From the figure:-

$$tan\phi = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \frac{dw}{dz}$$

Differentiate the equation with respect to the s

$$sec^2\phi \, \frac{d\phi}{ds} = \frac{d^2w}{dz^2} \, \frac{dz}{ds}$$

From the figure 6

$$\frac{dz}{ds} = cos\phi$$

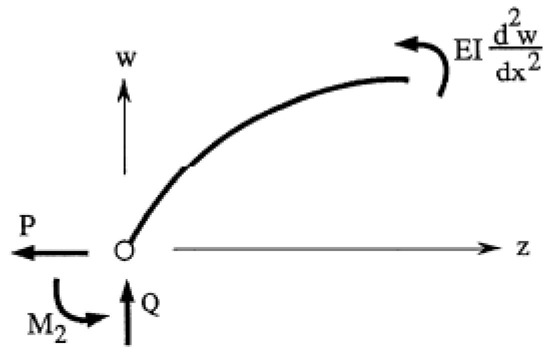

Figure 7:

$$\frac{d\phi}{ds} = \frac{d^2w}{dz^2}\, cos^3\phi$$

If ϕ is small then

$$cos\phi = 1$$

$$\frac{d\phi}{ds} = \frac{d^2w}{dz^2}$$

Now we know that

$$\frac{d\phi}{ds} = \frac{M}{EI}$$

So

$$\frac{M}{EI} = \frac{d^2w}{dz^2} \text{ .... } 2.1.1$$

Now, from the figure 6

$$\Sigma M = 0$$

$$M_1 + M_2 - QL = 0$$

$$Q = \frac{M_1 + M_2}{L} \text{ .... } 2.1.2$$

2.1.1 If P is negative, P < 0

$$\Sigma M_B = 0$$

$$M + M_2 + Pw - Qz = 0$$

Here Q is the Shear force

$$M = -M_2 - Pw + Qz = 0 \text{ .... } 2.1.1.1$$

Then from equation 2.1.1, 2.1.2 and 2.1.1.1

$$\frac{d^2w}{dz^2} = \frac{1}{EI}\,(-M_2 - Pw + \frac{M_1 + M_2}{L}\ z) \text{ .... } 2.1.1.2$$

Now

$$\frac{d^2w}{dz^2} + \frac{pw}{EI} = \frac{1}{EI}\,(-M_2 + \frac{M_1 + M_2}{L}\ z) \text{ .... } 2.1.1.3$$

The solution of the above given differential equation is

$$w = w_h + w_p \text{ ....} 2.1.1.4$$

Here

$$w_h = solution\ of\ the\ homogeneous = \frac{d^2w}{dz^2} + \frac{Pw}{EI}$$

$$w_p = particular\ solution\ of\ the\ equation\ in\ the\ form\ of = az + b, Here\ a\ and\ b\ are\ constant$$

Then
for the solution of the homogeneous equation is

$$\frac{d^2w}{dz^2} + \frac{Pw}{EI} = 0$$

Now, we can also write

$$(D^2 + \frac{P}{EI})w = 0$$

$$m^2 + \frac{P}{EI} = 0$$

To solve this

$$m_1 = \sqrt{\frac{P}{EI}}\ i$$

$$m_2 = -\sqrt{\frac{P}{EI}}\ i$$

Here m_1 and m_2 are the roots
Here We can take $m_1 = -m_2$ and $\sqrt{\frac{P}{EI}} = \beta$

Then

$$w_h = c_1 \ \cos\beta z + c_2 \ \sin\beta z \text{ .... } 2.1.1.5$$

Now the particular solution of the equation is

$$w_p = az + b \text{ .... } 2.1.1.6$$

Then Differentiate the equation

$$w'_p = a$$

$$w''_p = 0$$

put the value of w'_p and w''_p in the equation 2.1.1.3
Then

$$\frac{P}{EI}\ (az + b) = \frac{1}{EI}\ \left(-M_2 + \frac{M_1 + M_2}{L}\ z\right)$$

$$P\ (az + b) = -M_2 + \frac{M_1 + M_2}{L}\ z$$

Compare the equation of both side
Then

$$P\ a = \frac{M_1 + M_2}{L}$$

$$a = \frac{M_1 + M_2}{P\ L}$$

and

$$P\ b = -M_2$$

$$b = \frac{-M_2}{P}$$

Put the value of a and b in the equation 2.1.1.6
Then

$$w_p = \frac{M_1 + M_2}{PL}\ z - \frac{M_2}{P} \text{ .... } 2.1.1.7$$

Put the value of w_h and w_p from the equation 2.1.1.5 and 2.1.1.7 in the equation 2.1.1.4

$$w = c_1 \ \cos\beta z + c_2 \ \sin\beta z + \frac{M_1 + M_2}{PL}\ z - \frac{M_2}{P} \text{ .... } 2.1.1.8$$

Then from the figure 6,

$$w(z = 0) = 0$$

Then

$$c_1 = \frac{M_2}{P}$$

and

$$w(z=L)=0$$

Then

$$c_1\ \cos\beta L + c_2\ \sin\beta L + \frac{M_1+M_2}{P} - \frac{M_2}{P} = 0$$

$$\frac{M_2}{P}\ \cos\beta L + c_2\ \sin\beta L + \frac{M1}{P} = 0$$

$$c_2 = -\frac{M_2\ \cos\beta L}{P\ \sin\beta L} - \frac{M_1}{P\ \sin\beta L}$$

Now, put the all value of $c_1 and c_2$ in the equation 2.1.1.8

$$w = \frac{M_2}{P}\ \cos\beta z + \left(-\frac{M_2\ \cos\beta L}{P\ \ sin\beta L} - \frac{M_1}{P\ \sin\beta L}\right)\ \sin\beta z + \frac{M_1+M_2}{PL}\ z - \frac{M_2}{P}$$

$$w = \frac{M_1}{P}\left(\frac{-\sin\beta z}{\sin\beta L} + \frac{z}{L}\right) + \frac{M_2}{P}\left(\cos\beta z - \frac{\sin\beta z\ \cos\beta L}{\sin\beta L} + \frac{z}{L} - 1\right)$$

$$w = \frac{M_1}{P}\left(\frac{z}{L} - \frac{\sin\beta z}{\sin\beta L}\right) + \frac{M_2}{P}\left(\frac{\cos\beta z\ \sin\beta l - \sin\beta z\ \cos\beta L}{\sin\beta L} + \frac{z-L}{L}\right)$$

We know that

$$\cos y\ \sin x - \sin y\ \cos x = \sin(x-y)$$

Then

$$w = \frac{M_1}{P}\left(\frac{z}{L} - \frac{\sin\beta z}{\sin\beta L}\right) + \frac{M_2}{P}\left(\frac{\sin\beta(L-z)}{\sin\beta L} + \frac{z-L}{L}\right) \text{ .... } 2.1.1.9$$

2.1.2 If P is equal to zero, P=0

From the equation 2.1.1.3

$$\frac{d^2w}{dz^2} = \frac{1}{EI}\ \left(-M_2 + \frac{M_1+M_2}{L}\ z\right) \text{ .... } 2.1.2.1$$

integrate the above equation

$$\frac{dw}{dz} = \frac{1}{EI}\ \left(-M_2\ z + \left(\frac{M_1+M_2}{L}\right)\ \frac{z^2}{2}\right) + c_1 \text{ .... } 2.1.2.2$$

$$w = \frac{1}{EI}\ \left(-M_2\ \frac{z^2}{2} + \left(\frac{M_1+M_2}{L}\right)\ \frac{z^3}{6}\right) + c_1\ z + c_2 \text{ .... } 2.1.2.3$$

at $w(z=0)=0$

$$c_2 = 0$$

at $w(z=L)=0$

$$\frac{1}{EI}\left[-M_2\frac{L^2}{2}+\left(\frac{M_1+M_2}{L}\right)\frac{L^3}{6}\right]+c_1\ L=0$$

$$c_1\ L=-\frac{1}{EI}\left[-M_2\frac{L^2}{2}+\left(\frac{M_1+M_2}{L}\right)\frac{L^3}{6}\right]$$

$$c_1\ L=-\frac{1}{EI}\left[M_2\left(-\frac{L^2}{2}+\frac{L^2}{6}\right)+M_1\frac{L^2}{6}\right]$$

$$c_1=-\frac{1}{EI}\left[M_1\frac{L}{6}-M_2\frac{2L}{6}\right]$$

Now put the value of c_1 and c_2 in the equation 2.1.2.3

$$w=\frac{1}{EI}\left[-M_2\frac{z^2}{2}+\left(\frac{M_1+M_2}{L}\right)\frac{z^3}{6}\right]-\frac{1}{EI}\left[-M_2\ \frac{2L}{6}+M_1\frac{L}{6}\right]$$

$$w=\frac{1}{6EI}\left[M_2\left(-3\ z^2+\frac{z^3}{L}+2\ L\right)+M_1\left(\frac{z^3}{L}-L\right)\right]$$

or

$$w=\frac{M_1L}{6EI}\ z\left(\frac{z^2}{L^2}-1\right)-\frac{M_2L}{6EI}(L-z)\left[\frac{(L-z)^2}{L^2}-1\right]\ \text{......... 2.1.2.4}$$

2.1.3 If P is positive, P $>$ 0

From the equation 2.1.1.3

$$\frac{d^2w}{dz^2}-\frac{pw}{EI}=\frac{1}{EI}\ \left(-M_2+\frac{M_1+M_2}{L}\ z\right)\ \text{.................. 2.1.3.1}$$

The solution of the above given differential equation is

$$w=w_h+w_p\ \text{..........2.1.3.2}$$

Here

$$w_h=solution\ of\ the\ homogeneous=\frac{d^2w}{dz^2}-\frac{Pw}{EI}$$

$w_p = particular\ solution\ of\ the\ equation\ in\ the\ form\ of = az+b, Here\ a\ and\ b\ are\ constant$

Then
for the solution of the homogeneous equation is

$$\frac{d^2w}{dz^2} - \frac{Pw}{EI} = 0$$

Now, we can also write

$$(D^2 - \frac{P}{EI})w = 0$$

$$m^2 - \frac{P}{EI} = 0$$

To solve this

$$m_1 = \sqrt{\frac{P}{EI}}$$

$$m_2 = -\sqrt{\frac{P}{EI}}$$

Here m_1 and m_2 are the roots

Here We can take $m_1 = -m_2$

If we take $m_1 = m$ then $m_2 = -m$

Then, the solution of the homogeneous equation is

$$w_h = c_1\ e^{m_1 z} + c_2\ e^{m_2 z}$$

and we can also write

$$w_h = c_1\ e^{mz} + c_2\ e^{-mz} \text{ .... 2.1.3.3}$$

Now the particular solution of the equation is

$$w_p = az + b \text{ .... 2.1.3.4}$$

Then Differentiate the equation

$$w_p' = a$$

$$w_p'' = 0$$

put the value of w_p' and w_p'' in the equation 2.1.3.1
Then

$$-\frac{P}{EI}\ (az + b) = \frac{1}{EI}\ \left(-M_2 + \frac{M_1 + M_2}{L}\ z\right)$$

$$-P\ (az+b) = -M_2 + \frac{M_1+M_2}{L}\ z$$

Compare the equation of both side
Then

$$-P\ a = \frac{M_1+M_2}{L}$$

$$a = -\left(\frac{M_1+M_2}{P\ L}\right)$$

and

$$P\ b = M_2$$

$$b = \frac{M_2}{P}$$

Put the value of a and b in the equation 2.1.3.4
Then

$$w_p = -\frac{M_1+M_2}{PL}\ z + \frac{M_2}{P}\ \text{...........}\ 2.1.3.5$$

Put the value of w_h and w_p from the equation 2.1.3.3 and 2.1.3.5 in the equation 2.1.3.2

$$w = c_1\ e^{mz} + c_2\ e^{-mz} - \frac{M_1+M_2}{PL}\ z + \frac{M_2}{P}\ \text{........}\ 2.1.3.6$$

Now
at $w(z=0) = 0$

$$c_1 + c_2 + \frac{M_2}{P} = 0$$

$$c_1 = -c_2 - \frac{M_2}{P}\ \text{...........}\ 2.1.3.7$$

and
at $w(z=L) = 0$

$$c_1\ e^{mL} + c_2\ e^{-mL} + \frac{M_2}{P} - \frac{M_1+M_2}{P} = 0$$

Put the value of the c_1 from the equation 2.1.3.7 in this equation

$$-\left(\frac{M_2}{P} + c_2\right)\ e^{mL} + c_2\ e^{-mL} + \frac{M_2}{P} - \frac{M_1+M_2}{P} = 0$$

$$c_2\left(-e^{mL} + e^{-mL}\right) - \frac{M_2}{P}\ e^{mL} - \frac{M_1}{P} = 0\ \text{.............}\ 2.1.3.8$$

Now, we know that

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Then, from the equation 2.1.3.7

$$c_2\left(-2\sinh(mL)\right) = \frac{M_2\ e^{mL} + M_1}{P}$$

$$c_2 = -\frac{M_2\ e^{mL} + M_1}{2P\sinh(mL)} \text{ .... } 2.1.3.9$$

Now put the value of the c_2 from the equation 2.1.3.9 in the equation 2.1.3.7
Then

$$c_1 = \frac{M_2\ e^{mL} + M_1}{2P\sinh(mL)} - \frac{M_2}{P}$$

Now, put the value of the c_1 and c_2 in the equation 2.1.3.6
Then

$$w = \left(\frac{M_2\ e^{mL} + M_1}{2P\sinh(mL)} - \frac{M_2}{P}\right)\ e^{mz} - \left(\frac{M_2\ e^{mL} + M_1}{2P\sinh(mL)}\right)\ e^{-mz} - \frac{M_1 + M_2}{PL}\ z + \frac{M_2}{P}$$

$$w = \frac{M_2}{P}\left[\frac{e^{mL}\ e^{mz}}{2\sinh(mL)} - e^{mz} - \frac{e^{mL}\ e^{-mz}}{2\sinh(mL)} + 1 - \frac{z}{L}\right] + \frac{M_1}{P}\left[\frac{e^{mz}}{2\sinh(mL)} - \frac{e^{-mz}}{2\sinh(mL)} - \frac{z}{L}\right]$$

$$w = \frac{M_2}{P}\left[\frac{e^{mL}}{2\sinh(mL)}\left(e^{mz} - e^{-mz}\right) - e^{mz} + \frac{L-z}{L}\right] + \frac{M_1}{P}\left[\frac{e^{mz} - e^{-mz}}{2\sinh(mL)} - \frac{z}{L}\right]$$

$$w = \frac{M_2}{P}\left[\frac{e^{mL}}{2\sinh(mL)}\left(2\sinh(mz)\right) - e^{mz} + \frac{L-z}{L}\right] + \frac{M_1}{P}\left[\frac{2\sinh(mz)}{2\sinh(mL)} - \frac{z}{L}\right]$$

$$w = \frac{M_2}{P}\left[\frac{e^{mL}\ \sinh(mz) - e^{mz}\ \sinh(mL)}{\sinh(mL)} + \frac{L-z}{L}\right] + \frac{M_1}{P}\left[\frac{\sinh(mz)}{\sinh(mL)} - \frac{z}{L}\right]$$

Now, we know that

$$\cosh x - \sinh x = e^{-x}$$

$$\cosh x + \sinh x = e^{x}$$

Then

$$w = \frac{M_2}{P}\left[\frac{[\cosh(mL) + \sinh(mL)]\ \sinh(mz) - [\cosh(mz) + \sinh(mz)]\ \sinh(mL)}{\sinh(mL)} + \frac{L-z}{L}\right] + \frac{M_1}{P}\left[\frac{\sinh(mz)}{\sinh(mL)} - \frac{z}{L}\right] \tag{1}$$

$$w = \frac{M_2}{P}\left[\frac{\cosh(mL)\ \sinh(mz) - \cosh(mz)\ \sinh(mL)}{\sinh(mL)} + \frac{L-z}{L}\right] + \frac{M_1}{P}\left[\frac{\sinh(mz)}{\sinh(mL)} - \frac{z}{L}\right]$$

Now, we know that

$$\cosh(y)\ \sinh(x) - \cosh(x)\ \sinh(y) = \sinh(x-y)$$

Then

$$w = \frac{M_1}{P}\left[\frac{\sinh(mz)}{\sinh(mL)} - \frac{z}{L}\right] + \frac{M_2}{P}\left[\frac{L-z}{L} + \frac{\sinh(z-L)m}{\sinh(mL)}\right] \text{ .... 2.1.3.10}$$

2.2 Relation between Moment and angle

The relation between Moments and Angles are different for the three cases. From the figure:-

$$\phi_1 = \left.\frac{dw}{dz}\right|_{z=L} \text{ .... 2.2.1}$$

$$\phi_2 = \left.\frac{dw}{dz}\right|_{z=0} \text{ .... 2.2.2}$$

2.2.1 If P is negative, P < 0

From the equation 2.1.1.9

$$w = \frac{M_1}{P}\left(\frac{z}{L} - \frac{\sin\beta z}{\sin\beta L}\right) + \frac{M_2}{P}\left(\frac{\sin\beta(L-z)}{\sin\beta L} + \frac{z-L}{L}\right)$$

Now differentiate the equation with respect to the z

$$\frac{dw}{dz} = \frac{M_1}{P}\left(\frac{1}{L} - \beta\frac{\cos\beta z}{\sin\beta L}\right) + \frac{M_2}{P}\left(\frac{1}{L} - \beta\frac{\cos\beta(L-z)}{\sin\beta L}\right)$$

Now

$$\phi_1 = \frac{M_1}{P}\left(\frac{1}{L} - \beta\frac{\cos\beta L}{\sin\beta L}\right) + \frac{M_2}{P}\left(\frac{1}{L} - \frac{\beta}{\sin\beta L}\right)$$

$$P\ L\ \phi_1 = M_1\left(1 - \beta\ L\ \frac{\cos\beta L}{\sin\beta L}\right) + M_2\left(1 - \frac{\beta\ L}{\sin\beta L}\right) \text{ .... 2..2.1.1}$$

and

$$\phi_2 = \frac{M_1}{P}\left(\frac{1}{L} - \frac{\beta}{\sin\beta L}\right) + \frac{M_2}{P}\left(\frac{1}{L} - \beta\ \frac{\cos\beta L}{\sin\beta L}\right)$$

$$P\ L\ \phi_2 = M_1 \left(1 - \frac{\beta\ L}{\sin \beta L}\right) + M_2 \left(1 - \beta\ L\ \frac{\cos \beta L}{\sin \beta L}\right) \text{ .... 2.2.1.2}$$

Now, we can assume that $\beta\ L = w$

Then, From equation 2.2.1.1 and 2.2.1.2

$$P\ L\ \phi_1 = M_1 \left(1 - w\ \frac{\cos w}{\sin w}\right) + M_2 \left(1 - \frac{w}{\sin w}\right) \text{ .... 2..2.1.3}$$

$$P\ L\ \phi_2 = M_1 \left(1 - \frac{w}{\sin w}\right) + M_2 \left(1 - w\ \frac{\cos w}{\sin w}\right) \text{ .... 2.2.1.4}$$

we can write the equation 2.2.1.3 and 2.2.1.4 in the form of Matrix, Then

$$\text{P L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 - \frac{w\ \cos w}{\sin w} & 1 - \frac{w}{\sin w} \\ 1 - \frac{w}{\sin w} & 1 - \frac{w\ \cos w}{\sin w} \end{bmatrix} \quad \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

Let's assume that

$$\text{A} = \begin{bmatrix} 1 - \frac{w\ \cos w}{\sin w} & 1 - \frac{w}{\sin w} \\ 1 - \frac{w}{\sin w} & 1 - \frac{w\ \cos w}{\sin w} \end{bmatrix}$$

Then

$$\text{P L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \text{A} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

$$PL \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \ A^{-1} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \text{ .... 2.2.1.5}$$

Here,

$$A^{-1} = \frac{1}{\left(1 - \frac{w \cos w}{\sin w}\right)^2 - \left(1 - \frac{w}{\sin w}\right)^2} \begin{bmatrix} 1 - \frac{w\ \cos w}{\sin w} & -\left(1 - \frac{w}{\sin w}\right) \\ -\left(1 - \frac{w}{\sin w}\right) & 1 - \frac{w\ \cos w}{\sin w} \end{bmatrix}$$

$$A^{-1} = \frac{1}{1 + \frac{w^2\ \cos^2 w}{\sin^2 w} - 2\ w \frac{\cos w}{\sin w} - 1 - \frac{w^2}{\sin^2 w} + \frac{2\ w}{\sin w}} \begin{bmatrix} \frac{\sin w - w\ \cos w}{\sin w} & -\left(\frac{\sin w - w}{\sin w}\right) \\ -\left(\frac{\sin w - w}{\sin w}\right) & \frac{\sin w - w\ \cos w}{\sin w} \end{bmatrix}$$

$$A^{-1} = \frac{1}{\frac{w^2}{\sin^2 w}\left(\cos^2 w - 1\right) + \frac{2w}{\sin w}\left(1 - \cos w\right))} \begin{bmatrix} \frac{\sin w - w \ \cos w}{\sin w} & -\left(\frac{\sin w - w}{\sin w}\right) \\ -\left(\frac{\sin w - w}{\sin w}\right) & \frac{\sin w - w \ \cos w}{\sin w} \end{bmatrix}$$

$$A^{-1} = \frac{\sin w}{-w^2 \ \sin w + 2w - 2w \ \cos w} \begin{bmatrix} \frac{\sin w - w \ \cos w}{\sin w} & -\left(\frac{\sin w - w}{\sin w}\right) \\ -\left(\frac{\sin w - w}{\sin w}\right) & \frac{\sin w - w \ \cos w}{\sin w} \end{bmatrix}$$

$$A^{-1} = \frac{1}{w\left(-w \ \sin w + 2 - 2 \ \cos w\right)} \begin{bmatrix} \sin w - w \ \cos w & -\sin w + w \\ -\sin w + w & \sin w - w \ \cos w \end{bmatrix}$$

Put the value of the A^{-1} in the 2.2.1.5

Then

$$\frac{P \ L^2}{EI} \ \frac{EI}{L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \ \frac{1}{w\left(-w \ \sin w + 2 - 2 \ \cos w\right)} \begin{bmatrix} \sin w - w \ \cos w & -\sin w + w \\ -\sin w + w & \sin w - w \ \cos w \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

Here $\frac{PL^2}{EI} = \beta \ L = w^2$

Then

$$w^2 \ \frac{EI}{L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \ \frac{1}{w\left(-w \ \sin w + 2 - 2 \ \cos w\right)} \begin{bmatrix} \sin w - w \ \cos w & -\sin w + w \\ -\sin w + w & \sin w - w \ \cos w \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

$$\frac{EI}{L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \ \frac{w}{\left(-w \ \sin w + 2 - 2 \ \cos w\right)} \begin{bmatrix} \sin w - w \ \cos w & -\sin w + w \\ -\sin w + w & \sin w - w \ \cos w \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} w\left(\frac{\sin w - w \ \cos w}{2 - 2\cos w - w \sin w}\right) & w\left(\frac{w - \sin w}{2 - 2\cos w - w \sin w}\right) \\ w\left(\frac{w - \sin w}{2 - 2\cos w - w \sin w}\right) & w\left(\frac{\sin w - w \ \cos w}{2 - 2\cos w - w \sin w}\right) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

We can also write

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} c_1 & c_2 \\ c_2 & c_1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

or

$$M_1 = \frac{E\ I}{L}\ (c_1\phi_1 + c_2\phi_2)\ \text{............}\ 2.2.1.6$$

$$M_2 = \frac{E\ I}{L}\ (c_2\phi_1 + c_1\phi_2)\ \text{......}\ 2.2.1.7$$

Now, here

$$c_1 = w\left(\frac{\sin w - w\ \cos w}{2 - 2\cos w - w\sin w}\right)\ \text{..............}\ 2.2.1.8$$

$$c_2 = w\left(\frac{w - \sin w}{2 - 2\cos w - w\sin w}\right)\ \text{.........}\ 2.2.1.9$$

2.2.2 If P is equal to zero, P = 0

From the equation 2.1.2.2

$$\frac{dw}{dz} = \frac{1}{EI}\ \left[-M_2\ z + \left(\frac{M_1 + M_2}{L}\right)\ \frac{z^2}{2}\right] + c_1$$

here

$$c_1 = -\frac{1}{EI}\left[M_1\frac{L}{6} - M_2\frac{2L}{6}\right]$$

Then

$$\frac{dw}{dz} = \frac{1}{EI}\ \left[-M_2\ z + \left(\frac{M_1 + M_2}{L}\right)\ \frac{z^2}{2}\right] - \frac{1}{EI}\left[M_1\frac{L}{6} - M_2\frac{2L}{6}\right]$$

$$\phi_1 = \frac{1}{EI}\left[-M_2 L + \left(\frac{M_1 + M_2}{L}\right)\frac{L^2}{2}\right] - \frac{1}{EI}\left[M_1\frac{L}{6} - M_2\frac{2L}{6}\right]$$

$$\phi_1 = \frac{1}{EI}\left[M_1\left(\frac{L}{2} - \frac{L}{6}\right) + M_2\left(-L + \frac{L}{2} + \frac{2L}{6}\right)\right]$$

$$6\ EI\ \phi_1 = M_1(2L) + M_2(-L)$$

$$\frac{6\ EI}{L}\ \phi_1 = 2\ M_1 - M_2\ \text{............}\ 2.2.2.1$$

and

$$\phi_2 = -\frac{1}{EI}\left(M_1 \frac{L}{6} - M_2 \frac{2L}{6}\right)$$

$$\frac{6\ EI}{L}\ \phi_2 = -M_1 + 2\ M_2 \text{ .... 2.2.2.2}$$

Then, from the equation 2.2.2.1 and 2.2.2.2, we can write it in the matrix form

$$\tfrac{6EI}{L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

Let's assume that

$$\text{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Then

$$\tfrac{6EI}{L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \text{A} \quad \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

Then

$$\frac{6\ EI}{L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}\ A^{-1} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \text{ .... 2.2.2.3}$$

Here

$$A^{-1} = \frac{1}{4-1} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ ....2.2.2.4}$$

Put the value of A^{-1} from the equation 2.2.2.4 in the equation 2.2.2.3

$$\frac{6\ EI}{L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}\ \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{2\ EI}{L} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

We can also write

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} c_1 & c_2 \\ c_2 & c_1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

or

$$M_1 = \frac{E\ I}{L} \left(c_1\phi_1 + c_2\phi_2 \right) \text{ .... 2.2.2.5}$$

$$M_2 = \frac{E\ I}{L} \left(c_2\phi_1 + c_1\phi_2 \right) \text{ ...... 2.2.2.6}$$

2.2.3 If P is positive, P > 0

From the equation 2.1.3.10

$$w = \frac{M_1}{P} \left[\frac{\sinh(mz)}{\sinh(mL)} - \frac{z}{L} \right] + \frac{M_2}{P} \left[\frac{L-z}{L} + \frac{\sinh(z-L)m}{\sinh(mL)} \right]$$

Now differentiate the equation with respect to the z

$$\frac{dw}{dz} = \frac{M_1}{P} \left(\frac{m\ \cosh(mz)}{\sinh(mL)} - \frac{1}{L} \right) + \frac{M_2}{P} \left(\frac{m\ \cosh m(z-L)}{\sinh(mL)} - \frac{1}{L} \right)$$

Now

$$\phi_1 = \frac{M_1}{P} \left(m\ \frac{\cosh(mL)}{\sinh(mL)} - \frac{1}{L} \right) + \frac{M_2}{P} \left(\frac{m}{\sinh(mL)} - \frac{1}{L} \right)$$

$$P\ L\ \phi_1 = M_1 \left(mL\ \frac{\cosh(mL)}{\sinh(mL)} - 1 \right) + M_2 \left(\frac{mL}{\sinh(mL)} - 1 \right) \text{ .... 2.2.3.1}$$

and

$$\phi_2 = \frac{M_1}{P} \left(\frac{m}{\sinh(mL)} - \frac{1}{L} \right) + \frac{M_2}{P} \left(\frac{m\ \cosh(mL)}{\sinh(mL)} - \frac{1}{L} \right)$$

$$P\ L\ \phi_2 = M_1 \left(\frac{mL}{\sinh(mL)} - 1 \right) + M_2 \left(\frac{mL\ \cosh(mL)}{\sinh(mL)} - 1 \right) \text{ .... 2.2.3.2}$$

Now, we can assume that $m\ L = w$

Then, From equation 2.2.3.1 and 2.2.3.2

$$P\ L\ \phi_1 = M_1 \left(w\ \frac{\cosh(w)}{\sinh(w)} - 1 \right) + M_2 \left(\frac{w}{\sinh(w)} - 1 \right) \text{ .... 2.2.3.3}$$

$$P\ L\ \phi_2 = M_1 \left(\frac{w}{\sinh(w)} - 1 \right) + M_2 \left(\frac{w\ \cosh(w)}{\sinh(w)} - 1 \right) \text{ .... 2.2.3.4}$$

we can write the equation 2.2.3.3 and 2.2.3.4 in the form of Matrix, Then

$$\text{P L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} w\ \frac{\cosh(w)}{\sinh(w)} - 1 & \frac{w}{\sinh(w)} - 1 \\ \frac{w}{\sinh(w)} - 1 & \frac{w\ \cosh(w)}{\sinh(w)} - 1 \end{bmatrix} \quad \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

Let's assume that

$$\text{A} = \begin{bmatrix} w\ \frac{\cosh(w)}{\sinh(w)} - 1 & \frac{w}{\sinh(w)} - 1 \\ \frac{w}{\sinh(w)} - 1 & \frac{w\ \cosh(w)}{\sinh(w)} - 1 \end{bmatrix}$$

Then

$$\text{P L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \text{A} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

$$PL \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}\ A^{-1} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \text{ .... 2.2.3.5}$$

Here,

$$A^{-1} = \frac{1}{\left(w\ \frac{\cosh w}{\sinh w} - 1\right)^2 - \left(\frac{w}{\sinh w} - 1\right)^2} \begin{bmatrix} w\ \frac{\cosh(w)}{\sinh(w)} - 1 & -\left(\frac{w}{\sinh(w)} - 1\right) \\ -\left(\frac{w}{\sinh(w)} - 1\right) & \frac{w\ \cosh(w)}{\sinh(w)} - 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\left(w^2 \frac{\cosh^2 w}{\sinh^2 w} + 1 - 2\ w\ \frac{\cosh w}{\sinh w} - \frac{w^2}{\sinh^2 w} - 1 + \frac{2w}{\sinh w}\right)} \begin{bmatrix} w\ \frac{\cosh(w)}{\sinh(w)} - 1 & -\left(\frac{w}{\sinh(w)} - 1\right) \\ -\left(\frac{w}{\sinh(w)} - 1\right) & \frac{w\ \cosh(w)}{\sinh(w)} - 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\left(\frac{w^2}{\sinh^2 w}\left(\cosh^2 w - 1\right) - 2\ w\ \frac{\cosh w}{\sinh w} + \frac{2w}{\sinh w}\right)} \begin{bmatrix} w\ \frac{\cosh(w)}{\sinh(w)} - 1 & -\left(\frac{w}{\sinh(w)} - 1\right) \\ -\left(\frac{w}{\sinh(w)} - 1\right) & \frac{w\ \cosh(w)}{\sinh(w)} - 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\left(w^2 - 2\ w\ \frac{\cosh w}{\sinh w} + \frac{2w}{\sinh w}\right)} \begin{bmatrix} w\ \frac{\cosh(w)}{\sinh(w)} - 1 & -\left(\frac{w}{\sinh(w)} - 1\right) \\ -\left(\frac{w}{\sinh(w)} - 1\right) & \frac{w\ \cosh(w)}{\sinh(w)} - 1 \end{bmatrix}$$

Put the value of the A^{-1} in the 2.2.1.5

Then

$$\frac{P\ L^2}{EI}\ \frac{EI}{L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}\ \frac{1}{\left(w^2 - 2\ w\ \frac{\cosh w}{\sinh w} + \frac{2w}{\sinh w}\right)} \begin{bmatrix} w\ \frac{\cosh(w)}{\sinh(w)} - 1 & -\left(\frac{w}{\sinh(w)} - 1\right) \\ -\left(\frac{w}{\sinh(w)} - 1\right) & \frac{w\ \cosh(w)}{\sinh(w)} - 1 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

Here $\frac{PL^2}{EI} = mL = w^2$

Then

$$w^2\ \frac{EI}{L} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}\ \frac{1}{w\left(w - 2\ \frac{\cosh w}{\sinh w} + \frac{2}{\sinh w}\right)} \begin{bmatrix} w\ \frac{\cosh(w)}{\sinh(w)} - 1 & -\left(\frac{w}{\sinh(w)} - 1\right) \\ -\left(\frac{w}{\sinh(w)} - 1\right) & \frac{w\ \cosh(w)}{\sinh(w)} - 1 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{L} \frac{w}{\left(w - 2\ \frac{\cosh w}{\sinh w} + \frac{2}{\sinh w}\right)} \begin{bmatrix} w\ \frac{\cosh(w)}{\sinh(w)} - 1 & -\left(\frac{w}{\sinh(w)} - 1\right) \\ -\left(\frac{w}{\sinh(w)} - 1\right) & \frac{w\ \cosh(w)}{\sinh(w)} - 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{L} \frac{w\ \sinh w}{(w\ \sinh w - 2\ \cosh w + 2)} \begin{bmatrix} \frac{w\ \cosh(w) - \sinh w}{\sinh(w)} & -\left(\frac{w - \sinh w}{\sinh(w)}\right) \\ -\left(\frac{w - \sinh w}{\sinh(w)}\right) & \frac{w\ \cosh(w) - \sinh w}{\sinh(w)} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{L} \frac{w}{(w\ \sinh w - 2\ \cosh w + 2)} \begin{bmatrix} w\ \cosh(w) - \sinh w & -(w - \sinh w) \\ -(w - \sinh w) & w\ \cosh(w) - \sinh w \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{w(w\ \cosh(w) - \sinh w)}{2+2\ \cosh w + w \sinh w} & \frac{w(\sinh w - w)}{2+2\ \cosh w + w \sinh w} \\ \frac{w(\sinh w - w)}{2+2\ \cosh w + w \sinh w} & \frac{w(w\ \cosh(w) - \sinh w)}{2+2\ \cosh w + w \sinh w} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

We can also write

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \frac{EI}{L} \begin{bmatrix} c_1 & c_2 \\ c_2 & c_1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

or

$$M_1 = \frac{E\ I}{L} (c_1 \phi_1 + c_2 \phi_2) \text{ .... 2.2.3.6}$$

$$M_2 = \frac{E\ I}{L} (c_2 \phi_1 + c_1 \phi_2) \text{ ...... 2.2.3.7}$$

Now, Here

$$c_1 = \frac{w\,(w\ \cosh(w) - \sinh w)}{2 + 2\ \cosh w + w \sinh w} \text{ .... 2.2.3.8}$$

$$c_2 = \frac{w\,(\sinh w - w)}{2 + 2\ \cosh w + w \sinh w} \text{ .... 2.2.3.9}$$

2.3 Include the Bowing Function

Bowing is a result of the applied load inducing compressive and tensile stresses within the material. In simple terms, one side of the member experiences compressive stresses, while the other side experiences tensile stresses. The combination of these stresses causes the member to bend or bow.

When an axial load is applied to such a member, it tends to deform by bending into a curve rather than remaining straight. If we consider the bowing of the beam then the increasing length of the beam $c_b\ L$ due to bowing. Here c_b is the bowing coefficient. If u is the axial displacement then the the axial force is :-

$$P = EA\left(\frac{u}{L} + c_b\right)$$

In the figure 6, if arc length is ds and the projection of this segment is dz then the increases length is:-

$$c_b L = L_{arc} - L_{chord} = \int_0^L (ds - dz)$$

Here, from the figure 6

$$(ds)^2 = (dw)^2 + (dz)^2$$

Then

$$c_b\ L = \int_0^L \left(\sqrt{dw^2 + dz^2} - dz\right)$$

$$c_b\ L = \int_0^L \left(dz\sqrt{\frac{dw^2}{dz^2} + 1} - dz\right) \text{ ..... 2.3.1}$$

We can write

$$\sqrt{1 + x^2} = 1 + \frac{x^2}{2} + \frac{1}{2}(\frac{-1}{2})x^4 + ...$$

or

$$\sqrt{1 + x^2} = 1 + \frac{x^2}{2} + \frac{1}{2}$$

Then, from the equation 2.3.1

$$c_b\ L = \int_0^L \left(dz(1 + \frac{(dw)^2}{(dz)^2}) - dz\right)$$

Now

$$c_b\ L = \int_0^L \left(\frac{dw}{dz}\right)^2 dz \text{ .... 2.3.2}$$

2.3.1 If P is negative, P < 0

From the equation 2.1.1.9

$$w = \frac{M_1}{P}\left(\frac{z}{L} - \frac{\sin \beta z}{\sin \beta L}\right) + \frac{M_2}{P}\left(\frac{\sin \beta(L - z)}{\sin \beta L} + \frac{z - L}{L}\right)$$

Differentiate above equation with respect to z

$$\frac{dw}{dz} = \frac{M_1}{P}\left(\frac{1}{L} - \beta\frac{\cos \beta z}{\sin \beta L}\right) + \frac{M_2}{P}\left(-\beta\frac{\cos \beta(L - z)}{\sin \beta L} + \frac{1}{L}\right)$$

put the value of the above equation in the 2.3.2

$$c_b\, L = \frac{1}{2}\int_0^L \left[\frac{M_1}{P}\left(\frac{1}{L} - \beta\frac{\cos\beta z}{\sin\beta L}\right) + \frac{M_2}{P}\left(-\beta\frac{\cos\beta(L-z)}{\sin\beta L} + \frac{1}{L}\right)\right]^2\, dz$$

$$c_b\, L = \frac{1}{2}\int_0^L \left[\frac{M_1^2}{P^2}\left(\frac{1}{L} - \beta\frac{\cos\beta z}{\sin\beta L}\right)^2 + \frac{M_2^2}{P^2}\left(\frac{1}{L} - \beta\frac{\cos\beta(L-z)}{\sin\beta L}\right)^2 \right.$$
$$\left. + \frac{2M_1M_2}{P^2}\left(\frac{1}{L} - \beta\frac{\cos\beta z}{\sin\beta L}\right)\left(\frac{1}{L} - \beta\frac{\cos\beta(L-z)}{\sin\beta L}\right)\right]\, dz$$

$$c_b\, L = \frac{1}{2}\int_0^L \left[\frac{M_1^2}{P^2}\left(\frac{1}{L^2} + \beta^2\frac{\cos^2\beta z}{\sin^2\beta L} - \frac{2\beta\cos\beta z}{L\sin\beta L}\right) + \frac{M_2^2}{P^2}\left(\beta^2\frac{\cos^2\beta(L-z)}{\sin^2\beta L} + \frac{1}{L^2} - \frac{2\beta\cos\beta(L-z)}{L\sin\beta L}\right)\right.$$
$$\left. + \frac{2M_1M_2}{P^2}\left(\frac{1}{L^2} - \beta\frac{\cos\beta z}{L\sin\beta L} - \beta\frac{\cos\beta(L-z)}{L\sin\beta L} + \beta^2\frac{\cos\beta z\cos\beta(L-z)}{\sin^2\beta L}\right)\right]\, dz$$

$$c_bL = \frac{1}{2}\int_0^L \left[\frac{M_1^2}{P^2}\left(\frac{1}{L^2} + \beta^2\left(\frac{1+\cos 2\beta z}{2\sin^2\beta L}\right) - \frac{2\beta\cos\beta z}{L\sin\beta L}\right) + \right.$$
$$\frac{M_2^2}{P^2}\left(\beta^2\left(\frac{1+\cos 2\beta(L-z)}{2\sin^2\beta L}\right) + \frac{1}{L^2} - \frac{2\beta\cos\beta(L-z)}{L\sin\beta L}\right)$$
$$\left. + \frac{2M_1M_2}{P^2}\left(\frac{1}{L^2} - \beta\frac{\cos\beta z}{L\sin\beta L} - \beta\frac{\cos\beta(L-z)}{L\sin\beta L} + \beta^2\left(\frac{\cos\beta L + \cos(2\beta z - \beta L)}{2\sin^2\beta L}\right)\right)\right] dz$$

$$c_bL = \frac{1}{2}\left[\frac{M_1^2}{P^2}\left(\frac{1}{L^2}z + \frac{\beta^2}{2\sin^2\beta L}\left(z + \frac{\sin 2\beta z}{2\beta}\right) - \frac{2\beta\sin\beta z}{L\beta\sin\beta L}\right) + \right.$$
$$\frac{M_2^2}{P^2}\left(\frac{\beta^2}{2\sin^2\beta L}\left(z + \frac{\sin 2\beta(L-z)}{-2\beta}\right) + \frac{1}{L^2}z - \frac{2\beta\sin\beta(L-z)}{-\beta L\sin\beta L}\right)$$
$$\left. + \frac{2M_1M_2}{P^2}\left(\frac{1}{L^2}z - \frac{\beta}{L\sin\beta L}\frac{\sin\beta z}{\beta} - \frac{\beta}{L\sin\beta L}\frac{\sin\beta(L-z)}{-\beta} + \frac{\beta^2}{2\sin^2\beta L}\left(z\cos\beta L + \frac{\sin(2\beta z - \beta L)}{2\beta}\right)\right)\right]_0^L$$

$$c_bL = \frac{1}{2}\left[\frac{M_1^2}{P^2}\left(\frac{1}{L^2}L + \frac{\beta^2}{2\sin^2\beta L}\left(L + \frac{2\sin\beta L\ \cos\beta L}{2\beta}\right) - \frac{2\sin\beta L}{L\sin\beta L}\right) + \right.$$
$$\frac{M_2^2}{P^2}\left(\frac{\beta^2}{2\sin^2\beta L}(L) + \frac{1}{L^2}L - \left(\frac{\beta^2}{2\sin^2\beta L}\frac{2\sin\beta L\ \cos\beta L}{-2\beta} + \frac{2\sin\beta L}{L\sin\beta L}\right)\right) + \frac{2M_1M_2}{P^2}$$
$$\left.\left(\frac{L}{L^2} - \frac{\sin\beta L}{L\sin\beta L} + \frac{\beta^2}{2\sin^2\beta L}\left(L\cos\beta L + \frac{\sin(\beta L)}{2\beta}\right) - \left(\frac{\sin\beta L}{L\sin\beta L} + \frac{\beta^2}{2\sin^2\beta L}\left(-\frac{\sin\beta L}{2\beta}\right)\right)\right)\right]$$

$$c_bL = \frac{1}{2}\left[\frac{M_1^2}{P^2}\left(\frac{1}{L}+\frac{\beta^2\ L}{2\sin^2\beta L}+\frac{\beta\cos\beta L}{2\sin\beta L}-\frac{2}{L}\right)+\right.$$
$$\frac{M_2^2}{P^2}\left(\frac{\beta^2 L}{2\sin^2\beta L}+\frac{1}{L}+\frac{\beta\cos\beta L}{2\sin\beta L}-\frac{2}{L}\right)+\frac{2M_1M_2}{P^2}$$
$$\left.\left(\frac{1}{L}-\frac{1}{L}+\frac{\beta^2\ L\ \cos\beta L}{2\sin^2\beta L}+\frac{\beta}{4\sin\beta L}-\frac{1}{L}+\frac{\beta}{4\sin\beta L}\right)\right]$$

we can take $\beta L = w$

Then

$$c_bL = \frac{1}{2}\left[\frac{M_1^2}{P^2}\left(-\frac{1}{L}+\frac{w^2}{2\ L\ \sin^2 w}+\frac{w\cos w}{2\ L\sin w}\right)+\frac{M_2^2}{P^2}\left(-\frac{1}{L}+\frac{w^2}{2\ L\ \sin^2 w}+\frac{w\cos w}{2\ L\sin w}\right)\right.$$
$$\left.+\frac{2M_1M_2}{P^2}\left(-\frac{1}{L}+\frac{w^2\cos w}{2\ L\sin^2 w}+\frac{w}{2\ L\sin w}\right)\right]$$

$$c_b = \frac{1}{2L}\left[\frac{M_1^2}{P^2}\left(-\frac{1}{L}+\frac{w^2}{2\ L\ \sin^2 w}+\frac{w\cos w}{2\ L\sin w}\right)+\frac{M_2^2}{P^2}\left(-\frac{1}{L}+\frac{w^2}{2\ L\ \sin^2 w}+\frac{w\cos w}{2\ L\sin w}\right)\right.$$
$$\left.+\frac{2M_1M_2}{P^2}\left(-\frac{1}{L}+\frac{w^2\cos w}{2\ L\sin^2 w}+\frac{w}{2\ L\sin w}\right)\right]$$

Let's assume that:-

$$A = \frac{1}{2L}\left[-\frac{1}{L}+\frac{w^2}{2\ L\ \sin^2 w}+\frac{w\cos w}{2\ L\sin w}\right]$$

$$A = \frac{-2\sin^2 w + w^2 + w\cos w\sin w}{4L^2\sin^2 w} \text{ .... 2.3.1.1}$$

and

$$B = \frac{1}{2L}\left[-\frac{1}{L}+\frac{w^2\cos w}{2\ L\sin^2 w}+\frac{w}{2\ L\sin w}\right]$$

$$B = \frac{-2\sin^2 w + w^2\cos w + w\sin w}{4L^2\sin^2 w} \text{ .... 2.3.1.2}$$

Now

$$c_b = \frac{M_1^2}{P^2} A + \frac{M_2^2}{P^2} A + \frac{2M_1M_2}{P^2} B$$

Now take the value from the equation 2.2.1.6 and 2.2.1.7 of the M_1 and M_2 Then

$$\frac{M_1^2}{P^2} = \frac{1}{P^2} \frac{E^2I^2}{L^2} \left(c_1\phi_1 + c_2\phi_2\right)^2$$

$$\frac{M_2^2}{P^2} = \frac{1}{P^2} \frac{E^2I^2}{L^2} \left(c_2\phi_1 + c_1\phi_2\right)^2$$

$$2\frac{M_1M_2}{P^2} = \frac{2}{P^2} \frac{E^2I^2}{L^2} \left(c_1\phi_1 + c_2\phi_2\right)\left(c_2\phi_1 + c_1\phi_2\right)$$

Now we know that

$$\frac{P}{EI} = \beta^2$$

Then

$$\frac{M_1^2}{P^2} = \frac{1}{\beta^2} \frac{1}{L^2} \left(c_1\phi_1 + c_2\phi_2\right)^2$$

$$\frac{M_2^2}{P^2} = \frac{1}{\beta^2} \frac{1}{L^2} \left(c_2\phi_1 + c_1\phi_2\right)^2$$

$$2\frac{M_1M_2}{P^2} = \frac{2}{\beta^2L^2} \left(c_1\phi_1 + c_2\phi_2\right)\left(c_2\phi_1 + c_1\phi_2\right)$$

or

$$\frac{M_1^2}{P^2} = \frac{1}{w^2} \left(c_1\phi_1 + c_2\phi_2\right)^2$$

$$\frac{M_2^2}{P^2} = \frac{1}{w^2} \left(c_2\phi_1 + c_1\phi_2\right)^2$$

$$2\frac{M_1M_2}{P^2} = \frac{2}{w^2} \left(c_1\phi_1 + c_2\phi_2\right)\left(c_2\phi_1 + c_1\phi_2\right)$$

Now

$$c_b = \frac{1}{w^2} \left[\left(c_1\phi_1 + c_2\phi_2\right)^2 \ A + \left(c_2\phi_1 + c_1\phi_2\right)^2 \ A + 2\left(c_1\phi_1 + c_2\phi_2\right)\left(c_2\phi_1 + c_1\phi_2\right) \ B\right]$$

$$c_b = \frac{1}{w^2}\left[\left(c_1^2\phi_1^2 + c_2^2\phi_2^2 + 2c_1\ c_2\phi_1\phi_2\right)A + \left(c_2^2\phi_1^2 + c_1^2\phi_2^2 + 2c_1\ c_2\phi_1\phi_2\right)A + 2\left(c_1c_2\phi_1^2 + c_1c_2\phi_2^2 + c_2^2\phi_1\phi_2 + c_1^2\phi_1\phi_2\right)B\right]$$

$$c_b = \frac{1}{w^2}\left[\phi_1^2\left(c_1^2A + c_2^2A + 2B\ c_1\ c_2\right) + \phi_2^2\left(c_1^2A + c_2^2A + 2B\ c_1\ c_2\right) + 2\phi_1\ \phi_2\left(2Ac_1\ c_2 + B\ c_1^2 + Bc_2^2\right)\right]$$

Now, let's assume that

$$\frac{c_1^2A + c_2^2A + 2B\ c_1\ c_2}{w^2} = b_1 + b_2 \text{ ....2.3.1.3}$$

$$\frac{2Ac_1\ c_2 + B\ c_1^2 + Bc_2^2}{w^2} = b_1 - b_2 \text{ .... 2.3.1.4}$$

Here, b_1 and b_2 are the bowing function
Then

$$c_b = \left[\phi_1^2(b_1 + b_2) + \phi_2^2(b_1 + b_2) + 2\phi_1\ \phi_2(b_1 - b_2)\right]$$

$$c_b = \left[b_1(\phi_1^2 + \phi_2^2 + 2\phi_1\phi_2) + b_2(\phi_1^2 - \phi_2^2 + 2\phi_1\phi_2)\right]$$

$$c_b = b_1(\phi_1 + \phi_2)^2 + b_2(\phi_1 - \phi_2)^2 \text{ ..... 2.3.1.5}$$

Now solve this equation 2.3.1.3 and 2.3.1.4 for b_1 and b_2, then

$$2b_1 = \frac{1}{w^2}\left[A(c_1^2 + 2c_1c_2 + c_2^2) + B(2c_1c_2 + c_1^2 + c_2^2)\right]$$

$$b_1 = \frac{1}{2w^2}\left[A(c_1 + c_2)^2 + B(c_1 + c_2)^2\right]$$

$$b_1 = \frac{1}{2w^2}(c_1 + c_2)^2(A + B)$$

Now, for b_2

$$b_2 = \frac{c_1^2A + c_2^2A + 2B\ c_1c_2}{w^2} - b_1$$

$$b_2 = \frac{c_1^2A + c_2^2A + 2B\ c_1c_2}{w^2} - \frac{1}{2w^2}(c_1 + c_2)^2(A + B)$$

$$b_2 = \frac{1}{2w^2}\left[2c_1^2A + 2c_2^2A + 4B\ c_1c_2 - \left(A(c_1^2+c_2^2+2c_1c_2) + B(c_1^2+c_2^2+2c_1c_2)\right)\right]$$

$$b_2 = \frac{1}{2w^2}\left[c_1^2A + c_2^2A + 2Bc_1c_2 - 2Ac_1c_2 - Bc_1^2 - Bc_2^2\right]$$

$$b_2 = \frac{1}{2w^2}\left[A(c_1-c_2)^2 - B(c_1-c_2)^2\right]$$

Now

$$b_2 = \frac{1}{2w^2}(c_1-c_2)^2(A-B)$$

Now, put the value of the c_1 & c_2 from the equation 2.2.1.8 & 2.2.1.9 and the value of the A & B from the equation 2.3.1.1 & 2.3.1.2

Then

$$b_1 = \frac{(c_1+c_2)(c_2-2)}{8w^2} \text{ .... 2.3.1.6}$$

$$b_2 = \frac{c_2}{8(c_1+c_2)} \text{ .... 2.3.1.7}$$

2.3.2 If P is equal to zero, P = 0

from the equation 2.1.2.2

$$\frac{dw}{dz} = \frac{1}{EI}\left(-M_2\ z + \left(\frac{M_1+M_2}{L}\right)\ \frac{z^2}{2}\right) + c_1 \text{ .... 2.3.2.1}$$

Here

$$c_1 = -\frac{1}{EI}\left[M_1\frac{L}{6} - M_2\frac{2L}{6}\right]$$

Put the value of the c_1 in the equation 2.3.2.1, Then

$$\frac{dw}{dz} = \frac{1}{EI}\left(-M_2\ z + \left(\frac{M_1+M_2}{L}\right)\ \frac{z^2}{2}\right) - \frac{1}{EI}\left[M_1\frac{L}{6} - M_2\frac{2L}{6}\right]$$

or

$$\frac{dw}{dz} = \frac{1}{EI}\left[M_1\left(\frac{z^2}{2L} - \frac{L}{6}\right) + M_2\left(-z + \frac{z^2}{2L} + \frac{2L}{6}\right)\right]$$

put the value of the above equation in the 2.3.2

$$c_b\ L = \frac{1}{2}\int_0^L \left(\frac{1}{EI}\right)^2 \left[M_1\left(\frac{z^2}{2L} - \frac{L}{6}\right) + M_2\left(-z + \frac{z^2}{2L} + \frac{2L}{6}\right)\right]^2 dz$$

$$c_b\ L = \frac{1}{2(E\ I)^2}\int_0^L \left[M_1^2\left(\frac{z^2}{2L} - \frac{L}{6}\right)^2 + M_2^2\left(-z + \frac{z^2}{2L} + \frac{2L}{6}\right)^2 \right.$$
$$\left. +2M_1M_2\left(\frac{z^2}{2L} - \frac{L}{6}\right)\left(-z + \frac{z^2}{2L} + \frac{2L}{6}\right)\right] dz$$

$$c_b\ L = \frac{1}{2(E\ I)^2}\int_0^L \left[M_1^2\left(\frac{z^4}{4L^2} + \frac{L^2}{6^2} - 2\frac{z^2}{2L}\frac{L}{6}\right) + M_2^2\left(z^2 + \frac{z^4}{4L^2} + \frac{4L^2}{6^2} - 2\frac{z^3}{2L} + 2\frac{z^2}{2L}\frac{2L}{6} - 2z\frac{2L}{6}\right)\right.$$
$$\left. -2M_1M_2\left(\frac{-z^3}{2L} + \frac{z^4}{4L^2} + \frac{z^2}{6} + \frac{zL}{6} - \frac{z^2}{12} - \frac{2L^2}{36}\right)\right]\ dz$$

$$c_b\ L = \frac{1}{2(E\ I)^2}\left[M_1^2\left(\frac{z^5}{20L^2} + z\frac{L^2}{6^2} - 2\frac{z^3}{6L}\frac{L}{6}\right) + M_2^2\left(\frac{z^3}{3} + \frac{z^5}{20L^2} + z\frac{4L^2}{6^2} - 2\frac{z^4}{8L} + 2\frac{z^3}{6L}\frac{2L}{6} - 2z^2\frac{L}{6}\right)\right.$$
$$\left. -2M_1M_2\left(\frac{-z^4}{8L} + \frac{z^5}{20L^2} + \frac{z^3}{18} + \frac{z^2L}{12} - \frac{z^3}{36} - \frac{2zL^2}{36}\right)\right]_0^L$$

$$c_b\ L = \frac{1}{2(E\ I)^2}\left[M_1^2\left(\frac{L^5}{20L^2} + L\frac{L^2}{6^2} - 2\frac{L^3}{6L}\frac{L}{6}\right) + M_2^2\left(\frac{L^3}{3} + \frac{L^5}{20L^2} + L\frac{4L^2}{6^2} - 2\frac{L^4}{8L} + 2\frac{L^3}{6L}\frac{2L}{6} - 2L^2\frac{L}{6}\right)\right.$$
$$\left. -2M_1M_2\left(\frac{-L^4}{8L} + \frac{L^5}{20L^2} + \frac{L^3}{18} + L^2\frac{L}{12} - \frac{L^3}{36} - 2L\frac{L^2}{36}\right)\right]$$

$$c_b\ L = \frac{1}{2(E\ I)^2}\left[M_1^2\left(\frac{L^3}{20} + \frac{L^3}{36} - 2\frac{L^3}{36}\right) + M_2^2\left(\frac{L^3}{3} + \frac{L^3}{20} + \frac{4L^3}{36} - 2\frac{L^3}{8} + 4\frac{L^3}{36} - 2\frac{L^3}{6}\right)\right.$$
$$\left. -2M_1M_2\left(\frac{-L^3}{8} + \frac{L^3}{20} + \frac{L^3}{18} + \frac{L^3}{12} - \frac{L^3}{36} - 2\frac{L^3}{36}\right)\right]$$

$$c_b\ L = \frac{L^3}{2(E\ I)^2}\left[M_1^2\left(\frac{1}{20}+\frac{1}{36}-2\frac{1}{36}\right)+M_2^2\left(\frac{1}{3}+\frac{1}{20}+\frac{4}{36}-\frac{2}{8}+\frac{4}{36}-\frac{2}{6}\right)\right.$$
$$\left.-2M_1M_2\left(\frac{-1}{8}+\frac{1}{20}+\frac{1}{18}+\frac{1}{12}-\frac{1}{36}-\frac{2}{36}\right)\right]$$

$$c_b = \frac{L^2}{2\left(EI\right)^2)}\left[\frac{M_1^2}{45}+\frac{M_2^2}{45}-2M_1M_2\frac{7}{360}\right]$$

Now take the value of the M_1 and M_2 from the equation 2.2.2.5 and 2.2.2.6

$$c_b = \frac{L^2}{2\left(EI\right)^2)}\left[\left(\frac{EI}{L}\right)^2\frac{\left(c_1\phi_1+c_2\phi_2\right)^2}{45}+\left(\frac{EI}{L}\right)^2\frac{\left(c_2\phi_1+c_1\phi_2\right)^2}{45}\right.$$
$$\left.-2\left(\frac{EI}{L}\right)^2\left(c_1\phi_1+c_2\phi_2\right)\left(c_2\phi_1+c_1\phi_2\right)\frac{7}{360}\right]$$

$$c_b = \frac{1}{2}\left[\frac{\left(c_1^2\phi_1^2+c_2^2\phi_2^2+2c_1c_2\phi_1\phi_2\right)}{45}+\frac{\left(c_2^2\phi_1^2+c_1^2\phi_2^2+2c_1c_2\phi_1\phi_2\right)}{45}\right.$$
$$\left.-14\frac{\phi_1^2c_1c_2+c_1^2\phi_1\phi_2+c_2^2\phi_1\phi_2+\phi_2^2c_1c_2}{360}\right]$$

$$c_b = \frac{1}{2}\left[\phi_1^2\left(\frac{c_1^2}{45}+\frac{c_2^2}{45}-\frac{14c_1c_2}{360}\right)+\phi_2^2\left(\frac{c_1^2}{45}+\frac{c_2^2}{45}-\frac{14c_1c_2}{360}\right)+\phi_1\phi_2\left(\frac{4c_1c_2}{45}-\frac{14c_1^2}{360}-\frac{14c_2^2}{360}\right)\right]$$

Let's assume that:-

$$b_1+b_2 = \frac{1}{2}\left(\frac{c_1^2}{45}+\frac{c_2^2}{45}-\frac{14c_1c_2}{360}\right) \text{ ...... } 2.3.2.2$$

$$b_1-b_2 = \frac{1}{2}\left(\frac{2c_1c_2}{45}-\frac{7c_1^2}{360}-\frac{7c_2^2}{360}\right) \text{ .... } 2.3.2.3$$

Now, we can write

$$c_b = \frac{1}{2}\left[\phi_1^2\left(b_1+b_2\right)+\phi_2^2\left(b_1+b_2\right)+2\phi_1\phi_2\left(b_1-b_2\right)\right]$$

$$c_b = \frac{1}{2}\left[b_1\left(\phi_1^2+\phi_2^2+2\phi_1\phi_2\right)+b_2\left(\phi_1^2+\phi_2^2-2\phi_1\phi_1\right)\right]$$

$$c_b = \frac{1}{2}\left[b_1\left(\phi_1+\phi_2\right)^2 + b_2\left(\phi_1-\phi_2\right)^2\right]$$

Here, b_1 and b_2 are the bowing function

Now, solve the equation 2.3.2.2 and 2.3.2.3 for the b_1 and b_2

$$b_1 = \frac{c_1^2 + c_2^2 + 2c_1c_2}{1440}$$

$$b_2 = \frac{1}{2}\left[\frac{1}{48}\left(c_1^2+c_2^2\right) - c_1c_2\frac{1}{24}\right]$$

If we take the value of the $c_1 = 4$ and $c_2 = 2$ Then, we find out

$$b_1 = \frac{1}{40}$$

$$b_2 = \frac{1}{24}$$

2.3.3 If P is positive, P > 0

From the equation 2.1.3.10

$$w = \frac{M_1}{P}\left[\frac{\sinh(mz)}{\sinh(mL)} - \frac{z}{L}\right] + \frac{M_2}{P}\left[\frac{L-z}{L} + \frac{\sinh(z-L)m}{\sinh(mL)}\right]$$

Differentiate above equation with respect to z

$$\frac{dw}{dz} = \frac{M_1}{P}\left(m\frac{\cosh mz}{\sinh mL} - \frac{1}{L}\right) + \frac{M_2}{P}\left(m\frac{\cosh m(L-z)}{\sinh mL} - \frac{1}{L}\right)$$

put the value of the above equation in the 2.3.2

$$c_b\ L = \frac{1}{2}\int_0^L \left[\frac{M_1}{P}\left(m\frac{\cosh mz}{\sinh mL} - \frac{1}{L}\right) + \frac{M_2}{P}\left(m\frac{\cosh m(L-z)}{\sinh mL} - \frac{1}{L}\right)\right]^2 dz$$

$$\begin{aligned} c_b\ L = \frac{1}{2}\int_0^L &\left[\frac{M_1^2}{P^2}\left(m\frac{\cosh mz}{\sinh mL} - \frac{1}{L}\right)^2 + \frac{M_2^2}{P^2}\left(m\frac{\cosh m(L-z)}{\sinh mL} - \frac{1}{L}\right)^2\right. \\ &\left. + \frac{2M_1M_2}{P^2}\left(m\frac{\cosh mz}{\sinh mL} - \frac{1}{L}\right)\left(m\frac{\cosh m(L-z)}{\sinh mL} - \frac{1}{L}\right)\right] dz \end{aligned}$$

$$
\begin{aligned}
c_b\, L = \frac{1}{2}\int_0^L \Bigg[& \frac{M_1^2}{P^2}\left(m^2\frac{\cosh^2 mz}{\sinh^2 mL}+\frac{1}{L^2}-2m\frac{\cosh mz}{L\sinh mL}\right)+ \\
& \frac{M_2^2}{P^2}\left(m^2\frac{\cosh^2 m(L-z)}{\sinh^2 mL}+\frac{1}{L^2}-2m\frac{\cosh m(L-z)}{L\sinh mL}\right) \\
& +\frac{2M_1M_2}{P^2}\left(m^2\frac{\cosh mz\cosh m(L-z)}{\sinh^2 mL}+\frac{1}{L^2}-m\frac{\cosh mz}{L\sinh mL}-m\frac{\cosh m(L-z)}{L\sinh mL}\right)\Bigg]\,dz
\end{aligned}
$$

$$
\begin{aligned}
c_b\, L = \frac{1}{2P^2}\int_0^L \Bigg[& M_1^2\left(m^2\left(\frac{1+\cosh 2mz}{2\sinh^2 mL}\right)+\frac{1}{L^2}-2m\frac{\cosh mz}{L\sinh mL}\right)+ \\
& M_2^2\left(m^2\left(\frac{1+\cosh 2m(L-z)}{2\sinh^2 mL}\right)+\frac{1}{L^2}-2m\frac{\cosh m(L-z)}{L\sinh mL}\right) \\
& +2M_1M_2\left(m^2\left(\frac{\cosh mL+\cosh(2mz-mL)}{2\sinh^2 mL}\right)+\frac{1}{L^2}-m\frac{\cosh mz}{L\sinh mL}-m\frac{\cosh m(L-z)}{L\sinh mL}\right)\Bigg]\,dz
\end{aligned}
$$

$$
\begin{aligned}
c_b\, L = \frac{1}{2P^2}\Bigg[& M_1^2\left(\frac{m^2}{2\sinh^2 mL}\left(z+\frac{\sinh 2mz}{2m}\right)+\frac{z}{L^2}-2m\frac{\sinh mz}{mL\sinh mL}\right)+ \\
& M_2^2\left(\frac{m^2}{2\sinh^2 mL}\left(z+\frac{\sinh 2m(L-z)}{-2m}\right)+\frac{z}{L^2}-2m\frac{\sinh m(L-z)}{-mL\sinh mL}\right) \\
& +2M_1M_2\left(\frac{m^2}{2\sinh^2 mL}\left(z\cosh mL+\frac{\sinh(2mz-mL)}{2m}\right)+\frac{z}{L^2}-m\frac{\sinh mz}{mL\sinh mL}-m\frac{\sinh m(L-z)}{-mL\sinh mL}\right)\Bigg]_0^L
\end{aligned}
$$

$$
\begin{aligned}
c_b\, L = \frac{1}{2P^2}\Bigg[& M_1^2\left(\frac{m^2}{2\sinh^2 mL}\left(L+\frac{2\sinh mL\cosh mL}{2m}\right)+\frac{L}{L^2}-2\frac{\sinh mL}{L\sinh mL}\right)+ \\
& M_2^2\left(\frac{Lm^2}{2\sinh^2 mL}+\frac{L}{L^2}-\left(\frac{-m\cosh mL}{2\sinh mL}+2\frac{\sinh mL}{L\sinh mL}\right)\right) \\
& +2M_1M_2\left(\frac{m^2}{2\sinh^2 mL}\left(L\cosh mL+\frac{\sinh mL}{2m}\right)+\frac{L}{L^2}-\frac{\sinh mL}{L\sinh mL}-\left(\frac{-m}{4\sinh mL}+\frac{\sinh mL}{L\sinh mL}\right)\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
c_b\, L = \frac{1}{2P^2}\Bigg[& M_1^2\left(\frac{Lm^2}{2\sinh^2 mL}+\frac{m\cosh mL}{2\sinh mL}+\frac{1}{L}-\frac{2}{L}\right)+M_2^2\left(\frac{Lm^2}{2\sinh^2 mL}+\frac{1}{L}+\frac{m\cosh mL}{2\sinh mL}-\frac{2}{L}\right) \\
& +2M_1M_2\left(\frac{m^2L\cosh mL}{2\sinh^2 mL}+\frac{m}{4\sinh mL}+\frac{1}{L}-\frac{1}{L}+\frac{m}{4\sinh mL}-\frac{1}{L}\right)\Bigg]
\end{aligned}
$$

$$c_b\ L = \frac{1}{2P^2}\left[M_1^2\left(\frac{Lm^2}{2\sinh^2 mL} + \frac{m\cosh mL}{2\sinh mL} - \frac{1}{L}\right) + M_2^2\left(\frac{Lm^2}{2\sinh^2 mL} + \frac{m\cosh mL}{2\sinh mL} - \frac{1}{L}\right)\right.$$
$$\left. +2M_1M_2\left(\frac{m^2L\cosh mL}{2\sinh^2 mL} + \frac{m}{2\sinh mL} - \frac{1}{L}\right)\right]$$

we can take $mL = w$

Then

$$c_b = \frac{1}{2LP^2}\left[M_1^2\left(\frac{w^2}{2L\sinh^2 w} + \frac{w\cosh w}{2L\sinh w} - \frac{1}{L}\right) + M_2^2\left(\frac{w^2}{2L\sinh^2 w} + \frac{w\cosh w}{2L\sinh w} - \frac{1}{L}\right)\right.$$
$$\left. +2M_1M_2\left(\frac{w^2\cosh w}{2L\sinh^2 w} + \frac{w}{2L\sinh w} - \frac{1}{L}\right)\right]$$

Let's assume that:-

$$A = \frac{1}{2L}\left[\frac{w^2}{2L\sinh^2 w} + \frac{w\cosh w}{2L\sinh w} - \frac{1}{L}\right]$$

$$A = \frac{w^2 + w\cosh w\ \sinh w - 2\sinh^2 w}{4\ L^2\sinh^2 w} \quad \text{......... 2.3.3.1}$$

and

$$B = \frac{1}{2L}\left[\frac{w^2\cosh w}{2L\sinh^2 w} + \frac{w}{2L\sinh w} - \frac{1}{L}\right]$$

$$B = \frac{w^2\cosh w + w\sinh w - 2\ \sinh^2 w}{4\ L^2\ \sinh^2 w} \quad \text{........... 2.3.3.2}$$

Then

$$c_b = \frac{1}{2P^2}\left[M_1^2A + M_2A + 2M_1M_2B\right] \quad \text{...... 2.3.3.3}$$

Now take the value of the M_1 and M_2 from the equation 2.2.3.6 and 2.2.3.7

$$\frac{M_1^2}{P^2} = \frac{E^2I^2}{P^2L^2}\left(c_1\phi_1 + c_2\phi_2\right)^2$$

$$\frac{M_2^2}{P^2} = \frac{E^2I^2}{P^2L^2}\left(c_1\phi_2 + c_2\phi_1\right)^2$$

$$2\frac{M_1M_2}{P^2}=\frac{2E^2I^2}{P^2L^2}\left(c_1\phi_1+c_2\phi_2\right)\left(c_1\phi_2+c_2\phi_1\right)$$

Now, we know that:-

$$\frac{E^2I^2}{P^2L^2}=\frac{1}{w^2}$$

Then

$$\frac{M_1^2}{P^2}=\frac{1}{w^2}\left(c_1\phi_1+c_2\phi_2\right)^2$$

$$\frac{M_2^2}{P^2}=\frac{1}{w^2}\left(c_1\phi_2+c_2\phi_1\right)^2$$

$$2\frac{M_1M_2}{P^2}=\frac{1}{w^2}\left(c_1\phi_1+c_2\phi_2\right)\left(c_1\phi_2+c_2\phi_1\right)$$

Now, put the value in the equation 2.3.3.3

$$c_b=\frac{1}{w^2}\left[\left(c_1\phi_1+c_2\phi_2\right)^2\ A+\left(c_2\phi_1+c_1\phi_2\right)^2\ A+2\left(c_1\phi_1+c_2\phi_2\right)\left(c_2\phi_1+c_1\phi_2\right)\ B\right]$$

$$\begin{aligned}c_b=\frac{1}{w^2}\Big[&\left(c_1^2\phi_1^2+c_2^2\phi_2^2+2c_1\ c_2\phi_1\phi_2\right)A+\\&\left(c_2^2\phi_1^2+c_1^2\phi_2^2+2c_1\ c_2\phi_1\phi_2\right)A\\&+2\left(c_1c_2\phi_1^2+c_1c_2\phi_2^2+c_2^2\phi_1\phi_2+c_1^2\phi_1\phi_2\right)B\Big]\end{aligned}$$

$$c_b=\frac{1}{w^2}\left[\phi_1^2\left(c_1^2A+c_2^2A+2Bc_1\ c_2\right)+\phi_2^2\left(c_1^2A+c_2^2A+2Bc_1\ c_2\right)+\phi_1\ \phi_2\left(4Ac_1\ c_2+2Bc_1^2+2Bc_2^2\right)\right]$$

Now, let's assume that

$$\frac{c_1^2A+c_2^2A+2B\ c_1\ c_2}{w^2}=b_1+b_2\ ....2.3.3.4$$

$$\frac{2Ac_1\ c_2+B\ c_1^2+Bc_2^2}{w^2}=b_1-b_2\ ....\ 2.3.3.5$$

Here, b_1 and b_2 are the bowing function
Then

$$c_b=\left[\phi_1^2\left(b_1+b_2\right)+\phi_2^2\left(b_1+b_2\right)+2\phi_1\ \phi_2\left(b_1-b_2\right)\right]$$

$$c_b = \left[b_1(\phi_1^2 + \phi_2^2 + 2\phi_1\phi_2) + b_2(\phi_1^2 - \phi_2^2 + 2\phi_1\phi_2)\right]$$

$$c_b = b_1(\phi_1 + \phi_2)^2 + b_2(\phi_1 - \phi_2)^2 \text{ ..... } 2.3.3.6$$

Now solve the equation 2.3.3.4 and 2.3.3.5 for b_1 and b_2, then

$$2b_1 = \frac{1}{w^2}\left[A(c_1^2 + 2c_1c_2 + c_2^2) + B(2c_1c_2 + c_1^2 + c_2^2)\right]$$

$$b_1 = \frac{1}{2w^2}\left[A(c_1 + c_2)^2 + B(c_1 + c_2)^2\right]$$

$$b_1 = \frac{1}{2w^2}(c_1 + c_2)^2\,(A + B)$$

Now, for b_2

$$b_2 = \frac{c_1^2A + c_2^2A + 2B\ c_1c_2}{w^2} - b_1$$

$$b_2 = \frac{c_1^2A + c_2^2A + 2B\ c_1c_2}{w^2} - \frac{1}{2w^2}(c_1 + c_2)^2\,(A + B)$$

$$b_2 = \frac{1}{2w^2}\left[2c_1^2A + 2c_2^2A + 4B\ c_1c_2 - \big(A(c_1^2 + c_2^2 + 2c_1c_2) + B(c_1^2 + c_2^2 + 2c_1c_2)\big)\right]$$

$$b_2 = \frac{1}{2w^2}\left[c_1^2A + c_2^2A + 2Bc_1c_2 - 2Ac_1c_2 - Bc_1^2 - Bc_2^2\right]$$

$$b_2 = \frac{1}{2w^2}\left[A(c_1 - c_2)^2 - B(c_1 - c_2)^2\right]$$

Now

$$b_2 = \frac{1}{2w^2}(c_1 - c_2)^2(A - B)$$

Now, put the value of the c_1 & c_2 from the equation 2.2.3.8 & 2.2.3.9 and the value of the A & B from the equation 2.3.3.1 & 2.3.3.2

Then

$$b_1 = \frac{(c_1 + c_2)(c_2 - 2)}{8w^2} \text{ .... } 2.3.3.7$$

$$b_2 = \frac{c_2}{8(c_1 + c_2)} \text{ .... } 2.3.3.8$$

2.3.4 Normalised axial load, p

$$w = \sqrt{\frac{P}{EI}}L$$

or

$$w^2 = \pi^2 \frac{P}{P_e} = \pi^2 p$$

Here

$$P_e = \pi^2 \frac{EI}{L^2}$$

and

$$p = Normalised\ axial\ load$$

2.4 Derivatives

Let's take the derivatives of the stability function and the Bowing function with respect to the normalised axial load p.

2.4.1 Derivation of the Stability Function

The stability function are

$$c_1 = w\frac{(\sin w - w\cos w)}{2 - 2\cos w - w\sin w} \text{ .... 2.4.1.1}$$

and

$$c_2 = w\frac{(w - \sin w)}{2 - 2\cos w - w\sin w} \text{ .... 2.4.1.2}$$

Now, differentiate the equation 2.4.1.1 with respect to p

$$c_1' = \frac{[w'(\sin w - w\cos w) + w(w'\cos w - w'\cos w + w\sin w)](2 - 2\cos w - w\sin w)}{(2 - 2\cos w - w\sin w)^2} - \frac{w(\sin w - w\cos w)(2\sin w - w'\sin w - w\cos w)}{(2 - 2\cos w - w\sin w)^2}$$

Here

$$w' = \frac{\pi^2}{2w}$$

Now, after solving the equation, we find out

$$c_1' = 2\pi^2(b_1 + b_2)$$

Now, differentiate the equation 2.4.1.2 with respect to p

$$c_1' = \frac{[w'\,(w - \sin w) + w\,(w' - w' \cos w)]\,(2 - 2\cos w - w \sin w)}{(2 - 2\cos w - w\sin w)^2} - \frac{w\,(w - \sin w)\,(2\sin w - w' \sin w - w\cos w)}{(2 - 2\cos w - w \sin w)^2}$$

Now, after solving the equation, we find out

$$c_1' = 2\pi^2(b_1 - b_2)$$

2.4.2 Derivation of the Bowing Function

The stability function are

$$b_1 = \frac{(c_1 + c_2)\,(c_2 - 2)}{8w^2} \text{ .... 2.4.2.1}$$

and

$$b_2 = \frac{c_2}{8\,(c_1 + c_2)} \text{ .... 2.4.2.2}$$

Now, differentiate the equation 2.4.2.1 with respect to p

$$b_1' \quad = \quad \frac{[(c_1' + c_2'))\,(c_2 - 2) + c_2'\,(c_1 + c_2)]\,8w^2 - (16\;w\;w')\,(c_1 + c_2)\,(c_2 - 2)}{(8\;w^2)^2}$$

Now, after solving the equation, we find out

$$b_1' = -\frac{\pi^2}{4w^2}\,[(b_1 - b_2)\,(c_1 + c_2) + 2b_1 c_2]$$

Now, differentiate the equation 2.4.2.1 with respect to p

$$b_2' = \frac{c_2'\,[8\,(c_1 + c_2)] - 8\,(c_1' + c_2')\,c_2}{[8\,(c_1 + c_2)]^2}$$

Now, after solving the equation, we find out

$$b_2' = -\frac{\pi^2}{4\left(c_1 + c_2\right)}\left(16 b_1 b_2 - b_1 + b_2\right)$$

3 Comparison in the Linear and Non-linear Euler Bernoulli Beam Theory

The difference between linear Euler Euler-Bernoulli beam theory and nonlinear Euler-Bernoulli beam theory depends on the assumption, governing equation, and stress-strain relationship. Here the difference between both theories is given below:-

3.1 Linear Beam Theory

1. This theory assumes that the deformations are small compared to the dimensions of the beam.
2. This theory assumes that the material of the beam behaves linearly and follows Hooke's law.
3. Linear strain-stress relationships are used, allowing for simple and straightforward analysis.
4. The governing equations are based on the assumption that the strain is directly proportional to the curvature of the beam.
5. The cross-sectional plane of the beam remains perpendicular to the longitudinal axis(no wrapping and torsion).
6. It provides a simplified modal that neglects geometric and material nonlinearities. This theory is suitable for slender beams under small to moderate loads and is widely used in engineering applications due to its simplicity.

3.2 Non-Linear Beam Theory

1. This theory considers that the deformation of the beam is not small compared to the dimension of the beam.
2. According to this theory, the material does not follow Hook's law.
3. It also considers material nonlinearity, where the stress-strain relationship of the beam material is nonlinear. The strain-stress relationships may involve higher-order terms, which capture the nonlinear effects more accurately.

4. The Governing equations of the nonlinear beam theory are derived by considering the equilibrium and compatibility equations for large deformation.

5. It considers geometric nonlinearity, which means that the beam's shape and its orientation change significantly during the deformation of the beam.

6. Nonlinear beam theory is necessary for beams under large displacements, highly flexible or soft materials, and situations where the linear theory fails to capture the behavior accurately.

In summary, linear Euler-Bernoulli beam theory assumes linear elasticity and small deformations, neglecting geometric and material nonlinearities, while nonlinear Euler-Bernoulli beam theory accounts for large deformations and/or nonlinear material behavior, considering both geometric and material nonlinearities.

3.3 Graph

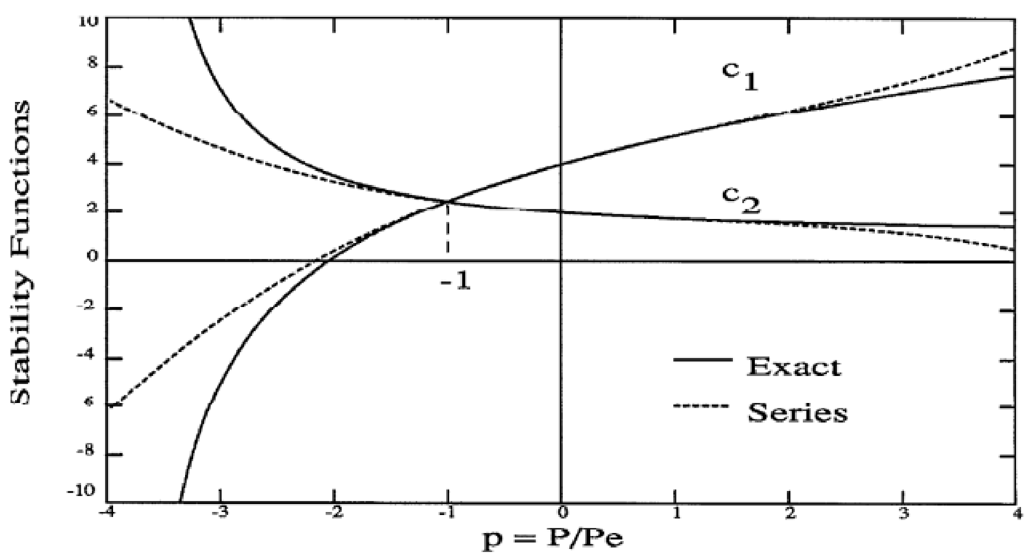

Figure 8:

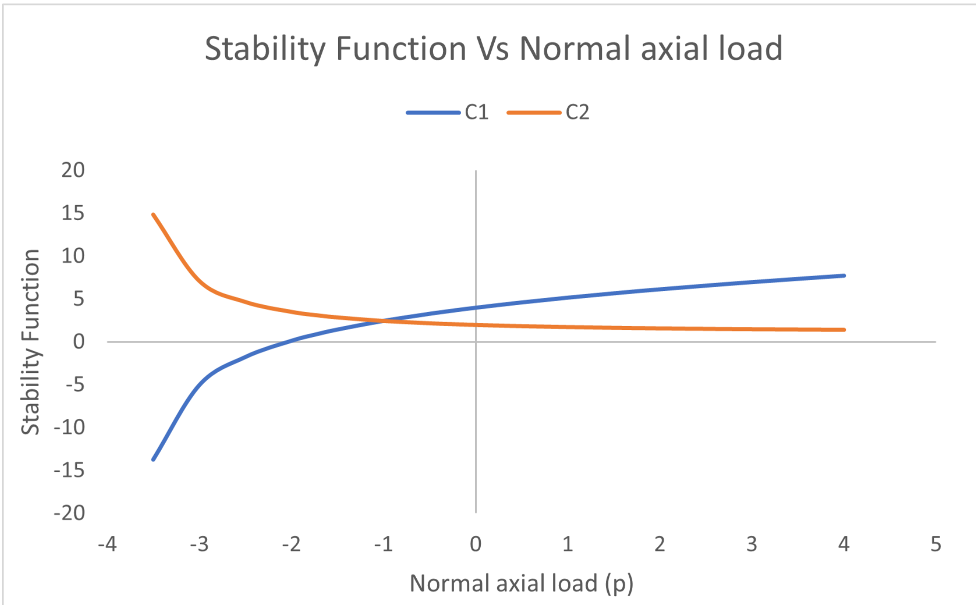

Figure 9:

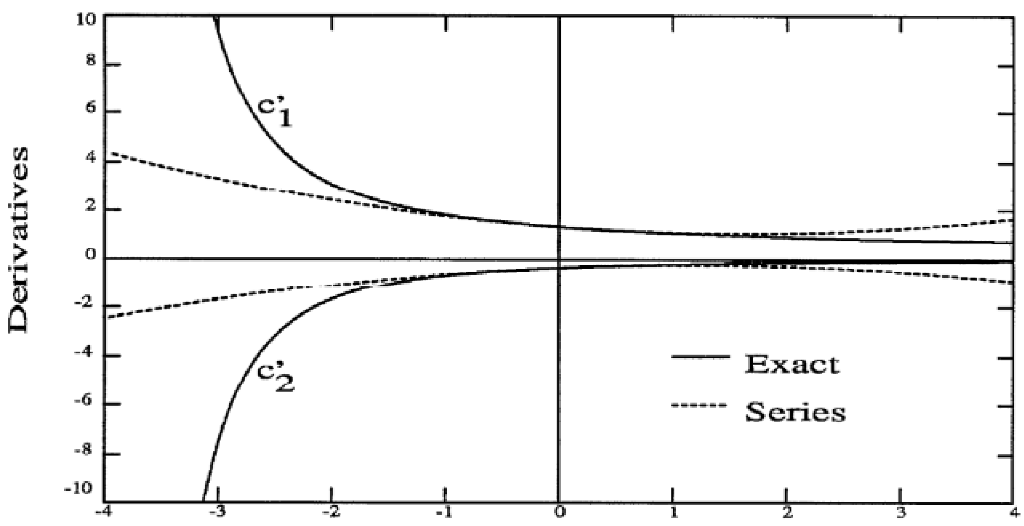

Figure 10:

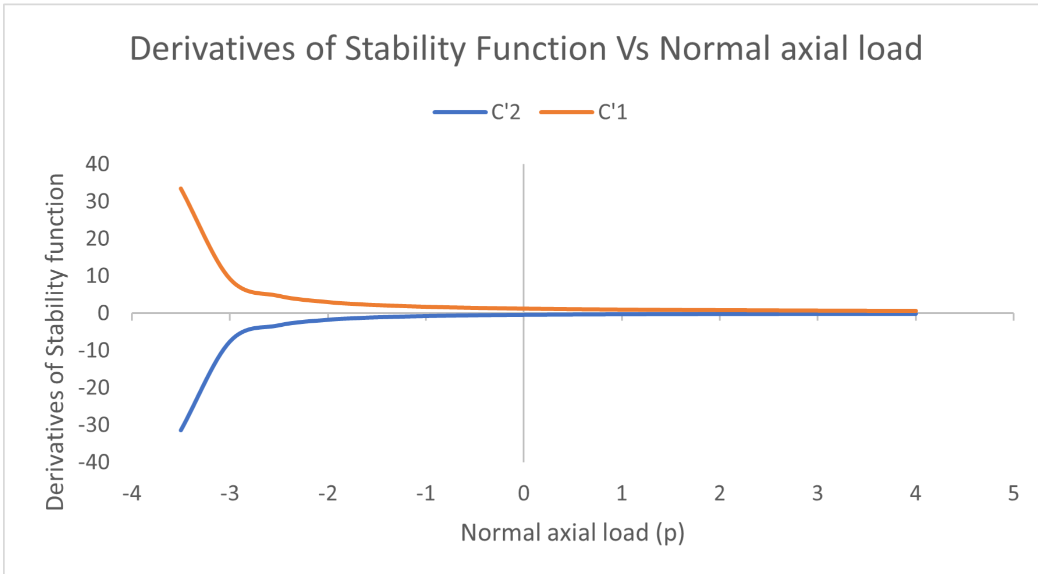

Figure 11:

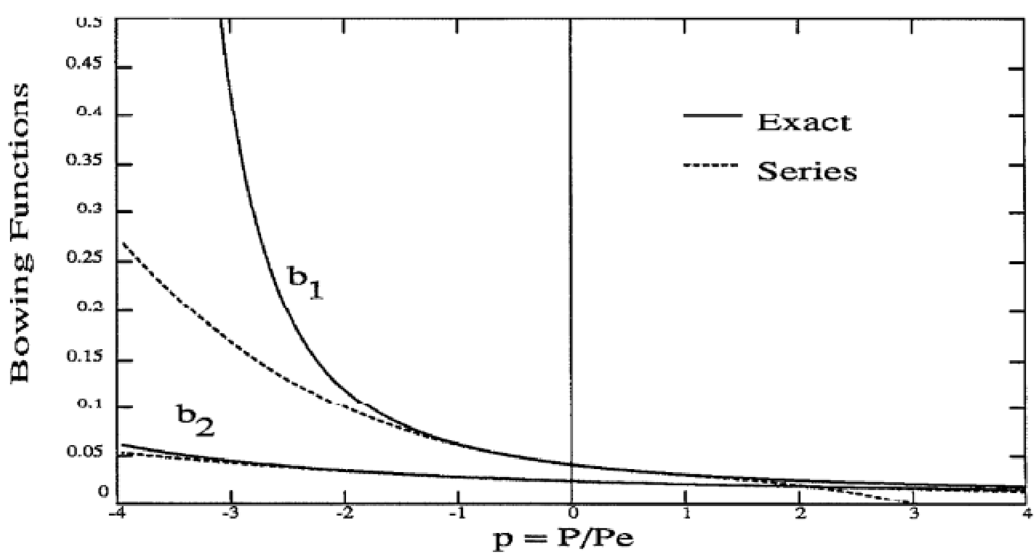

Figure 12:

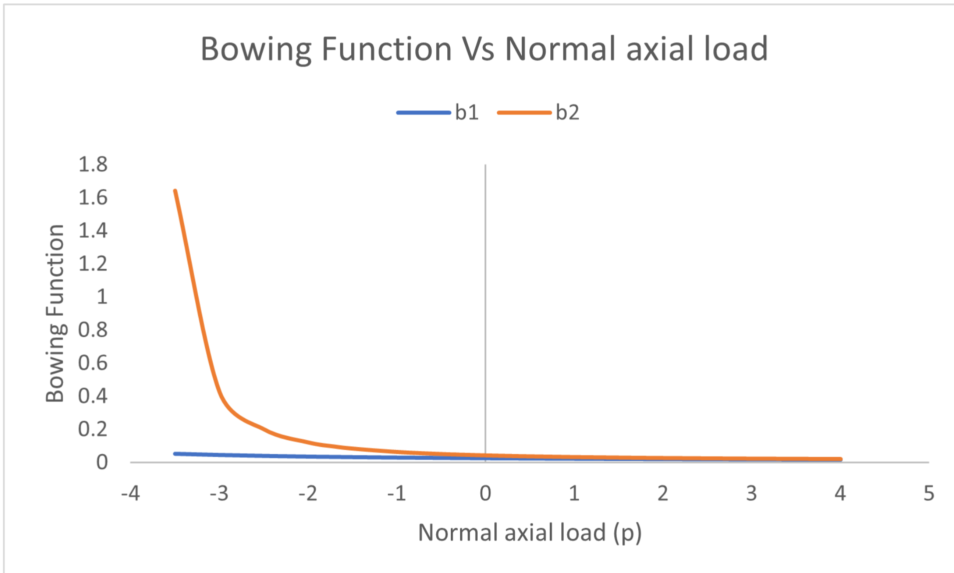

Figure 13:

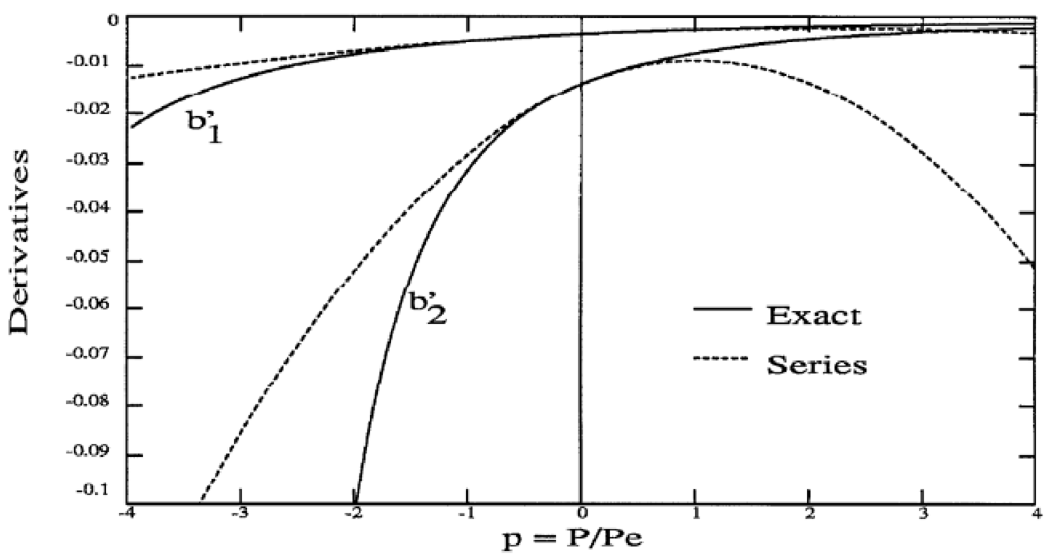

Figure 14:

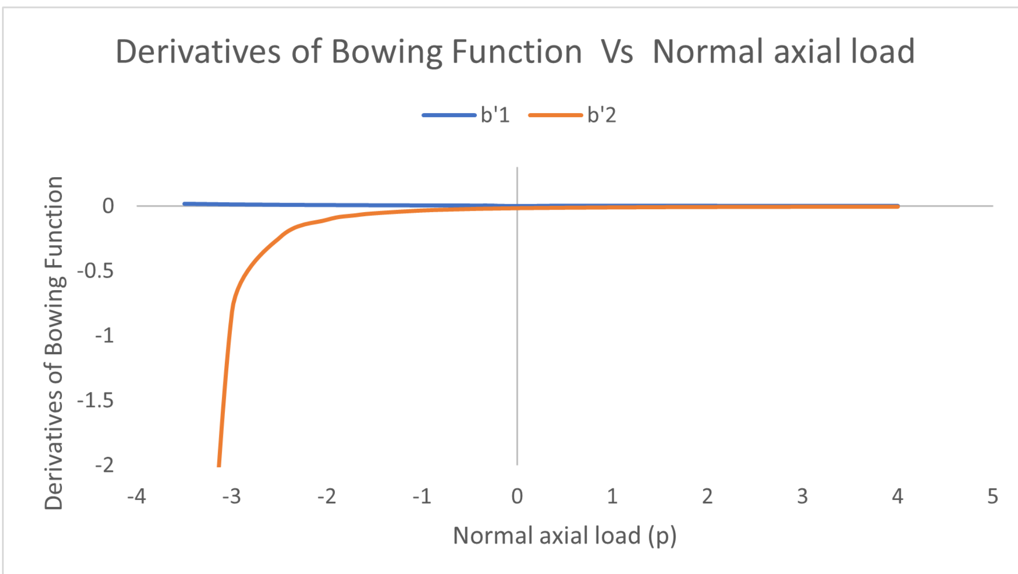
Derivatives of Bowing Function Vs Normal axial load
b'1
b'2
0
-0.5
-1
-1.5
-2
-4
-3
-2
-1
0
1
2
3
4
5
Derivatives of Bowing Function
Normal axial load (p)

Figure 15: