Interpolation

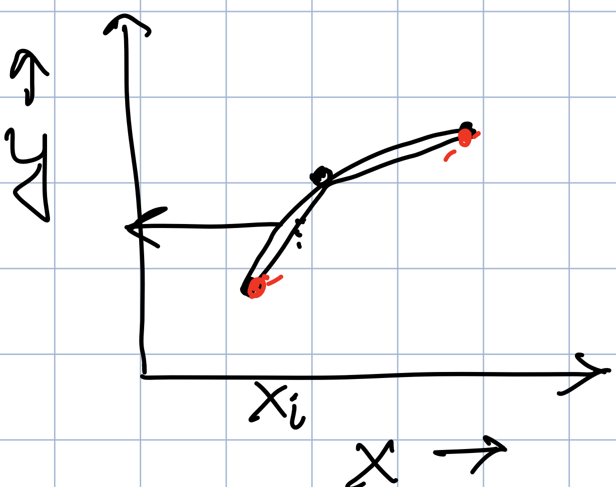
- Given a set of data, interpolant is a fn that matches the given data and passes through it exactly.

$(x_i, y_i) \quad i = 1, \ldots n$

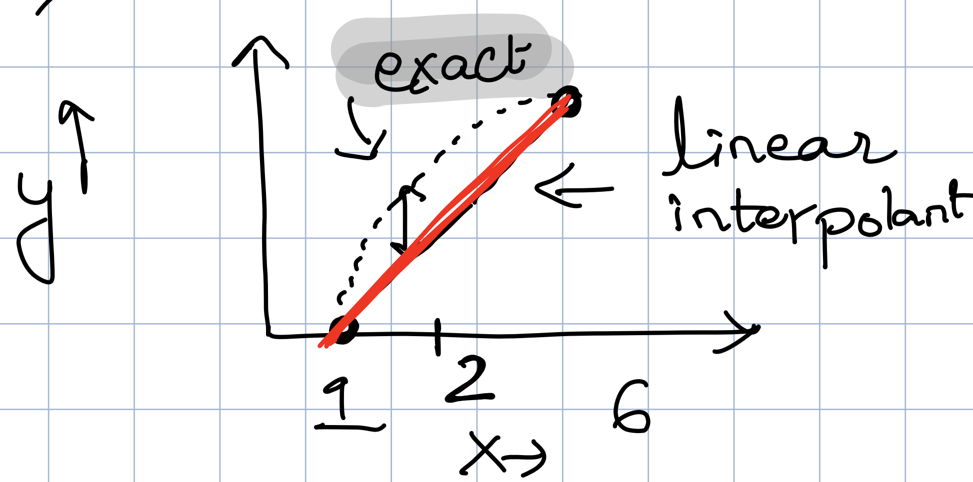
Independent variable

$x_1 < x_2, \ldots\ldots < x_n$

$f(x_i) = y_i \quad i = 1, \ldots\ldots n$



x_i	y_i
1	0
6	1.79
2	?



Polynomial Interpolation

$$P_n(x) = a_1 + a_2 x + a_3 x^2 + \dots a_{n+1} x^n$$

$a_1, \dots a_{n+1} \rightarrow$ Coefficients.

For dataset (x_i, y_i)

$$i = 1, \dots n+1$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ \vdots & & & & \\ 1 & x_{n+1} & x_{n+1}^2 & \dots & x_{n+1}^n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{bmatrix}$$

$\downarrow$

Vandermonde matrix

- ill-conditioned at lage n.

Newton's divided difference Interpolation polynomial

- Points are not evenly spaced.

Given datapts (x_i, y_i) $i = 1, \ldots n+1$

$$y = y_1 + (x - x_1)\, y[x_2, x_1]$$

linear Interpolant $\qquad n = 1$

$$y[x_2, x_1] = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y = y_1 + \frac{(x - x_1)(y_2 - y_1)}{x_2 - x_1}$$

linear Int.

Quadratic Interpolant

$$y = y_1 + (x - x_1)\, y[x_2, x_1] + (x - x_1)(x - x_2)\, y[x_3, x_2, x_1]$$

$$y[x_3, x_2, x_1]$$

$$= \frac{y[x_3, x_2] - y[x_2, x_1]}{x_3 - x_1}$$

$$= \frac{\dfrac{y_3 - y_2}{x_3 - x_2} - \dfrac{y_2 - y_1}{x_2 - x_1}}{x_3 - x_1}$$

General n^{th} degree polynomial

$$y = y_1 + (x - x_1)\, y[x_2, x_1] +$$

$$\text{-------} \; (x - x_1 \; (x - x_2) \ldots (x - x_n)$$

$$* \; y[x_{n+1}, x_n, \ldots x_1]$$

$$\frac{y[x_{n+1}, x_n, \ldots x_2] - y[x_n, \ldots x_1]}{x_{n+1} - x_1}$$

Verify by Substitution

At $x = x_1$, $y = y_1$

At $x = x_2$,

$$y = y_1 + (x - x_1) * y[x_2, x_1]$$

$$= y_1 + \frac{(\cancel{x_2 - x_1})(y_2 - y_1)}{\cancel{x_2 - x_1}}$$

$$= y_1 + y_2 - y_1$$

$$= y_2$$

At $x = x_3$

$$y = y_1 + \frac{(x_3 - x_1)(y_2 - y_1)}{(x_2 - x_1)} +$$

$$(x_3 - x_1)(x_3 - x_2)\left(\frac{y[x_3, x_2] - y[x_2, x_1]}{x_3 - x_1}\right)$$

$$y_1 + \frac{(x_3 - x_1)(y_2 - y_1)}{(x_2 - x_1)} +$$

$$(x_3 - x_2)\left(\frac{y_3 - y_2}{x_3 - x_2} - \frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$= y_3$$

<u>Verify</u> !!

Example,

i	x_i	y_i
1	1	7
2	3	53
3	4	157
4	6	857

$n = 3$

$n+1 = 4$

3rd degree Polynomial

① $P_3(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3^2 & 3^3 \\ 1 & 4 & 4^2 & 4^3 \\ 1 & 6 & 6^2 & 6^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 53 \\ 157 \\ 857 \end{bmatrix}$$

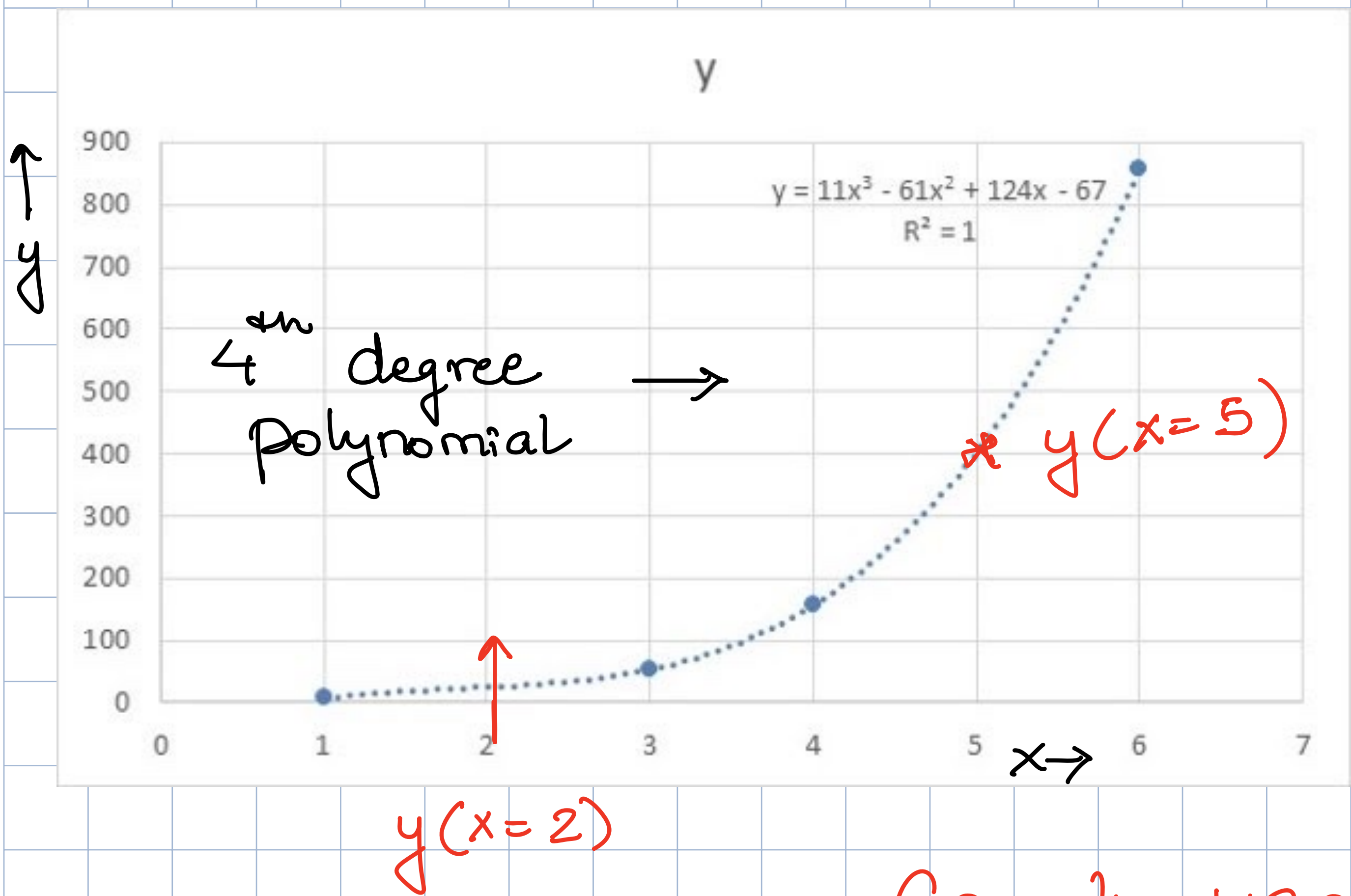
Solve by GE to Obtain

$a_1, a_2, a_3, a_4.$

Cond # $= 2.2 \times 10^3$

$a_1 = -67$ $a_3 = -61$

$a_2 = 124$ $a_4 = 11$



Cannot use this to predict y(x=7).

Applying Newton's divided diff. Interpolation polynomial.

$$y = y_1 + (x-x_1)\,y[x_2,x_1] + (x-x_1)(x-x_2) * y[x_3,x_2,x_1] + (x-x_1)(x-x_2)(x-x_3)\,y[x_4,x_3,x_2,x_1]$$

$$y[x_3,x_2,x_1] = \frac{y[x_3,x_2] - y[x_2,x_1]}{x_3 - x_1}$$

i	x_i	y_i	$y[x_{i+1}, x_i]$	$y[x_{i+2}, x_{i+1}, x_i]$
1	1	7	23	$\frac{104-23}{4-1} = 27$
2	3	53	104	82
3	4	157	350	
4	6	857		

$y[x_4, x_3]$

$$y[x_{i+3}, x_{i+2}, x_{i+1}, x_i]$$

$$\frac{82-27}{6-1} = \frac{55}{5} = 11$$

$$y[x_3, x_2] = (y_3 - y_2)/(x_3 - x_2)$$

i	x_i	y_i	$y[x_{i+1}, x_i]$	$y[x_{i+2}, x_{i+1}, x_i]$
1	1	7	23	$\frac{104-23}{4-1} = 27$
2	3	53	104	82
3	4	157	350	
4	6	857		

$$y[x_2, x_1] = \frac{y_2 - y_1}{x_2 - x_1} = \frac{53-7}{3-1} = 23$$

$$y[x_3, x_2] = \frac{y_3 - y_2}{x_3 - x_2} = \frac{157-53}{4-3} = 104$$

$$y[x_4, x_3] = \frac{y_4 - y_3}{x_4 - x_3} = \frac{857-157}{6-4} = 350$$

$$y[x_3, x_2, x_1] = \frac{y[x_3, x_2] - y[x_2, x_1]}{x_3 - x_1}$$

$$= \frac{104 - 23}{4 - 1}$$

$$= 27$$

$$y = y_1 + (x - x_1)\, y[x_2, x_1] + (x - x_1)(x - x_2) * y[x_3, x_2, x_1]$$
$$+ (x - x_1)(x - x_2)(x - x_3)\, y[x_4, x_3, x_2, x_1]$$

$$y = 7 + (x - 1)\, 23 + (x-1)(x-3)\, 27 + 11(x-1)(x-3)(x-4)$$

$$= 11x^3 - 61x^2 + 124x - 67$$

$$y[x_4, x_3, x_2, x_1] = \frac{y[x_4, x_3, x_2] - y[x_3, x_2, x_1]}{x_4 - x_1}$$

$$\frac{y[x_4, x_3] - y[x_3, x_2]}{x_4 - x_2}$$

$$= \frac{350 - 104}{6 - 3} = 82$$

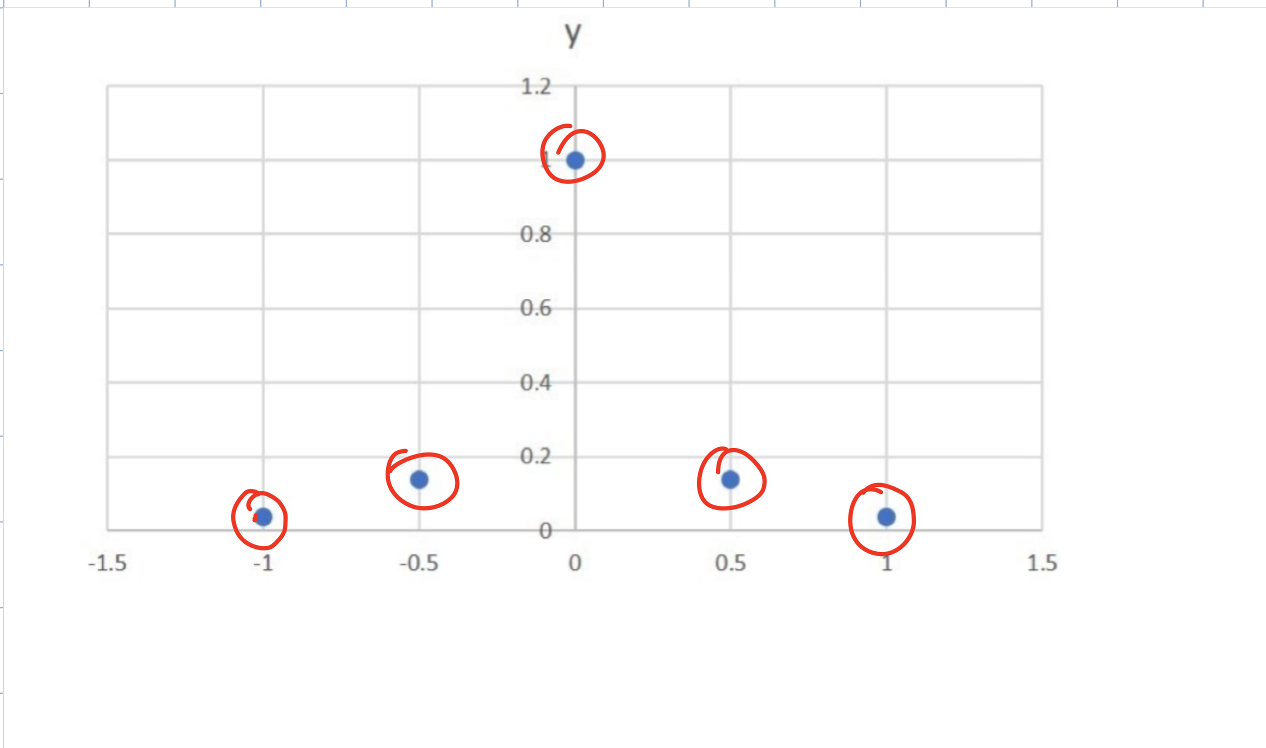
$$27$$

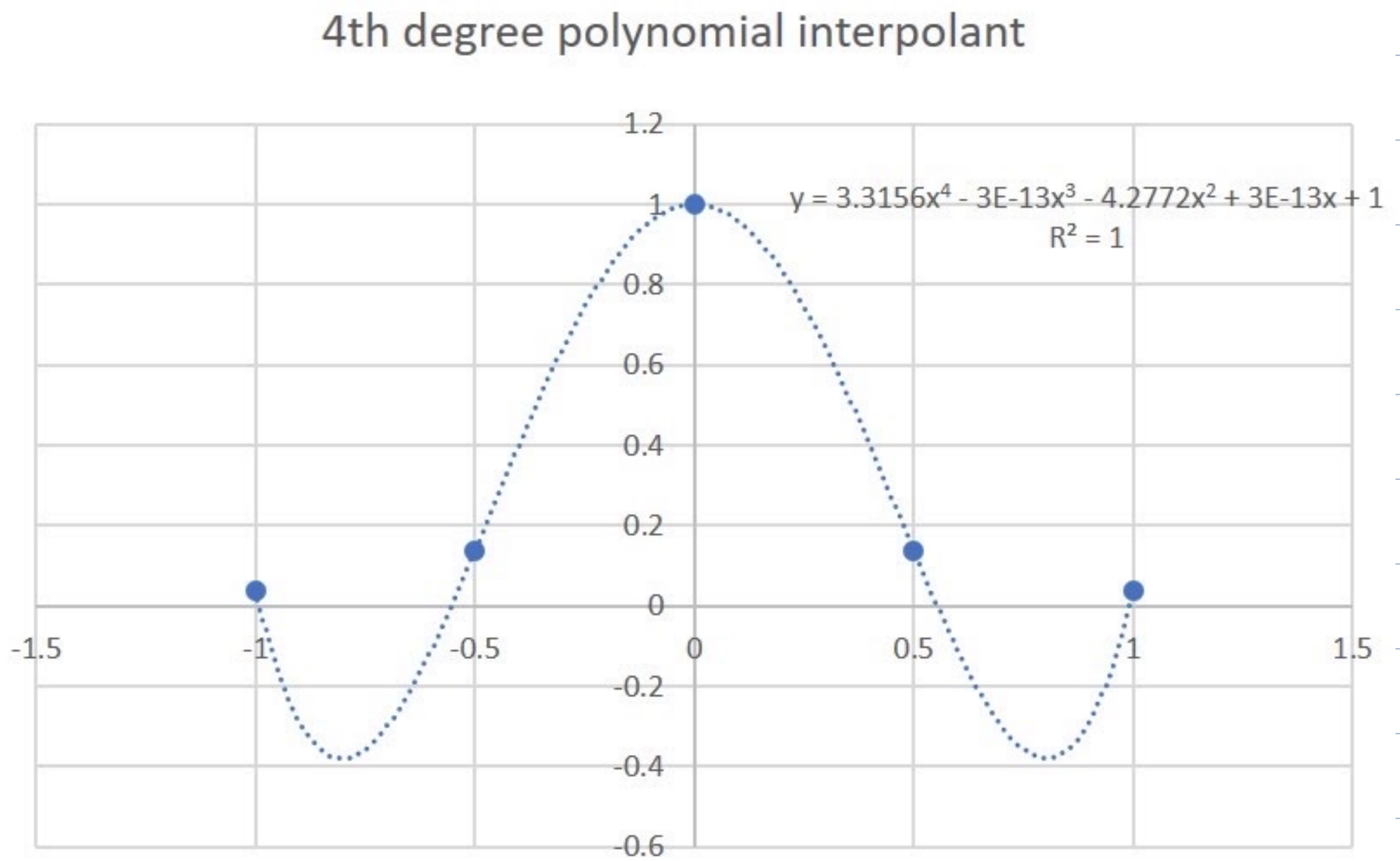
$$\frac{82 - 27}{6 - 1} = 11$$

Issues with Polynomial Interpolation.

Consider the following data

x	y
-1	0.038
-0.5	0.138
0	1
0.5	0.138
1	0.038





Polynomial Interpolation should NOT be used here.

Newton's Interpolation Polynomial when pts. are equally spaced.

(x_1, y_1) - - - - - - - - (x_{n+1}, y_{n+1})

$$x_{i+1} - x_i = \Delta x = \text{Constant}$$

Polynomial of degree n

Define, $\alpha = \dfrac{x - x_1}{\Delta x}$

$$\Delta y_1 = y_2 - y_1$$

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta(y_2 - y_1)$$

$$= \Delta y_2 - \Delta y_1$$

$$= (y_3 - y_2) - (y_2 - y_1)$$

$$= y_3 - 2y_2 + y_1$$

$$y = y_1 + (x - x_1)\, y[x_2, x_1] + (x - x_1)(x - x_2)\, y[x_3, x_2, x_1] + \cdots + (x - x_1)(x - x_2) \cdots (x - x_n)\, y[x_{n+1}, x_n, \cdots x_1]$$

$$y[x_2, x_1] = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{\Delta x}$$

$$(x - x_1)\, y[x_2, x_1] = \frac{(x - x_1)}{\Delta x}(y_2 - y_1)$$

$$= \alpha \, \Delta y_1$$

$$y[x_3, x_2, x_1] = \frac{y[x_3, x_2] - y[x_2, x_1]}{x_3 - x_1}$$

$$= \left(\frac{y_3 - y_2}{\Delta x} - \frac{y_2 - y_1}{\Delta x} \right) * \frac{1}{2\Delta x}$$

$$= \frac{y_3 - 2y_2 + y_1}{2(\Delta x)^2}$$

$$\frac{(x - x_1)(x - x_2)(y_3 - 2y_2 + y_1)}{2(\Delta x)^2}$$

$$= \frac{x - x_1}{\Delta x} \quad \frac{x - x_2}{\Delta x} \quad \frac{\Delta^2 y_1}{2}$$

$$\alpha \qquad \alpha - 1$$

$$\frac{x - x_2}{\Delta x} = \frac{x - x_1 - x_2 + x_1}{x_2 - x_1}$$

$$= \frac{x - x_1}{x_2 - x_1} - \left(\frac{x_2 - x_1}{x_2 - x_1}\right)$$

$$= \alpha - 1$$

For n^{th} degree polynomial,

$$y = y_1 + \alpha \Delta y_1 + \frac{\alpha(\alpha-1)}{2} \Delta^2 y_1$$

$$+ \cdots\cdots \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} \Delta^n y_1$$

Δx = Constant.

Linear Interpolant: $y = y_1 + \alpha \Delta y_1$

Quadratic "

$$y = y_1 + \alpha \Delta y_1 + \frac{\alpha(\alpha-1)}{2!} \Delta^2 y_1$$

Verify by Substitution

For example,

$x = x_1$, $y = y_1 \Rightarrow \alpha = 0$

$x = x_2$, $y = y_2 \Rightarrow \alpha - 1 = 0$

Verify by Substitution

$$y = y_1 + \alpha \Delta y_1 + \frac{(\alpha)(\alpha-1)}{2}\Delta^2 y_1 + \dots \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}\Delta^n y_1$$

Linear Interpolant, (x_1, y_1) (x_2, y_2)

$n+1 = 2$

$$\alpha = \frac{x - x_1}{\Delta x}$$

Polynomial of degree 1.

For $x = x_1$ $\quad \alpha = 0$

$x = x_2$ $\quad \alpha = \dfrac{x_2 - x_1}{\Delta x} = 1$

$x = x_1$, $\quad \alpha - 1 = -1$

$x = x_2$, $\quad \alpha - 1 = 0$

$\Rightarrow$

$$y = y_1 + \alpha \Delta y_1 + \cancel{(\alpha)(\alpha-1)\Delta^2 y_1} + \dots \cancel{\frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}\Delta^n y_1}$$

$$y = y_1 + \alpha \Delta y_1$$

$$x = x_1, \quad y = y_1, \quad \alpha = 0$$

$$x = x_2, \quad y = y_1 + 1(y_2 - y_1)$$
$$= y_2$$

Verify that for quadratic polynomial interpolant for data pt. (x_1, y_1), (x_2, y_2) and (x_3, y_3)

$$y = y_1 + \alpha(\Delta y_1) + \frac{(\alpha)(\alpha-1)}{2} \Delta^2 y_1$$

Example,

i	x_i	y_i
1	1	7
2	2	17
3	3	53
4	4	157

$n = 4$
Polynomial $= 3^{rd}$ degree.

$$y = y_1 + \alpha \Delta y_1 + \frac{\alpha(\alpha-1)}{2} \Delta^2 y_1 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} \Delta^3 y_1$$

i	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
1	1	7	$\Delta y_1 = 10$	$\Delta^2 y_1 = 26$	$\Delta^3 y_1 = 42$
2	2	17	$\Delta y_2 = 36$	$\Delta^2 y_2 = 68$	
3	3	53	$\Delta y_3 = 104$		
4	4	157			

$$y = 7 + 10\alpha + \frac{26}{2} \alpha(\alpha-1) + \frac{42}{6} \alpha(\alpha-1)(\alpha-2)$$

$$\alpha = \frac{x - x_1}{\Delta x} = x - 1$$

Lagrangian Interpolation

$$y = \sum_{i=1}^{n+1} L_i(x)\, y_i$$ $n+1$ pts degree n

$$L_i(x) = \prod_{\substack{j=i \\ j \neq i}}^{n+1} \frac{x - x_j}{x_i - x_j}$$

For example $n = 1$

$$L_1(x) = \frac{x - x_2}{x_1 - x_2}$$

$$L_2(x) = \frac{x - x_1}{x_2 - x_1}$$

$$y = \frac{x - x_2}{x_1 - x_2} y_1 + \frac{x - x_1}{x_2 - x_1} y_2$$

$x = x_1, \quad y = y_1$

$x = x_2, \quad y = y_2$

Example,

i	x_i	y_i
1	1	7
2	2	17
3	3	53
4	4	157

3^{rd} degree polynomial

$$y(x) = L_1(x)\,y_1 + L_2(x)\,y_2 + L_3(x)\,y_3 + L_4(x)\,y_4.$$

$$L_1(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)}$$

$$L_2(x) = \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$$

Similarly $L_3(x)$, $L_4(x)$ forms can be written.

$$y(x) = 7\,\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} +$$

$$\frac{17(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} +$$

$$\frac{53(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} +$$

$$\frac{157(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$$

$$= -19 + 48x - 29x^2 + 7x^3$$

Inverse Interpolation

Typically, $y = a_0 + a_1 x + a_2 x^2$

is used to find values of y at different x.

What if, find value of x for a value of y?

This is known as Inverse Interpolation.

$$y_{given} = a_0 + a_1 x + a_2 x^2$$

Find x?

$$f(x) = y_{given} - a_0 - a_1 x - a_2 x^2$$

Root of $f(x)$ where $f(x^*) = 0$

Example

1. Estimate $\ln 2$ using linear Interpolation.

   Given (1) $\ln 1 = 0$
   $\ln 6 = 1.791754$

   (2) $\ln 1 = 0$
   $\ln 4 = 1.386294$

   Use Quadratic Interpolation

   Compare Newton divided diff Interpolation form and lagrange form.

2. Derive lagrange form Newton Interpolating polynomial for 1^{st} order.

3. Estimating error.
   additional data pt.
   $\ln 5 = 1.609438.$

Spline Interpolation.

- Polynomial → lead to over-fitting
- Piecewise Interpolation.

3rd degree polynomial

Oscillations.

5th degree poly.

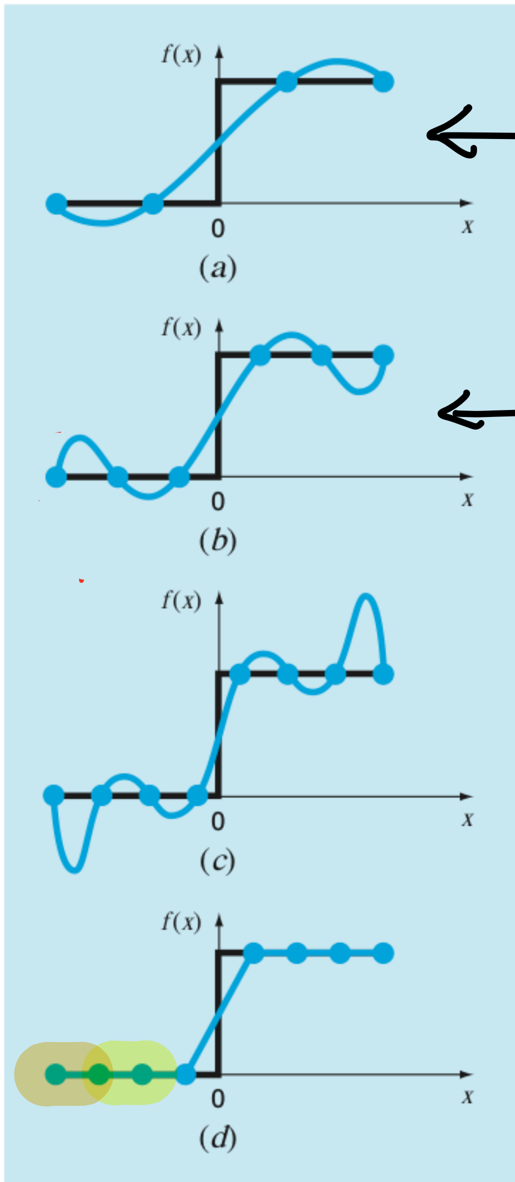
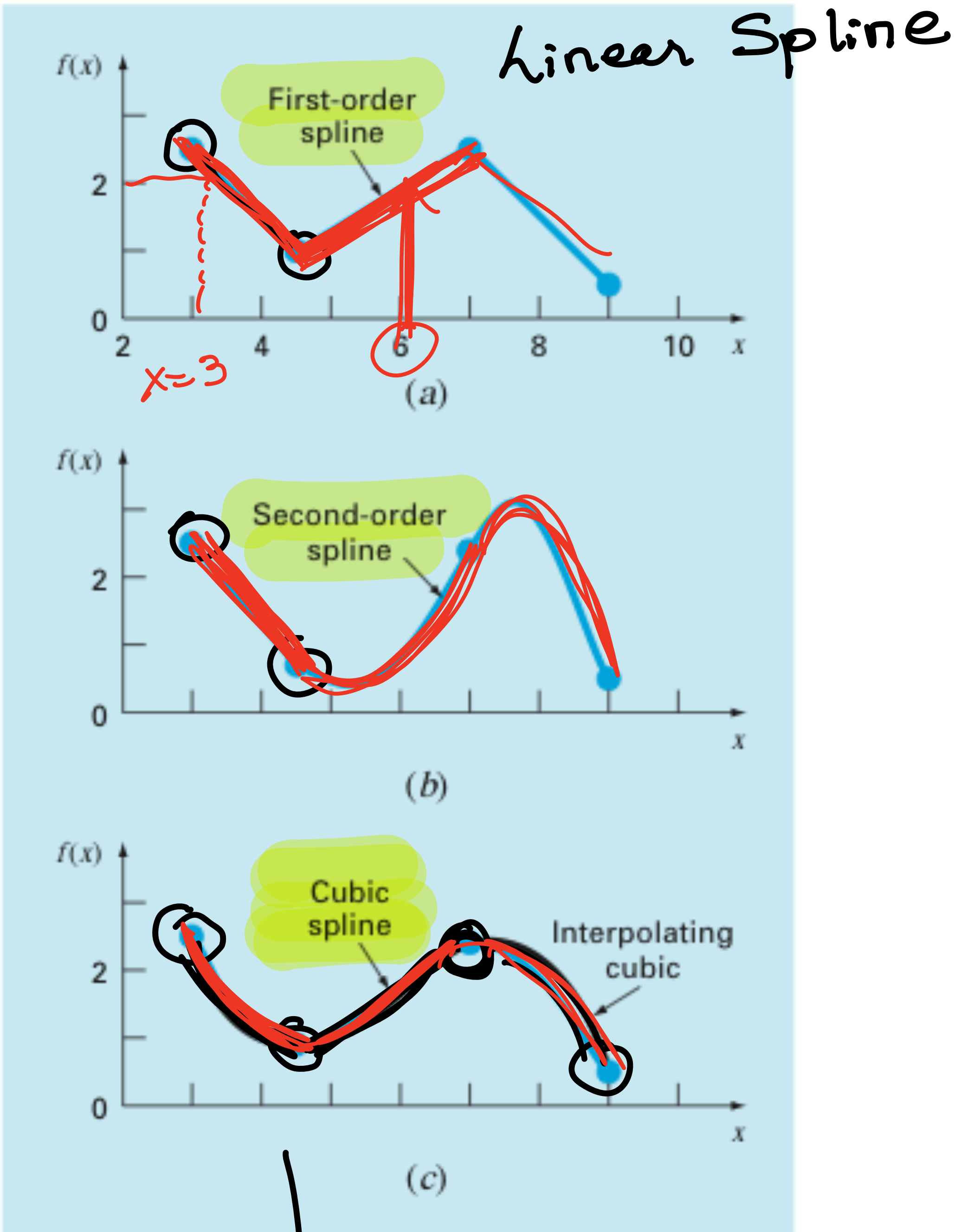


FIGURE 18.14
A visual representation of a situation where the splines are superior to higher-order interpolating polynomials. The function to be fit undergoes an abrupt increase at $x = 0$. Parts (a) through (c) indicate that the abrupt change induces oscillations in interpolating polynomials. In contrast, because it is limited to third-order curves with smooth transitions, a linear spline (d) provides a much more acceptable approximation.

FIGURE 18.16
Spline fits of a set of four points. (*a*) Linear spline, (*b*) quadratic spline, and (*c*) cubic spline, with a cubic interpolating polynomial also plotted.



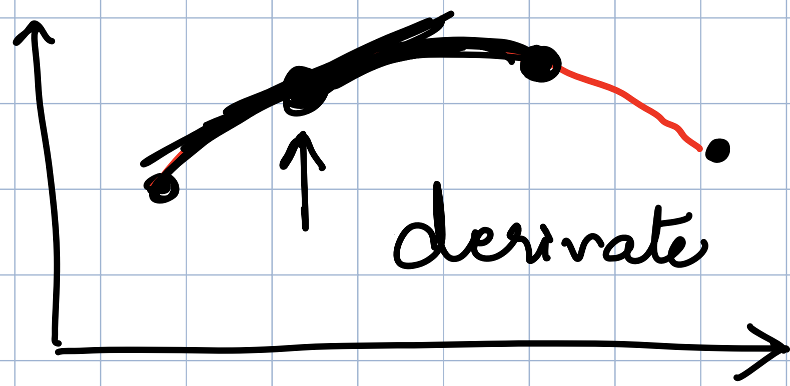
4 pts. ⇒ Polynomial Interpolant

3^{rd} degree ⇒ Cubic

Different from Cubic Spline.

Cubic Spline

$x_i, y_i \qquad i = 1, \ldots, n+1$



n splines.

derivate should be same.

→ Piecewise curve
→ Continuous Curve.
→ Pass through data pts.
→ Seperate Cubic polynomial for each interval.

→ k^{th} degree Spline, $k-1$ times, derivatives at the meeting pt. should be same for two connecting splines.

$$S_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$$

$$i = 1, \ldots, n$$

$$x \in [x_i, x_{i+1}]$$

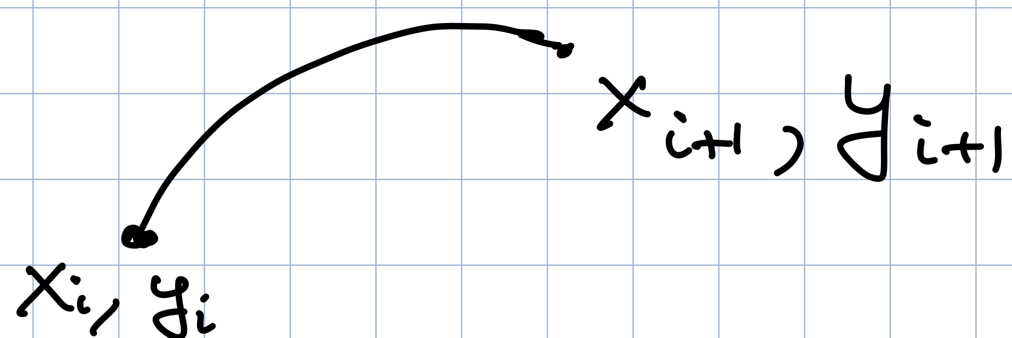
Each spline has 4 coeff.

$\Rightarrow$ $4n$ coefficients.

$$S_i(x_i) = y_i$$

$$S_i(x_{i+1}) = y_{i+1}$$

Cubic Splines match the data pts at both ends.



n Splines $\rightarrow$ $2n$ Conditions

$$S_i(x_i) = y_i$$

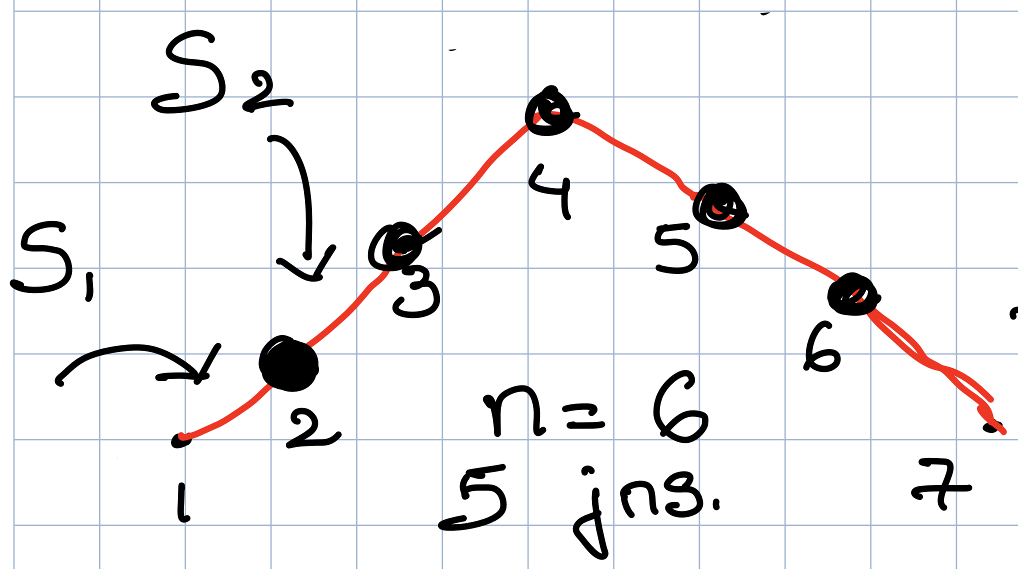
$$S_i(x_{i+1}) = y_{i+1}$$

Need $2n$ more Conditions.

Interpolant to be smooth at the boundaries, we require 1^{st} and 2^{nd} derivatives are continuous

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$$

$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$$



$n+1$ pts

n Spline

$n-1$ jns.

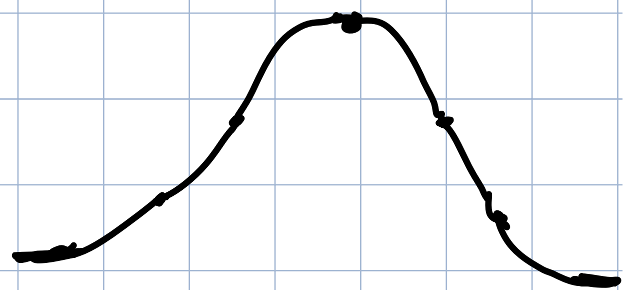
$2(n-1)$ Constraints from equality of 1^{st} & 2^{nd} derivative.

We need 2 more Cond.

Natural Condition

$S_1''(x_1) = 0$

$S_n''(x_{n+1}) = 0$



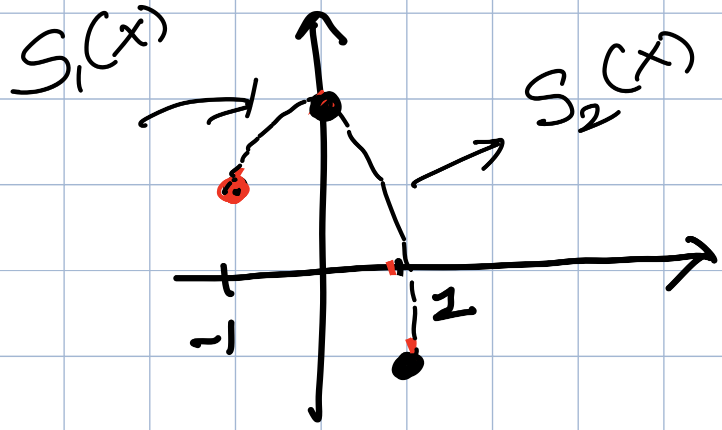
Alternative Conditions:

$S_1'(x_1) = f'(x_1)$

$S_n'(x_{n+1}) = f'(x_{n+1})$

Example

i	x	y
1	-1	1
2	0	2
3	1	-1



$$S_1(x) = ax^3 + bx^2 + cx + d \quad [-1,\ 0]$$

$$S_2(x) = ex^3 + fx^2 + gx + h \quad [0,\ 1]$$

(8 parameters)

$$S_1(-1) = 1$$
$$S_1(0) = 2$$
$$S_2(0) = 2$$
$$S_2(1) = -1$$

$$\left.\begin{aligned} S_1'(0) &= S_2'(0) \\ S_1''(0) &= S_2''(0) \end{aligned}\right\} \text{Smooth}$$

$$\left.\begin{aligned} S_1''(-1) &= 0 \\ S_2''(1) &= 0 \end{aligned}\right\} \text{Boundary}$$

$$S_1(x) = -x^3 - 3x^2 - x + 2$$
$$S_2(x) = x^3 - 3x^2 - x + 2$$