PyQBench: a Python library for benchmarking gate-based quantum computers
Supplemental material

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1. Mathematical preliminaries

Let $\mathcal{M}_{d_1,d_2}$ be the set of all matrices of dimension $d_1 \times d_2$ over the field $\mathbb{C}$. For simplicity, square matrices will be denoted by $\mathcal{M}_d$. By Ω_d, we will denote the set of quantum states, that is positive semidefinite operators having trace equal to one. The subset of $\mathcal{M}_d$ consisting of unitary matrices will be denoted by $\mathcal{U}_d$, while its subgroup of diagonal unitary operators will be denoted by $\mathcal{DU}_d$.

We will also need a linear mapping transforming $\mathcal{M}_{d_1}$ into $\mathcal{M}_{d_2}$, which will be denoted

$$\Phi : \mathcal{M}_{d_1} \to \mathcal{M}_{d_2}. \tag{1}$$

There exists a bijection between the set of linear mappings Φ and the set of matrices $\mathcal{M}_{d_1 d_2}$, known as the Choi-Jamiołkowski isomorphism. For a given linear mapping Φ the corresponding Choi operator $J(\Phi)$ is explicitly written as

$$J(\Phi) \coloneqq \sum_{i,j=0}^{d-1} \Phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|. \tag{2}$$

We also introduce a special subset of all mappings Φ, called quantum channels, which are completely positive and trace preserving (CPTP). In this work we will consider a special class of quantum channels, called unitary channels. A quantum channel Φ_U is said to be a unitary channel if it has the following form $\Phi_U(\cdot) = U \cdot U^\dagger$ for any $U \in \mathcal{U}_d$.

Let us recall a general form of a quantum measurement, so called Positive Operator Valued Measure (POVM). A POVM $\mathcal{P}$ is a collection of positive

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semidefinite operators $\{E_1, \ldots, E_m\}$, called effects, that sum up to the identity operator, *i.e.* $\sum_{i=1}^{m} E_i = \mathbb{1}$. In PyQBench, we are interested only in von Neumann measurements, that is measurements for which all the effects are rank-one projectors. Every such measurement can be parameterized by a unitary matrix $U \in \mathcal{U}_d$ with effects $\{|u_0\rangle\langle u_0|, \ldots, |u_{d-1}\rangle\langle u_{d-1}|\}$, are created by taking $|u_i\rangle$ as $(i+1)$-th column of the unitary matrix U. We will denote von Neumann measurements described by the matrix U by $\mathcal{P}_U$. The action of von Neumann measurement $\mathcal{P}_U$ on some state $\rho \in \Omega_d$ can be seen as a measure-and-prepare quantum channel as follows

$$\mathcal{P}_U : \rho \to \sum_{i=0}^{d-1} \langle u_i|\rho|u_i\rangle |i\rangle\langle i|. \tag{3}$$

Moreover, observe that each von Neumann measurement $\mathcal{P}_U$ poses a composition of a unitary channel $\Phi_{U^\dagger}$ and the maximally dephasing channel Δ, that means $\mathcal{P}_U = \Delta \circ \Phi_{U^\dagger}$.

We need to also briefly discuss about the distance between quantum operations. From [1, Theorem 1], the distance between measurements $\mathcal{P}_U$ and $\mathcal{P}_{\mathbf{1}}$ can be expressed in the notion of diamond norm, that is

$$\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond = \min_{E \in \mathcal{DU}_d} \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond. \tag{4}$$

To express the distance between unitary channels, we need to introduce the definition of numerical range [2]. The set

$$W(A) = \{\langle x|A|x\rangle : |x\rangle \in \mathbb{C}^d, \ \ \langle x|x\rangle = 1\} \tag{5}$$

is called the numerical range of a given matrix $A \in \mathcal{M}_d$. The detailed properties of the numerical range and its generalizations we can read on the website [3].

Due to the definition of $W(A)$, the distance between two unitary channels Φ_U and $\Phi_{\mathbf{1}}$ can be written as

$$\|\Phi_U - \Phi_{\mathbf{1}}\|_\diamond = 2\sqrt{1-\nu^2}, \tag{6}$$

where $\nu = \min_{x \in W(U^\dagger)} |x|$.

2. Discrimination task for Hadamard gate

For the discrimination task between von Neumann measurements $\mathcal{P}_U$ and $\mathcal{P}_{\mathbf{1}}$, where $U = H$ (the Hadamard gate), the key is to calculate the diamond norm $\|\mathcal{P}_H - \mathcal{P}_{\mathbf{1}}\|_\diamond$ and determine the discriminator $|\psi_0\rangle$. Using semidefinite programming [4], we obtain

$$\|\mathcal{P}_H - \mathcal{P}_{\mathbf{1}}\|_\diamond = \sqrt{2}. \tag{7}$$

From [5] we have

$$\|\mathcal{P}_H - \mathcal{P}_{\mathbf{1}}\|_\diamond = \|\Phi_{HE_0} - \Phi_{\mathbf{1}}\|_\diamond, \tag{8}$$

where Φ_U is a unitary channel and E_0 is of the form

$$E_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 \\ 0 & -1-i \end{pmatrix}. \tag{9}$$

Next, in order to construct the discriminator $|\psi_0\rangle$ we use Lemma 5 and the proof of Theorem 1 in [1]. We show that there exist states ρ_1 and ρ_2 of the form $\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and $\rho_2 = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$, respectively. Thus, we construct the quantum state ρ_0 as follows:

$$\rho_0 = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{10}$$

According to the Lemma 5 and the proof of Theorem 1 in [1] we assume that

$$|\psi_0\rangle = \left|\sqrt{\rho_0^\top}\right\rangle\!\!\Big\rangle. \tag{11}$$

It directly implies that

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{12}$$

Next, from Holevo-Helstrom theorem [6], we determine the final measurement $\mathcal{P}_{V_i}$. Let us consider

$$X = (\mathcal{P}_H \otimes \mathbb{1})\left(|\psi_0\rangle\langle\psi_0|\right) - (\mathcal{P}_{\mathbf{1}} \otimes \mathbb{1})\left(|\psi_0\rangle\langle\psi_0|\right). \tag{13}$$

From the Hahn-Jordan decomposition [6], let us note

$$X = P - Q, \tag{14}$$

where $P, Q \geq 0$. Let us define projectors Π_P and Π_Q onto $\mathrm{im}(P)$ and $\mathrm{im}(Q)$, respectively. Observe, that P and Q are block-diagonal. Then, Π_P and Π_Q have the following forms

$$\Pi_P = \begin{pmatrix} |x_p\rangle\langle x_p| & 0 \\ 0 & |y_p\rangle\langle y_p| \end{pmatrix}, \tag{15}$$

and

$$\Pi_Q = \begin{pmatrix} |x_q\rangle\langle x_q| & 0 \\ 0 & |y_q\rangle\langle y_q| \end{pmatrix}. \tag{16}$$

Hence, we define V_0 as

$$\begin{cases} |x_p\rangle = V_0|0\rangle \\ |x_q\rangle = V_0|1\rangle \end{cases} \tag{17}$$

and V_1 as

$$\begin{cases} |y_p\rangle = V_1|0\rangle \\ |y_q\rangle = V_1|1\rangle \end{cases} . \tag{18}$$

For the discrimination task between $\mathcal{P}_H$ and $\mathcal{P}_{\mathbf{1}}$ the explicit form of V_0 and V_1 is given as follows (see also `mathematics/optimal_final_measurement_discrimination.nb` in the source code repository):

$$V_0 = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \tag{19}$$

and

$$V_1 = \begin{pmatrix} -\beta & \alpha \\ \alpha & \beta \end{pmatrix}, \tag{20}$$

where

$$\alpha = \frac{\sqrt{2-\sqrt{2}}}{2} = \cos\left(\frac{3}{8}\pi\right), \tag{21}$$

and

$$\beta = \frac{\sqrt{2+\sqrt{2}}}{2} = \sin\left(\frac{3}{8}\pi\right). \tag{22}$$

3. Optimal probability for parameterized Fourier family

Let us focus on single-qubit von Neumann measurements $\mathcal{P}_{\mathbf{1}}$ and $\mathcal{P}_U$. Assume that the unitary matrix U is of the form

$$U = H \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} H^\dagger \tag{23}$$

where H is the Hadamard matrix of dimension two and $\phi \in [0, 2\pi)$. In this section we present theoretical probability of correct discrimination between these measurements. To do that, we will present an auxiliary lemma.

Lemma 1. *Let $U = H\mathrm{diag}(1, e^{i\phi})H^\dagger$, $\phi \in [0, 2\pi)$ and let Φ_U and $\Phi_{\mathbf{1}}$ be two unitary channels. Then, the following equation holds*

$$\min_{E \in \mathcal{DU}_2} \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond = \|\Phi_U - \Phi_{\mathbf{1}}\|_\diamond, \tag{24}$$

Proof. Recall that the distance between two unitary channels is given by $\|\Phi_U - \Phi_{\mathbf{1}}\|_\diamond = 2\sqrt{1-\nu^2}$, where $\nu = \min_{x\in W(U^\dagger)} |x|$ for any $U \in \mathcal{U}_d$. For $U = H \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} H^\dagger$ the readers briefly observe that $\nu^2 = 1 - \frac{|1-e^{-i\phi}|^2}{4} = 1 - \frac{|1-e^{i\phi}|^2}{4}$. So,

$$\|\Phi_U - \Phi_{\mathbf{1}}\|_\diamond = |1 - e^{i\phi}|. \tag{25}$$

It implies that it is enough to prove

$$\min_{E\in\mathcal{DU}_2} \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond = \ |1 - e^{i\phi}|. \tag{26}$$

This condition is equivalent to show that for every $E \in \mathcal{DU}_2$

$$\nu_E \leq \frac{|1+e^{i\phi}|}{2}, \tag{27}$$

where $\nu_E = \min_{x\in W(U^\dagger E)} |x|$.

The celebrated Hausdorff-Töplitz theorem [7, 8] states that $W(A)$ of any matrix $A \in \mathcal{M}_d$ is a convex set, and therefore we have

$$W(A) = \{\mathrm{tr}(A\rho) : \rho \in \Omega_d\}. \tag{28}$$

So, we can assume that

$$\min_{|x\rangle\in\mathbb{C}^2 : |x\rangle\langle x|=1} |\langle x|U^\dagger|x\rangle| = \min_{\rho\in\Omega_2} |\mathrm{tr}(U^\dagger\rho)|. \tag{29}$$

Then, we have

$$\nu_E = \min_{\rho\in\Omega_2} |\mathrm{tr}\,(\rho U E)|\,. \tag{30}$$

For that, our task is reduced to show that for every $E \in \mathcal{DU}_2$ there exists $\rho \in \Omega_2$ such that

$$|\mathrm{tr}\,(\rho U E)\,| \leq \frac{|1+e^{i\phi}|}{2}. \tag{31}$$

Let us define $E = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}$ and take $\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. From spectral theorem, let us decompose U as

$$U = \lambda_0 |x_0\rangle\langle x_0| + \lambda_1 |x_1\rangle\langle x_1|, \tag{32}$$

where for eigenvalue $\lambda_0 = 1$, the corresponding eigenvector is of the form $|x_0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, whereas for $\lambda_1 = e^{i\phi}$ we have $|x_1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. Then, for every $E \in \mathcal{DU}_2$ we have

$$\begin{aligned}|\mathrm{tr}(\rho U E)| &= \frac{1}{2}\left|\mathrm{tr}\left(H \mathrm{diag}(1, e^{i\phi}) H^\dagger E\right)\right| = \frac{1}{2}\left|\mathrm{tr}\left((|x_0\rangle\langle x_0| + e^{i\phi}|x_1\rangle\langle x_1|)E\right)\right| \\ &= \frac{1}{2}\left|\langle x_0|E|x_0\rangle + e^{i\phi}\langle x_1|E|x_1\rangle\right| = \frac{1}{2}\left|\frac{E_0+E_1}{2} + e^{i\phi}\frac{E_0+E_1}{2}\right| \\ &= \frac{\left|1+e^{i\phi}\right|}{2}\left|\frac{E_0+E_1}{2}\right| \le \frac{|1+e^{i\phi}|}{2},\end{aligned} \tag{33}$$

which completes the proof. □

Theorem 1. *The optimal probability of correct discrimination between von Neumann measurements $\mathcal{P}_U$ and $\mathcal{P}_{\mathbf{1}}$ for $U = H\mathrm{diag}(1, e^{i\phi})H^\dagger$, where $\phi \in [0, 2\pi)$ is given by*

$$p_{succ}(\mathcal{P}_U, \mathcal{P}_{\mathbf{1}}) = \frac{1}{2} + \frac{|1-e^{i\phi}|}{4}. \tag{34}$$

Proof. From Holevo-Helstrom theorem, we obtain

$$p_{\mathrm{succ}}(\mathcal{P}_U, \mathcal{P}_{\mathbf{1}}) = \frac{1}{2} + \frac{1}{4}\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond. \tag{35}$$

From [1, Theorem 1], we have

$$\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond = \min_{E \in \mathcal{DU}_d} \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond. \tag{36}$$

From Lemma 1, we know that for $U = H\mathrm{diag}(1, e^{i\phi})H^\dagger$, it also holds that

$$\min_{E \in \mathcal{DU}_2} \|\Phi_{UE} - \Phi_{\mathbf{1}}\|_\diamond = \|\Phi_U - \Phi_{\mathbf{1}}\|_\diamond, \tag{37}$$

which is exactly equal to

$$\|\Phi_U - \Phi_{\mathbf{1}}\|_\diamond = 2\sqrt{1-\nu^2} = |1-e^{i\phi}|. \tag{38}$$

It implies that

$$p_{\mathrm{succ}}(\mathcal{P}_U, \mathcal{P}_{\mathbf{1}}) = \frac{1}{2} + \frac{|1-e^{i\phi}|}{4}, \tag{39}$$

which completes the proof. □

4. Optimal discrimination strategy for parameterized Fourier family

In this Appendix we create the optimal theoretical strategy of discrimination between $\mathcal{P}_U$ and $\mathcal{P}_{\mathbf{1}}$. To indicate the optimal strategy, we will present two propositions. The first one is concentrated around the discriminator as the optimal input state of discrimination strategy, whereas the second one describes the optimal final measurement.

Proposition 1. *Consider the problem of discrimination between von Neumann measurements $\mathcal{P}_U$ and $\mathcal{P}_{\mathbf{1}}$, $U = H\mathrm{diag}(1, e^{i\phi})H^\dagger$ and $\phi \in [0, 2\pi)$. The discriminator has the form*

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|\mathbb{1}_2\rangle\rangle. \tag{40}$$

Proof. Let $U = H\mathrm{diag}(1, e^{i\phi})H^\dagger, \phi \in [0, 2\pi)$ be decomposed as

$$U = \lambda_0|x_0\rangle\langle x_0| + \lambda_1|x_1\rangle\langle x_1|, \tag{41}$$

where for eigenvalue $\lambda_0 = 1$, the corresponding eigenvector is of the form $|x_0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, whereas for $\lambda_1 = e^{i\phi}$ we have $|x_1\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$. For Hermitian-preserving maps [6] the diamond norm may be expressed as

$$\|\Phi\|_\diamond = \max_{\rho\in\Omega_d} \| \left(\mathbb{1} \otimes \sqrt{\rho}\right) J(\Phi) \left(\mathbb{1} \otimes \sqrt{\rho}\right) \|_1. \tag{42}$$

Hence, we obtain

$$\begin{aligned}
\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond &= \max_{\rho\in\Omega_2} \left\|\left(\mathbb{1} \otimes \sqrt{\rho}\right) J(\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}) \left(\mathbb{1} \otimes \sqrt{\rho}\right)\right\|_1 \\
&= \max_{\rho\in\Omega_2} \left\|\left(\mathbb{1} \otimes \sqrt{\rho}\right) \sum_{i=0}^{1} |i\rangle\langle i| \otimes \left(|u_i\rangle\langle u_i| - |i\rangle\langle i|\right)^\top \left(\mathbb{1} \otimes \sqrt{\rho}\right)\right\|_1 \\
&= \max_{\rho\in\Omega_2} \left\|\sum_{i=0}^{1} |i\rangle\langle i| \otimes \sqrt{\rho} \left(|u_i\rangle\langle u_i| - |i\rangle\langle i|\right)^\top \sqrt{\rho}\right\|_1 \\
&= \max_{\rho\in\Omega_2} \sum_{i=0}^{1} \left\|\sqrt{\rho} \left(|u_i\rangle\langle u_i| - |i\rangle\langle i|\right)^\top \sqrt{\rho}\right\|_1 .
\end{aligned} \tag{43}$$

One can prove that for all $\alpha, \beta \geq 0$, and unit vectors $|x\rangle, |y\rangle$ the following equation holds [6]

$$\|\alpha|x\rangle\langle x| - \beta|y\rangle\langle y|\|_1 = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta|\langle x|y\rangle|^2}. \tag{44}$$

By taking $|x\rangle = \frac{\sqrt{\rho}|\bar{u}_i\rangle}{\|\sqrt{\rho}|\bar{u}_i\rangle\|}$ and $|y\rangle = \frac{\sqrt{\rho}|i\rangle}{\|\sqrt{\rho}|i\rangle\|}$ we have

$$\|\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}\|_\diamond = \max_{\rho\in\Omega_2} \sum_{i=0}^{1} \sqrt{\left(\langle\bar{u}_i|\rho|\bar{u}_i\rangle + \langle i|\rho|i\rangle\right)^2 - 4|\langle\bar{u}_i|\rho|i\rangle|^2}. \tag{45}$$

Let us take $\rho_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain

$$\begin{aligned}
||\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}||_\diamond &\geq \sum_{i=0}^{1} \sqrt{\left(\langle\bar{u}_i|\rho_0|\bar{u}_i\rangle + \langle i|\rho_0|i\rangle\right)^2 - 4|\langle i|\rho_0|\bar{u}_i\rangle|^2} \\
&= \sum_{i=0}^{1} \sqrt{1 - |\langle i|U|i\rangle|^2} \\
&= \sum_{i=0}^{1} \sqrt{1 - |1\cdot\langle i|u_0\rangle\langle u_0|i\rangle + e^{i\phi}\cdot\langle i|u_1\rangle\langle u_1|i\rangle|^2} \\
&= \sum_{i=0}^{1} \sqrt{1 - \left|\frac{1+e^{i\phi}}{2}\right|^2} = 2\sqrt{1 - \left|\frac{1+e^{i\phi}}{2}\right|^2} \\
&= |1 - e^{i\phi}|.
\end{aligned} \tag{46}$$

Due to Theorem 1 and the following equality

$$\|\left(\mathbb{1}\otimes\sqrt{\rho}\right) J(\mathcal{P}_U - \mathcal{P}_{\mathbf{1}}) \left(\mathbb{1}\otimes\sqrt{\rho}\right)\|_1 = \left\|\left((\mathcal{P}_U - \mathcal{P}_{\mathbf{1}})\otimes\mathbb{1}\right)\left(|\sqrt{\rho}^\top\rangle\rangle\langle\langle\sqrt{\rho}^\top|\right)\right\|_1, \tag{47}$$

the discriminator $|\psi_0\rangle$ is equal to

$$|\psi_0\rangle = |\sqrt{\rho_0}^\top\rangle\rangle = \frac{1}{\sqrt{2}}|\mathbb{1}_2\rangle\rangle, \tag{48}$$

which completes the proof. $\square$

Proposition 2. *Consider the problem of discrimination between von Neumann measurements $\mathcal{P}_U$ and $\mathcal{P}_{\mathbf{1}}$, $U = H\mathrm{diag}(1, e^{i\phi})H^\dagger$ and $\phi \in [0, 2\pi)$. The controlled unitaries V_0 and V_1 have the form*

$$V_0 = \begin{pmatrix} i\sin\left(\frac{\pi-\phi}{4}\right) & -i\cos\left(\frac{\pi-\phi}{4}\right) \\ \cos\left(\frac{\pi-\phi}{4}\right) & \sin\left(\frac{\pi-\phi}{4}\right) \end{pmatrix}, \tag{49}$$

and

$$V_1 = \begin{pmatrix} -i\cos\left(\frac{\pi-\phi}{4}\right) & i\sin\left(\frac{\pi-\phi}{4}\right) \\ \sin\left(\frac{\pi-\phi}{4}\right) & \cos\left(\frac{\pi-\phi}{4}\right) \end{pmatrix}. \tag{50}$$

Proof. From Proposition 1 we obtain the exact form of discriminator given by

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|\mathbb{1}_2\rangle\rangle. \tag{51}$$

Repeating the procedure used to distinguish the von Neumann measurements in the Hadamard basis (see 2), we use the Hahn-Jordan decomposition and then we create the projective operators into the positive and negative part of X matrix. Hence, the explicit form of V_0 and V_1 is given as follows:

$$V_0 = \begin{pmatrix} i\sin\left(\frac{\pi-\phi}{4}\right) & -i\cos\left(\frac{\pi-\phi}{4}\right) \\ \cos\left(\frac{\pi-\phi}{4}\right) & \sin\left(\frac{\pi-\phi}{4}\right) \end{pmatrix}, \tag{52}$$

and

$$V_1 = \begin{pmatrix} -i\cos\left(\frac{\pi-\phi}{4}\right) & i\sin\left(\frac{\pi-\phi}{4}\right) \\ \sin\left(\frac{\pi-\phi}{4}\right) & \cos\left(\frac{\pi-\phi}{4}\right) \end{pmatrix}, \tag{53}$$

where $\phi \in [0, 2\pi)$. □

5. Output YAML files

In this appendix we present examples of YAML's files obtained from synchronous ans asynchronous experiments. We will start at synchronous case.

Listing 1: Defining experiment file

```
type: discrimination-fourier
qubits:
        - target: 0
          ancilla: 1
        - target: 1
          ancilla: 2
angles:
        start: 0
        stop: 2 * pi
        num_steps: 3
gateset: ibmq
method: direct_sum
num_shots: 100
```

Listing 2: Defining backend file

```
name: ibmq_quito
asynchronous: false
```

```
provider:
        hub: ibm-q
        group: open
        project: main
```

Listing 3: Results (synchronous)

```
metadata:
        experiments:
                type: discrimination-fourier
                qubits:
                - {target: 0, ancilla: 1}
                - {target: 1, ancilla: 2}
                angles: {start: 0.0, stop: 6.283185307179586, num_step
                gateset: ibmq
                method: direct_sum
                num_shots: 100
        backend_description:
                name: ibmq_quito
                asynchronous: false
                provider: {group: open, hub: ibm-q, project: main}
data:
- target: 0
  ancilla: 1
  phi: 0.0
  results_per_circuit:
  - name: id
        histogram: {'00': 28, '01': 26, '10': 21, '11': 25}
        mitigation_info:
                target: {prob_meas0_prep1: 0.0522000000000000024, prob_
                ancilla: {prob_meas0_prep1: 0.05900000000000005, prob_
        mitigated_histogram: {'00': 0.2637212373658018, '01': 0.258650
  - name: u
        histogram: {'00': 30, '01': 16, '10': 28, '11': 26}
        mitigation_info:
                target: {prob_meas0_prep1: 0.0522000000000000024, prob_
                ancilla: {prob_meas0_prep1: 0.05900000000000005, prob_
        mitigated_histogram: {'00': 0.2857378991684036, '01': 0.149752
- target: 0
  ancilla: 1
  phi: 3.141592653589793
  results_per_circuit:
```

```
– name: id
      histogram: {'00': 4, '01': 5, '10': 45, '11': 46}
      mitigation_info:
            target: {prob_meas0_prep1: 0.05220000000000024, prob_
            ancilla: {prob_meas0_prep1: 0.05900000000000005, prob_
      mitigated_histogram: {'00': 0.011053610583159325, '01': 0.0226
– name: u
      histogram: {'00': 56, '01': 43, '10': 1}
      mitigation_info:
            target: {prob_meas0_prep1: 0.05220000000000024, prob_
            ancilla: {prob_meas0_prep1: 0.05900000000000005, prob_
      mitigated_histogram: {'00': 0.5573987337172156, '01': 0.444247
– target: 0
  ancilla: 1
  phi: 6.283185307179586
  results_per_circuit:
  – name: id
      histogram: {'00': 36, '01': 18, '10': 25, '11': 21}
      mitigation_info:
            target: {prob_meas0_prep1: 0.05220000000000024, prob_
            ancilla: {prob_meas0_prep1: 0.05900000000000005, prob_
      mitigated_histogram: {'00': 0.3488190312089973, '01': 0.173552
  – name: u
      histogram: {'00': 32, '01': 27, '10': 24, '11': 17}
      mitigation_info:
            target: {prob_meas0_prep1: 0.05220000000000024, prob_
            ancilla: {prob_meas0_prep1: 0.05900000000000005, prob_
      mitigated_histogram: {'00': 0.3025357275361897, '01': 0.274136
– target: 1
  ancilla: 2
  phi: 0.0
  results_per_circuit:
  – name: id
      histogram: {'00': 27, '01': 20, '10': 24, '11': 29}
      mitigation_info:
            target: {prob_meas0_prep1: 0.05900000000000005, prob_m
            ancilla: {prob_meas0_prep1: 0.07540000000000002, prob_
      mitigated_histogram: {'00': 0.2594378169217188, '01': 0.193188
  – name: u
      histogram: {'00': 31, '01': 24, '10': 23, '11': 22}
      mitigation_info:
```

```
            target: {prob_meas0_prep1: 0.05900000000000005, prob_m
            ancilla: {prob_meas0_prep1: 0.07540000000000002, prob_
      mitigated_histogram: {'00': 0.30056875246775644, '01': 0.24382
- target: 1
  ancilla: 2
  phi: 3.141592653589793
  results_per_circuit:
  - name: id
      histogram: {'00': 5, '01': 4, '10': 50, '11': 41}
      mitigation_info:
            target: {prob_meas0_prep1: 0.05900000000000005, prob_m
            ancilla: {prob_meas0_prep1: 0.07540000000000002, prob_
      mitigated_histogram: {'00': 0.009552870928837118, '01': 0.0071
  - name: u
      histogram: {'00': 41, '01': 51, '10': 3, '11': 5}
      mitigation_info:
            target: {prob_meas0_prep1: 0.05900000000000005, prob_m
            ancilla: {prob_meas0_prep1: 0.07540000000000002, prob_
      mitigated_histogram: {'00': 0.4073387714165384, '01': 0.561461
- target: 1
  ancilla: 2
  phi: 6.283185307179586
  results_per_circuit:
  - name: id
      histogram: {'00': 30, '01': 28, '10': 23, '11': 19}
      mitigation_info:
            target: {prob_meas0_prep1: 0.05900000000000005, prob_m
            ancilla: {prob_meas0_prep1: 0.07540000000000002, prob_
      mitigated_histogram: {'00': 0.2868459834940102, '01': 0.291956
  - name: u
      histogram: {'00': 15, '01': 20, '10': 36, '11': 29}
      mitigation_info:
            target: {prob_meas0_prep1: 0.05900000000000005, prob_m
            ancilla: {prob_meas0_prep1: 0.07540000000000002, prob_
      mitigated_histogram: {'00': 0.1187719606657805, '01': 0.196208
```


For the same experiment file, we use the flag `asynchronous: true` to define asynchronous experiment.

Listing 4: Backend file

```
name: ibmq_quito
asynchronous: true
```

```
provider:
    hub: ibm-q
    group: open
    project: main
```

If the backend is asynchronous, the output will contain intermediate data containing, amongst others, job ids correlated with the circuit they correspond to.

Listing 5: Resolved results

```
metadata:
    experiments:
        type: discrimination-fourier
        qubits:
        - {target: 0, ancilla: 1}
        - {target: 1, ancilla: 2}
        angles: {start: 0.0, stop: 6.283185307179586, num_step
        gateset: ibmq
        method: direct_sum
        num_shots: 100
    backend_description:
        name: ibmq_quito
        asynchronous: true
        provider: {group: open, hub: ibm-q, project: main}
data:
- job_id: 63e7f17a17b7ed49ca24e05b
  keys:
  - [0, 1, id, 0.0]
  - [0, 1, u, 0.0]
  - [0, 1, id, 3.141592653589793]
  - [0, 1, u, 3.141592653589793]
  - [0, 1, id, 6.283185307179586]
  - [0, 1, u, 6.283185307179586]
  - [1, 2, id, 0.0]
  - [1, 2, u, 0.0]
  - [1, 2, id, 3.141592653589793]
  - [1, 2, u, 3.141592653589793]
  - [1, 2, id, 6.283185307179586]
  - [1, 2, u, 6.283185307179586]
```

Finally, if the status of jobs is `DONE`, we resolve the measurements from the submitted jobs obtaining the following file.

Listing 6: Results (asynchronous)


```
metadata:
  experiments:
        type: discrimination-fourier
        qubits:
        - target: 0
          ancilla: 1
        - target: 1
          ancilla: 2
        angles:
          start: 0.0
          stop: 6.283185307179586
          num_steps: 3
        gateset: ibmq
        method: direct_sum
        num_shots: 100
  backend_description:
        name: ibmq_quito
        asynchronous: true
        provider:
                group: open
                hub: ibm-q
                project: main
data:
- target: 0
  ancilla: 1
  phi: 0.0
  results_per_circuit:
  - name: id
        histogram:
                '00': 27
                '01': 28
                '10': 18
                '11': 27
        mitigation_info:
          target:
                prob_meas0_prep1: 0.052200000000000024
                prob_meas1_prep0: 0.0172
          ancilla:
                prob_meas0_prep1: 0.05900000000000005
```

```
            prob_meas1_prep0: 0.0202
        mitigated_histogram:
            '00': 0.254196166145997
            '01': 0.2790358060520916
            '10': 0.1732699847244092
            '11': 0.29349804307750227
  - name: u
        histogram:
            '00': 29
            '01': 17
            '10': 30
            '11': 24
        mitigation_info:
          target:
            prob_meas0_prep1: 0.05220000000000024
            prob_meas1_prep0: 0.0172
          ancilla:
            prob_meas0_prep1: 0.05900000000000005
            prob_meas1_prep0: 0.0202
        mitigated_histogram:
            '00': 0.2733793468261183
            '01': 0.1621115306717096
            '10': 0.3045273800167787
            '11': 0.2599817424853933
- target: 0
  ancilla: 1
  phi: 3.141592653589793
  results_per_circuit:
  - name: id
        histogram:
            '00': 3
            '01': 5
            '10': 37
            '11': 55
        mitigation_info:
          target:
            prob_meas0_prep1: 0.05220000000000024
            prob_meas1_prep0: 0.0172
          ancilla:
            prob_meas0_prep1: 0.05900000000000005
            prob_meas1_prep0: 0.0202
```

```
        mitigated_histogram:
                '00': 0.006189545789708441
                '01': 0.016616709640352317
                '10': 0.3675478279476653
                '11': 0.6096459166222741
  - name: u
        histogram:
                '00': 56
                '01': 42
                '10': 2
        mitigation_info:
          target:
                prob_meas0_prep1: 0.05220000000000024
                prob_meas1_prep0: 0.0172
          ancilla:
                prob_meas0_prep1: 0.05900000000000005
                prob_meas1_prep0: 0.0202
        mitigated_histogram:
                '00': 0.55731929321128
                '01': 0.43367489257574243
                '10': 0.009005814212977551
- target: 0
  ancilla: 1
  phi: 6.283185307179586
  results_per_circuit:
  - name: id
        histogram:
                '00': 18
                '01': 28
                '10': 30
                '11': 24
        mitigation_info:
          target:
                prob_meas0_prep1: 0.05220000000000024
                prob_meas1_prep0: 0.0172
          ancilla:
                prob_meas0_prep1: 0.05900000000000005
                prob_meas1_prep0: 0.0202
        mitigated_histogram:
                '00': 0.15258295844557557
                '01': 0.2829079190522524
```

```
                '10': 0.3071204587046501
                '11': 0.25738866379752195
    - name: u
          histogram:
                '00': 32
                '01': 28
                '10': 23
                '11': 17
          mitigation_info:
            target:
                prob_meas0_prep1: 0.05220000000000024
                prob_meas1_prep0: 0.0172
            ancilla:
                prob_meas0_prep1: 0.05900000000000005
                prob_meas1_prep0: 0.0202
          mitigated_histogram:
                '00': 0.3026150836796529
                '01': 0.28491749668524724
                '10': 0.23230862145681827
                '11': 0.18015879817828173
  - target: 1
    ancilla: 2
    phi: 0.0
    results_per_circuit:
    - name: id
          histogram:
                '00': 27
                '01': 16
                '10': 30
                '11': 27
          mitigation_info:
            target:
                prob_meas0_prep1: 0.05900000000000005
                prob_meas1_prep0: 0.0202
            ancilla:
                prob_meas0_prep1: 0.07540000000000002
                prob_meas1_prep0: 0.0528
          mitigated_histogram:
                '00': 0.256742095057232
                '01': 0.15000257115061383
                '10': 0.2982101204075811б
```

```
            '11': 0.29504521338457296
  - name: u
      histogram:
            '00': 34
            '01': 22
            '10': 25
            '11': 19
      mitigation_info:
       target:
            prob_meas0_prep1: 0.05900000000000005
            prob_meas1_prep0: 0.0202
        ancilla:
            prob_meas0_prep1: 0.07540000000000002
            prob_meas1_prep0: 0.0528
      mitigated_histogram:
            '00': 0.3325088211394024
            '01': 0.22335261496979697
            '10': 0.2441636375921354
            '11': 0.19997492629866526
- target: 1
  ancilla: 2
  phi: 3.141592653589793
  results_per_circuit:
  - name: id
      histogram:
            '00': 3
            '01': 9
            '10': 51
            '11': 37
      mitigation_info:
        target:
            prob_meas0_prep1: 0.05900000000000005
            prob_meas1_prep0: 0.0202
        ancilla:
            prob_meas0_prep1: 0.07540000000000002
            prob_meas1_prep0: 0.0528
      mitigated_histogram:
            '00': -0.016627023111853642
            '01': 0.06778554570877951
            '10': 0.53899887367658
            '11': 0.40984260372649417
```

```
  - name: u
      histogram:
          '00': 43
          '01': 45
          '10': 7
          '11': 5
      mitigation_info:
        target:
          prob_meas0_prep1: 0.05900000000000005
          prob_meas1_prep0: 0.0202
        ancilla:
          prob_meas0_prep1: 0.07540000000000002
          prob_meas1_prep0: 0.0528
      mitigated_histogram:
          '00': 0.42955729968594086
          '01': 0.49336080079582095
          '10': 0.04937406434533623
          '11': 0.02770783517290191
- target: 1
  ancilla: 2
  phi: 6.283185307179586
  results_per_circuit:
  - name: id
      histogram:
          '00': 22
          '01': 19
          '10': 35
          '11': 24
      mitigation_info:
        target:
          prob_meas0_prep1: 0.05900000000000005
          prob_meas1_prep0: 0.0202
        ancilla:
          prob_meas0_prep1: 0.07540000000000002
          prob_meas1_prep0: 0.0528
      mitigated_histogram:
          '00': 0.19592641048040849
          '01': 0.18787721420415215
          '10': 0.3590258049844047
          '11': 0.25717057033103463
  - name: u
```

```
        histogram:
        '00': 27
        '01': 24
        '10': 25
        '11': 24
mitigation_info:
  target:
        prob_meas0_prep1: 0.05900000000000005
        prob_meas1_prep0: 0.0202
  ancilla:
        prob_meas0_prep1: 0.07540000000000002
        prob_meas1_prep0: 0.0528
mitigated_histogram:
        '00': 0.2555586681758725
        '01': 0.2429501641251142
        '10': 0.24509293912212946
        '11': 0.2563982285768841
```

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