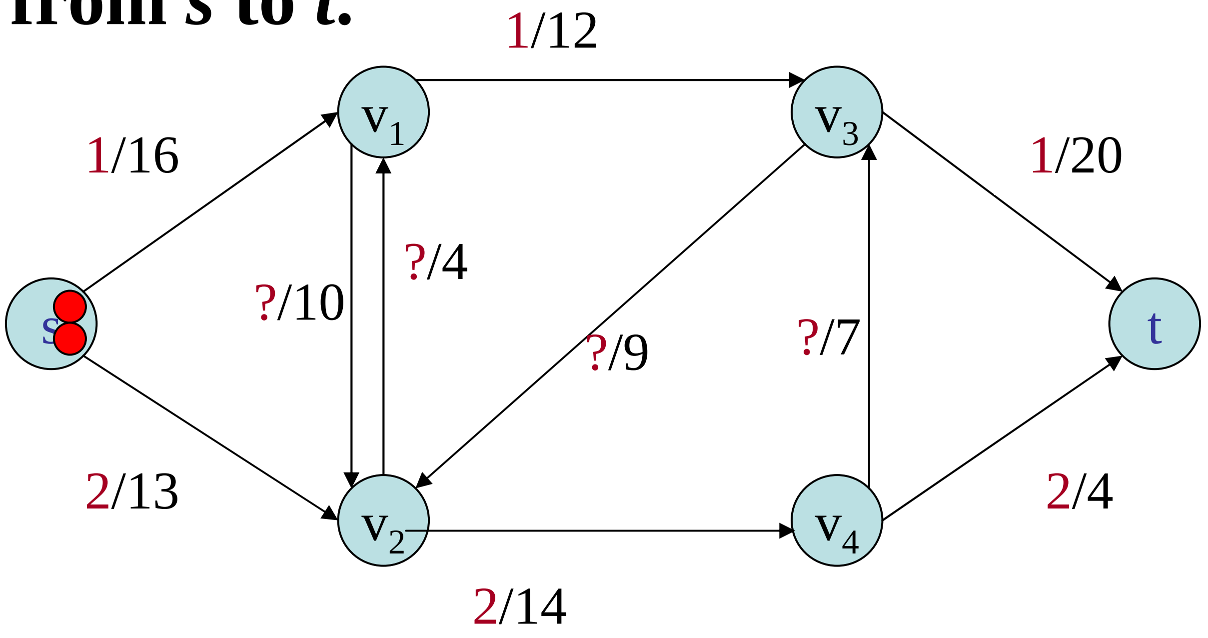
Graph Problems

- Critical Path Analysis
- Maximum Flow Problem
- Other Difficult Problems

Maximum Flow Problem

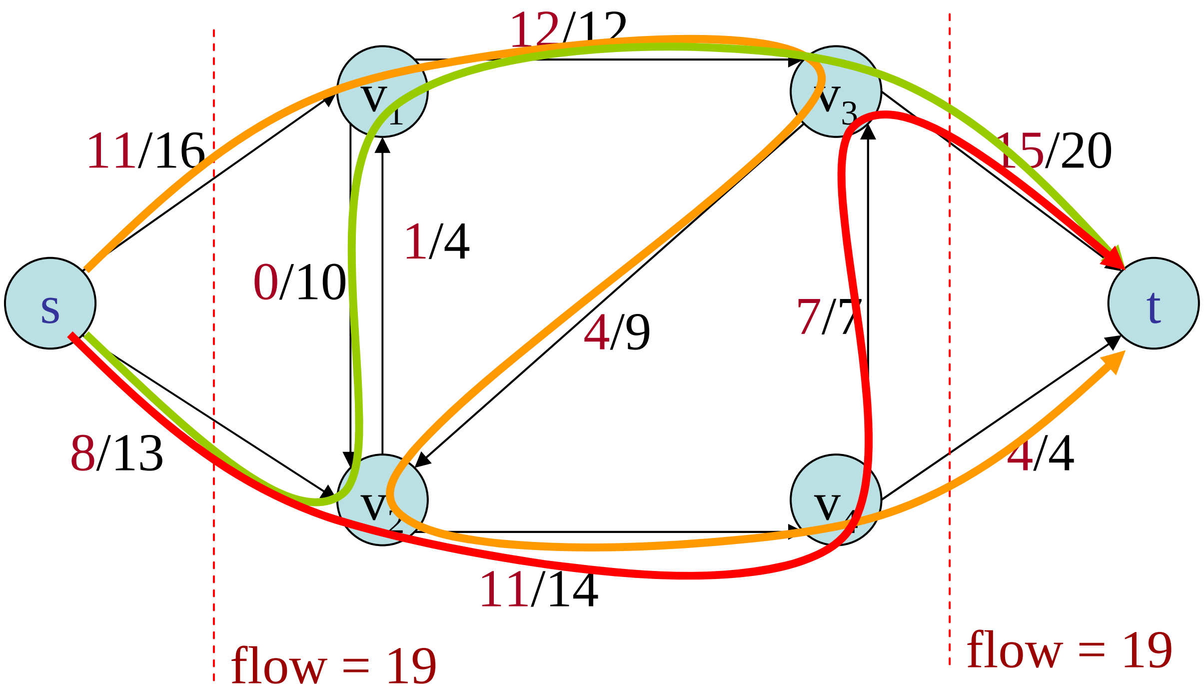
- We are given a **flow network** G with **source** *s* and **sink** *t*, and we wish to find a flow of *maximum value* from *s* to *t*.



flow f(u,v) / capacity c(u,v)

Maximum Flow Problem

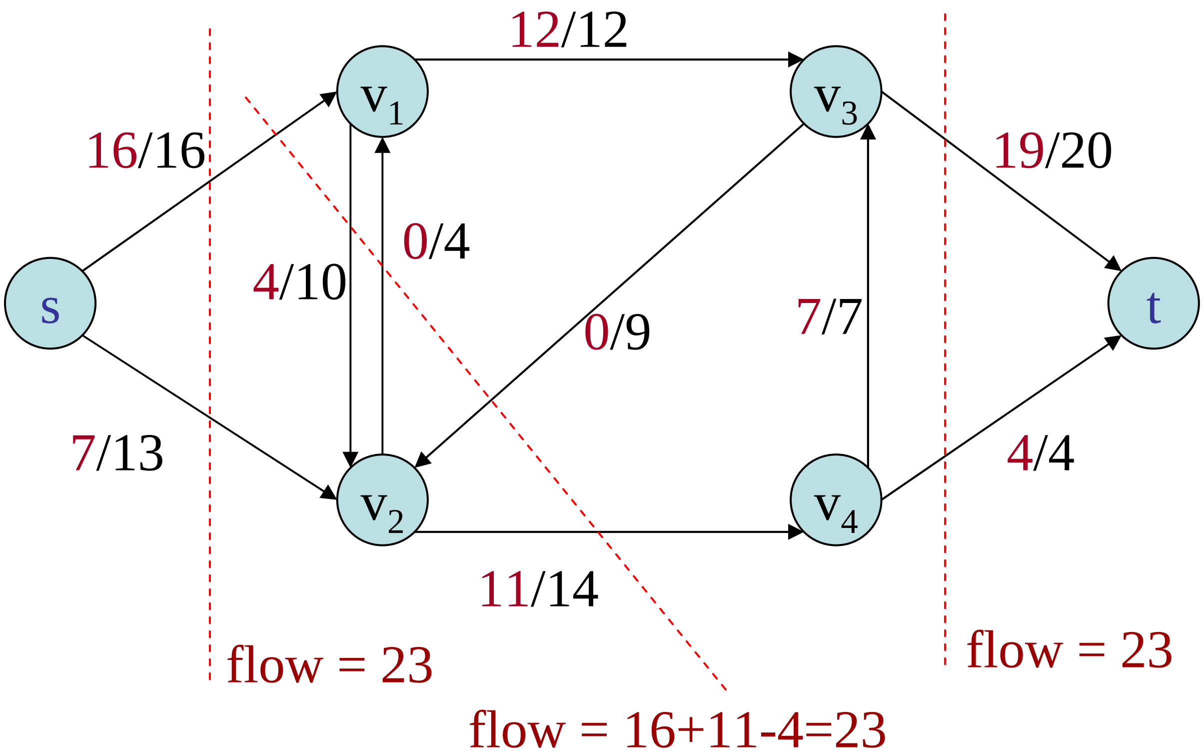
- ***Single-source single-sink*** **maximum flow problem**
- ***Maximum-flow min-cut*** **theorem**



flow f(u,v) / capacity c(u,v)

Maximum Flow Problem

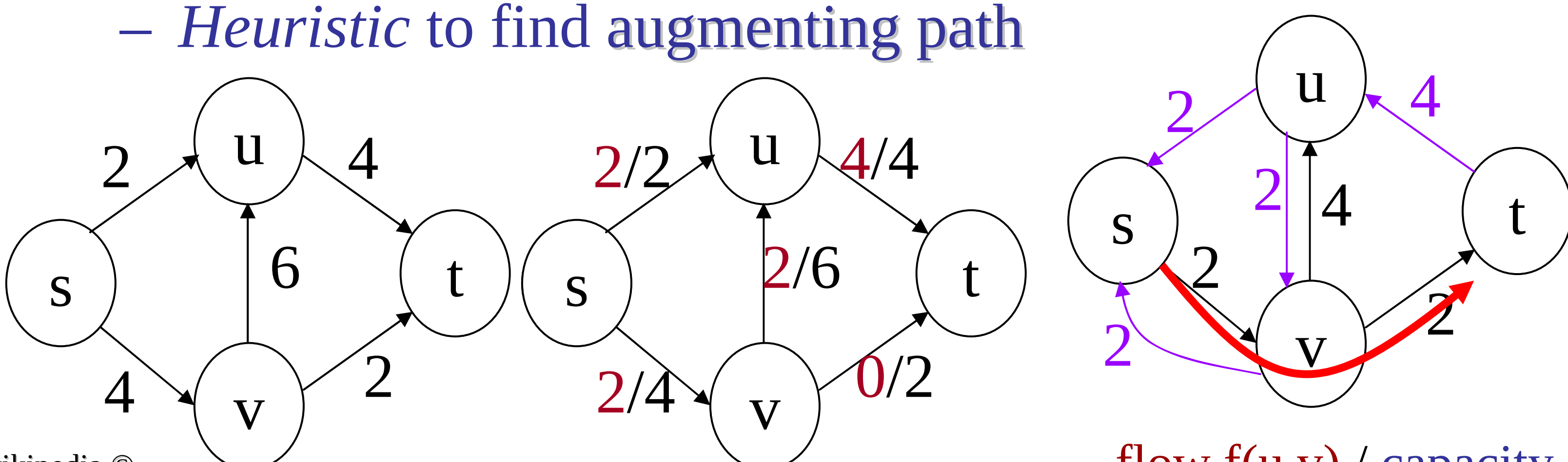
- ***What is the maximum flow?***



flow f(u,v) / capacity c(u,v)

Maximum Flow Problem: *Background*

- **A simplified model of Soviet railway traffic flow**
  - Formulated by T.E. Harris 1954
- ***Ford-Fulkerson* algorithm, 1955**
  - Residual graph
    - **residual capacity:** $c_f(u,v) = c(u,v) - f(u,v)$, $c_f(v,u) = c(v,u) - f(v,u)$
- ***Edmonds-Karp* algorithm, 1972**
  - *Heuristic* to find augmenting path



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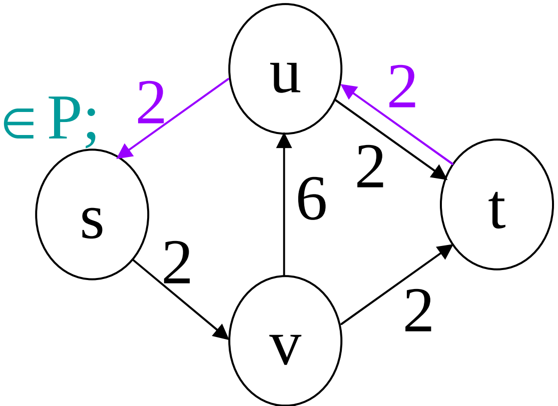
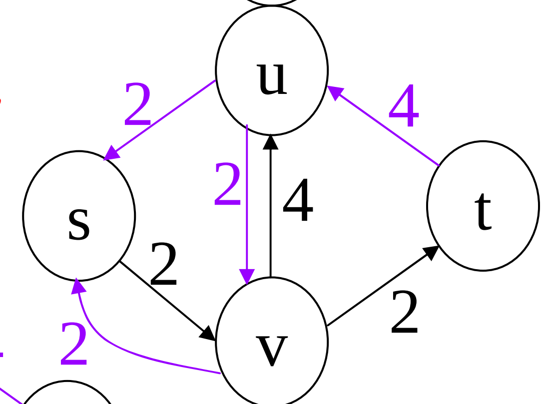
flow f(u,v) / capacity c(u,v)

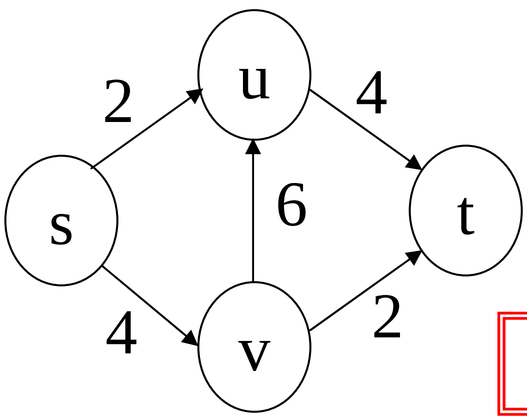
Ford-Fulkerson algorithm

1. Initialize $c_f(u,v)$ for every edge;
2. Find a path P from s to t ∍ $c_f(u,v)>0\ \forall(u,v)\in P$;
3. $c_f(P) = \mathit{min}\{c_f(u,v): (u,v)\in P\}$;
4. For each edge $(u,v)\in P$
   - **$c_f(u,v) = c_f(u,v) - c_f(P)$;**
   - **$c_f(v,u) = c_f(v,u) + c_f(P)$;**

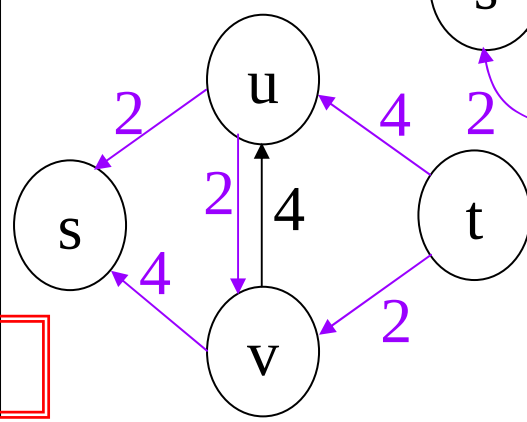




maximum flow
= 4 + 2 = 6



c_f	s	u	v	t
s	0	0	0	0
u	2	0	2	0
v	4	4	0	0
t	0	4	2	0

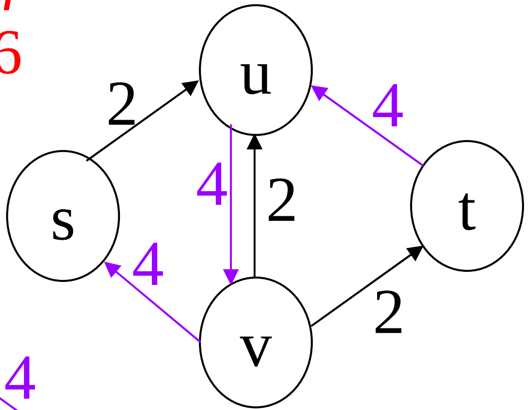


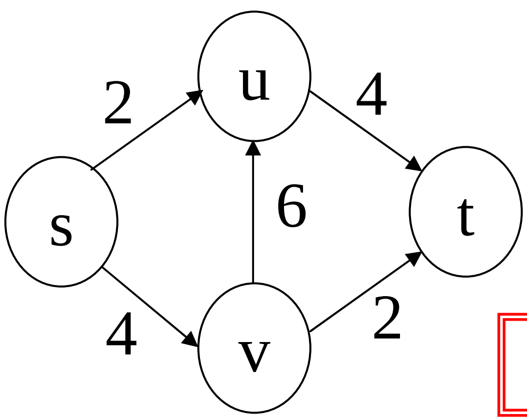
Edmonds-Karp algorithm

1. Initialize $c_f(u,v)$ for every edge;
2. Find a path P from s to t by a heuristic;
3. $c_f(P) = \mathit{min}\{c_f(u,v): (u,v) \in P\}$;
4. For each edge $(u,v) \in P$
   – **$c_f(u,v) = c_f(u,v) - c_f(P)$;**
   – **$c_f(v,u) = c_f(v,u) + c_f(P)$;**

Heuristic 1. *max-capacity first*

maximum flow
= 4 + 2 = 6





c_f	s	u	v	t
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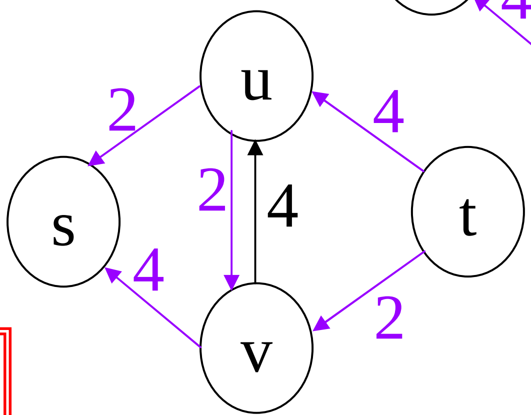

Edmonds-Karp algorithm

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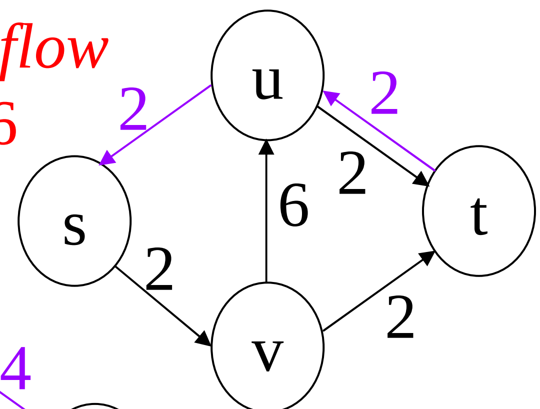
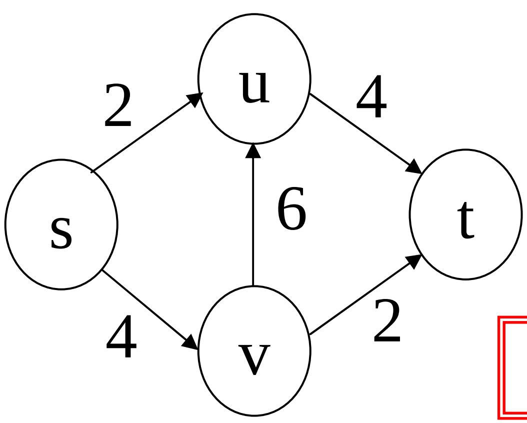
   - **$c_f(u,v) = c_f(u,v) - c_f(P)$;**
   - **$c_f(v,u) = c_f(v,u) + c_f(P)$;**

Heuristic 1. *max-capacity first*

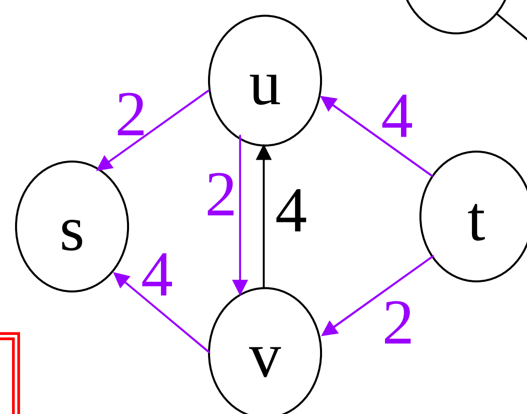
Heuristic 2. *breadth first*

maximum flow
= 4 + 2 = 6

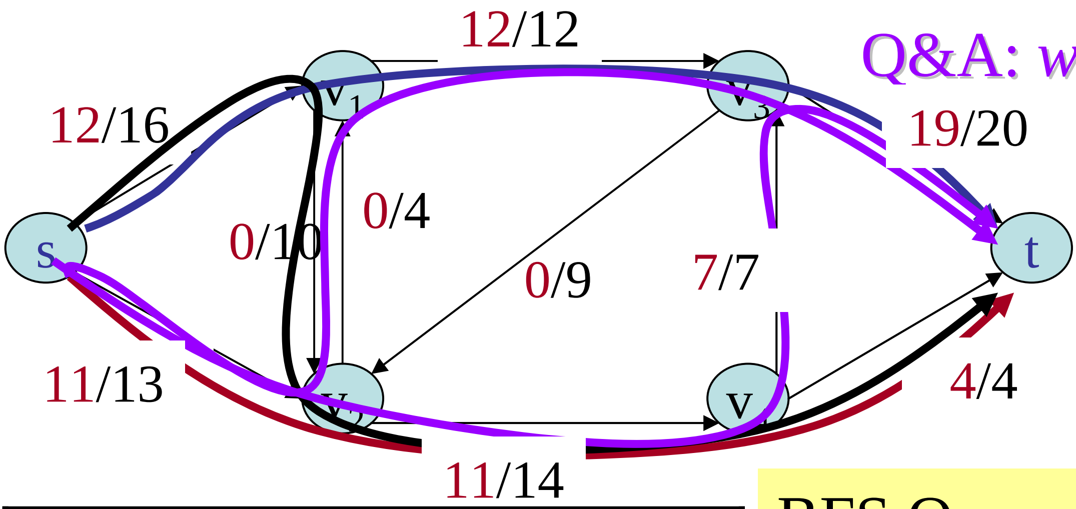




c_f	s	u	v	t
s	0	0	0	0
u	2	0	2	0
v	4	4	0	0
t	0	4	2	0



Maximum Flow Problem: *Example*



Q&A: what is the next path from s to t?

$t \leftarrow v_3 \leftarrow v_1 \leftarrow v_2$

$t \leftarrow v_3 \leftarrow v_4 \leftarrow v_2$

	s	**v_1**	**v_2**	**v_3**	**v_4**	**t**
s	0	16	13	0	0	0
v_1	0	0	10	12	0	0
v_2	0	4	0	0	14	0
v_3	0	0	9	0	0	20
v_4	0	0	0	7	0	4
t	0	0	0	0	0	0

BFS Queue: $t \leftarrow v_3 \leftarrow v_1$

$v_2 \leftarrow v_3 \leftarrow v_1$

$v_3 \leftarrow v_1 \leftarrow v_2$

$t \leftarrow v_4 \leftarrow v_2$

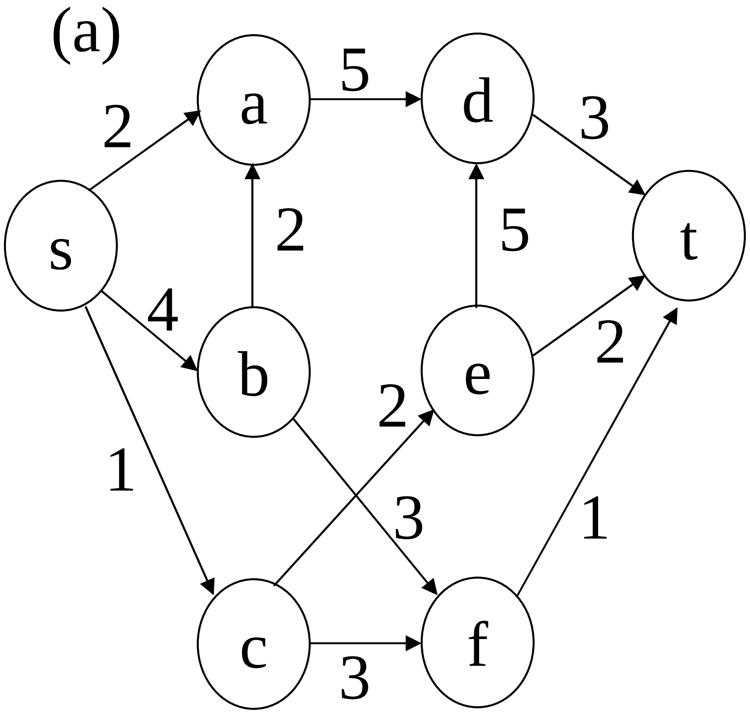
$v_3 \leftarrow v_4 \leftarrow v_2$

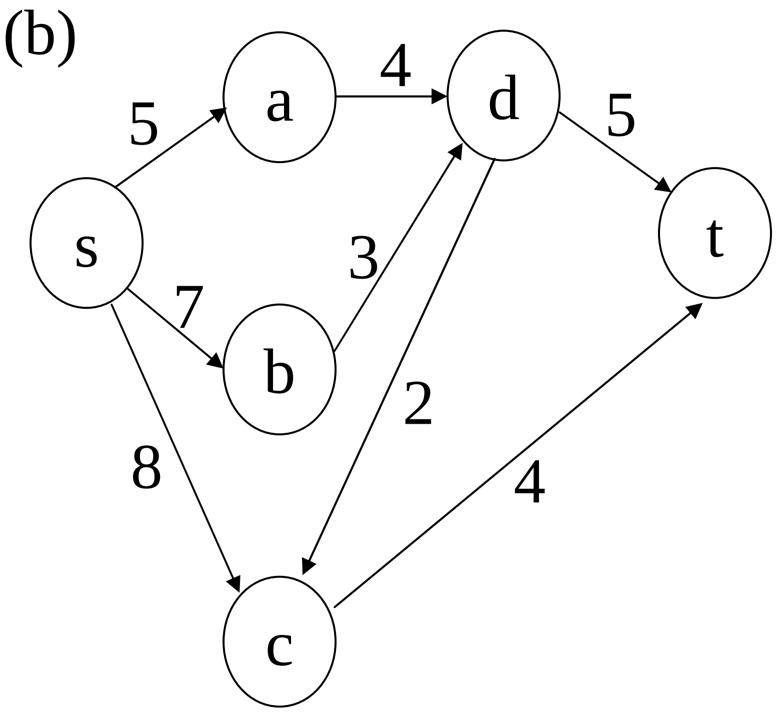
$t \leftarrow v_4 \leftarrow v_2 \leftarrow v_1$

$v_3 \leftarrow v_4 \leftarrow v_2 \leftarrow v_1$

Practice 12: *Maximum Flow*

- Find the maximum flows in the following networks





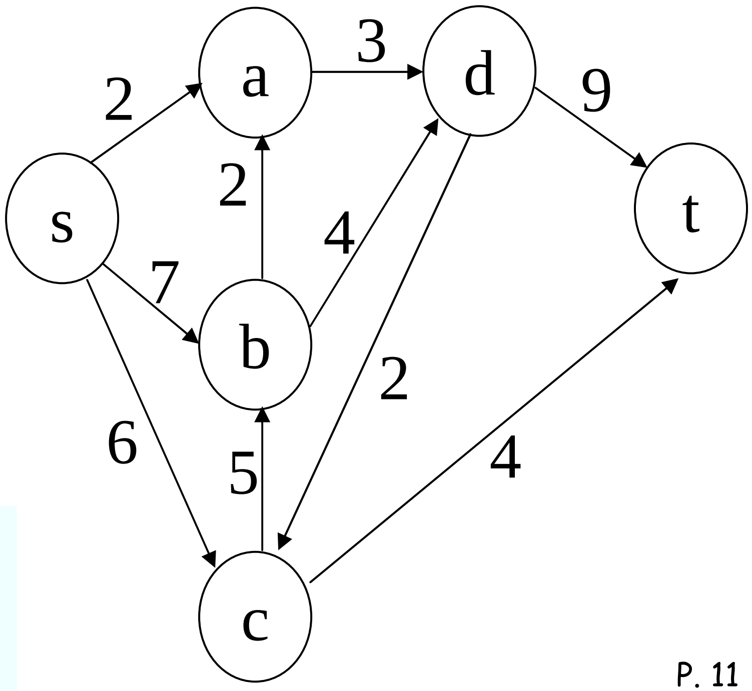
Self-exercise 8

1. Find the maximum flow by *max-capacity edge first*.

(a) What is the first path from s to t that creates a flow?

(b) What is the second path from s to t that creates a flow?

(c) What is the maximum flow?



If you have multiple choices,
visit the vertex with the smallest label first.