Autoregressive Model as a State Space Model

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1 Measurement Model

The measurement model is given by

$$Y = \nu + \lambda \eta + \varepsilon \quad \text{with} \quad \varepsilon \sim \mathcal{N}\left(0, \theta^2\right) \tag{1}$$

where Y, η, and ε are random variables and ν, λ, and θ^2 are model parameters. Y is the observed random variable, η is the latent random variable, and ε is the random measurement error while ν is the intercept, λ is the factor loading, and θ^2 is the variance of ε. For the autoregressive model some constraints are applied specifically $\nu = 0$, $\lambda = 1$, $\varepsilon = 0$. Such that

$$Y = \eta. \tag{2}$$

1.1 Example

The measurement model is given by

$$\begin{array}{ccccccc} Y & = & \nu & + & \lambda\eta & + & \varepsilon \\ Y & = & 0 & + & 1\eta & + & 0 \\ Y & = & & & \eta. & & \end{array} \tag{3}$$

2 Dynamic Structure

The dynamic structure is given by

$$\eta_t = \alpha + \boldsymbol{\beta} \boldsymbol{\eta}_l + \zeta_t \quad \text{with} \quad \zeta_t \sim \mathcal{N}\left(0, \psi^2\right) \tag{4}$$

where η_t, $\boldsymbol{\eta}_l$, and ζ_t are random variables and α, $\boldsymbol{\beta}$, and ψ^2 are model parameters. η_t is the latent variable at time t, $\boldsymbol{\eta}_l$ is a vector of latent variables at lags $l = \{1, \ldots, p\}$, and ζ_t is the dynamic noise at time t while α is the intercept, $\boldsymbol{\beta}$ is a row vector of autoregression coefficients, and ψ^2 is the variance of ζ_t.

2.1 Example

The random variables and the parameters are given below for $p = 2$.

$$\eta_t \tag{5}$$

$$\boldsymbol{\eta}_l = \begin{pmatrix} \eta_{t-1} \\ \eta_{t-2} \end{pmatrix} \tag{6}$$

$$\alpha \tag{7}$$

$$\beta = \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \tag{8}$$

$$\begin{array}{c|ccc|} \beta_1 & \eta_t & \text{regressed on} & \eta_{t-1} \\ \beta_2 & \eta_t & \text{regressed on} & \eta_{t-2} \end{array} \tag{9}$$

$$\psi^2 \tag{10}$$

3 Initial Condition

In the state space model, the initial value of the latent variable η given by $\eta_{|0}$ needs to be specified. We can sample from a particular distribution, for example, the normal distribution as follows

$$\eta_{|0} \sim \mathcal{N}\left(\mu_0, \sigma_0^2\right). \tag{11}$$