Vector Autoregressive Model as a State Space Model

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1 Measurement Model

The general form of the measurement model is given by

$$\mathbf{y} = \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon} \quad \text{with} \quad \boldsymbol{\varepsilon} \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Theta}\right) \tag{1}$$

where $\mathbf{y}$, $\boldsymbol{\eta}$, and $\boldsymbol{\varepsilon}$ are random variables and $\boldsymbol{\nu}$, $\boldsymbol{\Lambda}$, and $\boldsymbol{\Theta}$ are model parameters. $\mathbf{y}$ is a vector of observed random variables, $\boldsymbol{\eta}$ is a vector of latent random variables, and $\boldsymbol{\varepsilon}$ is a vector of random measurement errors while $\boldsymbol{\nu}$ is a vector of intercept, $\boldsymbol{\Lambda}$ is a matrix of factor loadings, and $\boldsymbol{\Theta}$ is the covariance matrix of $\boldsymbol{\varepsilon}$. For the vector autoregressive model some constraints are applied specifically $\boldsymbol{\nu} = \mathbf{0}$, $\boldsymbol{\Lambda} = \mathbf{I}$, $\boldsymbol{\varepsilon} = \mathbf{0}$. Such that

$$\mathbf{y} = \boldsymbol{\eta}. \tag{2}$$

1.1 Three Variable Model Example

Let $\mathbf{y} = \{Y_1, Y_2, Y_3\}'$. The measurement model is given by

$$\begin{array}{ccccccc}
\mathbf{y} & = & \boldsymbol{\nu} & + & \boldsymbol{\Lambda}\boldsymbol{\eta} & + & \boldsymbol{\varepsilon} \\
\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} & = & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & + & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} & + & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} & = & & & \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}. & &
\end{array} \tag{3}$$

2 Dynamic Structure

The general form of the dynamic structure is given by

$$\boldsymbol{\eta}_t = \boldsymbol{\alpha} + \boldsymbol{\beta}\boldsymbol{\eta}_l + \boldsymbol{\zeta}_t \quad \text{with} \quad \boldsymbol{\zeta}_t \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Psi}\right) \tag{4}$$

where $\boldsymbol{\eta}_t$, $\boldsymbol{\eta}_l$, and $\boldsymbol{\zeta}_t$ are random variables and $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\Psi}$ are model parameters. $\boldsymbol{\eta}_t$ is a vector of latent variables at time t, $\boldsymbol{\eta}_l$ is a vector of latent variables at lags $l = \{1, \ldots, p\}$, and $\boldsymbol{\zeta}_t$ is a vector of dynamic noise at time t while $\boldsymbol{\alpha}$ is a vector of intercepts, $\boldsymbol{\beta}$ is a matrix of autoregression and cross regression coefficients, and $\boldsymbol{\Psi}$ is the covariance matrix of $\boldsymbol{\zeta}_t$.

2.1 Three Variable Model Example

Let $\boldsymbol{\eta}_t = \{\eta_{1_t}, \eta_{2_t}, \eta_{3_t}\}'$. The random variables and the parameters are given below.

$$\boldsymbol{\eta}_t = \begin{pmatrix} \eta_{1_t} \\ \eta_{2_t} \\ \eta_{3_t} \end{pmatrix} \tag{5}$$

$$\boldsymbol{\eta}_l = \begin{pmatrix} \eta_{1_{t-1}} \\ \eta_{2_{t-1}} \\ \eta_{3_{t-1}} \\ \eta_{1_{t-2}} \\ \eta_{2_{t-2}} \\ \eta_{3_{t-2}} \end{pmatrix} \tag{6}$$

$$\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{7}$$

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_1 & \beta_2 & 0 & 0 & 0 & \beta_3 \\ 0 & \beta_4 & \beta_5 & 0 & 0 & 0 \\ 0 & 0 & \beta_6 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

$$\begin{array}{r|lcl|} \beta_1 & \eta_{1_t} & \text{regressed on} & \eta_{1_{t-1}} \\ \beta_2 & \eta_{1_t} & \text{regressed on} & \eta_{2_{t-1}} \\ \beta_3 & \eta_{1_t} & \text{regressed on} & \eta_{3_{t-2}} \\ \beta_4 & \eta_{2_t} & \text{regressed on} & \eta_{2_{t-1}} \\ \beta_5 & \eta_{2_t} & \text{regressed on} & \eta_{3_{t-1}} \\ \beta_6 & \eta_{3_t} & \text{regressed on} & \eta_{3_{t-1}} \end{array} \tag{9}$$

$$\boldsymbol{\Psi} = \begin{pmatrix} \psi_{1,1} & 0 & 0 \\ 0 & \psi_{2,2} & 0 \\ 0 & 0 & \psi_{3,3} \end{pmatrix} \tag{10}$$

3 Initial Condition

In the state space model, the initial values of the latent variables $\boldsymbol{\eta}$ given by $\boldsymbol{\eta}_{|0}$ need to be specified. We can sample from a particular distribution, for example, the multivariate normal distribution as follows

$$\boldsymbol{\eta}_{|0} \sim \mathcal{N}\left(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0\right) \tag{11}$$

where $\boldsymbol{\mu}_0$ is a vector of means and $\boldsymbol{\Sigma}_0$ is a covariance matrix.

3.1 Three Variable Model Example

The mean vector and covariance matrix are given by

$$\boldsymbol{\mu}_0 = \begin{pmatrix} \mu_{0_1} \\ \mu_{0_2} \\ \mu_{0_3} \end{pmatrix}, \quad \text{and} \tag{12}$$

$$\boldsymbol{\Sigma}_0 = \begin{pmatrix} \sigma_{0_{1,1}} & \sigma_{0_{1,2}} & \sigma_{0_{1,3}} \\ \sigma_{0_{2,1}} & \sigma_{0_{2,2}} & \sigma_{0_{2,3}} \\ \sigma_{0_{3,1}} & \sigma_{0_{3,2}} & \sigma_{0_{3,3}} \end{pmatrix}. \tag{13}$$