

Vector Autoregressive Model as a State Space Model

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1 Measurement Model

The general form of the measurement model is given by

$$\mathbf{y} = \boldsymbol{\nu} + \mathbf{\Lambda}\boldsymbol{\eta} + \boldsymbol{\varepsilon} \quad \text{with} \quad \boldsymbol{\varepsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Theta}) \quad (1)$$

where \mathbf{y} , $\boldsymbol{\eta}$, and $\boldsymbol{\varepsilon}$ are random variables and $\boldsymbol{\nu}$, $\mathbf{\Lambda}$, and $\boldsymbol{\Theta}$ are model parameters. \mathbf{y} is a vector of observed random variables, $\boldsymbol{\eta}$ is a vector of latent random variables, and $\boldsymbol{\varepsilon}$ is a vector of random measurement errors while $\boldsymbol{\nu}$ is a vector of intercept, $\mathbf{\Lambda}$ is a matrix of factor loadings, and $\boldsymbol{\Theta}$ is the covariance matrix of $\boldsymbol{\varepsilon}$. For the vector autoregressive model some constraints are applied specifically $\boldsymbol{\nu} = \mathbf{0}$, $\mathbf{\Lambda} = \mathbf{I}$, $\boldsymbol{\varepsilon} = \mathbf{0}$. Such that

$$\mathbf{y} = \boldsymbol{\eta}. \quad (2)$$

1.1 Three Variable Model Example

Let $\mathbf{y} = \{Y_1, Y_2, Y_3\}'$. The measurement model is given by

$$\begin{aligned} \begin{pmatrix} \mathbf{y} \\ Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\nu} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{\Lambda}\boldsymbol{\eta} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta} \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\varepsilon} \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} &= \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}. \end{aligned} \quad (3)$$

2 Dynamic Structure

The general form of the dynamic structure is given by

$$\boldsymbol{\eta}_t = \boldsymbol{\alpha} + \boldsymbol{\beta}\boldsymbol{\eta}_l + \boldsymbol{\zeta}_t \quad \text{with} \quad \boldsymbol{\zeta}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Psi}) \quad (4)$$

where $\boldsymbol{\eta}_t$, $\boldsymbol{\eta}_l$, and $\boldsymbol{\zeta}_t$ are random variables and $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\Psi}$ are model parameters. $\boldsymbol{\eta}_t$ is a vector of latent variables at time t , $\boldsymbol{\eta}_l$ is a vector of latent variables

at lags $l = \{1, \dots, p\}$, and ζ_t is a vector of dynamic noise at time t while α is a vector of intercepts, β is a matrix of autoregression and cross regression coefficients, and Ψ is the covariance matrix of ζ_t .

2.1 Three Variable Model Example

Let $\eta_t = \{\eta_{1t}, \eta_{2t}, \eta_{3t}\}'$. The random variables and the parameters are given below.

$$\eta_t = \begin{pmatrix} \eta_{1t} \\ \eta_{2t} \\ \eta_{3t} \end{pmatrix} \quad (5)$$

$$\eta_l = \begin{pmatrix} \eta_{1t-1} \\ \eta_{2t-1} \\ \eta_{3t-1} \\ \eta_{1t-2} \\ \eta_{2t-2} \\ \eta_{3t-2} \end{pmatrix} \quad (6)$$

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \quad (7)$$

$$\beta = \begin{pmatrix} \beta_1 & \beta_2 & 0 & 0 & 0 & \beta_3 \\ 0 & \beta_4 & \beta_5 & 0 & 0 & 0 \\ 0 & 0 & \beta_6 & 0 & 0 & 0 \end{pmatrix} \quad (8)$$

$$\begin{array}{l|l} \beta_1 & \eta_{1t} \text{ regressed on } \eta_{1t-1} \\ \beta_2 & \eta_{1t} \text{ regressed on } \eta_{2t-1} \\ \beta_3 & \eta_{1t} \text{ regressed on } \eta_{3t-2} \\ \beta_4 & \eta_{2t} \text{ regressed on } \eta_{2t-1} \\ \beta_5 & \eta_{2t} \text{ regressed on } \eta_{3t-1} \\ \beta_6 & \eta_{3t} \text{ regressed on } \eta_{3t-1} \end{array} \quad (9)$$

$$\Psi = \begin{pmatrix} \psi_{1,1} & 0 & 0 \\ 0 & \psi_{2,2} & 0 \\ 0 & 0 & \psi_{3,3} \end{pmatrix} \quad (10)$$

3 Initial Condition

In the state space model, the initial values of the latent variables η given by $\eta_{|0}$ need to be specified. We can sample from a particular distribution, for example, the multivariate normal distribution as follows

$$\eta_{|0} \sim \mathcal{N}(\mu_0, \Sigma_0) \quad (11)$$

where μ_0 is a vector of means and Σ_0 is a covariance matrix.

3.1 Three Variable Model Example

The mean vector and covariance matrix are given by

$$\boldsymbol{\mu}_0 = \begin{pmatrix} \mu_{0_1} \\ \mu_{0_2} \\ \mu_{0_3} \end{pmatrix}, \quad \text{and} \quad (12)$$

$$\boldsymbol{\Sigma}_0 = \begin{pmatrix} \sigma_{0_1,1} & \sigma_{0_1,2} & \sigma_{0_1,3} \\ \sigma_{0_2,1} & \sigma_{0_2,2} & \sigma_{0_2,3} \\ \sigma_{0_3,1} & \sigma_{0_3,2} & \sigma_{0_3,3} \end{pmatrix}. \quad (13)$$