1 Introduction

The goal of this iteration is to replace the stacked linear axes with a pair of rotating arms (figure 1). This document serves as theoretical foundation and proof-of-concept for this idea.

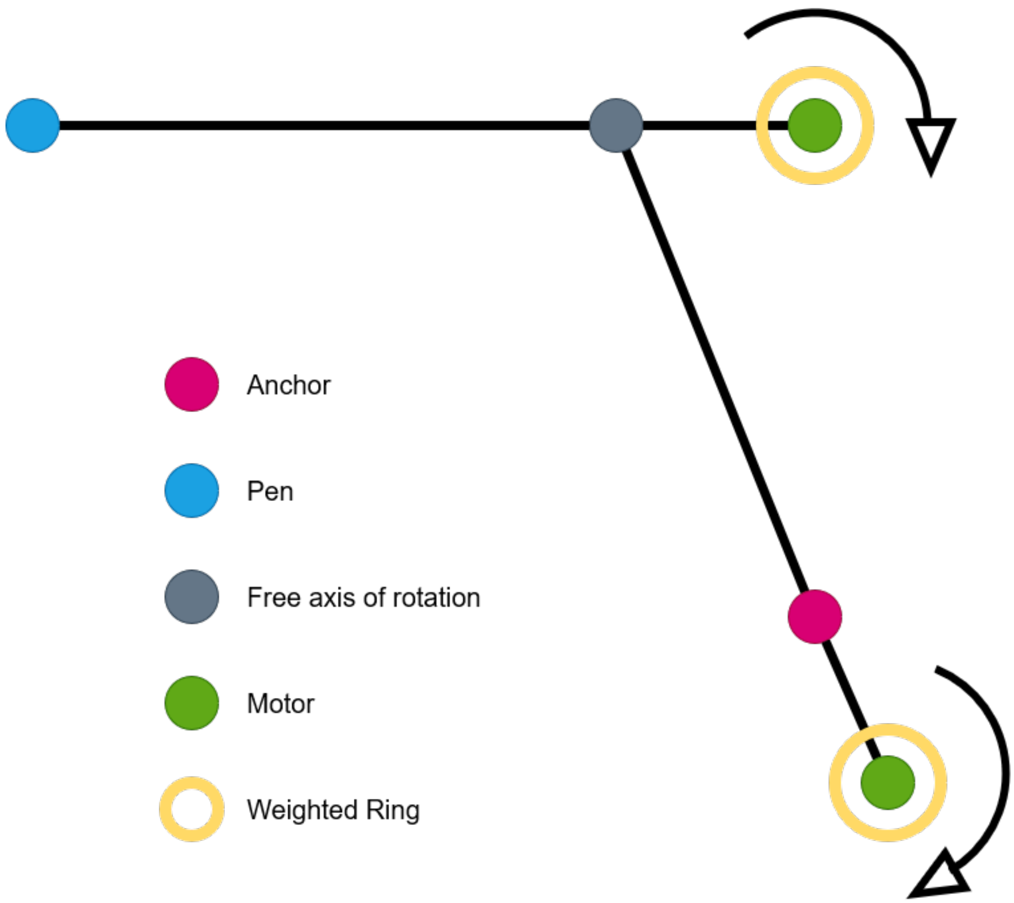

Figure 1: Birds-eye concept drawing of a gyroscopic fern.

2 Setup

To start, consider a single arm (figure 2). It has a motor on one end and an ambiguous mass on the other (e.g. a pen or another arm).

Assume the motor and supporting rod are massless. With the system at rest, the angular momentum is zero. Suppose the motor begins to spin such that the angular rotation about the rotor axis is ω. We want to predict the resulting arm motion. The conservation of angular momentum says that the change in angular momentum of a system (lacking an

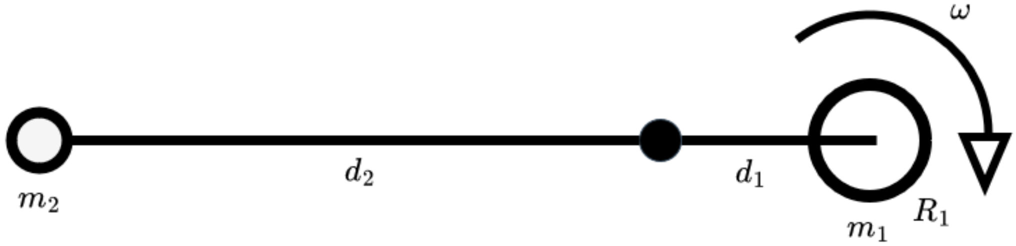

Figure 2: Body diagram for one arm.

external torques) is zero. To express the rotational velocity of the arm as a function of motor speed, we can equate the angular momentum of the motor with the angular momentum of the arm.

3 Calculation

3.1 Angular Momentum from the Spinning Ring

By definition...

$$\begin{aligned} L &= \vec{r} \times \vec{p} && (1) \\ &= \vec{r} \times m\vec{v} && (2) \\ &= \int_m \vec{r} \times \vec{v} dm && (3) \\ &= \int_\theta \vec{r} \times \rho \vec{v} d\theta && (4) \end{aligned}$$

Where...

$$\rho = \frac{m_1}{2\pi} \tag{5}$$

The components of this integral are shown in figure 3

Continuing...

$$L = \rho \int_0^{2\pi} |\vec{r}(\theta)||\vec{v}| sin(\phi(\theta)) d\theta \tag{6}$$

Where..

$$\phi = \angle \vec{v} - \angle \vec{r} \tag{7}$$

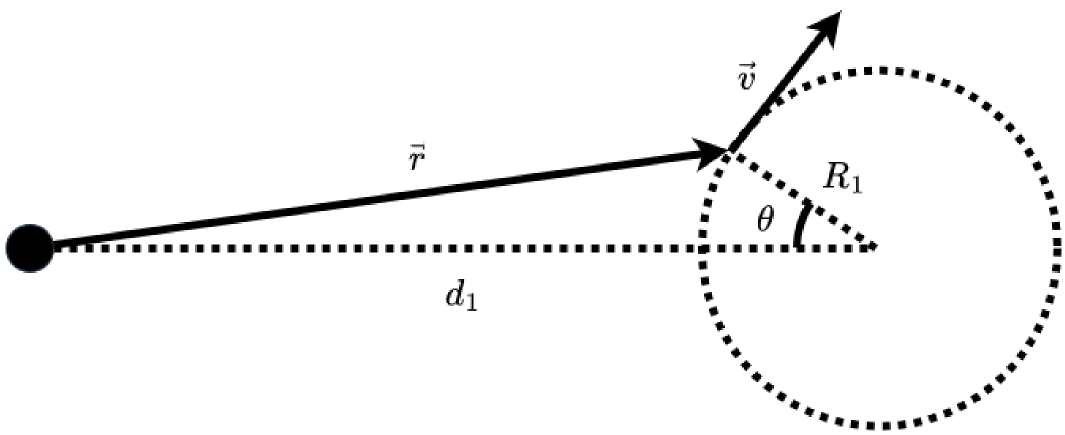

Figure 3: Components of angular momentum integration.

Solving for the needed quanitities (figure 4)

$$|\vec{r}(\theta)| = \sqrt{R_1^2 + d_1^2 - 2R_1 d_1 cos(\theta)} \tag{8}$$

$$|\vec{v}| = \omega r \tag{9}$$

$$\phi(\theta) = \frac{\pi}{2} - \theta - \alpha(\theta) \tag{10}$$

Where...

$$\alpha = \angle \vec{r} \tag{11}$$

$$= sin^{-1}(\frac{R_1 sin(\theta)}{|\vec{r}(\theta)|}) \tag{12}$$

Continuing...

$$sin(\phi(\theta)) = sin(\frac{\pi}{2} - (\theta + \alpha(\theta))) \tag{13}$$

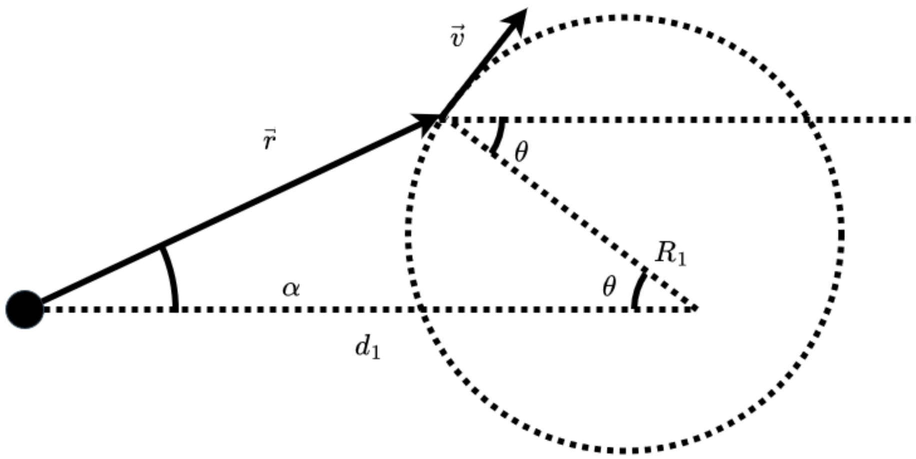

Figure 4: Components of angular momentum integration.

To simplify...

$$\begin{aligned}
sin(\phi(\theta)) &= cos(\theta + \alpha(\theta)) && (14)\\
&= cos(\theta)cos(\alpha(\theta)) - sin(\theta)sin(\alpha(\theta)) && (15)\\
&= cos(\theta)cos(sin^{-1}(\frac{R_1 sin(\theta)}{|\vec{r}(\theta)|})) - sin(\theta)sin(sin^{-1}(\frac{R_1 sin(\theta)}{|\vec{r}(\theta)|})) && (16)\\
&= cos(\theta)\sqrt{1 - \frac{R_1^2 sin^2(\theta)}{|\vec{r}(\theta)|^2}} - \frac{R_1}{|\vec{r}(\theta)|}sin^2(\theta) && (17)\\
&= cos(\theta)\sqrt{\frac{|\vec{r}(\theta)|^2 - R_1^2 sin^2(\theta)}{|\vec{r}(\theta)|^2}} - \frac{R_1}{|\vec{r}(\theta)|}sin^2(\theta) && (18)\\
&= cos(\theta)\sqrt{\frac{R_1^2 + d_1^2 - 2R_1 d_1 cos(\theta) - R_1^2 sin^2(\theta)}{|\vec{r}(\theta)|^2}} - \frac{R_1}{|\vec{r}(\theta)|}sin^2(\theta) && (19)\\
&= cos(\theta)\sqrt{\frac{R_1^2(1 - sin^2(\theta)) + d_1^2 - 2R_1 d_1 cos(\theta)}{|\vec{r}(\theta)|^2}} - \frac{R_1}{|\vec{r}(\theta)|}sin^2(\theta) && (20)\\
&= cos(\theta)\sqrt{\frac{R_1^2 cos^2(\theta) + d_1^2 - 2R_1 d_1 cos(\theta)}{|\vec{r}(\theta)|^2}} - \frac{R_1}{|\vec{r}(\theta)|}sin^2(\theta) && (21)\\
&= cos(\theta)\sqrt{\frac{(R_1 cos(\theta) - d_1)^2}{|\vec{r}(\theta)|^2}} - \frac{R_1}{|\vec{r}(\theta)|}sin^2(\theta) && (22)\\
&= \frac{d_1 cos(\theta) - R_1 cos^2(\theta) - R_1 sin^2(\theta)}{|\vec{r}(\theta)|} && (23)\\
&= \frac{d_1 cos(\theta) - R_1(cos^2(\theta) + sin^2(\theta))}{|\vec{r}(\theta)|} && (24)\\
&= \frac{d_1 cos(\theta) - R_1}{|\vec{r}(\theta)|} && (25)
\end{aligned}$$

Recall...

$$L = \rho \int_0^{2\pi} |\vec{r}(\theta)||\vec{v}|sin(\phi(\theta))d\theta \tag{26}$$

Finally...

$$\begin{align}
L &= \rho\omega R_1 \int_0^{2\pi} |\vec{r}(\theta)| \frac{d_1 cos(\theta) - R_1}{|\vec{r}(\theta)|} d\theta \tag{27}\\
&= \rho\omega R_1 \int_0^{2\pi} d_1 cos(\theta) - R_1 d\theta \tag{28}\\
&= \rho\omega R_1^2 2\pi \tag{29}\\
&= \frac{m}{2\pi}\omega R_1^2 2\pi \tag{30}\\
&= m\omega R_1^2 \tag{31}
\end{align}$$