1 Proof of Lemma 1 in the IJCAI Submission

Lemma 1. *Given any vector $\mathbf{x} \in \mathbb{R}^d$, and $m < d$. Sample m entries from $\mathbf{x}$ with replacement by running Algorithm 1 with inputs $\mathbf{x}$ and m, let $\{p_k > 0\}_{k=1}^d$ denote the corresponding sampling probabilities, and let $\mathbf{S} \in \mathbb{R}^{d\times m}$ denote the corresponding rescaled sampling matrix. Then, we have*

$$\mathbb{E}\left[\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T\right] = \sum_{k=1}^d \frac{x_k^2}{mp_k}\mathbf{e}_k\mathbf{e}_k^T + \frac{m-1}{m}\mathbf{x}\mathbf{x}^T; \tag{1}$$

$$\mathbb{E}\left[\mathbb{D}(\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T)\right] = \sum_{k=1}^d (\frac{1}{mp_k} + \frac{m-1}{m})\mathbf{x}_k^2\mathbf{e}_k\mathbf{e}_k^T; \tag{2}$$

$$\mathbb{E}\left[(\mathbb{D}(\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T))^2\right] = \sum_{k=1}^d \left[\frac{1}{m^3p_k^3} + \frac{7(m-1)}{m^3p_k^2} + \frac{6(m^2-3m+2)}{m^3p_k} + \frac{(m^3-6m^2+11m-6)}{m^3}\right] x_k^4\mathbf{e}_k\mathbf{e}_k^T; \tag{3}$$

$$\begin{aligned}&\mathbb{E}\left[\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T\mathbb{D}(\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T)\right] = (\mathbb{E}\left[\mathbb{D}(\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T)\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T\right])^T \\ &= \sum_{k=1}^d \left[\frac{1}{m^3p_k^3} + \frac{7(m-1)}{m^3p_k^2} + \frac{4(m^2-3m+2)}{m^3p_k}\right] x_k^4\mathbf{e}_k\mathbf{e}_k^T + \frac{2(m^2-3m+2)}{m^3}\mathbf{x}\mathbf{x}^T\mathbb{D}(\{\frac{x_k^2}{p_k}\}) \\ &+ \frac{(m^3-6m^2+11m-6)}{m^3}\mathbf{x}\mathbf{x}^T\mathbb{D}(\{x_k^2\});\end{aligned} \tag{4}$$

$$\begin{aligned}&\mathbb{E}\left[(\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T)^2\right] = \sum_{k=1}^d \left[\frac{4(m-1)}{m^3p_k^2} + \frac{1}{m^3p_k^3}\right] x_k^4\mathbf{e}_k\mathbf{e}_k^T \\ &+ \sum_{k=1}^d \left[\frac{\|\mathbf{x}\|_2^2(m^2-3m+2)}{m^3} + \frac{m-1}{m^3}\sum_{k=1}^d \frac{x_k^2}{p_k}\right]\frac{x_k^2}{p_k}\mathbf{e}_k\mathbf{e}_k^T \\ &+ \left[\frac{\|\mathbf{x}\|_2^2(m^3-6m^2+11m-6)}{m^3} + \frac{m^2-3m+2}{m^3}\sum_{k=1}^d\frac{x_k^2}{p_k}\right]\mathbf{x}\mathbf{x}^T + \frac{2(m^2-3m+2)}{m^3}\mathbf{x}\mathbf{x}^T\mathbb{D}(\{\frac{x_k^2}{p_k}\}) \\ &+ \frac{m-1}{m^3}\mathbf{x}\mathbf{x}^T\mathbb{D}(\{\frac{x_k^2}{p_k^2}\}) + \frac{2(m^2-3m+2)}{m^3}\mathbb{D}(\{\frac{x_k^2}{p_k}\})\mathbf{x}\mathbf{x}^T + \frac{m-1}{m^3}\mathbb{D}(\{\frac{x_k^2}{p_k^2}\})\mathbf{x}\mathbf{x}^T.\end{aligned} \tag{5}$$

The expectation is w.r.t. $\mathbf{S}$. $\{\mathbf{e}_k\}_{k=1}^d$ denote the standard basis vectors for $\mathbb{R}^d$. $\mathbb{D}(\{\frac{x_k^2}{p_k}\})$ means a square diagonal matrix with $\{\frac{x_k^2}{p_k}\}_{k=1}^d$ on its diagonal, which can be extended for other similar notations.

Proof. According to Algorithm 1, each column vector in the rescaled sampling matrix $\mathbf{S} \in \mathbb{R}^{d\times m}$ is sampled with replacement from $\{\mathbf{r}_k = \frac{1}{\sqrt{mp_k}}\mathbf{e}_k\}_{k=1}^d$ with corresponding probabilities $\{p_k\}_{k=1}^d$.

Firstly, we prove Eq. (1). By the definition, we expand

$$\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T = \sum_{j=1}^m \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\sum_{j=1}^m \mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \tag{6}$$

$$= \sum_{j=1}^m \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T + \sum_{i\neq j\in[m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T, \tag{7}$$

where the random variable t_j is in $[d]$.

Passing the expectation over $\mathbf{S}$ through the sum in Eq. (7), we have

$$\mathbb{E}\sum_{j=1}^m \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T = \sum_{j=1}^m\sum_{k=1}^d \mathbb{P}(t_j = k)\mathbf{r}_k\mathbf{r}_k^T\mathbf{x}\mathbf{x}^T\mathbf{r}_k\mathbf{r}_k^T$$

$$= \sum_{j=1}^{m} \sum_{k=1}^{d} p_k \frac{1}{m^2 p_k^2} \mathbf{e}_k \mathbf{e}_k^T \mathbf{x}\mathbf{x}^T \mathbf{e}_k \mathbf{e}_k^T = \sum_{k=1}^{d} \frac{x_k^2}{m p_k} \mathbf{e}_k \mathbf{e}_k^T, \tag{8}$$

and similarly

$$\mathbb{E} \sum_{i \neq j \in [m]} \mathbf{s}_{t_i} \mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \tag{9}$$

$$= \sum_{i \neq j \in [m]} \sum_{k=1}^{d} \sum_{q=1}^{d} \mathbb{P}(t_i = k) \mathbb{P}(t_j = q) \mathbf{r}_k \mathbf{r}_k^T \mathbf{x}\mathbf{x}^T \mathbf{r}_q \mathbf{r}_q^T \tag{10}$$

$$= \sum_{k=1}^{d} \sum_{q=1}^{d} x_k x_q \frac{m-1}{m} \mathbf{e}_k \mathbf{e}_q^T = \frac{m-1}{m} \mathbf{x}\mathbf{x}^T. \tag{11}$$

Now, combing Eq. (8) with Eq. (11) immediately proves Eq. (1).

Then, Eq. (2) can be proved by Eq. (1).

$$\mathbb{E}\left[\mathbb{D}(\mathbf{S}\mathbf{S}^T \mathbf{x}\mathbf{x}^T \mathbf{S}\mathbf{S}^T)\right] = \mathbb{D}(\mathbb{E}\left[\mathbf{S}\mathbf{S}^T \mathbf{x}\mathbf{x}^T \mathbf{S}\mathbf{S}^T\right]) \tag{12}$$

$$= \sum_{k=1}^{d} (\frac{1}{m p_k} + \frac{m-1}{m}) \mathbf{x}_k^2 \mathbf{e}_k \mathbf{e}_k^T. \tag{13}$$

Alternatively, $\mathbb{D}(\mathbf{S}\mathbf{S}^T \mathbf{x}\mathbf{x}^T \mathbf{S}\mathbf{S}^T)$ can be explicitly expanded by

$$\mathbb{D}(\mathbf{S}\mathbf{S}^T \mathbf{x}\mathbf{x}^T \mathbf{S}\mathbf{S}^T) = \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T. \tag{14}$$

Thus, the whole target expectations in Eq. (3) Eq. (4) and Eq. (5) can be explicitly expanded, and we can use similar ways to prove the remainder of the lemma.

To prove Eq. (3), we expand

$$\mathbb{E}\left[(\mathbb{D}(\mathbf{S}\mathbf{S}^T \mathbf{x}\mathbf{x}^T \mathbf{S}\mathbf{S}^T))^2\right] \tag{15}$$

$$= \mathbb{E}\left[(\sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T)^2\right] \tag{16}$$

$$= \mathbb{E}\left[\sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T\right] \tag{17}$$

$$= \mathbb{E} \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \tag{18}$$

$$+ \mathbb{E} \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{i \neq j \in [m]} \mathbf{s}_{t_i} \mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \tag{19}$$

$$+ \mathbb{E} \sum_{i \neq j \in [m]} \mathbf{s}_{t_i} \mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \tag{20}$$

$$+ \mathbb{E} \sum_{i \neq j \in [m]} \mathbf{s}_{t_i} \mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{i \neq j \in [m]} \mathbf{s}_{t_i} \mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T, \tag{21}$$

where the four terms in the last equations can be calculated as:

$$(18) = \mathbb{E} \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{j=1}^{m} \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T$$

$$
\begin{aligned}
&= \mathbb{E}\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \\
&+ \mathbb{E}\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \\
&= \sum_{k_1=1}^{d}\sum_{j=1}^{m}p_{k_1}\frac{1}{m^4p_{k_1}^4}\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T \\
&+ \mathbb{E}\sum_{i\neq j\in[m]}\sum_{k_1=1}^{d}\sum_{q=1}^{d}p_{k_1}p_q\frac{1}{m^4p_{k_1}^2p_q^2}\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{e}_q\mathbf{e}_q^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_q\mathbf{e}_q^T \\
&= \sum_{k=1}^{d}\frac{x_k^4}{m^3p_k^3}\mathbf{e}_k\mathbf{e}_k^T + \sum_{k=1}^{d}\frac{(m^2-m)x_k^4}{m^4p_k^2}\mathbf{e}_k\mathbf{e}_k^T \\
&= \sum_{k=1}^{d}\left(\frac{1}{m^3p_k^3}+\frac{m-1}{m^3p_k^2}\right)x_k^4\mathbf{e}_k\mathbf{e}_k^T; \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
(19) &= \mathbb{E}\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \\
&= \mathbb{E}\sum_{g\neq i\neq j\in[m]}\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \\
&+ \mathbb{E}\sum_{g=i\neq j\in[m]}\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \\
&+ \mathbb{E}\sum_{g=j\neq i\in[m]}\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \\
&= \sum_{k_1,k_2,k_3=1}^{d}\frac{m(m-1)(m-2)}{m^4p_{k_1}}\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{e}_{k_2}\mathbf{e}_{k_2}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_3}\mathbf{e}_{k_3}^T \\
&+ \sum_{k_1,k_3=1}^{d}\frac{m(m-1)}{m^4p_{k_1}^2}\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_3}\mathbf{e}_{k_3}^T \\
&+ \sum_{k_1,k_2=1}^{d}\frac{m(m-1)}{m^4p_{k_1}^2}\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{e}_{k_2}\mathbf{e}_{k_2}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T \\
&= \sum_{k_1=1}^{d}\frac{m(m-1)(m-2)}{m^4p_{k_1}}x_{k_1}^4\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T + \sum_{k_1=1}^{d}\frac{m(m-1)}{m^4p_{k_1}^2}x_{k_1}^4\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T + \sum_{k_1=1}^{d}\frac{m(m-1)}{m^4p_{k_1}^2}x_{k_1}^4\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T \\
&= \sum_{k=1}^{d}\left[\frac{m(m-1)(m-2)}{m^4p_k}x_k^4 + \frac{2m(m-1)x_k^4}{m^4p_k^2}\right]\mathbf{e}_k\mathbf{e}_k^T; \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
(20) &= \mathbb{E}\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{k=1}^{d}x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \\
&= \sum_{k=1}^{d}\left[\frac{m(m-1)(m-2)}{m^4p_k}x_k^4 + \frac{2m(m-1)x_k^4}{m^4p_k^2}\right]\mathbf{e}_k\mathbf{e}_k^T; \qquad (24)
\end{aligned}
$$

$$(21) = \mathbb{E} \sum_{i \neq j \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \sum_{i \neq j \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T$$

$$= \mathbb{E} \sum_{i \neq j \neq g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T$$

$$+ \mathbb{E} \sum_{i \neq j, i=g, j \neq h, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T$$

$$+ \mathbb{E} \sum_{i \neq j, i=h, j \neq g, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T$$

$$+ \mathbb{E} \sum_{i \neq j, i \neq g, j=h, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T$$

$$+ \mathbb{E} \sum_{i \neq j, i \neq h, j=g, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T$$

$$+ \mathbb{E} \sum_{i \neq j, i=g, j=h, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T$$

$$+ \mathbb{E} \sum_{i \neq j, i=h, j=g, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T$$

$$= \sum_{k=1}^{d} \left[\frac{m(m-1)(m-2)(m-3)}{m^4} x_k^4 + \frac{4m(m-1)(m-2)}{m^4 p_k} x_k^4 + \frac{2m(m-1)}{m^4 p_k^2} x_k^4 \right] \mathbf{e}_k\mathbf{e}_k^T. \quad (25)$$

$$(26)$$

Combing the above terms with simplification and reformulation completes the proof of Eq. (3). Now, we continue to prove Eq. (4).

$$\mathbb{E}\left[\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T\mathbb{D}(\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T)\right]$$

$$= \mathbb{E}\left[\sum_{j=1}^{m} \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{x} \sum_{j=1}^{m} \mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \sum_{j=1}^{m} \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \sum_{j=1}^{m} \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\right]$$

$$= \mathbb{E} \sum_{j=1}^{m} \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \sum_{j=1}^{m} \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \quad (27)$$

$$+ \mathbb{E} \sum_{j=1}^{m} \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \sum_{i \neq j \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \quad (28)$$

$$+ \mathbb{E} \sum_{i \neq j \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \sum_{j=1}^{m} \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \quad (29)$$

$$+ \mathbb{E} \sum_{i \neq j \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \sum_{i \neq j \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T, \quad (30)$$

where we calculate the four terms in the last equation as shown in below:

$$(27) = \mathbb{E} \sum_{j=1}^{m} \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \sum_{j=1}^{m} \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k\mathbf{e}_k^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T$$

$$= \mathbb{E}\sum_{j=1}^{m} \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T + \mathbb{E}\sum_{i\neq j\in[m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T$$

$$= \sum_{k_1=1}^{d}\sum_{j=1}^{m} p_{k_1}\frac{1}{m^4p_{k_1}^4}\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{x}\mathbf{x}^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T$$

$$+ \mathbb{E}\sum_{i\neq j\in[m]}\sum_{k_1=1}^{d}\sum_{q=1}^{d} p_{k_1}p_q\frac{1}{m^4p_{k_1}^2p_q^2}\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{x}\mathbf{x}^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{e}_q\mathbf{e}_q^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_q\mathbf{e}_q^T$$

$$= \sum_{k=1}^{d}\frac{x_k^4}{m^3p_k^3}\mathbf{e}_k\mathbf{e}_k^T + \sum_{k=1}^{d}\frac{(m^2-m)x_k^4}{m^4p_k^2}\mathbf{e}_k\mathbf{e}_k^T$$

$$= \sum_{k=1}^{d}(\frac{1}{m^3p_k^3} + \frac{m-1}{m^3p_k^2})x_k^4\mathbf{e}_k\mathbf{e}_k^T; \tag{31}$$

$$(28) = \mathbb{E}\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T$$

$$= \mathbb{E}\sum_{g\neq i\neq j\in[m]}\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T$$

$$+ \mathbb{E}\sum_{g=i\neq j\in[m]}\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T$$

$$+ \mathbb{E}\sum_{g=j\neq i\in[m]}\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T$$

$$= \sum_{k_1,k_2,k_3=1}^{d}\frac{m(m-1)(m-2)}{m^4p_{k_1}}\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{x}\mathbf{x}^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{e}_{k_2}\mathbf{e}_{k_2}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_3}\mathbf{e}_{k_3}^T$$

$$+ \sum_{k_1,k_3=1}^{d}\frac{m(m-1)}{m^4p_{k_1}^2}\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{x}\mathbf{x}^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_3}\mathbf{e}_{k_3}^T$$

$$+ \sum_{k_1,k_2=1}^{d}\frac{m(m-1)}{m^4p_{k_1}^2}\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{x}\mathbf{x}^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\mathbf{e}_{k_2}\mathbf{e}_{k_2}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T$$

$$= \sum_{k_1=1}^{d}\frac{m(m-1)(m-2)}{m^4p_{k_1}}x_{k_1}^4\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T + \sum_{k_1=1}^{d}\frac{m(m-1)}{m^4p_{k_1}^2}x_{k_1}^4\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T + \sum_{k_1=1}^{d}\frac{m(m-1)}{m^4p_{k_1}^2}x_{k_1}^4\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T$$

$$= \sum_{k=1}^{d}\left[\frac{m(m-1)(m-2)}{m^4p_k}x_k^4 + \frac{2m(m-1)}{m^4p_k^2}x_k^4\right]\mathbf{e}_k\mathbf{e}_k^T; \tag{32}$$

$$(29) = \mathbb{E}\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T$$

$$= \sum_{k=1}^{d}\left[\frac{m(m-1)(m-2)}{m^4p_k}x_k^4 + \frac{2m(m-1)}{m^4p_k^2}x_k^4\right]\mathbf{e}_k\mathbf{e}_k^T; \tag{33}$$

$$(30) = \mathbb{E}\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\sum_{k=1}^{d} x_k^2\mathbf{e}_k\mathbf{e}_k^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T$$

$$
\begin{aligned}
&= \mathbb{E} \sum_{i \neq j \neq g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T \\
&+ \mathbb{E} \sum_{i \neq j, i=g, j \neq h, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T \\
&+ \mathbb{E} \sum_{i \neq j, i=h, j \neq g, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T \\
&+ \mathbb{E} \sum_{i \neq j, i \neq g, j=h, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T \\
&+ \mathbb{E} \sum_{i \neq j, i \neq h, j=g, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T \\
&+ \mathbb{E} \sum_{i \neq j, i=g, j=h, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T \\
&+ \mathbb{E} \sum_{i \neq j, i=h, j=g, g \neq h \in [m]} \mathbf{s}_{t_i}\mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \mathbf{s}_{t_g}\mathbf{s}_{t_g}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{s}_{t_h}\mathbf{s}_{t_h}^T \\
&= \sum_{k_1,k_2,k_3,k_4=1}^{d} \frac{m(m-1)(m-2)(m-3)}{m^4} x_{k_1} x_{k_2} \mathbf{e}_{k_1}\mathbf{e}_{k_2}^T \mathbf{e}_{k_3}\mathbf{e}_{k_3}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{e}_{k_4}\mathbf{e}_{k_4}^T \\
&+ \sum_{k_1,k_2,k_4=1}^{d} \frac{m(m-1)(m-2)}{m^4 p_{k_1}} x_{k_1} x_{k_2} \mathbf{e}_{k_1}\mathbf{e}_{k_2}^T \mathbf{e}_{k_1}\mathbf{e}_{k_1}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{e}_{k_4}\mathbf{e}_{k_4}^T \\
&+ \sum_{k_1,k_2,k_3=1}^{d} \frac{m(m-1)(m-2)}{m^4 p_{k_1}} x_{k_1} x_{k_2} \mathbf{e}_{k_1}\mathbf{e}_{k_2}^T \mathbf{e}_{k_3}\mathbf{e}_{k_3}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{e}_{k_1}\mathbf{e}_{k_1}^T \\
&+ \sum_{k_1,k_2,k_3=1}^{d} \frac{m(m-1)(m-2)}{m^4 p_{k_2}} x_{k_1} x_{k_2} \mathbf{e}_{k_1}\mathbf{e}_{k_2}^T \mathbf{e}_{k_3}\mathbf{e}_{k_3}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{e}_{k_2}\mathbf{e}_{k_2}^T \\
&+ \sum_{k_1,k_2,k_4=1}^{d} \frac{m(m-1)(m-2)}{m^4 p_{k_2}} x_{k_1} x_{k_2} \mathbf{e}_{k_1}\mathbf{e}_{k_2}^T \mathbf{e}_{k_2}\mathbf{e}_{k_2}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{e}_{k_4}\mathbf{e}_{k_4}^T \\
&+ \sum_{k_1,k_2}^{d} \frac{m(m-1)}{m^4 p_{k_1} p_{k_2}} x_{k_1} x_{k_2} \mathbf{e}_{k_1}\mathbf{e}_{k_2}^T \mathbf{e}_{k_1}\mathbf{e}_{k_2}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{e}_{k_2}\mathbf{e}_{k_2}^T \\
&+ \sum_{k_1,k_2=1}^{d} \frac{m(m-1)}{m^4 p_{k_1} p_{k_2}} x_{k_1} x_{k_2} \mathbf{e}_{k_1}\mathbf{e}_{k_2}^T \mathbf{e}_{k_2}\mathbf{e}_{k_2}^T \sum_{k=1}^{d} x_k^2 \mathbf{e}_k \mathbf{e}_k^T \mathbf{e}_{k_1}\mathbf{e}_{k_1}^T \\
&= \sum_{k_1,k_2=1}^{d} \frac{m(m-1)(m-2)(m-3)}{m^4} x_{k_1} x_{k_2}^3 \mathbf{e}_{k_1}\mathbf{e}_{k_2}^T + \sum_{k_1=1}^{d} \frac{m(m-1)(m-2)}{m^4 p_{k_1}} x_{k_1}^4 \mathbf{e}_{k_1}\mathbf{e}_{k_1}^T \\
&+ \sum_{k_1=1}^{d} \frac{m(m-1)(m-2)}{m^4 p_{k_1}} x_{k_1}^4 \mathbf{e}_{k_1}\mathbf{e}_{k_1}^T + \sum_{k_1,k_2=1}^{d} \frac{m(m-1)(m-2)}{m^4 p_{k_2}} x_{k_1} x_{k_2}^3 \mathbf{e}_{k_1}\mathbf{e}_{k_2}^T \\
&+ \sum_{k_1,k_2=1}^{d} \frac{m(m-1)(m-2)}{m^4 p_{k_2}} x_{k_1} x_{k_2}^3 \mathbf{e}_{k_1}\mathbf{e}_{k_2}^T + \sum_{k_1=1}^{d} \frac{m(m-1)}{m^4 p_{k_1}^2} x_{k_1}^4 \mathbf{e}_{k_1}\mathbf{e}_{k_1}^T + \sum_{k_1=1}^{d} \frac{m(m-1)}{m^4 p_{k_1}^2} x_{k_1}^4 \mathbf{e}_{k_1}\mathbf{e}_{k_1}^T
\end{aligned}
$$

$$= \frac{m(m-1)(m-2)(m-3)}{m^4}\mathbf{x}\mathbf{x}^T\mathbb{D}(\{x_k^2\}) + \frac{m(m-1)(m-2)}{m^4}\sum_{k=1}^{d}\frac{x_k^4}{p_k}\mathbf{e}_k\mathbf{e}_k^T$$
$$+ \frac{m(m-1)(m-2)}{m^4}\sum_{k=1}^{d}\frac{x_k^4}{p_k}\mathbf{e}_k\mathbf{e}_k^T + \frac{m(m-1)(m-2)}{m^4}\mathbf{x}\mathbf{x}^T\mathbb{D}(\{\frac{x_k^2}{p_k}\})$$
$$+ \frac{m(m-1)(m-2)}{m^4}\mathbf{x}\mathbf{x}^T\mathbb{D}(\{\frac{x_k^2}{p_k}\}) + \frac{2m(m-1)}{m^4}\sum_{k=1}^{d}\frac{x_k^4}{p_k^2}\mathbf{e}_k\mathbf{e}_k^T. \tag{34}$$

Combing the above terms with simplification and reformulation completes the proof of Eq. (4).

Finally, we have to prove Eq. (5).

$$\mathbb{E}\left[(\mathbf{S}\mathbf{S}^T\mathbf{x}\mathbf{x}^T\mathbf{S}\mathbf{S}^T)^2\right] = \mathbb{E}\left[(\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\sum_{j=1}^{m}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T)^2\right]$$
$$= \mathbb{E}(\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T + \sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T)^2$$
$$= \mathbb{E}(\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T)^2 \tag{35}$$
$$+ \mathbb{E}(\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T)^2 \tag{36}$$
$$+ \mathbb{E}\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T \tag{37}$$
$$+ \mathbb{E}\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T, \tag{38}$$

$$(35) = \mathbb{E}\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T + \mathbb{E}\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T$$
$$= \sum_{k=1}^{d}\sum_{j=1}^{m}p_k\frac{1}{m^4p_k^4}\mathbf{e}_k\mathbf{e}_k^T\mathbf{x}\mathbf{x}^T\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_k\mathbf{e}_k^T\mathbf{x}\mathbf{x}^T\mathbf{e}_k\mathbf{e}_k^T$$
$$+ \mathbb{E}\sum_{i\neq j\in[m]}\sum_{k=1}^{d}\sum_{q=1}^{d}p_kp_q\frac{1}{m^4p_k^2p_q^2}\mathbf{e}_k\mathbf{e}_k^T\mathbf{x}\mathbf{x}^T\mathbf{e}_k\mathbf{e}_k^T\mathbf{e}_q\mathbf{e}_q^T\mathbf{x}\mathbf{x}^T\mathbf{e}_q\mathbf{e}_q^T$$
$$= \sum_{k=1}^{d}\frac{x_k^4}{m^3p_k^3}\mathbf{e}_k\mathbf{e}_k^T + \sum_{k=1}^{d}\frac{(m^2-m)x_k^4}{m^4p_k^2}\mathbf{e}_k\mathbf{e}_k^T$$
$$= \sum_{k=1}^{d}(\frac{1}{m^3p_k^3} + \frac{m-1}{m^3p_k^2})x_k^4\mathbf{e}_k\mathbf{e}_k^T; \tag{39}$$

$$(36) = \mathbb{E}\left[\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\right]$$
$$= \mathbb{E}\sum_{i\neq j\neq g\neq h\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_h}\mathbf{s}_{t_h}^T$$
$$+ \mathbb{E}\sum_{i\neq j, i=g, j\neq h, g\neq h\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_h}\mathbf{s}_{t_h}^T$$
$$+ \mathbb{E}\sum_{i\neq j, i=h, j\neq g, g\neq h\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_h}\mathbf{s}_{t_h}^T$$

$$
\begin{aligned}
&+\mathbb{E}\sum_{i\neq j,i\neq g,j=h,g\neq h\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_h}\mathbf{s}_{t_h}^T\\
&+\mathbb{E}\sum_{i\neq j,i\neq h,j=g,g\neq h\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_h}\mathbf{s}_{t_h}^T\\
&+\mathbb{E}\sum_{i\neq j,i=g,j=h,g\neq h\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_h}\mathbf{s}_{t_h}^T\\
&+\mathbb{E}\sum_{i\neq j,i=h,j=g,g\neq h\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{s}_{t_g}\mathbf{s}_{t_g}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_h}\mathbf{s}_{t_h}^T\\
&=\sum_{k_1,k_2,k_3,k_4=1}^{d}\frac{m(m-1)(m-2)(m-3)}{m^4}x_{k_1}x_{k_2}x_{k_3}x_{k_4}\mathbf{e}_{k_1}\mathbf{e}_{k_2}^T\mathbf{e}_{k_3}\mathbf{e}_{k_4}^T\\
&+\sum_{k_1,k_2,k_4=1}^{d}\frac{m(m-1)(m-2)}{m^4p_{k_1}}x_{k_1}^2x_{k_2}x_{k_4}\mathbf{e}_{k_1}\mathbf{e}_{k_2}^T\mathbf{e}_{k_1}\mathbf{e}_{k_4}^T\\
&+\sum_{k_1,k_2,k_3=1}^{d}\frac{m(m-1)(m-2)}{m^4p_{k_1}}x_{k_1}^2x_{k_2}x_{k_3}\mathbf{e}_{k_1}\mathbf{e}_{k_2}^T\mathbf{e}_{k_3}\mathbf{e}_{k_1}^T\\
&+\sum_{k_1,k_2,k_3=1}^{d}\frac{m(m-1)(m-2)}{m^4p_{k_2}}x_{k_1}x_{k_2}^2x_{k_3}\mathbf{e}_{k_1}\mathbf{e}_{k_2}^T\mathbf{e}_{k_3}\mathbf{e}_{k_2}^T\\
&+\sum_{k_1,k_2,k_4=1}^{d}\frac{m(m-1)(m-2)}{m^4p_{k_2}}x_{k_1}x_{k_2}^2x_{k_4}\mathbf{e}_{k_1}\mathbf{e}_{k_2}^T\mathbf{e}_{k_2}\mathbf{e}_{k_4}^T\\
&+\sum_{k_1,k_2}^{d}\frac{m(m-1)}{m^4p_{k_1}p_{k_2}}x_{k_1}^2x_{k_2}^2\mathbf{e}_{k_1}\mathbf{e}_{k_2}^T\mathbf{e}_{k_1}\mathbf{e}_{k_2}^T+\sum_{k_1,k_2=1}^{d}\frac{m(m-1)}{m^4p_{k_1}p_{k_2}}x_{k_1}^2x_{k_2}^2\mathbf{e}_{k_1}\mathbf{e}_{k_2}^T\mathbf{e}_{k_2}\mathbf{e}_{k_1}^T\\
&=\sum_{k_2=1}^{d}x_{k_2}^2\sum_{k_1,k_4=1}^{d}\frac{m(m-1)(m-2)(m-3)}{m^4}x_{k_1}x_{k_4}\mathbf{e}_{k_1}\mathbf{e}_{k_4}^T\\
&+\sum_{k_1,k_4=1}^{d}\frac{m(m-1)(m-2)}{m^4p_{k_1}}x_{k_1}^3x_{k_4}\mathbf{e}_{k_1}\mathbf{e}_{k_4}^T+\sum_{k_2=1}^{d}x_{k_2}^2\sum_{k_1=1}^{d}\frac{m(m-1)(m-2)}{m^4p_{k_1}}x_{k_1}^2\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\\
&+\sum_{k_1,k_2=1}^{d}\frac{m(m-1)(m-2)}{m^4p_{k_2}}x_{k_1}x_{k_2}^3\mathbf{e}_{k_1}\mathbf{e}_{k_2}^T+\sum_{k_2=1}^{d}\frac{x_{k_2}^2}{p_{k_2}}\sum_{k_1,k_4=1}^{d}\frac{m(m-1)(m-2)}{m^4}x_{k_1}x_{k_4}\mathbf{e}_{k_1}\mathbf{e}_{k_4}^T\\
&+\sum_{k_1=1}^{d}\frac{m(m-1)}{m^4p_{k_1}^2}x_{k_1}^4\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T+\sum_{k_2=1}^{d}\frac{x_{k_2}^2}{p_{k_2}}\sum_{k_1=1}^{d}\frac{m(m-1)}{m^4p_{k_1}}x_{k_1}^2\mathbf{e}_{k_1}\mathbf{e}_{k_1}^T\\
&=\frac{\|\mathbf{x}\|_2^2m(m-1)(m-2)(m-3)}{m^4}\mathbf{x}\mathbf{x}^T+\frac{m(m-1)(m-2)}{m^4}\mathbb{D}(\{\frac{x_k^2}{p_k}\})\mathbf{x}\mathbf{x}^T\\
&+\frac{\|\mathbf{x}\|_2^2m(m-1)(m-2)}{m^4}\sum_{k=1}^{d}\frac{x_k^2}{p_k}\mathbf{e}_k\mathbf{e}_k^T+\frac{m(m-1)(m-2)}{m^4}\mathbf{x}\mathbf{x}^T\mathbb{D}(\{\frac{x_k^2}{p_k}\})\\
&+\frac{m(m-1)(m-2)}{m^4}\sum_{k=1}^{d}\frac{x_k^2}{p_k}\mathbf{x}\mathbf{x}^T+\frac{m(m-1)}{m^4}\sum_{k=1}^{d}\frac{x_k^4}{p_k^2}\mathbf{e}_k\mathbf{e}_k^T+\frac{m(m-1)}{m^4}\sum_{k=1}^{d}\frac{x_k^2}{p_k}\sum_{k=1}^{d}\frac{x_k^2}{p_k}\mathbf{e}_k\mathbf{e}_k^T;
\end{aligned}
\tag{40}
$$

$$
(37)=\mathbb{E}\sum_{j=1}^{m}\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T\sum_{i\neq j\in[m]}\mathbf{s}_{t_i}\mathbf{s}_{t_i}^T\mathbf{x}\mathbf{x}^T\mathbf{s}_{t_j}\mathbf{s}_{t_j}^T
$$

$$
\begin{aligned}
&= \mathbb{E} \sum_{g \neq i \neq j \in [m]} \mathbf{s}_{t_g} \mathbf{s}_{t_g}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_g} \mathbf{s}_{t_g}^T \mathbf{s}_{t_i} \mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \\
&+ \mathbb{E} \sum_{g = i \neq j \in [m]} \mathbf{s}_{t_g} \mathbf{s}_{t_g}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_g} \mathbf{s}_{t_g}^T \mathbf{s}_{t_i} \mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \\
&+ \mathbb{E} \sum_{g = j \neq i \in [m]} \mathbf{s}_{t_g} \mathbf{s}_{t_g}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_g} \mathbf{s}_{t_g}^T \mathbf{s}_{t_i} \mathbf{s}_{t_i}^T \mathbf{x}\mathbf{x}^T \mathbf{s}_{t_j} \mathbf{s}_{t_j}^T \\
&= \sum_{k_1, k_2, k_3 = 1}^{d} \frac{m(m-1)(m-2)}{m^4 p_{k_1}} \mathbf{e}_{k_1} \mathbf{e}_{k_1}^T \mathbf{x}\mathbf{x}^T \mathbf{e}_{k_1} \mathbf{e}_{k_1}^T \mathbf{e}_{k_2} \mathbf{e}_{k_2}^T \mathbf{x}\mathbf{x}^T \mathbf{e}_{k_3} \mathbf{e}_{k_3}^T \\
&+ \sum_{k_1, k_3 = 1}^{d} \frac{m(m-1)}{m^4 p_{k_1}^2} \mathbf{e}_{k_1} \mathbf{e}_{k_1}^T \mathbf{x}\mathbf{x}^T \mathbf{e}_{k_1} \mathbf{e}_{k_1}^T \mathbf{e}_{k_1} \mathbf{e}_{k_1}^T \mathbf{x}\mathbf{x}^T \mathbf{e}_{k_3} \mathbf{e}_{k_3}^T \\
&+ \sum_{k_1, k_2 = 1}^{d} \frac{m(m-1)}{m^4 p_{k_1}^2} \mathbf{e}_{k_1} \mathbf{e}_{k_1}^T \mathbf{x}\mathbf{x}^T \mathbf{e}_{k_1} \mathbf{e}_{k_1}^T \mathbf{e}_{k_2} \mathbf{e}_{k_2}^T \mathbf{x}\mathbf{x}^T \mathbf{e}_{k_1} \mathbf{e}_{k_1}^T \\
&= \sum_{k_1, k_3 = 1}^{d} \frac{m(m-1)(m-2)}{m^4 p_{k_1}} x_{k_1}^3 x_{k_3} \mathbf{e}_{k_1} \mathbf{e}_{k_3}^T + \sum_{k_1, k_3 = 1}^{d} \frac{m(m-1)}{m^4 p_{k_1}^2} x_{k_1}^3 x_{k_3} \mathbf{e}_{k_1} \mathbf{e}_{k_3}^T \\
&+ \sum_{k_1 = 1}^{d} \frac{m(m-1)}{m^4 p_{k_1}^2} x_{k_1}^4 \mathbf{e}_{k_1} \mathbf{e}_{k_1}^T \\
&= \frac{m(m-1)(m-2)}{m^4} \mathbb{D}(\{\frac{x_k^2}{p_k}\}) \mathbf{x}\mathbf{x}^T + \frac{m(m-1)}{m^4} \mathbb{D}(\{\frac{x_k^2}{p_k^2}\}) \mathbf{x}\mathbf{x}^T \\
&+ \frac{m(m-1)}{m^4} \sum_{k=1}^{d} \frac{x_k^4}{p_k^2} \mathbf{e}_k \mathbf{e}_k^T; \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
(38) &= \frac{m(m-1)(m-2)}{m^4} \mathbf{x}\mathbf{x}^T \mathbb{D}(\{\frac{x_k^2}{p_k}\}) + \frac{m(m-1)}{m^4} \mathbf{x}\mathbf{x}^T \mathbb{D}(\{\frac{x_k^2}{p_k^2}\}) \\
&+ \frac{m(m-1)}{m^4} \sum_{k=1}^{d} \frac{x_k^4}{p_k^2} \mathbf{e}_k \mathbf{e}_k^T. \qquad (42)
\end{aligned}
$$

Combing the above terms with simplification and reformulation completes the proof of Eq. (5). To this end, we complete the whole proof. □