Inequity Aversion Pricing in Multi-Unit Markets

First Author[1] , **Second Author**[2] , **Third Author**[2,3] and **Fourth Author**[4]
[1]First Affiliation
[2]Second Affiliation
[3]Third Affiliation
[4]Fourth Affiliation
{first, second}@example.com, third@other.example.com, fourth@example.com

Abstract

We build upon previous models for differential pricing in social networks and fair price discrimination in markets, considering a setting in which multiple units of a single product must be sold to selected buyers so as to maximize the seller's revenue or the social welfare, while limiting the differences of the prices offered to social neighbors. We first consider the case of general social graph topologies, and provide optimal or nearly-optimal hardness and approximation results for the related optimization problems under various meaningful assumptions, including a polylogarithmic lower bound on the achievable revenue under the unique game conjecture. Then, we focus on topologies that are typical of social networks. Namely, we consider graphs where the node degrees follow a power-law distribution, and show that it is possible to obtain constant or good approximations for the seller's revenue maximization with high probability, thus improving upon the general case.

1 Introduction

The study of differential pricing in economics can be traced back to the beginning of the 20th century[Varian, 1989; Phlips, 1983; Stokey, 1979; Laffont *et al.*, 1998], and it is progressively attracting increasing interest. In fact, this feature is nowadays widely used in several settings, like in raising the price of the tickets as time approaches the related event, in assigning different prices for the same products in geographically apart markets, in offering discounts to premium customers, and so forth. Several works outlined various levels of price discrimination [Pigou, 1920], also trying to figure out proper forms that buyers can perceive as fair [Feldman *et al.*, 2012; Alon *et al.*, 2013; Amanatidis *et al.*, 2016; Flammini *et al.*, 2018]. Another related concept of customers' satisfaction widely investigated in markets is envy-freeness. Namely, buyers, given the bundle of goods they receive and the assigned prices, should not prefer other bundles or bundles provided to other buyers. Seminal works in the field are [Guruswami *et al.*, 2005; Hartline and Yan, 2011; Cheung and Swamy, 2008; Balcan *et al.*, 2008; Briest and Krysta, 2006] where authors describe logarithmic approximation algorithms for maximizing the seller's revenue under various assumptions, while hardness of approximations complementing the previous results can be found in [Briest, 2008; Chalermsook *et al.*, 2012; Chalermsook *et al.*, 2013b; Chalermsook *et al.*, 2013a; Demaine *et al.*, 2008]. Many variants can be also found in the works of [Chen and Deng, 2010; Chen *et al.*, 2011; Feldman *et al.*, 2012; Anshelevich *et al.*, 2017; Bilò *et al.*, 2017; Chen *et al.*, 2016]. None of these works however considered discriminatory pricing policies, except for a mild form called bundle pricing, in which non-proportional or lower average prices are assigned to bundles of bigger sizes. However, each bundle has a unique price for all the buyers.

1.1 Our Contribution

In this paper we build upon previous works dealing with explicit forms of differential pricing [Alon *et al.*, 2013; Amanatidis *et al.*, 2016; Flammini *et al.*, 2018] Namely, like in [Alon *et al.*, 2013; Amanatidis *et al.*, 2016] we assume that buyers are members of a social population and that the difference of the prices offered to social neighbors must be suitably bounded. Like in [Flammini *et al.*, 2018], we model the underlying scenario as a multi-unit market with relaxed fair price discrimination constraints. This type of markets has been largely investigated [Feldman *et al.*, 2012; Brânzei *et al.*, 2016; Monaco *et al.*, 2015; Flammini *et al.*, 2019], and is typical of many real-world situations were homogeneous items are on sale, like in commodity markets.

We study four different cases arising by considering ***i.*** social welfare or seller's revenue maximization, and ***ii.*** single-minded or general valuations. Our results comprise approximation algorithms and hardness of approximation results for all the above-mentioned cases. Furthermore, we consider specific topologies that are typical of social networks, i.e., graphs where the node degrees follow a power-law distribution, and we show constant approximations with high probability for the revenue on such a class.

Due to space constraints, we omit or sketch the less significant proofs. A complete version of those can be found at: https://github.com/ijcai19submissionsupplementalmaterial/Inequity-Aversion-Pricing-in-Multi-Unit-Markets/blob/master/appendix.pdf

	Single-Minded	General Valuations
Social Welfare	NP-hard	NP-hard (strong)
	FPTAS	2
Revenue	$O(\frac{\sqrt{\log n}}{\log^2 \log n})$	$O(\frac{\sqrt{\log n}}{\log^2 \log n})$
	$O(\log n)$	$O(\log n + \log m)$

Table 1: Hardness and approximation results.

1.2 Related Work

Envy-freeness in multi-unit markets has been investigated in [Feldman *et al.*, 2012], where the authors studied budgeted buyers with additive valuations and considering progressively stronger levels of price discrimination. After showing that the different levels correspond to increasing maximum revenues, they proved that the revenue maximization problem is NP-hard, and provided a polynomial time 2-approximation algorithm for item-pricing, that is without discrimination. Such an approximation has been improved to an FPTAS in [Colini-Baldeschi *et al.*, 2014].

Monaco et al. [Monaco *et al.*, 2015] investigated multi-unit markets with buyers having no budgets and gave hardness and approximation results for the revenue maximization problem in several cases arising by assuming different notions of envy-freeness, item- or bundle-pricing, and single-minded or general valuation buyers.

Social influence in markets has been recently considered in [Abebe *et al.*, 2017; Chevaleyre *et al.*, 2007; Flammini *et al.*, 2019], where the authors focused on a relaxed notion of social envy-freeness restricting envy only to social neighbors, and applied it to problems like cake cutting, distributed negotiation, and multi-unit markets. [Alon *et al.*, 2013; Amanatidis *et al.*, 2016; Bilò *et al.*, 2017; Bilò *et al.*, 2018] considered also the possibility of dismissing some of the buyers from the market, which translates into limiting the social awareness acting on the social graph topology.

Frameworks more related to the present paper have been studied in [Alon *et al.*, 2013; Amanatidis *et al.*, 2016; Flammini *et al.*, 2018]. In particular, in [Alon *et al.*, 2013] the authors defined a form of price discrimination constrained by a social graph, precisely requiring a difference bounded by an additive constant in the prices offered to two social neighbors. Several results on revenue maximization have been provided, also in case of buyers preselection. Furthermore, [Amanatidis *et al.*, 2016] extended the previous model to a setting allowing the assignment of more than one item per node, which corresponds to multi-unit markets with single-minded buyers and unlimited supply.

A related model of fair price discrimination in multi-unit markets has been investigated in [Flammini *et al.*, 2018]. While it extends the above frameworks by allowing general valuations, and thus with the possibility of allocating bundles of different sizes to a single buyer, it does not allow the exclusion of buyers by means of special ($\perp$) prices, and it does not consider any slackness in fair pricing. The authors investigated the computational complexity of the social welfare and the revenue maximization, providing hardness and approximation results under various assumptions on the buyers' valuations and on the social graph topology.

As already mentioned, we borrow the model of fair price discrimination from [Flammini *et al.*, 2018] and extend it with the features of buyers' preselection and additive slackness in fairness constraints considered in [Alon *et al.*, 2013; Amanatidis *et al.*, 2016].

2 Preliminaries

We define a multi-unit market as a tuple $(N, M, (v_i)_{i\in N})$, where $N = \{1, \dots, n\}$ is a set of n buyers, M is a set of m identical items and for every buyer $i \in N$, $v_i = (v_i(1), \dots, v_i(m))$ is a valuation function (or vector) expressing, for each natural number j, the maximum amount $v_i(j) \in \mathbb{R}$ that buyer i is willing to pay for a subset of items $X \subseteq M$ of size j. We assume $v_i(0) = 0$ and $v_i(j) \geq 0$ for every $i \in N$ and j, $1 \leq j \leq m$.

We consider both the *single-minded* case, in which every buyer i is interested only in bundles of a *preferred* bundle size m_i, and *general valuations*, i.e., the unrestricted case.

We adopt a classical pricing scheme, which is natural in case of identical items, usually referred as *item-pricing*. In such a scheme, the seller assigns a single non-negative price per item $p_i \in \mathbb{R}$ to each buyer i. Thus, buyer i owes $p_i \cdot |X|$ for a bundle of items X and her utility for receiving X is given by $u_i(X, p_i) = v_i(|X|) - p_i \cdot |X|$. We denote by $\overline{p} = (p_1, \dots, p_n)$ the vector of all the prices assigned to the buyers in the market.

We assume buyers to be individuals of a population and we represent this by means of a directed social graph $G = (N, E)$. Such a graph captures the notion of buyers' awareness of the prices proposed to other buyers, more precisely a buyer i is only aware of the prices that the seller proposes to her neighbors $N(i) = \{k \in N | (i, k) \in E\}$. As in previous models for fair price discrimination, we assume that arcs of G are weighted according to a given *slackness* function α specifying, for each arc $(i, k) \in E$, a slackness factor $\alpha(i, k) \geq 0$. Starting from G, it is possible to define the following concept of fair price discrimination.

Definition 1. A price vector $\overline{p}$ is *fair* with respect to the social graph $G = (N, E)$ if $p_i \leq p_k + \alpha(i, k)$ for every $(i, k) \in E$.

We define an *allocation vector* as an n-tuple $\overline{X} = (X_1, \dots, X_n)$ such that $X_i \subseteq M$ is the set of items sold to buyer i, and we call a pair $(\overline{X}, \overline{p})$ an *outcome*. $(\overline{X}, \overline{p})$ is a *feasible* outcome for market η if it satisfies the supply constraint $\sum_{i=1}^{n} |X_i| \leq m$. Moreover, a feasible outcome $(\overline{X}, \overline{p})$ is *envy-free* if $X_i \in argmax_{X\subseteq M} u_i(X, p_i)$ for every buyer $i \in N$. Notice that, for every $i \in N$, since $v_i(0) = 0$, envy-freeness implies the classical assumption of individual rationality of the buyers, that is $u_i(X_i, p_i) \geq 0$.

We are now ready to define the solutions to our markets, that is *fair outcomes*.

Definition 2. A feasible outcome $(\overline{X}, \overline{p})$ is *fair* under the social graph G if it is envy-free and its price vector is fair with respect to G.

We study the (fair) *pricing problems* of determining fair outcomes that maximize two fundamental metrics: ***i.*** The *social welfare* $sw(\overline{X}, \overline{p}) = \sum_{i=1}^{n} v_i(|X_i|)$. ***ii.*** The *seller's revenue* $r(\overline{X}, \overline{p}) = \sum_{i=1}^{n} p_i \cdot |X_i|$.

We use the notation $opt_{sw}(\eta, G)$ (resp. $opt_r(\eta, G)$) for the maximum possible social welfare (resp. revenue) achievable by an outcome for η fair under G, and $opt_{sw}(\eta)$ (resp. $opt_r(\eta)$) for the highest possible one achievable without price discrimination (or analogously by an outcome fair under the complete social graph).

By the individual rationality constraint, for any feasible outcome $(\overline{X}, \overline{p})$, it holds $sw(\overline{X}, \overline{p}) \geq r(\overline{X}, \overline{p})$, so that also $opt_{sw}(\eta) \geq opt_r(\eta)$ and $opt_{sw}(\eta, G) \geq opt_r(\eta, G)$.

As in previous models of fair price discrimination, we consider the additional option of preselecting subsets of buyers admitted to the market. In fact, this feature allows the seller to break transitivity chains of price dependencies in the social graph, considerably increasing the maximum revenue achievable in some cases. Formally, we model this by introducing a distinguished bottom price $\perp$ for buyers to be excluded, yielding a corresponding price vector $\overline{p} \in (\mathbb{R} \cup \{\perp\})^n$. The notion of fair pricing is then extended as follows:

Definition 3. A price vector $\overline{p}$ is *fair* with respect to the social graph $G = (N, E)$ and the slackness function α if $p_i \leq p_k + \alpha(i, k)$ for every $(i, k) \in E$ such that $p_i \neq \perp$ and $p_k \neq \perp$.

The social welfare and the seller's revenue metrics are then computed considering only buyers not receiving bottom prices.

The following preliminary result shows the effectiveness of allowing the exclusion of buyers in terms of achievable performance of fair outcomes.

Proposition 1. *Let η and G be respectively a market and its corresponding social graph. Allowing bottom prices in η can increase the optimal social welfare and revenue of fair outcomes by a multiplicative factor equal to m, and such a bound is tight.*

For the sake of brevity, we call (SINGLE,WELFARE)-pricing (resp. (GENERAL,WELFARE)-, (SINGLE,REVENUE)- and (GENERAL,REVENUE)-pricing) the pricing problem restricted to the instances of multi-unit markets with single-minded valuations and social welfare maximization (resp. general valuations and social welfare maximization, single-minded and revenue maximization, and general valuations and revenue maximization).

Let us finally stress that in multi-unit markets, while the size of the representation of an instance with general valuations is polynomial in m, as different valuations must be specified for every different bundle size, in single-minded instances the dependence is logarithmic in m, as for each buyer it is sufficient to specify the size of her unique preferred bundle, together with the corresponding valuation. Thus, in a quite counter-intuitive way, hardness results for single-minded buyers do not directly extend to general valuations, and vice versa approximation bounds for general valuations do not automatically transfer to single-minded instances.

3 Single-Minded

We first provide optimal results for the social welfare.

Theorem 4. (SINGLE,SOCIAL)*-pricing is NP-hard, but admits an FPTAS.*

Proof. (Sketch) Both the hardness and the FPTAS are obtained by showing a direct correspondence between this problem and KNAPSACK. □

Regarding revenue maximization, an approximation algorithm can be obtained with the same approach used in [Flammini *et al.*, 2018] for the same case without bottom prices.

Corollary 5. (SINGLE,REVENUE)*-pricing admits a* $(\frac{\log n}{1-\varepsilon})$*-approximation algorithm.*

The approximation factor above presented is nearly optimal, as the following negative result holds.

Theorem 6. *It is Unique-Game-hard to approximate* (SINGLE,REVENUE)*-pricing within a factor of* $o(\frac{\sqrt{\log n}}{\log^2 \log n})$

In order to prove our claim, we resort on known hardness results on the INDEPENDENT SET problem on graphs with maximum degree bounded by δ. Such a problem has be shown to be Unique-Game-hard to approximate within a factor of $o(\frac{\delta}{\log^2 \delta})$ [Austrin *et al.*, 2009] (see [Khot,] for details on the Unique-Game conjecture and hardness). We first exploit the bound on the node degrees of the input graph in order to find a *good partition* of the nodes. More precisely:

Definition 7. Let $H = (V, F)$ be a graph, and let $\mathcal{S} = \{S_1, \ldots, S_\kappa\}$ be a partition of V. We say that $\mathcal{S}$ is a *good partition* for G if:

i. *(Coloring)* $\forall S_i \in \mathcal{S},\ \forall u, v \in S_i, (u, v) \notin F.$

ii. Each node has at most one neighbor in each subset $\mathcal{S}$.

When the maximum degree of H is bounded by δ, the following result holds:

Lemma 8. *Any graph $H = (V, F)$ with node degrees at most δ admits a good partition $\mathcal{S}$ with $|\mathcal{S}| \leq \delta^2 + 1$, and such a partition can be found in polynomial time.*

We are ready to prove Theorem 6.

Proof. Consider the following reduction from INDEPENDENT SET instances $H = (V, F)$ with bounded degree δ to instances (η, G) of (SINGLE,REVENUE)-pricing:
i. Find a good partition $\mathcal{S} = \{S_1, \ldots S_D\}$ of G, with $D \leq \delta^2 + 1$.
ii. For each node $u \in S_d$ add a set $\mathcal{N}_u$ of 2^d single-minded buyers with valuation 2^{-d} only for bundles of size 1.
iii. For each $v \in V$ and pair of buyers $i, h \in \mathcal{N}_v$ add in the social graph G arcs (i, h), (h, i) with $\alpha(i, h) = \alpha(h, j) = 0$.
iv. For each edge $\{u, v\} \in F$, with $u \in S_d$, $v \in S_{d'}$ and $d < d'$, add in G all arcs (i, h) with $i \in \mathcal{N}_u$ and $h \in \mathcal{N}_v$.
v. Consider unlimited supply (or equivalently set it to $|V|2^D$).
We are going to prove our claim by showing that ($\Rightarrow$) if G admits an independent set of cardinality k, then the reduced instance admits revenue at least k; and ($\Leftarrow$) if the reduced instance admits revenue k then G has an independent set of cardinality at least $\frac{k}{2}$.

($\Rightarrow$) Let $I \subseteq V$ be an independent set of G with cardinality k. Consider the outcome $(\overline{X}, \overline{p})$ for the reduced instance in which $|X_i| = 1$ and $p_i = 2^{-d}$ if $i \in \mathcal{N}_v$, $v \in S_d$ and $v \in I$, otherwise $|X_i| =$ and $p_i = 2^{-d}$.

Notice that $\overline{p}$ is fair under G. In fact, by construction, denoted as $\Delta(v)$ the set of the neighbors of node v in H, the set of the neighbors of buyer $i \in \mathcal{N}_v$ in G is a subset of $\mathcal{N}_v \cup \bigcup_{u\in\Delta(v)} \mathcal{N}_u$. Then, if $v \notin I$, $p_i = \perp$ and no fairness constraints on p_i must hold. If instead $v \in I$, all neighbors of i in $\mathcal{N}_v$ get the same price, and, since I is independent, $p_k = \perp$ for all buyers k in $\bigcup_{u\in\Delta(v)} \mathcal{N}_u$. Therefore, $\overline{p}$ is fair. Furthermore, $(\overline{X}, \overline{p})$ is envy-free, since a price equal to their valuation is proposed to all buyers receiving a bundle of cardinality 1, while all the ones not receiving any item get price $\perp$. Finally observe that for each $v \in I$ with $v \in S_d$, there are 2^d buyers buying at price 2^{-d}, ensuring revenue 1 for each node in I. Therefore, $r(\overline{X}, \overline{p}) = k$.

($\Leftarrow$) Assume that the reduced instance admits an outcome $(\overline{X}, \overline{p})$ with revenue k. Since we are under the hypothesis of unlimited supply, without loss of generality we can assume that $|X_i| = 1$ for each buyer i such that $p_i \le v_i(1)$, since this can only increase the revenue. Similarly, as $\overline{p}$ is fair, if a price $p_i \neq \perp$ is proposed to a buyer $i \in \mathcal{N}_v$, then the price proposed to all the other buyers in $\mathcal{N}_v$ must be either p_i or $\perp$. Thus, we can assume that $p_k = p_i$ and a bundle is assigned to all the buyers in $k \in \mathcal{N}_v$. Under these assumptions $(\overline{X}, \overline{p})$ can be described by means of a vector $\overline{\pi} \in (\mathbb{R} \cup \{\perp\})^{|V|}$, where component π_v is equal to the price proposed to all buyers in $\mathcal{N}_v$.

After these preliminary remarks, consider the subset of nodes I built as follows: ***i.*** Consider all the subsets $S_d \in \mathcal{S}$ in an inverse order with respect to their index d. ***ii.*** For each $v \in S_d$ such that $\pi_v \neq \perp$, add v to I, set $\pi_v = \perp$ and set $\pi_u = \perp$ for all $u \in \Delta(v)$.

Since each time that we add a node to I we set $\pi_u = \perp$ for all its neighbors, it is not possible to add to I two adjacent nodes in G, and thus I is independent. We now prove that $|I| \ge \frac{k}{2}$. Let $\rho = \sum_{v\in V} \pi_v |\mathcal{N}_v|$, considering π_v as 0 if $\pi_v = \perp$. Clearly, before starting running the building procedure described above, $\rho = k$, as it coincides with the revenue of outcome $(\overline{X}, \overline{p})$. Moreover, after I is built, $\rho = 0$, since all $\pi_v = \perp$. Furthermore, after adding node $v \in S_d$ to I, ρ decreases by exactly $\sum_{u\in\{v\}\cup\Delta(v)} \pi_u |\mathcal{N}_u|$. By the fairness constraints and since we are considering subset of nodes in a decreasing order, in this step $\pi_u \le \pi_v$ for all $u \in \Delta(v)$, and as $v_i(1) = 2^{-d}$ for all the buyers $i \in \mathcal{N}_v$, we have that:

$$\sum_{u\in\{v\}\cup\Delta(v)} \pi_u |\mathcal{N}_u| \le 2^{-d} \sum_{u\in\{v\}\cup\Delta(v)} |\mathcal{N}_u| \le 2^{-d} \sum_{d'=1}^{d} 2^{d'} \le 2,$$

where the second inequality derives from the fact that a node v can't have more then one neighbour in the same subset S_d. We then have that ρ decreases by at most 2 each time a node is added to I, and thus $|I| \ge \frac{k}{2}$.

To conclude the proof, we remark that in order to have a polynomial number of buyers and consequently a polynomial time reduction it is sufficient to choose $\delta = \sqrt{\log n}$. □

4 General Valuations

In order to achieve good approximations for general valuations, we resort on a reduction to the single-minded case provided in [Flammini *et al.*, 2018].

Theorem 9. (GENERAL,SOCIAL)*-pricing admits a* 2*-approximation algorithm.*

Proof. (Sketch) The algorithm can be obtained by exploiting the same technique of Theorem 8 in [Flammini *et al.*, 2019]. More precisely, we avoid the pre-processing phase of the buyers, and we exploit $\perp$ prices to ensure that at least one of the two outcomes produced by the algorithm achieves a 2-approximation of the social welfare. Avoiding the pre-processing phase ensures that our algorithm returns not only a 2-approximation of the social welfare, but also a 2-approximation achievable with an unconstrained discriminatory pricing policy. □

Considering general valuations worsens the complexity of finding an outcome that maximizes social welfare.

Theorem 10. (GENERAL,SOCIAL)*-pricing is strongly NP-hard.*

In the same fashion of Corollary 5, it is possible to exploit the 2-approximation for the social welfare in order to obtain the following result.

Corollary 11. (GENERAL,REVENUE)*-pricing admits a* $2(\log n + \log m)$*-approximation algorithm.*

The hardness result provided in Theorem 6 directly extends to general valuations, as in the provided reduction m is polynomially bounded in the size of the instance.

Corollary 12. (GENERAL,REVENUE)*-pricing is Unique-Game-hard to approximate within a factor of* $o(\frac{\sqrt{\log n}}{\log^2 \log n})$.

5 Social Networks

We now focus on graph topologies that are typical of social networks. Namely, we assume that node degrees in G respect a power law distribution. This class of graphs, also called scale-free, has been largely investigated in the literature as the paradigmatic model of the web graph and other common graphs arising from social relationships. While in the previous sections good approximations bound have been already obtained for the social welfare without any restriction on the structure of the network, we here provide better results for the revenue maximization. Due to space constraints, many details of algorithms and proofs are only sketched.

Let $\overline{d} = (d_1, d_2, \ldots, d_n)$ be a non-decreasing sequence or vector of n strictly positive integers, whose sum is even. We assume that $\overline{d}$ respects a power law distribution. Namely, for any fixed integer $k > 0$, the number $n(k)$ of integers d_i with $d_i = k$ is proportional to $k^{-\gamma}$, where typically $2 < \gamma < 3$. In other words, $c \cdot n \cdot k^{-\gamma} \le n(k) \le c' \cdot n \cdot k^{-\gamma}$, for three given constants c, c' and γ such that $c < c'$. As it can be easily checked, the number of integers d_i with $d_i > k$ in $\overline{d}$ can be suitably upper bounded as $\sum_{h=k+1}^{n} n(h) = O\left(\frac{n}{k^{\gamma-1}}\right)$.

Let $\mathcal{G}_{n,\mathbf{d}}$ be the class of graphs with node set $N = \{1, 2, \ldots, n\}$, in which the sequence of node degrees listed in non-decreasing order coincides with $\overline{d}$. We assume that the social graph G is randomly drawn in $\mathcal{G}_{n,\mathbf{d}}$ uniformly selecting a permutation of buyers π in such a way that buyer i is associated to position $\pi(i)$ of the degree sequence, with corresponding degree $d_{\pi(i)}$.

Let us first focus on the single-minded case. Before providing nice approximations for power-law graphs, let us give the following key lemma, which will be useful in the sequel.

Lemma 13. *Given any family of graphs $\mathcal{G}$ and a fixed integer $k > 0$, if a k-coloring for any graph in $\mathcal{G}$ exists and can be determined in polynomial time, then* (SINGLE,REVENUE)*-pricing restricted to social graphs in $\mathcal{G}$ admits a $(k+\varepsilon)$-approximation algorithm, for any $\varepsilon > 0$.*

Proof. Once colored the nodes of the graph, consider the subset of buyers N_i with a fixed color i. Since N_i forms an independent set, a $(1+\varepsilon/k)$-approximation for the submarket containing only the buyers in N_i can be easily determined by completely ignoring the fair price discrimination constraints and running the FPTAS of knapsack on the equivalent knapsack instance with capacity m containing an object o_i for every buyer $i \in N$ with profit $z_i = v(m_i)$ and weight m_i. In fact, the returned solution can be directly translated to an outcome of the original problem with the same revenue, by assigning a preferred bundle of size m_i at price $v(m_i)/m_i$ per item to every buyer i corresponding to a selected object, and discarding the remaining buyers by means of bottom prices.

Starting from the above-collected outcomes, a $(k+\varepsilon)$-approximation can be determined simply by returning the best of them, say associated to a given color i, completed by assigning bottom prices to all the buyers not in N_i.

In fact, at least one set N_i contributes $r \geq opt_r(\eta, G)/k$ to the optimal revenue of a fair outcome for the initial instance (η, G), and the optimal solution for the submarket restricted to N_i has revenue at least r. □

As a direct consequence of the above lemma, constant approximation algorithms can be obtained for graphs with maximum degree bounded by a constant (thanks to the well-known greedy coloring algorithm), for planar graphs, bipartite graphs and for many other classes of graphs.

Unfortunately, power law graphs do not have a constant bounded degree, thus not allowing a direct application of the above lemma. However, their average degree is constant. More precisely, for any choice of the constant parameters c, c' and γ, there exists a low constant integer k such that the number of nodes with degree greater than k is at most $n/2$.

Starting from the above observation, let us consider the following algorithm, called POWER-LAW: once drawn $G \in \mathcal{G}_{n,\mathbf{d}}$ according to the above random process, consider the subset $N' \subseteq N$ of buyers of degree at most k, and then run the above algorithm for bounded degree graphs on the instance (η, G'), where G' is the subgraph induced by N'.

Lemma 14. *POWER-LAW executed on randomly drawn social graphs in $\mathcal{G}_{n,\mathbf{d}}$ has constant expected approximation ratio for* (SINGLE,REVENUE)*-pricing.*

Proof. We prove that the expected revenue of the above algorithm is $\Omega(opt_r(\eta, G))$.

Let k be the constant selected by the algorithm, i.e. such that the set N' of the buyers of degree at most k has cardinality $|N'| \geq n/2$. Let X_i be the random variable equal to 1 if buyer i has degree at most k in G, $X_i = 0$ otherwise, and let S be the random variable corresponding to the sum of the preferred valuations of the buyers of degree at most k in G, that is $S = \sum_{i \in N} v_i(m_i) X_i$. Then, since POWER-LAW exploiting a $(k+1)$-coloring returns a solution of revenue at least $S/(k+1+\varepsilon)$ and k is constant, it is sufficient to asymptotically bound the expected value $E(S)$ of S.

To this aim, by the linearity of expectation, we have that $E(S) = E(\sum_{i \in N} v(m_i) X_i) = \sum_{i \in N} v(m_i) \cdot E(X_i) = \sum_{i \in N} v(m_i) \cdot Prob(X_i = 1) \geq \sum_{i \in N} v(m_i)/2 \geq opt_r(\eta, G)/2$, thus proving the claim.

The proof for the case of limited supply holds just repeating the above argument restricting on the subset of the buyers that are allocated in an optimal fair outcome for η and G. □

Ideally, we would like to prove that the outcome returned by POWER-LAW has constant approximation not only in expectation, but also with high probability. Unfortunately, this is not guaranteed in general, as it can be easily checked in case a single buyer has a very high valuation for her preferred bundle, while all the others have negligible valuations. In this case, the probability that the returned solution has a constant approximation can be bounded only by $1/2$.

However, in case of unlimited supply, it is possible to obtain a bound with high probability by preprocessing the buyers with the highest valuations, so as to reduce the variance of the random variable S, when restricted to the remaining buyers with lower valuations. Namely, consider the following algorithm: once drawn $G \in \mathcal{G}_{n,\mathbf{d}}$, order the buyers non-increasingly with respect to their preferred valuations, and let P be the prefix of the first $l = 8 \ln n$ buyers; determine the optimal solution for the submarket containing only buyers in P and their induced subgraph G_P, and complete it by assigning bottom prices to all the remaining buyers; let $(X_1, \overline{p1})$ be the resulting outcome; run POWER-LAW and let $(X_2, \overline{p2})$ be the corresponding outcome; return the best of the two outcomes.

Notice that $(X_1, \overline{p1})$ can be easily computed in polynomial time. In fact, given the subset P^* of P of the buyers allocated in an optimal outcome for P, the prices yielding the maximum revenue for P^* can be determined as follows. Order the buyers non-decreasingly with respect to the ratios $v_i(m_i)/m_i$. For each buyer i considered in such an order, set p_i to be the maximum possible value such that $p_i \leq v_i(m_i)/m_i$ and $p_i \leq p_k + \alpha(i,k)$ for every buyer $k \in P^*$ with $k < i$ and $(i,k) \in E$. In other words, p_i is set to the maximum possible value compatible with the individual rationality of i and the fairness constraints for the pricing. Thus, $(X_1, \overline{p1})$ can be computed in such a way by probing all the possible subsets of P, whose number is polynomially bounded.

We are now able to prove the following theorem.

Theorem 15. *The above algorithm, run on randomly drawn social graphs in $\mathcal{G}_{n,\mathbf{d}}$, in case of unlimited supply returns constant approximation ratio for* (SINGLE,REVENUE)*-pricing with probability at least $1 - 1/n$.*

Proof. Again we show that the revenue of the algorithm is $\Omega(opt_r(\eta, G))$ with probability $1 - 1/n$.

If the contribute to $opt_r(\eta, G)$ due to the prefix P of the first $l = 8 \ln n$ buyers is higher with respect to the one of the remaining ones, then the algorithm returns a solution of revenue at least $opt_r(\eta, G)/2$.

On the other hand, if such a contribution is lower, consider the subset $N \setminus P$ of the remaining buyers not in the prefix. Let k be the constant selected by the algorithm, i.e. such that the set $N' \subseteq N \setminus P$ of the buyers of degree at most k has cardinality $|N'| \geq (n-l)/2$. Let X_i be the random variable equal to 1 if buyer i has degree at most k in G, $X_i = 0$ otherwise, and let S be the random variable corresponding to the sum of the preferred valuations of the buyers of $N \setminus P$ of degree at most k in G, that is $S = \sum_{i \in N \setminus P} v_i(m_i) X_i$. Again, the algorithm returns a solution of revenue at least $S/(k+1+\varepsilon)$ and k is constant, so that the expected value of S is $E(S) = E(\sum_{i \in N \setminus P} v(m_i) X_i) = \sum_{i \in N \setminus P} v(m_i) \cdot E(X_i) = \sum_{i \in N \setminus P} v(m_i) \cdot Prob(X_i = 1) \geq \sum_{i \in N \setminus P} v(m_i)/2 \geq \sum_{i \in N} v(m_i)/4 \geq \sum_{i \in N} opt_r(\eta, G)/4$.

We now show that $S = \Omega(opt_r(\eta, G))$ with high probability. To this aim, let us first observe that all buyers in $N \setminus P$, not being in the prefix P, have preferred valuations at most $(\sum_{i \in N} v(m_i))/l$. Moreover, it is possible to show that the variance of S is maximum when the overall sum of all the valuations of the buyers in $N \setminus P$ is compacted in a set N'' of l buyers, that is the preferred valuations of the buyers in N'' are all equal to $(\sum_{i \in N} v(m_i))/l$, and all the other buyers not in N'' have null valuations. Therefore, it is sufficient to show that the probability of S being at least half of its expected value is high in this specific case. Under such an assumption, such a probability corresponds to the one that at least $l/4$ buyers of N'' in the random extraction of G receive degree at most k. It is possible to check that the associated random variable S' follows a hypergeometric distribution with expectation $l/2$. Then, by the tail bounds of such a distribution, the probability of less than $l/4$ successes for S' is at most $e^{-l/8} = 1/n$, thus proving the claim. □

Unfortunately, the argument in the previous theorem doesn't work for limited supply, as the values of the outcomes can significantly differ from the sum of the buyers' preferred valuations. However, we can make the probability of having a constant approximation arbitrarily high at the expense of the running time by means of the following algorithm: for a fixed constant parameter l, once randomly drawn $G \in \mathcal{G}_{n,\mathbf{d}}$, find optimal outcomes for all the possible subsets of at most $8l$ buyers and let $(X_1, \overline{p1})$ be the best resulting outcome (completed with bottom prices for the other buyers); run POWER-LAW and let $(X_2, \overline{p2})$ be the corresponding outcome; return the best of the two outcomes.

Notice that, since l is constant, in the initial phase the number of considered sets of $8l$ buyers are polynomial, and for each of them an optimal outcome can be obtained in polynomial time by an exhaustive search. Therefore, the algorithm has running time polynomial in the input size, but exponential in the parameter l. A proof similar to the one of Theorem 15 restricted to the buyers allocated in an optimal outcome shows the following theorem.

Theorem 16. *The above algorithm run on randomly drawn social graphs in $\mathcal{G}_{n,\mathbf{d}}$ returns a constant approximation ratio for* (SINGLE,REVENUE)*-pricing with probability at least* $1 - e^{-l}$.

Due to space constraints, we only briefly discuss the case of general valuations. The basic POWER-LAW algorithm is modified simply by formulating the problem of finding an optimal outcome for the subset of buyers associated to each color as an equivalent instance of the multiple-choice-knapsack problem, in which objects are partitioned in classes and at most one object per class can be selected. In particular, we associate to every buyer i a class containing m objects, each corresponding to a bundle size j and having profit $v_i(j)$ and weight j; the knapsack capacity is set to m. Then, we run the FPTAS for such a problem and transform the solution in an outcome with the same revenue, similarly to the single-minded case. Again, in order to achieve a constant approximation ratio arbitrarily high probability, the algorithm is combined with a preliminary exhaustive search of the best outcomes for all the possible subsets of $8l$ buyers, where l is a constant parameter.

By similar arguments to the ones of Theorem 16, it is possible to show the following theorem.

Theorem 17. *The above algorithm run on randomly drawn social graphs in $\mathcal{G}_{n,\mathbf{d}}$ returns a constant approximation ratio for* (GENERAL,REVENUE)*-pricing with probability at least* $1 - e^{-l}$.

6 Conclusions

It would be nice to close the polylogarithmic gaps on the approximability of the maximum revenue on general social topologies. Moreover, it would be worth providing better probabilistic bounds for some of the approximation algorithms on power law graphs, or even a good approximation in the worst case. Finally, it would be interesting to consider more general markets and other relevant graph topologies.

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Supplemental material

Proof. (of Proposition 1)
Consider the following instance (η, G) with two single-minded buyers: buyer 1 with $v_1(1) = 1+\varepsilon$ and buyer 2 with $v_2(m) = m$; graph G contains only the arc $(1, 2)$. If we do not allow bottom prices, the best possible outcome is the one that sells only one item to buyer 1 at a price $1+\varepsilon$, obtaining social welfare and revenue $1+\varepsilon$. On the other hand, if bottom prices are allowed, we can set $p_1 = \perp$ and $p_2 = 1$, allocating m items to buyer 2 and achieving social welfare and revenue m.

In order to prove the this bound is tight, consider an optimal outcome $(\overline{X}_\perp, \overline{p}_\perp)$ for an instance (η, G) using bottom prices. Let v_{max} be equal to $\max_{i\in N, j\leq m}\{\frac{v_i(j)}{j}\}$. Clearly, $v_{max} \cdot m$ is an upper bound to $sw(\overline{X}_\perp, \overline{p}_\perp)$, and then also to $r(\overline{X}_\perp, \overline{p}_\perp)$. Now, let i_{max} and j_{max} be respectively a buyer and a bundle size such that $v_{i_{max}}(j_{max}) = v_{max}$. Consider the outcome $(\overline{X}, \overline{p})$ where $|X_{i_{max}}| = j_{max}$, $X_i = \emptyset$ for all the other buyers, and $p_1 = \ldots = p_n = v_{max}$. Since the utility of all the buyers are not strictly positive, X is envy free. Moreover, as all the proposed prices are coincident, $\overline{p}$ is fair under G. The claim then follows by observing that $\overline{p}$ does not have any entry equal to $\perp$ and $sw(\overline{X}, \overline{p}) = r(\overline{X}, \overline{p}) = v_{max}$, $\square$

Proof. (of Theorem 4)
In order to get the claimed FPTAS, we transform (η, G) into an equivalent instance $\mathcal{K} = (O, \overline{w}, \overline{z}, k)$ of KNAPSACK as follows:

- O contains an object o_i for each buyer i with profit $z_i = v_i(m_i)$ and weight $w_i = m_i$;
- the knapsack capacity k is set equal to m.

We now prove that $(\Rightarrow)$ for each solution $O^* \subseteq O$ of $\mathcal{K}$ with profit z^* there exists a fair outcome $(\overline{X}, \overline{p})$ with $sw(\overline{X}, \overline{p}) = z^*$, and $(\Leftarrow)$ vice versa.

$(\Rightarrow)$: Given a solution $O^* \subseteq O$ for $\mathcal{K}$ with profit z^*, consider the outcome $(\overline{X}, \overline{p})$ defined as follows:

$$|X_i| = \begin{cases} m_i & \text{if } o_i \in O^* \\ 0 & \text{otherwise} \end{cases}$$
$$p_i = \begin{cases} \min_{o_k\in O^*} v_k(m_k)/m_k & \text{if } o_i \in O^* \\ \perp & \text{otherwise} \end{cases}$$

By construction, $\sum_{i\in N} |X_i| = \sum_{o_i\in O^*} w_i \leq k = m$ and $\sum_{i\in N} v_i(X_i) = \sum_{o_i\in O^*} z_i$, that is $(\overline{X}, \overline{p})$ satisfies the supply constrains and has social welfare $sw(\overline{X}, \overline{p}) = z^*$. Furthermore, since each buyer with price different from $\perp$ gets her preferred bundle, $(\overline{X}, \overline{p})$ is also envy-free. Finally, all buyers are either proposed price $\perp$ or the unique price $\min_{o_k\in O^*} v_k(m_k)/m_k$, and thus $\overline{p}$ is fair under any social graph.

$(\Leftarrow)$: Given a fair outcome $(\overline{X}, \overline{p})$ with $sw(\overline{X}, \overline{p}) = z^*$, consider the solution $O^* \subseteq O$ for $\mathcal{K}$, where $o_i \in O^*$ if $|X_i| = m_i$. Then O^* is feasible as $\sum_{o_i\in O^*} w_i = \sum_{i\in N} |X_i| \leq m = k$ and has total profit equal to $\sum_{o_i\in O^*} z_i = \sum_{i\in N\, s.t.\, |X_i|=m_i} v_i(m_i) = sw(\overline{X}, \overline{p})$.

Having shown the above equivalence, it is possible to provide the claimed FPTAS for (SINGLE,SOCIAL)-pricing as follows: we transform in polynomial time the input instance (η, G) into the equivalent one of KNAPSACK, we run the well known FPTAS for such a problem, and finally transform in polynomial time the output to an outcome of the initial problem as described in $\Rightarrow$) to obtain a $(1+\varepsilon)$-approximation of the optimal social welfare, hence the claim. $\square$

Proof. (of Corollary 5)
Consider the output $(\overline{X}, \overline{p})$ of the algorithm described in Theorem 4, and let $N_{\overline{X}} \subseteq N$ be the set of buyers allocated in $(\overline{X}, \overline{p})$. Without loss of generality assume that buyers in $N_{\overline{X}}$ are ordered non-increasingly with respect to the ratios $v_i(m_i)/m_i$, and let h be the index maximizing $\frac{v_h(m_h)}{m_h}\sum_{i=1}^{h} m_i$. Consider the price vector $\overline{p}'$ with $p'_i = \frac{v_h(m_h)}{m_h}$ for all $i \in N_{\overline{X}}$ such that $i \leq h$, and $p'_i = \perp$ otherwise. Let $(\overline{X}', \overline{p}')$ be the outcome allocating bundles only to buyers with price different from $\perp$. Then, $(\overline{X}', \overline{p}')$ is a fair outcome for (η, G) of revenue $\frac{v_h(m_h)}{m_h}\sum_{i=1}^{h-1} |X_i| \geq \frac{\sum_{i=1}^{k-1} v_i(m_i)}{\log n}$. Furthermore $\overline{p}$ is fair under G, since the same price is proposed to all buyers i such that $p'_i \neq \perp$. $\square$

Proof. (of Lemma 8)
Consider the graph $H^2 = (V, F^2)$, where:

$$F^2 = F \cup \{\{u, v\} \in V^2 | k \in V, \{u, k\}, \{k, v\} \in E\}.$$

By construction H^2 has degree at most δ^2. By the Brooks' Theorem H^2 can be colored using at most $\delta^2 + 1$ colors, and such a coloring can be found in polynomial time. Let $\mathcal{S}$ be the partition of V induced by such a coloring. Since $F \subseteq F^2$, property ***i.*** holds. Suppose then by contradiction that property ***ii.*** does not hold for $\mathcal{S}$, that is there exists $w \in V$ that has two neighbors v, u belonging to $S_i \in \mathcal{S}$. Since v and u share a neighbor in H, $\{v, u\} \in F^2$, but this implies that $\mathcal{S}$ is not induced by a coloring of H^2: a contradiction. $\square$

Proof. (of Theorem 9)
Let us first refresh the reduction provide in [Flammini *et al.*, 2018]. Given a buyer $i \in N$, let $S_i = \{m_i^1, ..., m_i^\ell\}$ be the bundle sizes that are in the demand set of i for at least one positive price, listed in non-decreasing order. Let $m_{i_1} = m_i^1$ and $m_{i_j} = m_i^j - m_i^{j-1}$ for $2 \leq j \leq \ell$. The reduction transforms buyer i into ℓ single-minded *marginal buyers* $i_1, \ldots, i_\ell$, where i_j has preferred bundle size m_{i_j} and valuation $v_{i_j}(m_{i_j}) = v_i(m_i^j) - v_i(m_i^{j-1})$. The reduced social graph $G' = (N', E')$ is such that $(i_j, i'_{j'}) \in E'$ if and only if $(i, i') \in E$.

The authors have shown that the ratios $\frac{v_{i_j}(m_{i_j})}{m_{i_j}}$ are non-increasing in j. Moreover, if an approximation algorithm for single-minded buyers applied on a reduced instance allocates bundles only to prefixes of marginal buyers, then its solution can be transformed back into an outcome for the initial problem preserving the same approximation ratio.

Unfortunately, the FPTAS given in the previous section for single-minded instances does not have such a property.

Hence, we devise an ad-hoc procedure that 2-approximates the social welfare, while allocating prefixes of marginal buyers.

Given an instance (η, G) of (GENERAL,SOCIAL)-pricing, let (η', G') be the associated output of the reduction of [Flammini *et al.*, 2018]. Consider the following algorithm for maximizing $opt_{sw}(\eta', G)$:

- Sort all marginal buyers by the ratios $\frac{v_{i_j}(m_{i_j})}{m_{i_j}}$ in non-increasing order, where in case of ties the marginal buyers i_j of a same buyer i are listed in order of j; let $\pi(i_j)$ be the position in the order of each i_j.
- Compute the following two outcomes:
  - $(\overline{X}', \overline{p}')$: Let i'_h be the last marginal buyer in the order such that $\sum_{i_j|\pi(i_j)\leq\pi(i'_h)} m_{i_j} \leq m$. Set $|X'_{i_j}| = m_{i_j}$ and $p_{i_j} = \frac{v_{i'h}(m_{i'_h})}{m_{i'_h}}$ if $\pi(i_j) \leq \pi(i'_h)$, and $|X'_{i_j}| = 0$ and $p_{i_j} = \perp$ otherwise;
  - $(\overline{X}'', \overline{p}'')$: Let i''_l be the marginal buyer following i'_h in the order, that is such that $\pi(i''_l) = \pi(i'_h) + 1$ (if not existing all the marginal buyers are allocated in $(\overline{X}', \overline{p}')$, that is in turn an optimal solution). For all i''_j with $j \leq l$, set $|X_{i''_j}| = m_{i''_j}$ and $p_{i''_j} = 0$, while set price equal to $\perp$ and give not items to all the other buyers.
- Return argmax$\{sw(\overline{X}', \overline{p}'), sw(\overline{X}'', \overline{p}'')\}$.

Notice that both $(\overline{X}', \overline{p}')$ and $(\overline{X}'', \overline{p}'')$ are fair under G', as in both a unique price different from bottom is proposed. Furthermore, since for each buyer i_j with a non bottom price $p_{i_j} \leq \frac{v_{i_j}(m_{i_j})}{m_{i_j}}$ and $|X'_{i_j}| = |X''_{i_j}| = m_{i_j}$, both $\overline{X}'$ and $\overline{X}''$ are also envy-free.

Notice also that $sw(\overline{X}', \overline{p}')+sw(\overline{X}', \overline{p}')$ is an upper bound on $opt_{sw}(\mu', G')$, as the union of their allocated buyers corresponds to an optimal outcome for supply bigger than m. Thus choosing the best of the two solutions ensures approximation ratio equal to 2. The claim then follows by observing that both $(\overline{X}', \overline{p}')$ and $(\overline{X}'', \overline{p}'')$ allocate only prefixes of marginal buyers, hence by the properties of the reduction they can be turned back into corresponding outcomes for (μ, G) with the same approximation ratio. □

Proof. (of Theorem 10)
In a similar way as in [Flammini *et al.*, 2019], in order to prove this result we provide a polynomial-time reduction from DENSEST K-SUBGRAPH. In such a problem, given an undirected graph $H = (V, F)$ and an integer k, we are interested in determining a subset $S \subseteq V$ with $|S| \leq k$ that maximizes the number of edges in the subgraph induced by S. Let $\varphi = (|F| + 1)(|V| + 1)$, $\psi = (|F| + 1)$, and $\varepsilon = \frac{1}{k\psi+1}$. Given an instance $(H = (V, F), k)$ of DENSEST K-SUBGRAPH, consider the reduced instance (μ, G) built as follows:

- To each node $\mathbf{v} \in V$ associate one buyer $i_\mathbf{v}$ with valuation function
$$v_{i_\mathbf{v}}(j) = \begin{cases} 2\varphi & \text{if } j = \varphi \\ 2\varphi + (1+\varepsilon)\psi & \text{if } j = \varphi + \psi \\ 0 & \text{otherwise} \end{cases}$$
- To each node $\mathbf{e} \in F$ associate one buyer $i_\mathbf{e}$ with valuation function
$$v_{i_\mathbf{e}}(j) = \begin{cases} 2\varphi & \text{if } j = \varphi \\ 2\varphi + 1 & \text{if } j = \varphi + 1 \\ 0 & \text{otherwise} \end{cases}$$
- Set $m = (|V| + |F|)\varphi + k\psi + |F|$.
- The social graph $G = (I, E)$ is built as follows: if $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\} \in F$, then $(i_\mathbf{v}, i_\mathbf{e})$, $(i_\mathbf{u}, i_\mathbf{e}) \in E$, setting $\alpha_{\mathbf{ue}} = \alpha_{\mathbf{ve}} = 0$.

Let us analyze the contribution of each buyer to the social welfare. Clearly, if a price is not proposed to a generic buyer, her contribution is 0. If instead a price $p_i \leq 2$ is proposed, buyer i contributes at least 2φ, having a non-negative utility for a bundle of size at least φ. For a generic buyer $i_\mathbf{u}$, if $p_{i_\mathbf{u}} \leq 1 + \varepsilon$, it possible to allocate to her ψ more items, increasing in this way her contribution to the social welfare of exactly $(1 + \varepsilon)\psi$. For a generic buyer $i_\mathbf{e}$ instead, if $p_{i_\mathbf{e}}$ is dropped to 1, it is possible to allocate to her 1 more item, increasing her contribution to the social welfare of exactly 1.

Since $\varphi > (1+\varepsilon)(|V|)\psi+|F|$, an outcome that maximizes the social welfare must propose a price $p_i \leq 2$ to all the buyers, allocating at least φ items to all of them. Since $\psi > |F|$, with the remaining supply $kD+|F|$, we can possibly allocate ψ more items to at most k buyers $i_\mathbf{v}$, and one more item to each of the $|F|$ buyers $i_\mathbf{e}$. However, by the construction of G, setting $p_{i_\mathbf{e}} = 1$ for some $\mathbf{e} = (\mathbf{u}, \mathbf{v})$ forces us to propose price 1 also to buyers $i_\mathbf{u}, i_\mathbf{v}$. Then, in order for such buyers not to be envious, we must allocate ψ more items to each of them. We conclude by observing that the solution that maximizes the social welfare in (μ, G) allocates φ items to all buyers, ψ more items to each buyer $i_\mathbf{u}$ belonging to subset S, with $|S| \leq k$, and one more item to all the buyers $i_{(\mathbf{u},(\mathbf{v}))}$ such that $i_\mathbf{u}, i_\mathbf{v} \in S$. The social-welfare of an optimal solution is then equal to $2(|V|+|F|)\varphi+(1+\varepsilon)k\psi+h$, where h is the cardinality of the DENSEST K-SUBGRAPH of H. Finding an optimal S is thus equivalent to finding a DENSEST K-SUBGRAPH in (V, F). □