Supplemental material

Proof. (of Proposition 1)
Consider the following instance (η, G) with two single-minded buyers: buyer 1 with $v_1(1) = 1 + \varepsilon$ and buyer 2 with $v_2(m) = m$; graph G contains only the arc $(1, 2)$. If we do not allow bottom prices, the best possible outcome is the one that sells only one item to buyer 1 at a price $1 + \varepsilon$, obtaining social welfare and revenue $1+\varepsilon$. On the other hand, if bottom prices are allowed, we can set $p_1 = \perp$ and $p_2 = 1$, allocating m items to buyer 2 and achieving social welfare and revenue m.

In order to prove the this bound is tight, consider an optimal outcome $(\overline{X}_\perp, \overline{p}_\perp)$ for an instance (η, G) using bottom prices. Let v_{max} be equal to $\max_{i \in N, j \leq m}\{\frac{v_i(j)}{j}\}$. Clearly, $v_{max} \cdot m$ is an upper bound to $sw(\overline{X}_\perp, \overline{p}_\perp)$, and then also to $r(\overline{X}_\perp, \overline{p}_\perp)$. Now, let i_{max} and j_{max} be respectively a buyer and a bundle size such that $v_{i_{max}}(j_{max}) = v_{max}$. Consider the outcome $(\overline{X}, \overline{p})$ where $|X_{i_{max}}| = j_{max}$, $X_i = \emptyset$ for all the other buyers, and $p_1 = \ldots = p_n = v_{max}$. Since the utility of all the buyers are not strictly positive, X is envy free. Moreover, as all the proposed prices are coincident, $\overline{p}$ is fair under G. The claim then follows by observing that $\overline{p}$ does not have any entry equal to $\perp$ and $sw(\overline{X}, \overline{p}) = r(\overline{X}, \overline{p}) = v_{max}$, □

Proof. (of Theorem 4)
In order to get the claimed FPTAS, we transform (η, G) into an equivalent instance $\mathcal{K} = (O, \overline{w}, \overline{z}, k)$ of KNAPSACK as follows:

- O contains an object o_i for each buyer i with profit $z_i = v_i(m_i)$ and weight $w_i = m_i$;
- the knapsack capacity k is set equal to m.

We now prove that ($\Rightarrow$) for each solution $O^* \subseteq O$ of $\mathcal{K}$ with profit z^* there exists a fair outcome $(\overline{X}, \overline{p})$ with $sw(\overline{X}, \overline{p}) = z^*$, and ($\Leftarrow$) vice versa.

($\Rightarrow$): Given a solution $O^* \subseteq O$ for $\mathcal{K}$ with profit z^*, consider the outcome $(\overline{X}, \overline{p})$ defined as follows:

$$|X_i| = \begin{cases} m_i & \text{if } o_i \in O^* \\ 0 & \text{otherwise} \end{cases}$$
$$p_i = \begin{cases} \min_{o_k \in O^*} v_k(m_k)/m_k & \text{if } o_i \in O^* \\ \perp & \text{otherwise} \end{cases}$$

By construction, $\sum_{i \in N} |X_i| = \sum_{o_i \in O^*} w_i \leq k = m$ and $\sum_{i \in N} v_i(X_i) = \sum_{o_i \in O^*} z_i$, that is $(\overline{X}, \overline{p})$ satisfies the supply constrains and has social welfare $sw(\overline{X}, \overline{p}) = z^*$. Furthermore, since each buyer with price different from $\perp$ gets her preferred bundle, $(\overline{X}, \overline{p})$ is also envy-free. Finally, all buyers are either proposed price $\perp$ or the unique price $\min_{o_k \in O^*} v_k(m_k)/m_k$, and thus $\overline{p}$ is fair under any social graph.

($\Leftarrow$): Given a fair outcome $(\overline{X}, \overline{p})$ with $sw(\overline{X}, \overline{p}) = z^*$, consider the solution $O^* \subseteq O$ for $\mathcal{K}$, where $o_i \in O^*$ if $|X_i| = m_i$. Then O^* is feasible as $\sum_{o_i \in O^*} w_i = \sum_{i \in N} |X_i| \leq m = k$ and has total profit equal to $\sum_{o_i \in O^*} z_i = \sum_{i \in N \text{ s.t. } |X_i| = m_i} v_i(m_i) = sw(\overline{X}, \overline{p})$.

Having shown the above equivalence, it is possible to provide the claimed FPTAS for (SINGLE,SOCIAL)-pricing as follows: we transform in polynomial time the input instance (η, G) into the equivalent one of KNAPSACK, we run the well known FPTAS for such a problem, and finally transform in polynomial time the output to an outcome of the initial problem as described in $\Rightarrow$) to obtain a $(1 + \varepsilon)$-approximation of the optimal social welfare, hence the claim. □

Proof. (of Corollary 5)
Consider the output $(\overline{X}, \overline{p})$ of the algorithm described in Theorem 4, and let $N_{\overline{X}} \subseteq N$ be the set of buyers allocated in $(\overline{X}, \overline{p})$. Without loss of generality assume that buyers in $N_{\overline{X}}$ are ordered non-increasingly with respect to the ratios $v_i(m_i)/m_i$, and let h be the index maximizing $\frac{v_h(m_h)}{m_h} \sum_{i=1}^{h} m_i$. Consider the price vector $\overline{p}'$ with $p'_i = \frac{v_h(m_h)}{m_h}$ for all $i \in N_{\overline{X}}$ such that $i \leq h$, and $p'_i = \perp$ otherwise. Let $(\overline{X}', \overline{p}')$ be the outcome allocating bundles only to buyers with price different from $\perp$. Then, $(\overline{X}', \overline{p}')$ is a fair outcome for (η, G) of revenue $\frac{v_h(m_h)}{m_h} \sum_{i=1}^{h-1} |X_i| \geq \frac{\sum_{i=1}^{k-1} v_i(m_i)}{\log n}$. Furthermore $\overline{p}$ is fair under G, since the same price is proposed to all buyers i such that $p'_i \neq \perp$. □

Proof. (of Lemma 8)
Consider the graph $H^2 = (V, F^2)$, where:

$$F^2 = F \cup \{\{u, v\} \in V^2 | k \in V, \{u, k\}, \{k, v\} \in E\}.$$

By construction H^2 has degree at most δ^2. By the Brooks' Theorem H^2 can be colored using at most $\delta^2 + 1$ colors, and such a coloring can be found in polynomial time. Let $\mathcal{S}$ be the partition of V induced by such a coloring. Since $F \subseteq F^2$, property ***i.*** holds. Suppose then by contradiction that property ***ii.*** does not hold for $\mathcal{S}$, that is there exists $w \in V$ that has two neighbors v, u belonging to $S_i \in \mathcal{S}$. Since v and u share a neighbor in H, $\{v, u\} \in F^2$, but this implies that $\mathcal{S}$ is not induced by a coloring of H^2: a contradiction. □

Proof. (of Theorem 9)
Let us first refresh the reduction provide in [Flammini *et al.*, 2018]. Given a buyer $i \in N$, let $S_i = \{m_i^1, ..., m_i^\ell\}$ be the bundle sizes that are in the demand set of i for at least one positive price, listed in non-decreasing order. Let $m_{i_1} = m_i^1$ and $m_{i_j} = m_i^j - m_i^{j-1}$ for $2 \leq j \leq \ell$. The reduction transforms buyer i into ℓ single-minded *marginal buyers* $i_1, \ldots, i_\ell$, where i_j has preferred bundle size m_{i_j} and valuation $v_{i_j}(m_{i_j}) = v_i(m_i^j) - v_i(m_i^{j-1})$. The reduced social graph $G' = (N', E')$ is such that $(i_j, i'_{j'}) \in E'$ if and only if $(i, i') \in E$.

The authors have shown that the ratios $\frac{v_{i_j}(m_{i_j})}{m_{i_j}}$ are non-increasing in j. Moreover, if an approximation algorithm for single-minded buyers applied on a reduced instance allocates bundles only to prefixes of marginal buyers, then its solution can be transformed back into an outcome for the initial problem preserving the same approximation ratio.

Unfortunately, the FPTAS given in the previous section for single-minded instances does not have such a property.

Hence, we devise an ad-hoc procedure that 2-approximates the social welfare, while allocating prefixes of marginal buyers.

Given an instance (η, G) of (GENERAL,SOCIAL)-pricing, let (η', G') be the associated output of the reduction of [Flammini *et al.*, 2018]. Consider the following algorithm for maximizing $opt_{sw}(\eta', G)$:

- Sort all marginal buyers by the ratios $\frac{v_{i_j}(m_{i_j})}{m_{i_j}}$ in non-increasing order, where in case of ties the marginal buyers i_j of a same buyer i are listed in order of j; let $\pi(i_j)$ be the position in the order of each i_j.
- Compute the following two outcomes:
  - $(\overline{X}', \overline{p}')$: Let i'_h be the last marginal buyer in the order such that $\sum_{i_j|\pi(i_j)\leq\pi(i'_h)} m_{i_j} \leq m$. Set $|X'_{i_j}| = m_{i_j}$ and $p_{i_j} = \frac{v_{i'_h}(m_{i'_h})}{m_{i'_h}}$ if $\pi(i_j) \leq \pi(i'_h)$, and $|X'_{i_j}| = 0$ and $p_{i_j} = \perp$ otherwise;
  - $(\overline{X}'', \overline{p}'')$: Let i''_l be the marginal buyer following i'_h in the order, that is such that $\pi(i''_l) = \pi(i'_h) + 1$ (if not existing all the marginal buyers are allocated in $(\overline{X}', \overline{p}')$, that is in turn an optimal solution). For all i''_j with $j \leq l$, set $|X_{i''_j}| = m_{i''_j}$ and $p_{i''_j} = 0$, while set price equal to $\perp$ and give not items to all the other buyers.
- Return $\text{argmax}\{sw(\overline{X}', \overline{p}'), sw(\overline{X}'', \overline{p}'')\}$.

Notice that both $(\overline{X}', \overline{p}')$ and $(\overline{X}'', \overline{p}'')$ are fair under G', as in both a unique price different from bottom is proposed. Furthermore, since for each buyer i_j with a non bottom price $p_{i_j} \leq \frac{v_{i_j}(m_{i_j})}{m_{i_j}}$ and $|X'_{i_j}| = |X''_{i_j}| = m_{i_j}$, both $\overline{X}'$ and $\overline{X}''$ are also envy-free.

Notice also that $sw(\overline{X}', \overline{p}') + sw(\overline{X}', \overline{p}')$ is an upper bound on $opt_{sw}(\mu', G')$, as the union of their allocated buyers corresponds to an optimal outcome for supply bigger than m. Thus choosing the best of the two solutions ensures approximation ratio equal to 2. The claim then follows by observing that both $(\overline{X}', \overline{p}')$ and $(\overline{X}'', \overline{p}'')$ allocate only prefixes of marginal buyers, hence by the properties of the reduction they can be turned back into corresponding outcomes for (μ, G) with the same approximation ratio. □

Proof. (of Theorem 10)
In a similar way as in [Flammini *et al.*, 2019], in order to prove this result we provide a polynomial-time reduction from DENSEST K-SUBGRAPH. In such a problem, given an undirected graph $H = (V, F)$ and an integer k, we are interested in determining a subset $S \subseteq V$ with $|S| \leq k$ that maximizes the number of edges in the subgraph induced by S. Let $\varphi = (|F| + 1)(|V| + 1)$, $\psi = (|F| + 1)$, and $\varepsilon = \frac{1}{k\psi+1}$. Given an instance $(H = (V, F), k)$ of DENSEST K-SUBGRAPH, consider the reduced instance (μ, G) built as follows:

- To each node $\mathbf{v} \in V$ associate one buyer $i_{\mathbf{v}}$ with valuation function

$$v_{i_{\mathbf{v}}}(j) = \begin{cases} 2\varphi & \text{if } j = \varphi \\ 2\varphi + (1+\varepsilon)\psi & \text{if } j = \varphi + \psi \\ 0 & \text{otherwise} \end{cases}$$

- To each node $\mathbf{e} \in F$ associate one buyer $i_{\mathbf{e}}$ with valuation function

$$v_{i_{\mathbf{e}}}(j) = \begin{cases} 2\varphi & \text{if } j = \varphi \\ 2\varphi + 1 & \text{if } j = \varphi + 1 \\ 0 & \text{otherwise} \end{cases}$$

- Set $m = (|V| + |F|)\varphi + k\psi + |F|$.
- The social graph $G = (I, E)$ is built as follows: if $\mathbf{e} = \{\mathbf{u}, \mathbf{v}\} \in F$, then $(i_{\mathbf{v}}, i_{\mathbf{e}})$, $(i_{\mathbf{u}}, i_{\mathbf{e}}) \in E$, setting $\alpha_{\mathbf{ue}} = \alpha_{\mathbf{ve}} = 0$.

Let us analyze the contribution of each buyer to the social welfare. Clearly, if a price is not proposed to a generic buyer, her contribution is 0. If instead a price $p_i \leq 2$ is proposed, buyer i contributes at least 2φ, having a non-negative utility for a bundle of size at least φ. For a generic buyer $i_{\mathbf{u}}$, if $p_{i_{\mathbf{u}}} \leq 1 + \varepsilon$, it possible to allocate to her ψ more items, increasing in this way her contribution to the social welfare of exactly $(1 + \varepsilon)\psi$. For a generic buyer $i_{\mathbf{e}}$ instead, if $p_{i_{\mathbf{e}}}$ is dropped to 1, it is possible to allocate to her 1 more item, increasing her contribution to the social welfare of exactly 1.

Since $\varphi > (1+\varepsilon)(|V|)\psi + |F|$, an outcome that maximizes the social welfare must propose a price $p_i \leq 2$ to all the buyers, allocating at least φ items to all of them. Since $\psi > |F|$, with the remaining supply $kD + |F|$, we can possibly allocate ψ more items to at most k buyers $i_{\mathbf{v}}$, and one more item to each of the $|F|$ buyers $i_{\mathbf{e}}$. However, by the construction of G, setting $p_{i_{\mathbf{e}}} = 1$ for some $\mathbf{e} = (\mathbf{u}, \mathbf{v})$ forces us to propose price 1 also to buyers $i_{\mathbf{u}}, i_{\mathbf{v}}$. Then, in order for such buyers not to be envious, we must allocate ψ more items to each of them. We conclude by observing that the solution that maximizes the social welfare in (μ, G) allocates φ items to all buyers, ψ more items to each buyer $i_{\mathbf{u}}$ belonging to subset S, with $|S| \leq k$, and one more item to all the buyers $i_{(\mathbf{u},(\mathbf{v}))}$ such that $i_{\mathbf{u}}, i_{\mathbf{v}} \in S$. The social-welfare of an optimal solution is then equal to $2(|V|+|F|)\varphi+(1+\varepsilon)k\psi+h$, where h is the cardinality of the DENSEST K-SUBGRAPH of H. Finding an optimal S is thus equivalent to finding a DENSEST K-SUBGRAPH in (V, F). □