Supplementary Material: Minimizing Adaptive Regret with One Gradient per Iteration

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A Main Analysis

A.1 Proof of Theorem 1

We start our proof by investigating a new game between the learner and the adversary. In round t, firstly the learner is asked to choose an action $\mathbf{x}_t$ in $\mathcal{D}$; next, an adversary *reveals a loss function* $L_t^e(\cdot)$, as defined in (8), instead of $f_t(\cdot)$ to the learner. Then the learner suffers a loss $L_t^e(\mathbf{x}_t)$. By definition, the SAR incurred under this new scenario is:

$$\begin{aligned}&\mathrm{SAR}_{L_1^e,\dots,L_T^e}(\tau)\\ &= \max_{I\subseteq[T],|I|=\tau}\left\{\sum_{t\in I} L_t^e(\mathbf{x}_t)-\sum_{t\in I} L_t^e(\mathbf{x}_*^{I,e})\right\}.\end{aligned} \tag{26}$$

where $\mathbf{x}_*^{I,e} = \max_{\mathbf{x}\in\mathcal{D}}\sum_{t\in I} L_t^e(\mathbf{x})$. Our first observation is that for any algorithm, its SAR under the new problem, as defined in (26) , is an upper bound of that of the original problem, as defined in (1).

Lemma 3. *Given a parameter τ, let $\mathrm{SAR}_f(\tau)$ and $\mathrm{SAR}_{L^e}(\tau)$ be the strongly adaptive regret of any online learning algorithm defined in* (1) *and* (26) *respectively.* $\forall\tau < T$*, we have*

$$\mathrm{SAR}_f(\tau)\le \mathrm{SAR}_{L^e}(\tau). \tag{27}$$

Proof of Lemma 3

Given a parameter τ, consider a time interval I_*, such that

$$I_* = \arg\max_{I\subseteq[T],|I|=\tau}\left\{\sum_{t\in I} f_t(\mathbf{x}_t)-\sum_{t\in I} f_t(\mathbf{x}_*^I)\right\}. \tag{28}$$

where $\mathbf{x}_*^I = \max_{\mathbf{x}\in\mathcal{D}}\sum_{t\in I} f_t(\mathbf{x})$. By Lemma 1, we have $\forall t\in I_*$,

$$\begin{aligned}f(\mathbf{x}_t)-f(\mathbf{x}_*^{I_*})\le &-\frac{\gamma}{2}(\mathbf{x}_t-\mathbf{x}_*^{I_*})^\top\nabla_t\nabla_t^\top(\mathbf{x}_t-\mathbf{x}_*^{I_*})\\ &+\nabla_t^\top(\mathbf{x}_t-\mathbf{x}_*^{I_*})\end{aligned}$$

Combining with (8), we get

$$f(\mathbf{x}_t)-f(\mathbf{x}_*^{I_*})\le -L_t^e(\mathbf{x}_*^{I_*})$$

Since $L_t^e(\mathbf{x}_t)=0$, the equation above leads to

$$f(\mathbf{x}_t)-f(\mathbf{x}_*^{I_*})\le L_t^e(\mathbf{x}_t)-L_t^e(\mathbf{x}_*^{I_*}) \tag{29}$$

By adding things together over I, we have

$$\begin{aligned}\mathrm{SAR}_f(\tau)&=\sum_{t\in I_*}\Big(f(\mathbf{x}_t)-f(\mathbf{x}_*^{I_*})\Big)\\ &\le\sum_{t\in I_*}L_t^e(\mathbf{x}_t)-\sum_{t\in I_*}L_t^e(\mathbf{x}_*^{I_*})\\ &\le\sum_{t\in I_*}L_t^e(\mathbf{x}_t)-\sum_{t\in I_*}L_t^e(\mathbf{x}_*^{I_*,e})\\ &\le\max_{I\in[T],|I|=\tau}\sum_{t\in I}L_t^e(\mathbf{x}_t)-\sum_{t\in I}L_t^e(\mathbf{x}_*^{I,e})\\ &=\mathrm{SAR}_{L^e}(\tau)\end{aligned} \tag{30}$$

The second inequality is derived by the definition of $\mathbf{x}_t^I$ and $\mathbf{x}_t^{I,e}$, and the third inequality is obtained by the definition of SAR_{L^e}. The proof can be finished by noticing that the (30) holds $\forall\tau<T$.

Next, we bound the SAR of MARSL-exp under the new problem. To start with, we prove the exp-concavity of $L_t^e(\cdot)$:

Lemma 4. *Suppose Assumption 1 and 2 hold, then $L_t^e(\mathbf{u})$ is α' exp-concave, where*

$$\alpha'=\gamma/\left(1+2\gamma DG+\gamma^2D^2G^2\right).$$

Proof of Lemma 4

We first introduce the following Lemma for exp-concave functions [Hazan, 2016]:

Lemma 5. *A twice differentiable function f is α exp-concave if and only if $\nabla^2 f_t(\mathbf{x})\succeq\alpha\nabla f(\mathbf{x})\nabla f(\mathbf{x})^\top$.*

By (8), the gradient of $L_t^e(\mathbf{u})$ is

$$\nabla L_t^e(\mathbf{u})=\frac{\gamma}{2}\nabla_t\nabla_t^\top(\mathbf{u}-\mathbf{x}_t)+\nabla_t, \tag{31}$$

Thus,

$$\begin{aligned}
&\nabla L_t^e(\mathbf{u})(\nabla L_t^e(\mathbf{u}))^\top \\
=&\gamma^2 \nabla_t \nabla_t^\top (\mathbf{u}-\mathbf{x}_t)(\mathbf{u}-\mathbf{x}_t)^\top \nabla_t \nabla_t^\top \\
&+ \gamma \nabla_t \nabla_t^\top (\mathbf{u}-\mathbf{x}_t) \nabla_t^\top + \gamma \nabla_t (\mathbf{u}-\mathbf{x}_t)^\top \nabla_t \nabla_t^\top \\
&+ \nabla_t \nabla_t^\top \\
=&\gamma^2 \nabla_t \left[\nabla_t^\top (\mathbf{u}-\mathbf{x}_t)\right] \left[(\mathbf{u}-\mathbf{x}_t)^\top \nabla_t\right] \nabla_t^\top \\
&+ \gamma \nabla_t \left[\nabla_t^\top (\mathbf{u}-\mathbf{x}_t)\right] \nabla_t^\top + \gamma \nabla_t \left[(\mathbf{u}-\mathbf{x}_t)^\top \nabla_t\right] \nabla_t^\top \\
&+ \nabla_t \nabla_t^\top \\
\preceq& \left(\gamma^2 D^2 G^2 + 2\gamma D G + 1\right) \nabla_t \nabla_t^\top
\end{aligned}$$

On the other hand,

$$\nabla^2 L_t^e(\mathbf{u}) = \gamma \nabla_t \nabla_t^\top. \tag{32}$$

The proof is finished by using Lemma 5.

Lemma 4 indicates that the series of loss function $L_1^e(\cdot), ..., L_T^e(\cdot)$ are all exp-concave. Besides, by the definition of $\nabla L_t^e(\cdot)$, we have $\forall \mathbf{u} \in \mathcal{D}$:

$$||\nabla L_t^e(\mathbf{u})|| \leq \frac{5}{4} GD = G^e \tag{33}$$

Thus, by simply applying AFLH on the new problem, which is exactly as we do in MARSL-exp, according to Theorem 1.2 in [Hazan and Seshadhri, 2007], we can immediately obtain the following lemma:

Lemma 6. *Suppose Assumption 1 and 2 hold and all functions f_t are α-exp-concave. Then, MARSL-exp achieves:*

$$\begin{aligned}
&\mathrm{SAR}_{L_1^e,...,f_T^e}(\tau) \\
&\leq (\log T + 1)\left(\left(\frac{4+5d}{\alpha'} + dG^e D\right)\log T + 1\right) \\
&= O(\frac{1}{\alpha'}\log^2 T)
\end{aligned} \tag{34}$$

The Proof of Theorem 1 can be finished by combining Lemma 3 and Lemma 6.

A.2 Proof of Theorem 2

For strongly convex functions, following the proof of Theorem 1, we define the SAR under the loss function series $L_t^{sc}(\cdot), ..., L_t^{sc}(\cdot)$ as

$$\begin{aligned}
&\mathrm{SAR}^T_{L_1^{sc},...,L_T^{sc}}(\tau) \\
&= \max_{I \subseteq [T], |I| = \tau} \left\{ \sum_{t \in I} L_t^{sc}(\mathbf{x}_t) - \sum_{t \in I} L_t^c(\mathbf{x}_*^{I,sc}) \right\}.
\end{aligned} \tag{35}$$

where $\mathbf{x}_*^{I,sc} = \max_{\mathbf{x} \in \mathcal{D}} \sum_{t \in I} L_t^{sc}(\mathbf{x})$. We first note that it can be easily verified that the surrogate loss function series $L_1^{sc}(\cdot), \ldots, L_T^{sc}(\cdot)$ are all λ-strongly convex. Besides, we have $\forall \mathbf{u} \in \mathcal{D},\ t \in [T]$,

$$||\nabla L_t^{sc}(\mathbf{u})|| = ||\lambda(\mathbf{x}_t - \mathbf{u}) - \nabla_t|| \leq \lambda D + G = G^{sc}$$

By Lemma 2, it implies that all loss functions are λ/G^{sc}-exp-concave. Thus, according to Lemma 3, we derive the following lemma:

Lemma 7. *Given a parameter τ, let $\mathrm{SAR}_f(\tau)$ and $\mathrm{SAR}_{L^{sc}}(\tau)$ be the strongly adaptive regret of any online learning algorithm defined in* (1) *and* (35) *respectively. $\forall \tau < T$, we have*

$$\mathrm{SAR}_f(\tau) \leq \mathrm{SAR}_{L^e}(\tau). \tag{36}$$

Next, we focus on the SAR of our algorithm under the new problem. Following Lemma 4.5 in [Hazan and Seshadhri, 2007], we have

Lemma 8. *Suppose Assumption 1 and 2 hold and all functions $L_1^{sc}(\cdot), \ldots, L_T^{sc}(\cdot)$ are λ strongly convex and $\forall t \in [T]$, $||\nabla L_t^{sc}(\cdot)|| \leq G^{sc}$. Then, MARSL-sc attains*

$$\begin{aligned}
&\mathrm{SAR}_{L_1^{sc},...,L_T^{sc}}(\tau) \\
&\leq (\log T + 1)\left(\frac{(G^{sc})^2}{2\lambda}(\log T + 1) + 1\right) \\
&= O\left(\log^2 T\right)
\end{aligned} \tag{37}$$

The proof of Theorem 2 can be finished by combining Lemma 7 and 8.

A.3 Proof of Theorem 3

To begin with, following the proof of Theorem 1, we define the SAR under the loss function series $L_t^c(\cdot), ..., L_t^c(\cdot)$ as

$$\begin{aligned}
&\mathrm{SAR}_{L_1^c,...,L_T^c}(\tau) \\
&= \max_{I \subseteq [T], |I| = \tau} \left\{ \sum_{t \in I} L_t^c(\mathbf{x}_t) - \sum_{t \in I} L_t^c(\mathbf{x}_*^{I,c}) \right\}.
\end{aligned} \tag{38}$$

where $\mathbf{x}_*^{I,c} = \max_{\mathbf{x} \in \mathcal{D}} \sum_{t \in I} L_t^c(\mathbf{x})$. The following Lemma illustrates the relations between (38) and (1):

Lemma 9. *Given a parameter τ, let $SAR_f(\tau)$ and $SAR_{L^c}(\tau)$ be the strongly adaptive regret defined in* (1) *and* (38). *$\forall \tau < T$, we have*

$$\mathrm{SAR}_f(\tau) \leq 2GD\left(SAR_{L^c}(\tau)\right). \tag{39}$$

Proof of Lemma 9

Given a parameter τ, consider a time interval I_*, which is defined in (28). $\forall t \in I_*$, by the convexity of $f_t(\cdot)$, we have

$$f(\mathbf{x}_t) - f(\mathbf{x}_*^{I_*}) \leq \nabla f(\mathbf{x}_t)^\top (\mathbf{x}_t - \mathbf{x}_*^{I_*}) \tag{40}$$

By (21), we get

$$f(\mathbf{x}_t) - f(\mathbf{x}_*^{I_*}) \leq -2GDL_t^c(\mathbf{x}_*^{I_*}) + GD.$$

Since $L_t^c(\mathbf{x}_t) = 1/2$, the equation above leads to

$$f(\mathbf{x}_t) - f(\mathbf{x}_*^{I_*}) \leq 2GD\left(L_t^c(\mathbf{x}_t) - L_t^c(\mathbf{x}_*^{I_*})\right) \tag{41}$$

By summing the inequality above in both side over I_*, given τ, we have

$$\begin{aligned}
\mathrm{SAR}_f(\tau) &= \sum_{t\in I_*}\left(f(\mathbf{x}_t) - f(\mathbf{x}_*^{I_*})\right) \\
&\leq 2GD\left(\sum_{t\in I_*} L_t^c(\mathbf{x}_t) - \sum_{t\in I_*} L_t^c(\mathbf{x}_*^{I_*})\right) \\
&\leq 2GD\left(\sum_{t\in I_*} L_t^c(\mathbf{x}_t) - \sum_{t\in I_*} L_t^c(\mathbf{x}_*^{I_*,e})\right) \\
&\leq 2GD\left(\max_{I\in[T],|I|=\tau}\sum_{t\in I} L_t^c(\mathbf{x}_t) - \sum_{t\in I} L_t^c(\mathbf{x}_*^{I,c})\right) \\
&= \mathrm{SAR}_{L^c}(\tau)
\end{aligned}$$

The proof can be finished by noticing that the inequality above holds $\forall\tau < T$.

Next, we bound the SAR of our algorithm under the new problem. To start with, it is easy to verify that $L_t^c(\mathbf{u})$ is convex, since it is a linear function about $\mathbf{u}$. Besides, by the definition of $\nabla L_t^e(\cdot)$,we have $\forall\mathbf{u}\in\mathcal{D}$:

$$||\nabla L_t^e(\mathbf{u})||_2 \leq \frac{1}{2D} = G^c \tag{42}$$

Thus, by simply applying CBCE on the new problem, which as we do in MARSL-gc, according to Theorem 2 in [Jun *et al.*, 2016], we can immediately obtain the following lemma:

Lemma 10. *Suppose Assumption 1 and 2 hold, all functions* f_t *are* α *exp-concave. We have:*

$$\begin{aligned}
&\mathrm{SAR}^T_{L_1^c,\ldots,L_T^c}(\tau) \\
&\leq \sqrt{\tau}\left(\frac{12D^2G^c}{\sqrt{2}-1} + 8\sqrt{7\log T + 5}\right) \\
&= O\left(\sqrt{\tau\log T}\right)
\end{aligned} \tag{43}$$

The Proof of Theorem 1 can be finished by combining Lemma 3 and Lemma 10.