Proofs

Chapter 1

Proof for Theorem 1

Proof. The proof of Theorem 1 is shown below. □

To prove a mechanism that satisfies the Definition 2 is DSIC, we first use the sufficient and necessary characterisation of incentive compatible mechanisms for a setting where users can only misreport their valuation function (i.e., where the misreports of $\hat{T}_i^a$, $\hat{T}_i^s$, $\hat{T}_i^f$, $\hat{t}_i$, $\{\hat{a}_{i,r}\}_{r\in R}$ and $\{\hat{\Gamma}_l^i\}_{l\in L}$ are not considered) proposed by Bartal et al. (2003) .

Lemma 1. *In our setting, assuming the parameters $\hat{T}_i^a$, $\hat{T}_i^s$, $\hat{T}_i^f$ $\hat{t}_i$, $\{\hat{a}_{i,r}\}_{r\in R}$, $\{\hat{\Gamma}_l^i\}_{l\in L}$ in user i's bid $\hat{\theta}_i$ are truthful, a direct revelation mechanism is DSIC if and only if*

1. *The payment function $\tilde{p}_i(\lambda_i, \hat{\theta}^{\langle \hat{T}_i^a \rangle})$ is computed for every possible allocation $\tilde{t}_i$ for task i and does not depend on $\hat{\boldsymbol{v}}_i$.*
2. *The allocation function allocates λ_i for task i such that the value of $\hat{\boldsymbol{v}}_i - \tilde{p}_i$ is maximised (over all λ_i that can be allocated to i for any choice of $\hat{\boldsymbol{v}}_i$).*

Lemma 1 is an extension of Theorem 1 in (Bartal et al., 2003).

Then, we use this Lemma to prove the following theorem that is in settings where users can misreport $\hat{T}_i^a, \hat{T}_i^s, \hat{T}_i^f$, $\hat{t}_i$, $\{\hat{a}_{i,r}\}_{r\in R}$, $\{\hat{\Gamma}_l^i\}_{l\in L}$ besides $\hat{\boldsymbol{v}}_i$ (assuming limited misreports).

Theorem 2. *In our setting, a direct revelation mechanism is DSIC if and only if*

1. *The payment function $\tilde{p}_i(\lambda_i, \hat{\theta}^{\langle \hat{T}_i^a \rangle})$ is computed for every possible allocation $\tilde{t}_i$ for task i and does not depend on $\hat{\boldsymbol{v}}_i$ and monotonic.*
2. *The allocation function allocates λ_i for task i such that the value of $\hat{\boldsymbol{v}}_i - \tilde{p}_i$ is maximised (over all λ_i that can be allocated to i for any choice of $\hat{\boldsymbol{v}}_i$).*

Proof. At first, we show that the conditions in Theorem 2 are sufficient. According to Lemma 1, a user i cannot increase its utility by manipulating $\hat{\boldsymbol{v}}_i$. Therefore, we can assume that it truthfully reports its valuation coefficient $\hat{\boldsymbol{v}}_i$, and the only way to increase its utility is by decreasing the payment function. Since the payment function is monotonic, only misreporting $\hat{T}_i^a < T_i^a, \hat{T}_i^s < T_i^s, \hat{T}_i^f > T_i^f$, $\hat{t}_i < t_i$, $\hat{a}_{i,r} < a_{i,r}, r \in R$ or $\hat{\Gamma}_l^i < \Gamma_l^i, l \in L$ can reduce it. First of all, misreporting $\hat{T}_i^a < T_i^a, \hat{T}_i^s < T_i^s, \hat{T}_i^f > T_i^f$ is impossible because the limited misreports assumption we made earlier. Then, misreporting $\hat{t}_i < t_i$ only reduces u_i because of condition two (it only reduce the space of λ_i). Finally, misreporting $\hat{a}_{i,r} < a_{i,r}$ $r \in R$ or $\hat{\Gamma}_l^i < \Gamma_l^i$ $l \in L$ will lead to the failure of the task, which reduces u_i to negative. Hence, user i has no incentive to submit a non-truthful bid (i.e., $(\hat{T}_i^a, \hat{T}_i^s, \hat{T}_i^f, \hat{\boldsymbol{v}}_i, \hat{t}_i, \{\hat{a}_{i,r}\}_{r\in R}, \{\hat{\Gamma}_l^i\}_{l\in L}) \neq (T_i^a, T_i^s, T_i^f, \boldsymbol{v}_i, t_i, \{a_{i,r}\}_{r\in R}, \{\Gamma_l^i\}_{l\in L})$).

Then, we show that the conditions in theorem 2 are also necessary. We first assume to the contrary that the first condition does not hold, i.e., the payment function is not independent of $\hat{\boldsymbol{v}}_i$ or the payment function is not *monotonic*. In the former case, the mechanism is not DSIC according to Lemma 1. In the latter case, that is, there is some $T_i^{a\prime} < T_i^{a\prime\prime}$ such that the resource allocation is the same ($\lambda' = \lambda''$) but the payment satisfies $\tilde{p}_i(\lambda', T_i^{a\prime}) > \tilde{p}_i(\lambda'', T_i^{a\prime\prime})$, while $\hat{T}_i^s$ and $\hat{T}_i^f$ remain unchanged. On this occasion, a user whose true arrival time is $T_i^{a\prime}$ and who gets allocation λ' when reporting truthfully is incentivised to misreport $T_i^{a\prime}$ as $T_i^{a\prime\prime}$ because it can get the same allocation with less payment. Since this is contrary to the definition of DSIC, the first condition must be necessary.

Then, we assume that the first condition holds, but to the contrary that the second condition does not. For instance, for some user i with $\boldsymbol{v}_i = \boldsymbol{v}_i'$, there exists $\boldsymbol{v}_i''$, the mechanism allocates λ_i' and λ_i'' respectively such that $\boldsymbol{v}_i'\tilde{t}_i' - \tilde{p}_i(\lambda_i', \hat{\theta}^{\langle \hat{T}_i^a \rangle}) < \boldsymbol{v}_i''\tilde{t}_i'' - \tilde{p}_i(\lambda_i'', \hat{\theta}^{\langle \hat{T}_i^a \rangle})$. On this occasion, this user is incentivised to misreport $\boldsymbol{v}_i'$ as $\boldsymbol{v}_i''$. Hence, the second condition is also necessary.

From the above, Theorem 2 is proven. □

Finally, a mechanism that satisfies the Definition 2 is also IR for the following reasons. Since user i will always bid truthfully (i.e., $\hat{\boldsymbol{v}}(t) = \boldsymbol{v}(t)$), the final allocation actually maximises the utility of user i: $u_i = \boldsymbol{v}(\tilde{t}_i) - \tilde{p}_i$. In addition, the maximum of u_i should be greater or equal to zero as i can always get a utility of zero with no resource allocated according to condition 2 in Definition 2. From the above discussion, it is clear that i will never get a negative utility under such mechanisms.

Bibliography

Yair Bartal, Rica Gonen, and Noam Nisan. 2003. Incentive compatible multi unit combinatorial auctions. In *Proceedings of the 9th conference on Theoretical aspects of rationality and knowledge*. ACM, 72–87.