Solution to Atiyah and MacDonald
Chapter 2. Modules

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This is a solution to Exercise problems in Chapter 2 of "Introduction to Commutative Algebra" written by M. F. Atiyah and I. G. MacDonald. You can find the updated version and solutions to other chapters on my personal website: [`https://ijhlee0511.github.io`].

WARNING This solution is written for self-study purposes and to consolidate my understanding. **I do not take responsibility for any disadvantages resulting from the use of this solution. It is at your own risk.** If you find any typos or errors in this solution, please feel free to contact me via email at [`ijhlee0511@gmail.com`] or [`ijhlee0511@kaist.ac.kr`].

Exercises and Solutions

2.1. Show that $(\mathbf{Z}/m\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{Z}/n\mathbf{Z}) = 0$ of m, n are coprime.

Solution. There exist integers x and y such that $xm + yn = 1$. As a result, for any $a \in \mathbf{Z}/m\mathbf{Z}$ and $b \in \mathbf{Z}/n\mathbf{Z}$,

$$a \otimes b = (xm + yn)(a \otimes b) = xma \otimes b + a \otimes ynb = 0.$$

■

2.2. Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module. Show that $(A/\mathfrak{a}) \otimes_A M$ is isomorphic to $M/\mathfrak{a}M$.

Solution. By the natural inclusion and projection, a sequence $0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$ is exact. Tensoring the sequence, we get an exact sequence $\mathfrak{a} \otimes M \longrightarrow M \longrightarrow A/\mathfrak{a} \otimes M \longrightarrow 0$. Since the map $\mathfrak{a} \otimes M \longrightarrow M$ in the sequence is given by $x \otimes m \mapsto xm$, we get $A/\mathfrak{a} \otimes M \cong M/\mathrm{Im}(\mathfrak{a} \otimes M \longrightarrow M) \cong M/\mathfrak{a}M$. ■

2.3. Let A be a local ring, M and N finitely generated A-modules. Prove that if $M \otimes N = 0$, then $M = 0$ or $N = 0$.

Solution. Let $\mathfrak{m}$ be the unique maximal ideal. Suppose neither $M = 0$ nor $N = 0$. Then by Nakayama's lemma, $M/\mathfrak{m}M \neq 0$ and $N/\mathfrak{m}N \neq 0$. We can view $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ as finite dimensional vector spaces over $k := A/\mathfrak{m}$. Therefore, $V := M/\mathfrak{m}M \otimes_k N/\mathfrak{m}N$ is a k-vector space with dimension $(\dim_k M/\mathfrak{m}M) \times$

$(\dim_k N/\mathfrak{m}N)$[1], implying it is nonzero. Define a map $f : M/\mathfrak{m}M \times N/\mathfrak{m}N \longrightarrow V$ by $(\bar{m}, \bar{n}) \mapsto \bar{m} \otimes_k \bar{n}$. Viewing V as an A-module, this is naturally an A-bilinear map, so there is a surjective A-module homomorphism $f^* : M/\mathfrak{m}M \otimes_A N/\mathfrak{m}N \twoheadrightarrow V$ sending $\bar{m} \otimes_A \bar{n}$ to $\bar{m} \otimes_k \bar{n}$. From the natural surjection $N \twoheadrightarrow N/\mathfrak{m}N$ we get an exact sequence $M/\mathfrak{m}M \otimes_A N \to M/\mathfrak{m}M \otimes_A N/\mathfrak{m}N \to 0$; hence, there is a surjective A-module homomomrophism $M/\mathfrak{m}M \otimes_A N \twoheadrightarrow V$. However, by Exercise 2.2,

$$M/\mathfrak{m}M \otimes_A N = (A/\mathfrak{m} \otimes_A M) \otimes_A N = A/\mathfrak{m} \otimes_A (M \otimes_A N) = 0,$$

a contradiction. ■

2.4. Let M_i $(i \in I)$ be any family of A-modules, and let M be their direct sum. Prove that M is flat $\Leftrightarrow$ each M_i is flat.

Solution. Suppose M is flat. If $f : N' \to N$ is an injective A-module homormophim, then $f \otimes 1 : N \otimes M \to N' \otimes M$. However, $N \otimes M = \bigoplus_{i \in I} N \otimes M_i$ and $N' \otimes M = \bigoplus_{i \in I} N' \otimes M_i$. Observing $(f \otimes 1)(N \otimes M_i) \subseteq N' \otimes M_i$, the restriction $(f \otimes 1)|_{N \otimes M_i} : N \otimes M_i \to N' \otimes M_i$ is also injective. Therefore, each M_i is also flat.

Conversely, suppose each M_i is flat. If $0 \to N' \to N \to N'' \to 0$ is an exact sequence of A-modules, then $0 \to N' \otimes M_i \to N \otimes M_i \to N'' \otimes M_i \to 0$ is also exact. Therefore, the direct sum of exact sequences $0 \to \bigoplus_{i \in I} N' \otimes M_i \to \bigoplus_{i \in I} N \otimes M_i \to \bigoplus_{i \in I} N'' \otimes M_i \to 0$ is also exact. ■

2.5. Let $A[x]$ be the ring of polynomials in one indeterminate over a ring A. Prove that $A[x]$ is a flat A-algebra.

Solution. As an A-module, $A[x] = \bigoplus_{i=0}^{\infty} Ax^i$ where $Ax^i \cong A$ for each $i \in \mathbf{Z}_{\geqslant 0}$. Since A is clearly flat, its direct sum $A[x] \cong A^{\mathbf{Z}_{\geqslant 0}}$ is also flat by Exercise 2.4. ■

2.6. For any A-module, let $M[x]$ denote the set of all polynomials in x with coefficients in M, that is to say expressions of the form

$$m_0 + m_1 x + \cdots + m_r x^r \qquad (m_i \in M).$$

Defining the product of an element of $A[x]$ and an element of $M[x]$ in the obvious way, show that $M[x]$ is an $A[x]$-module.

Show that $M[x] \cong A[x] \otimes_A M$.

Solution. $M[x]$ is clearly an abelian group since M is itself an abelian group. Precisely, its abelian group structure is the same with the direct sum $M[x] \cong \bigoplus_{i=1}^{\infty} Mx^i \cong \bigoplus_{i=1}^{\infty} M$. Also, rules for scalar multiplication by $A[x]$ straightforwardly (but too tedious to type every minor detail) hold.

Firstly, let's construct an A-module isomorphism $A[x] \otimes_A M \to M[x]$, and show that it preserves $A[x]$-scalar multiplication later. Define a map $f : A[x] \times M \to M[x]$ by

$$(a_0 + a_1 x + \cdots + a_r x^r, m) \mapsto a_0 m + (a_1 m)x + \cdots + (a_r m)x^r.$$

[1] There are a lot of ways to show this, but one may use the fact that every n-dimensional vector space is isomorphic to k^n. In general, $R^n \otimes_R R^m = R^{nm}$ for any commutative ring R by Proposition 2.14.

It is easy to verify f is A-bilinear. Therefore, there is a unique A-module homomorphism $\phi : A[x] \otimes_A M \to M[x]$ sending $a_i x^i \otimes m$ to $(a_i m)x^i$. Now, define an A-module homomrphism $\psi : M[x] \to A[x] \otimes_A M$ by $m_i x^i \mapsto x^i \otimes m_i$ for each $i \in \mathbf{Z}_{\geqslant 0}$.[2] Then $\psi \circ \phi = \mathrm{id}_{A[x]\otimes M}$ and $\phi \circ \psi = \mathrm{id}_{M[x]}$,[3] so $\phi : A[x] \otimes_A M \to M[x]$ is an A-module isomorphism. We claim that ϕ actually respects scalar multiplication by $A[x]$. For $b_0 + b_1 x + \cdots + b_s x^s \in A$, we have

$$\begin{aligned} f((b_0 + b_1 x + \cdots + b_s x^s)(a_i x^i \otimes m)) &= f((a_i b_0 x^i + a_i b_1 x^{i+1} + \cdots + a_i b_s x^{s+i}) \otimes m) \\ &= a_i b_0 m x^i + a_i b_1 m x^{i+1} + \cdots + a_i b_s m x^{s+i} \\ &= (b_0 + b_1 x + \cdots + b_s x^s)(a_i m x^i) \\ &= (b_0 + b_1 x + \cdots + b_s x^s) f(a_i x^i \otimes m), \end{aligned}$$

for each $i \in \mathbf{Z}_{\geqslant 0}$. Therefore, ϕ is also an $A[x]$-module isomorphism. ■

2.7. Let $\mathfrak{p}$ be a prime ideal in A. Show that $\mathfrak{p}[x]$ is a prime ideal in $A[x]$. If $\mathfrak{m}$ is a maximal ideal in A, is $\mathfrak{m}[x]$ a maximal ideal in $A[x]$?

Solution. Since $\mathfrak{p}[x]$ is the kernel of $A[x] \twoheadrightarrow (A/\mathfrak{p})[x]$, we have $A[x]/\mathfrak{p}[x] \cong (A/\mathfrak{p})[x]$. Suppose there are $a_0 + \cdots + a_r x^r$ and $b_0 + \cdots + b_r x^r$ in $(A/\mathfrak{p})[x]$ with non-zero a_r and b_r so that $(a_0 + \cdots + a_r x^r)(b_0 + \cdots + b_r x^r) = 0$. However, we get $a_r b_r = 0$, a contradiction for $A/\mathfrak{p}$ is an integral domain. As a reulst, $\mathfrak{p}[x]$ is a prime ideal in $A[x]$.

Let k be a field, $A = k[y]$, and $\mathfrak{m} = (y)$. Then $k[y][x]/(y)[x] \cong (k[y]/(y))[x] \cong k[x]$, which is clearly not a field in general. Therefore, $\mathfrak{m}[x]$ is not a maximal ideal. ■

2.8. i) If M and N are flat A-modules, then so is $M \otimes_A N$.

ii) If B is a flat A-algebra and N is a flat B-module, then N is flat as an A-module.

Solution. i) If $f : L' \to L$ is an injective A-module homomorhpism, then $(1 \otimes f) : N \otimes L' \to N \otimes L$ is injective. Also, $1 \otimes (1 \otimes f) : M \otimes (N \otimes L') \to M \otimes (N \otimes L)$ is injective. By the identification $M \otimes (N \otimes L') = (M \otimes N) \otimes L'$ and $M \otimes (N \otimes L) = (M \otimes N) \otimes L$, it induces an injective map $1' \otimes f : (M \otimes N) \otimes L' \to (M \otimes N) \otimes L$ given by $(m \otimes n) \otimes l' \mapsto (m \otimes n) \otimes l$. This shows $M \otimes_A N$ is flat.

ii) If $f : M' \to M$ is an injective A-module homomorphism then $(1 \otimes f) : B \otimes_A M' \to B \otimes_A M$ is injective. However, we can regard this injective map as a B-module homomorphism, since $b_1 b_2 \otimes m' \mapsto b_1 b_2 \otimes f(m') = b_1(b_2 \otimes f(m'))$. Therefore, we get an injective B-module homomorhpism $N \otimes_B (B \otimes_A M') \to N \otimes_B (B \otimes_A M)$, but by the canonical isomorphism in 2.15 of the main text, we get $N \otimes_A M' \to N \otimes_A M$ given by $n \otimes m' \mapsto n \otimes f(m')$. As a result, N is flat as an A-module. ■

2.9. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M' and M'' are finitely generated, then so is M.

[2] Since $M[x] \cong \bigoplus_i M x^i$ as an A-module, this assignment uniquely determines ψ, which is well-defined.

[3] Checking only for generators suffices to show this.

Solution. Regard M' as a submodule of M. Since $M/M' \cong M''$, there are $x_1, \ldots, x_n \in M$ so that $x_1 + M', \ldots, x_n + M'$ generate M/M'. That is, every element of M belongs to some coset, which is a linear combination of $x_1 + M', \ldots, x_n + M'$. Let $y_1, \ldots, y_m$ be generators of M'. Then $x_1, \ldots, x_n, y_1, \ldots, y_m$ generate M. ■

2.10. Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module, and let $u : M \to N$ be a homomorphism. If the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

Solution. Since N is finitely generated, so is $N/\mathfrak{a}N$. Suppose $n_1 + \mathfrak{a}N, \ldots, n_r + \mathfrak{a}N$ generate $N/\mathfrak{a}N$ where $n_i = u(m_i)$ for some $m_i \in M$. Then $N = \mathfrak{a}N + \sum_{i=1}^r An_i$, so

$$N/\left(\textstyle\sum_{i=1}^r An_i\right) \cong \mathfrak{a}N/\left(\mathfrak{a}N \cap \textstyle\sum_{i=1}^r An_i\right) \cong \mathfrak{a}\left(N/\textstyle\sum_{i=1}^r An_i\right).$$

By Nakayama's lemma, we get $N = \sum_{i=1}^r An_i$. Therefore, u is surjective. ■

2.11. Let A be a ring $\neq 0$. Show that $A^m \cong A^n \Rightarrow m = n$.

If $\phi : A^m \to A^n$ is surjective, then $m \geqslant n$.

If $\phi : A^m \to A^n$ is injective, is it always the case $m \leqslant n$?

Solution. Let $\mathfrak{m}$ be a maximal dieal of A and let $\phi : A^m \to A^n$ be an isomorphism. Then $1 \otimes \phi : (A/\mathfrak{m}) \otimes A^m \to (A/\mathfrak{m}) \otimes A^n$ is an isomorphism between vector spaces of dimensions m and n over the field $k = A/\mathfrak{m}$, since $(A/\mathfrak{m}) \otimes A^m \cong k^m$ and $(A/\mathfrak{m}) \otimes A^n \cong k^n$ by Exercise 2.2. Hence, $m = n$.

If ϕ is surjective, then the tensored morphism $1 \otimes \phi : k^m \to k^n$ is again surjective by Proposition 2.18. Therefore, $m \geq n$ by linear algebra.

The injectivity part is a very famous problem, and there are a lot of good answers for it. One way with a structural approach involves exterior algebra as Corollary 5.11 of [1]. However, I can not figure out a better answer than the following solution [2] in MathOverflow, which uses only Proposition 2.4. Suppose there is an injective A-module homomorphism $\phi : A^m \to A^n$ with $m > n$. Identifying A^n with $\{(a_1, \ldots, a_n, 0, \ldots, 0) \in A^m \mid a_i \in A\} \subseteq A^m$, we can regard it as an A-module embedding $\phi : A^m \hookrightarrow A^m$; i.e., $\phi(A^m) \subseteq A^m$. Therefore, by Proposition 2.4, ϕ satisfies an equation of the form

$$p(\phi) = \phi^d + a_1\phi^{d-1} + \cdots + a_d = 0$$

where a_i are in A and $p(x) \in A[x]$. Suppose the polynomial p has the minimum degree (well ordering principle). If $a_d = 0$, then $\phi(\phi^{d-1} + a_1\phi^{d-2} + \cdots + a_{d-1})(v) = 0$ for all $v \in A^m$. However, by the injectivity of ϕ, we get $\phi^{d-1} + a_1\phi^{d-2} + \cdots + a_{d-1} = 0$, a contradiction. Therefore a_d is nonzero. However, the m-th coordinate of $p(\phi)(0, \ldots, 0, 1)$ is a_d, contradicting to the assumption $\phi(A^m) \subseteq A^n$. This shows $m \leq n$. ■

2.12. Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Show that $\mathrm{Ker}(\phi)$ is finitely generated.

Solution. Let $e_1, \ldots, e_n$ be a basis of A^n and choose $u_i \in M$ such that $\phi(u_i) = e_i$ $(1 \leq i \leq n)$. Let N be the submodule of M generated by $u_1, \ldots, u_n$. Then every element of x of M must be in some coset $y + \mathrm{Ker}(\phi)$ for some $y \in N$ if and only if $\phi(x) = \phi(y)$. This shows $N + \mathrm{Ker}(\phi) = M$. If $r_1 u_1 + \cdots + r_n u_n \in N$ is in $\mathrm{Ker}(\phi)$, then $r_1 e_1 + \cdots + r_n e_n = 0$ in A^n. This implies $r_1 = \cdots = r_n = 0$, so $N \cap \mathrm{Ker}(\phi) = 0$. As a result, $M = N \oplus \mathrm{Ker}(\phi)$. Since M is finitely generated, its quotient $M/N \cong \mathrm{Ker}(\phi)$ is also finitely generated. ■

2.13. Let $f : A \to B$ be a ring homomorphism, and let N be a B-module. Regarding N as an A-module by restriction of scalars, form the B-module $N_B = B \otimes_A N$. Show that the homomorphism $g : N \to N_B$ which maps y to $1 \otimes y$ is injective and that $g(N)$ is a direct summand of N_B.

Solution. We shall show $1 \otimes y \in B \otimes_A N$ is zero only if $y = 0$. Define $p : N_B \to N$ by $p(b \otimes y) = by$. It is easy to see that p is a B-module homomorphism. However, $p(1 \otimes y) = 0$ only if $y = 0$, so g is injective. Since p is clearly surjective, we have $N_B/\mathrm{Ker}(p) \cong N \cong \mathrm{Im}(g)$. Let $p^* : N_B/\mathrm{Ker}(p) \to N$ be the induced map by p. Then $p^*(1 \otimes y + \mathrm{Ker}(p)) = y$ for all $y \in N$, so $\mathrm{Im}(g) + \mathrm{Ker}(p) = N_B$ since every element of N must be in some coset $1 \otimes y + \mathrm{Ker}(p)$. However, because $p \circ g = \mathrm{id}_N$, we get $\mathrm{Im}(g) \cap \mathrm{Ker}(p) = 0$. This shows $N_B \cong \mathrm{Im}(g) \oplus \mathrm{Ker}(p) \cong N \oplus \mathrm{Ker}(p)$. ■

2.14. A partially ordered set I is said to be a *directed set* if for each pair i, j in I there exists $k \in I$ such that $i \leqslant k$ and $j \leqslant k$.

Let A be a ring, let I be a directed set and let $(M_i)_{i \in I}$ be a family of A-modules indexed by I. For each pair i, j in I such that $i \leqslant j$, let $\mu_{ij} : M_i \to M_j$ be an A-homomorphism, and suppose that the following axioms are satisfied:

(1) μ_{ii} is the identity mapping of M_i, for all $i \in I$;
(2) $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leqslant j \leqslant k$.

Then the modules M_i and homomorphisms μ_{ij} are said to form a *direct system* $\mathbf{M} = (M_i, \mu_{ij})$ over the directed set I.

We shall construct an A-module M called the *direct limit* of the direct system $\mathbf{M}$. Let C be the direct sum of the M_i, and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leqslant j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of μ to M_i.

The module M, or more correctly the pair consisting of M and the family of homomorphisms $\mu_i : M_i \to M$, is called the *direct limit* of the direct system M, and is written $\varinjlim M_i$ From the construction it is clear that $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leqslant j$.

Solution. There is nothing to do. ■

2.15. In the situation of Exercise 14, show that every element of M can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$.

Show also that if $\mu_i(x_i) = 0$ then there exists $j \geqslant i$ such that $\mu_{ij}(x_i) = 0$ in M_j.

Solution. In the construction of M in Exercise 2.14, there are finitely many $i_1, \ldots, i_n \in I$ so that $y = \mu_{i_1}(x_{i_1}) + \cdots + \mu_{i_n}(x_{i_n})$. Choose some $j \geq i_1, \ldots, i_n$ and let $x_j := \mu_{i_1 j}(x_{i_1}) + \cdots + \mu_{i_n j}(x_{i_n})$. Then $\mu_j(x_j) = y$.

Suppose $\mu_i(x_i) = 0$. Then x_i is in $\mathrm{Ker}(\mu)$, where $\mu : C := \bigoplus_{i \in I} M_i \to M$ is the projection. Since $\mathrm{Ker}(\mu)$ is a submodule of C generated by $\{y_i - \mu_{ij}(y_i) \mid i, j \in I,\ j \geq i\}$, we have $x_i = \sum_{p=1}^{n}(y_p - \mu_{i_p j_p}(y_p))$ where $y_p \in M_{i_p}$ and $i_p \leqslant j_p$. However, every term not in M_i must be eliminated by other terms, so $x_i = y_{i'} - \mu_{i'j}(y_{i'})$ for some $y_{i'} \in M_{i'}$. The only possible way is $y_{i'} = x_i$ and $\mu_{ij}(x_i) = 0$. ∎

2.16. Show that the direct limit is characterized (up to isomorphism) by the following property. Let N be an A-module and for each $i \in I$ let $\alpha_i : M_i \to N$ be an A-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leqslant j$. Then there exists a unique homomorphism $\alpha : M \to N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.

Solution. Rephrasing the property as a commutative diagram is as follows:

$$
\begin{tikzcd}
M_i \arrow[rr, "\mu_{ij}"] \arrow[dr, "\mu_i"] \arrow[ddr, bend right, "\alpha_i"'] & & M_j \arrow[dl, "\mu_j"'] \arrow[ddl, bend left, "\alpha_j"] \\
& M \arrow[d, dashed, "\exists!\alpha"] & \\
& N &
\end{tikzcd}
$$

for $j \geqslant i$, where each triangle in the diagram commutes. This property uniquely characterizes M up to 'unique' isomorphism. Suppose our given (M, μ_i) satisfies the property, and another module (M', μ_i') satisfies the same property. Trivially, if we plug M to N, the unique morphism α is the identity id_M. However, by the assumption, there are unique morphisms β and γ so that each triangle of the following diagram commutes

$$
\begin{tikzcd}
M_i \arrow[dd, "\mu_{ij}"'] \arrow[dr, "\mu_i"'] \arrow[drr, bend left, "\mu_i'"] \arrow[drrr, bend left=30, "\mu_i"] & & & \\
& M \arrow[r, dashed, "\exists!\beta"] & M' \arrow[r, dashed, "\exists!\gamma"] & M \\
M_j \arrow[ur, "\mu_j"] \arrow[urr, bend right, "\mu_j'"'] \arrow[urrr, bend right=30, "\mu_j"'] & & &
\end{tikzcd}
$$

for each $j \geqslant i$. Therefore we get $\gamma \circ \beta = \mathrm{id}_M$. Ditto $\beta \circ \gamma = \mathrm{id}_{M'}$, and this shows M and M' are isomorphic by 'unique' isomorphisms β and γ.

Now let's show (M, μ_i) actually satisfy the property. Define an A-module homomorphism $f : \bigoplus_{i \in I} M_i \to N$ as $(x_i)_{i \in I} \mapsto \sum_{i \in I} \alpha_i(x_i)$. Since $\alpha_i(x_i) = \alpha_j(\mu_{ij}(x_i))$ by the assumption, $f(x_i - \mu_{ij}(x_i)) = 0$. Therefore $\mathrm{Ker}(\mu) \subseteq \mathrm{Ker}(f)$, so we get the induced A-module homomorphism $\alpha : M \to N$ satisfying $\alpha \circ \mu = f$. By the construction, $\alpha(\mu_i(x_i)) = f(x_i) = \alpha_i(x_i)$ for any $x_i \in M_i$, so α satisfies the desired property. To show the uniqueness of α, suppose $\alpha' : M \to N$ also satisfies the same property. By Exercise 2.15, every element of M can be written in the form $\mu_i(x_i)$ for some $x_i \in M_i$. But $\alpha'(\mu_i(x_i)) = \alpha_i(x_i) = \alpha(\mu_i(x_i))$. This ends the proof. ∎

2.17. Let $(M_i)_{i \in I}$ be a family of submodules of an A-module, such that for each pair of indices i, j in I there exists $k \in I$ such that $M_i + M_j \subseteq M_k$. Define $i \leqslant j$

to mean $M_i \subseteq M_j$ and let $\mu_{ij} : M_i \to M_j$ be the embedding of M_i in M_j. Show that
$$\varinjlim M_i = \sum M_i = \bigcup M_i.$$
In particular, any A-module is the direct limit of its finitely generated submodules.

Solution. For any $m, n \in \sum M_i$, notice $m + n$ belongs to some ambient module M_k, so $\bigcup M_i = \sum M_i$. Let $\mu_i : M_i \to \sum M_i$ be the natural inclusion, and suppose N be an A-module and for each $i \in I$ let $\alpha_i : M_i \to N$ is an A-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leqslant j$. Then an A-module homomorphism $\alpha : \sum M_i \to N$ given by $(x_i)_{i\in I} \mapsto \sum \alpha_i(x_i)$ satisfies $\alpha(\mu_i(x_i)) = \alpha_i(x_i)$ for arbitrary $x_i \in M_i$. Since μ_i is nothing but inclusion, such α satisfying $\alpha = \alpha_i \circ x_i$ for any $i \in I$ is unique. This shows that $\varinjlim M_i = \sum M_i$ by Exercise 2.16.

In particular, let M be any A-module, and $(M_i)_{i\in I}$ be the collection of all finitely generated submodules of M. For each $i, j \in I$, $M_i + M_j$ is also finitely generated; hence I and $(M_i)_{i\in I}$ satisfies the desired property. Moreover, for any $x \in M$, Ax is a finitely generated submodule itself, so $M = \sum M_i = \varinjlim M_i$. ■

2.18. Let $\mathbf{M} = (M_i, \mu_{ij})$, $\mathbf{N} = (N_i, \nu_{ij})$ be direct systems of A-modules over the same directed set. Let M, N be the direct limits and $\mu_i : M_i \to M$, $\nu_i : N_i \to N$ the associated homomorphisms.

A *homomorphism* $\mathbf{\Phi} : \mathbf{M} \to \mathbf{N}$ is by definition a family of A-module homomorphisms $\phi_i : M_i \to N_i$, such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ whenever $i \leqslant j$. Show that $\mathbf{\Phi}$ defines a unique homomorphism $\phi = \varinjlim \phi_i : M \to N$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.

Solution. Let $\alpha_i \coloneqq \nu_i \circ \phi_i$ for each $i \in I$. Then by the assumption, we get
$$\begin{aligned}\alpha_i &= \nu_i \circ \phi_i \\ &= \nu_j \circ \nu_{ij} \circ \phi_i \\ &= \nu_j \circ \phi_j \circ \mu_{ij} \\ &= \alpha_j \circ \mu_{ij},\end{aligned}$$
whenever $i \leqslant j$. By Exercise 2.16, this implies that there exists a unique homomorphism $\phi : M \to N$ so that the following diagram commutes:
$$\begin{tikzcd}
M_j \arrow[r, "\phi_j"] \arrow[dr, "\mu_j"] & N_j \arrow[dr, "\nu_j"] & \\
 & M \arrow[r, dashed, "\exists!\phi"] & N \\
M_i \arrow[uu, "\mu_{ij}"] \arrow[ur, "\mu_i"] \arrow[r, "\phi_i"'] & N_i \arrow[ur, "\nu_i"'] &
\end{tikzcd}$$
whenever $i \leqslant j$. This ends the proof. ■

2.19. A sequence of direct systems and homomorphisms
$$\mathbf{M} \to \mathbf{N} \to \mathbf{P}$$

is *exact* if the corresponding sequence of modules and module homomorphisms is exact for each $i \in I$. Show that the sequence $M \to N \to P$ of direct limits is then exact.

Solution. For the notations, let $\mathbf{M} = (M_i, \mu_{ij})$, $\mathbf{N} = (N_i, \nu_{ij})$, and $\mathbf{P} = (P_i, \pi_{ij})$ be direct systems of A-modules over the same directed set I. Let M, N, and P be the direct limits and $\mu_i : M_i \to M$, $\nu_i : N_i \to N$, and $\pi_i : P_i \to P$ be the associated homomorphisms. Let $\mathbf{\Phi} : \mathbf{M} \to \mathbf{N}$ and $\mathbf{\Psi} : \mathbf{N} \to \mathbf{P}$ denote homomorphisms of direct systems so that the given sequence is exact where $\phi_i : M_i \to N_i$, and $\psi_i : N_i \to P_i$ are associated homomorphisms. By Exercise 2.18, they define a unique homomorphism $\phi = \varinjlim \phi_i : M \to N$ and $\psi = \varinjlim \psi_i : N \to P$ such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ and $\psi \circ \nu_i = \pi_i \circ \psi_i$. Then the following diagram commutes:

$$
\begin{tikzcd}
M_i \arrow[rr, "\phi_i"] \arrow[dr, "\mu_{ij}"] \arrow[dd, "\mu_i"'] & & N_i \arrow[rr, "\psi_i"] \arrow[dr, "\nu_{ij}"] \arrow[dd, "\nu_i"'] & & P_i \arrow[dr, "\pi_{ij}"] \arrow[dd, "\pi_i"'] & \\
& M_j \arrow[rr, "\phi_j"] \arrow[dl, "\mu_j"] & & N_j \arrow[rr, "\psi_j"] \arrow[dl, "\nu_j"] & & P_j \arrow[dl, "\pi_j"] \\
M \arrow[rr, "\phi"] & & N \arrow[rr, "\psi"] & & P &
\end{tikzcd}
$$

whenever $i \leqslant j$, where the first and second rows are exact by the assumption. We claim that $\mathrm{Im}(\phi) = \mathrm{Ker}(\psi)$; i.e., the third row is exact. By Exercise 2.15, for any $m \in M$, there are some $i \in I$ and some $m_i \in M_i$ so that $m = \mu_i(m_i)$. Then $\psi(\phi(m)) = (\psi \circ \phi \circ \mu_i)(m_i) = (\pi_i \circ \psi_i \circ \phi_i)(m_i) = \pi_i(0) = 0$, so $\mathrm{Im}(\phi) \subseteq \mathrm{Ker}(\psi)$. For the reverse inclusion, suppose n is in $\mathrm{Ker}(\psi)$. By Exercise 2.15 again, there exists some $i \in I$ and some $n_i \in N_i$ such that $n = \nu_i(n_i)$. However, $0 = \psi(n) = \psi(\nu_i(n_i)) = \pi_i(\psi_i(n_i))$, so there exists some $j \geqslant i$ such that $\pi_{ij}(\psi_i(n_i)) = 0$ by the second statement of Exercise 2.15. Notice $\psi_j(\nu_{ij}(n_i)) = \pi_{ij}(\psi_i(n_i)) = 0$. Thus, there exists some $m_j \in M_j$ such that $\phi_j(m_j) = \nu_{ij}(n_i)$ due to the assumption that $\mathrm{Im}(\phi_j) = \mathrm{Ker}(\psi_j)$. As a result,

$$\phi(\mu_j(m_j)) = \nu_j(\phi_j(m_j)) = \nu_j(\nu_{ij}(n_i)) = \nu_i(n_i) = n.$$

This shows $\mathrm{Im}(\phi) = \mathrm{Ker}(\psi)$. ∎

2.20. Keeping the same notation as in Exercise 14, let N be any A-module. Then $(M_i \otimes N, \mu_{ij} \otimes 1)$ is a direct system; let $P = \varinjlim (M_i \otimes N)$ be its direct limit. For each $i \in I$ we have a homomorphism $\mu_i \otimes 1 : M_i \otimes N \to M \otimes N$, hence by Exercise 16 a homomorphism $\psi : P \to M \otimes N$. Show that ψ is an isomorphism, so that

$$\varinjlim (M_i \otimes N) \cong (\varinjlim M_i) \otimes N.$$

Solution. For the notation, let $\mu'_i : M_i \otimes N \to P$ denote the canonical A-module homomorphism characterizing the direct limit P. Then $\psi \circ \mu'_i = \mu_i \otimes 1$ for all $i \in I$. For each $i \in I$, let $g_i : M_i \times N \to M_i \otimes N$ be the canonical bilinear mapping given by $(m_i, n) \mapsto m_i \otimes n$. Fixing $n \in N$, we get an A-module homomorphisms $g_i(-, n) : M_i \to M_i \otimes N$, and it is easy to see that they form a homomorphism $(M_i, \mu_{ij}) \to (M_i \otimes N, \mu_{ij} \otimes 1)$ between two directed system. Therefore, by Exercise 2.18, they define a unique homomorphism $g(-, n) : M \to P$ such that $g(-, n) \circ \mu_i = \mu'_i \circ g_i(-, n)$. We claim that $g(m, -) : N \to P$ is also an A-module homomorphism for each fixed $m \in M$. By Exercise 2.15, there exist some $i \in I$ and some $m_i \in M_i$ so that $\mu_i(m_i) = m$. Then for any $n_1, n_2 \in N$ and $a \in A$, we have

$$\begin{aligned} g(m, n_1 + an_2) &= g(\mu_i(m_i), n_1 + an_2) \\ &= (g(-, n_1 + an_2) \circ \mu_i)(m_i) \\ &= \mu_i'(g_i(m_i, n_1 + an_2)) \\ &= \mu_i'(g_i(m_i, n_1) + a g_i(m_i, n_2)) \\ &= \mu_i'(g_i(m_i, n_1)) + a\mu_i'(g_i(m_i, n_2)) \\ &= g(m, n_1) + a g(m, n_2), \end{aligned}$$

assuming $m = \mu_i(m_i)$ for some $i \in I$ and some $m_i \in M_i$. We finally get a bilinear map $g : M \times N \to P$, and hence we obtain the corresponding A-module homomorphism $\phi : M \otimes N \to P$ such that $\phi(m \otimes n) = g(m, n)$.

Now we claim that ϕ and ψ are mutually inverse. For any $m \in M$ and $n \in N$, assuming $m = \mu_i(m_i)$ for some $i \in I$ and some $m_i \in M_i$,

$$\begin{aligned} \psi(\phi(m \otimes n)) &= \psi(g(m, n)) \\ &= \psi(g(\mu_i(m_i), n)) \\ &= (\psi \circ \mu_i')(g_i(m_i, n)) \\ &= (\mu_i \otimes 1)(g_i(m_i, n)) \\ &= (\mu_i \otimes 1)(m_i \otimes n) \\ &= \mu_i(m_i) \otimes n \\ &= m \otimes n, \end{aligned}$$

so $\psi \circ \phi = \mathrm{id}_{M \otimes N}$. For the converse, for given $p \in P$, there is some $i \in I$ and some $x_i \in M_i \otimes N$ so that $p = \mu_i'(x_i)$. We may write $x_i = \sum_{j=1}^k m_{ij} \otimes n_j$ for some $k \in \mathbf{Z}_{\geqslant 0}$, $m_{i1}, \ldots, m_{ik} \in M_i$, and $n_1, \ldots, n_k \in N$. Then,

$$\begin{aligned} \phi(\psi(p)) &= (\phi \circ \psi \circ \mu_i')(x_i) \\ &= \sum_{j=1}^k (\phi \circ (\mu_i \otimes 1))(m_{ij} \otimes n_j) \\ &= \sum_{j=1}^k \phi(\mu_i(m_{ij}) \otimes n_j) \\ &= \sum_{j=1}^k g(\mu_i(m_{ij}), n_j) \\ &= \sum_{j=1}^k \mu_i'(g_i(m_{ij}, n_j)) \\ &= \sum_{j=1}^k \mu_i'(m_{ij} \otimes n_j) \\ &= p. \end{aligned}$$

As a result, $\varinjlim(M_i \otimes N) \cong (\varinjlim M_i) \otimes N$. ■

2.21. Let $(A_i)_{i\in I}$ be a family of rings indexed by a directed set I, and for each pair $i \leqslant j$ in I let $\alpha_{ij} : A_i \to A_j$ be a ring homomorphism, satisfying conditions (1) and (2) of Exercise 14. Regarding each A_i as a $\mathbf{Z}$-module we can then form the direct limit $A = \varinjlim A_i$. Show that A inherits a ring structure from the A_i so that the mappings $A_i \to A$ are ring homomorphisms. The ring A is the direct limit of the system (A_i, α_{ij}).

If $A = 0$ prove that $A_i = 0$ for some $i \in I$.

Solution. We define multiplication of A as follows. For any $a, b \in A$, by Exercise 2.15, there are some $i \in I$ and some $x_i, y_i \in A_i$ such that $a = \alpha_i(x_i)$ and $b = \alpha_i(y_i)$. (We can say x_i, y_i lie on same A_i since I is a directed set; precisely, if $x_{i_1} \in A_{i_1}$ and $y_{i_2} \in A_{i_2}$, then there exists $i \in I$ such that $i_1 \leqslant i$ and $i_2 \leqslant i$, and let x_i and y_i be $\alpha_{i_1 i}(x_{i_1})$ and $\alpha_{i_2 i}(y_{i_2})$, respectively) Then define ab as $\alpha_i(x_i y_i)$. To show it is well-defined, suppose $a = \alpha_j(x_j)$ and $b = \alpha_j(y_j)$ for some $j \in I$ and some $x_j, y_j \in A_j$. There exists some $k \in I$ such that $i \leqslant k$ and $j \leqslant k$, so

$$\alpha_k(\alpha_{ik}(x_i) - \alpha_{jk}(x_j)) = 0 \quad \text{and} \quad \alpha_k(\alpha_{ik}(y_i) - \alpha_{jk}(y_j)) = 0.$$

Then by Exercise 2.15, there is some $k' \geqslant k$ so that[4]

$$\alpha_{kk'}(\alpha_{ik}(x_i) - \alpha_{jk}(x_j)) = 0 \quad \text{and} \quad \alpha_{kk'}(\alpha_{ik}(y_i) - \alpha_{jk}(y_j)) = 0.$$

Observe

$$\alpha_{ik}(x_i y_i) - \alpha_{jk}(x_j y_j) = \alpha_{ik}(x_i)(\alpha_{ik}(y_i) - \alpha_{jk}(y_j)) + \alpha_{jk}(y_j)(\alpha_{ik}(x_i) - \alpha_{jk}(x_j)).$$

Plugging it into the ring homomorphism $\alpha_{kk'}$, we obtain

$$\alpha_{kk'}(\alpha_{ik}(x_i y_i) - \alpha_{jk}(x_j y_j)) = 0.$$

This demonstrates that $\alpha_i(x_i y_i) = \alpha_j(x_j y_j)$ in A, ensuring the well-defined nature of the multiplication. For $i \in I$, let 1_i denote the multiplicative identity of A_i. As $\alpha_{ij}(1_i) = 1_j$, we can deduce that $\alpha_i(1_i) = \alpha_j(1_j)$ for all $i, j \in I$. Let 1 represent $\alpha_i(1_i)$. Consequently, for any element $a = \alpha_i(x_i)$ in A, $1a = \alpha_i(1_i x_i) = \alpha_i(x_i) = a$. This confirms that the ring structure with which we have endowed A makes each α_i a ring homomorphism.

Now suppose $A = 0$. Then for any $i \in I$ and $a_i \in A_i$, a_i is in $\mathrm{Ker}(\mu)$, where $\mu : C := \bigoplus_{i\in I} M_i \to M$ is the projection. Then as the solution of Exercise 2.15, there exists some $j \in I$ such that $\mu_{ij}(1_i) = 0$. Since μ_{ij} must sends 1_i to 1_j, it implies $A_j = 0$. ∎

2.22. Let (A_i, α_{ij}) be a direct system of rings and let $\mathfrak{N}_i$ be the nilradical of A_i. Show that $\varinjlim \mathfrak{N}_i$ is the nilradical of $\varinjlim A_i$.

If each A_i us an integral domain, then $\varinjlim A_i$ is an integral domain.

Solution. Let $\mathfrak{N}$ denote the nilradical of $\varinjlim A_i$. Since $\alpha_{ij}(\mathfrak{N}_i) \subseteq \mathfrak{N}_j$ for each $i \leqslant j$, the inclusion $\iota_i : \mathfrak{N}_i \hookrightarrow A_i$ induces the corresponding homomorphism $\iota : \varinjlim \mathfrak{N}_i \to \varinjlim A_i$, and ι is injective by Exercise 2.19. Therefore we can regard

[4]Rigorously speaking, to show $\alpha_{kk'}$ sends 'both' of them 0, we should repeat the same argument as showing that we can assume x_i and y_i lie on the same A_i.

$\varinjlim \mathfrak{N}_i$ as a subset of $\varinjlim A_i$ via ι. If $x_i \in A_i$ is nilpotent, then $\alpha_i(x_i)$ is also nilpotent, so $\varinjlim \mathfrak{N}_i \subseteq \mathfrak{N}$ by Exercise 2.15. Conversely, suppose $x \in \varinjlim A_i$ is nilpotent; i.e., $x^r = 0$ for some $r \in \mathbf{Z}_{\geqslant 0}$. By Exercise 2.15 again, it implies there exists some i, j such that $x = \alpha_i(x_i)$ and $\alpha_{ij}(x_i)^r = 0$. Then $\alpha_{ij}(x_i)$ is in $\mathfrak{N}_j$, so $x = \alpha_j(\alpha_{ij}(x_i))$ is in $\varinjlim \mathfrak{N}_i$. ■

2.23. Let $(B_\lambda)_{\lambda\in\Lambda}$ be a family of A-algebras. For each finite subset J of Λ let B_J denote the tensor product (over A) of the B_λ for $\lambda \in J$. If J' is another finite subset of A and $J \subseteq J'$, there is a canonical A-algebra homomorphism $B_J \to B_{J'}$. Let B denote the direct limit of the rings B_J as J runs through all finite subsets of Λ. The ring B has a natural A-algebra structure for which the homomorphisms $B_J \to B$ are A-algebra homomorphisms. The A-algebra B is the *tensor product* of the family $(B_\lambda)_{\lambda\in\Lambda}$.

Solution. There is nothing to do. ■

2.24. If M is an A-module, the following are equivalent:

i) M is flat;
ii) $\mathrm{Tor}_n^A(M, N) = 0$ for all $n > 0$ and all A-modules N;
iii) $\mathrm{Tor}_1^A(M, N) = 0$ for all A-modules N.

Remark. An A-module P is ***projective*** if and only if for every surjective A-module homomorphism $p : M \to M''$ and any A-module homomorphism $h : P \to M''$, there exists a lifting g; that is, there exists a homomorphism g makes the following diagram commute:

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "h"] & \\ M \arrow[r, "p"'] & M'' \arrow[r] & 0 \end{tikzcd}$$

It is easy to see that an A-module P is projective if and only if $\mathrm{Hom}_A(P, -)$ is an exact functor; that is, for every exact sequence of A-modules

$$0 \to M'' \to M \to M' \to 0,$$

the sequence

$$0 \to \mathrm{Hom}(P, M'') \to \mathrm{Hom}(P, M) \to \mathrm{Hom}(P, M') \to 0$$

is also exact. For instance, every free A-module is projective ([3], Theorem 3.5).

For an A-module N, a ***projective resolution*** of N is an exact sequence

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} N \to 0$$

in which each P_n is projective. If P_n is free, then the sequence is called ***free resolution*** of N. It is well known that every A-module N has a free resolution (the proof is actually not difficult, see Proposition 6.2 of [3]); hence, every A-module has a projective resolution. For a given projective resolution of N, remove N

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} 0$$

and form the following sequence by tensoring it with M:

$$\cdots \to M \otimes_A P_2 \xrightarrow{1_M \otimes \partial_2} M \otimes_A P_1 \xrightarrow{1_M \otimes \partial_1} M \otimes_A P_0 \xrightarrow{1_M \otimes \partial_0} 0.$$

It is not an exact sequence in general, but it is easy to see that $\operatorname{Im}(1_M \otimes \partial_{n+1}) \subseteq \operatorname{Ker}(1_M \otimes \partial_n)$ for all $n \geqslant 0$. Such sequence is called a ***chain complex***. For $n \geq 0$, the A-module $\operatorname{Tor}_n^A(M, N)$ is the ***homology*** of this complex at position n; that is, $\operatorname{Tor}_n^A(M, N) = \operatorname{Ker}(1_M \otimes \partial_n)/\operatorname{Im}(1_M \otimes \partial_{n+1})$ for $n > 0$, and $\operatorname{Tor}_0^A(M, N) = \operatorname{Coker}(1_M \otimes \partial_1) \cong M \otimes_A N$. Surprisingly, $\operatorname{Tor}_n^A(M, N)$ does not depend on the choice of projective resolution of N ([3], Proposition 6.20).

One of the most fundamental properties (in some context it is treated as an axiom for derived functors, which is the general notion of Tor functor; see Definition 2.1.1 of [4]) of Tor functor is as follows. If $0 \to N' \to N \to N''$ is an exact sequence of A-modules, then for an A-module M there is a long exact sequence, called Tor exact sequence ([3], Theorem 6.27),

$$\cdots \to \operatorname{Tor}_n^A(M, N') \to \operatorname{Tor}_n^A(M, N) \to \operatorname{Tor}_n^A(M, N'') \to \\ \operatorname{Tor}_{n-1}^A(M, N') \to \operatorname{Tor}_{n-1}^A(M, N) \to \operatorname{Tor}_{n-1}^A(M, N'') \to \cdots$$

which ends with

$$\cdots \to \operatorname{Tor}_0^A(M, N') \to \operatorname{Tor}_0^A(M, N) \to \operatorname{Tor}_0^A(M, N'') \to 0.$$

Recall $\operatorname{Tor}_0^A(M, N) \cong M \otimes_A N$.

Solution. [(i) $\Rightarrow$ (ii)] Suppose an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to N \to 0$$

is a free resolution of N, and by tensoring with M we get

$$\cdots \to F_2 \otimes M \to F_1 \otimes M \to F_0 \otimes M \to N \otimes M \to 0.$$

Since M is flat, the resulting sequence is exact and therefore its homology groups, which are the $\operatorname{Tor}_n^A(M, N)$, are zero for $n > 0$.

[(ii) $\Rightarrow$ (iii)] It is trivial.

[(iii) $\Rightarrow$ (i)] Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence. Then from the Tor exact sequence,

$$\operatorname{Tor}_1^A(M, N'') \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0$$

is exact. Since $\operatorname{Tor}_1^A(M, N'') = 0$ it follows that M is flat. ■

2.25. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence, with N'' flat. Then N' is flat $\Leftrightarrow$ N is flat.

Solution. From the Tor exact sequence, we get an exact sequence

$$\cdots \to \operatorname{Tor}_2(M, N'') \to \operatorname{Tor}_1(M, N') \to \operatorname{Tor}_1(M, N) \to \operatorname{Tor}_1(M, N'') \to \cdots$$

for all A-modules M. If N'' and N' are flat, then $0 \to \operatorname{Tor}_1(M, N) \to 0$ is exact, implying $\operatorname{Tor}_1(M, N) = 0$. Therefore N is flat. If N'' and N are flat, then $0 \to \operatorname{Tor}_1(M, N') \to 0$ is exact. As a result N' is flat. ■

2.26. Let N be an A-module. Then N is flat $\Leftrightarrow$ $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$ in A.

Solution. If N is flat, then $\mathrm{Tor}_1(M, N) = 0$ for all A-modules M by Exercise 2.24. To show the converse, firstly we claim that N is flat if $\mathrm{Tor}_1(M, N) = 0$ for all finitely generated A-modules M. Let $0 \to M' \to M \to .M'' \to 0$ be an exact sequence of finitely generated A-modules. Then form the Tor exact sequence, we get an exact sequence

$$\mathrm{Tor}_1(M'', N) \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0.$$

Since $\mathrm{Tor}_1(M'', N) = 0$ by the assumption, we conclude that for any injective homomorphism $f : M' \to M$ the corresponding homomorphism $f \otimes 1 : M' \otimes N \to M \otimes N$ is injective. Hence N is flat by Proposition 2.19, and this shows the claim holds. Now suppose $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$ in A. If M is finitely generated, let $x_1, \dots, x_n$ be a set of generators of M, and let M_i be the submodule generated by $x_1, \dots, x_i$. Observe that for a given cyclic module Ax, a map $f : A \to Ax$ given by $1 \mapsto x$ is an A-module homomorphism, implying $Ax \cong A/\mathrm{Ker}(f)$. Since M_i/M_{i-1} is generated by a single element for $2 \le i \le n$, $M_i/M_{i-1} \cong A/\mathfrak{a}_i$ for some ideal $\mathfrak{a}_i$. Consider the exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$$

for $2 \le i \le n$. Since $\mathrm{Tor}(M_1, N) = 0$ and $\mathrm{Tor}_1(M_i/M_{i-1}, N) = 0$ by the hypothesis for $2 \le i \le n$, proceeding by induction on i we get $\mathrm{Tor}_1(M, N) = \mathrm{Tor}_1(M_n, N) = 0$. This ends the proof. ■

2.27. A ring A is *absolutely flat* if every A-module is flat. Prove that the following are equivalent:

i) A is absolutely flat
ii) Every principal ideal is idempotent.
iii) Every finitely generated ideal is a direct summand of A.

Solution. [i) $\Rightarrow$ ii)] Let $x \in A$. Since $A/(x)$ is a flat A-module, the map $\alpha : (x) \otimes A/(x) \to A \otimes A/(x) = A/(x)$ induced by the inclusion $(x) \hookrightarrow A$ is injective. Since $\alpha(x \otimes \bar{a}) = x\bar{a} = 0$, we get $(x) \otimes A/(x) = 0$. However, $(x) \otimes A/(x) = (x)/(x^2)$ by Exercise 2.2, so $(x) = (x^2)$.

[ii) $\Rightarrow$ iii)] Let $x \in A$. Then $x = ax^2$ for some $a \in A$, hence $e = ax$ is idempotent, and $(x) = (e)$ because $x = xe$. Now if e, f are idempotents, then $(e, f) = (e + f - ef)$ since $e(e + f - ef) = e$ and $f(e + f - ef) = f$. Therefore every finitely generated ideal is principal, and generated by an idempotent e, hence is a direct summand because $A = (e) \oplus (1 - e)$.[5]

[iii) $\Rightarrow$ i)] Clearly A is an flat A-module, so every finitely generated ideal of A is flat by Exercise 2.4. Since $A/\mathfrak{a}$ is a direct summand of A for any finitely generated ideal $\mathfrak{a}$, we have $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for any A-module N. By Exercise 2.26, every A-module is flat. ■

[5]Consider maps $A \to (e) \oplus (1 - e)$ and $(e) \oplus (1 - e) \to A$ given by $a \mapsto (ae, a(1 - e))$ and $(a, b) \mapsto a + b$, respectively. Then they are two-sided inverses of each other.

2.28. A Boolean ring is absolutely flat. The ring of Chapter 1, Exercise 7 is absolutely flat. Every homomorphic image of an absolutely flat ring is absolutely flat. If a local ring is absolutely flat, then it is a field.

If A is absolutely flat, every non-unit in A is a zero-divisor.

Solution. Every principal ideal of a Boolean ring is clearly idempotent, which makes it absolutely flat.

Let A be the ring of Chapter 1, Exercise 7; i.e., A is a nonzero ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Suppose $x^n = x$ and $y^m = y$ for some $n, m > 1$. Then $x = x(x^{n-1} + y^{m-1} - x^{m-1}y^{m-1})$ and $y = y(x^{n-1} + y^{m-1} - x^{m-1}y^{m-1})$, so $(x, y) = (x^{n-1} + y^{m-1} - x^{m-1}y^{m-1})$. Therefore every finitely generated ideal is principal. Since $x = x^{n-2}x^2$, we have $(x) = (x^2)$, so A is absolutely flat.

Let $\phi : A \to B$ be a ring homomorphism where A is absolutely flat. Then $\phi(A) \cong A/\mathrm{Ker}(\phi)$. For any principal ideal $(\bar{x})$ of $A/\mathrm{Ker}(\phi)$, clearly $(\bar{x}) = (\bar{x})^2$ because $(x) = (x)^2$ in A. Therefore every homomorphic image of an absolutely flat ring is absolutely flat.

Suppose A is a local ring which is absolutely flat. For any $x \in A$, since every principal ideal is idempotent, we have $x = ax^2$ for some $a \in A$. Then $e = ax$ is idempotent, but a local ring contains no idempotent neither 0 nor 1. Therefore, if x is nonzero, x is a unit, so A is a field.

Now suppose A is absolutely flat and x is a non-unit in A. Since every principal ideal is idempotent, there is some $a \in A$ so that $x(1-ax) = x - ax^2 = 0$. If x is not a zero divisor, then $1 - ax = 0$, leading to a contradiction. This shows that every non-unit in A is a zero-divisor. ∎

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