Notes for Analysis I and II (4e) by Terence Tao

ProFatXuanAll

February 14, 2026

Contents

Preface

These are my notes on the book "Analysis I" and "Analysis II," 4th edition, written by Terence Tao. My notes cover statements I found important or interesting, including almost all axioms, corollaries, exercises, lemmata, proofs, propositions, remarks, and theories in both books. Errata on Tao's website (https://terrytao.wordpress.com/books/analysis-i and https://terrytao.wordpress.com/books/analysis-ii) are used to correct errors in both books. I have written proofs for exercises. Most exercises cannot be completed without referencing solutions from online forums. So, many thanks to the people who their provided solutions at https://math.stackexchange.com, https://taoanalysis.wordpress.com, and many other sources.

I wrap each statement in a LaTeX environment. Statements shows up in the books already have an environment wrapped around them. These statements are wrapped in the same environment as in the book, but with different typesets (names). The following table provides the mapping between typesets and the meaning of each environment. The only environment does not show up in the books is "additional corollaries." It is used to wrap statements that I added to help proving other complex statements.

Environment Typesets	Meaning
Ax.	Axiom
A.Cor.	Additional Corollary
Ch.	Chapter
Cor.	Corollary
Def.	Definition
E.g.	Example
Ex.	Exercise
Lem.	Lemma
Note	Note
Prop.	Proposition
Sec.	Section
Thm.	Theorem

Table 1: Mapping between environment typesets and meaning.

Part I

Analysis-I

Chapter I.1

Introduction

Note. *circularity*: Using an advanced fact to prove a more elementary fact, and then later using the elementary fact to prove the advanced fact. When do a mathematics proofs, one should avoid *circularity*.

Note. From a logical point of view, there is no difference between a lemma, proposition, theorem, or corollary - they are all claims waiting to be proved. However, we use these terms to suggest different levels of importance and difficulty. A lemma is an easily proved claim which is helpful for proving other propositions and theorems but is usually not particularly interesting in its own right. A proposition is a statement which is interesting in its own right. A theorem is a more important statement than a proposition which says something definitive on the subject and often takes more effort to prove than a proposition or lemma. A corollary is an immediate consequence of a proposition or theorem that was proven recently.

Chapter I.2

Natural Number

I.2.1 The Peano axioms

Note. We now present one standard way to define the natural numbers, in terms of the *Peano axioms*, which were first laid out by Giuseppe Peano (1858–1932). This is not the only way to define the natural numbers. For instance, another approach is to talk about the cardinality of finite sets. For instance, one could take a set of five elements and define 5 to be the number of elements in that set.

Note. In some texts, the natural numbers start at 1 instead of 0, but this is a matter of notational convention more than anything else. In this text, we shall refer to the set $\{1, 2, 3, ...\}$ as the set of positive integers $\mathbb{Z}^+$ rather than the natural numbers. Natural numbers are sometimes also known as *whole numbers*.

Note. In mathematics, we try not to define a variable more than once in any given setting, as it can often lead to confusion; many of the statements which were true for the old value of the variable can now become false, and vice versa.

Ax. I.2.1. 0 is a natural number.

Ax. I.2.2. If n is a natural number, then $n{+}{+}$ is a natural number.

Ax. I.2.3. 0 is not the successor of any natural number; i.e., we have $n{+}{+} \neq 0$ for every natural number n.

Ax. I.2.4. Different natural numbers must have different successors; i.e., if n and m are natural numbers and $n \neq m$, then $n{+}{+} \neq m{+}{+}$. Equivalently, if $n{+}{+} = m{+}{+}$, then we must have $n = m$.

Ax. I.2.5 (Principle of mathematical induction)**.** Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n{+}{+})$ is also true. Then $P(n)$ is true for every natural number n.

Note. Ax. I.2.1 to I.2.5 are known as the *Peano axioms* for the natural numbers.

Note. "a priori" is Latin for "beforehand" - it refers to what one already knows or assumes to be true before one begins a proof or argument. The opposite is "a posteriori" - what one knows to be true after the proof or argument is concluded.

Note. Ax. I.2.5 should technically be called an *axiom schema* rather than an *axiom* - it is a template for producing an (infinite) number of axioms rather than being a single axiom in its own right.

Note. A remarkable accomplishment of modern analysis is that by starting from these five very primitive axioms and some additional axioms from set theory, we can build all the other number systems, create functions, and do all the algebra and calculus that we are used to.

I.2.2 Addition

Def. I.2.2.1 (Addition of natural numbers)**.** Let m be a natural number. To add zero to m, we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m. Then we can add $n{+}{+}$ to m by defining $(n{+}{+}) + m := (n + m){+}{+}$.

A.Cor. I.2.2.1. The sum $n + m$ of two natural numbers n, m is again a natural number.

Proof of A.Cor. I.2.2.1. Let m, n be a natural number. We induct on n. For $n = 0$, by Def. I.2.2.1, we have $0 + m = m$, which is a natural number by the definition of m. So the base case holds. Suppose inductively that for some natural number n, we know that $n + m$ is a natural number. We want to show that $(n{+}{+}) + m$ is also a natural number. By Def. I.2.2.1, we have $(n{+}{+}) + m = (n + m){+}{+}$. By the induction hypothesis, we know that $n + m$ is a natural number. Thus, by Ax. I.2.2, we know that $(n + m){+}{+}$ is again a natural number. This closes the induction. □

Lem. I.2.2.2. For any natural number n, we have $n + 0 = n$.

Proof of Lem. I.2.2.2. We induct on n. The base case $0 + 0 = 0$ follows since we know that $0 + m = m$ for every natural number m (Def. I.2.2.1), and 0 is a natural number (Ax. I.2.1). Now suppose inductively that $n + 0 = n$. We wish to show that $(n{+}{+}) + 0 = n{+}{+}$. But by Def. I.2.2.1, $(n{+}{+}) + 0$ is equal to $(n + 0){+}{+}$, which is equal to $n{+}{+}$ since $n + 0 = n$. This closes the induction. □

Lem. I.2.2.3. For any natural numbers n and m, we have $n + (m{+}{+}) = (n + m){+}{+}$.

Proof of Lem. I.2.2.3. We induct on n (keeping m fixed). For $n = 0$, we must prove $0 + (m{+}{+}) = (0 + m){+}{+}$. But by Def. I.2.2.1, $0 + (m{+}{+}) = m{+}{+}$ and $0 + m = m$, so both sides are equal to $m{+}{+}$ and are thus equal to each other. So the base case holds. Suppose inductively that $n + (m{+}{+}) = (n + m){+}{+}$. We need to show that $(n{+}{+}) + (m{+}{+}) = ((n{+}{+}) + m){+}{+}$. The left-hand side is $(n + (m{+}{+})){+}{+}$ by Def. I.2.2.1, which is equal to

$((n+m)\text{++})\text{++}$ by the inductive hypothesis. Similarly, we have $(n\text{++}) + m = (n+m)\text{++}$ by Def. I.2.2.1, and so the right-hand side is also equal to $((n+m)\text{++})\text{++}$. Thus, both sides are equal, and we have closed the induction. $\square$

A.Cor. I.2.2.2. For any natural number n, we have $n\text{++} = n + 1$.

Proof of A.Cor. I.2.2.2. Since n is a natural number, by Ax. I.2.2 we know that $n\text{++}$ is also a natural number. Thus, we can apply Lem. I.2.2.2 to derive the following fact:

$$\begin{aligned} n\text{++} &= (n\text{++}) + 0 && \text{(by Lem. I.2.2.2)} \\ &= (n+0)\text{++} && \text{(by Def. I.2.2.1)} \\ &= n + (0\text{++}) && \text{(by Lem. I.2.2.3)} \\ &= n + 1. \end{aligned}$$

$\square$

Prop. I.2.2.4 (Addition is commutative)**.** For any natural numbers n and m, we have $n + m = m + n$.

Proof of Prop. I.2.2.4. We induct on n. For $n = 0$, we need to show $0 + m = m + 0$. But by Def. I.2.2.1, $0 + m = m$, while by Lem. I.2.2.2, $m + 0 = m$. Thus, both sides are equal, and the base case holds. Suppose inductively that $n + m = m + n$. We must prove that $(n\text{++}) + m = m + (n\text{++})$ to close the induction. By Def. I.2.2.1, $(n\text{++}) + m = (n+m)\text{++}$. By Lem. I.2.2.3, $m + (n\text{++}) = (m+n)\text{++}$, but this is equal to $(n+m)\text{++}$ by the induction hypothesis. Thus, $(n\text{++}) + m = m + (n\text{++})$, and we have closed the induction. $\square$

Prop. I.2.2.5 (Addition is associative)**.** For any natural numbers a, b, c, we have $(a+b)+c = a + (b + c)$.

Proof of Prop. I.2.2.5. We induct on c and keep both a and b fixed. For $c = 0$, we have

$$\begin{aligned} (a+b)+0 &= a + b && \text{(by Lem. I.2.2.2)} \\ &= a + (b+0). && \text{(by Lem. I.2.2.2)} \end{aligned}$$

Thus, the base case holds. Suppose inductively that $(a+b)+c = a+(b+c)$ for some natural number c. We want to show that $(a+b) + (c\text{++}) = a + (b + (c\text{++}))$. But this is true since

$$\begin{aligned} (a+b) + (c\text{++}) &= ((a+b)+c)\text{++} && \text{(by Lem. I.2.2.3)} \\ &= (a+(b+c))\text{++} && \text{(by the induction hypothesis)} \\ &= a + (b+c)\text{++} && \text{(by Lem. I.2.2.3)} \\ &= a + (b + (c\text{++})). && \text{(by Lem. I.2.2.3)} \end{aligned}$$

This closes the induction. $\square$

Note. Because of associativity showed in Prop. I.2.2.5, we can write sums such as $a+b+c$ without having to worry about which order the numbers are being added together.

Prop. I.2.2.6 (Cancellation law)**.** Let a, b, c be natural numbers such that $a+b=a+c$. Then we have $b=c$.

Proof of Prop. I.2.2.6. We induct on a. For $a=0$, we have $0+b=0+c$, which by Def. I.2.2.1 implies that $b=c$ as desired. Suppose inductively that we have the cancellation law for a (so that $a+b=a+c$ implies $b=c$); we now have to prove the cancellation law for $a{++}$. In other words, we assume that $(a{++})+b=(a{++})+c$ and need to show that $b=c$. By Def. I.2.2.1, we have $(a{++})+b=(a+b){++}$ and $(a{++})+c=(a+c){++}$, and so we have $(a+b){++}=(a+c){++}$. By Ax. I.2.4, we have $a+b=a+c$. Since we already have the cancellation law for a, we thus have $b=c$ as desired. This closes the induction. □

Def. I.2.2.7 (Positive natural numbers)**.** A natural number n is said to be *positive* iff it is not equal to 0.

Prop. I.2.2.8. If a is a positive natural number and b is a natural number, then $a+b$ is positive (and hence $b+a$ is also, by Prop. I.2.2.4).

Proof of Prop. I.2.2.8. We induct on b. If $b=0$, then $a+b=a+0=a$, which is positive, proving the base case. Suppose inductively that $a+b$ is positive. Then $a+(b{++})=(a+b){++}$, which cannot be zero by Ax. I.2.3, and is hence positive by Def. I.2.2.7. This closes the induction. □

Cor. I.2.2.9. If a and b are natural numbers such that $a+b=0$, then we have $a=0$ and $b=0$.

Proof of Cor. I.2.2.9. Suppose for the sake of contradiction that $a \neq 0$ or $b \neq 0$. If $a \neq 0$, then a is positive, and hence $a+b=0$ is positive by Prop. I.2.2.8, a contradiction. Similarly, if $b \neq 0$, then b is positive, and again $a+b=0$ is positive by Prop. I.2.2.8, a contradiction. Thus, a and b must both be zero. □

Lem. I.2.2.10. Let a be a positive natural number. Then there exists exactly one natural number b such that $b{++}=a$.

Proof of Lem. I.2.2.10. Let $P(n)$ be the statement "either we have $n=0$, or there exists a natural number m, such that $m{++}=n$." We induct on n to show that $P(n)$ is true for any natural number n. Clearly, $P(0)$ is true. So suppose inductively that $P(n)$ is true for some natural number n. We want to show that $P(n{++})$ is true. By Ax. I.2.3, we know that $n{++} \neq 0$. So we have to show that there exists a natural number m, such that $m{++}=n{++}$. By Ax. I.2.4, we see that $m=n$. Thus, $P(n{++})$ is true, closing the induction.

Now we prove the existence of b. From the first part of the proof, we know that $P(a)$ is true. Since a is a positive natural number, by Def. I.2.2.7, we know that $a \neq 0$. Thus, there must exist a natural number b such that $b{++}=a$.

Finally, we prove the uniqueness of b. Suppose that there exists another natural number c such that $c{++}=a$. But this means $b{++}=c{++}$. Thus, by Ax. I.2.4, we have $b=c$. □

Def. I.2.2.11 (Ordering of the natural numbers)**.** Let n and m be natural numbers. We say that n is *greater than or equal to* m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

A.Cor. I.2.2.3. We have $n{+}{+} > n$ for any natural number n. Therefore, there is no largest natural number n, because the next number $n{+}{+}$ is always larger.

Proof of A.Cor. I.2.2.3. Let n be a natural number. By A.Cor. I.2.2.2, we have $n{+}{+} = n+1$. Since 1 is a natural number, by Def. I.2.2.11, we have $n{+}{+} \geq n$. To show that $n{+}{+} > n$, by Def. I.2.2.11, we only need to show that $n{+}{+} \neq n$.

We induct on n to show that $n{+}{+} \neq n$ for any natural number n. For $n = 0$, we have $0{+}{+} \neq 0$, by Ax. I.2.3. Thus, the base case holds. Suppose inductively that $n{+}{+} \neq n$ for some natural number n. Then we have

$$\begin{aligned} & n{+}{+} \neq n && \text{(by the induction hypothesis)} \\ \implies & (n{+}{+}){+}{+} \neq n{+}{+}. && \text{(by Ax. I.2.4)} \end{aligned}$$

This closes the induction. We conclude that $n{+}{+} > n$ for any natural number n. □

A.Cor. I.2.2.4. We have $n \geq 0$ for every natural number n. If n is a positive natural number, then we have $n > 0$.

Proof of A.Cor. I.2.2.4. Let n be a natural number. By Def. I.2.2.1, we have $n = 0 + n$. Thus, by Def. I.2.2.11, we have $n \geq 0$.

Now suppose that n is a positive natural number. By Def. I.2.2.7, this means $n \neq 0$. From first paragraph we see that $n \geq 0$. Thus, by Def. I.2.2.11, we have $n > 0$. □

Prop. I.2.2.12 (Basic properties of order for natural numbers)**.** Let a, b, c be natural numbers. Then

(a) (Order is reflexive) $a \geq a$.

(b) (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$.

(c) (Order is anti-symmetric) If $a \geq b$ and $b \geq a$, then $a = b$.

(d) (Addition preserves order) $a \geq b$ iff $a + c \geq b + c$.

(e) $a < b$ iff $a{+}{+} \leq b$.

(f) $a < b$ iff $b = a + d$ for some *positive* number d.

Proof of Prop. I.2.2.12(a). We have

$$\begin{cases} 0 \text{ is a natural number} \\ a = a + 0 \end{cases} \qquad \text{(by Ax. I.2.1 and Lem. I.2.2.2)}$$

$$\Longrightarrow a \geq a. \qquad \text{(by Def. I.2.2.11)}$$

□

Proof of Prop. I.2.2.12(b). Suppose that $a \geq b$ and $b \geq c$. By Def. I.2.2.11, there exist some natural numbers d and e, such that $a = b + d$ and $b = c + e$. Then we have

$$\begin{aligned}
&\begin{cases} a = b + d = (c + e) + d = c + (e + d) \\ e + d \text{ is a natural number} \end{cases} && \text{(by Prop. I.2.2.5 and A.Cor. I.2.2.1)} \\
\Longrightarrow\ & a \geq c. && \text{(by Def. I.2.2.11)}
\end{aligned}$$

□

Proof of Prop. I.2.2.12(c). Suppose that $a \geq b$ and $b \geq a$. By Def. I.2.2.11, there exist some natural numbers c and d, such that $a = b + c$ and $b = a + d$. Then we have

$$\begin{aligned}
&\begin{cases} a = a + 0 \\ a = b + c = (a + d) + c = a + (d + c) \end{cases} && \text{(by Lem. I.2.2.2 and Prop. I.2.2.5)} \\
\Longrightarrow\ & 0 = d + c && \text{(by Prop. I.2.2.6)} \\
\Longrightarrow\ & d = c = 0 && \text{(by Cor. I.2.2.9)} \\
\Longrightarrow\ & a = b + 0 = b. && \text{(by Lem. I.2.2.2)}
\end{aligned}$$

□

Proof of Prop. I.2.2.12(d). We have

$$\begin{aligned}
& a \geq b \\
\Longrightarrow\ & a = b + d \text{ for some natural number } d && \text{(by Def. I.2.2.11)} \\
\Longrightarrow\ & a + c = c + a = c + (b + d) && \text{(by Prop. I.2.2.4)} \\
& \quad = (c + b) + d = (b + c) + d && \text{(by Prop. I.2.2.4 and I.2.2.5)} \\
\Longrightarrow\ & a + c \geq b + c, && \text{(by Def. I.2.2.11)}
\end{aligned}$$

and

$$\begin{aligned}
& a + c \geq b + c \\
\Longrightarrow\ & a + c = (b + c) + d \text{ for some natural number } d && \text{(by Def. I.2.2.11)} \\
\Longrightarrow\ & c + a = (c + b) + d = c + (b + d) && \text{(by Prop. I.2.2.4 and I.2.2.5)} \\
\Longrightarrow\ & a = b + d && \text{(by Prop. I.2.2.6)} \\
\Longrightarrow\ & a \geq b. && \text{(by Def. I.2.2.11)}
\end{aligned}$$

Thus, we conclude that $a \geq b \iff a + c \geq b + c$. □

Proof of Prop. I.2.2.12(e). First, suppose that $a < b$. Then we have

$$\begin{aligned}
&a < b \\
\implies &\begin{cases} a \leq b \\ a \neq b \end{cases} && \text{(by Def. I.2.2.11)} \\
\implies &\begin{cases} b = a + c \text{ for some natural number } c \\ a \neq b \end{cases} && \text{(by Def. I.2.2.11)} \\
\implies &c \neq 0 && \text{(by Lem. I.2.2.2)} \\
\implies &c \text{ is a positive natural number} && \text{(by Prop. I.2.2.8)} \\
\implies &\text{there exists a natural number } d \text{ such that } d{+\!+} = c && \text{(by Lem. I.2.2.10)} \\
\implies &b = a + (d{+\!+}) = (a + d){+\!+} = (a{+\!+}) + d && \text{(by Def. I.2.2.1 and Lem. I.2.2.3)} \\
\implies &a{+\!+} \leq b. && \text{(by Def. I.2.2.11)}
\end{aligned}$$

Now suppose that $a{+\!+} \leq b$. Then we have

$$\begin{aligned}
&a{+\!+} \leq b \\
\implies &b = (a{+\!+}) + c \text{ for some natural number } c && \text{(by Def. I.2.2.11)} \\
\implies &b = (a{+\!+}) + c = (a + c){+\!+} = a + (c{+\!+}) && \text{(by Def. I.2.2.1 and Lem. I.2.2.3)} \\
\implies &a \leq b. && \text{(by Def. I.2.2.11)}
\end{aligned}$$

By Ax. I.2.3, we know that $c{+\!+} \neq 0$. Thus, by Prop. I.2.2.6, we must have $b \neq a$. (If $b = a$, then we would have $b = a + (c{+\!+}) = a + 0$, which implies $c{+\!+} = 0$, a contradiction.) By Def. I.2.2.11, this means $a < b$. From all proofs above, we conclude that $a < b \iff a{+\!+} \leq b$. □

Proof of Prop. I.2.2.12(f). We have

$$\begin{aligned}
&a < b \\
\iff &a{+\!+} \leq b && \text{(by Prop. I.2.2.12(e))} \\
\iff &b = (a{+\!+}) + c \text{ for some natural number } c && \text{(by Def. I.2.2.11)} \\
\iff &b = (a{+\!+}) + c = (a + c){+\!+} = a + (c{+\!+}) && \text{(by Def. I.2.2.1 and Lem. I.2.2.3)} \\
\iff &b = a + d \text{ for some positive natural number } d. && \text{(by Ax. I.2.3 and Lem. I.2.2.10)}
\end{aligned}$$

□

Prop. I.2.2.13 (Trichotomy of order for natural numbers)**.** Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b$, $a = b$, or $a > b$.

Proof of Prop. I.2.2.13. First, we show that we cannot have more than one of the statements $a < b$, $a = b$, $a > b$ holding simultaneously. If $a < b$, then $a \neq b$ by Def. I.2.2.11, and if $a > b$, then $a \neq b$ by Def. I.2.2.11. If $a > b$ and $a < b$, then by Prop. I.2.2.12(c), we have $a = b$, a contradiction. Thus, no more than one of the statements is true.

Now we show that at least one of the statements is true. We keep b fixed and induct on a. When $a = 0$, by Def. I.2.2.1, we have $b = 0 + b$. Thus, by A.Cor. I.2.2.4, we have $0 \leq b$ for any natural number b. So we have either $0 = b$ or $0 < b$, which proves the base case. Suppose we have proven the proposition for a, and now we prove the proposition for $a{+}{+}$. From the trichotomy for a, there are three cases: $a < b$, $a = b$, and $a > b$.

- If $a > b$, then by Prop. I.2.2.12(d), we have $a{+}{+} \geq b{+}{+}$. Since $b{+}{+} > b$, by Prop. I.2.2.12(b), we have $a{+}{+} \geq b$. Then we must have $a{+}{+} > b$. Otherwise, by Def. I.2.2.11, we would have $a{+}{+} = b$, and by Prop. I.2.2.12(e), this implies $a < b$ and contradicts $a > b$ (the first paragraph proves the contradiction).

- If $a = b$, then by Ax. I.2.4, we have $a{+}{+} = b{+}{+}$. Since $b{+}{+} > b$, by Prop. I.2.2.12(b), we have $a{+}{+} \geq b$. Then we must have $a{+}{+} > b$. Otherwise, by Def. I.2.2.11, we would have $a{+}{+} = b$, and by Prop. I.2.2.12(e), this implies $a < b$ and contradicts $a = b$ (the first paragraph proves the contradiction).

- If $a < b$, then by Prop. I.2.2.12(e), we have $a{+}{+} \leq b$. Thus, either $a{+}{+} = b$ or $a{+}{+} < b$, and in either case, we are done.

This closes the induction. □

Prop. I.2.2.14 (Strong principle of induction)**.** Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true since, in this case, the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

Proof of Prop. I.2.2.14. Let n be a natural number, and let $Q(n)$ be the statement "$P(m)$ is true for any natural number m satisfying $m_0 \leq m < n$." We induct on n to show that $Q(n)$ is true for any natural number n.

For $n = 0$, we want to show that $Q(0)$ is true. However, we know that $0 \leq m_0$ for any natural number m_0. Thus, we have either $0 = m_0$ or $0 < m_0$. So we split it into two cases:

- If $0 < m_0$, then the statement "$P(m)$ is true for any natural number m satisfying $m_0 \leq m < n$" is vacuously true, since there does not exist a natural number m satisfying $0 < m_0 \leq m < n = 0$. Thus, $Q(0)$ is true in this case.

- If $0 = m_0$, then the statement "$P(m)$ is true for any natural number m satisfying $m_0 \leq m < n$" is vacuously true, since there does not exist a natural number m satisfying $0 = m_0 \leq m < n = 0$. Hence, $Q(0)$ is true in this case.

From all cases above, we see that $Q(0)$ is true. Thus, the base case holds.

Suppose inductively that $Q(n)$ is true for some natural number n. We need to show that $Q(n{+}{+})$ is true. Using the induction hypothesis $Q(n)$ and the hypothesis of P, we see that $P(n)$ is true. Since $n < n{+}{+}$, we know that $P(m)$ is true for any natural number m satisfying $m_0 \leq m \leq n < n{+}{+}$. So $P(m)$ is true for any natural number m satisfying $m_0 \leq m < n{+}{+}$, which in turn implies that $Q(n{+}{+})$ is true. This closes the induction. Hence, we can conclude that $Q(n)$ is true for any natural number n.

Since $Q(n)$ is true for any natural number n, by the hypothesis of P, we know that $P(n)$ is true for any natural number n. In particular, we see that $P(n)$ is true for any natural number n satisfying $n \geq m_0$. □

Rmk. I.2.2.15. In applications we usually use Prop. I.2.2.14 with $m_0 = 0$ or $m_0 = 1$.

— Exercises —

Ex. I.2.2.1. Prove Prop. I.2.2.5.

Proof of Ex. I.2.2.1. See Prop. I.2.2.5. □

Ex. I.2.2.2. Prove Lem. I.2.2.10.

Proof of Ex. I.2.2.2. See Lem. I.2.2.10. □

Ex. I.2.2.3. Prove Prop. I.2.2.12.

Proof of Ex. I.2.2.3. See Prop. I.2.2.12. □

Ex. I.2.2.4. Justify the three statements marked in the proof of Prop. I.2.2.13.

Proof of Ex. I.2.2.4. See Prop. I.2.2.13. □

Ex. I.2.2.5. Prove Prop. I.2.2.14.

Proof of Ex. I.2.2.5. See Prop. I.2.2.14. □

Ex. I.2.2.6 (Principle of backwards induction)**.** Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+}{+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for any natural numbers $m \leq n$; this is known as the *principle of backwards induction.*

Proof of Ex. I.2.2.6. We induct on n. For $n = 0$, the only natural number m satisfying $m \leq n = 0$ is 0. By the given hypothesis, $P(0)$ is true. Therefore, the base case holds trivially.

Suppose inductively that for some natural number n, we have the implication "if $P(n)$ is true, then $P(m)$ is true for any natural number m satisfying $m \leq n$." We want to show the implication "if $P(n{+}{+})$ is true, then $P(m)$ is true for any natural number m satisfying

$m \leq n{+}{+}$" is also true. But when $P(n{+}{+})$ is true, we know that $P(n)$ is true by the hypothesis of P. Thus, we can apply the induction hypothesis to derive "$P(m)$ is true for any natural number m satisfying $m \leq n$." Combining the statement "$P(n{+}{+})$ is true," we see that the statement "$P(m)$ is true for any natural number m satisfying $m \leq n{+}{+}$" is true. This closes the induction. □

Ex. I.2.2.7. Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m)$ is true, $P(m{+}{+})$ is true. Show that if $P(n)$ is true, then $P(m)$ is true for any natural number m satisfying $m \geq n$. This principle is sometimes referred to as *the principle of induction starting from the base case n.*

Proof of Ex. I.2.2.7. Suppose that $P(n)$ is true. Let $Q(k) = P(n + k)$ for every natural number k. We induct on k to show that $Q(k)$ is true for every natural number k.

For $k = 0$, we have $Q(0) = P(n + 0) = P(n)$ by Lem. I.2.2.2. Since $P(n)$ is true by the given hypothesis, we know that $Q(0)$ is true. Thus, the base case holds.

Suppose inductively that $Q(k)$ is true for some natural number k. We want to show that $Q(k{+}{+})$ is true. By Lem. I.2.2.3, we have $Q(k{+}{+}) = P(n + (k{+}{+})) = P((n + k){+}{+})$. By the induction hypothesis, we know that $Q(k) = P(n + k)$ is true. Thus, we can use the hypothesis of P to show that $P((n + k){+}{+})$ is also true. This closes the induction. We conclude that $P(n + k)$ is true for every natural number k.

By Def. I.2.2.11, we know that for every natural number m, we have $m \geq n \iff m = n + k$ for some natural number k. Thus, $P(m)$ is true for every natural number m satisfying $m \geq n$. □

I.2.3 Multiplication

Def. I.2.3.1 (Multiplication of natural numbers)**.** Let m be a natural number. To multiply zero by m, we define $0 \times m := 0$. Now suppose inductively that we have defined how to multiply n by m. Then we can multiply $n{+}{+}$ by m by defining $(n{+}{+}) \times m := (n \times m) + m$.

A.Cor. I.2.3.1. The product of two natural numbers is a natural number.

Proof of A.Cor. I.2.3.1. Let n and m be two natural numbers. We induct on n. For $n = 0$, by Def. I.2.3.1, we have $0 \times m = 0$, which is a natural number by Ax. I.2.1. So the base case holds. Suppose inductively that for some natural number n, we know that $n \times m$ is a natural number. We want to show that $(n{+}{+}) \times m$ is a natural number. By Def. I.2.3.1, $(n{+}{+}) \times m = (n \times m) + m$. By the induction hypothesis, $n \times m$ is a natural number. By A.Cor. I.2.2.1, $(n \times m) + m$ is a natural number. Thus, $(n{+}{+}) \times m$ is a natural number. This closes the induction. □

A.Cor. I.2.3.2. Let n be a natural number. Then $n \times 0 = 0$.

Proof of A.Cor. I.2.3.2. We induct on n. For $n = 0$, by Def. I.2.3.1, we have $0 \times 0 = 0$. So the base case holds. Suppose inductively that for some natural number n, we have $n \times 0 = 0$. Then for $n{+}{+}$, we have

$$\begin{aligned}
(n{+}{+}) \times 0 &= (n \times 0) + 0 && \text{(by Def. I.2.3.1)} \\
&= 0 + 0 && \text{(by the induction hypothesis)} \\
&= 0. && \text{(by Def. I.2.2.1)}
\end{aligned}$$

This closes the induction. □

A.Cor. I.2.3.3. Let n and m be natural numbers. Then $n \times (m{+}{+}) = (n \times m) + n$.

Proof of A.Cor. I.2.3.3. We induct on n and fix m. For $n = 0$, by Def. I.2.3.1, we have $0 \times (m{+}{+}) = 0$. So the base case holds. Suppose inductively that for some natural number n, we have $n \times (m{+}{+}) = (n \times m) + n$. Then for $n{+}{+}$, we have

$$\begin{aligned}
(n{+}{+}) \times (m{+}{+}) &= (n \times (m{+}{+})) + (m{+}{+}) && \text{(by Def. I.2.3.1)} \\
&= ((n \times m) + n) + (m{+}{+}) && \text{(by the induction hypothesis)} \\
&= (n \times m) + (n + (m{+}{+})) && \text{(by Prop. I.2.2.5)} \\
&= (n \times m) + ((n + m){+}{+}) && \text{(by Lem. I.2.2.3)} \\
&= (n \times m) + ((m + n){+}{+}) && \text{(by Prop. I.2.2.4)} \\
&= (n \times m) + (m + (n{+}{+})) && \text{(by Lem. I.2.2.3)} \\
&= ((n \times m) + m) + (n{+}{+}) && \text{(by Prop. I.2.2.5)} \\
&= ((n{+}{+}) \times m) + (n{+}{+}). && \text{(by Def. I.2.3.1)}
\end{aligned}$$

This closes the induction. □

Lem. I.2.3.2 (Multiplication is commutative)**.** Let n and m be natural numbers. Then $n \times m = m \times n$.

Proof of Lem. I.2.3.2. We induct on n and fix m. For $n = 0$, by Def. I.2.3.1, we have $0 \times m = 0$, and by A.Cor. I.2.3.2, we have $m \times 0 = 0$. So the base case holds. Suppose inductively that for some natural number n, we have $n \times m = m \times n$. Then for $n{+}{+}$, we have

$$\begin{aligned}
(n{+}{+}) \times m &= (n \times m) + m && \text{(by Def. I.2.3.1)} \\
&= (m \times n) + m && \text{(by the induction hypothesis)} \\
&= m \times (n{+}{+}). && \text{(by A.Cor. I.2.3.3)}
\end{aligned}$$

This closes the induction. □

Note. We will now abbreviate $n \times m$ as nm, and use the convention that multiplication takes precedence over addition. Thus, for instance, $ab + c$ means $(a \times b) + c$, not $a \times (b + c)$.

Lem. I.2.3.3 (Positive natural numbers have no zero divisors)**.** Let n and m be natural numbers. Then $n \times m = 0$ iff at least one of n or m equals zero. In particular, if n and m are both positive, then nm is also positive.

Proof of Lem. I.2.3.3. First, suppose that $n \times m = 0$. Suppose for the sake of contradiction that $n \neq 0 \neq m$. By Def. I.2.2.7, this means n and m are positive natural numbers. Then by Lem. I.2.2.10, there exist some natural numbers a and b such that $n = a{+}{+}$ and $m = b{+}{+}$. Thus, we have

$$\begin{aligned} n \times m &= (a{+}{+}) \times (b{+}{+}) \\ &= a \times (b{+}{+}) + (b{+}{+}). && \text{(by Def. I.2.3.1)} \end{aligned}$$

By Ax. I.2.3, we know that $b{+}{+} \neq 0$. Thus, by Prop. I.2.2.8, we know that $n \times m$ is a positive natural number. But this contradicts $n \times m = 0$. Thus, we must have either $n = 0$ or $m = 0$.

Now suppose that $n = 0$ or $m = 0$. If $n = 0$, then we have $n \times m = 0 \times m = 0$ by Def. I.2.3.1. If $m = 0$, then we have $n \times m = n \times 0 = 0$ by A.Cor. I.2.3.2. In either case, we have $n \times m = 0$.

From all proofs above, we conclude that $n \times m = 0 \iff (n = 0) \vee (m = 0)$. Thus, we have $n \times m \neq 0 \iff (n \neq 0) \wedge (m \neq 0)$. By Def. I.2.2.7, we see that n and m are positive natural numbers iff $n \times m \neq 0$. ☐

A.Cor. I.2.3.4. Let n be a natural number. Then $n1 = 1n = n$.

Proof of A.Cor. I.2.3.4. By Lem. I.2.3.2, we know that $n1 = 1n$. Thus, we only need to show that $n1 = n$. We induct on n. For $n = 0$, we have $0 \times 1 = 0$ by Lem. I.2.3.3. So the base case holds. Suppose inductively that $n1 = n$ is true for some natural number n. Then for $n + 1$, we have $(n + 1) \times 1 = n1 + 1$ by Def. I.2.3.1. By the induction hypothesis, we have $n1 = n$. Thus, we have $(n + 1) \times 1 = n + 1$, and this closes the induction. ☐

Prop. I.2.3.4 (Distributive law)**.** For any natural numbers a, b, c, we have $a(b+c) = ab+ac$ and $(b + c)a = ba + ca$.

Proof of Prop. I.2.3.4. Since multiplication is commutative, we only need to show the first identity $a(b + c) = ab + ac$. We keep a and b fixed, and we induct on c. Let's prove the base case $c = 0$, i.e., $a(b + 0) = ab + a0$. The left-hand side is ab, while the right-hand side is $ab + 0 = ab$, so we are done with the base case. Now let us suppose inductively that $a(b + c) = ab + ac$, and let us prove that $a(b + (c{+}{+})) = ab + a(c{+}{+})$. The left-hand side is $a((b + c){+}{+}) = a(b + c) + a$ by A.Cor. I.2.3.3, while the right-hand side is $ab+ac+a = a(b+c)+a$ by the induction hypothesis, and so we can close the induction. ☐

Prop. I.2.3.5 (Multiplication is associative)**.** For any natural numbers a, b, c, we have $(a \times b) \times c = a \times (b \times c)$.

Proof of Prop. I.2.3.5. We keep a and b fixed, and we induct on c. For $c = 0$, by A.Cor. I.2.3.2, we have $(a \times b) \times 0 = 0 = a \times 0 = a \times (b \times 0)$. So the base case holds. Suppose inductively that for some natural number c, we have $(a \times b) \times c = a \times (b \times c)$. Then for $c{+}{+}$, we have

$$\begin{aligned}
(a \times b) \times (c{+}{+}) &= (a \times b) \times c + a \times b && \text{(by A.Cor. I.2.3.3)} \\
&= a \times (b \times c) + a \times b && \text{(by the induction hypothesis)} \\
&= a \times (b \times c + b) && \text{(by Prop. I.2.3.4)} \\
&= a \times (b \times (c{+}{+})). && \text{(by A.Cor. I.2.3.3)}
\end{aligned}$$

This closes the induction. □

Prop. I.2.3.6 (Multiplication preserves order)**.** If a, b, c are natural numbers such that $a < b$, and c is positive, then $ac < bc$.

Proof of Prop. I.2.3.6. Since $a < b$, we have $b = a + d$ for some positive d by Prop. I.2.2.12(f). Multiplying by c and using the distributive law (Prop. I.2.3.4), we obtain $bc = ac + dc$. Since d is positive, and c is positive, dc is positive (Lem. I.2.3.3), and hence $ac < bc$ (by Def. I.2.2.11), as desired. □

Cor. I.2.3.7 (Cancellation law)**.** Let a, b, c be natural numbers such that $ac = bc$ and c is non-zero. Then $a = b$.

Proof of Cor. I.2.3.7. By the trichotomy of order (Prop. I.2.2.13), we have three cases: $a < b$, $a = b$, $a > b$. Suppose first that $a < b$, then by Prop. I.2.3.6, we have $ac < bc$, a contradiction. We can obtain a similar contradiction when $a > b$. Thus, the only possibility is that $a = b$, as desired. □

Rmk. I.2.3.8. Just as Prop. I.2.2.6 will allow for a "virtual subtraction" which will eventually let us define genuine subtraction, Cor. I.2.3.7 provides a "virtual division" which will be needed to define genuine division later on.

Prop. I.2.3.9 (Euclid's division lemma)**.** Let n be a natural number, and let q be a positive natural number. Then there exist natural numbers m and r such that $0 \leq r < q$ and $n = mq + r$.

Proof of Prop. I.2.3.9. We induct on n and fix q. For $n = 0$, let $r = m = 0$. Then we have

$$\begin{aligned}
mq + r &= 0q + 0 \\
&= 0 + 0 && \text{(by Def. I.2.3.1)} \\
&= 0, && \text{(by Def. I.2.2.1)}
\end{aligned}$$

and

$$0 \leq 0 = r \qquad \text{(by Prop. I.2.2.12(a))}$$

$$< q. \qquad \text{(by Def. I.2.2.11)}$$

So the base case holds. Suppose inductively that for some natural number n, there exist some natural numbers m and r such that $n = mq + r$ and $0 \leq r < q$. Then for $n{+}{+}$, we have

$$\begin{aligned} n{+}{+} &= (mq + r){+}{+} && \text{(by the induction hypothesis)} \\ &= mq + (r{+}{+}). && \text{(by Lem. I.2.2.3)} \end{aligned}$$

Since $r < q$, we have $r{+}{+} \leq q$ by Prop. I.2.2.12(e). Now we split into two cases:

- If $r{+}{+} < q$, then we have $0 \leq r < r{+}{+} < q$, and we are done in this case.
- If $r{+}{+} = q$, then by Def. I.2.3.1, we have

  $$n{+}{+} = mq + (r{+}{+}) = mq + q = (m{+}{+}) \times q = (m{+}{+}) \times q + r',$$

  where $r' = 0$ and $0 \leq r' < q$ by A.Cor. I.2.2.4, and we are also done in this case.

From all cases above, we can find some natural numbers m and r such that $n{+}{+} = mq + r$ and $0 \leq r < q$. This closes the induction. □

Rmk. I.2.3.10. In other words, we can divide a natural number n by a positive number q to obtain a quotient m (another natural number) and a remainder r (less than q). This algorithm marks the beginning of *number theory*, which is a beautiful and important subject that is beyond this text's scope.

Def. I.2.3.11 (Exponentiation for natural numbers)**.** Let m be a natural number. To raise m to the power 0, we define $m^0 := 1$; in particular, we define $0^0 := 1$. Now suppose recursively that m^n has been defined for some natural number n, then we define $m^{n{+}{+}} := m^n \times m$.

A.Cor. I.2.3.5. For any natural number n, we have $n^1 = n$.

Proof of A.Cor. I.2.3.5. We have

$$\begin{aligned} n^1 &= n^0 \times n = 1 \times n && \text{(by Def. I.2.3.11)} \\ &= n. && \text{(by A.Cor. I.2.3.4)} \end{aligned}$$

□

— Exercises —

Ex. I.2.3.1. Prove Lem. I.2.3.2.

Proof of Ex. I.2.3.1. See Lem. I.2.3.2 □

Ex. I.2.3.2. Prove Lem. I.2.3.3

Proof of Ex. I.2.3.2. See Lem. I.2.3.3 $\square$

Ex. I.2.3.3. Prove Prop. I.2.3.5

Proof of Ex. I.2.3.3. See Prop. I.2.3.5 $\square$

Ex. I.2.3.4. Prove the identity $(a+b)^2 = a^2 + 2ab + b^2$ for all natural numbers a and b.

Proof of Ex. I.2.3.4. We have

$$\begin{aligned}
(a+b)^2 &= (a+b)^1 \times (a+b) && \text{(by Def. I.2.3.11)}\\
&= (a+b) \times (a+b) && \text{(by A.Cor. I.2.3.5)}\\
&= a(a+b) + b(a+b) = aa + ab + ba + bb && \text{(by Prop. I.2.3.4)}\\
&= a^1 \times a + ab + ba + b^1 \times b && \text{(by A.Cor. I.2.3.5)}\\
&= a^2 + ab + ba + b^2 && \text{(by Def. I.2.3.11)}\\
&= a^2 + ab + ab + b^2 && \text{(by Lem. I.2.3.2)}\\
&= a^2 + 1 \times ab + 1 \times ab + b^2 && \text{(by A.Cor. I.2.3.4)}\\
&= a^2 + (1+1) \times ab + b^2 = a^2 + 2ab + b^2. && \text{(by Prop. I.2.3.4)}
\end{aligned}$$

$\square$

Ex. I.2.3.5. Prove Prop. I.2.3.9

Proof of Ex. I.2.3.5. See Prop. I.2.3.9 $\square$

Chapter I.3

Set Theory

I.3.1 Fundamentals

Def. I.3.1.1. We define a *set* A to be any unordered collection of objects. If x is an object, we say that x *is an element of* A or $x \in A$ if x lies in the collection; otherwise, we say that $x \notin A$.

Ax. I.3.1 (Sets are objects)**.** If A is a set, then A is also an object. In particular, given two sets A and B, it is meaningful to ask whether A is also an element of B.

Rmk. I.3.1.3. There is a special case of set theory, called "pure set theory," in which *all* objects are sets; for instance, the number 0 might be identified with the empty set $\emptyset = \{\}$, the number 1 might be identified with $\{0\} = \{\{\}\}$, the number 2 might be identified with $\{0, 1\} = \{\{\}, \{\{\}\}\}$, and so forth. From a logical point of view, pure set theory is a simpler theory, since one only has to deal with sets and not with objects; however, from a conceptual point of view, it is often easier to deal with impure set theories in which some objects are not considered to be set. The two types of theories are more or less equivalent for the purposes of doing mathematics, and so we shall take an agnostic position as to whether all objects are sets or not.

Def. I.3.1.4 (Equality of sets)**.** Two sets A and B are *equal*, $A = B$, iff every element of A is an element of B and vice versa. To put it another way, $A = B$ iff every element x of A belongs also to B, and every element y of B belongs also to A.

A.Cor. I.3.1.1. The definition of equality in Def. I.3.1.4 is reflexive, symmetric, and transitive.

Proof of A.Cor. I.3.1.1. Let A, B, C be sets. We first show that Def. I.3.1.4 is reflexive. Since $x \in A \iff x \in A$ for all $x \in A$, by Def. I.3.1.4, we have $A = A$. Thus, Def. I.3.1.4 is reflexive.

Next we show that Def. I.3.1.4 is symmetric. Suppose that $A = B$. By Def. I.3.1.4, we have $x \in A \iff x \in B$ for all $x \in A$. But this means $y \in B \iff y \in A$ for all $y \in B$. Thus, by Def. I.3.1.4, we have $B = A$. So Def. I.3.1.4 is symmetric.

Finally, we show that Def. I.3.1.4 is transitive. Suppose that $A = B$ and $B = C$. By Def. I.3.1.4, we have $x \in A \iff x \in B$ for all $x \in A$, and $y \in B \iff y \in C$ for all $y \in B$. Then we have $x \in A \iff x \in C$ for all $x \in A$, and, by Def. I.3.1.4, we have $A = C$. Thus, Def. I.3.1.4 is transitive. $\square$

Note. Observe that if $x \in A$ and $A = B$, then $x \in B$, by Def. I.3.1.4. Thus, the "is an element of" relation $\in$ obeys the axiom of substitution. Because of this, any new operation we define on sets will also obey the axiom of substitution, as long as we can define that operation purely in terms of the relation $\in$.

Note. Next, we turn to the issue of exactly which objects are sets and which objects are not. The situation is analogous to how we defined the natural numbers in Ch. I.2; we started with a single natural number, 0, and started building more numbers out of 0 using the increment operation. We will try something similar here, starting with a single set, the *empty set*, and building more sets out of the empty set by various operations. We begin by postulating the existence of the empty set.

Ax. I.3.2 (Empty set)**.** There exists a set $\emptyset$, known as the *empty set*, which contains no elements, i.e., for every object x we have $x \notin \emptyset$.

Note. The empty set is also denoted $\{\}$.

A.Cor. I.3.1.2. There can only be one empty set; if there were two sets $\emptyset$ and $\emptyset'$ which were both empty, then they would be equal to each other.

Proof of A.Cor. I.3.1.2. Suppose there exist two empty sets $\emptyset$ and $\emptyset'$. Then the statement "every element x of $\emptyset$ is also an element of $\emptyset'$" is vacuously true, since, by Ax. I.3.2, there does not exist any element of $\emptyset$. Similarly, the statement "every element x of $\emptyset'$ is also an element of $\emptyset$" is vacuously true. Thus, we see that x is an element of $\emptyset$ iff x is an element $\emptyset'$. By Def. I.3.1.4, this means $\emptyset = \emptyset'$. We conclude that empty set is unique. $\square$

Note. If a set is not equal to the empty set, we call it *non-empty.*

Lem. I.3.1.6 (Single choice)**.** Let A be a non-empty set. Then there exists an object x such that $x \in A$.

Proof of Lem. I.3.1.6. We prove by contradiction. Suppose there does not exist any object x such that $x \in A$. Then for all objects x, we have $x \notin A$. Also, by Ax. I.3.2, we have $x \notin \emptyset$. Thus, $x \in A \iff x \in \emptyset$ (both statements are equally false), and so $A = \emptyset$ by Def. I.3.1.4, a contradiction. $\square$

Rmk. I.3.1.7. Lem. I.3.1.6 asserts that given any non-empty set A, we are allowed to "choose" an element x of A which demonstrates this non-emptyness. Later on (in Lem. I.3.5.12) we will show that given any finite number of non-empty sets, say $A_1, \ldots, A_n$, it is possible to choose one element $x_1, \ldots, x_n$ from each set $A_1, \ldots, A_n$; this is known as "finite choice." However, in order to choose elements from an infinite number of sets, we need an additional axiom, the *axiom of choice* (Ax. I.8.1).

Rmk. I.3.1.8. Note that the empty set is *not* the same thing as the natural number 0. One is a set; the other is a number. However, it is true that the *cardinality* of the empty set is 0. See Ex. I.3.6.2.

Ax. I.3.3 (Singleton sets and pair sets)**.** If a is an object, then there exists a set $\{a\}$ whose only element is a, i.e., for every object y, we have $y \in \{a\}$ iff $y = a$; we refer to $\{a\}$ as the *singleton set* whose element is a. Furthermore, if a and b are objects, then there exists a set $\{a, b\}$ whose only elements are a and b; i.e., for every object y, we have $y \in \{a, b\}$ iff $y = a$ or $y = b$; we refer to this set as the *pair set* formed by a and b.

Rmk. I.3.1.9. There is only one singleton set for each object a. Similarly, given any two objects a and b, there is only one pair set formed by a and b. Also, Def. I.3.1.4 also ensures that $\{a, b\} = \{b, a\}$ and $\{a, a\} = \{a\}$. Thus, the singleton set axiom is in fact redundant, being a consequence of the pair set axiom. Conversely, the pair set axiom will follow from the singleton set axiom and the pairwise union axiom (Ax. I.3.4) below (see Lem. I.3.1.13). One may wonder why we don't go further and create triplet axioms, quadruplet axioms, etc.; however there will be no need for this once we introduce the pairwise union axiom (Ax. I.3.4) below.

Proof of Rmk. I.3.1.9. We first show the uniqueness of singleton set. Suppose there exist two sets A and A' which are singleton sets of object a. Then by Ax. I.3.3, we have "$x \in A \iff x = a$" and "$x \in A' \iff x = a$." But this means $x \in A \iff x \in A'$. Thus, by Def. I.3.1.4, we have $A = A'$, and therefore the uniqueness of singleton set is proved.

Next we show the uniqueness of pair set. Suppose there exist two sets X and X' which are pair sets of object a and b. Then by Ax. I.3.3, we have "$x \in X \iff (x = a) \vee (x = b)$" and "$x \in X' \iff (x = a) \vee (x = b)$." But this means $x \in X \iff x \in X'$. Thus, by Def. I.3.1.4, we have $X = X'$, and therefore the uniqueness of pair set is proved.

Next we show that $\{a, b\} = \{b, a\}$. Since

$$\begin{aligned}
&x \in \{a, b\} \\
\iff &(x = a) \vee (x = b) && \text{(by Ax. I.3.3)} \\
\iff &(x = b) \vee (x = a) \\
\iff &x \in \{b, a\}, && \text{(by Ax. I.3.3)}
\end{aligned}$$

by Def. I.3.1.4, we know that $\{a, b\} = \{b, a\}$.

Finally we show that $\{a, a\} = \{a\}$. Since

$$x \in \{a, a\}$$

$$
\begin{aligned}
&\iff (x = a) \vee (x = a) && \text{(by Ax. I.3.3)} \\
&\iff x = a \\
&\iff x \in \{a\}, && \text{(by Ax. I.3.3)}
\end{aligned}
$$

by Def. I.3.1.4, we know that $\{a, a\} = \{a\}$. □

E.g. I.3.1.10. Since $\emptyset$ is a set (and hence an object), so is the singleton set $\{\emptyset\}$, i.e., the set whose only element is $\emptyset$, is a set (and it is not the same set as $\emptyset$, $\{\emptyset\} \neq \emptyset$). Similarly, the singleton set $\{\{\emptyset\}\}$ and the pair set $\{\emptyset, \{\emptyset\}\}$ are also sets. These three sets are not equal to each other.

Ax. I.3.4 (Pairwise union)**.** Given any two sets A, B, there exists a set $A \cup B$, called the *union* of A and B, whose elements consist of all the elements which belong to A or B or both. In other words, for any object x,

$$x \in A \cup B \iff (x \in A) \vee (x \in B).$$

Rmk. I.3.1.12. If A, B, A' are sets, and A is equal to A', then $A \cup B$ is equal to $A' \cup B$. Similarly, if B' is a set which is equal to B, then $A \cup B$ is equal to $A \cup B'$. Thus, the operation of union obeys the axiom of substitution, and is thus well-defined on sets.

Proof of Rmk. I.3.1.12. First, suppose $A = A'$. Then we have

$$
\begin{aligned}
& x \in A \cup B \\
\iff & (x \in A) \vee (x \in B) && \text{(by Ax. I.3.4)} \\
\iff & (x \in A') \vee (x \in B) && \text{(by Def. I.3.1.4)} \\
\iff & x \in A' \cup B. && \text{(by Ax. I.3.4)}
\end{aligned}
$$

Thus, by Def. I.3.1.4, we have $A \cup B = A' \cup B$.

Now suppose $B = B'$. Then we have

$$
\begin{aligned}
& x \in A \cup B \\
\iff & (x \in A) \vee (x \in B) && \text{(by Ax. I.3.4)} \\
\iff & (x \in A) \vee (x \in B') && \text{(by Def. I.3.1.4)} \\
\iff & x \in A \cup B'. && \text{(by Ax. I.3.4)}
\end{aligned}
$$

Thus, by Def. I.3.1.4, we have $A \cup B = A \cup B'$. □

Lem. I.3.1.13. If a and b are objects, then $\{a, b\} = \{a\} \cup \{b\}$. If A, B, C are sets, then the union operation is commutative (i.e., $A \cup B = B \cup A$) and associative (i.e., $(A \cup B) \cup C = A \cup (B \cup C)$). Also, we have $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.

Proof of Lem. I.3.1.13. We first show that $\{a,b\} = \{a\} \cup \{b\}$. By Ax. I.3.3, we know the sets $\{a\}, \{b\}, \{a,b\}$ exist. By Ax. I.3.4, we know the set $\{a\} \cup \{b\}$ exists. Then we have

$$\begin{aligned} & x \in \{a,b\} \\ \iff & (x = a) \vee (x = b) && \text{(by Ax. I.3.3)} \\ \iff & (x \in \{a\}) \vee (x \in \{b\}) && \text{(by Ax. I.3.3)} \\ \iff & x \in \{a\} \cup \{b\}. && \text{(by Ax. I.3.4)} \end{aligned}$$

Thus, by Def. I.3.1.4, we have $\{a,b\} = \{a\} \cup \{b\}$.

Next we show the commutative identity of union sets. Suppose that A, B are sets. By Ax. I.3.4, we know that both $A \cup B$ and $B \cup A$ exist. Then we have

$$\begin{aligned} & x \in A \cup B \\ \iff & (x \in A) \vee (x \in B) && \text{(by Ax. I.3.4)} \\ \iff & (x \in B) \vee (x \in A) \\ \iff & x \in B \cup A. && \text{(by Ax. I.3.4)} \end{aligned}$$

Thus, by Def. I.3.1.4, we have $A \cup B = B \cup A$.

Next we show the associativity identity of union sets. Suppose A, B, C are sets. By Ax. I.3.4, we know that both $A \cup B$ and $B \cup C$ exist. Thus, by Ax. I.3.4 again, we know that both $(A \cup B) \cup C$ and $A \cup (B \cup C)$ exist. Then we have

$$\begin{aligned} & x \in (A \cup B) \cup C \\ \iff & (x \in (A \cup B)) \vee (x \in C) && \text{(by Ax. I.3.4)} \\ \iff & ((x \in A) \vee (x \in B)) \vee (x \in C) && \text{(by Ax. I.3.4)} \\ \iff & (x \in A) \vee ((x \in B) \vee (x \in C)) \\ \iff & (x \in A) \vee (x \in B \cup B) && \text{(by Ax. I.3.4)} \\ \iff & x \in A \cup (B \cup C). && \text{(by Ax. I.3.4)} \end{aligned}$$

Thus, by Def. I.3.1.4, we have $(A \cup B) \cup C = A \cup (B \cup C)$.

Finally we show that $A \cup A = A \cup \emptyset = \emptyset \cup A = A$ for any set A. By Ax. I.3.4, we know that the sets $A \cup A$, $A \cup \emptyset$, and $\emptyset \cup A$ exist. Then we have

$$\begin{aligned} & x \in A \cup A \\ \iff & (x \in A) \vee (x \in A) && \text{(by Ax. I.3.4)} \\ \iff & x \in A \\ \iff & (x \in A) \vee (x \in \emptyset) && \text{(by Ax. I.3.2)} \\ \iff & x \in A \cup \emptyset. && \text{(by Ax. I.3.4)} \end{aligned}$$

Thus, by Def. I.3.1.4, we see that $A \cup A = A = A \cup \emptyset$. Using commutative law of union, we see that $A \cup \emptyset = \emptyset \cup A$. $\square$

Note. Because of Lem. I.3.1.13, we do not need to use parentheses to denote multiple unions, thus for instance we can write $A \cup B \cup C$ instead of $(A \cup B) \cup C$ or $A \cup (B \cup C)$. Similarly, for unions of four sets, $A \cup B \cup C \cup D$, etc.

Rmk. I.3.1.14. While the operation of union has some similarities with addition, the two operations are *not* identical.

Note. Ax. I.3.4 allows us to define triplet sets, quadruplet sets, and so forth: if a, b, c are three objects, we define $\{a, b, c\} := \{a\} \cup \{b\} \cup \{c\}$; if a, b, c, d are four objects, then we define $\{a, b, c, d\} := \{a\} \cup \{b\} \cup \{c\} \cup \{d\}$, and so forth. On the other hand, we are not yet in a position to define sets consisting of n objects for any given natural number n; this would require iterating the above construction "n times," but the concept of n-fold iteration has not yet been rigorously defined. For similar reasons, we cannot yet define sets consisting of infinitely many objects, because that would require iterating the axiom of pairwise union (Ax. I.3.4) infinitely often, and it is not clear at this stage that one can do this rigorously. Later on, we will introduce other axioms of set theory which allow one to construct arbitrarily large, and even infinite, sets.

Def. I.3.1.15 (Subsets)**.** Let A, B be sets. We say that A is a *subset* of B, denoted $A \subseteq B$, iff every element of A is also an element of B, i.e.

$$\text{For any object } x, x \in A \implies x \in B.$$

We say that A is a *proper subset* of B, denoted $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

Rmk. I.3.1.16. Because these definitions involve only the notions of equality and the "is an element of" relation, both of which already obey the axiom of substitution, the notion of subset also automatically obeys the axiom of substitution. Thus, for instance if $A \subseteq B$ and $A = A'$, then $A' \subseteq B$.

E.g. I.3.1.17. Given any set A, we always have $A \subseteq A$ and $\emptyset \subseteq A$.

Proof of E.g. I.3.1.17. The statement "$x \in A \implies x \in A$" is a tautology. Therefore, by Def. I.3.1.15, we have $A \subseteq A$. The statement "$x \in \emptyset \implies x \in A$" is vacuously true, since $x \in \emptyset$ is false for all object x (Ax. I.3.2). Thus, by Def. I.3.1.15, we have $\emptyset \subseteq A$. ☐

Prop. I.3.1.18 (Sets are partially ordered by set inclusion)**.** Let A, B, C be sets.

- If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- $A \subseteq B$ and $B \subseteq A$ iff $A = B$.
- If $A \subsetneq B$ and $B \subsetneq C$, then $A \subsetneq C$.

Proof of Prop. I.3.1.18. First, we show that $(A \subseteq B) \wedge (B \subseteq C) \implies A \subseteq C$. This is true since

$$\begin{aligned}
&\begin{cases} A \subseteq B \\ B \subseteq C \end{cases} \\
\implies &\begin{cases} x \in A \implies x \in B \\ y \in B \implies y \in C \end{cases} && \text{(by Def. I.3.1.15)} \\
\implies &(x \in A \implies x \in B \implies x \in C) \\
\implies &A \subseteq C. && \text{(by Def. I.3.1.15)}
\end{aligned}$$

Next we show that $(A \subseteq B) \wedge (B \subseteq A) \iff A = B$. This is true since

$$\begin{aligned}
&\begin{cases} A \subseteq B \\ B \subseteq A \end{cases} \\
\iff &\begin{cases} x \in A \implies x \in B \\ y \in B \implies y \in A \end{cases} && \text{(by Def. I.3.1.15)} \\
\iff &A = B. && \text{(by Def. I.3.1.4)}
\end{aligned}$$

Finally we show that $(A \subsetneq B) \wedge (B \subsetneq C) \implies A \subsetneq C$. Suppose that $(A \subsetneq B) \wedge (B \subsetneq C)$. Then we have

$$\begin{aligned}
&\begin{cases} A \subsetneq B \\ B \subsetneq C \end{cases} \\
\implies &\begin{cases} A \subseteq B \\ B \subseteq C \end{cases} && \text{(by Def. I.3.1.15)} \\
\implies &A \subseteq C. && \text{(from the proof above)}
\end{aligned}$$

Thus, by Def. I.3.1.15, we only need to show that $A \neq C$. Suppose for the sake of contradiction that $A = C$. Then from the proof above we have $C \subseteq A$. Since $A \subseteq B$, from the proof above we have $C \subseteq B$. Then we must have $B = C$, since we have $B \subseteq C$. But this contradicts $B \subsetneq C$. Thus, we must have $A \neq C$. □

Rmk. I.3.1.20. There is one important difference between the subset relation $\subsetneq$ and the less than relation $<$. Given any two distinct natural numbers n, m, we know that one of them is smaller than the other (Prop. I.2.2.13); however, given two distinct sets, it is not in general true that one of them is a subset of the other. We say that sets are only *partially ordered*, whereas the natural numbers are *totally ordered* (see Def. I.8.5.1 and I.8.5.3).

Rmk. I.3.1.21. We should also caution that the subset relation $\subseteq$ is not the same as the element relation $\in$. It is important to distinguish sets from their elements, as they can have different properties. For instance, it is possible to have an infinite set consisting of finite numbers (the set $\mathbb{N}$ of natural numbers is one such example), and it is also possible to have a finite set consisting of infinite objects (consider for instance the finite set $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, which has four elements, all of which are infinite).

Ax. I.3.5 (Axiom of specification)**.** Let A be a set, and for each $x \in A$, let $P(x)$ be a property pertaining to x (i.e., $P(x)$ is either a true statement or a false statement). Then there exists a set, called $\{x \in A : P(x) \text{ is true}\}$ (or simply $\{x \in A : P(x)\}$ for short), whose elements are precisely the elements x in A for which $P(x)$ is true. In other words, for any object y,

$$y \in \{x \in A : P(x) \text{ is true}\} \iff (y \in A \text{ and } P(y) \text{ is true}).$$

Note. Ax. I.3.5 is also known as the *axiom of separation*. We sometimes write $\{x \in A \mid P(x)\}$ instead of $\{x \in A : P(x)\}$; this is useful when we are using the colon ":" to denote something else.

Def. I.3.1.23 (Intersections)**.** The *intersection* $S_1 \cap S_2$ of two sets is defined to be the set

$$S_1 \cap S_2 \coloneqq \{x \in S_1 : x \in S_2\}.$$

In other words, $S_1 \cap S_2$ consists of all the elements which belong to both S_1 and S_2. Thus, for all objects x,

$$x \in S_1 \cap S_2 \iff x \in S_1 \text{ and } x \in S_2.$$

Note. Two sets A, B are said to be *disjoint* if $A \cap B = \emptyset$. This is not the same concept as being *distinct*, $A \neq B$. Meanwhile, the sets $\emptyset$ and $\emptyset$ are disjoint but not distinct.

Def. I.3.1.27 (Difference sets)**.** Given two sets A and B, we define the set $A - B$ or $A \setminus B$ to be the set A with any elements of B removed:

$$A \setminus B \coloneqq \{x \in A : x \notin B\}.$$

Prop. I.3.1.28 (Sets form a boolean algebra)**.** Let A, B, C be sets, and let X be a set containing A, B, C as subsets.

(a) (Minimal element) We have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

(b) (Maximal element) We have $A \cup X = X$ and $A \cap X = A$.

(c) (Identity) We have $A \cap A = A$ and $A \cup A = A$.

(d) (Commutativity) We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

(e) (Associativity) We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

(f) (Distributivity) We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(g) (Partition) We have $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \emptyset$.

(h) (De Morgan laws) We have $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

Proof of Prop. I.3.1.28(a). By Lem. I.3.1.13, we have $A \cup \emptyset = A$. Then we have

$$\begin{aligned}
& x \in A \cap \emptyset \\
\iff & (x \in A) \wedge (x \in \emptyset) && \text{(by Def. I.3.1.27)} \\
\iff & x \in \emptyset. && \text{(by Ax. I.3.2)}
\end{aligned}$$

Thus, by Def. I.3.1.4, we have $A \cap \emptyset = \emptyset$. □

Proof of Prop. I.3.1.28(b). We have

$$\begin{aligned}
& \begin{cases} x \in A \cup X \\ A \subseteq X \end{cases} \\
\iff & \begin{cases} (x \in A) \vee (x \in X) \\ x \in A \implies x \in X \end{cases} && \text{(by Ax. I.3.4 and Def. I.3.1.15)} \\
\iff & \begin{cases} x \in X \\ x \in A \implies x \in X \end{cases}. && \text{(by Def. I.3.1.15)}
\end{aligned}$$

Thus, by Def. I.3.1.4, we have $A \cup X = X$. Similarly, we have

$$\begin{aligned}
& \begin{cases} x \in A \cap X \\ A \subseteq X \end{cases} \\
\iff & \begin{cases} (x \in A) \wedge (x \in X) \\ x \in A \implies x \in X \end{cases} && \text{(by Def. I.3.1.15 and I.3.1.23)} \\
\iff & \begin{cases} x \in A \\ x \in A \implies x \in X \end{cases}. && \text{(by Def. I.3.1.15)}
\end{aligned}$$

Thus, by Def. I.3.1.4, we have $A \cap X = A$. □

Proof of Prop. I.3.1.28(c). By Lem. I.3.1.13, we have $A \cup A = A$. Then we have

$$\begin{aligned}
& x \in A \cap A \\
\iff & (x \in A) \wedge (x \in A) && \text{(by Def. I.3.1.23)} \\
\iff & x \in A.
\end{aligned}$$

Thus, by Def. I.3.1.4, we have $A \cap A = A$. □

Proof of Prop. I.3.1.28(d). By Lem. I.3.1.13, we have $A \cup B = B \cup A$. Then we have

$$\begin{aligned}
& x \in A \cap B \\
\iff & (x \in A) \wedge (x \in B) && \text{(by Def. I.3.1.23)} \\
\iff & (x \in B) \wedge (x \in A) \\
\iff & x \in B \cap A. && \text{(by Def. I.3.1.23)}
\end{aligned}$$

Thus, by Def. I.3.1.4, we have $A \cap B = B \cap A$. □

Proof of Prop. I.3.1.28(e). By Lem. I.3.1.13, we have $(A \cup B) \cup C = A \cup (B \cup C)$. Then we have

$$\begin{aligned}
& x \in (A \cap B) \cap C \\
\iff & (x \in A \cap B) \wedge (x \in C) && \text{(by Def. I.3.1.23)} \\
\iff & ((x \in A) \wedge (x \in B)) \wedge (x \in C) && \text{(by Def. I.3.1.23)} \\
\iff & (x \in A) \wedge ((x \in B) \wedge (x \in C)) \\
\iff & (x \in A) \wedge (x \in B \cap C) && \text{(by Def. I.3.1.23)} \\
\iff & x \in A \cap (B \cap C). && \text{(by Def. I.3.1.23)}
\end{aligned}$$

Thus, by Def. I.3.1.4, we have $(A \cap B) \cap C = A \cap (B \cap C)$. □

Proof of Prop. I.3.1.28(f). We have

$$\begin{aligned}
& x \in A \cap (B \cup C) \\
\iff & (x \in A) \wedge (x \in B \cup C) && \text{(by Def. I.3.1.23)} \\
\iff & (x \in A) \wedge ((x \in B) \vee (x \in C)) && \text{(by Ax. I.3.4)} \\
\iff & ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C)) \\
\iff & (x \in A \cap B) \vee (x \in A \cap C) && \text{(by Def. I.3.1.23)} \\
\iff & x \in (A \cap B) \cup (A \cap C), && \text{(by Ax. I.3.4)}
\end{aligned}$$

and

$$\begin{aligned}
& x \in A \cup (B \cap C) \\
\iff & (x \in A) \vee (x \in B \cap C) && \text{(by Ax. I.3.4)} \\
\iff & (x \in A) \vee ((x \in B) \wedge (x \in C)) && \text{(by Def. I.3.1.23)} \\
\iff & ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C)) \\
\iff & (x \in A \cup B) \wedge (x \in A \cup C) && \text{(by Ax. I.3.4)} \\
\iff & x \in (A \cup B) \cap (A \cup C). && \text{(by Def. I.3.1.23)}
\end{aligned}$$

Thus, by Def. I.3.1.4, we know that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. □

Proof of Prop. I.3.1.28(g). We have

$$\begin{aligned}
& x \in A \cup (X \setminus A) \\
\iff & (x \in A) \vee (x \in X \setminus A) && \text{(by Ax. I.3.4)} \\
\iff & (x \in A) \vee ((x \in X) \wedge (x \notin A)) && \text{(by Def. I.3.1.27)} \\
\iff & ((x \in A) \vee (x \in X)) \wedge ((x \in A) \vee (x \notin A)) \\
\iff & (x \in A) \vee (x \in X) \\
\iff & x \in A \cup X, && \text{(by Ax. I.3.4)}
\end{aligned}$$

and

$$\begin{aligned}
& x \in A \cap (X \setminus A) \\
\iff & (x \in A) \wedge (x \in X \setminus A) && \text{(by Def. I.3.1.23)} \\
\iff & (x \in A) \wedge ((x \in X) \wedge (x \notin A)) && \text{(by Def. I.3.1.27)} \\
\iff & (x \in A) \wedge (x \notin A) \\
\iff & x \in \emptyset. && \text{(by Ax. I.3.2)}
\end{aligned}$$

Thus, by Def. I.3.1.4, we have $A \cup (X \setminus A) = A \cup X$, and $A \cap (X \setminus A) = \emptyset$. □

Proof of Prop. I.3.1.28(h). We have

$$\begin{aligned}
& x \in X \setminus (A \cup B) \\
\iff & (x \in X) \wedge (x \notin A \cup B) && \text{(by Def. I.3.1.27)} \\
\iff & (x \in X) \wedge ((x \notin A) \wedge (x \notin B)) && \text{(by Ax. I.3.4)} \\
\iff & ((x \in X) \wedge (x \notin A)) \wedge ((x \in X) \wedge (x \notin B)) \\
\iff & (x \in X \setminus A) \wedge (x \in X \setminus B) && \text{(by Def. I.3.1.27)} \\
\iff & x \in (X \setminus A) \cap (X \setminus B), && \text{(by Def. I.3.1.23)}
\end{aligned}$$

and

$$\begin{aligned}
& x \in X \setminus (A \cap B) \\
\iff & (x \in X) \wedge (x \notin A \cap B) && \text{(by Def. I.3.1.27)} \\
\iff & (x \in X) \wedge ((x \notin A) \vee (x \notin B)) && \text{(by Def. I.3.1.23)} \\
\iff & ((x \in X) \wedge (x \notin A)) \vee ((x \in X) \wedge (x \notin B)) \\
\iff & (x \in X \setminus A) \vee (x \in X \setminus B) && \text{(by Def. I.3.1.27)} \\
\iff & x \in (X \setminus A) \cup (X \setminus B). && \text{(by Ax. I.3.4)}
\end{aligned}$$

Thus, by Def. I.3.1.4, we have $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$, and $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. □

Rmk. I.3.1.29. The de Morgan laws are named after the logician Augustus De Morgan (1806–1871), who identified them as one of the basic laws of set theory.

Rmk. I.3.1.30. The reader may observe a certain symmetry in the above laws between $\cup$ and $\cap$, and between X and $\emptyset$. This is an example of *duality* - two distinct properties or objects being dual to each other. In this case, the duality is manifested by the complementation relation $A \mapsto X \setminus A$; the de Morgan laws assert that this relation converts unions into intersections and vice versa. (It also interchanges X and the empty set.) Prop. I.3.1.28 are collectively known as the *laws of Boolean algebra*, after the mathematician George Boole (1815–1864), and are also applicable to a number of other objects other than sets; it plays a particularly important role in logic.

Ax. I.3.6 (Replacement)**.** Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement $P(x, y)$ pertaining to x and y, such that for each $x \in A$ there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y) \text{ is true for some } x \in A\}$, such that for any object z,

$$z \in \{y : P(x, y) \text{ is true for some } x \in A\} \iff P(x, z) \text{ is true for some } x \in A.$$

Note. The keyword here is "suppose"; We have to assume that there exists a set $E = \{x \in A : \exists!\ y \text{ such that } P(x, y) \text{ is true}\}$. E must exist so that we can apply Ax. I.3.6. This means that we assert the existence of a function $f : E \to \{y : P(x, y) \text{ is true for some } x\}$.

Note. We often abbreviate a set of the form

$$\{y : y = f(x) \text{ for some } x \in A\}$$

as $\{f(x) : x \in A\}$ or $\{f(x) \mid x \in A\}$. We can of course combine the axiom of replacement with the axiom of specification, thus for instance we can create sets such as $\{f(x) : x \in A; P(x) \text{ is true}\}$ by starting with the set A, using the axiom of specification to create the set $\{x \in A : P(x) \text{ is true}\}$, and then applying the axiom of replacement to create $\{f(x) : x \in A; P(x) \text{ is true}\}$.

Ax. I.3.7 (Infinity)**.** There exists a set $\mathbb{N}$, whose elements are called natural numbers, as well as an object 0 in $\mathbb{N}$, and an object $n{+}{+}$ assigned to every natural number $n \in \mathbb{N}$, such that the Peano axioms (Ax. I.2.1 to I.2.5) hold.

Note. Formally, one can refer to $\mathbb{N}$ as "the set of natural numbers," but we will often abbreviate this to "the natural numbers" for short. We will adopt similar abbreviations later in the text; for instance the set of integers $\mathbb{Z}$ will often be abbreviated to "the integers."

— Exercises —

Ex. I.3.1.1. Show that the definition of equality in Def. I.3.1.4 is reflexive, symmetric, and transitive.

Proof of Ex. I.3.1.1. See A.Cor. I.3.1.1. □

Ex. I.3.1.2. Using only Def. I.3.1.4, Ax. I.3.1, Ax. I.3.2, and Ax. I.3.3, prove that the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset,\{\emptyset\}\}$ are all distinct (i.e., no two of them are equal to each other).

Proof of Ex. I.3.1.2. We first show that $\emptyset \neq \{\emptyset\}$, $\emptyset \neq \{\{\emptyset\}\}$, and $\emptyset \neq \{\emptyset,\{\emptyset\}\}$. This is true since

$$\begin{cases} \emptyset \in \{\emptyset\} \\ \emptyset \notin \emptyset \end{cases} \implies \emptyset \neq \{\emptyset\}; \qquad \text{(by Def. I.3.1.4 and Ax. I.3.2 and I.3.3)}$$

$$\begin{cases} \{\emptyset\} \in \{\{\emptyset\}\} \\ \{\emptyset\} \notin \emptyset \end{cases} \implies \emptyset \neq \{\{\emptyset\}\}; \qquad \text{(by Def. I.3.1.4 and Ax. I.3.2 and I.3.3)}$$

$$\begin{cases} \emptyset \in \{\emptyset,\{\emptyset\}\} \\ \emptyset \notin \emptyset \end{cases} \implies \emptyset \neq \{\emptyset,\{\emptyset\}\}. \qquad \text{(by Def. I.3.1.4 and Ax. I.3.2 and I.3.3)}$$

Next we show that $\{\emptyset\} \neq \{\{\emptyset\}\}$ and $\{\emptyset\} \neq \{\emptyset,\{\emptyset\}\}$. This is true since

$$\begin{aligned} &\begin{cases} \emptyset \neq \{\emptyset\} \\ \emptyset \in \{\emptyset\} \\ \{\emptyset\} \in \{\{\emptyset\}\} \end{cases} && \text{(by Ax. I.3.3)} \\ &\implies \emptyset \notin \{\{\emptyset\}\} && \text{(by Ax. I.3.3)} \\ &\implies \{\emptyset\} \neq \{\{\emptyset\}\}; && \text{(by Def. I.3.1.4)} \\ &\begin{cases} \emptyset \neq \{\emptyset\} \\ \emptyset \in \{\emptyset\} \\ \{\emptyset\} \in \{\emptyset,\{\emptyset\}\} \end{cases} && \text{(by Ax. I.3.3)} \\ &\implies \{\emptyset\} \notin \{\emptyset\} && \text{(by Ax. I.3.3)} \\ &\implies \{\emptyset\} \neq \{\emptyset,\{\emptyset\}\}. && \text{(by Def. I.3.1.4)} \end{aligned}$$

Finally we show that $\{\{\emptyset\}\} \neq \{\emptyset,\{\emptyset\}\}$. This is true since

$$\begin{aligned} &\begin{cases} \emptyset \neq \{\emptyset\} \\ \{\emptyset\} \in \{\{\emptyset\}\} \\ \emptyset \in \{\emptyset,\{\emptyset\}\} \end{cases} && \text{(by Ax. I.3.3)} \\ &\implies \{\{\emptyset\}\} \neq \{\emptyset,\{\emptyset\}\}. && \text{(by Def. I.3.1.4)} \end{aligned}$$

□

Ex. I.3.1.3. Prove the remaining claims in Lem. I.3.1.13.

Proof of Ex. I.3.1.3. See Lem. I.3.1.13. □

Ex. I.3.1.4. Prove the remaining claims in Prop. I.3.1.18.

Proof of Ex. I.3.1.4. See Prop. I.3.1.18. □

Ex. I.3.1.5. Let A, B be sets. Show that the three statements $A \subseteq B$, $A \cup B = B$, $A \cap B = A$ are logically equivalent (any one of them implies the other two).

Proof of Ex. I.3.1.5. We first show that $A \subseteq B \iff A \cup B = B$. By Prop. I.3.1.28(b), we see that $A \subseteq B \implies A \cup B = B$. So we only need to show that $A \cup B = B \implies A \subseteq B$. Suppose that $A \cup B = B$. Suppose for the sake of contradiction that $A \not\subseteq B$. Then by Def. I.3.1.15, there exists an object $x \in A$, such that $x \notin B$. Since $x \in A$, by Ax. I.3.4, we know that $x \in A \cup B$. But, by Def. I.3.1.4, we know that $A \cup B = B$ implies $x \in B$, a contradiction. Thus, we must have $A \subseteq B$. We conclude that $A \subseteq B \iff A \cup B = B$.

Now we show that $A \subseteq B \iff A \cap B = A$. By Prop. I.3.1.28(b), we see that $A \subseteq B \implies A \cap B = A$. So we only need to show that $A \cap B = A \implies A \subseteq B$. Suppose that $A \cap B = A$. Then we have

$$\begin{aligned} & x \in A = A \cap B \\ \implies & (x \in A) \wedge (x \in B) && \text{(by Def. I.3.1.23)} \\ \implies & x \in B. \end{aligned}$$

By Def. I.3.1.15, this means $A \subseteq B$. We conclude that $A \subseteq B \iff A \cap B = A$. □

Ex. I.3.1.6. Prove Prop. I.3.1.28.

Proof of Ex. I.3.1.6. See Prop. I.3.1.28. □

Ex. I.3.1.7. Let A, B, C be sets. Show that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Furthermore, show that $C \subseteq A$ and $C \subseteq B$ iff $C \subseteq A \cap B$. In a similar spirit, show that $A \subseteq A \cup B$ and $B \subseteq A \cup B$, and furthermore that $A \subseteq C$ and $B \subseteq C$ iff $A \cup B \subseteq C$.

Proof of Ex. I.3.1.7. We first show that $A \cap B \subseteq A$ and $A \cap B \subseteq B$. We have

$$\begin{aligned} & x \in A \cap B \\ \implies & (x \in A) \wedge (x \in B) && \text{(by Def. I.3.1.23)} \\ \implies & x \in A. \end{aligned}$$

Thus, by Def. I.3.1.15, we have $A \cap B \subseteq A$. Using an identical argument, we see that $B \cap A \subseteq B$. By Prop. I.3.1.28(d), we see that $B \cap A = A \cap B \subseteq B$.

Next we show that $(C \subseteq A) \wedge (C \subseteq B) \iff C \subseteq A \cap B$. This is true since

$$\begin{aligned} & (C \subseteq A) \wedge (C \subseteq B) \\ \iff & (x \in C \implies x \in A) \wedge (x \in C \implies x \in B) && \text{(by Def. I.3.1.15)} \\ \iff & (x \in C \implies x \in A \wedge x \in B) \end{aligned}$$

$$\begin{aligned} &\iff (x \in C \implies x \in A \cap B) && \text{(by Def. I.3.1.23)} \\ &\iff (C \subseteq A \cap B). && \text{(by Def. I.3.1.15)} \end{aligned}$$

Next we show that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. We have

$$\begin{aligned} &x \in A \\ \implies &(x \in A) \vee (x \in B) \\ \implies &x \in A \cup B. && \text{(by Ax. I.3.4)} \end{aligned}$$

Thus, by Def. I.3.1.15, we have $A \subseteq A \cup B$. Using an identical argument, we see that $B \subseteq B \cup A$. By Prop. I.3.1.28(d), we have $B \subseteq A \cup B = B \cup A$.

Finally we show that $(A \subseteq C) \wedge (B \subseteq C) \iff A \cup B \subseteq C$. This is true since

$$\begin{aligned} &(A \subseteq C) \wedge (B \subseteq C) \\ \iff &(x \in A \implies x \in C) \wedge (x \in B \implies x \in C) && \text{(by Def. I.3.1.15)} \\ \iff &(x \in A \vee x \in B \implies x \in C) \\ \iff &(x \in A \cup B \implies x \in C) && \text{(by Ax. I.3.4)} \\ \iff &(A \cup B \subseteq C). && \text{(by Def. I.3.1.15)} \end{aligned}$$

□

Ex. I.3.1.8. Let A, B be sets. Prove the *absorption laws* $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$.

Proof of Ex. I.3.1.8. We have

$$\begin{aligned} &x \in A \cap (A \cup B) \\ \iff &(x \in A) \wedge (x \in A \cup B) && \text{(by Def. I.3.1.23)} \\ \iff &(x \in A) \wedge ((x \in A) \vee (x \in B)) && \text{(by Ax. I.3.4)} \\ \iff &x \in A, \end{aligned}$$

and

$$\begin{aligned} &x \in A \cup (A \cap B) \\ \iff &(x \in A) \vee (x \in A \cap B) && \text{(by Ax. I.3.4)} \\ \iff &(x \in A) \vee ((x \in A) \wedge (x \in B)) && \text{(by Def. I.3.1.23)} \\ \iff &x \in A. \end{aligned}$$

Thus, by Def. I.3.1.4, we have $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$. □

Ex. I.3.1.9. Let A, B, X be sets such that $A \cup B = X$ and $A \cap B = \emptyset$. Show that $A = X \setminus B$ and $B = X \setminus A$.

Proof of Ex. I.3.1.9. Since $X = A \cup B$, we have $X \setminus B = (A \cup B) \setminus B$. To show that $A = X \setminus B$, it suffices to show that $A = (A \cup B) \setminus B$. But we have

$$\begin{aligned} & x \in (A \cup B) \setminus B \\ \iff & (x \in (A \cup B)) \wedge (x \notin B) && \text{(by Def. I.3.1.27)} \\ \iff & ((x \in A) \vee (x \in B)) \wedge (x \notin B) && \text{(by Ax. I.3.4)} \\ \iff & ((x \in A) \wedge (x \notin B)) \vee ((x \in B) \wedge (x \notin B)) \\ \iff & (x \in A) \wedge (x \notin B) \\ \iff & x \in A. && (A \cap B = \emptyset) \end{aligned}$$

Thus, by Def. I.3.1.4, we have $A = (A \cup B) \setminus B$. Hence, $A = X \setminus B$.

By Prop. I.3.1.28(d), we have $A \cup B = B \cup A = X$ and $A \cap B = B \cap A = \emptyset$. Thus, we can use the same argument above to show that $X \setminus A = B$. □

Ex. I.3.1.10. Let A, B be sets. Show that the three sets $A \setminus B$, $A \cap B$, and $B \setminus A$ are disjoint, and that their union is $A \cup B$.

Proof of Ex. I.3.1.10. We have

$$\begin{aligned} & x \in (A \setminus B) \cap (A \cap B) \\ \iff & (x \in A \setminus B) \wedge (x \in A \cap B) && \text{(by Def. I.3.1.23)} \\ \iff & ((x \in A) \wedge (x \notin B)) \wedge ((x \in A) \wedge (x \in B)) && \text{(by Def. I.3.1.23 and I.3.1.27)} \\ \iff & (x \in B) \wedge (x \notin B) \\ \iff & x \in \emptyset. && \text{(by Ax. I.3.2)} \end{aligned}$$

Thus, by Def. I.3.1.4, we have $(A \setminus B) \cap (A \cap B) = \emptyset$. Hence, $A \setminus B$ and $A \cap B$ are disjoint. By Prop. I.3.1.28(d), we have $B \cap A = A \cap B$. Thus, we can use an identical argument above to show that $B \setminus A$ and $A \cap B$ are disjoint. Now we have

$$\begin{aligned} & x \in (A \setminus B) \cap (B \setminus A) \\ \iff & (x \in A \setminus B) \wedge (x \in B \setminus A) && \text{(by Def. I.3.1.23)} \\ \iff & ((x \in A) \wedge (x \notin B)) \wedge ((x \in B) \wedge (x \notin A)) && \text{(by Def. I.3.1.27)} \\ \iff & (x \in A) \wedge (x \notin A) \\ \iff & x \in \emptyset. && \text{(by Ax. I.3.2)} \end{aligned}$$

Hence, by Def. I.3.1.4, we have $(A \setminus B) \cap (B \setminus A) = \emptyset$. Thus, $A \setminus B$ and $B \setminus A$ are disjoint. This means

$$\begin{aligned} & x \in (A \setminus B) \cup (A \cap B) \cup (B \setminus A) \\ \iff & (x \in A \setminus B) \vee (x \in A \cap B) \vee (x \in B \setminus A) && \text{(by Ax. I.3.4)} \\ \iff & ((x \in A) \wedge (x \notin B)) \vee (x \in A \cap B) \vee ((x \in B) \wedge (x \notin A)) && \text{(by Def. I.3.1.27)} \end{aligned}$$

$$
\begin{aligned}
&\iff ((x \in A) \wedge (x \notin B)) \vee ((x \in A) \wedge (x \in B)) \vee ((x \in B) \wedge (x \notin A)) && \text{(by Def. I.3.1.23)} \\
&\iff ((x \in A) \wedge ((x \in B) \vee (x \notin B))) \vee ((x \in B) \wedge (x \notin A)) \\
&\iff (x \in A) \vee ((x \in B) \wedge (x \notin A)) \\
&\iff ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \notin A)) \\
&\iff (x \in A) \vee (x \in B) \\
&\iff x \in A \cup B. && \text{(by Ax. I.3.4)}
\end{aligned}
$$

Thus, by Def. I.3.1.4, we have $(A \setminus B) \cup (A \cap B) \cup (B \setminus A) = A \cup B$. □

Ex. I.3.1.11. Show that the axiom of replacement implies the axiom of specification.

Proof of Ex. I.3.1.11. To show that Ax. I.3.6 implies Ax. I.3.5, we need to show that every set created by Ax. I.3.5 can also be created by Ax. I.3.6. So let $\{x \in A : Q(x)\}$ be a set created by Ax. I.3.5, where A is a set, and $Q(x)$ is a property of object $x \in A$. Let $P(x, y)$ be the statement "$Q(x)$ is true, and $x = y$" for any $x \in A$ and any object y. Since A is a set, we know that each $x \in A$ is unique. Thus, $P(x, y)$ is true iff $x = y$. Therefore, the statement "there is at most one object y for which $P(x, y)$ is true" is true. Hence, we can use Ax. I.3.6 to create a set $\{y : P(x, y) \text{ is true for some } x \in A\}$. Then we have

$$
\begin{aligned}
& z \in \{y : P(x, y) \text{ is true for some } x \in A\} \\
\iff & P(x, z) \text{ is true for some } x \in A && \text{(by Ax. I.3.6)} \\
\iff & Q(x) \text{ is true, and } x = z \text{ for some } x \in A \\
\iff & z \in \{x \in A : Q(x) \text{ is true}\} && \text{(by Ax. I.3.5)}
\end{aligned}
$$

Thus, by Def. I.3.1.4, we have

$$\{y : P(x, y) \text{ is true for some } x \in A\} = \{x \in A : Q(x) \text{ is true}\}.$$

This means any set created by Ax. I.3.5 can be created by Ax. I.3.6. We conclude that Ax. I.3.6 implies Ax. I.3.5. □

I.3.2 Russell's paradox

Ax. I.3.8 (Universal specification)**.** (Dangerous!) Suppose for every object x we have a property $P(x)$ pertaining to x (so that for every x, $P(x)$ is either a true statement or a false statement). Then there exists a set $\{x : P(x) \text{ is true}\}$ such that for every object y,

$$y \in \{x : P(x) \text{ is true}\} \iff P(y) \text{ is true}.$$

Note. Compare to Ax. I.3.5, an object x does not need to be in a set A to apply this axiom. So Ax. I.3.8 is more powerful than Ax. I.3.5.

Note. Ax. I.3.8 is also known as the *axiom of comprehension*. It asserts that every property corresponds to a set. Ax. I.3.8 also implies most of the axioms in Sec. I.3.1 (see Ex. I.3.2.1). Unfortunately, this axiom cannot be introduced into set theory, because it creates a logical contradiction known as *Russell's paradox*, discovered by the philosopher and logician Bertrand Russell (1872–1970) in 1901. The paradox runs as follows. Let $P(x)$ be the statement

$$P(x) \iff \text{“}x \text{ is a set, and } x \notin x\text{”};$$

i.e., $P(x)$ is true only when x is a set which does not contain itself. Now use the axiom of universal specification to create the set

$$\Omega := \{x : P(x) \text{ is true}\} = \{x : x \text{ is a set and } x \notin x\},$$

i.e., the set of all sets which do not contain themselves. Now ask the question: does Ω contain itself, i.e. is $\Omega \in \Omega$? If Ω did contain itself, then by definition of Ω this means that $P(\Omega)$ is true, i.e., Ω is a set and $\Omega \notin \Omega$. On the other hand, if Ω did not contain itself, then by definition of P, $P(\Omega)$ would be true, and hence by the definition of Ω, $\Omega \in \Omega$. Thus, in either case we have both $\Omega \in \Omega$ and $\Omega \notin \Omega$, which is absurd.

Note. The problem with Ax. I.3.8 is that it creates sets which are far too "large." Since sets are themselves objects (Ax. I.3.1), this means that sets are allowed to contain themselves, which is a somewhat silly state of affairs. One way to informally resolve this issue is to think of objects as being arranged in a hierarchy. At the bottom of the hierarchy are the *primitive objects* - the objects that are not sets. Then on the next rung of the hierarchy there are sets whose elements consist only of primitive objects, let's call these "primitive sets" for now. Then there are sets whose elements consist only of primitive objects and primitive sets, and we can form sets out of these objects, and so forth. The point is that at each stage of the hierarchy we only see sets whose elements consist of objects at lower stages of the hierarchy, and so at no stage do we ever construct a set which contains itself.

Note. In pure set theory, there will be no primitive objects, but there will be one primitive set $\emptyset$ on the next rung of the hierarchy.

Ax. I.3.9 (Regularity)**.** If A is a non-empty set, then there is at least one element x of A which is either not a set, or is disjoint from A.

Note. The point of Ax. I.3.9 (which is also known as the *axiom of foundation*) is that it is asserting that at least one of the elements of A is so low on the hierarchy of objects that it does not contain any of the other elements of A. One particular consequence of Ax. I.3.9 is that sets are no longer allowed to contain themselves (Ex. I.3.2.2).

— Exercises —

Ex. I.3.2.1. Show that the universal specification axiom, Ax. I.3.8, if assumed to be true, would imply Ax. I.3.2 to I.3.6. (If we assume that all natural numbers are objects, we also obtain Ax. I.3.7.) Thus, Ax. I.3.8, if permitted, would simplify the foundations of set theory tremendously (and can be viewed as one basis for an intuitive model of set theory known as "naive set theory"). Unfortunately, as we have seen, Ax. I.3.8 is "too good to be true"!

Proof of Ex. I.3.2.1. We first show that Ax. I.3.8 implies Ax. I.3.2. Using Ax. I.3.8, we can create a set $E = \{x : P(x)\}$, where $P(x)$ is a property which is false for all object x. Then we must have $x \notin E$ for every object x. Hence, we can construct empty set using Ax. I.3.8. Therefore, Ax. I.3.8 implies Ax. I.3.2.

Next we show that Ax. I.3.8 implies Ax. I.3.3. Suppose that a, b are objects. Using Ax. I.3.8, we can create a set $A = \{x : P(x)\}$, where $P(x)$ is a property which is false for all object x other than a. Then the only element in the set A is a. Hence, we can construct any singleton set using Ax. I.3.8. Using Ax. I.3.8 again, we can create a set $B = \{x : Q(x)\}$, where $Q(x)$ is a property which is false for all object x other than a, b. Then a, b are the only elements in the set B. Hence, we can construct any pair set using Ax. I.3.8. Therefore, Ax. I.3.8 implies Ax. I.3.3.

Next we show that Ax. I.3.8 implies Ax. I.3.4. Suppose that A, B are sets. By Ax. I.3.8, there exists a set $C = \{x : P(x)\}$, where $P(x) = (x \in A) \vee (x \in B)$. Clearly, we have $C = A \cup B$. Therefore, Ax. I.3.8 implies Ax. I.3.4.

Next we show that Ax. I.3.8 implies Ax. I.3.5. Suppose that A is a set. Using Ax. I.3.8, we can create a property $P(x)$, which is true only for some object x. Then, using Ax. I.3.8 again, we can create a set $B = \{x : Q(x)\}$, where $Q(x) = (x \in A) \wedge P(x)$. Clearly, we have $B = \{x \in A : Q(x)\}$. Therefore, Ax. I.3.8 implies Ax. I.3.5.

Next we show that Ax. I.3.8 implies Ax. I.3.6. Suppose that A is a set, and $P(x, y)$ is a property, such that for each $x \in A$, there is at most one object y, such that $P(x, y)$ is true. By Ax. I.3.8, there exists a set $B = \{y : Q(y)\}$, where $Q(y)$ is the statement "$P(x, y)$ is true for some $x \in A$." Clearly, we have $B = \{y : P(x, y) \text{ is true for some } x \in A\}$. Therefore Ax. I.3.8 implies Ax. I.3.6.

Finally, suppose all natural numbers are objects. Using Ax. I.3.8, we can create a set $N = \{x : P(x)\}$, where $P(x)$ is the statement "x is a natural number, and x satisfies Ax. I.2.1 to I.2.5." Clearly, we have $N = \mathbb{N}$. Therefore, Ax. I.3.8 implies Ax. I.3.7. ☐

Ex. I.3.2.2. Use the axiom of regularity, Ax. I.3.9 (and the singleton set axiom, Ax. I.3.3) to show that if A is a set, then $A \notin A$. Furthermore, show that if A and B are two sets, then either $A \notin B$ or $B \notin A$ (or both).

Proof of Ex. I.3.2.2. We fisrt show that if A is a set, then $A \notin A$. Suppose for the sake of contradiction that there exists a set A with the property $A \in A$. By Ax. I.3.3, there exist a singleton set $\{A\}$, and $A \in \{A\}$ is true. Then $A \in A \cap \{A\}$ is true. But by Ax. I.3.9, the only element A in $\{A\}$ must be disjoint from $\{A\}$, which means $A \cap \{A\} = \emptyset$, a contradiction. Thus, there does not exist a set A with the property $A \in A$, i.e., we must have $A \notin A$ for any set A.

Now we show that if A, B are two sets, then $(A \notin B) \vee (B \notin A)$ is true. Since

$$(A \notin B) \vee (B \notin A) \iff (\neg(A \in B)) \vee (\neg(B \in A)) \iff \neg((A \in B) \wedge (B \in A)),$$

it suffices to show that $(A \in B) \wedge (B \in A)$ is false. So suppose for the sake of contradiction that $(A \in B) \wedge (B \in A)$ is true. By Ax. I.3.3, we can create a pair set $\{A, B\}$. By Ax. I.3.9, we know that there exists one element in $\{A, B\}$, such that either it is not a set, or it is disjoint from $\{A, B\}$. Since A, B are sets, we must have either $A \cap \{A, B\} = \emptyset$ or $B \cap \{A, B\} = \emptyset$. But since $(A \in B) \wedge (B \in A)$, we must have $A \in B \cap \{A, B\}$ and $B \in A \cap \{A, B\}$, a contradiction. Thus, $(A \in B) \wedge (B \in A)$ is false. □

Ex. I.3.2.3. Show (assuming the other axioms of set theory) that the universal specification axiom, Ax. I.3.8, is equivalent to an axiom postulating the existence of a "universal set" Ω consisting of all objects (i.e., for all objects x, we have $x \in \Omega$). In other words, if Ax. I.3.8 is true, then a universal set exists, and conversely, if a universal set exists, then Ax. I.3.8 is true. (This may explain why Ax. I.3.8 is called the axiom of universal specification.) Note that if a universal set Ω existed, then we would have $\Omega \in \Omega$ by Ax. I.3.1, contradicting Ex. I.3.2.2. Thus, the axiom of foundation specifically rules out the axiom of universal specification.

Proof of Ex. I.3.2.3. If Ax. I.3.8 is true, then there exists a set $\Omega = \{x : x \text{ is a object}\}$, and $\Omega \in \Omega$. Thus, Ax. I.3.8 implies a universal set exists. If a universal set Ω exists, then by Ax. I.3.5, we must have a set $A = \{x \in \Omega : P(x)\}$, where $P(x)$ is a property of object $x \in \Omega$. Clearly, we have $A = \{x : P(x)\}$ since Ω is consist of all objects. Thus, a universal set exists implies Ax. I.3.8 is true. We conclude that Ax. I.3.8 is true iff a universal set exists. □

I.3.3 Functions

Def. I.3.3.1 (Functions)**.** Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$, such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true (this is sometimes known as the *vertical line test*). Then we define the *function* $f : X \to Y$ *defined by* P *on the domain* X *and codomain* Y to be the object which, given any input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x) \in Y$ for which $P(x, f(x))$ is true. Thus, for any $x \in X$ and $y \in Y$,

$$y = f(x) \iff P(x, y) \text{ is true.}$$

Note. Implicit in Def. I.3.3.1 is the assumption that whenever one is given two sets X, Y and a property P obeying the vertical line test, one can form a function object. Strictly speaking, this assumption of the existence of the function as a mathematical object should be stated as an explicit axiom; however we will not do so here, as it turns out to be redundant. (More precisely, in view of Ex. I.3.5.10 below, it is always possible to encode a function f as an ordered triple $(X, Y, \{(x, f(x)) : x \in X\})$ consisting of the domain, codomain, and graph of the function, which gives a way to build functions as objects using the operations provided by the preceding axioms.)

Note. Functions are also referred to as *maps* or *transformations*, depending on the context. They are also sometimes called *morphisms*, although to be more precise, a morphism refers to a more general class of object, which may or may not correspond to actual functions, depending on the context.

Note. One common way to define a function is simply to specify its domain, its codomain, and how one generates the output $f(x)$ from each input; this is known as an *explicit* definition of a function. In other cases we only define a function f by specifying what property $P(x, y)$ links the input x with the output $f(x)$; this is an *implicit* definition of a function. An implicit definition is only valid if we know that for every input there is exactly one output which obeys the implicit relation.

Note. In many cases we omit specifying the domain and codomain of a function for brevity. However, too much of this abbreviation can be dangerous; sometimes it is important to know what the domain and codomain of the function is.

A.Cor. I.3.3.1. We observe that functions obey the axiom of substitution: if $x = x'$, then $f(x) = f(x')$. In other words, equal inputs imply equal outputs. On the other hand, unequal inputs do not necessarily ensure unequal outputs. For example, *constant function* simply assign each input with the same output.

Proof of A.Cor. I.3.3.1. Suppose that $f : X \to Y$ is a function defined by P on the domain X and codomain Y. Let $x, x' \in X$ where $x = x'$. By Def. I.3.3.1, we know that $f(x)$ is the unique object for which $P(x, f(x))$ is true. Similarly, we know that $f(x')$ is the unique object for which $P(x', f(x'))$ is true. Since $x = x'$, we know that $P(x', f(x))$ must be true. Then the uniqueness of $f(x')$ implies $f(x) = f(x')$. □

Rmk. I.3.3.5. We are now using parentheses () to denote several different things in mathematics; on one hand, we are using them to clarify the order of operations, but on the other hand we also use parentheses to enclose the argument of a function $f(x)$ or of a property such as $P(x)$. However, the two usages of parentheses usually are unambiguous from context. For instance, if a is a number, then $a(b + c)$ denotes the expression $a \times (b + c)$, whereas if f is a function, then $f(b + c)$ denotes the output of f when the input is $b + c$. Sometimes the argument of a function is denoted by subscripting instead of parentheses; for instance, a sequence of natural numbers $a_0, a_1, a_2, a_3, \ldots$ is, strictly speaking, a function from $\mathbb{N}$ to $\mathbb{N}$, but is denoted by $n \mapsto a_n$ rather than $n \mapsto a(n)$.

Rmk. I.3.3.6. Strictly speaking, functions are not necessarily sets, and sets are not necessarily functions; it does not make sense to ask whether an object x is an element of a function f, and it does not make sense to apply a set A to an input x to create an output $A(x)$. On the other hand, it is possible to start with a function $f : X \to Y$ and construct its *graph* $\{(x, f(x)) : x \in X\}$, which describes the function completely once the domain X and codomain Y are specified. See Sec. I.3.5.

Def. I.3.3.7 (Equality of functions)**.** Two functions $f : X \to Y, g : X' \to Y'$ are said to be *equal* iff they have the same domain and codomain (i.e., $X = X'$ and $Y = Y'$), and

$f(x) = g(x)$ for all $x \in X$. (If $f(x)$ and $g(x)$ agree for some values of x, but not others, then we do not consider f and g to be equal.) According to this definition, two functions that have different domains or different codomains are, strictly speaking, distinct functions. However, when it is safe to do so without causing confusion, it is sometimes useful to "abuse notation" by identifying together functions of different domains or codomains if their values agree on their common domain of definition; this is analogous to the practice of "overloading" an operator in software engineering. See the discussion after Def. I.9.4.1 for an instance of this.

E.g. I.3.3.9. A rather boring example of a function is the *empty function* $f : \emptyset \to X$ from the empty set to a given set X. Since the empty set has no elements, we do not need to specify what f does to any input. Nevertheless, just as the empty set is a set, the empty function is a function, albeit not a particularly interesting one. Note that for each set X, there is only one function from $\emptyset$ to X, since Def. I.3.3.7 asserts that all functions from $\emptyset$ to X are equal.

Def. I.3.3.10 (Composition)**.** Let $f : X \to Y$ and $g : Y \to Z$ be two functions, such that the codomain of f is the same set as the domain of g. We then define the *composition* $g \circ f : X \to Z$ of the two functions g and f to be the function defined explicitly by the formula

$$(g \circ f)(x) \coloneqq g(f(x)).$$

If the codomain of f does not match the domain of g, we leave the composition $g \circ f$ undefined.

Note. Composition is not commutative: $f \circ g$ and $g \circ f$ are not necessarily the same function.

Lem. I.3.3.12 (Composition is associative)**.** Let $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.

Proof of Lem. I.3.3.12. Since $g \circ h$ is a function from X to Z, $f \circ (g \circ h)$ is a function from X to W. Similarly, $f \circ g$ is a function from Y to W, and hence $(f \circ g) \circ h$ is a function from X to W. Thus, $f \circ (g \circ h)$ and $(f \circ g) \circ h$ have the same domain and codomain. In order to check that they are equal, we see from Def. I.3.3.7 that we have to verify that $(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x)$ for all $x \in X$. But by Def. I.3.3.10,

$$\begin{aligned}
(f \circ (g \circ h))(x) &= f((g \circ h)(x)) \\
&= f(g(h(x))) \\
&= (f \circ g)(h(x)) \\
&= ((f \circ g) \circ h)(x),
\end{aligned}$$

as desired. $\square$

Rmk. I.3.3.13. Note that while g appears to the left of f in the expression $g \circ f$, the function $g \circ f$ applies the right-most function f first, before applying g. This is often confusing at first; it arises because we traditionally place a function f to the left of its input x rather than to the right. (There are some alternate mathematical notations in which the function is placed to the right of the input, thus we would write xf instead of $f(x)$, but this notation has often proven to be more confusing than clarifying, and has not as yet become particularly popular.)

Def. I.3.3.14 (One-to-one function)**.** A function f is *one-to-one* (or *injective*) if different elements map to different elements:

$$x \neq x' \implies f(x) \neq f(x').$$

Equivalently, a function is one-to-one if

$$f(x) = f(x') \implies x = x'.$$

Note. The notion of a one-to-one function depends not just on what the function does, but also what its domain is.

Rmk. I.3.3.16. If a function $f : X \to Y$ is not one-to-one, then one can find distinct x and x' in the domain X such that $f(x) = f(x')$, thus one can find two inputs which map to one output. Because of this, we say that f is *two-to-one* instead of *one-to-one*.

Def. I.3.3.17 (Onto functions)**.** A function f is *onto* (or *surjective*) if $f(X) = Y$, i.e., every element in Y comes from applying f to some element in X:

$$\text{For every } y \in Y, \text{there exists } x \in X \text{ such that } f(x) = y.$$

Note. The notion of an onto function depends not just on what the function does, but also what its codomain is.

Rmk. I.3.3.19. The concepts of injectivity and surjectivity are in many ways dual to each other. See Ex. I.3.3.2, I.3.3.4 and I.3.3.5 for some evidence of this.

Def. I.3.3.20 (Bijective functions)**.** A function $f : X \to Y$ which is both one-to-one and onto is also called *bijective* or *invertible*.

If f is bijective, then for every $y \in Y$, there is exactly one x such that $f(x) = y$ (there is at least one because of surjectivity, and at most one because of injectivity). This value of x is denoted $f^{-1}(y)$; thus f^{-1} is a function from Y to X. We call f^{-1} the *inverse* of f.

Note. The notion of a bijective function depends not just on what the function does, but also what its domain and codomain are.

Rmk. I.3.3.23. If a function $x \mapsto f(x)$ is bijective, then we sometimes call f a *perfect matching* or a *one-to-one correspondence* (not to be confused with the notion of a one-to-one function), and denote the action of f using the notation $x \leftrightarrow f(x)$ instead of $x \mapsto f(x)$.

— Exercises —

Ex. I.3.3.1. Show that the definition of equality in Def. I.3.3.7 is reflexive, symmetric, and transitive. Also verify the substitution property: if $f, \tilde{f} : X \to Y$ and $g, \tilde{g} : Y \to Z$ are functions such that $f = \tilde{f}$ and $g = \tilde{g}$, then $g \circ f = \tilde{g} \circ \tilde{f}$. Of course, these statements are immediate from the axioms of equality applied directly to the functions in question, but the point of the exercise is to show that they can also be established by instead applying the axioms of equality to elements of the domain and codomain of these functions, rather than to the functions itself.

Proof of Ex. I.3.3.1. We first show that Def. I.3.3.7 is reflexive. Suppose that $f : X \to Y$ is a function. Then we have

$$\begin{aligned}
&\begin{cases} X = X \\ Y = Y \\ \forall x \in X, f(x) = f(x) \end{cases} && \text{(by A.Cor. I.3.1.1 and Def. I.3.3.1)} \\
\Longrightarrow\ & f = f. && \text{(by Def. I.3.3.7)}
\end{aligned}$$

Thus, Def. I.3.3.7 is reflexive.

Next we show that Def. I.3.3.7 is symmetric. Suppose that $f : X \to Y, g : X' \to Y'$ are functions such that $f = g$. Then we have

$$\begin{aligned}
& f = g \\
\Longleftrightarrow\ & \begin{cases} X = X' \\ Y = Y' \\ \forall x \in X, f(x) = g(x) \end{cases} && \text{(by Def. I.3.3.7)} \\
\Longleftrightarrow\ & \begin{cases} X' = X \\ Y' = Y \\ \forall x \in X, g(x) = f(x) \end{cases} && \text{(by A.Cor. I.3.1.1)} \\
\Longleftrightarrow\ & g = f. && \text{(by Def. I.3.3.7)}
\end{aligned}$$

Thus, Def. I.3.3.7 is symmetric.

Next we show that Def. I.3.3.7 is transitive. Suppose that $f : X_f \to Y_f, g : X_g \to Y_g, h : X_h \to Y_h$ are functions such that $(f = g) \wedge (g = h)$. Then we have

$$\begin{cases} f = g \\ g = h \end{cases}$$

$$\begin{aligned}
&\implies \begin{cases} X_f = X_g \\ Y_f = Y_g \\ \forall x \in X_f, f(x) = g(x) \\ X_g = X_h \\ Y_g = Y_h \\ \forall x \in X_g, g(x) = h(x) \end{cases} && \text{(by Def. I.3.3.7)} \\
&\implies \begin{cases} X_f = X_h \\ Y_f = Y_h \\ \forall x \in X_f, f(x) = g(x) \end{cases} && \text{(by A.Cor. I.3.1.1)} \\
&\implies f = h. && \text{(by Def. I.3.3.7)}
\end{aligned}$$

Thus, Def. I.3.3.7 is transitive.

Finally we show that Axiom of substitution holds for composition. Suppose that $f : X \to Y, \tilde{f} : X \to Y, g : Y \to Z, \tilde{g} : Y \to Z$ are functions such that $(f = \tilde{f}) \wedge (g = \tilde{g})$. By Def. I.3.3.10, $g \circ f : X \to Z$ and $\tilde{g} \circ \tilde{f} : X \to Z$ are well-defined. Since $X = X$ and $Z = Z$, to prove that $g \circ f = \tilde{g} \circ \tilde{f}$, by Def. I.3.3.7, we only need to show that $(g \circ f)(x) = \left(\tilde{g} \circ \tilde{f}\right)(x)$ for all $x \in X$. But this is true since.

$$\begin{aligned}
\forall x \in X, (g \circ f)(x) &= g(f(x)) && \text{(by Def. I.3.3.10)} \\
&= g\left(\tilde{f}(x)\right) && \text{(by Def. I.3.3.7)} \\
&= \tilde{g}\left(\tilde{f}(x)\right) && \text{(by Def. I.3.3.7)} \\
&= \left(\tilde{g} \circ \tilde{f}\right)(x). && \text{(by Def. I.3.3.10)}
\end{aligned}$$

□

Ex. I.3.3.2. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$; similarly, show that if f and g are both surjective, then so is $g \circ f$.

Proof of Ex. I.3.3.2. We first show that f, g are injective implies $g \circ f$ is injective. Suppose that $f : X \to Y, g : Y \to Z$ are injective functions. Then we have

$$\begin{aligned}
&\forall x, x' \in X : x \neq x' \\
&\implies f(x) \neq f(x') && \text{(by Def. I.3.3.14)} \\
&\implies g(f(x)) \neq g(f(x')) && \text{(by Def. I.3.3.14)} \\
&\implies (g \circ f)(x) \neq (g \circ f)(x'). && \text{(by Def. I.3.3.10)}
\end{aligned}$$

Thus, by Def. I.3.3.14, $g \circ f$ is injective.

Now we show that f, g are surjective implies $g \circ f$ is surjective. Suppose that $f : X \to Y, g : Y \to Z$ are surjective functions. Then we have

$$\begin{aligned} &\begin{cases} \forall z \in Z, \exists y \in Y : z = g(y) \\ \forall w \in Y, \exists x \in X : w = f(x) \end{cases} && \text{(by Def. I.3.3.17)} \\ \implies &\forall z \in Z, \exists x \in X : z = g(f(x)) = (g \circ f)(x). && \text{(by Def. I.3.3.10)} \end{aligned}$$

Thus, by Def. I.3.3.17, $g \circ f$ is surjective. □

Ex. I.3.3.3. When is the empty function into a given set X injective? surjective? bijective?

Proof of Ex. I.3.3.3. Suppose that $f : \emptyset \to X$ is the empty function for the given set X. f is always injective since the statement "for all $x, x' \in \emptyset$, $f(x) = f(x') \implies x = x'$" is vacuously true by Ax. I.3.2. For surjective we can split into two cases:

- If $X \neq \emptyset$, then f is not surjective, since $\forall y \in X$, $\nexists x \in \emptyset$ such that $f(x) = y$.
- If $X = \emptyset$, then f is surjective, since $\forall y \in \emptyset$, $\exists x \in \emptyset$ such that $f(x) = y$ (which is vacuously true).

From the proof above we see that the empty function f is bijective iff $X = \emptyset$. □

Ex. I.3.3.4. In this section we give some cancellation laws for composition. Let $f : X \to Y$, $\tilde{f} : X \to Y$, $g : Y \to Z$, and $\tilde{g} : Y \to Z$ be functions. Show that if $g \circ f = g \circ \tilde{f}$ and g is injective, then $f = \tilde{f}$. Is the same statement true if g is not injective? Show that if $g \circ f = \tilde{g} \circ f$ and f is surjective, then $g = \tilde{g}$. Is the same statement true if f is not surjective?

Proof of Ex. I.3.3.4. We first show that g is injective, and $g \circ f = g \circ \tilde{f}$ implies $f = \tilde{f}$. Suppose that $f : X \to Y, \tilde{f} : X \to Y, g : Y \to Z$ are functions, such that g is injective and $g \circ f = g \circ \tilde{f}$. Then we have

$$\begin{aligned} &g \circ f = g \circ \tilde{f} \\ \implies &\forall x \in X, (g \circ f)(x) = \left(g \circ \tilde{f}\right)(x) && \text{(by Def. I.3.3.7)} \\ \implies &\forall x \in X, g(f(x)) = g\left(\tilde{f}(x)\right) && \text{(by Def. I.3.3.10)} \\ \implies &\forall x \in X, f(x) = \tilde{f}(x) && \text{(by Def. I.3.3.14)} \\ \implies &f = \tilde{f}. && \text{(by Def. I.3.3.7)} \end{aligned}$$

The statement is not true when g is not injective. For example, define $f = x \mapsto x, \tilde{f} = x \mapsto |x|, g = x \mapsto x^2$. Then we see that $g \circ f = x \mapsto x^2 = x \mapsto |x|^2 = g \circ \tilde{f}$. But $f(-1) = -1 \neq |-1| = \tilde{f}(1)$ implies $f \neq \tilde{f}$.

Now we show that f is surjective, and $g \circ f = \tilde{g} \circ f$ implies $g = \tilde{g}$. Suppose that$f : X \to Y, g : Y \to Z, \tilde{g} : Y \to Z$ are functions, such that f is surjective and $g \circ f = \tilde{g} \circ f$. Then we have

$$\begin{aligned}
&\forall y \in Y, \exists x \in X : y = f(x) && \text{(by Def. I.3.3.17)}\\
\implies &\forall y \in Y, \exists x \in X : g(y) = g(f(x)) && \text{(by Def. I.3.3.1)}\\
&\quad = (g \circ f)(x) && \text{(by Def. I.3.3.10)}\\
&\quad = (\tilde{g} \circ f)(x) && \text{(by Def. I.3.3.7)}\\
&\quad = \tilde{g}(f(x)) = \tilde{g}(y) && \text{(by Def. I.3.3.10)}\\
\implies &g = \tilde{g}. && \text{(by Def. I.3.3.7)}
\end{aligned}$$

The statement is not true when f is not surjective. For example, define $f = x \mapsto |x|, g = x \mapsto x, \tilde{g} = x \mapsto |x|$. Then we see that $g \circ f = x \mapsto |x| = x \mapsto |(|x|)| = \tilde{g} \circ f$. But $g(-1) = -1 \neq |-1| = \tilde{g}(-1)$ implies $g \neq \tilde{g}$. □

Ex. I.3.3.5. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if $g \circ f$ is injective, then f must be injective. Is it true that g must also be injective? Show that if $g \circ f$ is surjective, then g must be surjective. Is it true that f must also be surjective?

Proof of Ex. I.3.3.5. We first show that $g \circ f$ is injective implies f is injective. Suppose $f : X \to Y, g : Y \to Z$ are functions such that $g \circ f$ is injective. Then we have

$$\begin{aligned}
&\forall x, x' \in X, x \neq x'\\
\implies &g(f(x)) = (g \circ f)(x) \neq (g \circ f)(x') = g(f(x')) && \text{(by Def. I.3.3.10 and I.3.3.14)}\\
\implies &f(x) \neq f(x'). && \text{(by A.Cor. I.3.3.1)}
\end{aligned}$$

Thus, by Def. I.3.3.14, f is injective. And we don't need g to be injective to finish the proof.

Now we show that $g \circ f$ is surjective implies g is surjective. Suppose $f : X \to Y, g : Y \to Z$ are functions such that $g \circ f$ is surjective. Then we have

$$\begin{aligned}
&\forall z \in Z, \exists x \in X : z = (g \circ f)(x) = g(f(x)) && \text{(by Def. I.3.3.10 and I.3.3.17)}\\
\implies &\forall z \in Z, \exists f(x) \in Y : z = g(f(x)) && \text{(by Def. I.3.3.1)}\\
\implies &g \text{ is surjective.} && \text{(by Def. I.3.3.17)}
\end{aligned}$$

And we don't need f to be surjective to finish the proof. □

Ex. I.3.3.6. Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Verify the cancellation laws $f^{-1}(f(x)) = x$ for all $x \in X$ and $f\big(f^{-1}(y)\big) = y$ for all $y \in Y$. Conclude that f^{-1} is also invertible, and has f as its inverse (thus $\big(f^{-1}\big)^{-1} = f$).

Proof of Ex. I.3.3.6. We first show that $f^{-1}(f(x)) = x$ for all $x \in X$.

$$\forall x \in X, \exists! y \in Y : \begin{cases} f(x) = y \\ f^{-1}(y) = x \end{cases} \qquad \text{(by Def. I.3.3.20)}$$

$$\implies \forall x \in X, \exists! y \in Y : f^{-1}(f(x)) = f(y) = x. \qquad \text{(by A.Cor. I.3.3.1)}$$

Next we show that $f\big(f^{-1}(y)\big) = y$ for all $y \in Y$.

$$\begin{aligned} &\forall y \in Y, \exists! x \in X : \begin{cases} f^{-1}(y) = x \\ f(x) = y \end{cases} && \text{(by Def. I.3.3.20)} \\ \implies &\forall y \in Y, \exists! x \in X : f\big(f^{-1}(y)\big) = f(x) = y. && \text{(by A.Cor. I.3.3.1)} \end{aligned}$$

Next we show that f^{-1} is bijective. Since

$$\begin{aligned} &\forall y, y' \in Y, y \neq y' \\ \implies &\begin{cases} \exists! x \in X : f(x) = y \\ \exists! x' \in X : f(x') = y' \\ x \neq x' \end{cases} && \text{(by Def. I.3.3.20)} \\ \implies &x = f^{-1}(y) \neq f^{-1}(y') = x', && \text{(by Def. I.3.3.20)} \end{aligned}$$

we know that f^{-1} is injective by Def. I.3.3.14. Since

$$\begin{aligned} &\forall x \in X, \exists! y \in Y : f(x) = y && \text{(by Def. I.3.3.1)} \\ \implies &\forall x \in X, \exists y \in Y : f^{-1}(y) = x, && \text{(by Def. I.3.3.20)} \end{aligned}$$

we know that f^{-1} is surjective by Def. I.3.3.17. Since f^{-1} is both injective and surjective, we know that f^{-1} is bijective by Def. I.3.3.20.

Finally we show that $\left(f^{-1}\right)^{-1} = f$. Clearly, $\left(f^{-1}\right)^{-1} : X \to Y$ has the same domain and codomain as f. Since

$$\begin{aligned} &\forall x \in X, \exists! y \in Y : \begin{cases} f(x) = y \\ f^{-1}(y) = x \\ \left(f^{-1}\right)^{-1}(x) = y \end{cases} && \text{(by Def. I.3.3.20)} \\ \implies &\forall x \in X, f(x) = \left(f^{-1}\right)^{-1}(x), \end{aligned}$$

we know that $f = \left(f^{-1}\right)^{-1}$ by Def. I.3.3.7. □

Ex. I.3.3.7. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are bijective, then so is $g \circ f$, and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof of Ex. I.3.3.7. Let $f : X \to Y, g : Y \to Z$ be bijectives. First, we show that $g \circ f$ is bijective. This is true since

$$\begin{aligned} &f, g \text{ are bijective} \\ \implies &(f, g \text{ are injective}) \wedge (f, g \text{ are surjective}) && \text{(by Def. I.3.3.20)} \end{aligned}$$

$$\begin{aligned} &\implies (g \circ f \text{ is injective}) \wedge (g \circ f \text{ is surjective}) && \text{(by Ex. I.3.3.2)} \\ &\implies g \circ f \text{ is bijective.} && \text{(by Def. I.3.3.20)} \end{aligned}$$

Now we show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. From the proof above we know that $g \circ f$ is bijective, so $(g \circ f)^{-1}$ is well-defined. Since $g \circ f$ has domain X and codomain Z, by Def. I.3.3.20, we know that $(g \circ f)^{-1}$ has domain Z and codomain X. By Def. I.3.3.20, we know that $g^{-1} : Z \to Y$ and $f^{-1} : Y \to X$ are well-defined. Thus, by Def. I.3.3.10, we know that $f^{-1} \circ g^{-1} : Z \to X$ is well-defined. Since $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ both have the same domain and codomain, by Def. I.3.3.7, we only need to show that $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$ for every $z \in Z$. Since

$$\begin{aligned} \forall x \in X, \big(\big(f^{-1} \circ g^{-1}\big) \circ (g \circ f)\big)(x) &= \big(f^{-1} \circ \big(g^{-1} \circ (g \circ f)\big)\big)(x) && \text{(by Lem. I.3.3.12)} \\ &= f^{-1}\big(g^{-1}(g(f(x)))\big) && \text{(by Def. I.3.3.10)} \\ &= f^{-1}(f(x)) && \text{(by Ex. I.3.3.6)} \\ &= x, && \text{(by Ex. I.3.3.6)} \end{aligned}$$

we see that

$$\begin{aligned} &\forall z \in Z, \big(f^{-1} \circ g^{-1}\big)(z) \\ &\quad = \big(f^{-1} \circ g^{-1}\big)\big(\big((g \circ f) \circ (g \circ f)^{-1}\big)(z)\big) && \text{(by Ex. I.3.3.6)} \\ &\quad = \big(\big(\big(f^{-1} \circ g^{-1}\big) \circ (g \circ f)\big) \circ (g \circ f)^{-1}\big)(z) && \text{(by Lem. I.3.3.12)} \\ &\quad = \big(\big(f^{-1} \circ g^{-1}\big) \circ (g \circ f)\big)\big((g \circ f)^{-1}(z)\big) && \text{(by Def. I.3.3.10)} \\ &\quad = (g \circ f)^{-1}(z). && \text{(from the proof above)} \end{aligned}$$

Thus, by Def. I.3.3.7, we conclude that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

Ex. I.3.3.8. If X is a subset of Y, let $\iota_{X \to Y} : X \to Y$ be the *inclusive map from* X *to* Y, defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X \to Y}(x) \coloneqq x$ for all $x \in X$. The map $\iota_{X \to X}$ is, in particular, called the *identity map* on X.

(a) Show that if $X \subseteq Y \subseteq Z$ then $\iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}$.

(b) Show that if $f : A \to B$ is any function, then $f = f \circ \iota_{A \to A} = \iota_{B \to B} \circ f$.

(c) Show that if $f : A \to B$ is a bijective function, then $f \circ f^{-1} = \iota_{B \to B}$ and $f^{-1} \circ f = \iota_{A \to A}$.

(d) Show that if X and Y are disjoint sets, and $f : X \to Z$ and $g : Y \to Z$ are functions, then there is a unique function $h : X \cup Y \to Z$ such that $h \circ \iota_{X \to X \cup Y} = f$ and $h \circ \iota_{Y \to X \cup Y} = g$.

Proof of Ex. I.3.3.8(a). If X, Y, Z are sets such that $X \subseteq Y \subseteq Z$, then we have

$$\forall x \in X, (\iota_{Y \to Z} \circ \iota_{X \to Y})(x) = \iota_{Y \to Z}(\iota_{X \to Y}(x)) \qquad \text{(by Def. I.3.3.10)}$$

$$\begin{aligned}
&= \iota_{Y\to Z}(x) && \text{(by Ex. I.3.3.8)}\\
&= x && \text{(by Ex. I.3.3.8)}\\
&= \iota_{X\to Z}(x). && \text{(by Ex. I.3.3.8)}
\end{aligned}$$

Thus, by Def. I.3.3.7 and I.3.3.10, we have $\iota_{Y\to Z}\circ\iota_{X\to Y} = \iota_{X\to Z}$. □

Proof of Ex. I.3.3.8(b). Let $f : A \to B$ be a function. Then we have

$$\begin{aligned}
\forall a\in A, f(a) &= \begin{cases} f(\iota_{A\to A}(a)) \\ \iota_{B\to B}(f(a)) \end{cases} && \text{(by Ex. I.3.3.8)}\\
&= \begin{cases} (f\circ\iota_{A\to A})(a) \\ (\iota_{B\to B}\circ f)(a) \end{cases}. && \text{(by Def. I.3.3.10)}
\end{aligned}$$

Thus, by Def. I.3.3.7 and I.3.3.10, we have $f = f\circ\iota_{A\to A} = \iota_{B\to B}\circ f$. □

Proof of Ex. I.3.3.8(c). Suppose $f : A\to B$ is bijective. Then we have

$$\begin{aligned}
\forall a\in A, a &= f^{-1}(f(a)) && \text{(by Ex. I.3.3.6)}\\
&= (f^{-1}\circ f)(a) && \text{(by Def. I.3.3.10)}\\
&= \iota_{A\to A}(a). && \text{(by Ex. I.3.3.8)}\\
\forall b\in B, b &= f(f^{-1}(b)) && \text{(by Ex. I.3.3.6)}\\
&= (f\circ f^{-1})(b) && \text{(by Def. I.3.3.10)}\\
&= \iota_{B\to B}(b). && \text{(by Ex. I.3.3.8)}
\end{aligned}$$

Thus, by Def. I.3.3.7 and I.3.3.10, we have $f^{-1}\circ f = \iota_{A\to A}$ and $f\circ f^{-1} = \iota_{B\to B}$. □

Proof of Ex. I.3.3.8(d). Suppose that X, Y, Z are sets such that $X\cap Y = \emptyset$. Let $f : X \to Z, g : Y \to Z$ be functions. We now define a function $h : X\cup Y \to Z$ as follow:

$$\forall w\in X\cup Y, h(w) = \begin{cases} f(w) & \text{if } w\in X \\ g(w) & \text{if } w\in Y \end{cases}.$$

This function is well-defined since $X\cap Y = \emptyset$. Thus, each $w\in X\cup Y$ can either be in X or Y but not both. Then we have

$$\begin{aligned}
\forall w\in X, h(w) &= h(\iota_{X\to X\cup Y}(w)) && \text{(by Ex. I.3.3.8)}\\
&= (h\circ\iota_{X\to X\cup Y})(w) && \text{(by Def. I.3.3.10)}\\
&= f(w). && \text{(by the definition of } h)\\
\forall w\in Y, h(w) &= h(\iota_{Y\to X\cup Y}(w)) && \text{(by Ex. I.3.3.8)}\\
&= (h\circ\iota_{Y\to X\cup Y})(w) && \text{(by Def. I.3.3.10)}
\end{aligned}$$

$$= g(w). \qquad \text{(by the definition of } h\text{)}$$

Thus, by Def. I.3.3.7 and I.3.3.10, we have $h \circ \iota_{X \to X \cup Y} = f$ and $h \circ \iota_{Y \to X \cup Y} = g$.

Now suppose there exists another function $h' : X \cup Y \to Z$ such that $h' \circ \iota_{X \to X \cup Y} = f$ and $h' \circ \iota_{Y \to X \cup Y} = g$. Then we have

$$\begin{aligned}
\forall x \in X, f(x) &= (h' \circ \iota_{X \to X \cup Y})(x) \\
&= h'(\iota_{X \to X \cup Y}(x)) && \text{(by Def. I.3.3.10)} \\
&= h'(x) \\
&= (h \circ \iota_{X \to X \cup Y})(x) \\
&= h(\iota_{X \to X \cup Y}(x)) && \text{(by Def. I.3.3.10)} \\
&= h(x). \\
\forall y \in Y : g(y) &= (h' \circ \iota_{Y \to X \cup Y})(y) \\
&= h'(\iota_{Y \to X \cup Y}(y)) && \text{(by Def. I.3.3.10)} \\
&= h'(y) \\
&= (h \circ \iota_{Y \to X \cup Y})(y) \\
&= h(\iota_{Y \to X \cup Y}(y)) && \text{(by Def. I.3.3.10)} \\
&= h(y).
\end{aligned}$$

Thus, by Def. I.3.3.7, we have $h = h'$, so h is unique. □

I.3.4 Images and inverse images

Def. I.3.4.1 (Images of sets)**.** If $f : X \to Y$ is a function from X to Y, and S is a subset of X, we define $f(S)$ to be the set

$$f(S) \coloneqq \{f(x) : x \in S\};$$

this set is a subset of Y, and is sometimes called the *image* of S under the map f. We sometimes call $f(S)$ the *forward image* of S to distinguish it from the concept of the *inverse image* $f^{-1}(S)$ of S.

A.Cor. I.3.4.1. Let $f : X \to Y$ be a function and let $S \subseteq X$. The set $f(S)$ is well-defined thanks to the axiom of replacement (Ax. I.3.6). One can also define $f(S)$ using the axiom of specification (Ax. I.3.5) instead of replacement.

Proof of A.Cor. I.3.4.1. First, we use Def. I.3.4.1 to create the set $f(X) = \{f(x) : x \in X\}$. Now we use Ax. I.3.5 to create the set

$$E = \{y \in f(X) \mid \exists x \in S : f(x) = y\}.$$

By Def. I.3.1.4 and I.3.4.1, we see that $E = f(S)$. Thus, we can define $f(S)$ by Ax. I.3.5 instead of Ax. I.3.6. □

Def. I.3.4.4 (Inverse images)**.** If U is a subset of Y, we define the set $f^{-1}(U)$ to be the set

$$f^{-1}(U) := \{x \in X : f(x) \in U\}.$$

In other words, $f^{-1}(U)$ consists of all the elements of X which map into U:

$$f(x) \in U \iff x \in f^{-1}(U).$$

We call $f^{-1}(U)$ the *inverse image* of U.

Rmk. I.3.4.7. If f is a bijective function, then we have defined f^{-1} in two slightly different ways, but this is not an issue because both definitions are equivalent (Ex. I.3.4.1).

Ax. I.3.10 (Power set axiom)**.** Let X and Y be sets. Then there exists a set, denoted Y^X, which consists of all the functions from X to Y, thus

$$f \in Y^X \iff (f \text{ is a function with domain } X \text{ and codomain } Y).$$

Note. The reason we use the notation Y^X to denote this set is that if Y has n elements and X has m elements, then one can show that Y^X has n^m elements. See Prop. I.3.6.14(f).

Lem. I.3.4.9. Let X be a set. Then the set

$$\{Y : Y \text{ is a subset of } X\}$$

is a set.

Proof of Lem. I.3.4.9. Suppose that X is a set. By Ax. I.3.10, there exists a set $\{0,1\}^X$ which consists of functions with domain X and codomain $\{0,1\}$. By Ax. I.3.6, we can replace each $f \in \{0,1\}^X$ with $f^{-1}(\{1\})$, i.e., there exists a set

$$S = \left\{ f^{-1}(\{1\}) : f \in \{0,1\}^X \right\}.$$

Next we claim that S is consist of subsets of X. Let $Y \in S$. We want to show that Y is a subset of X. By the definition of S, we know that $Y = f^{-1}(\{1\})$ for some $f \in \{0,1\}^X$. By Def. I.3.4.4, we know that $Y = \{x \in X : f(x) \in \{1\}\}$, which is clearly a subset of X (Def. I.3.1.15). Thus, our claim is true.

Finally we show that S is consist of all subsets of X, and therefore Lem. I.3.4.9 is true. Let Y be a subset of X. We want to show that $Y \in S$, and we will use Ex. I.3.3.8(d) to achieve this. We define two functions $f : X \setminus Y \to \{0,1\}$ and $g : Y \to \{0,1\}$ as follow:

$$\forall z \in X \setminus Y, f(z) = 0;$$
$$\forall z \in Y, g(z) = 1.$$

Since every element in $X \setminus Y$ is mapped by f to the unique element 0 in $\{0,1\}$, we know that f obey the vertical line test, and therefore f is well-defined by Def. I.3.3.1. Similarly, g

is well-defined. By Prop. I.3.1.28(g), we know that $(X \setminus Y) \cap Y = \emptyset$ and $(X \setminus Y) \cup Y = X$. Thus, we can apply Ex. I.3.3.8(d) to create a function $h : X \to \{0,1\}$ as follow:

$$\forall z \in X, h(z) = \begin{cases} f(z) = 0 & \text{if } z \in X \setminus Y \\ g(z) = 1 & \text{if } z \in Y \end{cases}.$$

Clearly, we have $h \in \{0,1\}^X$ and $h^{-1}(\{1\}) = Y$. But by the definition of S, we know that $h^{-1}(\{1\}) \in S$, therefore $Y \in S$. □

Rmk. I.3.4.10. The set $\{Y : Y \text{ is a subset of } X\}$ is know as the *power set* of X and is denoted 2^X.

Ax. I.3.11 (Union)**.** Let A be a set, all of whose elements are themselves sets. Then there exists a set $\bigcup A$ whose elements are precisely those objects which are elements of the elements of A, thus for all objects x

$$x \in \bigcup A \iff (x \in S \text{ for some } S \in A)$$

Note. The axiom of union (Ax. I.3.11), combined with the axiom of pair set (Ax. I.3.3), implies the axiom of pairwise union (Ax. I.3.4) (see Ex. I.3.4.8). Another important consequence of Ax. I.3.11 is that if one has some set I, and for every element $\alpha \in I$ we have some set A_α, then we can form the union set $\bigcup_{\alpha \in I} A_\alpha$ by defining

$$\bigcup_{\alpha \in I} A_\alpha \coloneqq \bigcup \{A_\alpha : \alpha \in I\},$$

which is a set thanks to the axiom of replacement (Ax. I.3.6) and the axiom of union (Ax. I.3.11). More generally, we see that for any object y,

$$y \in \bigcup_{\alpha \in I} A_\alpha \iff (y \in A_\alpha \text{ for some } \alpha \in I).$$

In situations like this, we often refer to I as an *index set*, and the elements α of this index set as *labels*; the sets A_α are then called a *family of sets*, and are *indexed* by the labels $\alpha \in I$. Note that if I was empty, then $\bigcup_{\alpha \in I} A_\alpha$ would automatically also be empty.

Note. We can similarly form intersections of families of sets, as long as the index set is non-empty. More specifically, given any non-empty set I, and given an assignment of a set A_α to each $\alpha \in I$, we can define the intersection $\bigcap_{\alpha \in I} A_\alpha$ by first choosing some element β of I (which we can do since I is non-empty), and setting

$$\bigcap_{\alpha \in I} A_\alpha \coloneqq \{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\},$$

which is a set by the axiom of specification (Ax. I.3.5). This definition may look like it depends on the choice of β, but it does not. Observe that for any object y,

$$y \in \bigcap_{\alpha \in I} A_\alpha \iff (y \in A_\alpha \text{ for all } \alpha \in I).$$

Rmk. I.3.4.12. The axioms of set theory that we have introduced (Ax. I.3.1 to I.3.11, excluding the dangerous Ax. I.3.8) are known as the *Zermelo-Fraenkel axioms of set theory*, after Ernst Zermelo (1871–1953) and Abraham Fraenkel (1891–1965). There is one further axiom we will eventually need, the famous *axiom of choice* (see Sec. I.8.4), giving rise to the *Zermelo-Fraenkel-Choice (ZFC) axioms of set theory*, but we will not need this axiom for some time.

— Exercises —

Ex. I.3.4.1. Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Let V be any subset of Y. Prove that the forward image of V under f^{-1} is the same set as the inverse image of V under f; thus the fact that both sets are denoted by $f^{-1}(V)$ will not lead to any inconsistency.

Proof of Ex. I.3.4.1. Let A be the set of the forward image of V under f^{-1}. Let B be the set of the inverse image of V under f. Then we have

$$\begin{aligned}
&x \in A \\
\iff &\exists y \in V : f^{-1}(y) = x && \text{(by Def. I.3.4.1)} \\
\iff &\exists y \in V : f(x) = y && \text{(by Def. I.3.3.20)} \\
\iff &x \in B. && \text{(by Def. I.3.4.4)}
\end{aligned}$$

Thus, by Def. I.3.1.4, we have $A = B$. □

Ex. I.3.4.2. Let $f : X \to Y$ be a function from one set X to another set Y, let S be a subset of X, and let U be a subset of Y. What, in general, can one say about $f^{-1}(f(S))$ and S? What about $f(f^{-1}(U))$ and U?

Proof of Ex. I.3.4.2. We first show that $S \subseteq f^{-1}(f(S))$. Since

$$\begin{aligned}
&x \in S \\
\implies &f(x) \in f(S) && \text{(by Def. I.3.4.1)} \\
\implies &x \in f^{-1}(f(S)), && \text{(by Def. I.3.4.4)}
\end{aligned}$$

we have $S \subseteq f^{-1}(f(S))$ by Def. I.3.1.15.

Now we show that $f(f^{-1}(U)) \subseteq U$. Since

$$y \in f(f^{-1}(U))$$

$$
\begin{aligned}
&\implies \exists x \in f^{-1}(U) : f(x) = y && \text{(by Def. I.3.4.1)}\\
&\implies y \in U, && \text{(by Def. I.3.4.4)}
\end{aligned}
$$

we have $f(f^{-1}(U)) \subseteq U$ by Def. I.3.1.15. □

Ex. I.3.4.3. Let A, B be two subsets of a set X, and let $f : X \to Y$ be a function. Show that $f(A \cap B) \subseteq f(A) \cap f(B)$, that $f(A) \setminus f(B) \subseteq f(A \setminus B)$, $f(A \cup B) = f(A) \cup f(B)$. For the first two statements, is it true that the $\subseteq$ relation can be imporved to $=$?

Proof of Ex. I.3.4.3. We first show that $f(A \cap B) \subseteq f(A) \cap f(B)$. Since

$$
\begin{aligned}
& y \in f(A \cap B) \\
\implies & y \in \{f(x) : x \in A \cap B\} && \text{(by Def. I.3.4.1)}\\
\implies & y \in \{f(x) : (x \in A) \wedge (x \in B)\} && \text{(by Def. I.3.1.23)}\\
\implies & (y \in \{f(x) : x \in A\}) \wedge (y \in \{f(x) : x \in B\}) \\
\implies & (y \in f(A)) \wedge (y \in f(B)) && \text{(by Def. I.3.4.1)}\\
\implies & y \in f(A) \cap f(B), && \text{(by Def. I.3.1.23)}
\end{aligned}
$$

we have $f(A \cap B) \subseteq f(A) \cap f(B)$ by Def. I.3.1.15. We do not have $f(A \cap B) = f(A) \cap f(B)$ in general. Consider the example $f = x \mapsto x^2$, $A = \{1\}$, and $B = \{-1\}$. We have $f(A \cap B) = f(\emptyset) = \emptyset \neq \{1\} = \{1\} \cap \{1\} = f(A) \cap f(B)$.

Next we show that $f(A) \setminus f(B) \subseteq f(A \setminus B)$. Since

$$
\begin{aligned}
& y \in f(A) \setminus f(B) \\
\implies & (y \in f(A)) \wedge (y \notin f(B)) && \text{(by Def. I.3.1.27)}\\
\implies & (y \in \{f(x) : x \in A\}) \wedge (y \notin \{f(x) : x \in B\}) && \text{(by Def. I.3.4.1)}\\
\implies & y \in \{f(x) : (x \in A) \wedge (x \notin B)\} \\
\implies & y \in \{f(x) : x \in A \setminus B\} && \text{(by Def. I.3.1.27)}\\
\implies & y \in f(A \setminus B), && \text{(by Def. I.3.4.1)}
\end{aligned}
$$

we have $f(A) \setminus f(B) \subseteq f(A \setminus B)$ by Def. I.3.1.15. We do not have $f(A) \setminus f(B) = f(A \setminus B)$ in general. Consider the example $f = x \mapsto x^2$, $A = \{1\}$, and $B = \{-1\}$. We have $f(A) \setminus f(B) = \{1\} \setminus \{1\} = \emptyset \neq \{1\} = f(\{1\}) = f(A \setminus B)$.

Finally we show that $f(A \cup B) = f(A) \cup f(B)$. Since

$$
\begin{aligned}
& y \in f(A \cup B) \\
\iff & y \in \{f(x) : x \in A \cup B\} && \text{(by Def. I.3.4.1)}\\
\iff & y \in \{f(x) : (x \in A) \vee (x \in B)\} && \text{(by Ax. I.3.4)}\\
\iff & (y \in \{f(x) : x \in A\}) \vee (y \in \{f(x) : x \in B\}) \\
\iff & (y \in f(A)) \vee (y \in f(B)) && \text{(by Def. I.3.4.1)}\\
\iff & y \in f(A) \cup f(B), && \text{(by Ax. I.3.4)}
\end{aligned}
$$

we have $f(A \cup B) = f(A) \cup f(B)$ by Def. I.3.1.4. □

Ex. I.3.4.4. Let $f : X \to Y$ be a function from one set X to another set Y, and let U, V be subsets of Y. Show that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$, that $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$, and that $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$.

Proof of Ex. I.3.4.4. We first show that $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$. Since

$$\begin{aligned}
&x \in f^{-1}(U \cup V) \\
\iff & f(x) \in U \cup V && \text{(by Def. I.3.4.4)} \\
\iff & (f(x) \in U) \vee (f(x) \in V) && \text{(by Ax. I.3.4)} \\
\iff & \big(x \in f^{-1}(U)\big) \vee \big(x \in f^{-1}(V)\big) && \text{(by Def. I.3.4.4)} \\
\iff & x \in f^{-1}(U) \cup f^{-1}(V), && \text{(by Ax. I.3.4)}
\end{aligned}$$

we have $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ by Def. I.3.1.4.

Next we show that $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$. Since

$$\begin{aligned}
&x \in f^{-1}(U \cap V) \\
\iff & f(x) \in U \cap V && \text{(by Def. I.3.4.4)} \\
\iff & (f(x) \in U) \wedge (f(x) \in V) && \text{(by Def. I.3.1.23)} \\
\iff & \big(x \in f^{-1}(U)\big) \wedge \big(x \in f^{-1}(V)\big) && \text{(by Def. I.3.4.4)} \\
\iff & x \in f^{-1}(U) \cap f^{-1}(V), && \text{(by Def. I.3.1.23)}
\end{aligned}$$

we have $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$ by Def. I.3.1.4.

Finally we show that $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$. Since

$$\begin{aligned}
&x \in f^{-1}(U \setminus V) \\
\iff & f(x) \in U \setminus V && \text{(by Def. I.3.4.4)} \\
\iff & (f(x) \in U) \wedge (f(x) \notin V) && \text{(by Def. I.3.1.23)} \\
\iff & \big(x \in f^{-1}(U)\big) \wedge \big(x \notin f^{-1}(V)\big) && \text{(by Def. I.3.4.4)} \\
\iff & x \in f^{-1}(U) \setminus f^{-1}(V), && \text{(by Def. I.3.1.23)}
\end{aligned}$$

we have $f^{-1}(U \setminus V) = f^{-1}(U) \setminus f^{-1}(V)$ by Def. I.3.1.4. □

Ex. I.3.4.5. Let $f : X \to Y$ be a function from one set X to another set Y. Show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ iff f is surjective. Show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ iff f is injective.

Proof of Ex. I.3.4.5. We first show that $f\big(f^{-1}(S)\big) = S$ for every $S \subseteq Y$ iff f is surjective. By Ex. I.3.4.2, we have $f\big(f^{-1}(S)\big) \subseteq S$ for every $S \subseteq Y$. Thus, it suffices to show that $S \subseteq f\big(f^{-1}(S)\big)$ for every $S \subseteq Y$ iff f is surjective. This is true since

$$f \text{ is surjective}$$

$$\begin{aligned}
&\iff \forall y \in Y, \exists x \in X : f(x) = y && \text{(by Def. I.3.3.17)}\\
&\iff \forall S \subseteq Y, \forall y \in S, \exists x \in X : f(x) = y && \text{(by Def. I.3.1.15)}\\
&\iff \forall S \subseteq Y, \forall y \in S, \exists x \in f^{-1}(S) : f(x) = y && \text{(by Def. I.3.4.4)}\\
&\iff \forall S \subseteq Y, \forall y \in S, y \in f\big(f^{-1}(S)\big) && \text{(by Def. I.3.4.1)}\\
&\iff \forall S \subseteq Y, S \subseteq f\big(f^{-1}(S)\big). && \text{(by Def. I.3.1.15)}
\end{aligned}$$

Now we show that $f^{-1}(f(S)) = S$ for all $S \subseteq X$ iff f is injective. By Ex. I.3.4.2, we have $S \subseteq f^{-1}(f(S))$ for every $S \subseteq X$. Thus, it suffices to show that $f^{-1}(f(S)) \subseteq S$ for every $S \subseteq X$ iff f is injective.

We start by showing that if $f^{-1}(f(S)) \subseteq S$ for every $S \subseteq X$, then f is injective. So suppose that $f^{-1}(f(S)) \subseteq S$ for every $S \subseteq X$. Let $x_1, x_2 \in X$ where $x_1 \neq x_2$. Then by Ax. I.3.3 and Def. I.3.1.15, we know that $\{x_1\} \subseteq X$ and $\{x_2\} \subseteq X$. Thus, we can apply the hypothesis and Ex. I.3.4.2 to derive

$$\{x_1\} \subseteq f^{-1}(f(\{x_1\})) \subseteq \{x_1\} \quad \text{and} \quad \{x_2\} \subseteq f^{-1}(f(\{x_2\})) \subseteq \{x_2\}.$$

By Prop. I.3.1.18, this means $\{x_1\} = f^{-1}(f(\{x_1\}))$ and $\{x_2\} = f^{-1}(f(\{x_2\}))$. By Def. I.3.4.1, we see that $f(\{x_1\})$ and $f(\{x_2\})$ are singleton sets. Thus, by Def. I.3.4.4, we must have $f(x_1) \neq f(x_2)$, otherwise we would have $f(\{x_1\}) = f(\{x_2\})$, which implies $\{x_1\} = \{x_2\}$, a contradiction. Therefore, f is injective by Def. I.3.3.14.

Now we show that f is injective implies $f^{-1}(f(S)) \subseteq S$ for every $S \subseteq X$. Suppose that f is injective. Let $S \subseteq X$. Suppose for the sake of contradiction that $f^{-1}(f(S)) \nsubseteq S$. Then by Def. I.3.1.15, there exists an $x \in f^{-1}(f(S))$ such that $x \notin S$. Fix one such x. By Def. I.3.4.4, we know that there exists a $y \in f(S)$ such that $f(x) = y$. Fix one such y. Since $y \in f(S)$, by Def. I.3.4.1, there exists an $x' \in S$ such that $f(x') = y$. But f is injective implies $x = x'$, which means $x \in S$, a contradiction. Thus, we must have $f^{-1}(f(S)) \subseteq S$. □

Ex. I.3.4.6. Prove Lem. I.3.4.9.

Proof of Ex. I.3.4.6. See Lem. I.3.4.9. □

Ex. I.3.4.7. Let X, Y be sets. Define a *partial function* from X to Y to be any function $f : X' \to Y'$ whose domain X' is a subset of X, and whose codomain Y' is a subset of Y. Show that the collection of all partial functions from X to Y is itself a set.

Proof of Ex. I.3.4.7. Suppose that X, Y are sets. Then by Lem. I.3.4.9, both the sets $A = \big\{X' : X' \subseteq X\big\}$ and $B = \big\{Y' : Y' \subseteq Y\big\}$ exist. Now we have

$$\begin{aligned}
C_1 &= \Big\{Y'^{X'} : \big(X' \in A\big) \wedge \big(Y' \in B\big)\Big\}; && \text{(by Ax. I.3.5, I.3.6 and I.3.10)}\\
C_2 &= \bigcup C_1 = \Big\{f \in Y'^{X'} : Y'^{X'} \in C_1\Big\}. && \text{(by Ax. I.3.11)}
\end{aligned}$$

If $f : X' \to Y'$ is a partial function whose domain $X' \subseteq X$ and whose codomain $Y' \subseteq Y$, then we have $Y'^{X'} \in C_1$, and thus $f \in C_2$. □

Ex. I.3.4.8. Show that Ax. I.3.4 can be deduced from Ax. I.3.1, I.3.3 and I.3.11.

Proof of Ex. I.3.4.8. Let A, B be sets (the existence of A, B are guaranteed by Ax. I.3.1). By Ax. I.3.3, there exists a set $\{A, B\}$ whose only elements are A and B. By Ax. I.3.11, we can create a set $\bigcup\{A, B\}$. Now we claim that $A \cup B = \bigcup\{A, B\}$. Since

$$\begin{aligned}
&x \in \bigcup\{A, B\} \\
\iff &\exists C \in \{A, B\} : x \in C && \text{(by Ax. I.3.11)} \\
\iff &(x \in A) \vee (x \in B) && \text{(by Ax. I.3.3)} \\
\iff &x \in A \cup B, && \text{(by Ax. I.3.4)}
\end{aligned}$$

we see that $A \cup B = \bigcup\{A, B\}$ by Def. I.3.1.4. □

Ex. I.3.4.9. Show that if β and β' are two elements of a set I, and to each $\alpha \in I$ we assign a set A_α, then

$$\{x \in A_\beta : \forall\alpha \in I, x \in A_\alpha\} = \left\{x \in A_{\beta'} : \forall\alpha \in I, x \in A_\alpha\right\},$$

and so the definition of $\bigcap\limits_{\alpha \in I} A_\alpha$ does not depend on β.

Proof of Ex. I.3.4.9. Let B, B' be sets

$$\begin{aligned}
B &= \{x \in A_\beta : \forall\alpha \in I, x \in A_\alpha\} \\
B' &= \left\{x \in A_{\beta'} : \forall\alpha \in I, x \in A_\alpha\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
&x \in B \\
\iff &(x \in A_\beta) \wedge (\forall\alpha \in I, x \in A_\alpha) \\
\iff &\forall\alpha \in I, x \in A_\alpha && (\beta \in I) \\
\iff &(x \in A_{\beta'}) \wedge (\forall\alpha \in I, x \in A_\alpha) && (\beta' \in I) \\
\iff &x \in B',
\end{aligned}$$

we have $B = B'$ by Def. I.3.1.4. □

Ex. I.3.4.10. Suppose that I and J are two sets, and for all $\alpha \in I \cup J$ let A_α be a set. Show that $\left(\bigcup\limits_{\alpha \in I} A_\alpha\right) \cup \left(\bigcup\limits_{\alpha \in J} A_\alpha\right) = \bigcup\limits_{\alpha \in I \cup J} A_\alpha$. If I and J are non-empty, show that $\left(\bigcap\limits_{\alpha \in I} A_\alpha\right) \cap \left(\bigcap\limits_{\alpha \in J} A_\alpha\right) = \bigcap\limits_{\alpha \in I \cup J} A_\alpha$.

Proof of Ex. I.3.4.10. Since

$$\begin{aligned}
& x \in \left(\bigcup_{\alpha\in I} A_\alpha\right) \cup \left(\bigcup_{\alpha\in J} A_\alpha\right) \\
\iff & \left(x \in \bigcup_{\alpha\in I} A_\alpha\right) \vee \left(x \in \bigcup_{\alpha\in J} A_\alpha\right) && \text{(by Ax. I.3.4)} \\
\iff & (\exists \alpha \in I : x \in A_\alpha) \vee (\exists \alpha \in J : x \in A_\alpha) && \text{(by Ax. I.3.11)} \\
\iff & \exists \alpha : ((\alpha \in I) \vee (\alpha \in J)) \wedge (x \in A_\alpha) \\
\iff & \exists \alpha \in I \cup J : x \in A_\alpha && \text{(by Ax. I.3.4)} \\
\iff & x \in \bigcup_{\alpha\in I\cup J} A_\alpha, && \text{(by Ax. I.3.11)}
\end{aligned}$$

and

$$\begin{aligned}
& x \in \left(\bigcap_{\alpha\in I} A_\alpha\right) \cap \left(\bigcap_{\alpha\in J} A_\alpha\right) \\
\iff & \left(x \in \bigcap_{\alpha\in I} A_\alpha\right) \wedge \left(x \in \bigcap_{\alpha\in J} A_\alpha\right) && \text{(by Def. I.3.1.23)} \\
\iff & (\forall \alpha \in I, x \in A_\alpha) \wedge (\forall \alpha \in J, x \in A_\alpha) && \text{(by Ex. I.3.4.9)} \\
\iff & \forall \alpha, ((\alpha \in I) \vee (\alpha \in J)) \wedge (x \in A_\alpha) \\
\iff & \forall \alpha \in I \cup J, x \in A_\alpha && \text{(by Ax. I.3.4)} \\
\iff & x \in \bigcap_{\alpha\in I\cup J} A_\alpha, && \text{(by Ex. I.3.4.9)}
\end{aligned}$$

we have $\left(\bigcup_{\alpha\in I} A_\alpha\right) \cup \left(\bigcup_{\alpha\in J} A_\alpha\right) = \bigcup_{\alpha\in I\cup J} A_\alpha$ and $\left(\bigcap_{\alpha\in I} A_\alpha\right) \cap \left(\bigcap_{\alpha\in J} A_\alpha\right) = \bigcap_{\alpha\in I\cup J} A_\alpha$ by Def. I.3.1.4. □

Ex. I.3.4.11. Let X be a set, let I be a non-empty set, and for all $\alpha \in I$ let A_α be a subset of X. Show that

$$X \setminus \bigcup_{\alpha\in I} A_\alpha = \bigcap_{\alpha\in I} (X \setminus A_\alpha)$$

and

$$X \setminus \bigcap_{\alpha\in I} A_\alpha = \bigcup_{\alpha\in I} (X \setminus A_\alpha).$$

This should be compared with de Morgan's laws in Prop. I.3.1.28 (although one cannot derive the above identities directly from de Morgan's laws, as I could be infinite).

Proof of Ex. I.3.4.11. Since

$$\begin{aligned}
& x \in X \setminus \bigcup_{\alpha \in I} A_\alpha \\
\iff & (x \in X) \wedge \left(x \notin \bigcup_{\alpha \in I} A_\alpha \right) && \text{(by Def. I.3.1.27)} \\
\iff & (x \in X) \wedge \neg(\exists \alpha \in I : x \in A_\alpha) && \text{(by Ax. I.3.11)} \\
\iff & (x \in X) \wedge (\forall \alpha \in I, x \notin A_\alpha) \\
\iff & \forall \alpha \in I, (x \in X) \wedge (x \notin A_\alpha) \\
\iff & \forall \alpha \in I, x \in X \setminus A_\alpha && \text{(by Def. I.3.1.27)} \\
\iff & x \in \bigcap_{\alpha \in I} (X \setminus A_\alpha), && \text{(by Ex. I.3.4.9)}
\end{aligned}$$

and

$$\begin{aligned}
& x \in X \setminus \bigcap_{\alpha \in I} A_\alpha \\
\iff & (x \in X) \wedge \left(x \notin \bigcap_{\alpha \in I} A_\alpha \right) && \text{(by Def. I.3.1.27)} \\
\iff & (x \in X) \wedge \neg(\forall \alpha \in I, x \in A_\alpha) && \text{(by Ex. I.3.4.9)} \\
\iff & (x \in X) \wedge (\exists \alpha \in I : x \notin A_\alpha) \\
\iff & \exists \alpha \in I : (x \in X) \wedge (x \notin A_\alpha) \\
\iff & \exists \alpha \in I : x \in X \setminus A_\alpha && \text{(by Def. I.3.1.27)} \\
\iff & x \in \bigcup_{\alpha \in I} (X \setminus A_\alpha), && \text{(by Ax. I.3.11)}
\end{aligned}$$

we have $X \setminus \bigcup_{\alpha \in I} A_\alpha = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$ and $X \setminus \bigcap_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$ by Def. I.3.1.4. □

I.3.5 Cartesian products

Def. I.3.5.1 (Ordered pair)**.** If x and y are any objects (possibly equal), we define the *ordered pair* (x, y) to be a new object, consisting of x as its first component and y as its second component. Two ordered pairs (x, y) and (x', y') are considered equal iff both their components match, i.e.

$$(x, y) = (x', y') \iff (x = x' \text{ and } y = y').$$

This is consistent with the usual axioms of equality (Ex. I.3.5.3).

Rmk. I.3.5.2. Strictly speaking, Def. I.3.5.1 is partly an axiom, because we have simply postulated that given any two objects x and y, that an object of the form (x, y) exists. However, it is possible to define an ordered pair using the axioms of set theory in such a way that we do not need any further postulates (see Ex. I.3.5.1).

Rmk. I.3.5.3. We have now "overloaded" the parenthesis symbols () once again; they now are not only used to denote grouping of operators and arguments of functions, but also to enclose ordered pairs. This is usually not a problem in practice as one can still determine what usage the symbols () were intended for from context.

Def. I.3.5.4 (Cartesian product)**.** If X and Y are sets, then we define the *Cartesian product* $X \times Y$ to be the collection of ordered pairs, whose first component lies in X and second component lies in Y, thus

$$X \times Y \coloneqq \{(x, y) : x \in X, y \in Y\}$$

or equivalently,

$$a \in X \times Y \iff (a = (x, y) \text{ for some } x \in X \text{ and } y \in Y).$$

Rmk. I.3.5.5. One can show that the Cartesian product $X \times Y$ is indeed a set; see Ex. I.3.5.1.

Note. Let $f : X \times Y \to Z$ be a function whose domain $X \times Y$ is a Cartesian product of two other sets X and Y. Then f can either be thought of as a function of one variable, mapping the single input of an ordered pair (x, y) in $X \times Y$ to an output $f(x, y)$ in Z, or as a function of two variables, mapping an input $x \in X$ and another input $y \in Y$ to a single output $f(x, y)$ in Z. While the two notions are technically different, we will not bother to distinguish the two, and think of f simultaneously as a function of one variable with domain $X \times Y$ and as a function of two variables with domains X and Y. Thus, for instance the addition operation $+$ on the natural numbers can now be re-interpreted as a function $+ : N \times N \to N$, defined by $(x, y) \mapsto x + y$.

Def. I.3.5.7 (Ordered n-tuple and n-fold Cartesian product)**.** Let n be a natural number. An *ordered n-tuple* $(x_i)_{1 \le i \le n}$ (also denoted $(x_1, \ldots, x_n)$) is a collection of objects x_i, one for every natural number i between 1 and n; we refer to x_i as the i^{th} *component* of the ordered n-tuple. Two ordered n-tuples $(x_i)_{1 \le i \le n}$ and $(y_i)_{1 \le i \le n}$ are said to be equal iff $x_i = y_i$ for all $1 \le i \le n$. If $(X_i)_{1 \le i \le n}$ is an ordered n-tuple of sets, we define their *Cartesian product* $\prod_{1 \le i \le n} X_i$ (also denoted $\prod_{i=1}^{n} X_i$ or $X_1 \times \cdots \times X_n$) by

$$\prod_{1 \le i \le n} X_i \coloneqq \{(x_i)_{1 \le i \le n} : x_i \in X_i \text{ for all } 1 \le i \le n\}.$$

Note. Again, Def. I.3.5.7 simply postulates that an ordered n-tuple and a Cartesian product always exist when needed, but using the axioms of set theory one can explicitly construct these objects (Ex. I.3.5.2).

Rmk. I.3.5.8. One can show that $\prod_{1\leq i\leq n} X_i$ is indeed a set. Indeed, from the power set axiom we can consider the set of all functions $i \mapsto x_i$ from the domain $\{1 \leq i \leq n\}$ to the codomain $\bigcup_{1\leq i\leq n} X_i$, and then we can restrict using the axiom of specification to restrict to those functions $i \mapsto x_i$ for which $x_i \in X_i$ for all $1 \leq i \leq n$. One can generalize this construction to infinite Cartesian products, see Def. I.8.4.1.

Note. Strictly speaking, the sets $X_1 \times X_2 \times X_3$, $(X_1 \times X_2) \times X_3$, and $X_1 \times (X_2 \times X_3)$ are distinct. However, they are clearly very related to each other (for instance, there are obvious bijections between any two of the three sets), and it is common in practice to neglect the minor distinctions between these sets and pretend that they are in fact equal. Thus, a function $f : X_1 \times X_2 \times X_3 \to Y$ can be thought of as a function of one variable $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$, or as a function of three variables $x_1 \in X_1$, $x_2 \in X_2$, $x_3 \in X_3$, or as a function of two variables $x_1 \in X_1$, $(x_2, x_3) \in X_2 \times X_3$, and so forth; we will not bother to distinguish between these different perspectives.

Rmk. I.3.5.10. An ordered n-tuple $x_1, \ldots, x_n$ of objects is also called an *ordered sequence* of n elements, or a *finite sequence* for short. In Ch. I.5 we shall also introduce the very useful concept of an *infinite sequence*.

E.g. I.3.5.11. If x is an object, then (x) is a 1-tuple, which we shall identify with x itself (even though the two are, strictly speaking, not the same object). Then if X_1 is any set, then the Cartesian product $\prod_{1\leq i\leq 1} X_i$ is just X_1. Also, the *empty Cartesian product* $\prod_{1\leq i\leq 0} X_i$ gives, not the empty set $\{\}$, but rather the singleton set $\{()\}$ whose only element is the 0-*tuple* $()$, also known as the *empty tuple*.

If n is a natural number, we often write X^n as shorthand for the n-fold Cartesian product $X^n := \prod_{1\leq i\leq n} X$. Thus, X^1 is essentially the same set as X (if we ignore the distinction between an object x and the 1-tuple (x)), while X^2 is the Cartesian product $X \times X$. The set X^0 is a singleton set $\{()\}$.

Proof of E.g. I.3.5.11. First, we show that $\prod_{1\leq i\leq 1} X_i = X_1$. This is true since

$$\prod_{1\leq i\leq 1} X_i = \{(x) : x \in X_1\} = \{x : x \in X_1\} = X_1.$$

Next we show that $(x_i)_{1\leq i\leq 0} = ()$. By Ex. I.3.5.2, we see that $x : \{i \in \mathbb{N} : 1 \leq i \leq 0\} \to Y$ is a surjective function, where Y is an arbitrary set. Clearly, the domain of x is the empty set.

Since the i^{th} component of x and $()$ do not exist, we see that the statement "for $1 \leq i \leq n$, the i^{th} component of x and $()$ are the same" is vacuously true for arbitrary natural number n. Thus, we have $(x_i)_{1 \leq i \leq 0} = ()$.

Next we show that $\prod_{1 \leq i \leq 0} X_i = \{()\}$. This is true since

$$\begin{aligned}
\prod_{1 \leq i \leq 0} X_i &= \{(x_i)_{1 \leq i \leq 0} : x_i \in X_i \text{ for all } 1 \leq i \leq 0\} && \text{(by Def. I.3.5.7)} \\
&= \{() : x_i \in X_i \text{ for all } 1 \leq i \leq 0\} && \text{(from the proof above)} \\
&= \{()\}. && \text{(by Ax. I.3.2)}
\end{aligned}$$

Next we show that $X^1 = X$. This is true since

$$\begin{aligned}
X^1 &= \prod_{1 \leq i \leq 1} X && \text{(by definition)} \\
&= X. && \text{(from the proof above)}
\end{aligned}$$

Finally we show that $X^0 = \{()\}$. This is true since

$$\begin{aligned}
X^0 &= \prod_{1 \leq i \leq 0} X && \text{(by definition)} \\
&= \{()\}. && \text{(from the proof above)}
\end{aligned}$$

□

Lem. I.3.5.12 (Finite choice)**.** Let $n \geq 1$ be a natural number, and for each natural number $1 \leq i \leq n$, let X_i be a non-empty set. Then there exists an n-tuple $(x_i)_{1 \leq i \leq n}$ such that $x_i \in X_i$ for all $1 \leq i \leq n$. In other words, if each X_i is non-empty, then the set $\prod_{1 \leq i \leq n} X_i$ is also non-empty.

Proof of Lem. I.3.5.12. We induct on n (starting with the base case $n = 1$; the claim is also vacuously true with $n = 0$ but is not particularly interesting in that case). When $n = 1$ the claim follows from Lem. I.3.1.6. Now suppose inductively that the claim has already been proven for some n; we will now prove it for $n{+\!+}$. Let $X_1, \ldots, X_{n++}$ be a collection of non-empty sets. By the induction hypothesis, we can find an n-tuple $(x_i)_{1 \leq i \leq n}$ such that $x_i \in X_i$ for all $1 \leq i \leq n$. Also, since X_{n++} is non-empty, by Lem. I.3.1.6 we may find an object a such that $a \in X_{n++}$. If we thus define the $n{+\!+}$-tuple $(y_i)_{1 \leq i \leq n++}$ by setting $y_i := x_i$ when $1 \leq i \leq n$ and $y_i := a$ when $i = n{+\!+}$ it is clear that $y_i \in X_i$ for all $1 \leq i \leq n{+\!+}$, thus closing the induction. □

Rmk. I.3.5.13. It is intuitively plausible that this lemma should be extended to allow for an infinite number of choices, but this cannot be done automatically; it requires an additional axiom, the *axiom of choice*. See Sec. I.8.4.

— Exercises —

Ex. I.3.5.1. (a) Suppose we *define* the ordered pair (x, y) for any objects x and y by the formula $(x, y) := \{\{x\}, \{x, y\}\}$ (thus using several applications of Ax. I.3.3). Show that such a definition indeed obeys the Def. I.3.5.1. Thus, this definition can be validly used as a definition of an ordered pair.

(b) For an additional challenge, show that the alternate definition $(x, y) := \{x, \{x, y\}\}$ also verifies Def. I.3.5.1 and is thus also an acceptable definition of ordered pair.

(c) Show that regardless of the definition of ordered pair, the Cartesian product $X \times Y$ is a set.

Proof of Ex. I.3.5.1(a). Suppose that x, x', y, y' are objects. By definition, we have

$$(x, y) = \{\{x\}, \{x, y\}\}$$
$$(x', y') = \{\{x'\}, \{x', y'\}\}.$$

Then we have

$$\begin{aligned}
&\begin{cases} x = x' \\ y = y' \end{cases} \\
\iff &\begin{cases} x = x' \\ (x = y = y' = x') \vee (x \neq y = y' \neq x') \end{cases} \\
\iff &\begin{cases} \{x\} = \{x'\} \\ \{x, y\} = \{x', y'\} \end{cases} && \text{(by Ax. I.3.3)} \\
\iff &\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\} && \text{(by Ax. I.3.3)} \\
\iff &(x, y) = (x', y'). && \text{(by Ex. I.3.5.1(a))}
\end{aligned}$$

Thus, Ex. I.3.5.1(a) is a valid definition of ordered pairs. □

Proof of Ex. I.3.5.1(b). Suppose that x, x', y, y' are objects. By definition, we have

$$(x, y) = \{x, \{x, y\}\}$$
$$(x', y') = \{x', \{x', y'\}\}.$$

Then we have

$$\begin{aligned}
&\begin{cases} x = x' \\ y = y' \end{cases} \\
\iff &\begin{cases} x = x' \\ (x = y = y' = x') \vee (x \neq y = y' \neq x') \end{cases}
\end{aligned}$$

$$\begin{aligned}
&\iff \begin{cases} x = x' \\ \{x, y\} = \{x', y'\} \end{cases} && \text{(by Ax. I.3.3)} \\
&\iff \{x, \{x, y\}\} = \{x', \{x', y'\}\} && \text{(by Ax. I.3.3)} \\
&\iff (x, y) = (x', y'). && \text{(by Ex. I.3.5.1(b))}
\end{aligned}$$

Thus, Ex. I.3.5.1(b) is a valid definition of ordered pairs. □

Proof of Ex. I.3.5.1(c). We use Ex. I.3.5.1(a) as in the definition of ordered pairs. Suppose that X, Y are sets. For each $x \in X$, we can use Ax. I.3.3 to create singleton set $\{x\}$. Using Ax. I.3.3 again, we can create pair sets $\{x, y\}$ and $\{\{x\}, \{x, y\}\}$ for each $x \in X$ and each $y \in Y$. Since Y is a set, each object in Y is uniquely identified (Def. I.3.1.1). Thus, for each $x \in X$ and each $y \in Y$, the statement "$P_y(x, y') \coloneqq y' = y$" is only true for one $y' \in Y$, namely y. Thus, for each $y \in Y$, we can use Ax. I.3.6 to create the set S_y using P_y:

$$\begin{aligned}
S_y &= \big\{\{\{x\}, \{x, y'\}\} : P_y(x, y') \text{ is true for some } x \in X\big\} && \text{(by Ax. I.3.6)} \\
&= \{\{\{x\}, \{x, y\}\} : x \in X\} \\
&= \{(x, y) : x \in X\}. && \text{(by Ex. I.3.5.1).}
\end{aligned}$$

By Ax. I.3.11, we see that $\bigcup_{y \in Y} S_y = \{(x, y) : x \in X, y \in Y\}$, which equals to $X \times Y$ (Def. I.3.5.4). Thus, $X \times Y$ is a set. □

Ex. I.3.5.2. Suppose we *define* an ordered n-tuple to be a surjective function

$$x : \{i \in \mathbb{N} : 1 \leq i \leq n\} \to X$$

whose codomain is some arbitrary set X (so different ordered n-tuples are allowed to have different codomains); we then write x_i for $x(i)$, and also write x as $(x_i)_{1 \leq i \leq n}$. Using this definition, verify that we have $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$ iff $x_i = y_i$ for all $1 \leq i \leq n$. Also, show that if $(X_i)_{1 \leq i \leq n}$ are an ordered n-tuple of sets, then the Cartesian product, as defined in Def. I.3.5.7, is indeed a set. (Technically, this construction of ordered n-tuple is not compatible with the construction of ordered pair in Ex. I.3.5.1, but this does not cause a difficulty in practice; for instance, one can use the definition of an ordered 2-tuple here to replace the construction in Ex. I.3.5.1, or one can make a rather pedantic distinction between an ordered 2-tuple and an ordered pair in one's mathematical arguments.)

Proof of Ex. I.3.5.2. First, we show that Ex. I.3.5.2 is a valid definition of n-tuple. Let $n \in \mathbb{N}$, let X be a set, and let $I = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. Let $x : I \to X, y : I \to X$ be two surjective functions. By definition, we have $x = (x_i)_{1 \leq i \leq n}$ and $y = (y_i)_{1 \leq i \leq n}$. Then we have

$$\begin{aligned}
&(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n} \\
&\iff x = y && \text{(by Ex. I.3.5.2)} \\
&\iff \forall i \in I, x(i) = y(i) && \text{(by Def. I.3.3.1)}
\end{aligned}$$

$$\iff \forall i \in I, x_i = y_i. \qquad \text{(by Ex. I.3.5.2)}$$

Thus, Ex. I.3.5.2 is a valid definition of n-tuple.

Now we show that $\prod_{i=1}^{n} X_i$ defined in Def. I.3.5.7 is indeed a set. Suppose that $(X_i)_{1\le i\le n}$ is an ordered n-tuple of sets. Let $I = \{i \in \mathbb{N} : 1 \le i \le n\}$. By definition, we have a surjective function $X : I \to Y$, where Y is a set, and $X(i) = X_i \in Y$ for each $i \in I$. Then we can create the following set

$$\begin{aligned}
\bigcup\{X(i) : i \in I\} &= \bigcup_{i\in I} X(i) && \text{(by Ax. I.3.11)}\\
&= \bigcup_{i\in I} X_i. && \text{(by Ex. I.3.5.2)}
\end{aligned}$$

By Ex. I.3.4.7, there exists a set of all partial function with domain I and codomain $\bigcup_{i\in I} X_i$, and we denote this set A. By Ax. I.3.5, there exists a set $B = \{x \in A : \forall i \in I, x_i \in X_i\}$. Clearly, every $x \in B$ has the form $x = (x_i)_{1\le i\le n}$. And every n-tuple $(x_i)_{1\le i\le n}$ with $x_i \in X_i$ for all $i \in I$ is also in B, thanks to Ex. I.3.4.7. Thus, by Def. I.3.5.7, we see that $B = \prod_{i=1}^{n} X_i$. Therefore, $\prod_{i=1}^{n} X_i$ is indeed a set. $\square$

Ex. I.3.5.3. Show that the definitions of equality for ordered pair and ordered n-tuple are consistent with the reflexivity, symmetry, and transitivity axioms in the sense that if these axioms of equality are already assumed to hold for the individual components x, y of an ordered pair (x, y), then they hold for an ordered pair itself.

Proof of Ex. I.3.5.3. Let $(x, y), (x', y'), (x'', y'')$ be ordered pairs. We first show that Def. I.3.5.1 is reflexive. Since

$$\begin{aligned}
&(x = x) \wedge (y = y)\\
\implies &(x, y) = (x, y), && \text{(by Def. I.3.5.1)}
\end{aligned}$$

we see that Def. I.3.5.1 is reflexive.

Next we show that Def. I.3.5.1 is symmetry. Since

$$\begin{aligned}
&(x, y) = (x', y')\\
\iff &(x = x') \wedge (y = y') && \text{(by Def. I.3.5.1)}\\
\iff &(x' = x) \wedge (y' = y)\\
\iff &(x', y') = (x, y), && \text{(by Def. I.3.5.1)}
\end{aligned}$$

we see that Def. I.3.5.1 is symmetry.

Next we show that Def. I.3.5.1 is transitive. Since

$$
\begin{aligned}
&((x,y)=(x',y')) \wedge ((x',y')=(x'',y'')) \\
\Longrightarrow\ &(x=x') \wedge (y=y') \wedge (x'=x'') \wedge (y'=y'') && \text{(by Def. I.3.5.1)} \\
\Longrightarrow\ &(x=x'') \wedge (y=y'') \\
\Longrightarrow\ &(x,y)=(x'',y''), && \text{(by Def. I.3.5.1)}
\end{aligned}
$$

we see that Def. I.3.5.1 is transitive.

Let $i, n \in \mathbb{N}$, and let $(x_i)_{1\le i\le n}, (y_i)_{1\le i\le n}, (z_i)_{1\le i\le n}$ be ordered n-tuples. Next we show that Def. I.3.5.7 is reflexive. Since

$$
\begin{aligned}
&\forall 1\le i\le n, x_i=x_i \\
\Longrightarrow\ &(x_i)_{1\le i\le n}=(x_i)_{1\le i\le n}, && \text{(by Def. I.3.5.7)}
\end{aligned}
$$

we see that Def. I.3.5.7 is reflexive.

Next we show that Def. I.3.5.7 is symmetry. Since

$$
\begin{aligned}
&(x_i)_{1\le i\le n}=(y_i)_{1\le i\le n} \\
\Longleftrightarrow\ &\forall 1\le i\le n, x_i=y_i && \text{(by Def. I.3.5.7)} \\
\Longleftrightarrow\ &\forall 1\le i\le n, y_i=x_i \\
\Longleftrightarrow\ &(y_i)_{1\le i\le n}=(x_i)_{1\le i\le n}, && \text{(by Def. I.3.5.7)}
\end{aligned}
$$

we see that Def. I.3.5.7 is symmetry.

Finally we show that Def. I.3.5.7 is transitive. Since

$$
\begin{aligned}
&((x_i)_{1\le i\le n}=(y_i)_{1\le i\le n}) \wedge ((y_i)_{1\le i\le n}=(z_i)_{1\le i\le n}) \\
\Longrightarrow\ &\forall 1\le i\le n, (x_i=y_i) \wedge (y_i=z_i) && \text{(by Def. I.3.5.7)} \\
\Longrightarrow\ &\forall 1\le i\le n, x_i=z_i \\
\Longrightarrow\ &(x_i)_{1\le i\le n}=(z_i)_{1\le i\le n}, && \text{(by Def. I.3.5.7)}
\end{aligned}
$$

we see that Def. I.3.5.7 is transitive. □

Ex. I.3.5.4. Let A, B, C be sets. Show that $A\times(B\cup C)=(A\times B)\cup(A\times C)$, that $A\times(B\cap C)=(A\times B)\cap(A\times C)$, and that $A\times(B\setminus C)=(A\times B)\setminus(A\times C)$.

Proof of Ex. I.3.5.4. We first show that $A\times(B\cup C)=(A\times B)\cup(A\times C)$. Since

$$
\begin{aligned}
&(x,y)\in A\times(B\cup C) \\
\Longleftrightarrow\ &(x\in A) \wedge (y\in B\cup C) && \text{(by Def. I.3.5.4)} \\
\Longleftrightarrow\ &(x\in A) \wedge ((y\in B) \vee (y\in C)) && \text{(by Ax. I.3.4)} \\
\Longleftrightarrow\ &((x\in A) \wedge (y\in B)) \vee ((x\in A) \wedge (y\in C))
\end{aligned}
$$

$$
\begin{aligned}
&\iff ((x,y) \in A \times B) \vee ((x,y) \in A \times C) && \text{(by Def. I.3.5.4)}\\
&\iff (x,y) \in (A \times B) \cup (A \times C), && \text{(by Ax. I.3.4)}
\end{aligned}
$$

we have $A \times (B \cup C) = (A \times B) \cup (A \times C)$ by Def. I.3.1.4.

Next we show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. Since

$$
\begin{aligned}
&(x,y) \in A \times (B \cap C)\\
\iff &(x \in A) \wedge (y \in B \cap C) && \text{(by Def. I.3.5.4)}\\
\iff &(x \in A) \wedge ((y \in B) \wedge (y \in C)) && \text{(by Def. I.3.1.23)}\\
\iff &((x \in A) \wedge (y \in B)) \wedge ((x \in A) \wedge (y \in C))\\
\iff &((x,y) \in A \times B) \wedge ((x,y) \in A \times C) && \text{(by Def. I.3.5.4)}\\
\iff &(x,y) \in (A \times B) \cap (A \times C), && \text{(by Def. I.3.1.23)}
\end{aligned}
$$

we have $A \times (B \cap C) = (A \times B) \cap (A \times C)$ by Def. I.3.1.4.

Finally we show that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$. Since

$$
\begin{aligned}
&(x,y) \in A \times (B \setminus C)\\
\iff &(x \in A) \wedge (y \in B \setminus C) && \text{(by Def. I.3.5.4)}\\
\iff &(x \in A) \wedge ((y \in B) \wedge (y \notin C)) && \text{(by Def. I.3.1.27)}\\
\iff &((x \in A) \wedge (y \in B)) \wedge ((x \in A) \wedge (y \notin C))\\
\iff &((x,y) \in A \times B) \wedge ((x,y) \notin A \times C) && \text{(by Def. I.3.5.4)}\\
\iff &(x,y) \in (A \times B) \setminus (A \times C), && \text{(by Def. I.3.1.27)}
\end{aligned}
$$

we have $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$ by Def. I.3.1.4. $\square$

Ex. I.3.5.5. Let A, B, C, D be sets. Show that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. Is it true that $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$? Is it true that $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$?

Proof of Ex. I.3.5.5. We first show that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. Since

$$
\begin{aligned}
&(x,y) \in (A \times B) \cap (C \times D)\\
\iff &((x,y) \in A \times B) \wedge ((x,y) \in C \times D) && \text{(by Def. I.3.1.23)}\\
\iff &((x \in A) \wedge (y \in B)) \wedge ((x \in C) \wedge (y \in D)) && \text{(by Def. I.3.5.4)}\\
\iff &((x \in A) \wedge (x \in C)) \wedge ((y \in B) \wedge (y \in D))\\
\iff &(x \in A \cap C) \wedge (y \in B \cap D) && \text{(by Def. I.3.1.23)}\\
\iff &(x,y) \in (A \cap C) \times (B \cap D), && \text{(by Def. I.3.5.4)}
\end{aligned}
$$

we have $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ by Def. I.3.1.4.

We do not have $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$. Let $A = \{1\}, B = \{2\}, C = \{3\}, D = \{4\}$. Then we have

$$
(A \times B) \cup (C \times D) = \{(1,2)\} \cup \{(3,4)\} \qquad \text{(by Def. I.3.5.4)}
$$

$$
\begin{aligned}
&= \{(1,2),(3,4)\}. && \text{(by Ax. I.3.4)}\\
(A \cup C) \times (B \cup D) &= \{1,3\} \times \{2,4\} && \text{(by Ax. I.3.4)}\\
&= \{(1,2),(1,4),(3,2),(3,4)\}. && \text{(by Def. I.3.5.4)}
\end{aligned}
$$

Clearly, $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$ by Def. I.3.1.4.

We do not have $(A \times B) \setminus (C \times D) = (A \setminus C) \times (B \setminus D)$. Let $A = \{1,2\}, B = \{3,4\}, C = \{1\}, D = \{3\}$. Then we have

$$
\begin{aligned}
(A \times B) \setminus (C \times D) &= \{(1,3),(1,4),(2,3),(2,4)\} \setminus \{(1,3)\} && \text{(by Def. I.3.5.4)}\\
&= \{(1,4),(2,3),(2,4)\}. && \text{(by Def. I.3.1.27)}\\
(A \setminus C) \times (B \setminus D) &= \{2\} \times \{4\} && \text{(by Def. I.3.1.27)}\\
&= \{(2,4)\}. && \text{(by Def. I.3.5.4)}
\end{aligned}
$$

Clearly, $(A \times B) \setminus (C \times D) \neq (A \setminus C) \times (B \setminus D)$ by Def. I.3.1.4. $\square$

Ex. I.3.5.6. Let A, B, C, D be non-empty sets. Show that $A \times B \subseteq C \times D$ iff $A \subseteq C$ and $B \subseteq D$, and that $A \times B = C \times D$ iff $A = C$ and $B = D$. What happens if the hypotheses that the A, B, C, D are all non-empty are removed?

Proof of Ex. I.3.5.6. We first show that if A, B, C, D are non-empty sets, then $A \times B \subseteq C \times D \iff (A \subseteq C) \wedge (B \subseteq D)$. This is true since

$$
\begin{aligned}
& A \times B \subseteq C \times D \\
\iff & ((x,y) \in A \times B \implies (x,y) \in C \times D) && \text{(by Def. I.3.1.15)}\\
\iff & (((x \in A) \wedge (y \in B)) \implies ((x \in C) \wedge (y \in D))) && \text{(by Def. I.3.5.4)}\\
\iff & (x \in A \implies x \in C) \wedge (y \in B \implies y \in D) && (A \neq \emptyset \wedge B \neq \emptyset)\\
\iff & (A \subseteq C) \wedge (B \subseteq D). && \text{(by Def. I.3.1.15)}
\end{aligned}
$$

This statement is not true when $(A = \emptyset) \vee (B = \emptyset)$. For example, if $A = \{1\}$ and $B = C = D = \emptyset$, then $A \times B = \{()\} \subseteq C \times D = \{()\}$, but $A \not\subseteq C$.

Next we show that if A, B, C, D are non-empty sets, then $A \times B = C \times D \iff (A = C) \wedge (B = D)$. This is true since

$$
\begin{aligned}
& A \times B = C \times D \\
\iff & ((x,y) \in A \times B \iff (x,y) \in C \times D) && \text{(by Def. I.3.1.4)}\\
\iff & (((x \in A) \wedge (y \in B)) \iff ((x \in C) \wedge (y \in D))) && \text{(by Def. I.3.5.4)}\\
\iff & (x \in A \iff x \in C) \wedge (y \in B \iff y \in D) && (A, B, C, D \text{ are non-empty})\\
\iff & (A = C) \wedge (B = D). && \text{(by Def. I.3.1.4)}
\end{aligned}
$$

This statement is not true when $((A = \emptyset) \wedge (C = \emptyset)) \vee ((B = \emptyset) \wedge (D = \emptyset))$. For example, if $A = \{1\}$, $C = \{2\}$ and $B = D = \emptyset$, then $A \times B = \{()\} = C \times D$, but $A \neq C$. $\square$

Ex. I.3.5.7. Let X, Y be sets, and let $\pi_{X\times Y\to X} : X \times Y \to X$ and $\pi_{X\times Y\to Y} : X \times Y \to Y$ be the maps $\pi_{X\times Y\to X}(x,y) := x$ and $\pi_{X\times Y\to Y}(x,y) := y$; these maps are known as the *co-ordinate functions* on $X\times Y$. Show that for any functions $f : Z \to X$ and $g : Z \to Y$, there exists a unique function $h : Z \to X \times Y$ such that $\pi_{X\times Y\to X} \circ h = f$ and $\pi_{X\times Y\to Y} \circ h = g$. (Compare this to Ex. I.3.3.8(d), and to Ex. I.3.1.7.) This function h is known as the *direct sum* of f and g and is denoted $h = f \oplus g$.

Proof of Ex. I.3.5.7. We first show the existence of such function h. Suppose that X, Y, Z are sets, and $f : Z \to X, g : Z \to Y$ are functions. Let $\pi_{X\times Y\to X} : X \times Y \to X, \pi_{X\times Y\to Y} : X \times Y \to Y$ be functions where $\pi_{X\times Y\to X}(x, y) = x$ and $\pi_{X\times Y\to Y}(x, y) = y$. Both $\pi_{X\times Y\to X}$ and $\pi_{X\times Y\to Y}$ are well-defined by Ax. I.3.6. We now define a function $h : Z \to X \times Y$ where

$$\forall z \in Z, h(z) = (f(z), g(z)).$$

To show that h is well-defined, by Def. I.3.3.1, we have to show that h pass the vertical line test. Since f, g are functions, by Def. I.3.3.1, we know that for each $z \in Z$, $f(z)$ and $g(z)$ are unique objects in X and Y, respectively. Thus, by Ex. I.3.5.3, $(f(z), g(z)) \in X \times Y$ is unique for each $z \in Z$. Therefore, $h(z) \in X \times Y$ is unique for each $z \in Z$, Thus, h is well-defined. Now we have

$$\begin{aligned}
\forall z \in Z, (\pi_{X\times Y\to X} \circ h)(z) &= \pi_{X\times Y\to X}(h(z)) && \text{(by Def. I.3.3.10)}\\
&= \pi_{X\times Y\to X}(f(z), g(z))\\
&= f(z).\\
\forall z \in Z, (\pi_{X\times Y\to Y} \circ h)(z) &= \pi_{X\times Y\to Y}(h(z)) && \text{(by Def. I.3.3.10)}\\
&= \pi_{X\times Y\to Y}(f(z), g(z))\\
&= g(z).
\end{aligned}$$

Thus, by Def. I.3.3.7, we have $\pi_{X\times Y\to X} \circ h = f$ and $\pi_{X\times Y\to Y} \circ h = g$.

Now we show the uniqueness of such function h. Suppose that there exists another function $h' : Z \to X \times Y$ such that $\pi_{X\times Y\to X} \circ h' = f$ and $\pi_{X\times Y\to Y} \circ h' = g$. Then we have

$$\begin{aligned}
&\forall z \in Z, \begin{cases} f(z) = \big(\pi_{X\times Y\to X} \circ h'\big)(z) = \pi_{X\times Y\to X}\big(h'(z)\big)\\ g(z) = \big(\pi_{X\times Y\to Y} \circ h'\big)(z) = \pi_{X\times Y\to Y}\big(h'(z)\big)\end{cases} && \text{(by Def. I.3.3.7 and I.3.3.10)}\\
&\implies \forall z \in Z, h'(z) = (f(z), g(z)) = h(z)\\
&\implies h' = h. && \text{(by Def. I.3.3.7)}
\end{aligned}$$

Thus, h is unique. $\square$

Ex. I.3.5.8. Let $X_1, \ldots, X_n$ be sets. Show that the Cartesian product $\displaystyle\prod_{i=1}^{n} X_i$ is empty iff at least one of the X_i is empty.

Proof of Ex. I.3.5.8. We have

$$\begin{aligned}
\emptyset = \prod_{i=1}^{n} X_i &= \{(x_i)_{1\le i\le n} : x_i \in X_i \text{ for all } 1 \le i \le n\} && \text{(by Def. I.3.5.7)}\\
\iff \exists i \in \mathbb{N} &: (1 \le i \le n) \wedge (\forall x_i, x_i \notin X_i) && \text{(by Def. I.3.5.7)}\\
\iff \exists i \in N &: (1 \le i \le n) \wedge (X_i = \emptyset). && \text{(by Ax. I.3.2)}
\end{aligned}$$

□

Ex. I.3.5.9. Suppose that I and J are two sets, and for all $\alpha \in I$ let A_α be a set, and for all $\beta \in J$ let B_β be a set. Show that $\left(\bigcup_{\alpha\in I} A_\alpha\right) \cap \left(\bigcup_{\beta\in J} B_\beta\right) = \bigcup_{(\alpha,\beta)\in I\times J} (A_\alpha \cap B_\beta)$.

Proof of Ex. I.3.5.9. Since

$$\begin{aligned}
& x \in \left(\bigcup_{\alpha\in I} A_\alpha\right) \cap \left(\bigcup_{\beta\in J} B_\beta\right)\\
\iff & \left(x \in \bigcup_{\alpha\in I} A_\alpha\right) \wedge \left(x \in \bigcup_{\beta\in J} B_\beta\right) && \text{(by Def. I.3.1.23)}\\
\iff & (\exists \alpha \in I : x \in A_\alpha) \wedge (\exists \beta \in J : x \in B_\beta) && \text{(by Ax. I.3.11)}\\
\iff & \exists (\alpha,\beta) \in I \times J : (x \in A_\alpha) \wedge (x \in B_\beta) && \text{(by Def. I.3.5.4)}\\
\iff & \exists (\alpha,\beta) \in I \times J : x \in A_\alpha \cap B_\beta && \text{(by Def. I.3.1.23)}\\
\iff & x \in \bigcup_{(\alpha,\beta)\in I\times J} (A_\alpha \cap B_\beta), && \text{(by Ax. I.3.11)}
\end{aligned}$$

we have $\left(\bigcup_{\alpha\in I} A_\alpha\right) \cap \left(\bigcup_{\beta\in J} B_\beta\right) = \bigcup_{(\alpha,\beta)\in I\times J} (A_\alpha \cap B_\beta)$ by Def. I.3.1.4. □

Ex. I.3.5.10. If $f : X \to Y$ is a function, define the *graph* of f to be the subset of $X \times Y$ defined by $\{(x, f(x)) : x \in X\}$. Show that two functions $f : X \to Y$, $\tilde{f} : X \to Y$ are equal iff they have the same graph. Conversely, if G is any subset of $X \times Y$ with the property that for each $x \in X$, the set $\{y \in Y : (x, y) \in G\}$ has exactly one element (or in other words, G obeys the *vertical line test*), show that there is exactly one function $f : X \to Y$ whose graph is equal to G.

Proof of Ex. I.3.5.10. The first statement is true since

$$\begin{aligned}
& f = \tilde{f}\\
\iff & \forall x \in X, f(x) = \tilde{f}(x) && \text{(by Def. I.3.3.7)}
\end{aligned}$$

$$\iff \forall x \in X, (x, f(x)) = \left(x, \tilde{f}(x)\right) \qquad \text{(by Def. I.3.5.1)}$$
$$\iff \{(x, f(x)) : x \in X\} = \left\{\left(x, \tilde{f}(x)\right) : x \in X\right\}. \qquad \text{(by Ax. I.3.6)}$$

Next we show that there exists a function whose graph is equal to G. For each $x \in X$, we use Ax. I.3.6 to create the set $S_x = \{y \in Y : (x, y) \in G\}$. Define $P(x, y)$ to be the statement "$y \in S_x$" for each $(x, y) \in X \times Y$. Then, by the hypothesis of G, we know that for each $x \in X$, there is only one $y \in Y$ such that $P(x, y)$ is true. Thus, we can use Def. I.3.3.1 to create a function $f : X \to Y$ such that

$$\forall (x, y) \in X \times Y, y = f(x) \iff P(x, y) \iff y \in S_x \iff (x, y) \in G.$$

By definition, the graph of f is $\{(x, f(x)) : x \in X\}$. Thus, we have

$$\forall (x, y) \in X \times Y, (x, y) \in \{(x, f(x)) : x \in X\} \iff y = f(x) \iff (x, y) \in G.$$

By Def. I.3.1.4, this means $G = \{(x, f(x)) : x \in X\}$. Thus, f is a function whose graph is equal to G.

Now we show that f is unique. Suppose that there exists another function $\tilde{f} : X \to Y$ such that the graph of $\tilde{f}$ is equal to G. But then f and $\tilde{f}$ have the same graph, and by the first part of the proof we see that $f = \tilde{f}$. Thus, f is unique. □

Ex. I.3.5.11. Show that Ax. I.3.10 can in fact be deduced from Lem. I.3.4.9 and the other axioms of set theory, and thus Lem. I.3.4.9 can be used as an alternate formulation of the power set axiom.

Proof of Ex. I.3.5.11. Suppose that X, Y are sets. Then by Ex. I.3.5.1(c), $X \times Y$ is a set. Thus, we can use Lem. I.3.4.9 to create a set of subsets of $X \times Y$, i.e., $S_1 = \{S : S \subseteq X \times Y\}$. If $f : X \to Y$ is a function, then by Def. I.3.3.1, every element on X must be assigned exactly one object in Y. Thus, we use Ax. I.3.5 to rule out those sets violating the vertical line test, and we derive the following set

$$S_2 = \{G \in S_1 \mid \forall x \in X, \exists! y \in Y : (x, y) \in G\}.$$

By Ex. I.3.5.10, we see that every element $G \in S_2$ is a graph, and we know that there exists exactly one function $f : X \to Y$ whose graph is G. Thus, we can use Ax. I.3.6 to create the following set

$$S_3 = \{f : X \to Y \mid \text{the graph of } f \text{ is in } S_2\}.$$

Now we claim that every function $f : X \to Y$ is an element of S_3, and thus by Ax. I.3.10 we have $S_3 = X^Y$, and X^Y is indeed a set. But by Ex. I.3.5.10, every function with domain X and codomain Y is uniquely identified by one graph and vice versa. Thus, the claim is true. □

Ex. I.3.5.12. Let X be an arbitrary set, let $f : \mathbb{N} \times X \to X$ be a function, and let $c \in X$. Show that there exists a function $a : \mathbb{N} \to X$ such that

$$a(0) = c$$

and

$$a(n{+}{+}) = f(n, a(n)) \text{ for all } n \in \mathbb{N},$$

and furthermore that this function is unique. For an additional challenge, prove this result without using any properties of the natural numbers other than the Peano axioms directly.

Proof of Ex. I.3.5.12. For each $N \in \mathbb{N}$, let $P(N)$ be the statement "there exists an unique function $a_N : \{n \in \mathbb{N} : n \leq N\} \to X$, such that $a_N(0) = c$, and $a_N(n{+}{+}) = f(n, a_N(n))$ for all $n \in \mathbb{N}$ such that $n < N$." We induct on N to prove that $P(N)$ is true for all $N \in \mathbb{N}$.

For $N = 0$, we define $a_0 : \{0\} \to X$ to be the function $a_0(0) = c$. By A.Cor. I.2.2.4, we see that $\{0\} = \{n \in \mathbb{N} : n \leq 0\}$, and the statement "$a_0(n{+}{+}) = f(n, a_0(n))$ for all $n \in \mathbb{N}$ such that $n < 0$" is vacuously true. Thus, to ensure that the base case holds, we only need to show that a_0 is unique. Let $b_0 : \{0\} \to X$ be another function satisfying $b_0(0) = c$, and $b_0(n{+}{+}) = f(n, b_0(n))$ for all $n \in \mathbb{N}$ such that $n < 0$. Since 0 is the only element in $\{0\}$, and $b_0(0) = c = a_0(0)$, we must have $b_0 = a_0$ by Def. I.3.3.7. Thus, a_0 is unique, and the base case holds.

Suppose inductively that $P(N)$ is true for some $N \in \mathbb{N}$. We want to show that $P(N{+}{+})$ is true. By the induction hypothesis, there exists an unique function $a_N : \{n \in \mathbb{N} : n \leq N\} \to X$, such that $a_N(0) = c$, and $a_N(n{+}{+}) = f(n, a_N(n))$ for all $n \in \mathbb{N}$ such that $n < N$. Now we define a function $a_{N++} : \{n \in \mathbb{N} : n \leq N{+}{+}\} \to X$ as follow:

$$\forall n \in \mathbb{N} \text{ such that } n \leq N{+}{+}, a_{N++}(n) = \begin{cases} a_N(n) & \text{if } n \neq N{+}{+} \\ f(m, a_N(m)) & \text{if } n = N{+}{+} \text{ and } m{+}{+} = n \end{cases}.$$

Since each $n \in \{n \in \mathbb{N} : n \leq N{+}{+}\}$ is assigned with an unique object in X, we see that a_{N++} is well-defined. Since $0 \neq N{+}{+}$ (Ax. I.2.3), we have $a_{N++}(0) = a_N(0) = c$. We claim that we must have $a_{N++}(n) = f(n, a_{N++}(n))$ for each natural number $n < N{+}{+}$. So let $n \in N$ where $n < N{+}{+}$. We split into two cases:

- If $n{+}{+} \neq N{+}{+}$, then we have

$$\begin{aligned} a_{N++}(n{+}{+}) &= a_N(n{+}{+}) && (n{+}{+} \neq N{+}{+}) \\ &= f(n, a_N(n)) && \text{(by the induction hypothesis)} \\ &= f(n, a_{N++}(n)). && (n < N{+}{+}) \end{aligned}$$

- If $n{+}{+} = N{+}{+}$, then we have

$$\begin{aligned} a_{N++}(n{+}{+}) &= f(n, a_N(n)) && (n{+}{+} = N{+}{+}) \\ &= f(n, a_{N++}(n)). && (n < N{+}{+}) \end{aligned}$$

From all cases above, we see that $a_{N++}(n{+}{+}) = f(n, a_{N++}(n))$. Thus, our claim is true. To close the induction, we need to show that a_{N++} is unique. So suppose that there exists another function $b_{N++} : \{n \in \mathbb{N} : n \leq N{+}{+}\} \to X$, where $b_{N++}(0) = c$, and $b_{N++}(n{+}{+}) = f(n, b_{N++}(n))$ for all $n \in \mathbb{N}$ such that $n < N$. Clearly, we have $b_{N++}(0) = c = a_{N++}(0)$. If we have show that $b_{N++}(n) = a_{N++}(n)$ for some $n \in \mathbb{N}$ and $n < N{+}{+}$, then we see that

$$b_{N++}(n{+}{+}) = f(n, b_{N++}(n)) = f(n, a_{N++}(n)) = a_{N++}(n{+}{+}).$$

Thus, we have $b_{N++}(n) = a_{N++}(n)$ for all $n \in \{n \in \mathbb{N} : n \leq N{+}{+}\}$. By Def. I.3.1.4, this means $b_{N++} = a_{N++}$, and this closes the induction.

Now we define $a : \mathbb{N} \to X$ as follow:

$$\forall n \in \mathbb{N}, a(n) = a_n(n).$$

Since $P(n)$ is true for all $n \in \mathbb{N}$, we know that a_n is well-defined, and $a_n(n)$ is unique for each $n \in \mathbb{N}$. Thus, a is well-defined. Clearly, we have $a(0) = a_0(0) = c$. We claim that $a(n{+}{+}) = f(n, a(n))$ for all $n \in \mathbb{N}$. This is true since

$$\begin{aligned}
\forall n \in \mathbb{N}, a(n{+}{+}) &= a_{n++}(n{+}{+}) \\
&= f(n, a_{n++}(n)) \\
&= f(n, a_n(n)) && (n < n{+}{+}) \\
&= f(n, a(n)).
\end{aligned}$$

Now we show that a is unique. Suppose there exists another $b : \mathbb{N} \to X$, where $b(0) = c$, and $b(n{+}{+}) = f(n, b(n))$ for all $n \in \mathbb{N}$. Clealy, $b(0) = c = a(0)$. If $b(n) = a(n)$ is true for some $n \in \mathbb{N}$, then we have

$$b(n{+}{+}) = f(n, b(n)) = f(n, a(n)) = a(n{+}{+}).$$

Thus, by Ax. I.2.5, we know that $b(n) = a(n)$ for all $n \in \mathbb{N}$. By Def. I.3.1.4, this means $b = a$. Thus, a is unique.

Now we prove the additional challenge. We claim that for every natural number $N \in \mathbb{N}$, there exists a unique pair A_N, B_N of subsets of $\mathbb{N}$ which obeys the following properties:

(a) $A_N \cap B_N = \emptyset$;

(b) $A_N \cup B_N = \mathbb{N}$;

(c) $0 \in A_N$;

(d) $N{+}{+} \in B_N$;

(e) Whenever $n \in B_N$, we have $n{+}{+} \in B_N$;

(f) Whenever $n \in A_N$ and $n \neq N$, we have $n{+}{+} \in A_N$.

We induct on N to prove the claim.

For $N = 0$, let $A_0 = \{0\}$, and let $B_0 = \mathbb{N} \setminus A_0$. By Prop. I.3.1.28(g), we see that (a)(b) holds for A_0, B_0. By Ax. I.3.3, we have $0 \in A_0$ and $0{+\!+} = 1 \notin A_0$. Thus, $1 \in B_0$. So (c)(d) holds for A_0, B_0. If $n \in B_0$, then $n \in \mathbb{N}$ and $n{+\!+} \neq 0$ (Ax. I.2.4). Thus, we have $n{+\!+} \notin A_0$, and therefore $n{+\!+} \in B_0$. So (e) holds for A_0, B_0. Since there is no natural number n satisfying $n \in A_0$ and $n \neq 0$, we see that (f) holds for A_0, B_0. To ensure that the base case holds, we are left to show that A_0, B_0 are unique. So suppose that there exist another pair of sets A_0', B_0' such that (a)–(f) hold. By (c)(d), we know that $0 \in A_0'$ and $1 \in B_0'$. Thus, by (e), we see that $n \in B_0'$ for all $n \in \mathbb{N}$ and $n \geq 1$. Therefore, we have $A_0' = \{0\} = A_0$ by (a). By Ex. I.3.1.9, we have $B_0' = \mathbb{N} \setminus A_0' = \mathbb{N} \setminus A_0 = B_0$. Thus, A_0, B_0 is unique, and the base case holds.

Suppose inductively that, for some natural number N, there exists a unique pair of set A_N, B_N such that (a)–(f) hold. We define $A_{N++} = A_N \cup \{N{+\!+}\}$ and $B_{N++} = B_N \setminus \{N{+\!+}\}$. Since

$$\begin{aligned}
&A_{N++} \cap B_{N++} \\
&= (A_N \cup \{N{+\!+}\}) \cap (B_N \setminus \{N{+\!+}\}) \\
&= (A_N \cap (B_N \setminus \{N{+\!+}\})) \cup (\{N{+\!+}\} \cap (B_N \setminus \{N{+\!+}\})) && \text{(by Prop. I.3.1.28(f))} \\
&= (A_N \cap (B_N \setminus \{N{+\!+}\})) \cup \emptyset && \text{(by Prop. I.3.1.28(g))} \\
&= A_N \cap (B_N \setminus \{N{+\!+}\}) && \text{(by Prop. I.3.1.28(a))} \\
&\subseteq A_n \cap B_N && \text{(by Def. I.3.1.15)} \\
&= \emptyset, && \text{(by the induction hypothesis)}
\end{aligned}$$

we see that $A_{N++} \cap B_{N++} = \emptyset$ by Ax. I.3.2. Thus, (a) is true for A_{N++}, B_{N++}. Since

$$\begin{aligned}
&A_{N++} \cup B_{N++} \\
&= (A_N \cup \{N{+\!+}\}) \cup (B_N \setminus \{N{+\!+}\}) \\
&= A_N \cup (\{N{+\!+}\} \cup (B_N \setminus \{N{+\!+}\})) && \text{(by Prop. I.3.1.28(e))} \\
&= A_N \cup B_N && \text{(by Prop. I.3.1.28(g))} \\
&= \mathbb{N}, && \text{(by the induction hypothesis)}
\end{aligned}$$

we see that (b) is true for A_{N++}, B_{N++}. Since $A_{N++} = A_N \cup \{N{+\!+}\}$, we have $A_N \subseteq A_{N++}$ by Ex. I.3.1.7. By the induction hypothesis, we know that $0 \in A_N$. Thus, by Def. I.3.1.15, we have $0 \in A_{N++}$, and (c) is true for A_{N++}, B_{N++}. By the induction hypothesis, we have $N{+\!+} \in B_N$. Thus, by (e), we see that $(N{+\!+}){+\!+} \in B_N$. Since $B_{N++} = B_N \setminus \{N{+\!+}\}$, we see that $(N{+\!+}){+\!+} \in B_{N++}$. Thus, (d) is true for A_{N++}, B_{N++}. If $n \in B_{N++} = B_N \setminus \{N{+\!+}\}$, then we have $n \in B_N$ and $n \neq N{+\!+}$ by Def. I.3.1.27 and Ax. I.3.3. We must have $n{+\!+} \neq N{+\!+}$, for otherwise we would have $n{+\!+} = N{+\!+}$ and $n = N$ by Ax. I.2.4. But by induction hypotheses, we would have $N{+\!+} = n{+\!+} \in B_N$, a contradiction. Thus, $n \in B_{N++} \implies n{+\!+} \neq N{+\!+}$. Hence,

$$\forall n \in B_{N++}, (n \in B_n) \wedge (n{+\!+} \neq N{+\!+})$$

$$\implies \forall n \in B_{N++}, (n{++} \in B_N) \wedge (n{++} \neq N{++}) \qquad \text{(by the induction hypothesis)}$$
$$\implies \forall n \in B_{N++}, n{++} \in B_N \setminus \{N{++}\} = B_{N++}. \qquad \text{(by Def. I.3.1.27 and Ax. I.3.3)}$$

So (e) is true for A_{N++}, B_{N++}. Since

$$\begin{aligned}
&(n \in A_{N++} = A_N \cup \{N{++}\}) \wedge (n \neq N{++}) \\
\implies &((n \in A_N) \vee (n = N{++})) \wedge (n \neq N{++}) && \text{(by Ax. I.3.3 and I.3.4)} \\
\implies &((n \in A_N) \wedge (n \neq N)) \vee ((n = N{++}) \wedge (n \neq N{++})) \\
\implies &(n \in A_N) \wedge (n \neq N) \\
\implies &n{++} \in A_N && \text{(by the induction hypothesis)} \\
\implies &n{++} \in A_{N++}, && \text{(by Def. I.3.1.15)}
\end{aligned}$$

we see that (f) is true for A_{N++}, B_{N++}. To close the induction, we only need to show that the pair of sets A_{N++}, B_{N++} is unique. So suppose that there exist another pair of sets A'_{N++}, B'_{N++} such that (a)–(f) hold. Then, by (c) and (f), we see that $A'_{N++} = A_{N++}$. Thus, by (b) and Ex. I.3.1.9, we see that $B'_{N++} = B_{N++}$. Thus, the pair A_{N++}, B_{N++} is unique and this closes the induction.

From the construction of A_N we see that $A_N = \{n \in N : 0 \leq N\}$. Thus, we can simply apply A_N to define $a_N : N \to X$ as in the first part of the proof. The rest of the claims follows from the first part of the proof. □

Ex. I.3.5.13. Suppose we have a set $\mathbb{N}'$ of "alternative natural numbers," an "alternative zero" $0'$, and an "alternative increment operation" which takes any alternative natural number $n' \in N$ and returns another alternative natural number $n'{++}' \in \mathbb{N}'$, such that the Peano axioms (Ax. I.2.1 to I.2.5) all hold with the natural numbers, zero, and increment replaced by their alternative counterparts. Show that there exists a bijection $f : \mathbb{N} \to \mathbb{N}'$ from the natural numbers to the alternative natural numbers such that $f(0) = 0'$, and such that for any $n \in \mathbb{N}$ and $n' \in \mathbb{N}'$, we have $f(n) = n'$ iff $f(n{++}) = n'{++}'$.

Proof of Ex. I.3.5.13. Define $g : \mathbb{N} \times \mathbb{N}' \to \mathbb{N}'$ as follow:

$$\forall (n, n') \in \mathbb{N} \times \mathbb{N}', g(n, n') = n'{++}'.$$

By Ax. I.2.4, we see that g pass the vertical line test. Thus, g is well-defined. By Ex. I.3.5.12, there exists a unique function $a : \mathbb{N} \to \mathbb{N}'$ such that

$$a(0) = 0' \quad \text{and} \quad \forall n \in \mathbb{N}, a(n{++}) = g(n, a(n)) = a(n){++}'.$$

First, we show that a is injective. Let $n_1, n_2 \in \mathbb{N}$. We induct on n_1 to show that $n_1 \neq n_2 \implies a(n_1) \neq a(n_2)$. For $n_1 = 0$, we have $n_2 \neq 0$ and $a(n_1) = a(0) = 0'$. Since $n_2 \neq 0$, by Lem. I.2.2.10, there exists an $m \in \mathbb{N}$ such that $m{++} = n_2$. By the definition of a, we see that $a(n_2) = a(m{++}) = a(m){++}'$. By Ax. I.2.3, we know that $a(m){++}' \neq 0'$. Thus, we have $a(n_1) \neq a(n_2)$, and the base case holds. Suppose inductively that $n_1 \neq n_2 \implies a(n_1) \neq a(n_2)$ for some $n_1 \in \mathbb{N}$. We want to show that $n_1{++} \neq n_2 \implies a(n_1{++}) \neq a(n_2)$. We split into two cases:

- If $n_2 = 0$, then we can use the identical proof as in the base case, but we replace (n_1, n_2) with $(n_2, n_1{++})$ to see that $a(n_1{++}) \neq a(n_2)$.
- If $n_2 \neq 0$, then by Lem. I.2.2.10, there exists an $m \in \mathbb{N}$ such that $m{++} = n_2$. Thus,

$$\begin{aligned}
& n_1{++} \neq m{++} = n_2 && \text{(by Lem. I.2.2.10)}\\
\implies & n_1 \neq m && \text{(by Ax. I.2.4)}\\
\implies & a(n_1) \neq a(m) && \text{(by the induction hypothesis)}\\
\implies & a(n_1){++}' \neq a(m){++}' && \text{(by Ax. I.2.4)}\\
\implies & a(n_1{++}) \neq a(m{++}) = a(n_2). && \text{(by the definition of } a\text{)}
\end{aligned}$$

From all cases above, we see that $a(n_1{++}) \neq a(n_2)$. This closes the induction, and we see that a is injective by Def. I.3.3.14.

Next we claim that a is surjective, i.e. (Def. I.3.3.17), for each $n' \in \mathbb{N}'$, there exists an $n \in \mathbb{N}$ such that $a(n) = n'$. Since Ax. I.2.5 holds for $\mathbb{N}'$, we can induct on n' to prove the claim. For $n' = 0'$, we see that $a(0) = 0'$. So the base case holds. Suppose inductively that for some $n' \in \mathbb{N}'$, there exists an $n \in \mathbb{N}$ such that $a(n) = n'$. Then we have

$$\begin{aligned}
a(n{++}) &= a(n){++}' && \text{(by the definition of } a\text{)}\\
&= n'{++}', && \text{(by the induction hypothesis)}
\end{aligned}$$

and the claim is true for $n'{++}'$. This closes the induction, and thus a is surjective. Since a is both injective and surjective, we know that a is bijective by Def. I.3.3.20.

By the definition of a, we see that for all $n, n' \in \mathbb{N} \times \mathbb{N}'$, we have $a(n) = n' \iff a(n{++}) = n'{++}'$. Thus, by setting $f = a$, we are done. $\square$

I.3.6 Cardinality of sets

Note. In Ch. I.2 we defined the natural numbers axiomatically, assuming that they were equipped with a 0 and an increment operation, and assuming five axioms on these numbers. Philosophically, this is quite different from one of our main conceptualizations of natural numbers - that of *cardinality*, or measuring *how many* elements there are in a set. Indeed, the Peano axiom approach treats natural numbers more like *ordinals* than *cardinals*. (The *cardinals* are One, Two, Three, ..., and are used to count how many things there are in a set. The *ordinals* are First, Second, Third, ..., and are used to order a sequence of objects. There is a subtle difference between the two, especially when comparing infinite cardinals with infinite ordinals, but this is beyond the scope of this text.) We paid a lot of attention to what number came *next* after a given number n - which is an operation which is quite natural for ordinals, but less so for cardinals - but did not address the issue of whether these numbers could be used to *count* sets. The purpose of Sec. I.3.6 is to address this issue by noting that the natural numbers *can* be used to count the cardinality of sets, as long as the set is finite.

One way to define cardinality is to say that two sets have the same size if they have the same number of elements, but we have not yet defined what the "number of elements" in a set is. Besides, this runs into problems when a set is infinite.

The right way to define the concept of "two sets having the same size" is not immediately obvious, but can be worked out with some thought. One intuitive reason why two sets have the same size is that one can match the elements of the first set with the elements in the second set in a one-to-one correspondence (Indeed, this is how we first learn to count a set: we correspond the set we are trying to count with another set, such as a set of fingers on your hand). We will use this intuitive understanding as our rigorous basis for "having the same size."

Def. I.3.6.1 (Equal cardinality)**.** We say that two sets X and Y have *equal cardinality* iff there exists a bijection $f : X \to Y$ from X to Y.

Rmk. I.3.6.3. The fact that two sets have equal cardinality does not preclude one of the sets from containing the other. For instance, if X is the set of natural numbers and Y is the set of even natural numbers, then the map $f : X \to Y$ defined by $f(n) := 2n$ is a bijection from X to Y, and so X and Y have equal cardinality, despite Y being a subset of X and seeming intuitively as if it should only have "half" of the elements of X.

Prop. I.3.6.4. Let X, Y, Z be sets. Then X has equal cardinality with X. If X has equal cardinality with Y, then Y has equal cardinality with X. If X has equal cardinality with Y and Y has equal cardinality with Z, then X has equal cardinality with Z.

Proof of Prop. I.3.6.4. We first show that Def. I.3.6.1 is reflexive. Let $f : X \to X$ be a function where $f = x \mapsto x$. By Ax. I.3.6, f is well-defined. Since

$$\forall x_1, x_2 \in X, x_1 \neq x_2 \implies x_1 = f(x_1) \neq f(x_2) = x_2,$$

we know that f is injective (Def. I.3.3.14). Since

$$\forall x \in X, f(x) = x,$$

we know that f is surjective (Def. I.3.3.17). Thus, f is bijective (Def. I.3.3.20), and X has equal cardinality with X (Def. I.3.6.1).

Next we show that Def. I.3.6.1 is symmetric. Suppose that X has equal cardinality with Y. Then there exists a bijective function $f : X \to Y$ (Def. I.3.6.1). Since f is bijective, we know that $f^{-1} : Y \to X$ is also bijective (Ex. I.3.3.6). Thus, Y has equal cardinality with X (Def. I.3.6.1).

Finally we show that Def. I.3.6.1 is transitive. Suppose that X has equal cardinality with Y, and Y has equal cardinality with Z. Then there exist two bijective functions $f : X \to Y$ and $g : Y \to Z$ (Def. I.3.6.1). Since f and g are bijective, we know that $g \circ f : X \to Z$ is also bijective (Ex. I.3.3.7). Thus, X has equal cardinality with Z (Def. I.3.6.1). $\square$

Def. I.3.6.5. Let n be a natural number. A set X is said to have *cardinality* n, iff it has equal cardinality with $\{i \in \mathbb{N} : 1 \leq i \leq n\}$. We also say that X *has* n *elements* iff it has cardinality n.

Rmk. I.3.6.6. One can use the set $\{i \in \mathbb{N} : i < n\}$ instead of $\{i \in \mathbb{N} : 1 \leq i \leq n\}$, since these two sets clearly have equal cardinality.

Proof of Rmk. I.3.6.6. Let $A = \{i \in \mathbb{N} : i < n\}$, and let $B = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. Let $f : A \to B$ be the function defined by $f = n \mapsto n{+\!+}$. By Ax. I.2.4 and Def. I.3.3.14, we see that f is injective. Since $0 \notin B$, by Lem. I.2.2.10, we know that for every $i \in B$, there exists an $j \in \mathbb{N}$ such that $j{+\!+} = i$. Then we have

$$\begin{aligned} & i \in B \\ \implies & j{+\!+} = i \leq n && \text{(by Ax. I.3.5)} \\ \implies & j < n && \text{(by Prop. I.2.2.12(e))} \\ \implies & j \in A. && \text{(by Ax. I.3.5)} \end{aligned}$$

Thus, $f(j)$ is well-defined by Def. I.3.3.1, and $f(j) = j{+\!+} = i$. This means f is surjective (Def. I.3.3.17). Since f is both injective and surjective, we know that f is bijective (Def. I.3.3.20). Thus, A, B have equal cardinality (Def. I.3.6.1). □

Prop. I.3.6.8 (Uniqueness of cardinality)**.** Let X be a set with some cardinality n. Then X cannot have any other cardinality, i.e., X cannot have cardinality m for any $m \neq n$.

Proof of Prop. I.3.6.8. We induct on n. First, suppose that $n = 0$. Then X must be empty (Ex. I.3.3.3), and so X cannot have any non-zero cardinality. Now suppose that the proposition is already proven for some n; we now prove it for $n{+\!+}$. Let X have cardinality $n{+\!+}$; and suppose that X also has some other cardinality $m \neq n{+\!+}$. By Lem. I.2.2.10, there exists a $p \in \mathbb{N}$ such that $p{+\!+} = m$. By Lem. I.3.6.9, X is non-empty, and if x is any element of X, then $X \setminus \{x\}$ has cardinality n and also has cardinality p, by Lem. I.3.6.9. By the induction hypothesis, this means that $n = p$, which implies that $p{+\!+} = m = n{+\!+}$, a contradiction. This closes the induction. □

Lem. I.3.6.9. Let X be a set with cardinality $n \geq 1$. Then X is non-empty, and if x is any element of X, then the set $X \setminus \{x\}$ (i.e., X with the element x removed) has cardinality m, where $m{+\!+} = n$. (Such m exists by Lem. I.2.2.10.)

Proof of Lem. I.3.6.9. If X is empty, then it clearly cannot have the same cardinality as the non-empty set $\{i \in \mathbb{N} : 1 \leq i \leq n\}$, as there is no bijection from the empty set to a non-empty set (Ex. I.3.3.3). So let x be an element of X. Since X has the same cardinality as $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ (Def. I.3.6.5), we thus have a bijection f from X to $\{i \in \mathbb{N} : 1 \leq i \leq n\}$. In particular, $f(x)$ is a natural number between 1 and n. Now define the function $g : X \setminus \{x\} \to \{i \in \mathbb{N} : 1 \leq i \leq m\}$ by the following rule:

$$\forall y \in X \setminus \{x\}, g(y) \coloneqq \begin{cases} f(y) & \text{if } f(y) < f(x) \\ k & \text{if } f(y) > f(x) \text{ and } k{+\!+} = f(y) \end{cases}.$$

Note that $f(y)$ cannot equal $f(x)$ since $y \neq x$ and f is a bijection. Also note that since $f(y) \geq 1$, k is well-defined by Lem. I.2.2.10.

We show that g is injective. Let $y_1, y_2 \in X \setminus \{x\}$ and $y_1 \neq y_2$. Now we split into four cases:

- If $f(y_1) < f(x)$ and $f(y_2) < f(x)$, then $g(y_1) = f(y_1) \neq f(y_2) = g(y_2)$ since f is bijective.
- If $f(y_1) < f(x)$ and $f(y_2) > f(x)$, then there exists a $k \in \mathbb{N}$ such that $k{+}{+} = f(y_2)$. Thus, we have

$$\begin{aligned}
&g(y_1) = f(y_1) < f(x) < f(y_2) = k{+}{+} \\
\implies &g(y_1) < f(x) < f(x){+}{+} \leq k{+}{+} && \text{(by A.Cor. I.2.2.3 and Prop. I.2.2.12(e))} \\
\implies &g(y_1) < f(x) \leq k = g(y_2) && \text{(by Prop. I.2.2.12(d))} \\
\implies &g(y_1) < g(y_2). && \text{(by Prop. I.2.2.12(b))}
\end{aligned}$$

  In particular, we have $g(y_1) \neq g(y_2)$ by Prop. I.2.2.13.
- If $f(y_1) > f(x)$ and $f(y_2) < f(x)$, then we can use the second case and switch the role of $f(y_1)$ and $f(y_2)$ to derive $g(y_2) \neq g(y_1)$.
- If $f(y_1) > f(x)$ and $f(y_2) > f(x)$, then there exist some $k_1, k_2 \in \mathbb{N}$ such that $k_1{+}{+} = f(y_1)$ and $k_2{+}{+} = f(y_2)$. Since f is bijective, we must have $f(y_1) \neq f(y_2)$. Therefore, by Ax. I.2.4, we have $g(y_1) = k_1 \neq k_2 = g(y_2)$.

From all cases above, we see that $g(y_1) \neq g(y_2)$. Therefore, g is injective (Def. I.3.3.14).

Next we show that g is surjective. Let $i \in \{i \in \mathbb{N} : 1 \leq i \leq m\}$. Since f is bijective and $i \leq m < n$, we know that $f^{-1}(i) \in X$ is well-defined. Now we split into two cases:

- If $1 \leq i < f(x)$, then we have $f^{-1}(i) \neq f^{-1}(f(x)) = x$ since f is bijective. Thus, by Def. I.3.1.27, we have $f^{-1}(i) \in X \setminus \{x\}$. By Def. I.3.3.1, we know that $g\big(f^{-1}(i)\big)$ is well-defined, and by the definition of g, we have $i = f\big(f^{-1}(i)\big) < f(x) \implies g\big(f^{-1}(i)\big) = f\big(f^{-1}(i)\big) = i$. Thus, we have found some $y \in X \setminus \{x\}$ such that $g(y) = i$.
- If $f(x) \leq i \leq m$, then by Prop. I.2.2.12(d)(e), we have $f(x) < i{+}{+} \leq m{+}{+} = n$. Since f is bijective, we know that $f^{-1}(i{+}{+})$ is well-defined, and $f(x) \neq i{+}{+} \implies f^{-1}(f(x)) = x \neq f^{-1}(i{+}{+})$. Thus, by Def. I.3.1.27, we have $f^{-1}(i{+}{+}) \in X \setminus \{x\}$. By Def. I.3.3.1, we know that $g\big(f^{-1}(i{+}{+})\big)$ is well-defined, and by the definition of g, we have $i{+}{+} = f\big(f^{-1}(i{+}{+})\big) > f(x) \implies g\big(f^{-1}(i{+}{+})\big) = i$. Thus, we have found some $y \in X \setminus \{x\}$ such that $g(y) = i$.

From all cases above, we see that there exists a $y \in X \setminus \{x\}$ such that $g(y) = i$. Therefore, g is surjective (Def. I.3.3.17). Since g is both injective and surjective, we know that g is bijective (Def. I.3.3.20).

By Def. I.3.6.5, this means $X \setminus \{x\}$ has equal cardinality with $\{i \in \mathbb{N} : 1 \leq i \leq m\}$. In particular, $X \setminus \{x\}$ has cardinality m, as desired. $\square$

Def. I.3.6.10 (Finite sets)**.** A set is *finite* iff it has cardinality n for some natural number n; otherwise, the set is called *infinite*. If X is a finite set, we use $\#(X)$ to denote the cardinality of X.

Thm. I.3.6.12. The set of natural numbers $\mathbb{N}$ is infinite.

Proof of Thm. I.3.6.12. Suppose for the sake of contradiction that the set of natural numbers $\mathbb{N}$ was finite, so it had some cardinality $\#(\mathbb{N}) = n$. Then there is a bijection f from $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ to $\mathbb{N}$. One can show that the sequence $f(1), f(2), \ldots, f(n)$ is bounded, or more precisely that there exists a natural number M such that $f(i) \leq M$ for all $1 \leq i \leq n$ (Ex. I.3.6.3). But then the natural number $M+1$ is not equal to any of the $f(i)$, contradicting the hypothesis that f is a bijection. □

Rmk. I.3.6.13. One can also use similar arguments to show that any unbounded set is infinite; for instance the rationals $\mathbb{Q}$ and the reals $\mathbb{R}$ are infinite. However, it is possible for some sets to be "more" infinite than others. See Sec. I.8.3.

Prop. I.3.6.14 (Cardinal arithmetic)**.** (a) Let X be a finite set, and let x be an object which is not an element of X. Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$.

(b) Let X and Y be finite sets. Then $X \cup Y$ is finite and $\#(X \cup Y) \leq \#(X) + \#(Y)$. If in addition X and Y are disjoint (i.e., $X \cap Y = \emptyset$), then $\#(X \cup Y) = \#(X) + \#(Y)$.

(c) Let X be a finite set, and let Y be a subset of X. Then Y is finite, and $\#(Y) \leq \#(X)$. If in addition $Y \neq X$ (i.e., Y is a proper subset of X), then we have $\#(Y) < \#(X)$.

(d) If X is a finite set, and $f : X \to Y$ is a function, then $f(X)$ is a finite set with $\#(f(X)) \leq \#(X)$. If in addition f is one-to-one, then $\#(f(X)) = \#(X)$.

(e) Let X and Y be finite sets. Then Cartesian product $X \times Y$ is finite and $\#(X \times Y) = \#(X) \times \#(Y)$.

(f) Let X and Y be finite sets. Then the set Y^X (defined in Ax. I.3.10) is finite and $\#\left(Y^X\right) = \#(Y)^{\#(X)}$.

Proof of Prop. I.3.6.14(a). Since X is finite, by Def. I.3.6.10, there exists an $n \in \mathbb{N}$ such that $\#(X) = n$. By Def. I.3.6.5, there exists a bijective function $f : X \to \{i \in \mathbb{N} : 1 \leq i \leq n\}$. Now we define a function $g : X \cup \{x\} \to \{i \in \mathbb{N} : 1 \leq i \leq n+1\}$ as follow:

$$\forall y \in X \cup \{x\}, g(y) = \begin{cases} f(y) & \text{if } y \neq x \\ n+1 & \text{if } y = x \end{cases}.$$

Since $x \notin X$, we see that g passes the vertical line test. Therefore, g is well-defined.

Now we claim that g is bijective. Since f is bijective, we know that for all $j \in \{i \in \mathbb{N} : 1 \leq i \leq n\}$, there exists a $y \in X$ such that $f(y) = j$. With that and $g(x) = n+1$, we see that g is surjective (Def. I.3.3.17). To show that g is bijective, we also need to show that g is injective (Def. I.3.3.20). So let $y, y' \in X \cup \{x\}$ where $y \neq y'$. Now we split into three cases:

- If $y \neq x \neq y'$, then we have $g(y) = f(y) \neq f(y') = g(y')$ since f is bijective.
- If $y \neq x = y'$, then by A.Cor. I.2.2.3, we have $g(y) = f(y) \leq n < n+1 = g(x) = g(y')$. By Prop. I.2.2.13, this means $g(y) \neq g(y')$.
- If $y = x \neq y'$, then by A.Cor. I.2.2.3, we have $g(y') = f(y') \leq n < n+1 = g(x) = g(y)$. By Prop. I.2.2.13, this means $g(y) \neq g(y')$.

For all cases above we have $g(y) \neq g(y')$. Thus, g is injective (Def. I.3.3.14). This means g is bijective. Therefore, by Def. I.3.6.10, we know that $X \cup \{x\}$ is finite, and $\#(X \cup \{x\}) = n + 1 = \#(X) + 1$. □

Proof of Prop. I.3.6.14(b). Since X is finite, by Def. I.3.6.10, there exists an $n \in \mathbb{N}$ such that $\#(X) = n$. We induct on n to show that $X \cup Y$ is finite and $\#(X \cup Y) \leq \#(X) + \#(Y)$. For $n = 0$, by Ex. I.3.6.2, we have $X = \emptyset$. By Prop. I.3.1.28(a), we have $\emptyset \cup Y = Y$. Since Y is finite, we know that $\emptyset \cup Y$ is finite. Thus, we have

$$\begin{aligned}
\#(\emptyset \cup Y) &= \#(Y) && \text{(by Prop. I.3.1.28(a))}\\
&= 0 + \#(Y) && \text{(by Def. I.2.2.1)}\\
&= \#(\emptyset) + \#(Y). && \text{(by Ex. I.3.6.2)}
\end{aligned}$$

So the base case holds. Suppose inductively that $X \cup Y$ is finite, and $\#(X \cup Y) \leq \#(X) + \#(Y)$ for some $\#(X) = n$. We want to show that when $\#(X) = n{+}{+}$, we have $X \cup Y$ is finite, and $\#(X \cup Y) \leq \#(X) + \#(Y)$. So let X be a set with cardinality $n{+}{+}$. We split into two cases:

- If $X \setminus Y = \emptyset$, then we have

  $$\begin{aligned}
  X \cup Y &= (X \setminus Y) \cup (X \cap Y) \cup (Y \setminus X) && \text{(by Ex. I.3.1.10)}\\
  &= (X \cap Y) \cup (Y \setminus X) && \text{(by Prop. I.3.1.28(a))}\\
  &\subseteq Y. && \text{(by Def. I.3.1.15)}
  \end{aligned}$$

  But by Ex. I.3.1.7, we have $Y \subseteq X \cup Y$. Thus, by Prop. I.3.1.18, we have $X \cup Y = Y$. Since Y is finite, $X \cup Y$ is also finite. By Def. I.2.2.11, this means $\#(X \cup Y) = \#(Y) \leq \#(X) + \#(Y)$.

- If $X \setminus Y \neq \emptyset$, then by Lem. I.3.1.6 and Def. I.3.1.27, there exists a $z \in X$ such that $z \notin Y$. This implies $(X \setminus \{z\}) \cup Y = (X \cup Y) \setminus \{z\}$. By Lem. I.3.6.9, we know that $\#(X \setminus \{z\}) = n$. Thus, by the induction hypothesis, we know that $(X \setminus \{z\}) \cup Y$ is finite, and $\#((X \setminus \{z\}) \cup Y) \leq \#(X \setminus \{z\}) + \#(Y)$. By Prop. I.3.6.14(a), we know that $X \cup Y = (X \setminus \{z\}) \cup Y \cup \{z\}$ is finite. Thus, we have

  $$\#(X \cup Y) = \#(((X \cup Y) \setminus \{z\}) \cup \{z\}) \qquad \text{(by Prop. I.3.1.28(e))}$$

$$\begin{aligned}
&= \#(((X \setminus \{z\}) \cup Y) \cup \{z\}) && \text{(by Prop. I.3.1.28(e))}\\
&= \#((X \setminus \{z\}) \cup Y) + 1 && \text{(by Prop. I.3.6.14(a))}\\
&\leq \#(X \setminus \{z\}) + \#(Y) + 1 && \text{(by Prop. I.2.2.12(d))}\\
&= \#((X \setminus \{z\}) \cup \{z\}) + \#(Y) && \text{(by Prop. I.3.6.14(a))}\\
&= \#(X) + \#(Y). && \text{(by Prop. I.3.1.28(e))}
\end{aligned}$$

From all cases above, we see that $X \cup Y$ is finite, and $\#(X \cup Y) \leq \#(X) + \#(Y)$. This closes the induction.

Now we show that $X \cap Y = \emptyset \implies \#(X \cup Y) = \#(X) + \#(Y)$. Suppose that $X \cap Y = \emptyset$. Since X, Y are finite, by Def. I.3.6.5, there exist two bijective functions $f : X \to \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\}$ and $g : Y \to \{i \in \mathbb{N} : 1 \leq i \leq \#(Y)\}$. Now we define a function $h : X \cup Y \to \{i \in \mathbb{N} : 1 \leq i \leq \#(X) + \#(Y)\}$ as follow:

$$\forall z \in X \cup Y, h(z) = \begin{cases} f(z) & \text{if } z \in X \\ g(z) + \#(X) & \text{if } z \in Y \end{cases}.$$

Since $X \cap Y = \emptyset$, we know that h passes the vertical line test. Therefore, h is well-defined. We claim that h is injective. Let $z_1, z_2 \in X \cup Y$ where $z_1 \neq z_2$. Since $X \cap Y = \emptyset$, we know that z_1 and z_2 can either be in X or Y but not both. Thus, we can split into four cases:

- If both z_1 and z_2 are in X, then we have $h(z_1) = f(z_1) \neq f(z_2) = h(z_2)$ since f is bijective.
- If both z_1 and z_2 are in Y, then we have $g(z_1) \neq g(z_2)$ since g is bijective. By Prop. I.2.2.13, this means $g(z_1) < g(z_2)$ or $g(z_1) > g(z_2)$. Thus, we can apply Prop. I.2.2.12(d) to derive $g(z_1)+\#(X) < g(z_2)+\#(X)$ or $g(z_1)+\#(X) > g(z_2)+\#(X)$. By Prop. I.2.2.13 again, we have $h(z_1) = g(z_1) + \#(X) \neq g(z_2) + \#(X) = h(z_2)$.
- If $z_1 \in X$ and $z_2 \in Y$, then by A.Cor. I.2.2.3 and Prop. I.2.2.12(b)(d), we have
$$1 \leq h(z_1) = f(z_1) \leq \#(X) < \#(X) + 1 \leq h(z_2) = g(z_2) + \#(X) \leq \#(X) + \#(Y).$$
By Prop. I.2.2.13, this means $h(z_1) \neq h(z_2)$.
- If $z_1 \in Y$ and $z_2 \in X$, then by A.Cor. I.2.2.3 and Prop. I.2.2.12(b)(d), we have
$$1 \leq h(z_2) = f(z_2) \leq \#(X) < \#(X) + 1 \leq h(z_1) = g(z_1) + \#(X) \leq \#(X) + \#(Y).$$
By Prop. I.2.2.13, this means $h(z_1) \neq h(z_2)$.

From all cases above, we see that $h(z_1) \neq h(z_2)$. Therefore, h is injective. Now we claim that h is surjective. Let $j \in \{i \in \mathbb{N} : 1 \leq i \leq \#(X) + \#(Y)\}$. We split into two cases:

- If $1 \leq j \leq \#(X)$, then there exists a $z \in X$ such that $f(z) = j$. This is true since f is bijective. By the definition of h, we know that $h(z) = f(z) = j$. Thus, we conclude that there exists a $z \in X \cup Y$ such that $h(z) = j$.

- If $\#(X)+1 \leq j \leq \#(X)+\#(Y)$, then by Def. I.2.2.11, we have $j = \#(X)+1+k$ for some $k \in \mathbb{N}$. By Prop. I.2.2.12(d), we know that $\#(X)+1 \leq \#(X)+1+k \leq \#(X)+\#(Y) \implies 1 \leq 1+k \leq \#(Y)$. Thus, there exists a $z \in Y$ such that $g(z) = 1+k$ since g is bijective. By the definition of h, we know that $h(z) = g(z)+\#(X) = k+1+\#(X) = j$. Thus, we conclude that there exists a $z \in X \cup Y$ such that $h(z) = j$.

From all cases above, we conclude that there exists a $z \in X \cup Y$ such that $h(z) = j$. Thus, h is surjective. Since h is both injective and surjective, we know that h is bijective (Def. I.3.3.20). By Def. I.3.6.5, this means $X \cup Y$ has cardinality $\#(X)+\#(Y)$. Thus, by Def. I.3.6.10, we have $\#(X \cup Y) = \#(X)+\#(Y)$. □

Proof of Prop. I.3.6.14(c). Since X is finite, by Def. I.3.6.10, there exists an $n \in \mathbb{N}$ such that $\#(X) = n$. We induct on n to show that any subset Y of X is finite, and $\#(Y) \leq \#(X)$. For $n = 0$, by Ex. I.3.6.2, we have $X = \emptyset$. The only subset of $\emptyset$ is $\emptyset$, and we have $\#(\emptyset) = 0 = \#(\emptyset)$. So the base case holds. Suppose inductively that for some $\#(X) = n$, we have "$Y \subseteq X$ implies Y is finite, and $\#(Y) \leq \#(X)$." We want to show that when $\#(X) = n{+}{+}$, we have "$Y \subseteq X$ implies Y is finite, and $\#(Y) \leq \#(X)$." So let X be a set with cardinality $n{+}{+}$. Let $Y \subseteq X$, and let $z \in X$. We split into two cases:

- If $z \in Y$, then we have $Y \setminus \{z\} \subseteq X \setminus \{z\}$. By Lem. I.3.6.9, we know that $\#(X \setminus \{z\}) = n$. Thus, we can use the induction hypothesis to derive $Y \setminus \{z\}$ is finite, and $\#(Y \setminus \{z\}) \leq \#(X \setminus \{z\})$. By Prop. I.3.1.28(g), we have $Y = (Y \setminus \{z\}) \cup \{z\}$. Thus, by Prop. I.3.6.14(a), we know that Y is finite. Therefore, we have

$$\begin{aligned}
\#(Y) &= \#((Y \setminus \{z\}) \cup \{z\}) && \text{(by Prop. I.3.1.28(g))} \\
&= \#(Y \setminus \{z\}) + 1 && \text{(by Prop. I.3.6.14(a))} \\
&\leq \#(X \setminus \{z\}) + 1 && \text{(by Prop. I.2.2.12(d))} \\
&= \#((X \setminus \{z\}) \cup \{z\}) && \text{(by Prop. I.3.6.14(a))} \\
&= \#(X). && \text{(by Prop. I.3.1.28(g))}
\end{aligned}$$

- If $z \notin Y$, then we have $Y \subseteq X \setminus \{z\}$. By Lem. I.3.6.9, we know that $\#(X \setminus \{z\}) = n$. Thus, we can use the induction hypothesis to derive Y is finite, and $\#(Y) \leq \#(X \setminus \{z\})$. Then we have

$$\begin{aligned}
\#(Y) &\leq \#(X \setminus \{z\}) && \text{(by the induction hypothesis)} \\
&< \#(X \setminus \{z\}) + 1 && \text{(by A.Cor. I.2.2.3)} \\
&= \#((X \setminus \{z\}) \cup \{z\}) && \text{(by Prop. I.3.6.14(a))} \\
&= \#(X). && \text{(by Prop. I.3.1.28(g))}
\end{aligned}$$

From all cases above, we conclude that Y is finite, and $\#(Y) \leq \#(X)$. This closes the induction.

Now suppose that $Y \neq X$. Then we know that $X \setminus Y \neq \emptyset$. Since $X \setminus Y \subseteq X$, from first part of the proof we know that $X \setminus Y$ is finite. Thus, by Ex. I.3.6.2, we have $X \setminus Y \neq \emptyset \iff \#(X \setminus Y) > 0$. Since

$$\begin{aligned} \#(X) &= \#(Y \cup (X \setminus Y)) && \text{(by Prop. I.3.1.28(g))} \\ &= \#(Y) + \#(X \setminus Y), && \text{(by Prop. I.3.6.14(b))} \end{aligned}$$

we have $\#(Y) < \#(X)$ by Prop. I.2.2.12(f). □

Proof of Prop. I.3.6.14(d). Since X is finite, by Def. I.3.6.10, there exists an $n \in \mathbb{N}$ such that $\#(X) = n$. We induct on n to show that $\#(f(X))$ is finite, and $\#(f(X)) \leq \#(X)$. For $n = 0$, we have

$$\begin{aligned} & X = \emptyset && \text{(by Ex. I.3.6.2)} \\ \Longrightarrow\ & f(X) = \emptyset && \text{(by Def. I.3.4.1)} \\ \Longrightarrow\ & \#(f(X)) = 0 = \#(X). && \text{(by Ex. I.3.6.2)} \end{aligned}$$

Thus, the base case holds. Suppose inductively that $f(X)$ is finite, and $\#(f(X)) \leq \#(X)$ for some $\#(X) = n$. We show that the statement is still true when $\#(X) = n{++}$. So let X be a set with cardinality $n{++}$. Let $x \in X$. By Lem. I.3.6.9, we have $\#(X \setminus \{x\}) = n$. Thus, we can use the induction hypothesis to derive "$f(X \setminus \{x\})$ is finite, and $\#(f(X \setminus \{x\})) \leq \#(X \setminus \{x\})$." Now we split into two cases:

- If $f(X \setminus \{x\}) = f(X)$, then $f(X)$ is finite, and we have

$$\begin{aligned} \#(f(X)) &= \#(f(X \setminus \{x\})) \\ &\leq \#(X \setminus \{x\}) && \text{(by the induction hypothesis)} \\ &< \#(X \setminus \{x\}) + 1 && \text{(by A.Cor. I.2.2.3)} \\ &= \#((X \setminus \{x\}) \cup \{x\}) && \text{(by Prop. I.3.6.14(a))} \\ &= \#(X). && \text{(by Prop. I.3.1.28(g))} \end{aligned}$$

- If $f(X \setminus \{x\}) \neq f(X)$, then by Ex. I.3.4.3 and Prop. I.3.6.14(a), we know that $f(X) = f(X \setminus \{x\}) \cup \{f(x)\}$ is finite. Thus,

$$\begin{aligned} \#(f(X)) &= \#(f(X \setminus \{x\}) \cup \{f(x)\}) && \text{(by Ex. I.3.4.3)} \\ &= \#(f(X \setminus \{x\})) + 1 && \text{(by Prop. I.3.6.14(a))} \\ &\leq \#(X \setminus \{x\}) + 1 && \text{(by the induction hypothesis)} \\ &= \#((X \setminus \{x\}) \cup \{x\}) && \text{(by Prop. I.3.6.14(a))} \\ &= \#(X). && \text{(by Prop. I.3.1.28(g))} \end{aligned}$$

From all cases above, we conclude that $f(X)$ is finite, and $\#(f(X)) \leq \#(X)$. This closes the induction.

Now suppose that f is injective. Define $g : X \to f(X)$ as follow:

$$\forall x \in X, g(x) = f(x).$$

Since f is a function, g is well-defined. By Def. I.3.4.1, we know that g is surjective. Since f is injective, by Def. I.3.3.14, we know that

$$\forall x_1, x_2 \in X, x_1 \neq x_2 \implies g(x_1) = f(x_1) \neq f(x_2) = g(x_2).$$

Thus, g is injective. By Def. I.3.3.20, this means g is bijective. Therefore, by Def. I.3.6.1, X and $f(X)$ have equal cardinality. From first part of the proof we know that $f(X)$ is finite. Thus, by Def. I.3.6.10, we have $\#(f(X)) = \#(X)$. ☐

Proof of Prop. I.3.6.14(e). We first show that for any object x, $\{x\} \times Y$ is finite, and $\#(\{x\} \times Y) = \#(Y)$. Define $f : \{x\} \times Y \to Y$ as follow:

$$\forall (z, y) \in \{x\} \times Y, f(z, y) = y.$$

By Ex. I.3.5.7, we see that $f = \pi_{\{x\}\times Y \to Y}$. We claim that f is injective. Let $(x_1, y_1), (x_2, y_2) \in \{x\} \times Y$ where $(x_1, y_1) \neq (x_2, y_2)$. Since

$$\begin{aligned}
&\begin{cases} (x_1, y_1), (x_2, y_2) \in \{x\} \times Y \\ (x_1, y_1) \neq (x_2, y_2) \end{cases} \\
\implies &\begin{cases} x_1, x_2 \in \{x\} \\ (x_1 \neq x_2) \vee (y_1 \neq y_2) \end{cases} && \text{(by Def. I.3.5.1 and I.3.5.4)} \\
\implies &\begin{cases} x_1 = x_2 = x \\ (x_1 \neq x_2) \vee (y_1 \neq y_2) \end{cases} && \text{(by Ax. I.3.3)} \\
\implies & y_1 \neq y_2 \\
\implies & y_1 = f(x_1, y_1) \neq f(x_2, y_2) = y_2,
\end{aligned}$$

we see that f is injective (Def. I.3.3.14). Now we show that f is surjective. Let $y \in Y$. Then we have

$$\begin{aligned}
&\begin{cases} x \in \{x\} \\ y \in Y \end{cases} && \text{(by Ax. I.3.3)} \\
\implies & (x, y) \in \{x\} \times Y && \text{(by Def. I.3.5.4)} \\
\implies & f(x, y) = y.
\end{aligned}$$

Thus, f is surjective (Def. I.3.3.17). Since f is both injective and surjective, we know that f is bijective (Def. I.3.3.20). Thus, by Def. I.3.6.1, $\{x\} \times Y$ and Y have equal cardinality. Since Y is finite, by Prop. I.3.6.4, we know that $\{x\} \times Y$ is also finite. Thus, by Def. I.3.6.10, we have $\#(\{x\} \times Y) = \#(Y)$.

Now we show that $X \times Y$ is finite, and $\#(X \times Y) = \#(X) \times \#(Y)$. By Def. I.3.6.10, there exists an $n \in \mathbb{N}$ such that $\#(X) = n$. We induct on n to show that $X \times Y$ is finite, and $\#(X \times Y) = \#(X) \times \#(Y)$. For $n = 0$, by Ex. I.3.6.2, we have $X = \emptyset$. Thus, by Def. I.3.5.4, we have $\emptyset \times Y = \emptyset$, and therefore $\emptyset \times Y$ is finite. Then we have

$$\begin{aligned}
\#(\emptyset \times Y) &= \#(\emptyset) && \text{(by Def. I.3.5.4)}\\
&= 0 && \text{(by Ex. I.3.6.2)}\\
&= 0 \times \#(Y) && \text{(by Def. I.2.3.1)}\\
&= \#(\emptyset) \times \#(Y). && \text{(by Ex. I.3.6.2)}
\end{aligned}$$

Thus, the base case holds. Suppose inductively that $X \times Y$ is finite, and $\#(X \times Y) = \#(X) \times \#(Y)$ for some $\#(X) = n$. We show that the statement is still true for $\#(X) = n{+}{+}$. So let X be a set with cardinality $n{+}{+}$. Let $x \in X$. By Lem. I.3.6.9, we have $\#(X \setminus \{x\}) = n$. Thus, by the induction hypothesis, we know that $(X \setminus \{x\}) \times Y$ is finite. From first part of the proof we know that $\{x\} \times Y$ is finite. Since

$$\begin{aligned}
X \times Y &= ((X \setminus \{x\}) \cup \{x\}) \times Y && \text{(by Prop. I.3.1.28(g))}\\
&= ((X \setminus \{x\}) \times Y) \cup (\{x\} \times Y), && \text{(by Ex. I.3.5.4)}
\end{aligned}$$

we know that $X \times Y$ is finite by Prop. I.3.6.14(b). Thus, we have

$$\begin{aligned}
\#(X \times Y) &= \#(((X \setminus \{x\}) \times Y) \cup (\{x\} \times Y)) && \text{(by Ex. I.3.5.4)}\\
&= \#((X \setminus \{x\}) \times Y) + \#(\{x\} \times Y) && \text{(by Prop. I.3.6.14(b))}\\
&= \#(X \setminus \{x\}) \times \#(Y) + \#(\{x\} \times Y) && \text{(by the induction hypothesis)}\\
&= \#(X \setminus \{x\}) \times \#(Y) + \#(Y) && \text{(from the proof above)}\\
&= (\#(X \setminus \{x\}) + 1) \times \#(Y) && \text{(by Def. I.2.3.1)}\\
&= \#((X \setminus \{x\}) \cup \{x\}) \times \#(Y) && \text{(by Prop. I.3.6.14(a))}\\
&= \#(X) \times \#(Y). && \text{(by Prop. I.3.1.28(g))}
\end{aligned}$$

This closes the induction. □

Proof of Prop. I.3.6.14(f). We first show that $Y^{\{x\}}$ is finite, and $\#\left(Y^{\{x\}}\right) = \#(Y)$ for any object x. Define $f : Y^{\{x\}} \to Y$ as follow:

$$\forall g \in Y^{\{x\}}, f(g) = g(x).$$

Since $g \in Y^{\{x\}}$ implies $g(x)$ is an unique object defined in Y, we know that $f(g)$ passes the vertical line test. Thus, f is well-defined (Def. I.3.3.1). We now show that f is bijective. We start by showing f is injective. Let $g_1, g_2 \in Y^{\{x\}}$ where $g_1 \neq g_2$. Since

$$\begin{aligned}
&g_1 \neq g_2\\
\implies &\exists z \in \{x\} : g_1(z) \neq g_2(z) && \text{(by Def. I.3.3.7)}
\end{aligned}$$

$$\begin{aligned}
&\implies g_1(x) \neq g_2(x) && \text{(by Ax. I.3.3)}\\
&\implies g_1(x) = f(g_1) \neq f(g_2) = g_2(x),
\end{aligned}$$

we see that f is injective (Def. I.3.3.14). Now we show that f is surjective. Let $y \in Y$. Let $g : \{x\} \to Y$ be defined by $g(x) = y$. By Ax. I.3.10, we know that $g \in Y^{\{x\}}$. Therefore, $f(g) = y$. By Def. I.3.3.17, we see that f is surjective. Since f is both injective and surjective, we know that f is bijective (Def. I.3.3.20). Thus, by Def. I.3.6.1, $Y^{\{x\}}$ and Y have equal cardinality. Since Y is finite, by Def. I.3.6.10, we know that $Y^{\{x\}}$ is finite, and we have $\#\left(Y^{\{x\}}\right) = \#(Y)$.

Now we show that Y^X is finite, and $\#\left(Y^X\right) = \#(Y)^{\#(X)}$. By Def. I.3.6.10, there exists an $n \in \mathbb{N}$ such that $\#(X) = n$. We induct on n to show that Y^X is finite and $\#\left(Y^X\right) = \#(Y)^{\#(X)}$. For $n = 0$, by Ex. I.3.6.2, we have $X = \emptyset$. By E.g. I.3.3.9, there is only one function in $Y^\emptyset$. Thus, by Ax. I.3.3, $Y^\emptyset$ is a singleton set, and therefore $Y^\emptyset$ is finite. By Def. I.3.6.10, we have $\#\left(Y^\emptyset\right) = 1$. Thus,

$$\begin{aligned}
\#(Y)^{\#(\emptyset)} &= \#(Y)^0 && \text{(by Ex. I.3.6.2)}\\
&= 1 && \text{(by Def. I.2.3.11)}\\
&= \#\left(Y^\emptyset\right). && \text{(by Def. I.3.6.10)}
\end{aligned}$$

So the base case holds.

Suppose inductively that the statement is true for some $\#(X) = n$. We show that the statement is still true for $\#(X) = n{+}{+}$. So let X be a set with cardinality $n{+}{+}$. Let $x \in X$. If $f \in Y^X$, then we define two functions $f_1 : X \setminus \{x\} \to Y$ and $f_2 : \{x\} \to Y$ as follow:

$$\forall z \in X \setminus \{x\}, f_1(z) = f(z) \quad \text{and} \quad \forall z \in \{x\}, f_2(z) = f(z).$$

Since f is a function, we know that f_1, f_2 are well-defined and are unique with respect to f. Now we define $h : Y^X \to Y^{X\setminus\{x\}} \times Y^{\{x\}}$ as follow:

$$\forall f \in Y^X : h(f) = (f_1, f_2).$$

Since f_1, f_2 are unique with respect to f, we know that h is well-defined. We claim that h is bijective. We start by showing h is injective. So let $f, g \in Y^X$ and $f \neq g$. Since

$$\begin{aligned}
&f \neq g\\
\implies &\exists z \in X : f(z) \neq g(z) && \text{(by Def. I.3.3.7)}\\
\implies &(\exists z \in X \setminus \{x\} : f(z) \neq g(z)) \lor (\exists z \in \{x\} : f(z) \neq g(z))\\
\implies &(f_1 \neq g_1) \lor (f_2 \neq g_2) && \text{(by Def. I.3.3.7)}\\
\implies &(f_1, f_2) \neq (g_1, g_2) && \text{(by Def. I.3.5.1)}\\
\implies &h(f) \neq h(g),
\end{aligned}$$

we see that h is injective (Def. I.3.3.14). Now we show that h is surjective. Let $p \in Y^{X \setminus \{x\}}$, and let $q \in Y^{\{x\}}$. Define $f : X \to Y$ as follow:

$$\forall z \in X : f(z) = \begin{cases} p(z) & \text{if } z \in X \setminus \{x\} \\ q(z) & \text{if } z \in \{x\} \end{cases}$$

Since p, q are functions, we know that f is well-defined. Thus, $f \in Y^X$. Since

$$\begin{aligned}
&\begin{cases} \forall z \in X \setminus \{x\}, f_1(z) = f(z) = p(z) \\ \forall z \in \{x\}, f_2(z) = f(z) = q(z) \end{cases} \\
&\implies (f_1 = p) \wedge (f_2 = q) && \text{(by Def. I.3.3.7)} \\
&\implies (f_1, f_2) = (p, q) && \text{(by Def. I.3.5.1)} \\
&\implies h(f) = (p, q),
\end{aligned}$$

we see that h is surjective (Def. I.3.3.17). Since h is both injective and surjective, we know that h is bijective (Def. I.3.3.20). Thus, by Def. I.3.6.1, we know that Y^X and $Y^{X \setminus \{x\}} \times Y^{\{x\}}$ have equal cardinality. By Lem. I.3.6.9, we know that $\#(X \setminus \{x\}) = n$. Thus, by the induction hypothesis, we know that $Y^{X \setminus \{x\}}$ is finite. From the first part of the proof we know that $Y^{\{x\}}$ is finite. Thus, by Prop. I.3.6.14(e), we know that $Y^{X \setminus \{x\}} \times Y^{\{x\}}$ is finite. By Def. I.3.6.10, this means Y^X is finite, and we have $\#\big(Y^X\big) = \#\Big(Y^{X \setminus \{x\}} \times Y^{\{x\}}\Big)$. We now close our induction as follow:

$$\begin{aligned}
\#\big(Y^X\big) &= \#\Big(Y^{X \setminus \{x\}} \times Y^{\{x\}}\Big) && \text{(from the proof above)} \\
&= \#\Big(Y^{X \setminus \{x\}}\Big) \times \#\Big(Y^{\{x\}}\Big) && \text{(by Prop. I.3.6.14(e))} \\
&= \#(Y)^{\#(X \setminus \{x\})} \times \#\Big(Y^{\{x\}}\Big) && \text{(by the induction hypothesis)} \\
&= \#(Y)^{\#(X \setminus \{x\})} \times \#(Y) && \text{(from the first part of the proof)} \\
&= \#(Y)^{\#(X \setminus \{x\}) + 1} && \text{(by Def. I.2.3.11)} \\
&= \#(Y)^{\#((X \setminus \{x\}) \cup \{x\})} && \text{(by Prop. I.3.6.14(a))} \\
&= \#(Y)^{\#(X)}. && \text{(by Prop. I.3.1.28(g))}
\end{aligned}$$

This closes the induction. □

Rmk. I.3.6.15. Prop. I.3.6.14 suggests that there is another way to define the arithmetic operations of natural numbers; not defined recursively as in Def. I.2.2.1, I.2.3.1 and I.2.3.11, but instead using the notions of union, Cartesian product, and power set. This is the basis of *cardinal arithmetic*, which is an alternative foundation to arithmetic than the Peano arithmetic we have developed here; we will not develop this arithmetic in this text, but we give some examples of how one would work with this arithmetic in Ex. I.3.6.5 and I.3.6.6.

— Exercises —

Ex. I.3.6.1. Prove Prop. I.3.6.4.

Proof of Ex. I.3.6.1. See Prop. I.3.6.4. □

Ex. I.3.6.2. Show that a set X has cardinality 0 iff X is the empty set.

Proof of Ex. I.3.6.2. By Def. I.3.6.5, we know that $\#(X) = 0$ iff there exists a bijective function $f : X \to \{i \in \mathbb{N} : 1 \leq i \leq 0\}$. Since $\{i \in \mathbb{N} : 1 \leq i \leq 0\} = \emptyset$, by Ex. I.3.3.3, we know that f is a bijective function iff $X = \emptyset$. Thus, we have $\#(X) = 0$ iff $X = \emptyset$. □

Ex. I.3.6.3. Let n be a natural number, and let $f : \{i \in \mathbb{N} : 1 \leq i \leq n\} \to \mathbb{N}$ be a function. Show that there exists a natural number M such that $f(i) \leq M$ for all $1 \leq i \leq n$. Thus, finite subsets of the natural numbers are bounded.

Proof of Ex. I.3.6.3. We induct on n. For $n = 0$, any function $f : \{i \in \mathbb{N} : 1 \leq i \leq 0\} \to \mathbb{N}$ is the empty function to $\mathbb{N}$ (E.g. I.3.3.9). Thus, the following statement is vacuously true:

$$\forall M \in \mathbb{N}, \forall i \in \emptyset, f(i) \leq M.$$

So the base case holds. Suppose inductively that for some n the statement is true. We want to show that the statement is still true for $n{+}{+}$. Let $f : \{i \in \mathbb{N} : 1 \leq i \leq n{+}{+}\} \to \mathbb{N}$ be a function. Define $g : \{i \in \mathbb{N} : 1 \leq i \leq n\} \to \mathbb{N}$ as follow:

$$\forall j \in \{i \in \mathbb{N} : 1 \leq i \leq n\}, g(j) = f(j).$$

By the induction hypothesis, we know that there exists an $M \in \mathbb{N}$ such that $g(i) \leq M$ for all $1 \leq i \leq n$. Thus, by the definition of g, we know that there exists an $M \in \mathbb{N}$ such that $f(i) \leq M$ for all $1 \leq i \leq n$. Fix one such M. Now we split into two cases:

- If $f(n{+}{+}) \leq M$, then we have $f(i) \leq M$ for all $1 \leq i \leq n{+}{+}$.
- If $f(n{+}{+}) > M$, then by Prop. I.2.2.12(b), we know that $f(i) < f(n{+}{+})$ for all $1 \leq i \leq n$. By setting $M' = f(n{+}{+})$, we see that $f(i) \leq M'$ for all $1 \leq i \leq n{+}{+}$.

From all cases above, we conclude that there exists an $M \in \mathbb{N}$ such that $f(u) \leq M$ for all $1 \leq i \leq n$. This closes the induction. □

Ex. I.3.6.4. Prove Prop. I.3.6.14.

Proof of Ex. I.3.6.4. See Prop. I.3.6.14. □

Ex. I.3.6.5. Let A, B be sets. Show that $A \times B$ and $B \times A$ have equal cardinality by constructing an explicit bijection between the two sets. Then use Prop. I.3.6.14 to conclude an alternate proof of Lem. I.2.3.2.

Proof of Ex. I.3.6.5. Define $f : A \times B \to B \times A$ as follow:

$$\forall (a, b) \in A \times B : f(a, b) = (b, a).$$

By Def. I.3.5.1, we know that f passes the vertical line test. Thus, f is well-defined. We now show that such f is bijective. We start by showing f is injective. Let $(a_1, b_1), (a_2, b_2) \in A \times B$ and $(a_1, b_1) \neq (a_2, b_2)$. Since

$$\begin{aligned}
&(a_1, b_1) \neq (a_2, b_2) \\
\implies &(a_1 \neq a_2) \vee (b_1 \neq b_2) && \text{(by Def. I.3.5.1)} \\
\implies &(b_1 \neq b_2) \vee (a_1 \neq a_2) \\
\implies &(b_1, a_1) \neq (b_2, a_2) && \text{(by Def. I.3.5.1)} \\
\implies &f(a_1, b_1) \neq f(a_2, b_2),
\end{aligned}$$

we know that f is injective (Def. I.3.3.14). Now we show that f is surjective. Let $(b, a) \in B \times A$. Clearly, $(a, b) \in A \times B$. Thus, we have $f(a, b) = (b, a)$. Therefore, f is surjective (Def. I.3.3.17). Since f is both injective and surjective, we know that f is bijective (Def. I.3.3.20). By Def. I.3.6.1, we conclude that $A \times B$ and $B \times A$ have equal cardinality.

Now we give an alternative proof of Lem. I.2.3.2. Let $n, m \in \mathbb{N}$, let $A = \{i \in \mathbb{N} : 1 \leq i \leq n\}$, and let $B = \{i \in \mathbb{N} : 1 \leq i \leq m\}$. By Def. I.3.6.10, we know that $\#(A) = n$ and $\#(B) = m$. From the proof above we know that $A \times B$ and $B \times A$ have equal cardinality. Thus, by Prop. I.3.6.8, we have $\#(A \times B) = \#(B \times A)$. Then we have

$$\begin{aligned}
n \times m &= \#(A) \times \#(B) \\
&= \#(A \times B) && \text{(by Prop. I.3.6.14(e))} \\
&= \#(B \times A) && \text{(by Prop. I.3.6.8)} \\
&= \#(B) \times \#(A) && \text{(by Prop. I.3.6.14(e))} \\
&= m \times n.
\end{aligned}$$

Thus, Lem. I.2.3.2 is true. □

Ex. I.3.6.6. Let A, B, C be sets. Show that the sets $(A^B)^C$ and $A^{B \times C}$ have equal cardinality by constructing an explicit bijection between the two sets. Conclude that $(a^b)^c = a^{bc}$ for any natural numbers a, b, c. Use a similar argument to also conclude $a^b \times a^c = a^{b+c}$.

Proof of Ex. I.3.6.6. We first show that $(A^B)^C$ and $A^{B \times C}$ have equal cardinality. By Def. I.3.5.4, $B \times C$ is a set, and by Ax. I.3.10, $A^B, (A^B)^C, A^{B \times C}$ are sets. Therefore, it makes sense to ask whether $(A^B)^C$ and $A^{B \times C}$ have equal cardinality. Define $f : (A^B)^C \to A^{B \times C}$ as follow:

$$\forall g \in (A^B)^C, f(g) = h_g,$$

where $h_g : B \times C \to A$ is defined as follow:

$$\forall (b, c) \in B \times C, h_g(b, c) = (g(c))(b).$$

We need to show that f is well-defined. So let $g \in (A^B)^C$, and let $(b, c) \in B \times C$. Define $h_g : B \times C \to A$ as above. Since g is a function with domain C and codomain A^B, we know that $g(c)$ is well-defined, and $g(c)$ is a function in A^B. Since $g(c)$ has domain B and codomain A, we know that $(g(c))(b)$ is well-defined, and $(g(c))(b)$ is an object in A. Thus, h_g passes the vertical line test, and h_g is well-defined. If $h'_g : B \times C \to A$ is another function satisfying $h'_g(b, c) = (g(c))(b)$ for any $(b, c) \in B \times C$, then we have $h_g = h'_g$ (Def. I.3.3.7). Thus, f passes the vertical line test, and f is well-defined.

We now show that f is bijective. We start by showing that f is injective. Let $g_1, g_2 \in (A^B)^C$ where $g_1 \neq g_2$. Since

$$\begin{aligned}
& g_1 \neq g_2 \\
\implies & \exists c \in C : g_1(c) \neq g_2(c) && \text{(by Def. I.3.3.7)} \\
\implies & \exists c \in C : (\exists b \in B : (g_1(c))(b) \neq (g_2(c))(b)) && \text{(by Def. I.3.3.7)} \\
\implies & \exists (b, c) \in B \times C : (g_1(c))(b) \neq (g_2(c))(b) && \text{(by Def. I.3.5.1)} \\
\implies & \exists (b, c) \in B \times C : h_{g_1}(b, c) \neq h_{g_2}(b, c) \\
\implies & h_{g_1} \neq h_{g_2} && \text{(by Def. I.3.3.7)} \\
\implies & f(g_1) \neq f(g_2),
\end{aligned}$$

we know that f is injective (Def. I.3.3.14). Next we show that f is surjective. Let $h \in A^{B \times C}$. Define $g : C \to A^B$ as follow:

$$\forall (b, c) \in B \times C, (g(c))(b) = h(b, c).$$

Since h is a function, g pass the vertical line test. Thus, g is well-defined. By the definition of f, we see that $f(g) = h_g = h$. Thus, f is surjective (Def. I.3.3.17). Since f is both injective and surjective, we know that f is bijective (Def. I.3.3.20). By Def. I.3.6.1, we conclude that $(A^B)^C$ and $A^{B \times C}$ have equal cardinality.

Now we show that $(a^b)^c = a^{bc}$ for any $a, b, c \in \mathbb{N}$. Suppose that A, B, C are finite sets with cardinality $\#(A) = a$, $\#(B) = b$, and $\#(C) = c$. Since B, C are finite, by Prop. I.3.6.14(e), we know that $B \times C$ is finite. By Prop. I.3.6.14(f), we know that $A^{B \times C}$ is finite. From the proof above we see that $(A^B)^C$ and $A^{B \times C}$ have equal cardinality. Thus, $(A^B)^C$ is finite, and by Prop. I.3.6.8, we have $\#\big((A^B)^C\big) = \#\big(A^{B \times C}\big)$. Then we have

$$\begin{aligned}
(a^b)^c &= \left(\#(A)^{\#(B)}\right)^{\#(C)} && \text{(by Def. I.3.6.10)} \\
&= \big(\#\big(A^B\big)\big)^{\#(C)} && \text{(by Prop. I.3.6.14(f))} \\
&= \#\Big(\big(A^B\big)^C\Big) && \text{(by Prop. I.3.6.14(f))} \\
&= \#\big(A^{B \times C}\big) && \text{(from the proof above)} \\
&= \#(A)^{\#(B \times C)} && \text{(by Prop. I.3.6.14(f))} \\
&= \#(A)^{\#(B) \times \#(C)} && \text{(by Prop. I.3.6.14(e))}
\end{aligned}$$

$$= a^{bc}. \qquad \text{(by Def. I.3.6.10)}$$

Next we show that $A^B \times A^C$ and $A^{B\cup C}$ have equal cardinality if $B \cap C = \emptyset$. So suppose that $B \cap C = \emptyset$. By Ax. I.3.10, $A^B, A^C, A^{B\cup C}$ are sets, and by Def. I.3.5.4, $A^B \times A^C$ is a set. Therefore, it makes sense to ask whether $A^B \times A^C$ and $A^{B\cup C}$ have equal cardinality. Define $f : A^B \times A^C \to A^{B\cup C}$ as follow:

$$\forall (g,h) \in A^B \times A^C, \forall x \in B \cup C, (f(g,h))(x) = \begin{cases} g(x) & \text{if } x \in B \\ h(x) & \text{if } x \in C \end{cases}.$$

Since g, h are functions and $B \cap C = \emptyset$, we know that $(f(g,h))(x)$ is well-defined. If there exist some $g' \in A^B$ and $h' \in A^C$, such that $(f(g,h))(x) = g'(x)$ if $x \in B$, and $(f(g,h))(x) = h'(x)$ if $x \in C$, then we have $g = g'$ and $h = h'$ (Def. I.3.3.7). Thus, f passes the vertical line test, and f is well-defined. We now show that f is bijective. We start by showing that f is injective. Let $(g_1,h_1),(g_2,h_2) \in A^B \times A^C$ and $(g_1,h_1) \neq (g_2,h_2)$. Since

$$\begin{aligned}
& (g_1,h_1) \neq (g_2,h_2) \\
\implies & (g_1 \neq g_2) \vee (h_1 \neq h_2) && \text{(by Def. I.3.5.1)} \\
\implies & (\exists x \in B : g_1(x) \neq g_2(x)) \vee (\exists x \in C : h_1(x) \neq h_2(x)) && \text{(by Def. I.3.3.7)} \\
\implies & \exists x \in B \cup C : (g_1(x) \neq g_2(x)) \vee (h_1(x) \neq h_2(x)) && \text{(by Ax. I.3.4)} \\
\implies & \exists x \in B \cup C : (f(g_1,h_1))(x) \neq (f(g_2,h_2))(x) \\
\implies & f(g_1,h_1) \neq f(g_2,h_2), && \text{(by Def. I.3.3.7)}
\end{aligned}$$

we know that f is injective (Def. I.3.3.14). Next we show that f is surjective. Let $k \in A^{B\cup C}$. Define functions $g : B \to A$ and $h : C \to A$ as follow:

$$\forall x \in B, g(x) = k(x) \quad \text{and} \quad \forall x \in C, h(x) = k(x).$$

By the definition of f, we see that $f(g,h) = k$. Thus, f is surjective (Def. I.3.3.17). Since f is both injective and surjective, we know that f is bijective (Def. I.3.3.20). By Def. I.3.6.1, we conclude that $A^B \times A^C$ and $A^{B\cup C}$ have equal cardinality if $B \cap C = \emptyset$.

Now we show that $a^b \times a^c = a^{b+c}$ for any $a, b, c \in \mathbb{N}$. Suppose that A, B, C are finite sets with cardinality $\#(A) = a$, $\#(B) = b$, and $\#(C) = c$. Suppose also that $B \cap C = \emptyset$. Since B, C are finite, by Prop. I.3.6.14(b), we know that $B \cup C$ is finite. By Prop. I.3.6.14(f), we know that $A^{B\cup C}$ is finite. From the proof above we know that $A^B \times A^C$ and $A^{B\cup C}$ have equal cardinality. Thus, by Prop. I.3.6.8, $A^B \times A^C$ is finite, and $\#(A^B \times A^C) = \#\big(A^{B\cup C}\big)$. Then we have

$$\begin{aligned}
a^b \times a^c &= \#(A)^{\#(B)} \times \#(A)^{\#(C)} && \text{(by Def. I.3.6.10)} \\
&= \#\big(A^B\big) \times \#\big(A^C\big) && \text{(by Prop. I.3.6.14(f))} \\
&= \#\big(A^B \times A^C\big) && \text{(by Prop. I.3.6.14(e))}
\end{aligned}$$

$$\begin{aligned}
&= \#\big(A^{B\cup C}\big) && \text{(from the proof above)}\\
&= \#(A)^{\#(B\cup C)} && \text{(by Prop. I.3.6.14(f))}\\
&= \#(A)^{\#(B)+\#(C)} && \text{(by Prop. I.3.6.14(b))}\\
&= a^{b+c}. && \text{(by Def. I.3.6.10)}
\end{aligned}$$

□

Ex. I.3.6.7. Let A and B be sets. Let us say that A has *lesser or equal* cardinality to B if there exists an injection $f : A \to B$ from A to B. Show that if A and B are finite sets, then A has lesser or equal cardinality to B iff $\#(A) \le \#(B)$.

Proof of Ex. I.3.6.7. First, suppose that A has lesser or equal cardinality to B. By definition, there exists a injective function $f : A \to B$. By Def. I.3.4.1, we have $f(A) \subseteq B$. Thus,

$$\begin{aligned}
\#(A) &= \#(f(A)) && \text{(by Prop. I.3.6.14(d))}\\
&\le \#(B). && \text{(by Prop. I.3.6.14(c))}
\end{aligned}$$

Now suppose that $\#(A) \le \#(B)$. By Def. I.3.6.5, there exists two bijective functions $g_A : A \to \{i \in \mathbb{N} : 1 \le i \le \#(A)\}$ and $g_B : B \to \{i \in \mathbb{N} : 1 \le i \le \#(B)\}$. Now we define $f : A \to B$ as follow:

$$\forall x \in A, f(x) = g_B^{-1}(g_A(x)).$$

Since both g_A and g_B are bijective, we know that $g_B^{-1}(g_A(x))$ is unique to each x. Thus, f pass the vertical line test, and f is well-defined. We claim that f is injective. So let $x_1, x_2 \in A$ and $x_1 \ne x_2$. Since

$$\begin{aligned}
&x_1 \ne x_2\\
\implies &g_A(x_1) \ne g_A(x_2) && \text{(by Def. I.3.3.20)}\\
\implies &g_B^{-1}(g_A(x_1)) \ne g_B^{-1}(g_A(x_2)) && \text{(by Def. I.3.3.20)}\\
\implies &f(x_1) \ne f(x_2),
\end{aligned}$$

we see that f is injective (Def. I.3.3.14). Thus, A has lesser or equal cardinality to B. We conclude that if A, B are finite sets, then A has lesser or equal cardinality to B iff $\#(A) \le \#(B)$. □

Ex. I.3.6.8. Let A, B be sets such that $A \ne \emptyset$ and there exists an injection $f : A \to B$ from A to B (i.e., A has lesser or equal cardinality to B). Show that there exists a surjection $g : B \to A$ from B to A. (The converse to this statement requires the axiom of choice; see Ex. I.8.4.3.)

Proof of Ex. I.3.6.8. Since f is injective, by Def. I.3.3.14, we know that if $b \in f(A)$, then there exists only one element $a \in A$ such that $f(a) = b$. So it makes sense to talk about the inverse of b when $b \in f(A)$, and we denote the inverse of b as $f^{-1}(b)$.

Now we construct a surjective function. Let $a \in A$. Define $g : B \to A$ as follow:

$$\forall x \in B, g(x) = \begin{cases} f^{-1}(x) & \text{if } x \in f(A) \\ a & \text{if } x \notin f(A) \end{cases}.$$

Clearly, g passes the vertical line test. Thus, g is well-defined. Now we show that g is surjective. Let $y \in A$. By Def. I.3.4.1, we have $f(y) \in f(A) \subseteq B$. Thus, by Ex. I.3.3.6, we have $g(f(y)) = f^{-1}(f(y)) = y$. Therefore, g is surjective (Def. I.3.3.17). $\square$

Ex. I.3.6.9. Let A and B be finite sets. Show that $A \cup B$ and $A \cap B$ are also finite sets, and that $\#(A) + \#(B) = \#(A \cup B) + \#(A \cap B)$.

Proof of Ex. I.3.6.9. By Prop. I.3.6.14(b), $A \cup B$ is finite. By Ex. I.3.1.7, we have $A \cap B \subseteq A$. Thus, by Prop. I.3.6.14(c), we know that $A \cap B$ is finite since A is finite. Using similar arguments, we can show that $A \setminus B$ and $B \setminus A$ are finite. Thus, we have

$$\begin{aligned}
&\#(A \cup B) + \#(A \cap B) \\
&= \#((A \setminus B) \cup (A \cap B) \cup (B \setminus A)) + \#(A \cap B) && \text{(by Ex. I.3.1.10)} \\
&= \#(A \setminus B) + \#(A \cap B) + \#(B \setminus A) + \#(A \cap B) && \text{(by Prop. I.3.6.14(b))} \\
&= \#((A \setminus B) \cup (A \cap B)) + \#((B \setminus A) \cup (A \cap B)) && \text{(by Prop. I.3.6.14(b))} \\
&= \#(A) + \#(B). && \text{(by Prop. I.3.1.28(g))}
\end{aligned}$$

$\square$

Ex. I.3.6.10. Let $A_1, \ldots, A_n$ be finite sets such that $\#\left(\bigcup_{i \in \{1,\ldots,n\}} A_i\right) > n$. Show that there exists $i \in \{1, \ldots, n\}$ such that $\#(A_i) \geq 2$. (This is known as the *pigeonhole principle*.)

Proof of Ex. I.3.6.10. We induct on n. We start with $n = 1$ since for $n = 0$ the statement is vacuously true. For $n = 1$, we have

$$\begin{aligned}
&\#\left(\bigcup_{i \in \{1,\ldots,1\}} A_i\right) > 1 \\
\implies &\#(A_1) > 1 && \text{(by Ax. I.3.11)} \\
\implies &\#(A_1) \geq 2. && \text{(by Prop. I.2.2.12(e))}
\end{aligned}$$

Thus, the base case holds. Suppose inductively that for some n the statement is true. We show that the statement is still true for $n{+}{+}$. By the induction hypothesis, we know that $\bigcup_{i \in \{1,\ldots,n\}} A_i$ is finite. Since A_{n++} is finite, by Prop. I.3.6.14(b), we know that $\left(\bigcup_{i \in \{1,\ldots,n\}} A_i\right) \cup A_{n++} =$

$\bigcup_{i\in\{1,\ldots,n{+\!+}\}} A_i$ is finite. Thus, we have

$$\begin{aligned}
\#\left(\bigcup_{i\in\{1,\ldots,n\}} A_i\right) + \#(A_{n{+\!+}}) &\geq \#\left(\left(\bigcup_{i\in\{1,\ldots,n\}} A_i\right) \cup A_{n{+\!+}}\right) && \text{(by Prop. I.3.6.14)}(b) \\
&= \#\left(\bigcup_{i\in\{1,\ldots,n{+\!+}\}} A_i\right) && \text{(by Ax. I.3.11)} \\
&> n{+\!+}.
\end{aligned}$$

Now we split into three cases:

- If $\#(A_{n{+\!+}}) = 0$, then we have

$$\begin{aligned}
\#\left(\bigcup_{i\in\{1,\ldots,n{+\!+}\}} A_i\right) &= \#\left(\bigcup_{i\in\{1,\ldots,n\}} A_i\right) && \text{(by Prop. I.3.1.28(a))} \\
&> n{+\!+} \\
&> n. && \text{(by A.Cor. I.2.2.3)}
\end{aligned}$$

  By the induction hypothesis, there exists an $i \in \{1, \ldots, n\}$ such that $\#(A_i) \geq 2$. Therefore, there exists an $i \in \{1, \ldots, n{+\!+}\}$ such that $\#(A_i) \geq 2$.

- If $\#(A_{n{+\!+}}) = 1$, then we have

$$\begin{aligned}
&\#\left(\bigcup_{i\in\{1,\ldots,n\}} A_i\right) + 1 > n{+\!+} \\
\implies &\#\left(\bigcup_{i\in\{1,\ldots,n\}} A_i\right) > n. && \text{(by Prop. I.2.2.12(b))}
\end{aligned}$$

  By the induction hypothesis, there exists an $i \in \{1, \ldots, n\}$ such that $\#(A_i) \geq 2$. Therefore, there exists an $i \in \{1, \ldots, n{+\!+}\}$ such that $\#(A_i) \geq 2$.

- If $\#(A_{n{+\!+}}) > 1$, then $\#(A_{n{+\!+}}) \geq 2$. Thus, there exists an $i \in \{1, \ldots, n{+\!+}\}$ such that $\#(A_i) \geq 2$.

From all cases above, we conclude that there exists an $i \in \{1, \ldots, n{+\!+}\}$ such that $\#(A_i) \geq 2$. This closes the induction. $\square$

Chapter I.4

Integers and rationals

I.4.1 The integers

Def. I.4.1.1 (Integers)**.** An *integer* is an expression of the form $a—b$, where a and b are natural numbers. Two integers are considered to be equal, $a—b = c—d$, iff $a + d = c + b$. We let $\mathbb{Z}$ denote the set of all integers.

Note. In the language of set theory, what we are doing here is starting with the space $\mathbb{N}\times\mathbb{N}$ of ordered pairs (a, b) of natural numbers. Then we place an *equivalence relation* $\sim$ on these pairs by declaring $(a, b) \sim (c, d)$ iff $a+d = c+b$. The set-theoretic interpretation of the symbol $a—b$ is that it is the space of all pairs equivalent to (a, b) : $a—b := \{(c, d) \in \mathbb{N} \times \mathbb{N} : (a, b) \sim (c, d)\}$. However, this interpretation plays no role in how we manipulate the integers and we will not refer to it again. A similar set-theoretic interpretation can be given to the construction of the rational numbers later in this chapter, or the real numbers in the next chapter.

A.Cor. I.4.1.1. The definition of equality on the integers is reflexive, symmetric, and transitive.

Proof of A.Cor. I.4.1.1. Let $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$. Since

$$\begin{aligned} & a + b = a + b && \text{(by A.Cor. I.2.2.1)} \\ \implies & a—b = a—b, && \text{(by Def. I.4.1.1)} \end{aligned}$$

we know that Def. I.4.1.1 is reflexive.

Next suppose that $a—b = c—d$. Then we have

$$\begin{aligned} & a—b = c—d \\ \implies & a + d = c + b && \text{(by Def. I.4.1.1)} \\ \implies & c + b = a + d \\ \implies & c—d = a—b. && \text{(by Def. I.4.1.1)} \end{aligned}$$

Thus, Def. I.4.1.1 is symmetric.

Finally suppose that a—$b = c$—d and c—$d = e$—f. Then we have

$$
\begin{aligned}
& (a\text{—}b = c\text{—}d) \wedge (c\text{—}d = e\text{—}f) \\
\implies & (a+d = c+b) \wedge (c+f = e+d) && \text{(by Def. I.4.1.1)} \\
\implies & (a+d+f = c+b+f) \wedge (c+f+b = e+d+b) && \text{(by Prop. I.2.2.6)} \\
\implies & (a+f+d = c+f+b) \wedge (c+f+b = e+b+d) && \text{(by Prop. I.2.2.4)} \\
\implies & a+f+d = e+b+d \\
\implies & a+f = e+b && \text{(by Prop. I.2.2.6)} \\
\implies & a\text{—}b = e\text{—}f. && \text{(by Def. I.4.1.1)}
\end{aligned}
$$

Thus, Def. I.4.1.1 is transitive. □

Def. I.4.1.2. The sum of two integers, $(a\text{—}b) + (c\text{—}d)$, is defined by the formula

$$(a\text{—}b) + (c\text{—}d) := (a+c)\text{—}(b+d).$$

The product of two integers, $(a\text{—}b) \times (c\text{—}d)$, is defined by

$$(a\text{—}b) \times (c\text{—}d) := (ac+bd)\text{—}(ad+bc).$$

Lem. I.4.1.3 (Addition and multiplication are well-defined)**.** Let a, b, a', b', c, d be natural numbers. If $(a\text{—}b) = (a'\text{—}b')$, then $(a\text{—}b) + (c\text{—}d) = (a'\text{—}b') + (c\text{—}d)$ and $(a\text{—}b) \times (c\text{—}d) = (a'\text{—}b') \times (c\text{—}d)$, and also $(c\text{—}d) + (a\text{—}b) = (c\text{—}d) + (a'\text{—}b')$ and $(c\text{—}d) \times (a\text{—}b) = (c\text{—}d) \times (a'\text{—}b')$. Thus, addition and multiplication are well-defined operations (equal inputs give equal outputs).

Proof of Lem. I.4.1.3. To prove that $(a\text{—}b) + (c\text{—}d) = (a'\text{—}b') + (c\text{—}d)$, we evaluate both sides as $(a+c)\text{—}(b+d)$ and $(a'+c)\text{—}(b'+d)$. Thus, we need to show that $a+c+b'+d = a'+c+b+d$. But since $(a\text{—}b) = (a'\text{—}b')$, we have $a+b' = a'+b$, and so by adding $c+d$ to both sides we obtain the claim.

Now we show that $(a\text{—}b) \times (c\text{—}d) = (a'\text{—}b') \times (c\text{—}d)$. We evaluate both sides to $(ac+bd)\text{—}(ad+bc)$ and $(a'c+b'd)\text{—}(a'd+b'c)$, so we have to show that $ac+bd+a'd+b'c = a'c+b'd+ad+bc$. But the left-hand side factors as $c(a+b')+d(a'+b)$, while the right-hand side factors as $c(a'+b)+d(a+b')$. Since $a+b' = a'+b$, the two sides are equal. The other two identities are proven similarly. □

A.Cor. I.4.1.2. The integers n—0 behave in the same way as the natural numbers n; indeed one can check that $(n\text{—}0)+(m\text{—}0) = (n+m)\text{—}0$ and $(n\text{—}0)\times(m\text{—}0) = nm\text{—}0$. Furthermore, $(n\text{—}0)$ is equal to $(m\text{—}0)$ iff $n = m$. (The mathematical term for this is that there is an *isomorphism* between the natural numbers n and those integers of the form n—0.) Thus, we may *identify* the natural numbers with integers by setting $n \equiv n\text{—}0$; this does not affect our definitions of addition or multiplication or equality since they are consistent with each other. In particular, 0 is equal to $0—0$ and 1 is equal to $1—0$. Of course, if we set n equal to $n—0$, then it will also be equal to any other integer which is equal to $n—0$.

Proof of A.Cor. I.4.1.2. Let $n, m \in \mathbb{N}$. First, we show that $(n—0)+(m—0) = (n+m)—0$. This is true since

$$\begin{aligned}(n—0) + (m—0) &= (n + m)—(0 + 0) && \text{(by Def. I.4.1.2)}\\ &= (n + m)—0. && \text{(by Def. I.2.2.1)}\end{aligned}$$

Next we show that $(n—0) \times (m—0) = nm—0$. This is true since

$$\begin{aligned}(n—0) \times (m—0) &= (nm + 0 \times 0)—(n \times 0 + 0 \times m) && \text{(by Def. I.4.1.2)}\\ &= (nm + 0)—(0 + 0) && \text{(by Def. I.2.3.1 and A.Cor. I.2.3.2)}\\ &= nm—0. && \text{(by Lem. I.2.2.2)}\end{aligned}$$

Finally we show that $(n—0) = (m—0) \iff n = m$. This is true since

$$\begin{aligned}&(n—0) = (m—0)\\ \iff &n + 0 = m + 0 && \text{(by Def. I.4.1.1)}\\ \iff &n = m. && \text{(by Def. I.2.2.1)}\end{aligned}$$

□

Note. We can now define incrementation on the integers by defining $x{+}{+} := x + 1$ for any integer x; this is of course consistent with our definition of the increment operation for natural numbers. However, this is no longer an important operation for us, as it has been now superceded by the more general notion of addition.

Def. I.4.1.4 (Negation of integers)**.** If $(a—b)$ is an integer, we define the negation $-(a—b)$ to be the integer $(b—a)$. In particular, if $n = n—0$ is a positive natural number, we can define its negation $-n = 0—n$.

A.Cor. I.4.1.3. The definition of negation on the integers is well-defined.

Proof of A.Cor. I.4.1.3. Let $a, b, a', b' \in \mathbb{N}$ and $a—b = a'—b'$. Then we have

$$\begin{aligned}&a—b = a'—b'\\ \implies &a + b' = a' + b && \text{(by Def. I.4.1.1)}\\ \implies &b' + a = b + a' && \text{(by Prop. I.2.2.4)}\\ \implies &b'—a' = b—a && \text{(by Def. I.4.1.1)}\\ \implies &-(a'—b') = -(a—b). && \text{(by Def. I.4.1.4)}\end{aligned}$$

Thus, Def. I.4.1.4 is well-defined. □

Lem. I.4.1.5 (Trichotomy of integers)**.** Let x be an integer. Then exactly one of the following three statements is true:

(a) x is zero.

(b) x is equal to a positive natural number n.

(c) x is the negation $-n$ of a positive natural number n.

Proof of Lem. I.4.1.5. We first show that at least one of (a), (b), (c) is true. By definition, $x = a\text{—}b$ for some natural numbers a, b. By Prop. I.2.2.13, we have three cases: $a > b$, $a = b$, or $a < b$. If $a > b$ then by Prop. I.2.2.12(f), $a = b + c$ for some positive natural number c, which means that $a\text{—}b = c\text{—}0 = c$, which is (b). If $a = b$, then $a\text{—}b = a\text{—}a = 0\text{—}0 = 0$, which is (a). If $a < b$, then $b > a$, so that $b\text{—}a = n$ for some natural number n by the previous reasoning, and thus $a\text{—}b = -n$, which is (c). Now we show that no more than one of (a), (b), (c) can hold at a time. By Def. I.2.2.7, a positive natural number is non-zero, so (a) and (b) cannot simultaneously be true. If (a) and (c) were simultaneously true, then $0 = -n$ for some positive natural n; thus $(0\text{—}0) = (0\text{—}n)$, so that $0 + n = 0 + 0$, so that $n = 0$, a contradiction. If (b) and (c) were simultaneously true, then $n = -m$ for some positive n, m, so that $(n\text{—}0) = (0\text{—}m)$, so that $n + m = 0 + 0$, which contradicts Prop. I.2.2.8. Thus, exactly one of (a), (b), (c) is true for any integer x. □

Note. If n is a positive natural number, we call n a *positive integer* and $-n$ a *negative integer*. Thus, by Lem. I.4.1.5, every integer is positive, zero, or negative, but not more than one of these at a time.

Note. One could well ask why we don't use Lem. I.4.1.5 to *define* the integers; i.e., why didn't we just say an integer is anything which is either a positive natural number, zero, or the negative of a natural number. The reason is that if we did so, the rules for adding and multiplying integers would split into many different cases (e.g., negative times positive equals negative; negative plus positive is either negative, positive, or zero, depending on which term is larger, etc.) and to verify all the properties would end up being much messier.

Prop. I.4.1.6 (Laws of algebra for integers)**.** Let x, y, z be integers. Then we have

$$
\begin{aligned}
x + y &= y + x \\
(x + y) + z &= x + (y + z) \\
x + 0 = 0 + x &= x \\
x + (-x) = (-x) + x &= 0 \\
xy &= yx \\
(xy)z &= x(yz) \\
x1 = 1x &= x \\
x(y + z) &= xy + xz \\
(y + z)x &= yx + zx.
\end{aligned}
$$

Proof of Prop. I.4.1.6. There are two ways to prove these identities. One is to use Lem. I.4.1.5 and split into a lot of cases depending on whether x, y, z are zero, positive, or negative. This becomes very messy. A shorter way is to write $x = (a—b)$, $y = (c—d)$, and $z = (e—f)$ for some natural numbers a, b, c, d, e, f, and expand these identities in terms of a, b, c, d, e, f and use the algebra of the natural numbers. This allows each identity to be proven in a few lines.

We first show that $x + y = y + x$.

$$\begin{aligned} x + y &= (a—b) + (c—d) \\ &= (a + c)—(b + d) && \text{(by Def. I.4.1.2)} \\ &= (c + a)—(d + b) && \text{(by Prop. I.2.2.4)} \\ &= (c—d) + (a—b) && \text{(by Def. I.4.1.2)} \\ &= y + x. \end{aligned}$$

Thus, the addition on integers is commutative.

Next we show that $(x + y) + z = x + (y + z)$.

$$\begin{aligned} (x + y) + z &= ((a—b) + (c—d)) + (e—f) \\ &= ((a + c)—(b + d)) + (e—f) && \text{(by Def. I.4.1.2)} \\ &= ((a + c) + e)—((b + d) + f) && \text{(by Def. I.4.1.2)} \\ &= (a + (c + e))—(b + (d + f)) && \text{(by Prop. I.2.2.5)} \\ &= (a—b) + ((c + e)—(d + f)) && \text{(by Def. I.4.1.2)} \\ &= (a—b) + ((c—d) + (e—f)) && \text{(by Def. I.4.1.2)} \\ &= x + (y + z). \end{aligned}$$

Thus, the addition on integers is associative.

Next we show that $x + 0 = 0 + x = x$. Since the addition on integers is commutative, we know that $x + 0 = 0 + x$. So we only need to show that $x + 0 = x$.

$$\begin{aligned} x + 0 &= (a—b) + (0—0) && \text{(by A.Cor. I.4.1.2)} \\ &= (a + 0)—(b + 0) && \text{(by Def. I.4.1.2)} \\ &= a—b && \text{(by Lem. I.2.2.2)} \\ &= x. \end{aligned}$$

Thus, 0 is the additive identity on integers.

Next we show that $x + (-x) = (-x) + x = 0$. Since the addition on integers is commutative, we know that $x + (-x) = (-x) + x$. So we only need to show that $x + (-x) = 0$.

$$\begin{aligned} x + (-x) &= (a—b) + (b—a) && \text{(by Def. I.4.1.4)} \\ &= (a + b)—(b + a) && \text{(by Def. I.4.1.2)} \\ &= (a + b)—(a + b) && \text{(by Prop. I.2.2.4)} \end{aligned}$$

$$\begin{aligned} &= 0 — 0 && \text{(by Def. I.4.1.1)} \\ &= 0. && \text{(by A.Cor. I.4.1.2)} \end{aligned}$$

Thus, the additive inverse of integer x is $-x$.

Next we show that $xy = yx$.

$$\begin{aligned} xy &= (a — b) \times (c — d) \\ &= (ac + bd) — (ad + bc) && \text{(by Def. I.4.1.2)} \\ &= (ca + db) — (da + cb) && \text{(by Lem. I.2.3.2)} \\ &= (ca + db) — (cb + da) && \text{(by Prop. I.2.2.4)} \\ &= (c — d) \times (a — b) && \text{(by Def. I.4.1.2)} \\ &= yx. \end{aligned}$$

Thus, the multiplication on integers is commutative.

Next we show that $(xy)z = x(yz)$.

$$\begin{aligned} &(xy)z \\ &= ((a — b) \times (c — d)) \times (e — f) \\ &= ((ac + bd) — (ad + bc)) \times (e — f) && \text{(by Def. I.4.1.2)} \\ &= ((ac + bd)e + (ad + bc)f) — ((ac + bd)f + (ad + bc)e) && \text{(by Def. I.4.1.2)} \\ &= ((ac)e + (bd)e + (ad)f + (bc)f) — ((ac)f + (bd)f + (ad)e + (bc)e) && \text{(by Prop. I.2.3.4)} \\ &= (a(ce) + b(de) + a(df) + b(cf)) — (a(cf) + b(df) + a(de) + b(ce)) && \text{(by Prop. I.2.3.5)} \\ &= (a(ce) + a(df) + b(cf) + b(de)) — (a(cf) + a(de) + b(ce) + b(df)) && \text{(by Prop. I.2.2.4)} \\ &= (a(ce + df) + b(cf + de)) — (a(cf + de) + b(ce + df)) && \text{(by Prop. I.2.3.4)} \\ &= (a — b) \times ((ce + df) — (cf + de)) && \text{(by Def. I.4.1.2)} \\ &= (a — b) \times ((c — d) \times (e — f)) && \text{(by Def. I.4.1.2)} \\ &= x(yz). \end{aligned}$$

Thus, the multiplication on integers is associative.

Next we show that $x1 = 1x = x$. Since the multiplication on integers is commutative, we know that $x1 = 1x$. So we only need to show that $x1 = x$.

$$\begin{aligned} x1 &= (a — b) \times (1 — 0) && \text{(by A.Cor. I.4.1.2)} \\ &= (a1 + b0) — (a0 + b1) && \text{(by Def. I.4.1.2)} \\ &= (a + 0) — (0 + b) && \text{(by A.Cor. I.2.3.2 and I.2.3.4)} \\ &= a — b && \text{(by Def. I.2.2.1 and Lem. I.2.2.2)} \\ &= x. \end{aligned}$$

Thus, 1 is the multiplicative identity on integers.

Next we show that $x(y+z) = xy + xz$.

$$\begin{aligned}
x(y+z) &= (a \text{—} b) \times ((c \text{—} d) + (e \text{—} f)) \\
&= (a \text{—} b) \times ((c+e) \text{—} (d+f)) && \text{(by Def. I.4.1.2)} \\
&= (a(c+e) + b(d+f)) \text{—} (a(d+f) + b(c+e)) && \text{(by Def. I.4.1.2)} \\
&= (ac + ae + bd + bf) \text{—} (ad + af + bc + be) && \text{(by Prop. I.2.3.4)} \\
&= (ac + bd + ae + bf) \text{—} (ad + bc + af + be) && \text{(by Prop. I.2.2.4)} \\
&= ((ac + bd) \text{—} (ad + bc)) + ((ae + bf) \text{—} (af + be)) && \text{(by Def. I.4.1.2)} \\
&= (a \text{—} b) \times (c \text{—} d) + (a \text{—} b) \times (e \text{—} f) && \text{(by Def. I.4.1.2)} \\
&= xy + xz.
\end{aligned}$$

Thus, the multiplication and addition on integers are left distributive.

Finally we show that $(y+z)x = yx + zx$.

$$\begin{aligned}
(y+z)x &= x(y+z) && \text{(multiplication is commutative)} \\
&= xy + xz && \text{(multiplication and addition are left distributive)} \\
&= yx + zx. && \text{(multiplication is commutative)}
\end{aligned}$$

Thus, the multiplication and addition on integers are right distributive. □

Rmk. I.4.1.7. The set of nine identities in Prop. I.4.1.6 have a name; they are asserting that the integers form a *commutative ring*. (If one deleted the identity $xy = yx$, then they would only assert that the integers form a *ring*). Note that some of these identities were already proven for the natural numbers, but this does not automatically mean that they also hold for the integers because the integers are a larger set than the natural numbers. On the other hand, this proposition supercedes many of the propositions derived earlier for natural numbers.

A.Cor. I.4.1.4. We now define the operation of *subtraction* $x - y$ of two integers by the formula

$$x - y := x + (-y).$$

We do not need to verify the substitution axiom for this operation, since we have defined subtraction in terms of two other operations on integers, namely addition and negation, and we have already verified that those operations are well-defined.

One can easily check now that if a and b are natural numbers, then

$$a - b = a + -b = (a \text{—} 0) + (0 \text{—} b) = a \text{—} b,$$

and so $a \text{—} b$ is just the same thing as $a - b$. Because of this we can now discard the — notation, and use the familiar operation of subtraction instead. (As remarked before, we could not use subtraction immediately because it would be circular.)

A.Cor. I.4.1.5. Let x, y be integers. Then $-x = (-1)x$ and $-xy = (-x)y = x(-y)$.

Proof of A.Cor. I.4.1.5. Let $x = (a—b)$ where $a, b \in \mathbb{N}$. Then we have

$$\begin{aligned} -x &= (b—a) && \text{(by Def. I.4.1.4)} \\ &= (1b—1a) && \text{(by A.Cor. I.2.3.4)} \\ &= (0 + 1b)—(0 + 1a) && \text{(by Def. I.2.2.1)} \\ &= (0a + 1b)—(0b + 1a) && \text{(by Def. I.2.3.1)} \\ &= (0—1) \times (a—b) && \text{(by Def. I.4.1.2)} \\ &= (-1) \times (a—b) && \text{(by Def. I.4.1.4)} \\ &= (-1)x. \end{aligned}$$

So we can use Prop. I.4.1.6 to show that

$$\begin{aligned} -xy &= (-1)(xy) && \text{(from the proof above)} \\ &= ((-1)x)y && \text{(by Prop. I.4.1.6)} \\ &= (-x)y && \text{(from the proof above)} \\ &= (-1)(yx) && \text{(by Prop. I.4.1.6)} \\ &= ((-1)y)x && \text{(by Prop. I.4.1.6)} \\ &= (-y)x && \text{(from the proof above)} \\ &= x(-y). && \text{(by Prop. I.4.1.6)} \end{aligned}$$

□

A.Cor. I.4.1.6. Let x be an integer. Then $x = -(-x)$.

Proof of A.Cor. I.4.1.6. Let $x = (a—b)$ where $a, b \in \mathbb{N}$. Then we have

$$\begin{aligned} -(-x) &= -(b—a) && \text{(by Def. I.4.1.4)} \\ &= (a—b) && \text{(by Def. I.4.1.4)} \\ &= x. \end{aligned}$$

□

A.Cor. I.4.1.7. Let x, y be integers. Then $(-x)(-y) = xy$.

Proof of A.Cor. I.4.1.7. We have

$$\begin{aligned} (-x)(-y) &= ((-1)x)(-y) && \text{(by A.Cor. I.4.1.5)} \\ &= (-1)(x(-y)) && \text{(by Prop. I.4.1.6)} \\ &= -(x(-y)) && \text{(by A.Cor. I.4.1.5)} \\ &= x(-(-y)) && \text{(by A.Cor. I.4.1.5)} \\ &= xy. && \text{(by A.Cor. I.4.1.6)} \end{aligned}$$

□

A.Cor. I.4.1.8. Let x, y be integers. Then we have $x = -x \iff x = 0$ and $x - y = 0 \iff x = y$.

Proof of A.Cor. I.4.1.8. First, we show that $x = -x \iff x = 0$. Let $x = a —b$ for some $a, b \in \mathbb{N}$. Then we have

$$\begin{aligned}
&x = -x \\
\iff\ & a—b = b—a && \text{(by Def. I.4.1.4)} \\
\iff\ & a + a = b + b && \text{(by Def. I.4.1.1)} \\
\iff\ & 2a = 2b && \text{(by Def. I.2.3.1)} \\
\iff\ & a = b && \text{(by Cor. I.2.3.7)} \\
\iff\ & a + 0 = 0 + b && \text{(by Def. I.2.2.1 and Lem. I.2.2.2)} \\
\iff\ & a—b = 0—0 && \text{(by Def. I.4.1.1)} \\
\iff\ & x = 0. && \text{(by A.Cor. I.4.1.2)}
\end{aligned}$$

Now we show that $x - y = 0 \iff x = y$. Let $x = a—b$ and $y = c—d$ for some $a, b, c, d \in \mathbb{N}$. Then we have

$$\begin{aligned}
&x - y = x + (-y) = 0 && \text{(by A.Cor. I.4.1.4)} \\
\iff\ & (a—b) + (d—c) = 0—0 && \text{(by A.Cor. I.4.1.2 and Def. I.4.1.4)} \\
\iff\ & (a + d)—(b + c) = 0—0 && \text{(by Def. I.4.1.2)} \\
\iff\ & a + d + 0 = b + c + 0 && \text{(by Def. I.4.1.1)} \\
\iff\ & a + d = b + c && \text{(by Lem. I.2.2.2)} \\
\iff\ & a + d = c + b && \text{(by Prop. I.2.2.4)} \\
\iff\ & a—b = c—d && \text{(by Def. I.4.1.1)} \\
\iff\ & x = y.
\end{aligned}$$

□

Prop. I.4.1.8 (Integers have no zero divisors)**.** Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$ (or both).

Proof of Prop. I.4.1.8. Suppose for the sake of contradiction that $(a \neq 0) \wedge (b \neq 0)$. By Lem. I.4.1.5 we can split into four cases:

- a is positive and b is positive. Then by Def. I.2.2.7 and A.Cor. I.4.1.2, we know that a, b are positive natural numbers. But by Lem. I.2.3.3, we must have $(a = 0) \vee (b = 0)$, a contradiction.

- a is negative and b is positive. Then by Def. I.4.1.4, $-a$ is positive and we have

$$(-a)b = ((-1)a)b \qquad \text{(by A.Cor. I.4.1.5)}$$

$$
\begin{aligned}
&= (-1)(ab) && \text{(by Prop. I.4.1.6)}\\
&= (-1)0 && \\
&= 0. && \text{(by Def. I.4.1.2)}
\end{aligned}
$$

Using the same argument as in the first case, we see that $(-a = 0) \vee (b = 0)$. But by A.Cor. I.4.1.6 and I.4.1.8, we know that $-a = 0 \implies -(-a) = a = -0 = 0$. Thus, we have $(a = 0) \vee (b = 0)$, a contradiction.

- a is positive and b is negative. We can use the same argument as the second case and switch the role of a, b to derive $(b = 0) \vee (a = 0)$. But again this contradicts $(a \neq 0) \wedge (b \neq 0)$.
- a is negative and b is negative. Then by Def. I.4.1.4, $-a$ and $-b$ are positive. By A.Cor. I.4.1.7, we know that $(-a)(-b) = ab$. This means $(-a = 0) \vee (-b = 0)$. Using the same argument as in the second case, we have $(a = 0) \vee (b = 0)$, a contradiction.

From all cases above, we derived contradiction. Thus, we must have $(a = 0) \vee (b = 0)$. □

Cor. I.4.1.9 (Cancellation law for integers)**.** If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.

Proof of Cor. I.4.1.9. We have

$$
\begin{aligned}
& ac = bc && \\
\implies & ac + (-bc) = bc + (-bc) = 0 && \text{(by Prop. I.4.1.6)}\\
\implies & ac + (-b)c = 0 && \text{(by A.Cor. I.4.1.5)}\\
\implies & (a + (-b))c = 0 && \text{(by Prop. I.4.1.6)}\\
\implies & (a + (-b) = 0) \vee (c = 0) && \text{(by Prop. I.4.1.8)}\\
\implies & a + (-b) = 0 && (c \neq 0)\\
\implies & a = b. && \text{(by A.Cor. I.4.1.8)}
\end{aligned}
$$

□

Def. I.4.1.10 (Ordering of the integers)**.** Let n and m be integers. We say that n is *greater than or equal* to m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is *strictly greater than* m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.

Lem. I.4.1.11 (Properties of order)**.** Let a, b, c be integers.

(a) $a > b$ iff $a - b$ is a positive natural number.

(b) (Addition preserves order) If $a > b$, then $a + c > b + c$.

(c) (Positive multiplication preserves order) If $a > b$ and c is positive, then $ac > bc$.

(d) (Negation reverses order) If $a > b$, then $-a < -b$.

(e) (Order is transitive) If $a > b$ and $b > c$, then $a > c$.

(f) (Order trichotomy) Exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.

Proof of Lem. I.4.1.11(a). We have

$$\begin{aligned} & a > b \\ \iff & (\exists m \in \mathbb{N} : a = b + m) \wedge (a \neq b) && \text{(by Def. I.4.1.10)} \\ \iff & \exists m \in \mathbb{N} : a - b = m \neq 0 && \text{(by Prop. I.4.1.6)} \\ \iff & a - b \text{ is a positive natural number.} && \text{(by Def. I.2.2.7)} \end{aligned}$$

□

Proof of Lem. I.4.1.11(b). We have

$$\begin{aligned} & a > b \\ \Longrightarrow & (\exists m \in \mathbb{N} : a = b + m) \wedge (a \neq b) && \text{(by Def. I.4.1.10)} \\ \Longrightarrow & (\exists m \in \mathbb{N} : a + c = b + m + c) \wedge (a + c \neq b + c) && \text{(by Lem. I.4.1.3)} \\ \Longrightarrow & (\exists m \in \mathbb{N} : a + c = b + c + m) \wedge (a + c \neq b + c) && \text{(by Prop. I.4.1.6)} \\ \Longrightarrow & a + c > b + c. && \text{(by Def. I.4.1.10)} \end{aligned}$$

□

Proof of Lem. I.4.1.11(c). We have

$$\begin{aligned} & a > b \\ \Longrightarrow & \exists m \in \mathbb{N} : (a - b = m) \wedge (m > 0) && \text{(by Lem. I.4.1.11(a))} \\ \Longrightarrow & \exists m \in \mathbb{N} : ((a - b)c = ac - bc = mc) \wedge (m > 0) && \text{(by Lem. I.4.1.3 and Prop. I.4.1.6)} \\ \Longrightarrow & \exists m \in \mathbb{N} : (ac - bc = mc) \wedge (mc > 0c = 0) && \text{(by Def. I.2.3.1 and Prop. I.2.3.6)} \\ \Longrightarrow & ac > bc. && \text{(by Lem. I.4.1.11(a))} \end{aligned}$$

□

Proof of Lem. I.4.1.11(d). We have

$$\begin{aligned} & a > b \\ \Longrightarrow & \exists m \in \mathbb{N} : (a - b = m) \wedge (m > 0) && \text{(by Lem. I.4.1.11(a))} \\ \Longrightarrow & \exists m \in \mathbb{N} : (-(-a) - b = m) \wedge (m > 0) && \text{(by A.Cor. I.4.1.6)} \\ \Longrightarrow & \exists m \in \mathbb{N} : ((-b) - (-a) = m) \wedge (m > 0) && \text{(by A.Cor. I.4.1.4 and Prop. I.4.1.6)} \\ \Longrightarrow & -b > -a. && \text{(by Lem. I.4.1.11(a))} \end{aligned}$$

□

Proof of Lem. I.4.1.11(e). We have

$$\begin{aligned}
&(a > b) \wedge (b > c) \\
\implies &\exists m, n \in \mathbb{N} : (a = b + m) \wedge (b = c + n) \wedge (m > 0) \wedge (n > 0) && \text{(by Lem. I.4.1.11(a))} \\
\implies &\exists m, n \in \mathbb{N} : (a = c + n + m) \wedge (m > 0) \wedge (n > 0) && \text{(by Lem. I.4.1.3)} \\
\implies &\exists m, n \in \mathbb{N} : (a = c + n + m) \wedge (n + m > 0) && \text{(by Prop. I.2.2.8)} \\
\implies &a > c. && \text{(by Lem. I.4.1.11(a))}
\end{aligned}$$

□

Proof of Lem. I.4.1.11(f). By Lem. I.4.1.5, $a - b$ can be exactly one of the following three statements:

- $a - b = 0$. Then by A.Cor. I.4.1.8, we have $a = b$.
- $a - b$ is a positive natural number. Then by Lem. I.4.1.11(a), we have $a > b$.
- $a - b = -n$ where n is a positive natural number. Then we have

$$\begin{aligned}
&a - b = -n \\
\iff &-(a - b) = -(-n) && \text{(by A.Cor. I.4.1.3)} \\
\iff &b - a = n, && \text{(by Def. I.4.1.4 and A.Cor. I.4.1.6)}
\end{aligned}$$

  and by Lem. I.4.1.11(a), we have $b > a$. By Def. I.4.1.10, we have $b > a \iff a < b$.

Thus, we conclude that exactly one of the statements $a = b$, $a > b$, or $a < b$ is true. □

A.Cor. I.4.1.9. Let $m, n \in \mathbb{Z}$. We define the following eight subsets of $\mathbb{Z}$:

$$\begin{aligned}
\mathbb{Z}_{\leq n} &:= \{i \in \mathbb{Z} : i \leq n\}; & \mathbb{Z}_{<n} &:= \{i \in \mathbb{Z} : i < n\}; & \mathbb{Z}^+ &:= \mathbb{Z}_{>0}; \\
\mathbb{Z}_{\geq n} &:= \{i \in \mathbb{Z} : i \geq n\}; & \mathbb{Z}_{>n} &:= \{i \in \mathbb{Z} : i > n\}; & \mathbb{Z}^- &:= \mathbb{Z}_{<0}; \\
\mathbb{Z}_{m \leq n} &:= \{i \in \mathbb{Z} : m \leq i \leq n\}; & \mathbb{Z}_{m<n} &:= \{i \in \mathbb{Z} : m < i < n\}.
\end{aligned}$$

In particular, we have $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

— Exercises —

Ex. I.4.1.1. Verify that the definition of equality on the integers is both reflexive and symmetric.

Proof of Ex. I.4.1.1. See A.Cor. I.4.1.1. □

Ex. I.4.1.2. Show that the definition of negation on the integers is well-defined in the sense that if $(a{-\!\!-}b) = (a'{-\!\!-}b')$, then $-(a{-\!\!-}b) = -(a'{-\!\!-}b')$ (so equal integers have equal negations).

Proof of Ex. I.4.1.2. See A.Cor. I.4.1.3. □

Ex. I.4.1.3. Show that $(-1) \times a = -a$ for every integer a.

Proof of Ex. I.4.1.3. See A.Cor. I.4.1.5. □

Ex. I.4.1.4. Prove the remaining identities in Prop. I.4.1.6.

Proof of Ex. I.4.1.4. See Prop. I.4.1.6. □

Ex. I.4.1.5. Prove Prop. I.4.1.8.

Proof of Ex. I.4.1.5. See Prop. I.4.1.8. □

Ex. I.4.1.6. Prove Cor. I.4.1.9.

Proof of Ex. I.4.1.6. See Cor. I.4.1.9. □

Ex. I.4.1.7. Prove Lem. I.4.1.11.

Proof of Ex. I.4.1.7. See Lem. I.4.1.11. □

Ex. I.4.1.8. Show that the principle of induction (Ax. I.2.5) does not apply directly to the integers. More precisely, give an example of a property $P(n)$ pertaining to an integer n such that $P(0)$ is true, and that $P(n)$ implies $P(n{+}{+})$ for all integers n, but that $P(n)$ is not true for all integers n. Thus, induction is not as useful a tool for dealing with the integers as it is with the natural numbers. (The situation becomes even worse with the rational and real numbers, which we shall define shortly.)

Proof of Ex. I.4.1.8. For the sake of contradiction, we claim that $n + 1 > 0$ for all $n \in \mathbb{Z}$. We use Ax. I.2.5 to prove the claim. For $n = 0$, by Def. I.4.1.10, we know that $(0 + 1 = 1) \wedge (1 \neq 0) \implies 1 > 0$. So the base case holds. Suppose inductively that $n+1 > 0$ for some $n \in \mathbb{N}$. Then for $n + 1$, we need to show that $(n + 1) + 1 > 0$. By the induction hypothesis, we know that $n + 1 > 0$. Then by Lem. I.4.1.11(b), we have $(n + 1) + 1 > 0 + 1 = 1$. Since $(n + 1) + 1 > 1$ and $1 > 0$, by Lem. I.4.1.11(e), we have $(n + 1) + 1 > 0$. This closes the induction.

Now let $n = -1$. By Prop. I.4.1.6, we have $(-1) + 1 = 0$. But by Def. I.4.1.10, this means $-1 > 0$, which implies $0 > 1$ by Lem. I.4.1.11(b). This contradict to $0 < 1$. Thus, one can not use induction on the integers as it is with the natural numbers. □

I.4.2 The rationals

Def. I.4.2.1. A *rational number* is an expression of the form $a//b$, where a and b are integers and b is non-zero; $a//0$ is not considered to be a rational number. Two rational numbers are considered to be equal, $a//b = c//d$, iff $ad = cb$. The set of all rational numbers is denoted $\mathbb{Q}$.

Note. There is no reasonable way we can divide by zero, since one cannot have both the identities $(a/b) \times b = a$ and $c \times 0 = 0$ hold simultaneously if b is allowed to be zero and a is non-zero. Similarly, the identities $a/a = 1$ and $2 \times (a/a) = (2 \times a)/a$ cannot hold simultaneously if $0/0$ is defined. However, we can eventually get a reasonable notion of dividing by a quantity which approaches zero - think of L'Hôpital's rule (see Sec. I.10.5), which suffices for doing things like defining differentiation.

A.Cor. I.4.2.1. The definition of equality for the rational numbers is reflexive, symmetric and transitive.

Proof of A.Cor. I.4.2.1. Let $a//b, c//d, e//f$ be rational numbers where $a, b, c, d, e, f \in \mathbb{Z}$ and $b, d, f \neq 0$. Since

$$\begin{aligned} & ab = ab && \text{(by Lem. I.4.1.3)} \\ \implies & a//b = a//b, && \text{(by Def. I.4.2.1)} \end{aligned}$$

we know that Def. I.4.2.1 is reflexive.

Next suppose that $a//b = c//d$. Then we have

$$\begin{aligned} & a//b = c//d \\ \implies & ad = cb && \text{(by Def. I.4.2.1)} \\ \implies & cb = ad && \text{(by Lem. I.4.1.3)} \\ \implies & c//d = a//b. && \text{(by Def. I.4.2.1)} \end{aligned}$$

Thus, Def. I.4.2.1 is symmetric.

Finally suppose that $a//b = c//d$ and $c//d = e//f$. Then we have

$$\begin{aligned} & (a//b = c//d) \wedge (c//d = e//f) \\ \implies & (ad = cb) \wedge (cf = ed) && \text{(by Def. I.4.2.1)} \\ \implies & (adf = cbf) \wedge (cfb = edb) && \text{(by Lem. I.4.1.3)} \\ \implies & (afd = cbf) \wedge (cbf = ebd) && \text{(by Prop. I.4.1.6)} \\ \implies & afd = ebd && \text{(by Lem. I.4.1.3)} \\ \implies & af = eb && \text{(by Cor. I.4.1.9)} \\ \implies & a//b = e//f. && \text{(by Def. I.4.2.1)} \end{aligned}$$

Thus, Def. I.4.2.1 is transitive. □

Def. I.4.2.2. If $a//b$ and $c//d$ are rational numbers, we define their sum

$$(a//b) + (c//d) \coloneqq (ad + bc)//(bd)$$

their product

$$(a//b) \times (c//d) \coloneqq (ac)//(bd)$$

and the negation

$$-(a//b) \coloneqq (-a)//b.$$

Note that if b and d are non-zero, then bd is also non-zero, by Prop. I.4.1.8, so the sum or product of two rational numbers remains a rational number.

Lem. I.4.2.3. The sum, product, and negation operations on rational numbers are well-defined, in the sense that if one replaces $a//b$ with another rational number $a'//b'$ which is equal to $a//b$, then the output of the above operations remains unchanged, and similarly for $c//d$.

Proof of Lem. I.4.2.3. We first show that the addition on rationals numbers is well-defined. Suppose $a//b = a'//b'$, so that b and b' are non-zero and $ab' = a'b$. We now show that $(a//b) + (c//d) = (a'//b') + (c//d)$. By Def. I.4.2.2, the left-hand side is $(ad + bc)//bd$ and the right-hand side is $(a'd + b'c)//b'd$. So by Def. I.4.2.1, we have to show that

$$(ad + bc)b'd = (a'd + b'c)bd,$$

which expands to

$$ab'd^2 + bb'cd = a'bd^2 + bb'cd.$$

But since $ab' = a'b$, the claim follows. Similarly, suppose $c//d = c'//d'$, so that d and d' are non-zero and $cd' = c'd$. We show that $(a//b)+(c//d) = (a//b)+(c'//d')$. By Def. I.4.2.2, the left-hand side is $(ad + bc)//bd$ and the right-hand side is $(ad' + bc')//bd'$. So by Def. I.4.2.1, we have to show that

$$(ad + bc)bd' = (ad' + bc')bd,$$

which expands to

$$abdd' + b^2cd' = abdd' + b^2c'd.$$

But since $cd' = c'd$, the claim follows.

Next we show that the multiplication on rationals numbers is well-defined. Suppose $a//b = a'//b'$, so that b and b' are non-zero and $ab' = a'b$. We now show that $(a//b)\times(c//d) = (a'//b') \times (c//d)$. By Def. I.4.2.2, the left-hand side is $(ac)//(bd)$ and the right-hand side is $(a'c)//(b'd)$. So by Def. I.4.2.1, we have to show that

$$(ac)(b'd) = (a'c)(bd),$$

which is equivalent to

$$ab'cd = a'bcd.$$

But since $ab' = a'b$, the claim follows. Similarly, suppose $c//d = c'//d'$, so that d and d' are non-zero and $cd' = c'd$. We now show that $(a//b) \times (c//d) = (a//b) \times (c'//d')$. By Def. I.4.2.2, the left-hand side is $(ac)//(bd)$ and the right-hand side is $(ac')//(bd')$. So by Def. I.4.2.1, we have to show that

$$(ac)(bd') = (ac')(bd),$$

which is equivalent to

$$abcd' = abc'd.$$

But since $cd' = c'd$, the claim follows.

Finally we show that the negation on rationals numbers is well-defined. Suppose $a//b = a'//b'$, so that b and b' are non-zero and $ab' = a'b$. We now show that $-(a//b) = -(a'//b')$. By Def. I.4.2.2, the left-hand side is $(-a)//b$ and the right-hand side is $(-a')//b'$. So by Def. I.4.2.1, we have to show that

$$(-a)b' = (-a')b,$$

which by A.Cor. I.4.1.5 is equivalent to

$$(-1)ab' = (-1)a'b.$$

But since $ab' = a'b$, the claim follows from Lem. I.4.1.3. □

A.Cor. I.4.2.2. The rational numbers $a//1$ behave in a manner identical to the integers a:

$$\begin{aligned}(a//1) + (b//1) &= (a+b)//1;\\ (a//1) \times (b//1) &= (ab//1);\\ -(a//1) &= (-a)//1.\end{aligned}$$

Also, $a//1$ and $b//1$ are only equal when a and b are equal. Because of this, we will identify a with $a//1$ for each integer a: $a \equiv a//1$; the above identities then guarantee that the arithmetic of the integers is consistent with the arithmetic of the rationals. Thus, just as we embedded the natural numbers inside the integers, we embed the integers inside the rational numbers. In particular, all natural numbers are rational numbers, for instance 0 is equal to $0//1$ and 1 is equal to $1//1$.

Observe that a rational number $a//b$ is equal to $0 = 0//1$ iff $a \times 1 = b \times 0$, i.e., if the numerator a is equal to 0. Thus, if a and b are non-zero then so is $a//b$.

Proof of A.Cor. I.4.2.2. Let $a, b \in \mathbb{Z}$. First, we show that $(a//1) + (b//1) = (a+b)//1$. This is true since

$$\begin{aligned}(a//1) + (b//1) &= (a1 + b1)//(1 \times 1) && \text{(by Def. I.4.2.2)}\\ &= (a+b)//1. && \text{(by Prop. I.4.1.6)}\end{aligned}$$

Next we show that $(a//1) \times (b//1) = (ab)//1$. This is true since

$$\begin{aligned} (a//1) \times (b//1) &= (ab)//(1 \times 1) && \text{(by Def. I.4.2.2)} \\ &= (ab)//1. && \text{(by Prop. I.4.1.6)} \end{aligned}$$

Next we show that $-(a//1) = (-a)//1$. This is true by Def. I.4.2.2.
Finally we show that $a//1 = b//1 \iff a = b$. This is true since

$$\begin{aligned} & a//1 = b//1 \\ \iff & a1 = b1 && \text{(by Def. I.4.2.1)} \\ \iff & a = b. && \text{(by Prop. I.4.1.6)} \end{aligned}$$

□

A.Cor. I.4.2.3. We now define a new operation on the rationals: reciprocal. If $x = a//b$ is a non-zero rational (so that $a, b \neq 0$) then we define the *reciprocal* x^{-1} of x to be the rational number $x^{-1} := b//a$. The reciprocal operation on rational numbers is consistent with Def. I.4.2.1: if two rational numbers $a//b, a'//b'$ are equal, then their reciprocals are also equal. In contrast to reciprocal, an operation such as "numerator" is not well-defined: the rationals $3//4$ and $6//8$ are equal, but have unequal numerators, so we have to be careful when referring to such terms as "the numerator of x." We however leave the reciprocal of 0 undefined.

Proof of A.Cor. I.4.2.3. By Def. I.4.2.1 and the definition of reciprocal we have $a, a', b, b' \neq 0$. Then we have

$$\begin{aligned} & a//b = a'//b' \\ \Longrightarrow\ & ab' = a'b && \text{(by Def. I.4.2.1)} \\ \Longrightarrow\ & b'a = ba' && \text{(by Prop. I.4.1.6)} \\ \Longrightarrow\ & b'//a' = b//a. && \text{(by Def. I.4.2.1)} \end{aligned}$$

□

Prop. I.4.2.4 (Laws of algebra for rationals)**.** Let x, y, z be rationals. Then the following laws of algebra hold:

$$\begin{aligned} x + y &= y + x \\ (x + y) + z &= x + (y + z) \\ x + 0 = 0 + x &= x \\ x + (-x) = (-x) + x &= 0 \\ xy &= yx \\ (xy)z &= x(yz) \end{aligned}$$

$$x1 = 1x = x$$
$$x(y+z) = xy + xz$$
$$(y+z)x = yx + zx.$$

If x is non-zero, we also have

$$xx^{-1} = x^{-1}x = 1.$$

Proof of Prop. I.4.2.4. To prove this identity, one writes $x = a//b, y = c//d, z = e//f$ for some integers a, c, e and non-zero integers b, d, f, and verifies each identity in turn using the algebra of the integers.

First, we show that $x + y = y + x$.

$$\begin{aligned}
x + y &= (a//b) + (c//d) \\
&= (ad + bc)//bd && \text{(by Def. I.4.2.2)} \\
&= (bc + ad)//bd && \text{(by Prop. I.4.1.6)} \\
&= (cb + da)//db && \text{(by Prop. I.4.1.6)} \\
&= (c//d) + (a//b) && \text{(by Def. I.4.2.2)} \\
&= y + x.
\end{aligned}$$

Thus, the addition on rationals is commutative.

Next we show that $(x + y) + z = x + (y + z)$.

$$\begin{aligned}
(x + y) + z &= ((a//b) + (c//d)) + (e//f) \\
&= ((ad + bc)//bd) + (e//f) && \text{(by Def. I.4.2.2)} \\
&= ((ad + bc)f + (bd)e)//((bd)f) && \text{(by Def. I.4.2.2)} \\
&= ((ad)f + (bc)f + (bd)e)//((bd)f) && \text{(by Prop. I.4.1.6)} \\
&= (a(df) + b(cf) + b(de))//(b(df)) && \text{(by Prop. I.4.1.6)} \\
&= (a(df) + b(cf + de))//(b(df)) && \text{(by Prop. I.4.1.6)} \\
&= (a//b) + ((cf + de)//df) && \text{(by Def. I.4.2.2)} \\
&= (a//b) + ((c//d) + (e//f)) && \text{(by Def. I.4.2.2)} \\
&= x + (y + z).
\end{aligned}$$

Thus, the addition on rationals is associative.

Next we show that $x + 0 = 0 + x = x$. Since the addition on rationals is commutative, we know that $x + 0 = 0 + x$. Thus, we only need to show that $x + 0 = x$.

$$\begin{aligned}
x + 0 &= (a//b) + (0//1) && \text{(by A.Cor. I.4.2.2)} \\
&= (a1 + b0)//b1 && \text{(by Def. I.4.2.2)} \\
&= (a + 0)//b && \text{(by Prop. I.4.1.6)} \\
&= a//b && \text{(by Prop. I.4.1.6)}
\end{aligned}$$

$$= x.$$

Thus, 0 is the additive identity on rationals.

Next we show that $x + (-x) = (-x) + x = 0$. Since the addition on rationals is commutative, we know that $x + (-x) = (-x) + x$. Thus, we only need to show that $x + (-x) = 0$.

$$\begin{aligned}
x + (-x) &= (a//b) + ((-a)//b) && \text{(by Def. I.4.2.2)}\\
&= (ab + b(-a))//b^2 && \text{(by Def. I.4.2.2)}\\
&= (ab + (-a)b)//b^2 && \text{(by Prop. I.4.1.6)}\\
&= (ab + (-(ab))//b^2 && \text{(by A.Cor. I.4.1.5)}\\
&= 0//b^2 && \text{(by Prop. I.4.1.6)}\\
&= 0. && \text{(by A.Cor. I.4.2.2)}
\end{aligned}$$

Thus, the additive inverse of rational x is $-x$.

Next we show that $xy = yx$.

$$\begin{aligned}
xy &= (a//b) \times (c//d)\\
&= ac//bd && \text{(by Def. I.4.2.2)}\\
&= ca//db && \text{(by Prop. I.4.1.6)}\\
&= (c//d) \times (a//b) && \text{(by Def. I.4.2.2)}\\
&= yx.
\end{aligned}$$

Thus, the multiplication on rationals is commutative.

Next we show that $(xy)z = x(yz)$.

$$\begin{aligned}
(xy)z &= ((a//b) \times (c//d)) \times (e//f)\\
&= (ac//bd) \times (e//f) && \text{(by Def. I.4.2.2)}\\
&= ((ac)e)//((bd)f) && \text{(by Def. I.4.2.2)}\\
&= (a(ce))//(b(df)) && \text{(by Prop. I.4.1.6)}\\
&= (a//b) \times (ce//df) && \text{(by Def. I.4.2.2)}\\
&= (a//b) \times ((c//d) \times (e//f)) && \text{(by Def. I.4.2.2)}\\
&= x(yz).
\end{aligned}$$

Thus, the multiplication on rationals is associative.

Next we show that $x1 = 1x = x$. Since the multiplication on rationals is commutative, we know that $x1 = 1x$. Thus, we only need to show that $x1 = x$.

$$\begin{aligned}
x1 &= (a//b) \times (1//1) && \text{(by A.Cor. I.4.2.2)}\\
&= a1//b1 && \text{(by Def. I.4.2.2)}
\end{aligned}$$

$$\begin{aligned}
&= a//b && \text{(by Prop. I.4.1.6)}\\
&= x.
\end{aligned}$$

Thus, 1 is the multiplicative identity on rationals.

Next we show that $x(y+z) = xy + xz$.

$$\begin{aligned}
x(y+z) &= (a//b) \times ((c//d) + (e//f)) \\
&= (a//b) \times ((cf + de)//df) && \text{(by Def. I.4.2.2)}\\
&= (a(cf+de))//(b(df)) && \text{(by Def. I.4.2.2)}\\
&= (b(a(cf+de)))//(b^2(df)) && \text{(by Lem. I.4.2.3)}\\
&= ((ba)(cf+de))//(b^2(df)) && \text{(by Prop. I.4.1.6)}\\
&= ((ba)(cf) + (ba)(de))//(b^2(df)) && \text{(by Prop. I.4.1.6)}\\
&= ((ab)(fc) + (ba)(ed))//(b^2(fd)) && \text{(by Prop. I.4.1.6)}\\
&= (a(bf)c + b(ae)d)//(b(bf)d) && \text{(by Prop. I.4.1.6)}\\
&= ((ac)(bf) + (bd)(ae))//((bd)(bf)) && \text{(by Prop. I.4.1.6)}\\
&= (ac//bd) + (ae//bf) && \text{(by Def. I.4.2.2)}\\
&= ((a//b) \times (c//d)) + ((a//b) \times (e//f)) && \text{(by Def. I.4.2.2)}\\
&= xy + xz.
\end{aligned}$$

Thus, the multiplication and addition on rationals are left distributive.

Next we show that $(y+z)x = yx + zx$.

$$\begin{aligned}
(y+z)x &= x(y+z) && \text{(multiplication is commutative)}\\
&= xy + xz && \text{(multiplication and addition are left distributive)}\\
&= yx + zx. && \text{(multiplication is commutative)}
\end{aligned}$$

Thus, the multiplication and addition on rationals are right distributive.

Finally we show that if $x \neq 0$, then $xx^{-1} = x^{-1}x = 1$. Since the multiplication on rationals is commutative, we know that $xx^{-1} = x^{-1}x$. Thus, we only need to show that $xx^{-1} = 1$.

$$\begin{aligned}
xx^{-1} &= (a//b) \times (b//a) && \text{(by A.Cor. I.4.2.3)}\\
&= ab//ba && \text{(by Def. I.4.2.2)}\\
&= ab//ab && \text{(by Prop. I.4.1.6)}\\
&= 1//1 && \text{(by Def. I.4.2.1)}\\
&= 1. && \text{(by A.Cor. I.4.2.2)}
\end{aligned}$$

Thus, the multiplicative inverse of rational x is x^{-1}. □

Rmk. I.4.2.5. The above set (Prop. I.4.2.4) of ten identities have a name; they are asserting that the rationals $\mathbb{Q}$ form a *field*. This is better than being a commutative ring because of the tenth identity $xx^{-1} = x^{-1}x = 1$. Note that Prop. I.4.2.4 supercedes Prop. I.4.1.6.

A.Cor. I.4.2.4. We can now define the *quotient* x/y of two rational numbers x and y, *provided that* y is non-zero, by the formula

$$x/y \coloneqq x \times y^{-1}.$$

Using the above formula, it is easy to see that $a/b = a//b$ for every integer a and every non-zero integer b. Thus, we can now discard the $//$ notation, and use the more customary a/b instead of $a//b$.

In a similar spirit, we define subtraction on the rationals by the formula

$$x - y \coloneqq x + (-y),$$

just as we did with the integers.

Proof of A.Cor. I.4.2.4. We have

$$\begin{aligned}
a/b &= (a//1)/(b//1) && \text{(by A.Cor. I.4.2.2)}\\
&= (a//1) \times (b//1)^{-1} && \text{(by A.Cor. I.4.2.4)}\\
&= (a//1) \times (1//b) && \text{(by A.Cor. I.4.2.3)}\\
&= a1//1b && \text{(by Def. I.4.2.2)}\\
&= a//b. && \text{(by Prop. I.4.1.6)}
\end{aligned}$$

□

A.Cor. I.4.2.5. Let $x = a/b$ be a rational number where $a, b \in \mathbb{Z}$ and $b \neq 0$. Then we have

$$-x = (-a)/b = a/(-b) = (-1)(a/b) = (-1)x$$

and $-(-x) = x$. If $y \in \mathbb{Q}$, then we have

$$-xy = (-x)y = x(-y).$$

If $y \neq 0$, then we have

$$-(x/y) = (-x)/y = x/(-y).$$

Proof of A.Cor. I.4.2.5. By Def. I.4.2.2 and A.Cor. I.4.2.4 we have $-x = (-a)/b$ and by Lem. I.4.2.3 we have $(-1)x = (-1)(a/b)$. We first show that $(-a)/b = a/(-b)$. This is true since

$$\begin{aligned}
(-a)/b &= ((-a)/b) \times 1 && \text{(by Prop. I.4.2.4)}\\
&= ((-a)/b) \times (1/1) && \text{(by A.Cor. I.4.2.2)}
\end{aligned}$$

$$\begin{aligned}
&= ((-a)/b) \times ((-1)/(-1)) && \text{(by Def. I.4.2.1)}\\
&= ((-a)(-1))/(b(-1)) && \text{(by Def. I.4.2.2)}\\
&= ((-1)(-a))/((-1)b) && \text{(by Prop. I.4.1.6)}\\
&= (-(-a))/(-b) && \text{(by A.Cor. I.4.1.5)}\\
&= a/(-b). && \text{(by A.Cor. I.4.1.6)}
\end{aligned}$$

Next we show that $-x = (-1)x$. This is true since

$$\begin{aligned}
-x &= -(a/b) && \text{(by A.Cor. I.4.2.4)}\\
&= (-a)/b && \text{(by Def. I.4.2.2)}\\
&= ((-1)a)/b && \text{(by A.Cor. I.4.1.5)}\\
&= ((-1)a)/(1b) && \text{(by Prop. I.4.1.6)}\\
&= ((-1)/1) \times (a/b) && \text{(by Def. I.4.2.2)}\\
&= (-1)(a/b) && \text{(by A.Cor. I.4.2.2)}\\
&= (-1)x. && \text{(by A.Cor. I.4.2.4)}
\end{aligned}$$

Next we show that $-(-x) = x$. This is true since

$$\begin{aligned}
-(-x) &= (-1)((-1)x) && \text{(from the proof above)}\\
&= ((-1)(-1))x && \text{(by Prop. I.4.2.4)}\\
&= 1x && \text{(by A.Cor. I.4.1.7)}\\
&= x. && \text{(by Prop. I.4.2.4)}
\end{aligned}$$

Next we show that $-xy = (-x)y = x(-y)$. This is true since

$$\begin{aligned}
-xy &= (-1)(xy) && \text{(from the proof above)}\\
&= ((-1)x)y && \text{(by Prop. I.4.2.4)}\\
&= (-x)y && \text{(from the proof above)}\\
&= (x(-1))y && \text{(by Prop. I.4.2.4)}\\
&= x((-1)y) && \text{(by Prop. I.4.2.4)}\\
&= x(-y). && \text{(from the proof above)}
\end{aligned}$$

Next we show that if $y \neq 0$, then $-(x/y) = (-x)/y$. Suppose that $y \neq 0$. Then we have

$$\begin{aligned}
-(x/y) &= -(x \times y^{-1}) && \text{(by A.Cor. I.4.2.4)}\\
&= (-1) \times (x \times y^{-1}) && \text{(from the proof above)}\\
&= ((-1) \times x) \times y^{-1} && \text{(by Prop. I.4.2.4)}\\
&= (-x) \times y^{-1} && \text{(from the proof above)}\\
&= (-x)/y. && \text{(by A.Cor. I.4.2.4)}
\end{aligned}$$

Finally we show that if $y \neq 0$, then $(-x)/y = x/(-y)$. Suppose that $y = c/d$ where $c, d \neq 0$. Then we have

$$\begin{aligned}
(-x)/y &= (-x) \times y^{-1} && \text{(by A.Cor. I.4.2.4)} \\
&= ((-1) \times x) \times y^{-1} && \text{(from the proof above)} \\
&= ((-1) \times x) \times (d/c) && \text{(by A.Cor. I.4.2.3)} \\
&= (x \times (-1)) \times (d/c) && \text{(by Prop. I.4.2.4)} \\
&= x \times ((-1) \times (d/c)) && \text{(by Prop. I.4.2.4)} \\
&= x \times (d/(-c)) && \text{(from the proof above)} \\
&= x \times ((-c)/d)^{-1} && \text{(by A.Cor. I.4.2.3)} \\
&= x \times (-y)^{-1} && \text{(by Def. I.4.2.2)} \\
&= x/(-y). && \text{(by A.Cor. I.4.2.4)}
\end{aligned}$$

□

Def. I.4.2.6. A rational number x is said to be *positive* iff we have $x = a/b$ for some positive integers a and b. It is said to be *negative* iff we have $x = -y$ for some positive rational y (i.e., $x = (-a)/b$ for some positive integers a and b).

Thus, for instance, every positive integer is a positive rational number, and every negative integer is a negative rational number, so our new definition is consistent with our old one.

Lem. I.4.2.7 (Trichotomy of rationals)**.** Let x be a rational number. Then exactly one of the following three statements is true:

(a) x is equal to 0.

(b) x is a positive rational number.

(c) x is a negative rational number.

Proof of Lem. I.4.2.7. We first show that at least one of (a), (b), (c) is true. Let $x = a/b$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. By Lem. I.4.1.11(f), a can only satisified one of the following three statements: $a = 0$, $a > 0$ and $a < 0$. Similarly, b can only satisified one of the following two statements: $b > 0$ and $b < 0$. We first consider a:

- If $a = 0$, then $x = 0/b = 0$.
- If $a > 0$, then we need to consider b:
  - If $b > 0$, then by Def. I.4.2.6 x is positive.
  - If $b < 0$, then by Def. I.4.1.4 $b = -c$ for some $c \in \mathbb{Z}^+$. Thus, by A.Cor. I.4.2.5 we have $a/b = a/(-c) = (-a)/c$, which means x is negative by Def. I.4.2.6.
- If $a < 0$, then by Def. I.4.1.4 $a = -c$ for some $c \in \mathbb{Z}^+$. Now we consider b:

- If $b > 0$, then $a/b = (-c)/b$, which means x is negative by Def. I.4.2.6.
- If $b < 0$, then by Def. I.4.1.4 $b = -d$ for some $d \in \mathbb{Z}^+$. Thus, by Def. I.4.2.1 we have $a/b = (-c)/(-d) = (-1)/(-1) \times (c/d) = c/d$, which means x is positive by Def. I.4.2.6.

From all cases above, we conclude that at least one of (a), (b), (c) is true.

Now we show that at most one of (a), (b), (c) is true.

- If x is both positive and equal to 0, then by A.Cor. I.4.2.2 and Def. I.4.2.6, $x = a/b = 0/1$, where $a, b \in \mathbb{Z}^+$. But by A.Cor. I.4.2.2, $a/b = 0/1$ means $a = 0$, contradicted to $a > 0$.
- If x is both negative and equal to 0, then by A.Cor. I.4.2.2 and Def. I.4.2.6, $x = (-a)/b = 0/1$, where $a, b \in \mathbb{Z}^+$. But by A.Cor. I.4.2.2 $(-a)/b = 0/1$ means $-a = 0$, so that $a = 0$ by A.Cor. I.4.1.8, contradicted to $a > 0$.
- If x is both positive and negative, then by Def. I.4.2.6, $x = a/b = (-c)/d$, where $a, b, c, d \in \mathbb{Z}^+$. By Def. I.4.2.1 $a/b = (-c)/d$ means $ad = b(-c) = (-1)(bc)$. By Lem. I.2.3.3 we know that ad and bc are positive. But by Def. I.4.1.4 we know that $(-1)(bc) = -(bc)$ is negative, which contradict to Lem. I.4.1.5.

From all cases above, we conclude that no more than one of (a), (b), (c) is true at the same time. □

Def. I.4.2.8 (Ordering of the rationals)**.** Let x and y be rational numbers. We say that $x > y$ iff $x - y$ is a positive rational number, and $x < y$ iff $x - y$ is a negative rational number. We write $x \geq y$ iff either $x > y$ or $x = y$, and similarly define $x \leq y$ iff either $x < y$ or $x = y$.

We define the following eight subsets of $\mathbb{Q}$:

$$\begin{aligned} &\mathbb{Q}_{\leq x} := \{q \in \mathbb{Q} : q \leq x\}; && \mathbb{Q}_{<x} := \{q \in \mathbb{Q} : q < x\}; && \mathbb{Q}^+ := \mathbb{Q}_{>0}; \\ &\mathbb{Q}_{\geq x} := \{q \in \mathbb{Q} : q \geq x\}; && \mathbb{Q}_{>x} := \{q \in \mathbb{Q} : q > x\}; && \mathbb{Q}^- := \mathbb{Q}_{<0}; \\ &\mathbb{Q}_{x \leq y} := \{q \in \mathbb{Q} : x \leq q \leq y\}; && \mathbb{Q}_{x<y} := \{q \in \mathbb{Q} : x < q < y\}. \end{aligned}$$

A.Cor. I.4.2.6. If x and y are two positive rationals, then $x + y$ is also a positive rational number. If x and y are two negative rationals, then $x + y$ is also a negative rational number.

Proof of A.Cor. I.4.2.6. We first show that if x and y are two positive rationals, then $x + y$ is also positive. By Def. I.4.2.6 we have $x = a/b$ and $y = c/d$ where $a, b, c, d \in \mathbb{Z}^+$. Then by Def. I.4.2.2 we have $x + y = (ad + bc)/bd$. By Lem. I.2.3.3 we know that $ad, bc, bd \in \mathbb{Z}^+$. Since $ad, bc \in \mathbb{Z}^+$, by Prop. I.2.2.8 we know that $ad + bc \in \mathbb{Z}^+$. Thus, by Def. I.4.2.6 we know that $x + y = (ad + bc)/bd$ is a positive rational number.

Now we show that if x and y are two negative rationals, then $x + y$ is also negative. By Def. I.4.2.6 we have $x = (-a)/b$ and $y = (-c)/d$ where $a, b, c, d \in \mathbb{Z}^+$. Then by Def. I.4.2.2 we have $x + y = ((-a)d + b(-c))/bd$. By A.Cor. I.4.1.5 and Prop. I.4.1.6 we have

$$(-a)d + b(-c) = (-1)(ad) + (-1)(cb) = (-1)(ad + cb) = -(ad + cb).$$

By Lem. I.2.3.3 we know that $ad, cb, bd \in \mathbb{Z}^+$. Since $ad, cb \in \mathbb{Z}^+$, by Prop. I.2.2.8 we have $ad + cb \in \mathbb{Z}^+$. Thus, by Def. I.4.1.4 we have $-(ad + cb) \in \mathbb{Z}^-$ and by Def. I.4.2.6 $x + y = -(ad + cb)/bd$ is a negative rational number. □

A.Cor. I.4.2.7. Let x and y be two rationals. If x and y are positive, then xy is positive. If x and y are negative, then xy is positive.

Proof of A.Cor. I.4.2.7. We first show that if x and y are two positive rationals, then xy is a positive rational number. By Def. I.4.2.6 we know that $x = a/b$ and $y = c/d$ where $a, b, c, d \in \mathbb{Z}^+$. By Def. I.4.2.2 we have $xy = ac/bd$. By Lem. I.2.3.3 we have $ac, bd \in \mathbb{Z}^+$, thus by Def. I.4.2.6 we know that xy is a positive rational number.

Now we show that if x and y are two negative rationals, then xy is a positive rational number. By Def. I.4.2.6 we know that $x = (-a)/b$ and $y = (-c)/d$ where $a, b, c, d \in \mathbb{Z}^+$. By Def. I.4.2.2 we have $xy = (-a)(-c)/bd$. By A.Cor. I.4.1.7 we have $(-a)(-c) = ac$. By Lem. I.2.3.3 we have $ac, bd \in \mathbb{Z}^+$, thus by Def. I.4.2.6 we know that xy is a positive rational number. □

A.Cor. I.4.2.8. Let x and y be two rationals. If x is negative and y is positive, then xy is negative. If x is positive and y is negative, then xy is negative.

Proof of A.Cor. I.4.2.8. By Prop. I.4.2.4 we know that $xy = yx$, thus we only need to show that if x is negative and y is positive, then xy is negative. By Def. I.4.2.6 we know that $x = (-a)/b$ and $y = c/d$ where $a, b, c, d \in \mathbb{Z}^+$. By Def. I.4.2.2 we have $xy = (-a)c/bd$. By A.Cor. I.4.1.5 we have $(-a)c = -(ac)$. By Lem. I.2.3.3 we know that $ac, bd \in \mathbb{Z}^+$, thus by Def. I.4.2.6 we know that xy is a negative rational number. □

A.Cor. I.4.2.9. x is a positive rational number iff $x > 0$. x is a negative rational number iff $x < 0$.

Proof of A.Cor. I.4.2.9. By Def. I.4.2.1 we have $x = x - 0$. Thus, x is a positive rational number iff $x - 0$ is a positive rational number, iff $x > 0$ (Def. I.4.2.8). Similarly, x is a negative rational number iff $x - 0$ is a negative rational number, iff $x < 0$ (Def. I.4.2.8). □

Prop. I.4.2.9 (Basic properties of order on the rationals)**.** Let x, y, z be rational numbers. Then the following properties hold.

(a) (Order trichotomy) Exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true.

(b) (Order is anti-symmetric) One has $x < y$ iff $y > x$.

(c) (Order is transitive) If $x < y$ and $y < z$, then $x < z$.

(d) (Addition preserves order) If $x < y$, then $x + z < y + z$.

(e) (Positive multiplication preserves order) If $x < y$ and z is positive, then $xz < yz$.

Proof of Prop. I.4.2.9(a). By Lem. I.4.2.7 $x - y$ satisfy exactly one of the following three statements:

- $x - y = 0$. Then by Prop. I.4.2.4 we have $x = y$.
- $x - y$ is a positive rational number. Then by Def. I.4.2.8 we have $x > y$.
- $x - y$ is a negative rational number. Then by Def. I.4.2.8 we have $x < y$.

□

Proof of Prop. I.4.2.9(b). Since

$$\begin{aligned}
x - y &= -(-(x-y)) && \text{(by A.Cor. I.4.2.5)}\\
&= -((-1)(x-y)) && \text{(by A.Cor. I.4.2.5)}\\
&= -((-1)(x+(-y))) && \text{(by A.Cor. I.4.2.4)}\\
&= -((-1)x+(-1)(-y)) && \text{(by Prop. I.4.2.4)}\\
&= -((-x)+y) && \text{(by A.Cor. I.4.2.5)}\\
&= -(y+(-x)) && \text{(by Prop. I.4.2.4)}\\
&= -(y-x) && \text{(by A.Cor. I.4.2.4)}\\
&= (-1)(y-x), && \text{(by A.Cor. I.4.2.5)}
\end{aligned}$$

we have

$$\begin{aligned}
& x < y \\
\iff & x - y \in \mathbb{Q}^- && \text{(by Def. I.4.2.8)}\\
\iff & (-1)(x-y) \in \mathbb{Q}^+ && \text{(by A.Cor. I.4.2.7 and I.4.2.8)}\\
\iff & (-1)(-1)(y-x) \in \mathbb{Q}^+ && \text{(by Lem. I.4.2.3)}\\
\iff & y - x \in \mathbb{Q}^+ && \text{(by A.Cor. I.4.2.5)}\\
\iff & y > x. && \text{(by Def. I.4.2.8)}
\end{aligned}$$

□

Proof of Prop. I.4.2.9(c). We have

$$\begin{aligned}
& (x < y) \wedge (y < z) \\
\implies & (x - y \in \mathbb{Q}^-) \wedge (y - z \in \mathbb{Q}^-) && \text{(by Def. I.4.2.8)}\\
\implies & (x - y) + (y - z) \in \mathbb{Q}^- && \text{(by A.Cor. I.4.2.6)}\\
\implies & x + z \in \mathbb{Q}^- && \text{(by Prop. I.4.2.4)}\\
\implies & x < z. && \text{(by Def. I.4.2.8)}
\end{aligned}$$

□

Proof of Prop. I.4.2.9(d). We have

$$\begin{aligned}
& x < y \\
\Longrightarrow\ & x - y \in \mathbb{Q}^- && \text{(by Def. I.4.2.8)} \\
\Longrightarrow\ & x + (-y) \in \mathbb{Q}^- && \text{(by A.Cor. I.4.2.4)} \\
\Longrightarrow\ & x + z + (-z) + (-y) \in \mathbb{Q}^- && \text{(by Prop. I.4.2.4)} \\
\Longrightarrow\ & x + z + (-y) + (-z) \in \mathbb{Q}^- && \text{(by Prop. I.4.2.4)} \\
\Longrightarrow\ & x + z + (-1)y + (-1)z \in \mathbb{Q}^- && \text{(by A.Cor. I.4.2.5)} \\
\Longrightarrow\ & x + z + (-1)(y + z) \in \mathbb{Q}^- && \text{(by Prop. I.4.2.4)} \\
\Longrightarrow\ & x + z + (-(y + z)) \in \mathbb{Q}^- && \text{(by A.Cor. I.4.2.5)} \\
\Longrightarrow\ & x + z - (y + z) \in \mathbb{Q}^- && \text{(by A.Cor. I.4.2.4)} \\
\Longrightarrow\ & x + z < y + z. && \text{(by Def. I.4.2.8)}
\end{aligned}$$

□

Proof of Prop. I.4.2.9(e). We have

$$\begin{aligned}
& x < y \\
\Longrightarrow\ & x - y \in \mathbb{Q}^- && \text{(by Def. I.4.2.8)} \\
\Longrightarrow\ & (x - y)z \in \mathbb{Q}^- && \text{(by A.Cor. I.4.2.8)} \\
\Longrightarrow\ & xz - yz \in \mathbb{Q}^- && \text{(by Prop. I.4.2.4)} \\
\Longrightarrow\ & xz < yz. && \text{(by Def. I.4.2.8)}
\end{aligned}$$

□

Rmk. I.4.2.10. The above five properties in Prop. I.4.2.9, combined with the field axioms in Prop. I.4.2.4, have a name: they assert that the rationals $\mathbb{Q}$ form an *ordered field*. It is important to keep in mind that Prop. I.4.2.9(e) only works when z is positive, see Ex. I.4.2.6.

— Exercises —

Ex. I.4.2.1. Show that the definition of equality for the rational numbers is reflexive, symmetric, and transitive.

Proof of Ex. I.4.2.1. See A.Cor. I.4.2.1. □

Ex. I.4.2.2. Prove the remaining components of Lem. I.4.2.3.

Proof of Ex. I.4.2.2. See Lem. I.4.2.3. □

Ex. I.4.2.3. Prove the remaining components of Prop. I.4.2.4.

Proof of Ex. I.4.2.3. See Prop. I.4.2.4. □

Ex. I.4.2.4. Prove Lem. I.4.2.7.

Proof of Ex. I.4.2.4. See Lem. I.4.2.7. □

Ex. I.4.2.5. Prove Prop. I.4.2.9.

Proof of Ex. I.4.2.5. See Prop. I.4.2.9. □

Ex. I.4.2.6. Show that if x, y, z are rational numbers such that $x < y$ and z is negative, then $xz > yz$.

Proof of Ex. I.4.2.6. We have

$$\begin{aligned} & x < y \\ \Longrightarrow\ & x - y \in \mathbb{Q}^- && \text{(by Def. I.4.2.8)} \\ \Longrightarrow\ & (x - y)z \in \mathbb{Q}^+ && \text{(by A.Cor. I.4.2.7)} \\ \Longrightarrow\ & xz - yz \in \mathbb{Q}^+ && \text{(by Prop. I.4.2.4)} \\ \Longrightarrow\ & xz > yz. && \text{(by Def. I.4.2.8)} \end{aligned}$$

□

I.4.3 Absolute value and exponentiation

Def. I.4.3.1 (Absolute value)**.** If x is a rational number, the *absolute value* $|x|$ of x is defined as follows. If x is positive, then $|x| := x$. If x is negative, then $|x| := -x$. If x is zero, then $|x| := 0$.

Def. I.4.3.2 (Distance)**.** Let x and y be rational numbers. The quantity $|x - y|$ is called the *distance between x and y* and is sometimes denoted $d(x, y)$, thus $d(x, y) := |x - y|$.

Prop. I.4.3.3 (Basic properties of absolute value and distance)**.** Let x, y, z be rational numbers.

(a) (Non-degeneracy of absolute value) We have $|x| \geq 0$. Also, $|x| = 0$ iff x is 0.

(b) (Triangle inequality for absolute value) We have $|x + y| \leq |x| + |y|$.

(c) We have the inequalities $-y \leq x \leq y$ iff $y \geq |x|$. In particular, we have $-|x| \leq x \leq |x|$.

(d) (Multiplicativity of absolute value) We have $|xy| = |x||y|$. In particular, $|-x| = |x|$.

(e) (Non-degeneracy of distance) We have $d(x, y) \geq 0$. Also, $d(x, y) = 0$ iff $x = y$.

(f) (Symmetry of distance) $d(x, y) = d(y, x)$.

(g) (Triangle inequality for distance) $d(x, z) \leq d(x, y) + d(y, z)$.

Proof of Prop. I.4.3.3(a). By Lem. I.4.2.7 we know that exactly one of the three statements is true:

- $x = 0$. Then by Def. I.4.3.1 we have $|x| = 0$.
- $x \in \mathbb{Q}^+$. Then by Def. I.4.3.1 we have $|x| = x \in \mathbb{Q}^+$. By A.Cor. I.4.2.9 we have $|x| = x > 0$.
- $x \in \mathbb{Q}^-$. Then by Def. I.4.3.1 we have $|x| = -x$. By A.Cor. I.4.2.5 and I.4.2.7 we know that $-x = (-1)x \in \mathbb{Q}^+$. Thus, by A.Cor. I.4.2.9 we have $|x| = -x > 0$.

From all cases above, we conclude that $|x| \geq 0$ and $|x| = 0 \iff x = 0$. $\square$

Proof of Prop. I.4.3.3(b). By Lem. I.4.2.7 exactly one of the following three statements is true:

- $x + y = 0$. By Prop. I.4.3.3(a) we have $|x + y| = 0 \leq |x|$ and $0 \leq |y|$. Thus

$$\begin{aligned} |x+y| &= 0 \leq |y| && \text{(by Prop. I.4.3.3(a))} \\ &= 0 + |y| && \text{(by Prop. I.4.2.4)} \\ &\leq |x| + |y|. && \text{(by Prop. I.4.2.9(d))} \end{aligned}$$

- $x + y \in \mathbb{Q}^+$. By A.Cor. I.4.2.6 we know that we cannot have both $x \in \mathbb{Q}^-$ and $y \in \mathbb{Q}^-$. So we can use Lem. I.4.2.7 to split into two further cases:
  - If exactly one of x, y is a negative rational number, say x, then by A.Cor. I.4.2.9 we have $x < 0$ and $y > 0$. Note that $y \neq 0$, otherwise we would have $x + y = x \in \mathbb{Q}^-$ by Prop. I.4.2.4, which contradicts to $x + y \in \mathbb{Q}^+$. Since $x \neq 0$, by Prop. I.4.3.3(a) we have $0 < |x|$. Thus

$$\begin{aligned} |x+y| &= x + y && \text{(by Def. I.4.3.1)} \\ &< 0 + y && \text{(by Prop. I.4.2.9(d))} \\ &< |x| + y && \text{(by Prop. I.4.2.9(d))} \\ &= |x| + |y|. && \text{(by Def. I.4.3.1)} \end{aligned}$$

  - If none of x, y are negative rational numbers, then by A.Cor. I.4.2.9 we have $x \geq 0$ and $y \geq 0$. Thus

$$\begin{aligned} |x+y| &= x + y && \text{(by Def. I.4.3.1)} \\ &= |x| + |y|. && \text{(by Def. I.4.3.1)} \end{aligned}$$

- $x + y \in \mathbb{Q}^-$. Then by A.Cor. I.4.2.7 we have $-(x + y) \in \mathbb{Q}^+$. Since

$$-(x+y) = (-1)(x+y) \qquad \text{(by A.Cor. I.4.2.5)}$$

$$
\begin{aligned}
&= (-1)x + (-1)y && \text{(by Prop. I.4.2.4)}\\
&= (-x) + (-y), && \text{(by A.Cor. I.4.2.5)}
\end{aligned}
$$

we can use the second case to derive $|-(x+y)| \leq |-x| + |-y|$. But by Def. I.4.3.1 we have $|x+y| = -(x+y) = |-(x+y)|$. And $|-x| = |x|$ since

$$
\begin{aligned}
|-x| &= \begin{cases} -x & \text{if } (-x) > 0 \\ 0 & \text{if } (-x) = 0 \\ -(-x) & \text{if } (-x) < 0 \end{cases} && \text{(by Def. I.4.3.1)}\\
&= \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases} && \text{(by A.Cor. I.4.2.5 and Ex. I.4.2.6)}\\
&= |x|. && \text{(by Def. I.4.3.1)}
\end{aligned}
$$

Similarly, $|-y| = |y|$. Thus, we have $|x+y| \leq |x| + |y|$.

For all cases above we conclude that $|x+y| \leq |x| + |y|$. □

Proof of Prop. I.4.3.3(c). We have

$$
\begin{aligned}
& -y \leq x \leq y \\
\iff & (x \leq y) \wedge (-x \leq y) && \text{(by Ex. I.4.2.6)}\\
\iff & |x| \leq y. && \text{(by Def. I.4.3.1)}
\end{aligned}
$$

In particular, we have $|x| \leq |x| \iff -|x| \leq x \leq |x|$. □

Proof of Prop. I.4.3.3(d). By Lem. I.4.2.7 we know that exactly one of the following three statements is true:

- $x = 0$. Then we have

$$
\begin{aligned}
|0y| &= |0| && \text{(by Def. I.4.2.2)}\\
&= 0 && \text{(by Def. I.4.3.1)}\\
&= 0|y| && \text{(by Def. I.4.2.2)}\\
&= |0||y|. && \text{(by Def. I.4.3.1)}
\end{aligned}
$$

- $x \in \mathbb{Q}^+$. By Lem. I.4.2.7 again we know that exactly one of the following three statements is true:
  - $y = 0$. By Prop. I.4.2.4 we know that $xy = yx$, thus this is the same case as $x = 0$.

  - $y \in \mathbb{Q}^+$. By A.Cor. I.4.2.7 we know that $xy \in \mathbb{Q}^+$. Thus

$$\begin{aligned} |xy| &= xy && \text{(by Def. I.4.3.1)} \\ &= |x||y|. && \text{(by Prop. I.4.3.3(a))} \end{aligned}$$

  - $y \in \mathbb{Q}^-$. By A.Cor. I.4.2.8 we know that $xy \in \mathbb{Q}^-$. Thus

$$\begin{aligned} |xy| &= -xy && \text{(by Def. I.4.3.1)} \\ &= x(-y) && \text{(by A.Cor. I.4.2.5)} \\ &= |x||y|. && \text{(by Def. I.4.3.1)} \end{aligned}$$

- $x \in \mathbb{Q}^-$. By Lem. I.4.2.7, exactly one of the following three statements is true:
  - $y = 0$. By Prop. I.4.2.4 we know that $xy = yx$, thus this is the same case as $x = 0$.
  - $y \in \mathbb{Q}^+$. By Prop. I.4.2.4 we know that $xy = yx$, thus this is the same case as $x \in \mathbb{Q}^+$ and $y \in \mathbb{Q}^-$.
  - $y \in \mathbb{Q}^-$. By A.Cor. I.4.2.7, xy is a positive. Thus

$$\begin{aligned} |xy| &= xy && \text{(by Def. I.4.3.1)} \\ &= (-1)(-1)xy && \text{(by Def. I.4.2.2)} \\ &= (-1)x(-1)y && \text{(by Prop. I.4.2.4)} \\ &= (-x)(-y) && \text{(by A.Cor. I.4.2.5)} \\ &= |x||y|. && \text{(by Def. I.4.3.1)} \end{aligned}$$

From all cases above, we conclude that $|xy| = |x||y|$. In particular, we have

$$\begin{aligned} |-x| &= |(-1)x| && \text{(by A.Cor. I.4.2.5)} \\ &= |-1||x| && \text{(from the proof above)} \\ &= -(-1)|x| && \text{(by Def. I.4.3.1)} \\ &= 1|x| && \text{(by A.Cor. I.4.1.6)} \\ &= |x|. && \text{(by Prop. I.4.2.4)} \end{aligned}$$

□

Proof of Prop. I.4.3.3(e). Since $x - y \in \mathbb{Q}$, by Prop. I.4.3.3(a) we have $d(x, y) = |x - y| \geq 0$ and

$$\begin{aligned} & d(x, y) = 0 \\ \iff & |x - y| = 0 && \text{(by Def. I.4.3.2)} \end{aligned}$$

$$
\begin{aligned}
&\iff x - y = 0 && \text{(by Prop. I.4.3.3(a))}\\
&\iff x = y. && \text{(by Prop. I.4.2.4)}
\end{aligned}
$$

□

Proof of Prop. I.4.3.3(f). We have

$$
\begin{aligned}
d(x,y) &= |x-y| && \text{(by Def. I.4.3.2)}\\
&= |-(x-y)| && \text{(by Prop. I.4.3.3(d))}\\
&= |y-x| && \text{(by Prop. I.4.2.4)}\\
&= d(y,x). && \text{(by Def. I.4.3.2)}
\end{aligned}
$$

□

Proof of Prop. I.4.3.3(g). We have

$$
\begin{aligned}
d(x,z) &= |x-z| && \text{(by Def. I.4.3.2)}\\
&= |x-y+y-z| && \text{(by Prop. I.4.2.4)}\\
&\leq |x-y| + |y-z| && \text{(by Prop. I.4.3.3(b))}\\
&= d(x,y) + d(y,z). && \text{(by Def. I.4.3.2)}
\end{aligned}
$$

□

A.Cor. I.4.3.1. Let x, y be rational numbers. Then $|x| - |y| \leq |x+y|$.

Proof of A.Cor. I.4.3.1.

$$
\begin{aligned}
&|x+y+(-y)| \leq |x+y| + |-y| && \text{(by Prop. I.4.3.3(b))}\\
\implies &|x| \leq |x+y| + |-y| && \text{(by Prop. I.4.2.4)}\\
\implies &|x| \leq |x+y| + |y| && \text{(by Prop. I.4.3.3(d))}\\
\implies &|x| + (-|y|) \leq |x+y| + |y| + (-|y|) && \text{(by Prop. I.4.2.9(d))}\\
\implies &|x| + (-|y|) \leq |x+y| && \text{(by Prop. I.4.2.4)}\\
\implies &|x| - |y| \leq |x+y|. && \text{(by A.Cor. I.4.2.4)}
\end{aligned}
$$

□

Def. I.4.3.4 (ε-closeness)**.** Let $\varepsilon \in \mathbb{Q}_{\geq 0}$, and let x, y be rational numbers. We say that y is *ε-close* to x iff we have $d(y,x) \leq \varepsilon$.

Rmk. I.4.3.5. This definition is not standard in mathematics textbooks; we will use it as "scaffolding" to construct the more important notions of limits (and of Cauchy sequences) later on, and once we have those more advanced notions we will discard the notion of ε-close.

Note. We do not bother defining a notion of ε-close when ε is zero or negative, because if ε is zero then x and y are only ε-close when they are equal, and when ε is negative then x and y are never ε-close.

Note. In any event it is a long-standing tradition in analysis that the Greek letters ε, δ should only denote small positive numbers.

Prop. I.4.3.7. Let x, y, z, w be rational numbers.

(a) If $x = y$, then x is ε-close to y for every $\varepsilon \in \mathbb{Q}^+$. Conversely, if x is ε-close to y for every $\varepsilon \in \mathbb{Q}^+$, then we have $x = y$.

(b) Let $\varepsilon \in \mathbb{Q}_{\geq 0}$. If x is ε-close to y, then y is ε-close to x.

(c) Let $\varepsilon, \delta \in \mathbb{Q}_{\geq 0}$. If x is ε-close to y, and y is δ-close to z, then x and z are $(\varepsilon + \delta)$-close.

(d) Let $\varepsilon, \delta \in \mathbb{Q}_{\geq 0}$. If x and y are ε-close, and z and w are δ-close, then $x + z$ and $y + w$ are $(\varepsilon + \delta)$-close, and $x - z$ and $y - w$ are also $(\varepsilon + \delta)$-close.

(e) Let $\varepsilon \in \mathbb{Q}_{\geq 0}$. If x and y are ε-close, they are also ε'-close for every $\varepsilon' > \varepsilon$.

(f) Let $\varepsilon \in \mathbb{Q}_{\geq 0}$. If y and z are both ε-close to x, and w is between y and z (i.e., $y \leq w \leq z$ or $z \leq w \leq y$), then w is also ε-close to x.

(g) Let $\varepsilon \in \mathbb{Q}_{\geq 0}$. If x and y are ε-close, and z is non-zero, then xz and yz are $\varepsilon|z|$-close.

(h) Let $\varepsilon, \delta \in \mathbb{Q}_{\geq 0}$. If x and y are ε-close, and z and w are δ-close, then xz and yw are $(\varepsilon|z| + \delta|x| + \varepsilon\delta)$-close.

Proof of Prop. I.4.3.7(a). We first show that if $x = y$, then x is ε-close to y for every $\varepsilon \in \mathbb{Q}^+$.

$$
\begin{aligned}
& x = y \\
\implies\ & x - y = 0 && \text{(by Prop. I.4.2.4)} \\
\implies\ & d(x, y) = |x - y| = 0 && \text{(by Def. I.4.3.2 and Prop. I.4.3.3(a))} \\
\implies\ & \forall \varepsilon \in \mathbb{Q}^+, d(x, y) \leq \varepsilon && \text{(by A.Cor. I.4.2.9)} \\
\implies\ & \forall \varepsilon \in \mathbb{Q}^+, x \text{ is } \varepsilon\text{-close to } y. && \text{(by Def. I.4.3.4)}
\end{aligned}
$$

Now we show that if x is ε-close to y for every $\varepsilon \in \mathbb{Q}^+$, then $x = y$. Suppose for the sake of contradiction that $x \neq y$. Then by Prop. I.4.3.3(e) we have $d(x, y) > 0$. But then we have $d(x, y) \in \mathbb{Q}^+$, so $d(x, y) < d(x, y)$, a contradiction. Thus, we must have $x = y$. □

Proof of Prop. I.4.3.7(b). We have

$$
\begin{aligned}
& x \text{ is } \varepsilon\text{-close to } y \\
\iff\ & d(x, y) \leq \varepsilon && \text{(by Def. I.4.3.4)} \\
\iff\ & d(y, x) \leq \varepsilon && \text{(by Prop. I.4.3.3(f))}
\end{aligned}
$$

$$\iff y \text{ is } \varepsilon\text{-close to } x. \qquad \text{(by Def. I.4.3.4)}$$

□

Proof of Prop. I.4.3.7(c). We have

$$
\begin{aligned}
& (x \text{ is } \varepsilon\text{-close to } y) \wedge (y \text{ is } \delta\text{-close to } z) \\
\implies & (d(x,y) \leq \varepsilon) \wedge (d(y,z) \leq \delta) && \text{(by Def. I.4.3.4)} \\
\implies & d(x,y) + d(y,z) \leq \varepsilon + d(y,z) \leq \varepsilon + \delta && \text{(by Prop. I.4.2.9(c)(d))} \\
\implies & d(x,z) \leq d(x,y) + d(y,z) \leq \varepsilon + \delta && \text{(by Prop. I.4.3.3(g))} \\
\implies & x \text{ is } (\varepsilon+\delta)\text{-close to } z. && \text{(by Def. I.4.3.4)}
\end{aligned}
$$

□

Proof of Prop. I.4.3.7(d). Since

$$
\begin{aligned}
& (x \text{ is } \varepsilon\text{-close to } y) \wedge (z \text{ is } \delta\text{-close to } w) \\
\implies & (d(x,y) \leq \varepsilon) \wedge (d(z,w) \leq \delta) && \text{(by Def. I.4.3.4)} \\
\implies & d(x,y) + d(z,w) \leq \varepsilon + d(z,w) \leq \varepsilon + \delta && \text{(by Prop. I.4.2.9(c)(d))} \\
\implies & |x-y| + |z-w| \leq \varepsilon + \delta, && \text{(by Def. I.4.3.2)}
\end{aligned}
$$

we have

$$
\begin{aligned}
& |x-y| + |z-w| \leq \varepsilon + \delta \\
\implies & |x-y+z-w| \leq |x-y| + |z-w| \leq \varepsilon + \delta && \text{(by Prop. I.4.3.3(b))} \\
\implies & |x+z-(y+w)| \leq \varepsilon + \delta && \text{(by Prop. I.4.2.4)} \\
\implies & d(x+z, y+w) \leq \varepsilon + \delta && \text{(by Def. I.4.3.2)} \\
\implies & (x+z) \text{ is } (\varepsilon+\delta)\text{-close to } (y+w) && \text{(by Def. I.4.3.4)}
\end{aligned}
$$

and

$$
\begin{aligned}
& |x-y| + |z-w| \leq \varepsilon + \delta \\
\implies & |x-y| + |-(z-w)| \leq \varepsilon + \delta && \text{(by Prop. I.4.3.3(d))} \\
\implies & |x-y| + |w-z| \leq \varepsilon + \delta && \text{(by Prop. I.4.2.4)} \\
\implies & |x-y+w-z| \leq |x-y| + |w-z| \leq \varepsilon + \delta && \text{(by Prop. I.4.3.3(b))} \\
\implies & |x-z-(y-w)| \leq \varepsilon + \delta && \text{(by Prop. I.4.2.4)} \\
\implies & d(x-z, y-w) \leq \varepsilon + \delta && \text{(by Def. I.4.3.2)} \\
\implies & (x-z) \text{ is } (\varepsilon+\delta)\text{-close to } (y-w). && \text{(by Def. I.4.3.4)}
\end{aligned}
$$

□

Proof of Prop. I.4.3.7(e). We have

$$\begin{aligned} & (x \text{ is } \varepsilon\text{-close to } y) \wedge (\varepsilon' > \varepsilon) && \\ \implies & (d(x,y) \leq \varepsilon) \wedge (\varepsilon' > \varepsilon) && \text{(by Def. I.4.3.4)} \\ \implies & d(x,y) < \varepsilon' && \text{(by Prop. I.4.2.9(b)(c))} \\ \implies & x \text{ is } \varepsilon'\text{-close to } y. && \text{(by Def. I.4.3.4)} \end{aligned}$$

□

Proof of Prop. I.4.3.7(f). We have

$$\begin{aligned} & (y \text{ is } \varepsilon\text{-close to } x) \wedge (z \text{ is } \varepsilon\text{-close to } x) && \\ \implies & (d(y,x) \leq \varepsilon) \wedge (d(z,x) \leq \varepsilon) && \text{(by Def. I.4.3.4)} \\ \implies & (|y-x| \leq \varepsilon) \wedge (|z-x| \leq \varepsilon) && \text{(by Def. I.4.3.2)} \\ \implies & (-\varepsilon \leq y-x \leq \varepsilon) \wedge (-\varepsilon \leq z-x \leq \varepsilon). && \text{(by Prop. I.4.3.3(c))} \end{aligned}$$

If $y \leq w \leq z$, then we have

$$\begin{aligned} & y \leq w \leq z && \\ \implies & y-x \leq w-x \leq z-x && \text{(by Prop. I.4.2.9(d))} \\ \implies & -\varepsilon \leq y-x \leq w-x \leq z-x \leq \varepsilon && \text{(by Prop. I.4.2.9(c))} \\ \implies & |w-x| \leq \varepsilon && \text{(by Prop. I.4.3.3(c))} \\ \implies & d(w,x) \leq \varepsilon && \text{(by Def. I.4.3.2)} \\ \implies & w \text{ is } \varepsilon\text{-close to } x. && \text{(by Def. I.4.3.4)} \end{aligned}$$

The case for $z \leq w \leq y$ can be proven similarly. □

Proof of Prop. I.4.3.7(g). We have

$$\begin{aligned} & (x \text{ is } \varepsilon\text{-close to } y) \wedge (z \neq 0) && \\ \implies & (|x-y| \leq \varepsilon) \wedge (z \neq 0) && \text{(by Def. I.4.3.2 and I.4.3.4)} \\ \implies & (|x-y| \leq \varepsilon) \wedge (|z| > 0) && \text{(by Prop. I.4.3.3(a))} \\ \implies & |x-y||z| \leq \varepsilon|z| && \text{(by Prop. I.4.2.9(e))} \\ \implies & |(x-y)z| \leq \varepsilon|z| && \text{(by Prop. I.4.3.3(d))} \\ \implies & |xz-yz| \leq \varepsilon|z| && \text{(by Prop. I.4.2.4)} \\ \implies & d(xz,yz) \leq \varepsilon|z| && \text{(by Def. I.4.3.2)} \\ \implies & xz \text{ is } (\varepsilon|z|)\text{-close to } yz. && \text{(by Def. I.4.3.4)} \end{aligned}$$

□

Proof of Prop. I.4.3.7(h). If we write $a := y - x$, then we have $y = x + a$ and that $|a| \leq \varepsilon$. Similarly, if we define $b := w - z$, then $w = z + b$ and $|b| \leq \delta$.

Since $y = x + a$ and $w = z + b$, by Prop. I.4.2.4 we have

$$yw = (x + a)(z + b) = xz + az + xb + ab.$$

Thus, by Prop. I.4.3.3(b)(d) we have

$$|yw - xz| = |az + bx + ab| \leq |az| + |bx| + |ab| = |a||z| + |b||x| + |a||b|.$$

Since $|a| \leq \varepsilon$ and $|b| \leq \delta$, we thus have

$$|yw - xz| \leq \varepsilon|z| + \delta|x| + \varepsilon\delta$$

and thus that yw and xz are $(\varepsilon|z| + \delta|x| + \varepsilon\delta)$-close. □

Rmk. I.4.3.8. One should compare statements (a)-(c) of Prop. I.4.3.7 with the reflexive, symmetric, and transitive axioms of equality. It is often useful to think of the notion of "ε-close" as an approximate substitute for that of equality in analysis.

Def. I.4.3.9 (Exponentiation to a natural number)**.** Let x be a rational number. To raise x to the power 0, we define $x^0 := 1$; in particular, we define $0^0 := 1$. Now suppose inductively that x^n has been defined for some natural number n, then we define $x^{n+1} := x^n \times x$.

Prop. I.4.3.10 (Properties of exponentiation, I)**.** Let x, y be rational numbers, and let n, m be natural numbers.

(a) We have $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$.

(b) Suppose $n > 0$. Then we have $x^n = 0$ iff $x = 0$.

(c) If $x \geq y \geq 0$, then $x^n \geq y^n \geq 0$. If $x > y \geq 0$ and $n > 0$, then $x^n > y^n \geq 0$.

(d) We have $|x^n| = |x|^n$.

Proof of Prop. I.4.3.10(a). We first show that $x^n x^m = x^{n+m}$. We induct on n. For $n = 0$, we have

$$\begin{aligned} x^0 x^m &= 1x^m && \text{(by Def. I.4.3.9)} \\ &= x^m && \text{(by Prop. I.4.2.4)} \\ &= x^{0+m}. && \text{(by Def. I.2.2.1)} \end{aligned}$$

So the base case holds. Suppose inductively that for some $n \in \mathbb{N}$ we have $x^n x^m = x^{n+m}$. Then for $n + 1$, we have

$$x^{n+1} x^m = (x^n x) x^m \qquad \text{(by Def. I.4.3.9)}$$

$$\begin{aligned}
&= x^n(xx^m) && \text{(by Prop. I.4.2.4)}\\
&= x^n(x^m x) && \text{(by Prop. I.4.2.4)}\\
&= (x^n x^m)x && \text{(by Prop. I.4.2.4)}\\
&= x^{n+m}x && \text{(by the induction hypothesis)}\\
&= x^{(n+m)+1} && \text{(by Def. I.4.3.9)}\\
&= x^{n+(m+1)} && \text{(by Prop. I.2.2.5)}\\
&= x^{n+(1+m)} && \text{(by Prop. I.2.2.4)}\\
&= x^{(n+1)+m}. && \text{(by Prop. I.2.2.5)}
\end{aligned}$$

This closes the induction.

Next we show that $(x^n)^m = x^{nm}$. We induct on m. For $m = 0$, we have

$$\begin{aligned}
(x^n)^0 &= 1 && \text{(by Def. I.4.3.9)}\\
&= x^0 && \text{(by Def. I.4.3.9)}\\
&= x^{n0}. && \text{(by A.Cor. I.2.3.2)}
\end{aligned}$$

So the base case holds. Suppose inductively that for some $m \in \mathbb{N}$ we have $(x^n)^m = x^{nm}$. Then for $m + 1$, we have

$$\begin{aligned}
(x^n)^{m+1} &= (x^n)^m(x^n) && \text{(by Def. I.4.3.9)}\\
&= x^{nm}x^n && \text{(by the induction hypothesis)}\\
&= x^{nm+n} && \text{(from the proof above)}\\
&= x^{n(m+1)}. && \text{(by A.Cor. I.2.3.3)}
\end{aligned}$$

This closes the induction.

Finally we show that $(xy)^n = x^n y^n$. We induct on n. For $n = 0$, we have

$$\begin{aligned}
(xy)^0 &= 1 && \text{(by Def. I.4.3.9)}\\
&= y^0 && \text{(by Def. I.4.3.9)}\\
&= 1y^0 && \text{(by Prop. I.4.2.4)}\\
&= x^0 y^0. && \text{(by Def. I.4.3.9)}
\end{aligned}$$

So the base case holds. Suppose inductively that for some $n \in \mathbb{N}$ we have $(xy)^n = x^n y^n$. Then for $n + 1$, we have

$$\begin{aligned}
(xy)^{n+1} &= (xy)^n(xy) && \text{(by Def. I.4.3.9)}\\
&= (x^n y^n)(xy) && \text{(by the induction hypothesis)}\\
&= x^n(y^n x)y && \text{(by Prop. I.4.2.4)}\\
&= x^n(xy^n)y && \text{(by Prop. I.4.2.4)}
\end{aligned}$$

$$
\begin{aligned}
&= (x^n x)(y^n y) && \text{(by Prop. I.4.2.4)}\\
&= x^{n+1} y^{n+1}. && \text{(by Def. I.4.3.9)}
\end{aligned}
$$

This closes the induction. $\square$

Proof of Prop. I.4.3.10(b). We induct on n and we start with $n = 1$. For $n = 1$, we have

$$
\begin{aligned}
&x^1 = 0\\
\iff &x^0 x = 0 && \text{(by Def. I.4.3.9)}\\
\iff &1x = 0 && \text{(by Def. I.4.3.9)}\\
\iff &x = 0. && \text{(by Prop. I.4.2.4)}
\end{aligned}
$$

So the base case holds. Suppose inductively that for some $n \in \mathbb{N} \setminus \{0\}$ we have $x^n = 0 \iff x = 0$. Then for $n + 1$, we have

$$
\begin{aligned}
&x^{n+1} = 0\\
\iff &x^n x = 0 && \text{(by Def. I.4.3.9)}\\
\iff &(x^n = 0) \vee (x = 0) && \text{(by A.Cor. I.4.2.7 and I.4.2.8)}\\
\iff &x = 0. && \text{(by the induction hypothesis)}
\end{aligned}
$$

This closes the induction. $\square$

Proof of Prop. I.4.3.10(c). We first show that if $x \geq y \geq 0$, then $x^n \geq y^n \geq 0$. We induct on n. For $n = 0$, we have

$$
\begin{aligned}
&x \geq y \geq 0\\
\implies &x^0 = 1 \geq y^0 = 1 \geq 0. && \text{(by Def. I.4.3.9)}
\end{aligned}
$$

So the base case holds. Suppose inductively that for some $n \in \mathbb{N}$ we have $x^n \geq y^n \geq 0$. Then for $n + 1$, we have

$$
\begin{aligned}
&(x^n \geq y^n \geq 0) \wedge (x \geq y \geq 0) && \text{(by the induction hypothesis)}\\
\implies &(x^n x \geq y^n x \geq 0x) \wedge (y^n x \geq y^n y \geq y^n 0) && \text{(by Prop. I.4.2.9(e))}\\
\implies &x^n x \geq y^n x \geq y^n y \geq y^n 0 && \text{(by Prop. I.4.2.9(c))}\\
\implies &x^{n+1} \geq y^{n+1} \geq 0. && \text{(by Def. I.4.2.2 and I.4.3.9)}
\end{aligned}
$$

This closes the induction.

Now we show that if $x > y \geq 0$ and $n > 0$, then $x^n > y^n \geq 0$. We split into two cases:

- If $y = 0$, then by Prop. I.4.3.10(b) we know that $x \neq 0 \iff x^n \neq 0$ and $y = 0 \iff y^n = 0$. By Prop. I.4.2.9(e) we know that $x > 0 \implies x^n > 0$. Thus, by Prop. I.4.2.9(c) we have $x^n > y^n = 0$.

- If $y > 0$, then we have $x > y > 0$. We induct on n to show that $x^n > y^n > 0$ and we start with $n = 1$. For $n = 1$, we have

$$\begin{aligned} &\begin{cases} x > y > 0 \\ x^1 = x^0 x = 1x = x \\ y^1 = y^0 y = 1y = y \end{cases} && \text{(by Prop. I.4.2.4 and Def. I.4.3.9)} \\ \Longrightarrow\ & x^1 > y^1 > 0. && \text{(by Def. I.4.3.9)} \end{aligned}$$

So the base case holds. Suppose inductively that for some $n \in \mathbb{N} \setminus \{0\}$ we have $x^n > y^n > 0$. Then for $n + 1$, we have

$$\begin{aligned} & (x^n > y^n > 0) \wedge (x > y > 0) && \text{(by the induction hypothesis)} \\ \Longrightarrow\ & (x^n x > y^n x > 0x) \wedge (y^n x > y^n y > y^n 0) && \text{(by Prop. I.4.2.9(e))} \\ \Longrightarrow\ & x^n x > y^n x > y^n y > y^n 0 && \text{(by Prop. I.4.2.9(c))} \\ \Longrightarrow\ & x^{n+1} > y^{n+1} > 0. && \text{(by Def. I.4.2.2 and I.4.3.9)} \end{aligned}$$

This closes the induction.

From all cases above, we conclude that $x > y \geq 0 \implies x^n > y^n \geq 0$. □

Proof of Prop. I.4.3.10(d). We induct on n. For $n = 0$, we have

$$\begin{aligned} \left|x^0\right| &= |1| && \text{(by Def. I.4.3.9)} \\ &= 1 && \text{(by Def. I.4.3.1)} \\ &= |x|^0. && \text{(by Def. I.4.3.9)} \end{aligned}$$

So the base case holds. Suppose inductively that for some $n \in \mathbb{N}$ we have $|x^n| = |x|^n$. Then for $n + 1$, we have

$$\begin{aligned} \left|x^{n+1}\right| &= |x^n x| && \text{(by Def. I.4.3.9)} \\ &= |x^n||x| && \text{(by Prop. I.4.3.3(d))} \\ &= |x|^n |x| && \text{(by the induction hypothesis)} \\ &= |x|^{n+1}. && \text{(by Def. I.4.3.9)} \end{aligned}$$

This closes the induction. □

Def. I.4.3.11 (Exponentiation to a negative number)**.** Let x be a non-zero rational number. Then for any negative integer $-n$, we define $x^{-n} := 1/x^n$.

Note that when $n = 1$, the definition of x^{-1} provided by Def. I.4.3.11 coincides with the reciprocal of x defined previously, so there is no incompatibility of notation caused by this new definition.

Prop. I.4.3.12 (Properties of exponentiation, II)**.** Let x, y be nonzero rational numbers, and let n, m be integers.

(a) We have $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$.

(b) If $x \geq y > 0$, then $x^n \geq y^n > 0$ if n is positive, and $0 < x^n \leq y^n$ if n is negative.

(c) If $x, y > 0$, $n \neq 0$, and $x^n = y^n$, then $x = y$.

(d) We have $|x^n| = |x|^n$.

Proof of Prop. I.4.3.12(a). We first show that $x^n x^m = x^{n+m}$. By Lem. I.4.1.11(f) exactly one of $n > 0$, $n = 0$ or $n < 0$ is true. Similarly, exactly one of $m > 0$, $m = 0$ or $m < 0$ is true, and exactly one of $n + m > 0$, $n + m = 0$ or $n + m < 0$ is true. Thus, we can split into following cases:

- If $n = 0$ or $m = 0$, then we have

$$\begin{aligned} x^0 x^m &= 1x^m && \text{(by Def. I.4.3.9)} \\ &= x^m && \text{(by Prop. I.4.2.4)} \\ &= x^{0+m} && \text{(by Prop. I.4.1.6)} \end{aligned}$$

  and

$$\begin{aligned} x^n x^0 &= x^0 x^n && \text{(by Prop. I.4.2.4)} \\ &= x^{0+n} && \text{(from the proof above)} \\ &= x^{n+0}. && \text{(by Prop. I.4.1.6)} \end{aligned}$$

- If $n > 0$ and $m > 0$, then by Prop. I.4.3.10(a) we have $x^n x^m = x^{n+m}$.
- If $n > 0$, $m < 0$ and $n + m \geq 0$, then by Lem. I.4.1.11(d) we have $-m > 0$. Thus

$$\begin{aligned} x^n x^m &= \frac{1x^n}{x^{-m}} && \text{(by Def. I.4.3.11)} \\ &= \frac{x^n}{x^{-m}} && \text{(by Prop. I.4.2.4)} \\ &= \frac{x^{n+m+(-m)}}{x^{-m}} && \text{(by Prop. I.4.1.6)} \\ &= \frac{x^{n+m}x^{-m}}{x^{-m}} && \text{(by Prop. I.4.3.10(a))} \\ &= x^{n+m}. && \text{(by Lem. I.4.2.3)} \end{aligned}$$

- If $n > 0$, $m < 0$ and $n + m < 0$, then by Lem. I.4.1.11(d) we have $-(n + m) > 0$. Thus

$$x^n x^m = \frac{1x^n}{x^{-m}} \qquad \text{(by Def. I.4.3.11)}$$

$$
\begin{aligned}
&= \frac{1x^n}{x^{n+(-n)+(-m)}} && \text{(by Prop. I.4.1.6)} \\
&= \frac{1x^n}{x^{n+(-(n+m))}} && \text{(by A.Cor. I.4.1.5)} \\
&= \frac{1x^n}{x^n x^{-(n+m)}} && \text{(by Prop. I.4.3.10(a))} \\
&= \frac{1}{x^{-(n+m)}} && \text{(by Lem. I.4.2.3)} \\
&= x^{n+m}. && \text{(by Def. I.4.3.11)}
\end{aligned}
$$

- If $n < 0$ and $m > 0$, then we can use the previous two cases to conclude that

$$
\begin{aligned}
x^n x^m &= x^m x^n && \text{(by Prop. I.4.2.4)} \\
&= x^{m+n} && \text{(from the previous two cases)} \\
&= x^{n+m}. && \text{(by Prop. I.4.1.6)}
\end{aligned}
$$

- If $n < 0$ and $m < 0$, then by Lem. I.4.1.11(d) we have $-n > 0$, $-m > 0$ and $-(n+m) > 0$. Thus

$$
\begin{aligned}
x^n x^m &= \frac{1}{x^{-n}x^{-m}} && \text{(by Def. I.4.3.11)} \\
&= \frac{1}{x^{(-n)+(-m)}} && \text{(by Prop. I.4.3.10(a))} \\
&= \frac{1}{x^{-(n+m)}} && \text{(by A.Cor. I.4.1.5)} \\
&= x^{n+m}. && \text{(by Def. I.4.3.11)}
\end{aligned}
$$

From all cases above, we conclude that $x^n x^m = x^{n+m}$.

Next we show that $(x^n)^m = x^{nm}$. By Lem. I.4.1.5 exactly one of the following two statements is true:

- $m \geq 0$. Then we induct on m to show that $(x^n)^m = x^{nm}$. For $m = 0$, we have

$$
\begin{aligned}
(x^n)^0 &= 1 && \text{(by Def. I.4.3.9)} \\
&= x^0 && \text{(by Def. I.4.3.9)} \\
&= x^{n0}. && \text{(by Prop. I.4.1.6)}
\end{aligned}
$$

So the base case holds. Suppose inductively that for some $m \in \mathbb{N}$ we have $(x^n)^m = x^{nm}$. Then for $m+1$, we have $m+1 > 0$ and

$$
\begin{aligned}
(x^n)^{m+1} &= (x^n)^m x^n && \text{(by Def. I.4.3.9)} \\
&= x^{nm} x^n && \text{(by the induction hypothesis)}
\end{aligned}
$$

$$\begin{aligned} &= x^{nm+n} && \text{(from the proof above)} \\ &= x^{n(m+1)}. && \text{(by A.Cor. I.2.3.3)} \end{aligned}$$

This closes the induction.

- $m < 0$. Then by A.Cor. I.4.2.7 we have $-m > 0$ and

$$\begin{aligned} (x^n)^m &= \frac{1}{(x^n)^{-m}} && \text{(by Def. I.4.3.11)} \\ &= \frac{1}{x^{n(-m)}} && \text{(from the first case)} \\ &= \frac{1}{x^{-nm}} && \text{(by A.Cor. I.4.1.5)} \\ &= x^{nm}. && \text{(by Def. I.4.3.11)} \end{aligned}$$

From all cases above, we conclude that $(x^n)^m = x^{nm}$.

Finally we show that $(xy)^n = x^n y^n$. By Lem. I.4.1.5 exactly one of the following two statements is true:

- $n \geq 0$. Then by Prop. I.4.3.10(a) we know that $(xy)^n = x^n y^n$.
- $n < 0$. Then by A.Cor. I.4.2.7 we have $-n > 0$ and

$$\begin{aligned} (xy)^n &= \frac{1}{(xy)^{-n}} && \text{(by Def. I.4.3.11)} \\ &= \frac{1}{x^{-n}y^{-n}} && \text{(by Prop. I.4.3.10(a))} \\ &= x^n y^n. && \text{(by Def. I.4.3.11)} \end{aligned}$$

From all cases above, we conclude that $(xy)^n = x^n y^n$. □

Proof of Prop. I.4.3.12(b). First, suppose that $n > 0$. Then by Prop. I.4.3.10(b)(c) we have $x^n \geq y^n > 0$. Now suppose that $n < 0$. Then by A.Cor. I.4.2.7 we have $-n > 0$ and

$$\begin{aligned} & x^{-n} \geq y^{-n} > 0 && \text{(by Prop. I.4.3.10(b)(c))} \\ \implies & \frac{1}{x^n} \geq \frac{1}{y^n} > 0 && \text{(by Def. I.4.3.11)} \\ \implies & \left(\frac{1}{x^n} \geq \frac{1}{y^n} > 0\right) \wedge (x^n > 0) \wedge (y^n > 0) && \text{(by A.Cor. I.4.2.8)} \\ \implies & \frac{x^n y^n}{x^n} \geq \frac{x^n y^n}{y^n} > 0(x^n y^n) && \text{(by Prop. I.4.2.9(e))} \\ \implies & y^n \geq x^n > 0. && \text{(by Lem. I.4.2.3)} \end{aligned}$$

□

Proof of Prop. I.4.3.12(c). Suppose for the sake of contradiction that $x \neq y$. Then by Prop. I.4.2.9(a) exactly one of the following two statements is true:

- $x > y$. By hypothesis we know that $n \neq 0$. Then by Lem. I.4.1.11(f) exactly one of the following two statements is true:
  - $n > 0$. But by Prop. I.4.3.10(c) we must have $x^n > y^n$, which contradicts to $x^n = y^n$.
  - $n < 0$. Then by A.Cor. I.4.2.7 we have $-n > 0$ and by Prop. I.4.3.10(b)(c) we have $x^{-n} > y^{-n} > 0$. Since

$$\begin{aligned}
x^n x^{-n} &= x^{n+(-n)} && \text{(by Prop. I.4.3.12(a))} \\
&= x^0 && \text{(by Prop. I.4.1.6)} \\
&= 1 && \text{(by Def. I.4.3.9)} \\
&> 0
\end{aligned}$$

  and $x^{-n} > 0$, by A.Cor. I.4.2.8 we know that $x^n > 0$. Similarly, we have $y^n > 0$. Thus, by A.Cor. I.4.2.7 $x^n y^n > 0$. But then we have

$$\begin{aligned}
&(x^{-n} > y^{-n} > 0) \wedge (x^n y^n > 0) \\
\Longrightarrow\ & x^{-n} x^n y^n > y^{-n} x^n y^n > 0 x^n y^n && \text{(by Prop. I.4.2.9(e))} \\
\Longrightarrow\ & x^{-n} x^n y^n > x^n y^{-n} y^n > 0 && \text{(by Prop. I.4.2.4)} \\
\Longrightarrow\ & x^{(-n)+n} y^n > x^n y^{(-n)+n} > 0 && \text{(by Prop. I.4.3.12(a))} \\
\Longrightarrow\ & x^0 y^n > x^n y^0 > 0 && \text{(by Prop. I.4.1.6)} \\
\Longrightarrow\ & 1 y^n > x^n 1 > 0 && \text{(by Def. I.4.3.9)} \\
\Longrightarrow\ & y^n > x^n > 0, && \text{(by Prop. I.4.2.4)}
\end{aligned}$$

  which contradicts to $x^n = y^n$.

- $x < y$. By Prop. I.4.2.9 we know that $x < y \implies y > x$, and we can use the first cases to derive either $y^n > x^n$ or $x^n > y^n$. But again this contradicts to $x^n = y^n$.

From all cases above, we derive contradictions, thus we must have $x = y$. □

Proof of Prop. I.4.3.12(d). First, we claim that $\left|x^{-1}\right| = |x|^{-1}$. By A.Cor. I.4.2.3 we know that $x^{-1} \neq 0$. By Prop. I.4.2.4 we have $xx^{-1} = 1$. Thus, by A.Cor. I.4.2.7 and I.4.2.8 both x, x^{-1} are positive or negative. So we use Prop. I.4.2.9(a) to split into two cases:

- If $x > 0$ and $x^{-1} > 0$, then we have

$$\begin{aligned}
\left|x^{-1}\right| &= x^{-1} && \text{(by Def. I.4.3.1)} \\
&= \frac{1}{x^1} && \text{(by Def. I.4.3.11)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{x} && \text{(by Def. I.4.3.9)}\\
&= \frac{1}{|x|} && \text{(by Def. I.4.3.1)}\\
&= \frac{1}{|x|^1} && \text{(by Def. I.4.3.9)}\\
&= |x|^{-1}. && \text{(by Def. I.4.3.11)}
\end{aligned}
$$

- If $x < 0$ and $x^{-1} < 0$, then we have

$$
\begin{aligned}
\left|x^{-1}\right| &= -(x^{-1}) && \text{(by Def. I.4.3.1)}\\
&= (-1)(x^{-1}) && \text{(by A.Cor. I.4.2.5)}\\
&= \frac{-1}{x} && \text{(by A.Cor. I.4.2.4)}\\
&= \frac{1}{-x} && \text{(by A.Cor. I.4.2.5)}\\
&= \frac{1}{|x|} && \text{(by Def. I.4.3.1)}\\
&= \frac{1}{|x|^1} && \text{(by Def. I.4.3.9)}\\
&= |x|^{-1}. && \text{(by Def. I.4.3.11)}
\end{aligned}
$$

From all cases above, we conclude that $\left|x^{-1}\right| = |x|^{-1}$.

Now we show that $|x^n| = |x|^n$. By Lem. I.4.1.11(f) exactly one of the following two statements is true:

- $n \geq 0$. Then by Prop. I.4.3.10(d) we have $|x^n| = |x|^n$.
- $n < 0$. Then by A.Cor. I.4.2.7 we have $-n > 0$ and

$$
\begin{aligned}
|x^n| &= \left|x^{-(-n)}\right| && \text{(by A.Cor. I.4.1.6)}\\
&= \left|x^{(-1)(-n)}\right| && \text{(by A.Cor. I.4.1.5)}\\
&= \left|(x^{-1})^{-n}\right| && \text{(by Prop. I.4.3.12(a))}\\
&= \left|x^{-1}\right|^{-n} && \text{(by Prop. I.4.3.10(d))}\\
&= (|x|^{-1})^{-n} && \text{(from the claim above)}\\
&= |x|^{(-1)(-n)} && \text{(by Prop. I.4.3.12(a))}\\
&= |x|^n. && \text{(by A.Cor. I.4.1.6)}
\end{aligned}
$$

From all cases above, we conclude that $|x^n| = |x|^n$. □

— Exercises —

Ex. I.4.3.1. Prove Prop. I.4.3.3.

Proof of Ex. I.4.3.1. See Prop. I.4.3.3. □

Ex. I.4.3.2. Prove the remaining claims in Prop. I.4.3.7.

Proof of Ex. I.4.3.2. See Prop. I.4.3.7. □

Ex. I.4.3.3. Prove Prop. I.4.3.10.

Proof of Ex. I.4.3.3. See Prop. I.4.3.10. □

Ex. I.4.3.4. Prove Prop. I.4.3.12.

Proof of Ex. I.4.3.4. See Prop. I.4.3.12. □

Ex. I.4.3.5. Prove that $2^N \geq N$ for all positive integers N.

Proof of Ex. I.4.3.5. We induct on N and start with $N = 1$. For $N = 1$, by Def. I.4.3.9 we have

$$2^1 = 2^0 \times 2 = 1 \times 2 = 2 \geq 1,$$

so the base case holds. Suppose inductively that for some $N \in \mathbb{N} \setminus \{0\}$ we have $2^N \geq N$. Then for $N + 1$, we have

$$\begin{aligned}
& 0 < N && \text{(by A.Cor. I.2.2.4)} \\
\Longrightarrow\ & N = 0 + N < N + N = 2N && \text{(by Prop. I.2.2.12(d))} \\
\Longrightarrow\ & N + 1 \leq 2N && \text{(by Prop. I.2.2.12(e))}
\end{aligned}$$

and

$$\begin{aligned}
& 2^N \geq N && \text{(by the induction hypothesis)} \\
\Longrightarrow\ & 2^N \times 2 \geq 2N \geq N + 1 && \text{(by Prop. I.4.2.9(c)(e))} \\
\Longrightarrow\ & 2^{N+1} \geq N + 1. && \text{(by Def. I.4.3.9)}
\end{aligned}$$

This closes the induction. □

I.4.4 Gaps in the rational numbers

A.Cor. I.4.4.1 (Euclidean algorithm)**.** Let $n \in \mathbb{Z}$ and let $q \in \mathbb{Z}^+$. Then there exist an unique pair of $(m, r) \in \mathbb{Z} \times \mathbb{N}$ such that $0 \leq r < q$ and $n = mq + r$.

Proof of A.Cor. I.4.4.1. We first show that there exists at least one pairs of $(m, r) \in \mathbb{Z} \times \mathbb{N}$ satisfy the statement. By Lem. I.4.1.5 exactly one of the following two statements is true:

- $n \geq 0$. Then by Prop. I.2.3.9 we know that there exists a pair of $(m, r) \in \mathbb{N} \times \mathbb{N}$ such that $0 \leq r < q$ and $n = mq + r$. By A.Cor. I.4.1.2 we have $m \in \mathbb{Z}$. Thus, there exists a pair of $(m, r) \in \mathbb{Z} \times \mathbb{N}$ such that $0 \leq r < q$ and $n = mq + r$.

- $n < 0$. Then by A.Cor. I.4.2.7 we have $-n > 0$ and by Prop. I.2.3.9 there exists a pair of $(m, r) \in \mathbb{N} \times \mathbb{N}$ such that $0 \leq r < q$ and $-n = mq + r$. Fix one such pair (m, r). Since $0 \leq r$, by Lem. I.4.1.11(d) we have $-r \leq 0$, and by Lem. I.4.1.11(b) we have $q - r \leq q$. Since $r < q$, by Lem. I.4.1.11(a) we have $q - r > 0$. Thus, by Lem. I.4.1.11(e) we have $0 < q - r \leq q$. Now observe that

$$\begin{aligned}
n &= -(-n) && \text{(by A.Cor. I.4.1.6)}\\
&= -(mq + r) && \text{(by Prop. I.2.3.9)}\\
&= -(mq + q + (-q) + r) && \text{(by Prop. I.4.1.6)}\\
&= -((m + 1)q + (r - q)) && \text{(by Prop. I.4.1.6 and A.Cor. I.4.1.4)}\\
&= (-1)((m + 1)q + (r - q)) && \text{(by A.Cor. I.4.1.5)}\\
&= (-1)(m + 1)q + (-1)(r - q) && \text{(by Prop. I.4.1.6)}\\
&= -(m + 1)q + (q - r). && \text{(by A.Cor. I.4.1.5 and Prop. I.4.1.6)}
\end{aligned}$$

  Since $-(m + 1) \in \mathbb{Z}$ and $q > q - r \geq 0$, by setting $m' = -m - 1$ and $r' = q - r$ we see that $n = m'q + r'$ satisfy the statement.

From all cases above, we conclude that at least one pairs of $(m, r) \in \mathbb{Z} \times \mathbb{N}$ satisfy the statement.

Now we show the uniqueness of such $(m, r) \in \mathbb{Z} \times \mathbb{N}$. Let $(m, r), (m', r') \in \mathbb{Z} \times \mathbb{N}$ such that

$$(n = mq + r = m'q + r') \wedge (0 \leq r < q) \wedge (0 \leq r' < q).$$

Suppose for the sake of contradiction that $r \neq r'$. By Lem. I.4.1.5 exactly one of the following two statements is true:

- $r > r'$. Let $a = r - r'$ By Lem. I.4.1.11(a) we know that $a \in \mathbb{Z}^+$. Then we have

$$\begin{aligned}
& mq + r = m'q + r' \\
\implies & r - r' = (m' - m)q && \text{(by Prop. I.4.1.6)}\\
\implies & m' - m > 0 && \text{(by A.Cor. I.4.2.8 and I.4.2.9)}
\end{aligned}$$

$$\begin{aligned}
&\Longrightarrow m' - m \geq 1 && \text{(by Prop. I.2.2.12(e))}\\
&\Longrightarrow (m' - m)q \geq q && \text{(by Prop. I.2.3.6)}\\
&\Longrightarrow r - r' \geq q\\
&\Longrightarrow r \geq q + r' \geq q. && \text{(by Lem. I.4.1.11(b))}
\end{aligned}$$

But this contradicts $r < q$.

- $r < r'$. By Def. I.4.1.10 we have $r' > r$. Using similar arguments as above, we derive a contradiction.

From all cases above, we derive contradictions, thus we must have $r = r'$. This means

$$\begin{aligned}
&mq + r = m'q + r'\\
\Longrightarrow\ &mq + r = m'q + r && \text{(by Lem. I.4.1.3)}\\
\Longrightarrow\ &mq = m'q && \text{(by Prop. I.4.1.6)}\\
\Longrightarrow\ &m = m'. && \text{(by Cor. I.4.1.9)}
\end{aligned}$$

Thus, such $(m, r) \in \mathbb{Z} \times \mathbb{N}$ is unique. □

Prop. I.4.4.1 (Interspersing of integers by rationals)**.** Let x be a rational number. Then there exists an integer n such that $n \leq x < n + 1$. In fact, this integer is unique (i.e., for each x there is only one n for which $n \leq x < n + 1$). In particular, there exists a natural number N such that $N > x$ (i.e., there is no such thing as a rational number which is larger than all the natural numbers).

Proof of Prop. I.4.4.1. By Def. I.4.2.1 we know that $x = a/b$ where $a, b \in \mathbb{Z}$ and $b > 0$. Since $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, by A.Cor. I.4.4.1 we know that there exists a pair of $(m, r) \in \mathbb{Z} \times \mathbb{N}$ such that $a = mb + r$ and $0 \leq r < b$. Then we have

$$\begin{aligned}
&(a = mb + r) \wedge (0 \leq r < b)\\
\Longrightarrow\ &\left(x = \frac{a}{b} = m + \frac{r}{b}\right) \wedge (0 \leq r < b) && \text{(by Prop. I.4.2.4)}\\
\Longrightarrow\ &\left(x = \frac{a}{b} = m + \frac{r}{b}\right) \wedge \left(0 \leq \frac{r}{b} < 1\right) && \text{(by Prop. I.4.2.9(e))}\\
\Longrightarrow\ &m \leq x = m + \frac{r}{b} < m + 1. && \text{(by Prop. I.4.2.9(d))}
\end{aligned}$$

Next we show the uniqueness of such n. Suppose there exists another $n' \in \mathbb{Z}$ such that $n' \leq x < n' + 1$. Then we have

$$\begin{aligned}
&(n \leq x < n + 1) \wedge (n' \leq x < n' + 1)\\
\Longrightarrow\ &(n < n' + 1) \wedge (n' < n + 1) && \text{(by Prop. I.4.2.9(c))}\\
\Longrightarrow\ &(n' + 1 - n > 0) \wedge (n + 1 - n' > 0) && \text{(by Prop. I.4.2.9(d))}
\end{aligned}$$

$$\implies (n' + 1 - n \geq 1) \wedge (n + 1 - n' \geq 1) \qquad \text{(by Prop. I.2.2.12(e))}$$
$$\implies (n' \geq n) \wedge (n \geq n') \qquad \text{(by Prop. I.4.2.9(d))}$$
$$\implies n' = n. \qquad \text{(by Prop. I.4.2.9(a))}$$

Thus, such n is unique.

Finally we show that there exists an $N \in \mathbb{N}$ such that $N > x$. By Prop. I.4.2.9(a) we split into three cases:

- $x = 0$. Then by setting $N = 1$ we have $N > x$.
- $x < 0$. Then by setting $N = 0$ we have $N > x$.
- $x > 0$. From the proof above we know that there exists an $n \in \mathbb{Z}$ such that $n \leq x < n+1$. Since $x > 0$, by Prop. I.4.2.9(c) we know that $n + 1 > 0$. Thus, $n + 1 \in \mathbb{N}$. By setting $N = n + 1$ we have $N > x$.

From all cases above, we conclude that there exists an $N \in \mathbb{N}$ such that $x < N$. □

Rmk. I.4.4.2. The integer n for which $n \leq x < n+1$ is sometimes referred to as the *integer part* of x and is sometimes denoted $n = \lfloor x \rfloor$.

Prop. I.4.4.3 (Interspersing of rationals by rationals)**.** If x and y are two rationals such that $x < y$, then there exists a third rational z such that $x < z < y$.

Proof of Prop. I.4.4.3. We set $z := (x + y)/2$. Since $x < y$, and $1/2 = 1//2$ is positive, we have from Prop. I.4.2.9 that $x/2 < y/2$. If we add $y/2$ to both sides using Prop. I.4.2.9 we obtain $x/2 + y/2 < y/2 + y/2$, i.e., $z < y$. If we instead add $x/2$ to both sides we obtain $x/2 + x/2 < y/2 + x/2$, i.e., $x < z$. Thus, $x < z < y$ as desired. □

Note. Despite the rationals having this denseness property, they are still incomplete; there are still an infinite number of "gaps" or "holes" between the rationals, although this denseness property does ensure that these holes are in some sense infinitely small.

A.Cor. I.4.4.2. Let $n \in \mathbb{N}$. Define n to be *even* if $n = 2m$ for some $m \in \mathbb{N}$, and *odd* if $n = 2m + 1$ for some $m \in \mathbb{N}$. Then every natural number is either even or odd, but not both.

Proof of A.Cor. I.4.4.2. We induct on n. For $n = 0$, by Def. I.2.3.1 and Lem. I.2.3.2 we have $0 = 0 \times 2 = 2 \times 0$. By Ax. I.2.3 we have $0 \neq 2m + 1$. Thus, 0 is even and is not odd, so the base case holds. Suppose inductively that for some $n \in N$, there exists some $m \in \mathbb{N}$ such that either $n = 2m$ or $n = 2m + 1$ is true, but not both. Then we need to show that the same statement holds for $n + 1$. By the induction hypothesis we can split into two cases:

- $n = 2m$ for some $m \in \mathbb{N}$. Then $n + 1 = 2m + 1$, which means $n + 1$ is odd.
- $n = 2m + 1$ for some $m \in \mathbb{N}$. Then by Prop. I.2.2.5 and I.2.3.4 we have $n + 1 = 2m + 2 = 2(m + 1)$, which means $n + 1$ is even.

Thus, $n+1$ is either even or odd. Now we show that we cannot have $n+1$ be both even and odd. Suppose for the sake of contradiction that $n+1$ is both even and odd. Then there exist $m,k\in\mathbb{N}$ such that $n+1=2m=2k+1$. By Ax. I.2.3 we know that $2m=n+1>0$, thus by Lem. I.2.3.3 we must have $m>0$. By Prop. I.2.2.12(e) we have $m\geq 1$, so by Lem. I.4.1.11(b) we have $m-1\geq 0$, and by definition $2(m-1)+1$ is odd. Now we use Prop. I.4.1.6 to rewrite $2m=2(m-1+1)=2(m-1)+2$. Therefore by Lem. I.2.2.10 $n+1=2m \implies n=2(m-1)+1$, which means n is odd. But by Lem. I.2.2.10 again we have $n+1=2k+1 \implies n=2k$, which means n is even, contradict to the induction hypothesis that n cannot be both even and odd. Therefore $n+1$ cannot be both even and odd. This closes the induction. ☐

A.Cor. I.4.4.3. Let n be a natural number. If n is even, then n^2 is also even. If n is odd, then n^2 is also odd.

Proof of A.Cor. I.4.4.3. We first show that n is even implies n^2 is even. Since n is even, by A.Cor. I.4.4.2 there exists an $m\in\mathbb{N}$ such that $n=2m$. By A.Cor. I.2.3.1 we have $m(2m)\in\mathbb{N}$. Since

$$\begin{aligned} n^2 &= (2m)^2 \\ &= (2m)(2m) && \text{(by Def. I.2.3.11)} \\ &= 2(m(2m)), && \text{(by Prop. I.2.3.5)} \end{aligned}$$

by A.Cor. I.4.4.2 this means n^2 is even.

Now we show that n is odd implies n^2 is odd. Since n is odd, by A.Cor. I.4.4.2 there exists an $m\in\mathbb{N}$ such that $n=2m+1$. By A.Cor. I.2.3.1 we know that $m(2m)\in\mathbb{N}$, thus by A.Cor. I.2.2.1 we have $m(2m)+2m\in\mathbb{N}$. Since

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= (2m)^2+2(2m)+1^2 && \text{(by Ex. I.2.3.4)} \\ &= (2m)(2m)+2(2m)+1 && \text{(by Def. I.2.3.11)} \\ &= 2(m(2m))+2(2m)+1 && \text{(by Prop. I.2.3.5)} \\ &= 2(m(2m)+2m)+1, && \text{(by Prop. I.2.3.4)} \end{aligned}$$

by A.Cor. I.4.4.2 this means n^2 is odd. ☐

Prop. I.4.4.4. There does not exist any rational number x for which $x^2=2$.

Proof of Prop. I.4.4.4. Suppose for the sake of contradiction that we had a rational number x for which $x^2=2$. Clearly, x is not zero by Prop. I.4.3.12(c). We may assume that x is positive, for if x were negative then we could just replace x by $-x$ (since $x^2=(-x)^2$). Thus, $x=p/q$ for some positive integers p,q, so $(p/q)^2=2$, which we can rearrange as $p^2=2q^2$. By A.Cor. I.4.4.2, every natural number is either even or odd, but not both. By A.Cor. I.4.4.3, if p is odd, then p^2 is also odd, which contradicts $p^2=2q^2$. Thus, p is even, i.e., $p = 2k$ for some natural number k. Since p is positive, k must also be positive. Inserting $p = 2k$ into $p^2 = 2q^2$ we obtain $4k^2 = 2q^2$, so that $q^2 = 2k^2$.

To summarize, we started with a pair (p, q) of positive integers such that $p^2 = 2q^2$, and ended up with a pair (q, k) of positive integers such that $q^2 = 2k^2$. Now we claim that $p^2 = 2q^2 \implies q < p$. We prove the claim by contradiction. Suppose for the sake of contradiction that $q \geq p$.

- If $q = p$, then we have $p^2 = q^2$. But $p^2 = 2q^2 = q^2 + q^2$ implies $q^2 = 0$, and by Prop. I.4.3.10(b) this means $q = 0$, which contradicts to $q > 0$.
- If $q > p$, then by Prop. I.4.3.10(c) we have $q^2 > p^2$. Since $q > 0$, by Prop. I.4.3.10(b) we have $q^2 > 0$. But by Prop. I.2.2.12(f) $p^2 = 2q^2 = q^2 + q^2$ implies $p^2 > q^2$, which contradicts to $q^2 > p^2$.

From all cases above, we derive contradiction. Thus, we must have $p^2 = 2q^2 \implies q < p$. If we rewrite $p' := q$ and $q' := k$, we thus can pass from one solution (p, q) to the equation $p^2 = 2q^2$ to a new solution (p', q') to the same equation which has a smaller value of p. But then we can repeat this procedure again and again, obtaining a sequence (p'', q''), (p''', q'''), etc. of solutions to $p^2 = 2q^2$, each one with a smaller value of p than the previous, and each one consisting of positive integers. But this contradicts the principle of infinite descent (see Ex. I.4.4.2). This contradiction shows that we could not have had a rational x for which $x^2 = 2$. □

Prop. I.4.4.5. For every rational number $\varepsilon > 0$, there exists a non-negative rational number x such that $x^2 < 2 < (x + \varepsilon)^2$.

Proof of Prop. I.4.4.5. Let $\varepsilon > 0$ be rational. Suppose for the sake of contradiction that there is no non-negative rational number x for which $x^2 < 2 < (x + \varepsilon)^2$. This means that whenever x is non-negative and $x^2 < 2$, we must also have $(x + \varepsilon)^2 < 2$ (note that $(x + \varepsilon)^2$ cannot equal 2, by Prop. I.4.4.4). Since $0^2 < 2$, we thus have $\varepsilon^2 < 2$, which then implies $(2\varepsilon)^2 < 2$, and indeed a simple induction shows that $(n\varepsilon)^2 < 2$ for every natural number n. (Note that $n\varepsilon$ is non-negative for every natural number n by A.Cor. I.4.2.7) But, by Prop. I.4.4.1 we can find an integer n such that $n > 2/\varepsilon$, which implies that $n\varepsilon > 2$, which implies that $(n\varepsilon)^2 > 4 > 2$, contradicting the claim that $(n\varepsilon)^2 < 2$ for all natural numbers n. This contradiction gives the proof. □

Note. Prop. I.4.4.5 indicates that, while the set $\mathbb{Q}$ of rationals does not actually have $\sqrt{2}$ as a member, we can get as close as we wish to $\sqrt{2}$. For instance, the sequence of rationals

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

seem to get closer and closer to $\sqrt{2}$, as their squares indicate:

$$1.96, 1.9881, 1.99396, 1.99996164, 1.9999899241, \ldots$$

Thus, it seems that we can create a square root of 2 by taking a "limit" of a sequence of rationals. This is how we shall construct the real numbers in Ch. I.5.

Note. There is another way to construct the real numbers, using something called "Dedekind cuts," which we will not pursue here. One can also proceed using infinite decimal expansions, but there are some sticky issues when doing so, e.g., one has to make $0.999\ldots$ equal to $1.000\ldots$, and this approach, despite being the most familiar, is actually more complicated than other approaches.

— Exercises —

Ex. I.4.4.1. Prove Prop. I.4.4.1.

Proof of Ex. I.4.4.1. See Prop. I.4.4.1. □

Ex. I.4.4.2. A definition: a sequence $a_0, a_1, a_2, \ldots$ of numbers (natural numbers, integers, rationals, or reals) is said to be in *infinite descent* if we have $a_n > a_{n+1}$ for all natural numbers n (i.e., $a_0 > a_1 > a_2 > \ldots$).

(a) Prove the *principle of infinite descent*: that it is not possible to have a sequence of *natural numbers* which is in infinite descent.

(b) Does the principle of infinite descent work if the sequence $a_1, a_2, a_3, \ldots$ is allowed to take integer values instead of natural number values? What about if it is allowed to take positive rational values instead of natural numbers? Explain.

Proof of Ex. I.4.4.2(a). Suppose for the sake of contradiction that there exists a sequence of natural numbers $a_0, a_1, \ldots$ is in infinite descent. Now we claim that if $k \in \mathbb{N}$, then $a_n \geq k$ for all $n \in \mathbb{N}$. We induct on k to prove the claim. For $k = 0$, by A.Cor. I.2.2.4 we have $a_n \geq 0$ for all $n \in \mathbb{N}$, so the base case holds. Suppose inductively that for some $k \in \mathbb{N}$ we have $a_n \geq k$ for all $n \in \mathbb{N}$. Then for $k+1$, we want to show that $a_n \geq k+1$ for all $n \in \mathbb{N}$. By the induction hypothesis we have $a_n \geq k$ for all $n \in \mathbb{N}$. Since the sequence $a_0, a_1, \ldots$ is in infinite descent, we know that $a_n > a_{n+1} \geq k$ for all $n \in \mathbb{N}$. Thus, we have

$$
\begin{aligned}
&\forall n \in \mathbb{N}, a_n > a_{n+1} \geq k && \text{(by the induction hypothesis)}\\
\implies &\forall n \in \mathbb{N}, a_n > k && \text{(by Prop. I.2.2.12(b))}\\
\implies &\forall n \in \mathbb{N}, a_n \geq k+1. && \text{(by Prop. I.2.2.12(e))}
\end{aligned}
$$

This closes the induction.

Now we show that such sequence does not exist. From the proof above we see that $a_1 \geq k$ for all $k \in \mathbb{N}$. If we set $k = a_0$, then we must have $a_1 \geq a_0$. Since the sequence is in infinite descent, we must have $a_0 > a_1$. But both $a_1 \geq a_0$ and $a_0 > a_1$ being true contradict to Prop. I.2.2.13. So we cannot have a sequence of natural number which is in infinite descent. □

Proof of Ex. I.4.4.2(b). By setting $a_n = -n$ for all $n \in \mathbb{N}$, we can always have $a_n > a_{n+1}$. So the principle of infinite descent does not work on integers.

Similarly, by setting $a_n = 1/n$ for all $n \in \mathbb{N}$, we can always have $a_n > a_{n+1}$. So the principle of infinite descent does not work on rationals. □

Ex. I.4.4.3. Fill in the gaps marked (why?) in the proof of Prop. I.4.4.4.

Proof of Ex. I.4.4.3. See Prop. I.4.4.4. □

Chapter I.5

The real numbers

Note. To review our progress to date, we have rigorously constructed three fundamental number systems: the natural number system $\mathbb{N}$, the integers $\mathbb{Z}$, and the rationals $\mathbb{Q}$. We defined the natural numbers using the five Peano axioms (Ax. I.2.1 to I.2.5), and postulated that such a number system existed; this is plausible, since the natural numbers correspond to the very intuitive and fundamental notion of *sequential counting.* Using that number system one could then recursively define addition and multiplication, and verify that they obeyed the usual laws of algebra. We then constructed the integers by taking formal differences of the natural numbers, $a{-\!\!-}b$. We then constructed the rationals by taking formal quotients of the integers, $a//b$, although we need to exclude division by zero in order to keep the laws of algebra reasonable. (You are of course free to design your own number system, possibly including one where division by zero is permitted; but you will have to give up one or more of the field axioms from Prop. I.4.2.4, among other things, and you will probably get a less useful number system in which to do any real-world problems.)

Note. The symbols $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ stand for "natural," "quotient," and "real" respectively. $\mathbb{Z}$ stands for "Zahlen," the German word for numbers. There is also the *complex numbers* $\mathbb{C}$, which obviously stands for "complex."

Note. *Formal* means "having the form of"; at the beginning of our construction the expression $a{-\!\!-}b$ did not actually *mean* the difference $a - b$, since the symbol ${-\!\!-}$ was meaningless. It only had the *form* of a difference. Later on we defined subtraction and verified that the formal difference was equal to the actual difference, so this eventually became a non-issue, and our symbol for formal differencing was discarded. Somewhat confusingly, this use of the term "formal" is unrelated to the notions of a formal argument and an informal argument.

Note. There is a fundamental area of mathematics where the rational number system does not suffice - that of *geometry* (the study of lengths, areas, etc.). For instance, a right-angled triangle with both sides equal to 1 gives a hypotenuse of $\sqrt{2}$, which is an *irrational* number, i.e., not a rational number; see Prop. I.4.4.4. Things get even worse when one starts to deal with the sub-field of geometry known as *trigonometry*, when one sees numbers such as π or cos(1), which turn out to be in some sense "even more" irrational than $\sqrt{2}$. (These numbers are known as *transcendental numbers*, but to discuss this further would be far beyond the scope of this text.) Thus, in order to have a number system which can adequately describe geometry - or even something as simple as measuring lengths on a line - one needs to replace the rational number system with the real number system. Since differential and integral calculus is also intimately tied up with geometry - think of slopes of tangents, or areas under a curve - calculus also requires the real number system in order to function properly.

Note. In the constructions of integers and rationals, the task was to introduce one more *algebraic* operation to the number system - e.g., one can get integers from naturals by introducing subtraction, and get the rationals from the integers by introducing division. But to get the reals from the rationals is to pass from a "discrete" system to a "continuous" one, and requires the introduction of a somewhat different notion - that of a *limit*.

Note. The limit is a concept which on one level is quite intuitive, but to pin down rigorously turns out to be quite difficult. (Even such great mathematicians as Euler and Newton had difficulty with this concept. It was only in the nineteenth century that mathematicians such as Cauchy and Dedekind figured out how to deal with limits rigorously.)

Note. The real number system will end up being a lot like the rational numbers, but will have some new operations - notably that of *supremum*, which can then be used to define limits and thence to everything else that calculus needs.

Note. The procedure we give here of obtaining the real numbers as the limit of sequences of rational numbers may seem rather complicated. However, it is in fact an instance of a very general and useful procedure, that of *completing* one metric space to form another. see Ex. II.1.4.8.

I.5.1 Cauchy sequences

Def. I.5.1.1 (Sequences)**.** Let m be an integer. A *sequence* $(a_n)_{n=m}^{\infty}$ *of rational numbers* is any function from the set $\mathbb{Z}_{\geq m}$ to $\mathbb{Q}$, i.e., a mapping which assigns to each integer n greater than or equal to m, a rational number a_n. More informally, a sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is a collection of rationals $a_m, a_{m+1}, a_{m+2}, \ldots$.

Def. I.5.1.3 (ε-steadiness)**.** Let $\varepsilon \in \mathbb{Q}^+$. A sequence $(a_n)_{n=m}^{\infty}$ is said to be ε*-steady* iff each pair a_j, a_k of sequence elements is ε-close for every natural number $j, k \in \mathbb{Z}_{\geq m}$. In other words, the sequence $a_m, a_{m+1}, a_{m+2}, \ldots$ is ε-steady iff $d(a_j, a_k) \leq \varepsilon$ for all $j, k \in \mathbb{Z}_{\geq m}$.

Rmk. I.5.1.4. Def. I.5.1.3 is not standard in the literature; we will not need it outside of this section; similarly for the concept of "eventual ε-steadiness" below. We have defined ε-steadiness for sequences whose index starts at m, but clearly we can make a similar notion for sequences whose indices start from any other number: a sequence $a_N, a_{N+1}, \ldots$ is ε-steady if one has $d(a_j, a_k) \leq \varepsilon$ for all $j, k \in \mathbb{Z}_{\geq N}$.

Note. The notion of ε-steadiness of a sequence is simple, but does not really capture the *limiting* behavior of a sequence, because it is too sensitive to the initial members of the sequence. So we need a more robust notion of steadiness that does not care about the initial members of a sequence.

Def. I.5.1.6 (Eventual ε-steadiness)**.** Let $\varepsilon \in \mathbb{Q}^+$. A sequence $(a_n)_{n=m}^\infty$ is said to be *eventually ε-steady* iff the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is ε-steady for some integer $N \geq m$. In other words, the sequence $a_m, a_{m+1}, a_{m+2}, \ldots$ is eventually ε-steady iff there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|a_j - a_k| \leq \varepsilon$ for all $j, k \in \mathbb{Z}_{\geq N}$.

Def. I.5.1.8 (Cauchy sequences)**.** A sequence $(a_n)_{n=m}^\infty$ of rational numbers is said to be a *Cauchy sequence* iff for every rational $\varepsilon \in \mathbb{Q}^+$, the sequence $(a_n)_{n=m}^\infty$ is eventually ε-steady. In other words, the sequence $a_m, a_{m+1}, a_{m+2}, \ldots$ is a Cauchy sequence iff for every $\varepsilon \in \mathbb{Q}^+$, there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|a_j - a_k| \leq \varepsilon$ for all $j, k \in \mathbb{Z}_{\geq N}$.

Rmk. I.5.1.9. At present, the parameter ε is restricted to be a positive rational; we cannot take ε to be an arbitrary positive real number, because the real numbers have not yet been constructed. However, once we do construct the real numbers, we shall see that Def. I.5.1.8 will not change if we require ε to be real instead of rational (Prop. I.6.1.4). In other words, we will eventually prove that a sequence is eventually ε-steady for every rational $\varepsilon \in \mathbb{Q}^+$ iff it is eventually ε-steady for every real $\varepsilon \in \mathbb{Q}^+$. This rather subtle distinction between a rational ε and a real ε turns out not to be very important in the long run, and the reader is advised not to pay too much attention as to what type of number ε should be.

Prop. I.5.1.11. The sequence $(a_n)_{n=1}^\infty$ defined by $a_n := 1/n$ (i.e., the sequence $1, 1/2, 1/3, \ldots$) is a Cauchy sequence.

Proof of Prop. I.5.1.11. We have to show that for every $\varepsilon \in \mathbb{Q}^+$, the sequence $a_1, a_2, \ldots$ is eventually ε-steady. So let $\varepsilon \in \mathbb{Q}^+$ be arbitrary. We now have to find a number $N \in \mathbb{Z}_{\geq 1}$ such that the sequence $a_N, a_{N+1}, \ldots$ is ε-steady. Let us see what this means. This means that $d(a_j, a_k) \leq \varepsilon$ for every $j, k \in \mathbb{Z}_{\geq N}$, i.e.

$$\left|\frac{1}{j} - \frac{1}{k}\right| \leq \varepsilon \text{ for every } j, k \in \mathbb{Z}_{\geq N}.$$

Now since $j, k \in \mathbb{Z}_{\geq N}$, by Prop. I.4.3.12(b) we know that $0 < 1/j, 1/k \leq 1/N$, so that

$$\begin{cases} \dfrac{-1}{N} < 0 < \dfrac{1}{j} \leq \dfrac{1}{N} \\ \dfrac{-1}{N} \leq \dfrac{-1}{k} < 0 < \dfrac{1}{N} \end{cases} \qquad \text{(by Ex. I.4.2.6)}$$

$$\Longrightarrow \begin{cases} \dfrac{1}{j} - \dfrac{1}{k} \leq \dfrac{1}{N} - \dfrac{1}{k} < \dfrac{1}{N} \\ \dfrac{-1}{N} < \dfrac{1}{j} - \dfrac{1}{N} \leq \dfrac{1}{j} - \dfrac{1}{k} \end{cases} \qquad \text{(by Prop. I.4.2.9(c)(d))}$$

$$\implies \frac{-1}{N} \leq \frac{1}{j} - \frac{1}{k} \leq \frac{1}{N}$$

$$\implies \left|\frac{1}{j} - \frac{1}{k}\right| \leq \frac{1}{N}. \qquad \text{(by Prop. I.4.3.3(c))}$$

So in order to force $|1/j - 1/k|$ to be less than or equal to ε, it would be sufficient for $1/N$ to be less than ε. So all we need to do is choose an N such that $1/N$ is less than ε, or in other words that N is greater than $1/\varepsilon$. But this can be done thanks to Prop. I.4.4.1. □

Note. As you can see, verifying from first principles (i.e., without using any of the machinery of limits, etc.) that a sequence is a Cauchy sequence requires some effort, even for a sequence as simple as $1/n$. The part about selecting an N can be particularly difficult for beginners - one has to think in reverse, working out what conditions on N would suffice to force the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ to be ε-steady, and then finding an N which obeys those conditions. Later we will develop some limit laws which allow us to determine when a sequence is Cauchy more easily.

Def. I.5.1.12 (Bounded sequences)**.** Let $M \in \mathbb{Q}_{\geq 0}$. A finite rational sequence $(a_n)_{n=m}^{k}$ is *bounded by* M iff $|a_i| \leq M$ for all $i \in \mathbb{Z}_{m \leq k}$. An infinite rational sequence $(a_n)_{n=m}^{\infty}$ is *bounded by* M iff $|a_i| \leq M$ for all $i \in \mathbb{Z}_{\geq m}$. A rational sequence is said to be *bounded* iff it is bounded by M for some $M \in \mathbb{Q}_{\geq 0}$.

Lem. I.5.1.14 (Finite sequences are bounded)**.** Every finite rational sequence $(a_n)_{n=m}^{k}$ is bounded.

Proof of Lem. I.5.1.14. We induct on k and we start with $k = m$. When $k = m$ the rational sequence $(a_n)_{n=m}^{m}$ is clearly bounded, for if we choose $M := |a_m|$ then clearly we have $|a_i| \leq M$ for all $i \in \mathbb{Z}_{m \leq k}$. Now suppose that we have already proved the lemma for some $k \geq m$; we now prove it for $k+1$, i.e., we prove every rational sequence $(a_n)_{n=m}^{k+1}$ is bounded. By the induction hypothesis we know that $(a_n)_{n=m}^{k}$ is bounded by some $M \in \mathbb{Q}_{\geq 0}$; in particular, it must be bounded by $M + |a_{k+1}|$. On the other hand, a_{k+1} is also bounded by $M + |a_{k+1}|$. Thus, $(a_n)_{n=m}^{k+1}$ is bounded by $M + |a_{k+1}|$, and is hence bounded. This closes the induction. □

Note. While Lem. I.5.1.14 shows that every finite rational sequence is bounded, no matter how long the finite sequence is, it does not say anything about whether an infinite rational sequence is bounded or not; infinity is not a natural number.

Lem. I.5.1.15 (Cauchy sequences are bounded)**.** Every rational Cauchy sequence $(a_n)_{n=m}^{\infty}$ is bounded.

Proof of Lem. I.5.1.15. Since $(a_n)_{n=m}^{\infty}$ is a rational Cauchy sequence, by Def. I.5.1.8 we know that $(a_n)_{n=m}^{\infty}$ is eventually ε-steady for all $\varepsilon \in \mathbb{Q}^+$. In particular, $(a_n)_{n=m}^{\infty}$ is eventually

1-steady. By Def. I.5.1.6 there exists an $N \in \mathbb{Z}_{\geq m}$ such that $(a_n)_{n=N}^{\infty}$ is 1-steady. Fix such N. Since $(a_n)_{n=N}^{\infty}$ is 1-steady, we have

$$\begin{aligned}
&\forall j \in \mathbb{Z}_{\geq N}, |a_j - a_N| \leq 1 && \text{(by Def. I.5.1.3)} \\
\implies &\forall j \in \mathbb{Z}_{\geq N}, |a_j - a_N| + |a_N| \leq 1 + |a_N| && \text{(by Prop. I.4.2.9(d))} \\
\implies &\forall j \in \mathbb{Z}_{\geq N}, |a_j - a_N + a_N| \leq |a_j - a_N| + |a_N| \leq 1 + |a_N| && \text{(by Prop. I.4.3.3(b))} \\
\implies &\forall j \in \mathbb{Z}_{\geq N}, |a_j| \leq 1 + |a_N|. && \text{(by Prop. I.4.2.4)}
\end{aligned}$$

Thus, by Def. I.5.1.12 $(a_n)_{n=N}^{\infty}$ is bounded by $1 + |a_N|$. Now we split into two cases:

- If $N = m$, then we see that $(a_n)_{n=m}^{\infty}$ is bounded by $1 + |a_N|$.
- If $N \neq m$, then we must have $m < N$. By Lem. I.5.1.14 we know that the finite rational sequence $(a_n)_{n=m}^{N-1}$ is bounded by some $M \in \mathbb{Q}_{\geq 0}$. So both $(a_n)_{n=m}^{N-1}$ and $(a_n)_{n=N}^{\infty}$ are bounded by $M + 1 + |a_N|$. Thus, $(a_n)_{n=m}^{\infty}$ is bounded by $M + 1 + |a_N|$.

From all cases above, we see that $(a_n)_{n=m}^{\infty}$ is bounded. Since $(a_n)_{n=m}^{\infty}$ was arbitrary, we conclude that every rational Cauchy sequences are bounded. □

— Exercises —

Ex. I.5.1.1. Prove Lem. I.5.1.15.

Proof of Ex. I.5.1.1. See Lem. I.5.1.15. □

I.5.2 Equivalent Cauchy sequences

Note. If we are to define the real numbers from the rationals as limits of rational Cauchy sequences, we have to know when two Cauchy sequences of rationals give the same limit, without first defining a real number (since that would be circular). To do this we use a similar set of definitions to those used to define a Cauchy sequence in the first place.

Def. I.5.2.1 (ε-close sequences)**.** Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be two rational sequences, and let $\varepsilon \in \mathbb{Q}^+$. We say that the sequence $(a_n)_{n=m}^{\infty}$ is *ε-close* to $(b_n)_{n=m}^{\infty}$ iff a_n is ε-close to b_n for each $n \in \mathbb{Z}_{\geq m}$. In other words, the sequence $a_m, a_{m+1}, a_{m+2}, \ldots$ is ε-close to the sequence $b_m, b_{m+1}, b_{m+2}, \ldots$ iff $|a_n - b_n| \leq \varepsilon$ for all $n = m, m+1, m+2, \ldots$.

Def. I.5.2.3 (Eventually ε-close sequences)**.** Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be two rational sequences, and let $\varepsilon \in \mathbb{Q}^+$. We say that the sequence $(a_n)_{n=m}^{\infty}$ is *eventually ε-close* to $(b_n)_{n=m}^{\infty}$ iff there exists an $N \in \mathbb{Z}_{\geq m}$ such that the sequences $(a_n)_{n=N}^{\infty}$ and $(b_n)_{n=N}^{\infty}$ are ε-close. In other words, $a_m, a_{m+1}, a_{m+2}, \ldots$ is eventually ε-close to $b_m, b_{m+1}, b_{m+2}, \ldots$ iff there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$.

Rmk. I.5.2.4. Again, the notations for ε-close sequences and eventually ε-close sequences are not standard in the literature, and we will not use them outside of this section.

Def. I.5.2.6 (Equivalent sequences)**.** Two rational sequences $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are *equivalent* iff for each $\varepsilon \in \mathbb{Q}^+$, the sequences $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close. In other words, $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are equivalent iff for every $\varepsilon \in \mathbb{Q}^+$, there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$.

Rmk. I.5.2.7. As with Def. I.5.1.8, the quantity $\varepsilon \in \mathbb{Q}^+$ is currently restricted to be a positive rational, rather than a positive real. However, we shall eventually see that it makes no difference whether ε ranges over the positive rationals or positive reals; see Ex. I.6.1.10.

A.Cor. I.5.2.1. Equivalence defined as Def. I.5.2.6 is reflexive, symmetric and transitive.

Proof of A.Cor. I.5.2.1. Let $(a_n)_{n=m}^{\infty}, (b_n)_{n=m}^{\infty}, (c_n)_{n=m}^{\infty}$ be rational sequences. We denote the equivalence relation defined in Def. I.5.2.6 as $\equiv$. First, we show that $\equiv$ is reflexive. Since for arbitrary $(a_n)_{n=m}^{\infty}$, we always have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{Q}^+, \forall n \geq m, |a_n - a_n| = |0| = 0 \leq \varepsilon \\
\implies &(a_n)_{n=m}^{\infty} \equiv (a_n)_{n=m}^{\infty}, && \text{(by Def. I.5.2.6)}
\end{aligned}$$

we see that $\equiv$ is reflexive.

Next we show that $\equiv$ is symmetric. Suppose that $(a_n)_{n=m}^{\infty} \equiv (b_n)_{n=m}^{\infty}$. Then we have

$$\begin{aligned}
&(a_n)_{n=m}^{\infty} \equiv (b_n)_{n=m}^{\infty} \\
\implies &\forall \varepsilon \in \mathbb{Q}^+, \exists N \in \mathbb{Z}_{\geq m} : \forall n \in \mathbb{Z}_{\geq N}, |a_n - b_n| \leq \varepsilon && \text{(by Def. I.5.2.6)} \\
\implies &\forall \varepsilon \in \mathbb{Q}^+, \exists N \in \mathbb{Z}_{\geq m} : \forall n \in \mathbb{Z}_{\geq N}, |b_n - a_n| \leq \varepsilon && \text{(by Prop. I.4.3.3(f))} \\
\implies &(b_n)_{n=m}^{\infty} \equiv (a_n)_{n=m}^{\infty}. && \text{(by Def. I.5.2.6)}
\end{aligned}$$

Thus, $\equiv$ is symmetric.

Finally we show that $\equiv$ is transitive. Suppose that $(a_n)_{n=m}^{\infty} \equiv (b_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty} \equiv (c_n)_{n=m}^{\infty}$. Then we have

$$\begin{aligned}
&((a_n)_{n=m}^{\infty} \equiv (b_n)_{n=m}^{\infty}) \wedge ((b_n)_{n=m}^{\infty} \equiv (c_n)_{n=m}^{\infty}) \\
\implies &\forall \varepsilon \in \mathbb{Q}^+, \exists N_1, N_2 \in \mathbb{Z}_{\geq m} : \begin{cases} \forall n \in \mathbb{Z}_{\geq N_1}, |a_n - b_n| \leq \dfrac{\varepsilon}{2} \\ \forall n \in \mathbb{Z}_{\geq N_2}, |b_n - c_n| \leq \dfrac{\varepsilon}{2} \end{cases} && \text{(by Def. I.5.2.6)} \\
\implies &\forall \varepsilon \in \mathbb{Q}^+, \exists N = \max(N_1, N_2) \in \mathbb{Z}_{\geq m} : && \text{(by Lem. I.4.1.11(f))} \\
&\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} |a_n - b_n| \leq \dfrac{\varepsilon}{2} \\ |b_n - c_n| \leq \dfrac{\varepsilon}{2} \end{cases} \\
\implies &\forall \varepsilon \in \mathbb{Q}^+, \exists N = \max(N_1, N_2) \in \mathbb{Z}_{\geq m} : \\
&\forall n \in \mathbb{Z}_{\geq N}, |a_n - b_n| + |b_n - c_n| \leq \varepsilon && \text{(by Prop. I.4.2.9(c)(d))} \\
\implies &\forall \varepsilon \in \mathbb{Q}^+, \exists N = \max(N_1, N_2) \in \mathbb{Z}_{\geq m} :
\end{aligned}$$

$$\forall n \in \mathbb{Z}_{\geq N}, |a_n - c_n| \leq |a_n - b_n| + |b_n - c_n| \leq \varepsilon \qquad \text{(by Prop. I.4.3.3(b))}$$
$$\implies (a_n)_{n=m}^{\infty} \equiv (c_n)_{n=m}^{\infty}. \qquad \text{(by Def. I.5.2.6)}$$

Thus, $\equiv$ is transitive. □

Prop. I.5.2.8. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be the sequences $a_n = 1 + 10^{-n}$ and $b_n = 1 - 10^{-n}$. Then the sequences $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ are equivalent.

Proof of Prop. I.5.2.8. We need to prove that for every $\varepsilon \in \mathbb{Q}^+$, the two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close to each other. So we fix an $\varepsilon \in \mathbb{Q}^+$. We need to find an $N \in \mathbb{Z}^+$ such that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are ε-close; in other words, we need to find an $N \in \mathbb{Z}^+$ such that

$$|a_n - b_n| \leq \varepsilon \text{ for all } n \in \mathbb{Z}_{\geq N}.$$

However, we have

$$|a_n - b_n| = \left|(1 + 10^{-n}) - (1 - 10^{-n})\right| = 2 \times 10^{-n}.$$

Since 10^{-n} is a decreasing function of n (i.e., $10^{-m} < 10^{-n}$ whenever $m > n$; this is easily proven by induction), and $n \in \mathbb{Z}_{\geq N}$, we have $2 \times 10^{-n} \leq 2 \times 10^{-N}$. Thus, we have

$$|a_n - b_n| \leq 2 \times 10^{-N} \text{ for all } n \in \mathbb{Z}_{\geq N}.$$

Thus, in order to obtain $|a_n - b_n| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$, it will be sufficient to choose N so that $2 \times 10^{-N} \leq \varepsilon$. This is easy to do using logarithms, but we have not yet developed logarithms yet, so we will use a cruder method. First, we observe 10^N is always greater than N for any $N \in \mathbb{Z}^+$ (see Ex. I.4.3.5). Thus, $10^{-N} \leq 1/N$, and so $2 \times 10^{-N} \leq 2/N$. Thus, to get $2 \times 10^{-N} \leq \varepsilon$, it will suffice to choose N so that $2/N \leq \varepsilon$, or equivalently that $N \geq 2/\varepsilon$. But by Prop. I.4.4.1 we can always choose such an N, and the claim follows. □

Rmk. I.5.2.9. Prop. I.5.2.8, in decimal notation, asserts that

$$1.0000\cdots = 0.9999\ldots.$$

— Exercises —

Ex. I.5.2.1. Show that if $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are equivalent sequences of rationals, then $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence iff $(b_n)_{n=m}^{\infty}$ is a Cauchy sequence.

Proof of Ex. I.5.2.1. First, suppose that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence. Let $\varepsilon \in \mathbb{Q}^+$. Since $(a_n)_{n=m}^{\infty}, (b_n)_{n=m}^{\infty}$ are equivalent, by Def. I.5.2.6 we have

$$\exists N_1 \in \mathbb{Z}_{\geq m} : \forall n \in \mathbb{Z}_{\geq N_1}, |a_n - b_n| \leq \frac{\varepsilon}{3}.$$

Fix such N_1. Since $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence, by Def. I.5.1.8 we have

$$\exists N_2 \in \mathbb{Z}_{\geq m} : \forall j, k \in \mathbb{Z}_{\geq N}, |a_j - a_k| \leq \frac{\varepsilon}{3}.$$

Fix such N_2. Now let $N = \max(N_1, N_2)$. By Lem. I.4.1.11(f) we know that N is well-defined. Then we have

$$\begin{aligned}\forall j,k \in \mathbb{Z}_{\geq N}, |b_j - b_k| &= |a_j - a_k + b_j - a_j + a_k - b_k| \\ &\leq |a_j - a_k| + |a_j - b_j| + |a_k - b_k| && \text{(by Prop. I.4.3.3(b))} \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. && \text{(by Prop. I.4.2.9(c)(d))}\end{aligned}$$

Since ε was arbitrary, we see that

$$\forall \varepsilon \in \mathbb{Q}^+, \exists N \in \mathbb{Z}_{\geq m} : \forall j,k \in \mathbb{Z}_{\geq N}, |b_j - b_k| \leq \varepsilon.$$

By Def. I.5.1.8 this means $(b_n)_{n=m}^{\infty}$ is a Cauchy sequence.

Using similar arguments, we can show that $(b_n)_{n=m}^{\infty}$ is a Cauchy sequence implies $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence. Thus, we conclude that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence iff $(b_n)_{n=m}^{\infty}$ is a Cauchy sequence. □

Ex. I.5.2.2. Let $\varepsilon \in \mathbb{Q}^+$. Show that if $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close, then $(a_n)_{n=m}^{\infty}$ is bounded iff $(b_n)_{n=m}^{\infty}$ is bounded.

Proof of Ex. I.5.2.2. First, suppose that $(a_n)_{n=m}^{\infty}$ is bounded. Since $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close, by Def. I.5.2.3 we have

$$\exists N \in \mathbb{Z}_{\geq m} : \forall n \in \mathbb{Z}_{\geq N}, |a_n - b_n| \leq \varepsilon.$$

Fix such N. Since $(a_n)_{n=m}^{\infty}$ is bounded, by Def. I.5.1.12 there exists some $M \in \mathbb{Q}_{\geq 0}$ such that $|a_n| \leq M$ for all $n \in \mathbb{Z}_{\geq m}$. Then we have

$$\begin{aligned}\forall n \in \mathbb{Z}_{\geq N}, |b_n| &= |-b_n| && \text{(by Prop. I.4.3.3(d))} \\ &= |a_n - b_n + a_n| \\ &\leq |a_n - b_n| + |a_n| && \text{(by Prop. I.4.3.3(b))} \\ &\leq \varepsilon + M. && \text{(by Prop. I.4.2.9(c)(d))}\end{aligned}$$

Thus, $(b_n)_{n=N}^{\infty}$ is bounded by $\varepsilon + M$. Now we split into two cases:

- If $N = m$, then we see that $(b_n)_{n=m}^{\infty}$ is bounded by $\varepsilon + M$.
- If $N \neq m$, then we must have $m < N$. By Lem. I.5.1.14 we know that the finite rational sequence $(b_n)_{n=m}^{N-1}$ is bounded by some $M' \in \mathbb{Q}_{\geq 0}$. So both $(b_n)_{n=m}^{N-1}$ and $(b_n)_{n=N}^{\infty}$ are bounded by $M' + \varepsilon + M$. Thus, $(b_n)_{n=m}^{\infty}$ is bounded by $M' + \varepsilon + M$.

From all cases above, we see that $(b_n)_{n=m}^{\infty}$ is bounded.

Using similar arguments, we can show that $(b_n)_{n=m}^{\infty}$ is bounded implies $(a_n)_{n=m}^{\infty}$ is bounded. Thus, we conclude that $(a_n)_{n=m}^{\infty}$ is bounded iff $(b_n)_{n=m}^{\infty}$ is bounded. □

I.5.3 The construction of the real numbers

Def. I.5.3.1 (Real numbers)**.** A *real number* is defined to be an object of the form $\operatorname*{LIM}_{n\to\infty} a_n$, where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Two real numbers $\operatorname*{LIM}_{n\to\infty} a_n$ and $\operatorname*{LIM}_{n\to\infty} b_n$ are said to be equal iff $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent rational Cauchy sequences. The set of all real numbers is denoted $\mathbb{R}$.

Note. We will refer to $\operatorname*{LIM}_{n\to\infty} a_n$ as the *formal limit* of the sequence $(a_n)_{n=1}^{\infty}$. Later on we will define a genuine notion of limit, and show that the formal limit of a Cauchy sequence is the same as the limit of that sequence; after that, we will not need formal limits ever again.

Prop. I.5.3.3 (Formal limits are well-defined)**.** Let $x = \operatorname*{LIM}_{n\to\infty} a_n, y = \operatorname*{LIM}_{n\to\infty} b_n, z = \operatorname*{LIM}_{n\to\infty} c_n$ be real numbers. Then, with the above definition of equality for real numbers, we have $x = x$. Also, if $x = y$, then $y = x$. Finally, if $x = y$ and $y = z$, then $x = z$.

Proof of Prop. I.5.3.3. Let $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}, (c_n)_{n=1}^{\infty}$ be rational Cauchy sequences such that $x = \operatorname*{LIM}_{n\to\infty} a_n, y = \operatorname*{LIM}_{n\to\infty} b_n, z = \operatorname*{LIM}_{n\to\infty} c_n$, respectively. Let $\equiv$ denote the equivalence of rational Cauchy sequences as defined in Def. I.5.2.6. First, we show that the equality defined in Def. I.5.3.1 is reflexive. This is true since

$$\begin{aligned}
&(a_n)_{n=1}^{\infty} \equiv (a_n)_{n=1}^{\infty} && \text{(by A.Cor. I.5.2.1)}\\
\Longrightarrow\ & x = \operatorname*{LIM}_{n\to\infty} a_n = \operatorname*{LIM}_{n\to\infty} a_n = x. && \text{(by Def. I.5.3.1)}
\end{aligned}$$

Next we show that the equality defined in Def. I.5.3.1 is symmetric. Suppose that $x = y$. Then we have

$$\begin{aligned}
&x = \operatorname*{LIM}_{n\to\infty} a_n = \operatorname*{LIM}_{n\to\infty} b_n = y\\
\Longrightarrow\ & (a_n)_{n=1}^{\infty} \equiv (b_n)_{n=1}^{\infty} && \text{(by Def. I.5.3.1)}\\
\Longrightarrow\ & (b_n)_{n=1}^{\infty} \equiv (a_n)_{n=1}^{\infty} && \text{(by A.Cor. I.5.2.1)}\\
\Longrightarrow\ & y = \operatorname*{LIM}_{n\to\infty} b_n = \operatorname*{LIM}_{n\to\infty} a_n = x. && \text{(by Def. I.5.3.1)}
\end{aligned}$$

Thus, the equality defined in Def. I.5.3.1 is symmetric.

Finally we show that the equality defined in Def. I.5.3.1 is transitive. Suppose that $x = y$ and $y = z$. Then we have

$$\begin{aligned}
&\begin{cases} x = \operatorname*{LIM}_{n\to\infty} a_n = \operatorname*{LIM}_{n\to\infty} b_n = y\\ y = \operatorname*{LIM}_{n\to\infty} b_n = \operatorname*{LIM}_{n\to\infty} c_n = z \end{cases}\\
\Longrightarrow\ & \begin{cases} (a_n)_{n=1}^{\infty} \equiv (b_n)_{n=1}^{\infty}\\ (b_n)_{n=1}^{\infty} \equiv (c_n)_{n=1}^{\infty} \end{cases} && \text{(by Def. I.5.3.1)}\\
\Longrightarrow\ & (a_n)_{n=1}^{\infty} \equiv (c_n)_{n=1}^{\infty} && \text{(by A.Cor. I.5.2.1)}
\end{aligned}$$

$$\Longrightarrow x = \operatorname*{LIM}_{n\to\infty} a_n = \operatorname*{LIM}_{n\to\infty} c_n = z. \qquad \text{(by Def. I.5.3.1)}$$

Thus, the equality defined in Def. I.5.3.1 is transitive. □

Note. Because of Prop. I.5.3.3, we know that our definition of equality between two real numbers is legitimate. Of course, when we define other operations on the reals, we have to check that they obey the axiom of substitution: two real number inputs which are equal should give equal outputs when applied to any operation on the real numbers.

Def. I.5.3.4 (Addition of reals)**.** Let $x = \operatorname*{LIM}_{n\to\infty} a_n$ and $y = \operatorname*{LIM}_{n\to\infty} b_n$ be real numbers. Then we define the sum $x+y$ to be $x + y \coloneqq \operatorname*{LIM}_{n\to\infty}(a_n + b_n)$.

Lem. I.5.3.6 (Sum of Cauchy sequences is Cauchy)**.** Let $x = \operatorname*{LIM}_{n\to\infty} a_n$ and $y = \operatorname*{LIM}_{n\to\infty} b_n$ be real numbers. Then $x+y$ is also a real number (i.e., $(a_n + b_n)_{n=1}^{\infty}$ is a Cauchy sequence of rationals).

Proof of Lem. I.5.3.6. We need to show that for every $\varepsilon \in \mathbb{Q}^+$, the sequence $(a_n + b_n)_{n=1}^{\infty}$ is eventually ε-steady. Now from hypothesis we know that $(a_n)_{n=1}^{\infty}$ is eventually ε-steady, and $(b_n)_{n=1}^{\infty}$ is eventually ε-steady, but it turns out that this is not quite enough (this can be used to imply that $(a_n + b_n)_{n=1}^{\infty}$ is eventually 2ε-steady, but that's not what we want). So we need to do a little trick, which is to play with the value of ε.

We know that $(a_n)_{n=1}^{\infty}$ is eventually δ-steady for every value of δ. This implies not only that $(a_n)_{n=1}^{\infty}$ is eventually ε-steady, but it is also eventually $\varepsilon/2$-steady. Similarly, the sequence $(b_n)_{n=1}^{\infty}$ is also eventually $\varepsilon/2$-steady. This will turn out to be enough to conclude that $(a_n + b_n)_{n=1}^{\infty}$ is eventually ε-steady.

Since $(a_n)_{n=1}^{\infty}$ is eventually $\varepsilon/2$-steady, we know that there exists an $N \in \mathbb{Z}^+$ such that $(a_n)_{n=N}^{\infty}$ is $\varepsilon/2$-steady, i.e., a_n and a_m are $\varepsilon/2$-close for every $n, m \in \mathbb{Z}_{\geq N}$. Similarly, there exists an $M \in \mathbb{Z}^+$ such that $(b_n)_{n=M}^{\infty}$ is $\varepsilon/2$-steady, i.e., b_n and b_m are $\varepsilon/2$-close for every $n, m \in \mathbb{Z}_{\geq M}$.

Let $\max(N, M)$ be the larger of N and M (we know from Prop. I.2.2.13 that one has to be greater than or equal to the other). If $n, m \geq \max(N, M)$, then we know that a_n and a_m are $\varepsilon/2$-close, and b_n and b_m are $\varepsilon/2$-close, and so by Prop. I.4.3.7(d) we see that $a_n + b_n$ and $a_m + b_m$ are ε-close for every $n, m \geq \max(N, M)$. This implies that the sequence $(a_n + b_n)_{n=1}^{\infty}$ is eventually ε-steady, as desired. □

Lem. I.5.3.7 (Sums of equivalent Cauchy sequences are equivalent)**.** Let $x = \operatorname*{LIM}_{n\to\infty} a_n, y = \operatorname*{LIM}_{n\to\infty} b_n, x' = \operatorname*{LIM}_{n\to\infty} a'_n$ be real numbers. Suppose that $x = x'$. Then we have $x + y = x' + y$.

Proof of Lem. I.5.3.7. Since x and x' are equal, we know that the rational Cauchy sequences $(a_n)_{n=1}^{\infty}$ and $(a'_n)_{n=1}^{\infty}$ are equivalent, so in other words they are eventually ε-close for each $\varepsilon \in \mathbb{Q}^+$. We need to show that the sequences $(a_n + b_n)_{n=1}^{\infty}$ and $(a'_n + b_n)_{n=1}^{\infty}$ are eventually ε-close for each $\varepsilon \in \mathbb{Q}^+$. But we already know that there is an $N \in \mathbb{Z}^+$ such that $(a_n)_{n=N}^{\infty}$ and $(a'_n)_{n=N}^{\infty}$ are ε-close, i.e., that a_n and a'_n are ε-close for each $n \in \mathbb{Z}_{\geq N}$. Since b_n is of course 0-close to b_n, we thus see from Prop. I.4.3.7(d) that $a_n + b_n$ and $a'_n + b_n$ are ε-close for each $n \in \mathbb{Z}_{\geq N}$. This implies that $(a_n + b_n)_{n=1}^{\infty}$ and $(a'_n + b_n)_{n=1}^{\infty}$ are eventually ε-close for each $\varepsilon \in \mathbb{Q}^+$, and we are done. $\square$

Rmk. I.5.3.8. Lem. I.5.3.7 verifies the axiom of substitution for the "x" variable in $x + y$, but one can similarly prove the axiom of substitution for the "y" variable. (A quick way is to observe from the definition of $x + y$ that we certainly have $x + y = y + x$, since $a_n + b_n = b_n + a_n$.)

Def. I.5.3.9 (Multiplication of reals)**.** Let $x = \operatorname*{LIM}_{n\to\infty} a_n$ and $y = \operatorname*{LIM}_{n\to\infty} b_n$ be real numbers. Then we define the product xy to be $xy \coloneqq \operatorname*{LIM}_{n\to\infty} a_n b_n$.

Prop. I.5.3.10 (Multiplication is well defined)**.** Let $x = \operatorname*{LIM}_{n\to\infty} a_n, y = \operatorname*{LIM}_{n\to\infty} b_n, x' = \operatorname*{LIM}_{n\to\infty} a'_n$ be real numbers. Then xy is also a real number. Furthermore, if $x = x'$, then $xy = x'y$.

Proof of Prop. I.5.3.10. Let $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}, (a'_n)_{n=1}^{\infty}$ be rational Cauchy sequences where $x = \operatorname*{LIM}_{n\to\infty} a_n$, $y = \operatorname*{LIM}_{n\to\infty} b_n$ and $x' = \operatorname*{LIM}_{n\to\infty} a'_n$. We first show that $xy \in \mathbb{R}$. Let $\varepsilon \in \mathbb{Q}^+$. Since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are rational Cauchy sequences, by Lem. I.5.1.15 we know that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are bounded by some $M_1, M_2 \in \mathbb{Q}_{\geq 0}$. Then by Prop. I.4.2.9(a) we know that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are bounded by $M = \max(M_1, M_2) + 1$. Clearly, $M \in \mathbb{Q}^+$, so $\dfrac{\varepsilon}{2M} \in \mathbb{Q}^+$. Then by Def. I.5.1.8 we have

$$\exists N_1 \in \mathbb{Z}^+ : \forall j, k \in \mathbb{Z}_{\geq N_1}, |a_j - a_k| \leq \frac{\varepsilon}{2M};$$
$$\exists N_2 \in \mathbb{Z}^+ : \forall j, k \in \mathbb{Z}_{\geq N_2}, |b_j - b_k| \leq \frac{\varepsilon}{2M}.$$

Let $N = \max(N_1, N_2)$. Clearly, $N \in \mathbb{Z}^+$. Then we have

$$\begin{aligned}
\forall j, k \in \mathbb{Z}_{\geq N}, |a_j b_j - a_k b_k| &= |a_j b_j - a_j b_k + a_j b_k - a_k b_k| && \text{(by Prop. I.4.2.4)}\\
&\leq |a_j b_j - a_j b_k| + |a_j b_k - a_k b_k| && \text{(by Prop. I.4.3.3(b))}\\
&= |a_j||b_j - b_k| + |b_k||a_j - a_k| && \text{(by Prop. I.4.3.3(d))}\\
&\leq M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} && \text{(by Prop. I.4.2.9(c)(e))}\\
&= \varepsilon.
\end{aligned}$$

Since ε was arbitrary, we can conclude that

$$\forall \varepsilon \in \mathbb{Q}^+, \exists N \in \mathbb{Z}^+ : \forall j, k \in \mathbb{Z}_{\geq N}, |a_j b_j - a_k b_k| \leq \varepsilon.$$

Thus, by Def. I.5.1.8 $(a_n b_n)_{n=1}^{\infty}$ is a rational Cauchy sequence and by Def. I.5.3.1 $xy \in \mathbb{R}$.

Now we show that $x = x' \implies xy = xy'$. Let $\varepsilon \in \mathbb{Q}^+$. Since $(a_n)_{n=1}^{\infty}, (a'_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ are rational Cauchy sequences, by Lem. I.5.1.15 we know that $(a_n)_{n=1}^{\infty}$ and $(a'_n)_{n=1}^{\infty}$ are bounded by some $M_1, M_2, M_3 \in \mathbb{Q}_{\geq 0}$. Then by Prop. I.4.2.9(a) we know that $(a_n)_{n=1}^{\infty}, (a'_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$

are bounded by $M = \max(M_1, M_2, M_3) + 1$. Clearly, $M \in \mathbb{Q}^+$, so $\dfrac{\varepsilon}{M} \in \mathbb{Q}^+$. Since $x = x'$, by Def. I.5.2.6 we know that

$$\exists N \in \mathbb{Z}^+ : \forall n \in \mathbb{Z}_{\geq N}, \left|a_n - a'_n\right| \leq \frac{\varepsilon}{M}.$$

Fix such N. Then we have

$$\begin{aligned}
& \forall n \in \mathbb{Z}_{\geq N}, \left|a_n - a'_n\right| \leq \frac{\varepsilon}{M} \\
\Longrightarrow\ & \forall n \in \mathbb{Z}_{\geq N}, |b_n|\left|a_n - a'_n\right| \leq |b_n|\frac{\varepsilon}{M} \leq M\frac{\varepsilon}{M} = \varepsilon && \text{(by Prop. I.4.2.9(c)(e))} \\
\Longrightarrow\ & \forall n \in \mathbb{Z}_{\geq N}, \left|b_n(a_n - a'_n)\right| \leq \varepsilon && \text{(by Prop. I.4.3.3(d))} \\
\Longrightarrow\ & \forall n \in \mathbb{Z}_{\geq N}, \left|a_n b_n - a'_n b_n\right| \leq \varepsilon. && \text{(by Prop. I.4.2.4)}
\end{aligned}$$

Since ε was arbitrary, we conclude that

$$\forall \varepsilon \in \mathbb{Q}^+, \exists N \in \mathbb{Z}^+ : \forall n \in \mathbb{Z}_{\geq N}, \left|a_n b_n - a'_n b_n\right| \leq \varepsilon.$$

Thus, by Def. I.5.2.6 and I.5.3.9 we have $xy = x'y$. □

Note. Of course we can prove a similar substitution rule when y is replaced by a real number y' which is equal to y.

A.Cor. I.5.3.1. At this point we embed the rationals back into the reals, by equating every rational number q with the real number $\operatorname*{LIM}_{n\to\infty} q$. This embedding is consistent with our definitions of addition and multiplication, since for any rational numbers a, b we have

$$\left(\operatorname*{LIM}_{n\to\infty} a\right) + \left(\operatorname*{LIM}_{n\to\infty} b\right) = \operatorname*{LIM}_{n\to\infty}(a+b) \quad \text{and} \quad \left(\operatorname*{LIM}_{n\to\infty} a\right) \times \left(\operatorname*{LIM}_{n\to\infty} b\right) = \operatorname*{LIM}_{n\to\infty}(ab);$$

this means that when one wants to add or multiply two rational numbers a, b it does not matter whether one thinks of these numbers as rationals or as the real numbers $\operatorname*{LIM}_{n\to\infty} a$, $\operatorname*{LIM}_{n\to\infty} b$. Also, this identification of rational numbers and real numbers is consistent with our definitions of equality (Ex. I.5.3.3).

Proof of A.Cor. I.5.3.1. First, we show that for any $q \in \mathbb{Q}$, $\operatorname*{LIM}_{n\to\infty} q$ is well-defined. So let $q \in \mathbb{Q}$. Let $(a_n)_{n=1}^{\infty}$ be a sequence where $a_n = q$ for all $n \in \mathbb{Z}^+$. Since

$$\begin{aligned}
& \forall \varepsilon \in \mathbb{Q}^+, \forall j, k \in \mathbb{Z}^+, |a_j - a_k| = |q - q| = 0 \leq \varepsilon \\
\Longrightarrow\ & \forall \varepsilon \in \mathbb{Q}^+, \exists N \in \mathbb{Z}^+ : \forall j, k \in \mathbb{Z}_{\geq N}, |a_j - a_k| \leq \varepsilon,
\end{aligned}$$

by Def. I.5.1.8 we see that $(a_n)_{n=1}^{\infty}$ is a rational Cauchy sequence. Thus, by Def. I.5.3.1 $\operatorname*{LIM}_{n\to\infty} a_n$ is well-defined. But $a_n = q$ for all $n \in \mathbb{Z}^+$, thus $\operatorname*{LIM}_{n\to\infty} q$ is well-defined.

Next we show that addition and multiplication of rationals (denoted as $+_{\mathbb{Q}}, \times_{\mathbb{Q}}$, respectively) are consistent with addition and multiplication of reals (denoted as $+_{\mathbb{R}}, \times_{\mathbb{R}}$, respectively). Let $a, b \in \mathbb{Q}$. Then we have

$$\begin{aligned} a +_{\mathbb{Q}} b &= \operatorname*{LIM}_{n\to\infty}(a +_{\mathbb{Q}} b) && \text{(by A.Cor. I.5.3.1)} \\ &= \left(\operatorname*{LIM}_{n\to\infty} a\right) +_{\mathbb{R}} \left(\operatorname*{LIM}_{n\to\infty} b\right) && \text{(by Def. I.5.3.4)} \\ &= a +_{\mathbb{R}} b && \text{(by A.Cor. I.5.3.1)} \end{aligned}$$

and

$$\begin{aligned} a \times_{\mathbb{Q}} b &= \operatorname*{LIM}_{n\to\infty}(a \times_{\mathbb{Q}} b) && \text{(by A.Cor. I.5.3.1)} \\ &= \left(\operatorname*{LIM}_{n\to\infty} a\right) \times_{\mathbb{R}} \left(\operatorname*{LIM}_{n\to\infty} b\right) && \text{(by Def. I.5.3.9)} \\ &= a \times_{\mathbb{R}} b. && \text{(by A.Cor. I.5.3.1)} \end{aligned}$$

Now we show that for any $a, b \in \mathbb{Q}$, we have $a = b$ iff $\operatorname*{LIM}_{n\to\infty} a = \operatorname*{LIM}_{n\to\infty} b$. If $a = b$, then we have $(a)_{n=1}^{\infty} = (b)_{n=1}^{\infty}$ and thus by Prop. I.5.3.3 we have $\operatorname*{LIM}_{n\to\infty} a = \operatorname*{LIM}_{n\to\infty} b$. So suppose that $\operatorname*{LIM}_{n\to\infty} a = \operatorname*{LIM}_{n\to\infty} b$. By A.Cor. I.5.3.1 this means $(a)_{n=1}^{\infty}$ and $(b)_{n=1}^{\infty}$ are equivalent rational Cauchy sequences. Then we have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{Q}^+, \exists N \in \mathbb{Z}^+ : \forall n \in \mathbb{N}, |a - b| \le \varepsilon && \text{(by Def. I.5.2.6)} \\ \implies &\forall \varepsilon \in \mathbb{Q}^+, |a - b| \le \varepsilon \\ \implies &a = b. && \text{(by Prop. I.4.3.7(a))} \end{aligned}$$

Thus, we conclude that for any $a, b \in \mathbb{Q}$, we have $a = b$ iff $\operatorname*{LIM}_{n\to\infty} a = \operatorname*{LIM}_{n\to\infty} b$. □

A.Cor. I.5.3.2. We can now easily define negation $-x$ for real numbers x by the formula

$$-x := (-1) \times x,$$

since -1 is a rational number and is hence real (A.Cor. I.5.3.1). Note that this is clearly consistent with our negation for rational numbers since we have $-q = (-1) \times q$ for all rational numbers q (A.Cor. I.5.3.1). Also, from our definitions it is clear that

$$-\operatorname*{LIM}_{n\to\infty} a_n = \operatorname*{LIM}_{n\to\infty}(-a_n).$$

Once we have addition and negation, we can define subtraction as usual by

$$x - y := x + (-y),$$

this implies

$$\operatorname*{LIM}_{n\to\infty} a_n - \operatorname*{LIM}_{n\to\infty} b_n = \operatorname*{LIM}_{n\to\infty}(a_n - b_n).$$

Proof of A.Cor. I.5.3.2. First, we show that $-\operatorname*{LIM}_{n\to\infty} a_n = \operatorname*{LIM}_{n\to\infty}(-a_n)$ for any rational Cauchy sequence $(a_n)_{n=1}^{\infty}$. This is true since

$$\begin{aligned}
-\operatorname*{LIM}_{n\to\infty} a_n &= (-1) \times \left(\operatorname*{LIM}_{n\to\infty} a_n\right) && \text{(by A.Cor. I.5.3.2)}\\
&= \left(\operatorname*{LIM}_{n\to\infty} -1\right) \times \left(\operatorname*{LIM}_{n\to\infty} a_n\right) && \text{(by A.Cor. I.5.3.1)}\\
&= \operatorname*{LIM}_{n\to\infty}((-1) \times a_n) && \text{(by Def. I.5.3.9)}\\
&= \operatorname*{LIM}_{n\to\infty}(-a_n). && \text{(by A.Cor. I.4.2.5)}
\end{aligned}$$

Now we show that $\operatorname*{LIM}_{n\to\infty} a_n - \operatorname*{LIM}_{n\to\infty} b_n = \operatorname*{LIM}_{n\to\infty}(a_n - b_n)$ for any rational Cauchy sequences $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$. This is true since

$$\begin{aligned}
\operatorname*{LIM}_{n\to\infty} a_n - \operatorname*{LIM}_{n\to\infty} b_n &= \operatorname*{LIM}_{n\to\infty} a_n + \left(-\operatorname*{LIM}_{n\to\infty} b_n\right) && \text{(by A.Cor. I.5.3.2)}\\
&= \operatorname*{LIM}_{n\to\infty} a_n + \operatorname*{LIM}_{n\to\infty}(-b_n) && \text{(from the proof above)}\\
&= \operatorname*{LIM}_{n\to\infty}(a_n + (-b_n)) && \text{(by Def. I.5.3.4)}\\
&= \operatorname*{LIM}_{n\to\infty}(a_n - b_n). && \text{(by A.Cor. I.4.2.4)}
\end{aligned}$$

□

Prop. I.5.3.11. All the laws of algebra from Prop. I.4.1.6 hold not only for the integers, but for the reals as well.

Proof of Prop. I.5.3.11. We illustrate this with one such rule: $x(y+z) = xy + xz$. Let $x = \operatorname*{LIM}_{n\to\infty} a_n$, $y = \operatorname*{LIM}_{n\to\infty} b_n$, and $z = \operatorname*{LIM}_{n\to\infty} c_n$ be real numbers. Then by definition, $xy = \operatorname*{LIM}_{n\to\infty} a_n b_n$ and $xz = \operatorname*{LIM}_{n\to\infty} a_n c_n$, and so $xy + xz = \operatorname*{LIM}_{n\to\infty}(a_n b_n + a_n c_n)$. A similar line of reasoning shows that $x(y+z) = \operatorname*{LIM}_{n\to\infty} a_n(b_n + c_n)$. But we already know that $a_n(b_n + c_n)$ is equal to $a_n b_n + a_n c_n$ for the rational numbers a_n, b_n, c_n, and the claim follows. The other laws of algebra are proven similarly. □

Def. I.5.3.12 (Sequences bounded away from zero)**.** A sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is said to be *bounded away from zero* iff there exists a $c \in \mathbb{Q}^+$ such that $|a_n| \geq c$ for all $n \in \mathbb{Z}_{\geq m}$.

Lem. I.5.3.14. Let x be a non-zero real number. Then $x = \operatorname*{LIM}_{n\to\infty} a_n$ for some rational Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is bounded away from zero.

Proof of Lem. I.5.3.14. Since x is real, we know that $x = \operatorname*{LIM}_{n\to\infty} b_n$ for some rational Cauchy sequence $(b_n)_{n=1}^{\infty}$. But we are not yet done, because we do not know that b_n is bounded away from zero. On the other hand, we are given that $x \neq 0 = \operatorname*{LIM}_{n\to\infty} 0$, which means that the sequence $(b_n)_{n=1}^{\infty}$ is not equivalent to $(0)_{n=1}^{\infty}$. Thus, the sequence $(b_n)_{n=1}^{\infty}$ cannot be eventually ε-close to $(0)_{n=1}^{\infty}$ for every $\varepsilon \in \mathbb{Q}^+$. Therefore we can find an $\varepsilon \in \mathbb{Q}^+$ such that $(b_n)_{n=1}^{\infty}$ is not eventually ε-close to $(0)_{n=1}^{\infty}$.

Let us fix this ε. We know that $(b_n)_{n=1}^{\infty}$ is a rational Cauchy sequence, so it is eventually ε-steady. Moreover, it is eventually $\varepsilon/2$-steady, since $\varepsilon/2 \in \mathbb{Q}^+$. Thus, there is an $N \in \mathbb{Z}^+$ such that $|b_n - b_m| \leq \varepsilon/2$ for all $n, m \in \mathbb{Z}_{\geq N}$.

On the other hand, we cannot have $|b_n| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$, since this would imply that $(b_n)_{n=1}^{\infty}$ is eventually ε-close to $(0)_{n=1}^{\infty}$. Thus, there must be some $n_0 \in \mathbb{Z}_{\geq N}$ for which $|b_{n_0}| > \varepsilon$. Since we already know that $|b_{n_0} - b_n| \leq \varepsilon/2$ for all $n \in \mathbb{Z}_{\geq N}$, we have

$$\begin{aligned}
&|b_{n_0}| - |b_{n_0} - b_n| \geq \varepsilon - \varepsilon/2 = \varepsilon/2 && \text{(by Prop. I.4.2.9(c)(d))}\\
\implies &|b_{n_0}| - |b_n - b_{n_0}| \geq \varepsilon/2 && \text{(by Prop. I.4.3.3(d))}\\
\implies &|b_{n_0} + (b_n - b_{n_0})| \geq \varepsilon/2 && \text{(by A.Cor. I.4.3.1)}\\
\implies &|b_n| \geq \varepsilon/2. && \text{(by Prop. I.4.2.4)}
\end{aligned}$$

Thus, we conclude from above that $|b_n| \geq \varepsilon/2$ for all $n \in \mathbb{Z}_{\geq N}$.

This almost proves that $(b_n)_{n=1}^{\infty}$ is bounded away from zero. Actually, what it does is show that $(b_n)_{n=1}^{\infty}$ is *eventually* bounded away from zero. But this is easily fixed, by defining a new sequence a_n, by setting $a_n := \varepsilon/2$ if $n \in \mathbb{Z}_{<N}$ and $a_n := b_n$ if $n \in \mathbb{Z}_{\geq N}$. Since b_n is a rational Cauchy sequence, it is not hard to verify that a_n is also a rational Cauchy sequence which is equivalent to b_n (because the two sequences are eventually the same), and so $x = \operatorname*{LIM}_{n\to\infty} a_n$. And since $|b_n| \geq \varepsilon/2$ for all $n \in \mathbb{Z}_{\geq N}$, we know that $|a_n| \geq \varepsilon/2$ for all $n \in \mathbb{Z}^+$ (splitting into the two cases $n \in \mathbb{Z}_{\geq N}$ and $n \in \mathbb{Z}_{<N}$ separately). Thus, we have a rational Cauchy sequence which is bounded away from zero (by $\varepsilon/2$ instead of ε, but that's still OK since $\varepsilon/2 \in \mathbb{Q}^+$), and which has x as a formal limit, and so we are done. □

Lem. I.5.3.15. Suppose that $(a_n)_{n=1}^{\infty}$ is a rational Cauchy sequence which is bounded away from zero. Then the sequence $(a_n^{-1})_{n=1}^{\infty}$ is also a rational Cauchy sequence.

Proof of Lem. I.5.3.15. Since $(a_n)_{n=1}^{\infty}$ is bounded away from zero, we know that there is a $c \in \mathbb{Q}^+$ such that $|a_n| \geq c$ for all $n \in \mathbb{Z}^+$. Now we need to show that $(a_n^{-1})_{n=1}^{\infty}$ is eventually ε-steady for each $\varepsilon \in \mathbb{Q}^+$. Thus, let us fix an $\varepsilon \in \mathbb{Q}^+$; our task is now to find an $N \in \mathbb{Z}^+$ such that $\left|a_n^{-1} - a_m^{-1}\right| \leq \varepsilon$ for all $n, m \in \mathbb{Z}_{\geq N}$. But

$$\left|a_n^{-1} - a_m^{-1}\right| = \left|\frac{a_m - a_n}{a_m a_n}\right| \leq \frac{|a_m - a_n|}{c^2}$$

(since $|a_m|, |a_n| \geq c$), and so to make $\left|a_n^{-1} - a_m^{-1}\right|$ less than or equal to ε, it will suffice to make $|a_m - a_n|$ less than or equal to $c^2\varepsilon$. But since $(a_n)_{n=1}^{\infty}$ is a rational Cauchy sequence, and $c^2\varepsilon \in \mathbb{Q}^+$, we can certainly find an N such that the sequence $(a_n)_{n=N}^{\infty}$ is $c^2\varepsilon$-steady, i.e., $|a_m - a_n| \leq c^2\varepsilon$ for all $n, m \in \mathbb{Z}_{\geq N}$. By what we have said above, this shows that $\left|a_n^{-1} - a_m^{-1}\right| \leq \varepsilon$ for all $m, n \in \mathbb{Z}_{\geq N}$, and hence the sequence $(a_n^{-1})_{n=1}^{\infty}$ is eventually ε-steady. Since we have proven this for every ε, we have that $(a_n^{-1})_{n=1}^{\infty}$ is a rational Cauchy sequence, as desired. □

Def. I.5.3.16 (Reciprocals of real numbers)**.** Let x be a non-zero real number. Let $(a_n)_{n=1}^{\infty}$ be a rational Cauchy sequence bounded away from zero such that $x = \operatorname*{LIM}_{n\to\infty} a_n$ (such a sequence exists by Lem. I.5.3.14). Then we define the reciprocal x^{-1} by the formula $x^{-1} := \operatorname*{LIM}_{n\to\infty} a_n^{-1}$. (From Lem. I.5.3.15 we know that x^{-1} is a real number.)

Lem. I.5.3.17 (Reciprocation is well defined)**.** Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two rational Cauchy sequences bounded away from zero such that $\operatorname*{LIM}_{n\to\infty} a_n = \operatorname*{LIM}_{n\to\infty} b_n$ (i.e., the two sequences are equivalent). Then $\operatorname*{LIM}_{n\to\infty} a_n^{-1} = \operatorname*{LIM}_{n\to\infty} b_n^{-1}$.

Proof of Lem. I.5.3.17. Consider the following product P of three real numbers:

$$P := \left(\operatorname*{LIM}_{n\to\infty} a_n^{-1}\right) \times \left(\operatorname*{LIM}_{n\to\infty} a_n\right) \times \left(\operatorname*{LIM}_{n\to\infty} b_n^{-1}\right).$$

If we multiply this out, we obtain

$$P = \operatorname*{LIM}_{n\to\infty} a_n^{-1} a_n b_n^{-1} = \operatorname*{LIM}_{n\to\infty} b_n^{-1}.$$

On the other hand, since $\operatorname*{LIM}_{n\to\infty} a_n = \operatorname*{LIM}_{n\to\infty} b_n$, we can write P in another way as

$$P = \left(\operatorname*{LIM}_{n\to\infty} a_n^{-1}\right) \times \left(\operatorname*{LIM}_{n\to\infty} b_n\right) \times \left(\operatorname*{LIM}_{n\to\infty} b_n^{-1}\right).$$

(cf. Prop. I.5.3.10). Multiplying things out again, we get

$$P = \operatorname*{LIM}_{n\to\infty} a_n^{-1} b_n b_n^{-1} = \operatorname*{LIM}_{n\to\infty} a_n^{-1}.$$

Comparing our different formulae for P we see that $\operatorname*{LIM}_{n\to\infty} a_n^{-1} = \operatorname*{LIM}_{n\to\infty} b_n^{-1}$, as desired. □

A.Cor. I.5.3.3. It is clear from Def. I.5.3.16 that $xx^{-1} = x^{-1}x = 1$; thus all the field axioms (Prop. I.4.2.4) apply to the reals as well as to the rationals. We of course cannot give 0 a reciprocal, since 0 multiplied by anything gives 0, not 1.

Proof of A.Cor. I.5.3.3. Let $(a_n)_{n=1}^{\infty}$ be a rational Cauchy sequence which is bounded away from zero. Then we have

$$\begin{aligned}
\left(\operatorname*{LIM}_{n\to\infty} a_n\right) \times \left(\operatorname*{LIM}_{n\to\infty} a_n\right)^{-1} &= \left(\operatorname*{LIM}_{n\to\infty} a_n\right) \times \left(\operatorname*{LIM}_{n\to\infty} a_n^{-1}\right) && \text{(by Def. I.5.3.16)}\\
&= \operatorname*{LIM}_{n\to\infty} (a_n \times a_n^{-1}) && \text{(by Def. I.5.3.9)}\\
&= \operatorname*{LIM}_{n\to\infty} 1 && \text{(by Prop. I.4.2.4)}\\
&= 1. && \text{(by A.Cor. I.5.3.1)}
\end{aligned}$$

□

A.Cor. I.5.3.4. If q is a non-zero rational, and hence equal to the real number $\operatorname*{LIM}_{n\to\infty} q$, then the reciprocal of $\operatorname*{LIM}_{n\to\infty} q$ is $\operatorname*{LIM}_{n\to\infty} q^{-1} = q^{-1}$; thus the operation of reciprocal on real numbers is consistent with the operation of reciprocal on rational numbers.

Proof of A.Cor. I.5.3.4. We have

$$\begin{aligned} q^{-1} &= \left(\operatorname*{LIM}_{n\to\infty} q\right)^{-1} && \text{(by A.Cor. I.5.3.1)} \\ &= \operatorname*{LIM}_{n\to\infty}(q^{-1}) && \text{(by Def. I.5.3.16)} \\ &= q^{-1}. && \text{(by A.Cor. I.5.3.1)} \end{aligned}$$

□

A.Cor. I.5.3.5. Once one has reciprocal, one can define division x/y of two real numbers x, y, provided y is non-zero, by the formula

$$x/y := x \times y^{-1},$$

just as we did with the rationals. In particular, we have the *cancellation law*: if x, y, z are real numbers such that $xz = yz$, and z is non-zero, then by dividing by z we conclude that $x = y$. This cancellation law does not work when z is zero.

Proof of A.Cor. I.5.3.5. Suppose that $xz = yz$ and $z \neq 0$. Then we have

$$\begin{aligned} x &= x1 && \text{(by Prop. I.5.3.11)} \\ &= x(zz^{-1}) && \text{(by A.Cor. I.5.3.3)} \\ &= (xz)z^{-1} && \text{(by Prop. I.5.3.11)} \\ &= (yz)z^{-1} && \\ &= y(zz^{-1}) && \text{(by Prop. I.5.3.11)} \\ &= y1 && \text{(by A.Cor. I.5.3.3)} \\ &= y. && \text{(by Prop. I.5.3.11)} \end{aligned}$$

□

— Exercises —

Ex. I.5.3.1. Prove Prop. I.5.3.3.

Proof of Ex. I.5.3.1. See Prop. I.5.3.3. □

Ex. I.5.3.2. Prove Prop. I.5.3.10.

Proof of Ex. I.5.3.2. See Prop. I.5.3.10. □

Ex. I.5.3.3. Let a, b be rational numbers. Show that $a = b$ iff $\underset{n\to\infty}{\mathrm{LIM}}\, a = \underset{n\to\infty}{\mathrm{LIM}}\, b$ (i.e., the rational Cauchy sequences $a, a, a, a, \ldots$ and $b, b, b, b \ldots$ equivalent iff $a = b$). This allows us to embed the rational numbers inside the real numbers in a well-defined manner.

Proof of Ex. I.5.3.3. See A.Cor. I.5.3.1 □

Ex. I.5.3.4. Let $(a_n)_{n=m}^{\infty}$ be a sequence of rational numbers which is bounded. Let $(b_n)_{n=m}^{\infty}$ be another sequence of rational numbers which is equivalent to $(a_n)_{n=m}^{\infty}$. Show that $(b_n)_{n=m}^{\infty}$ is also bounded.

Proof of Ex. I.5.3.4. Since $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are equivalent, by Def. I.5.2.6 we know that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close for every $\varepsilon \in \mathbb{Q}^+$. Thus, by Ex. I.5.2.2 $(a_n)_{n=m}^{\infty}$ is bounded iff $(b_n)_{n=m}^{\infty}$ is bounded. □

Ex. I.5.3.5. Show that $\underset{n\to\infty}{\mathrm{LIM}}\, 1/n = 0$.

Proof of Ex. I.5.3.5. By Def. I.5.3.1 we need to show that $(1/n)_{n=1}^{\infty}$ and $(0)_{n=1}^{\infty}$ are equivalent rational Cauchy sequences. By Prop. I.5.1.11 and A.Cor. I.5.3.1 we know that both $(1/n)_{n=1}^{\infty}$ and $(0)_{n=1}^{\infty}$ are rational Cauchy sequences. So we only need to show that both are eventually ε-close for arbitrary $\varepsilon \in \mathbb{Q}^+$. Let $\varepsilon \in \mathbb{Q}^+$. Clearly, $\dfrac{1}{\varepsilon} \in \mathbb{Q}^+$. By Prop. I.4.4.1 we know that there exists an $N \in \mathbb{Z}^+$ such that $\dfrac{1}{\varepsilon} \le N$, or $\dfrac{1}{N} \le \varepsilon$. Fix such N. Then we have

$$\begin{aligned}
\forall n \in \mathbb{Z}_{\ge N}, \left|\frac{1}{n} - 0\right| &= \frac{1}{n} && \text{(by Def. I.4.3.1)}\\
&\le \frac{1}{N} && \text{(by Prop. I.4.3.12(b))}\\
&< \varepsilon.
\end{aligned}$$

Since ε was arbitrary, we conclude that $(1/n)_{n=1}^{\infty}$ and $(0)_{n=1}^{\infty}$ are eventually ε-close for arbitrary $\varepsilon \in \mathbb{Q}^+$. □

I.5.4 Ordering the reals

Def. I.5.4.1. Let $(a_n)_{n=m}^{\infty}$ be a sequence of rationals. We say that this sequence is *positively bounded away from zero* iff we have a positive rational $c \in \mathbb{Q}^+$ such that $a_n \ge c$ for all $n \in \mathbb{Z}_{\ge m}$ (in particular, the sequence is entirely positive). The sequence is *negatively bounded away from zero* iff we have a negative rational $c \in \mathbb{Q}^-$ such that $a_n \le c$ for all $n \in \mathbb{Z}_{\ge m}$ (in particular, the sequence is entirely negative).

Note. It is clear that any sequence which is positively or negatively bounded away from zero, is bounded away from zero. Also, a sequence cannot be both positively bounded away from zero and negatively bounded away from zero at the same time.

Def. I.5.4.3. A real number x is said to be *positive* iff it can be written as $x = \underset{n\to\infty}{\mathrm{LIM}}\, a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is positively bounded away from zero. x is said to be *negative* iff it can be written as $x = \underset{n\to\infty}{\mathrm{LIM}}\, a_n$ for some sequence $(a_n)_{n=1}^{\infty}$ which is negatively bounded away from zero.

Prop. I.5.4.4 (Basic properties of positive reals)**.** For every real number x, exactly one of the following three statements is true:

(a) x is zero;

(b) x is positive;

(c) x is negative.

A real number x is negative iff $-x$ is positive. If x and y are positive, then so are $x+y$ and xy.

Proof of Prop. I.5.4.4. We first show that at least one of the three statements is true. Let $x \in \mathbb{R}$. By A.Cor. I.5.3.1 we know that 0 is the formal limit of $(0)_{n=1}^{\infty}$, thus by Def. I.5.3.1 and Prop. I.5.3.3 we can ask whether $x = 0$ or $x \neq 0$. If $x = 0$, then we are done. Otherwise by Lem. I.5.3.14 we know that x is the formal limit of a rational Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is bounded away from 0. In particular, by Def. I.5.3.12 we know that there exists a $c \in \mathbb{Q}^+$ such that $|a_n| \geq c$ for all $n \in \mathbb{Z}^+$. Fix such c. Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence and $c \in \mathbb{Q}^+$, by Def. I.5.1.8 we have

$$\exists N \in \mathbb{Z}^+ : \forall j,k \in \mathbb{Z}_{\geq N}, |a_j - a_k| \leq c.$$

Fix such N. Then we have

$$\begin{aligned}
&\forall j \in \mathbb{Z}_{\geq N}, |a_j - a_N| \leq c && (N \in \mathbb{Z}_{\geq N})\\
\Longrightarrow\ &\forall j \in \mathbb{Z}_{\geq N}, -c \leq a_j - a_N \leq c && (\text{by Prop. I.4.3.3(c)})\\
\Longrightarrow\ &\forall j \in \mathbb{Z}_{\geq N}, -c + a_N \leq a_j \leq c + a_N. && (\text{by Prop. I.4.2.9(c)(d)})
\end{aligned}$$

Since $|a_N| \geq c \in \mathbb{Q}^+$, by Def. I.4.3.1 we know that we have either $a_N \in \mathbb{Q}^+$ or $a_N \in \mathbb{Q}^-$. So we split into two cases:

- If $a_N \in \mathbb{Q}^+$, then by Def. I.4.3.1 we have $a_N \geq c$. Thus

$$\begin{aligned}
&\forall j \in \mathbb{Z}_{\geq N}, 0 \leq -c + a_N \leq a_j \leq c + a_N && (\text{by Prop. I.4.2.9(c)(d)})\\
\Longrightarrow\ &\forall j \in \mathbb{Z}_{\geq N}, 0 \leq a_j. && (\text{by Prop. I.4.2.9(c)})
\end{aligned}$$

  This means $a_j \in \mathbb{Q}^+$ for all $j \in \mathbb{Z}_{\geq N}$. If we now define $(b_n)_{n=1}^{\infty}$ to be the sequence where $b_n = c$ for all $n \in \mathbb{Z}^+ \cap \mathbb{Z}_{<N}$ and $b_n = a_n$ for all $n \in \mathbb{Z}_{\geq N}$, then we see that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent rational Cauchy sequences and $(b_n)_{n=1}^{\infty}$ is positively bounded away from zero. Thus, by Def. I.5.4.3 x is positive.

- If $a_N \in \mathbb{Q}^-$, then by Def. I.4.3.1 we have $-a_N \geq c$. By Ex. I.4.2.6 we have $a_N \leq -c$. Thus

$$\begin{aligned} &\forall j \in \mathbb{Z}_{\geq N}, -c + a_N \leq a_j \leq c + a_N \leq 0 && \text{(by Prop. I.4.2.9(c)(d))} \\ \implies &\forall j \in \mathbb{Z}_{\geq N}, a_j \leq 0. && \text{(by Prop. I.4.2.9(c))} \end{aligned}$$

  This means $a_j \in \mathbb{Q}^-$ for all $j \in \mathbb{Z}_{\geq N}$. If we now define $(b_n)_{n=1}^{\infty}$ to be the sequence where $b_n = -c$ for all $n \in \mathbb{Z}^+ \cap \mathbb{Z}_{<N}$ and $b_n = a_n$ for all $n \in \mathbb{Z}_{\geq N}$, then we see that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent rational Cauchy sequences and $(b_n)_{n=1}^{\infty}$ is negatively bounded away from zero. Thus, by Def. I.5.4.3 x is negative.

From all cases above, we see that x is either positive or negative. Thus, we conclude that at least one of the three statements is true.

Next we show that at most one of the three statements is true. Let $x \in \mathbb{R}$. Suppose for the sake of contradiction that one of the following three cases is true:

- We have both $x = 0$ and x is positive. Then from the first part of the proof we know that x is the formal limit of a rational Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is positively bounded away from 0. In particular, there exists a $c \in \mathbb{Q}^+$ such that $a_n \geq c$ for all $n \in \mathbb{Z}^+$. Fix such c. Since $x = 0$, by Def. I.5.2.6 we know that $(a_n)_{n=1}^{\infty}$ and $(0)_{n=1}^{\infty}$ are eventually ε-close for all $\varepsilon \in \mathbb{Q}^+$. Since $c \in \mathbb{Q}^+$, we know that $c/2 \in \mathbb{Q}^+$. Thus, $(a_n)_{n=1}^{\infty}$ and $(0)_{n=1}^{\infty}$ must be eventually $c/2$-close, i.e., there exists an $N \in \mathbb{Z}^+$ such that $a_n = |a_n - 0| \leq c/2 < c$ for all $n \in \mathbb{Z}_{\geq N}$. But this means for any $n \in \mathbb{Z}_{\geq N}$, we have both $a_n < c$ and $a_n \geq c$, which contradict to Prop. I.4.2.9(a).

- We have both $x = 0$ and x is negative. Then from the first part of the proof we know that x is the formal limit of a rational Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is negatively bounded away from 0. In particular, there exists a $c \in \mathbb{Q}^-$ such that $a_n \leq c$ for all $n \in \mathbb{Z}^+$. Fix such c. Since $x = 0$, by Def. I.5.2.6 we know that $(a_n)_{n=1}^{\infty}$ and $(0)_{n=1}^{\infty}$ are eventually ε-close for all $\varepsilon \in \mathbb{Q}^+$. Since $c \in \mathbb{Q}^-$, we know that $-c/2 \in \mathbb{Q}^+$. Thus, $(a_n)_{n=1}^{\infty}$ and $(0)_{n=1}^{\infty}$ must be eventually $-c/2$-close, i.e., there exists an $N \in \mathbb{Z}^+$ such that $-a_n = |a_n - 0| \leq -c/2 < -c$ for all $n \in \mathbb{Z}_{\geq N}$. But this means for any $n \in \mathbb{Z}_{\geq N}$, we have both $a_n > c$ and $a_n \leq c$, which contradict to Prop. I.4.2.9(a).

- We have x is both positive and negative. From the first part of the proof we know that x is the formal limit of a rational Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is positively bounded away from 0. Similarly, we know that x is the formal limit of a rational Cauchy sequence $(b_n)_{n=1}^{\infty}$ which is negatively bounded away from 0. By Def. I.5.4.1 we know that there exist some $c \in \mathbb{Q}^+$ and $d \in \mathbb{Q}^-$ such that $a_n \geq c$ and $b_n \leq d$ for all $n \in \mathbb{Z}^+$. Fix such c, d and observe that $c - d \in \mathbb{Q}^+$. Since $x = \operatorname*{LIM}_{n\to\infty} a_n = \operatorname*{LIM}_{n\to\infty} b_n$, by Prop. I.5.3.3 we know that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent rational Cauchy sequences. Thus, by Def. I.5.2.6 we have

$$\exists N \in \mathbb{Z}^+ : \forall n \in \mathbb{Z}_{\geq n}, |a_n - b_n| \leq \frac{c-d}{2} < c - d.$$

Fix such N. But then we have

$$\begin{aligned} & \forall n \in \mathbb{Z}_{\geq N}, (a_n \geq c) \wedge (b_n \leq d) \\ \implies & \forall n \in \mathbb{Z}_{\geq N}, (a_n \geq c) \wedge (-b_n \geq -d) && \text{(by Ex. I.4.2.6)} \\ \implies & \forall n \in \mathbb{Z}_{\geq N}, a_n - b_n = |a_n - b_n| \geq c - d, && \text{(by Prop. I.4.2.9(c)(d))} \end{aligned}$$

which contradict to Prop. I.4.2.9(a).

From all cases above, we derived contradictions. Thus, we conclude that at most one of the three statements is true.

Next we show that x is negative iff $-x$ is positive. Suppose that x is negative. From the first part of the proof we know that x is the formal limit of a rational Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is negatively bounded away from 0. This means $(-a_n)_{n=1}^{\infty}$ is positively bounded away from 0. But by A.Cor. I.5.3.2 we know that $-x = \operatorname*{LIM}_{n\to\infty} -a_n$, thus by Def. I.5.4.1 we know that $-x$ is positive. The converse argument holds by reversing the previous reasoning.

Finally we show that x, y are positive implies $x+y$ and xy are positive. From the first part of the proof we know that x and y are the formal limits of rational Cauchy sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$, respectively, which are positively bounded away from 0. Clearly, $(a_n + b_n)_{n=1}^{\infty}$ and $(a_n b_n)_{n=1}^{\infty}$ are positively bounded away from 0. But by Def. I.5.3.4 and I.5.3.9 we know that $x + y = \operatorname*{LIM}_{n\to\infty} a_n + b_n$ and $xy = \operatorname*{LIM}_{n\to\infty} a_n b_n$, thus by Def. I.5.4.1 we know that $x + y$ and xy are positive. □

Note. If q is a positive rational number, then the Cauchy sequence $q, q, q, \dots$ is positively bounded away from zero, and hence $\operatorname*{LIM}_{n\to\infty} q = q$ is a positive real number. Thus, the notion of positivity for rationals is consistent with that for reals. Similarly, the notion of negativity for rationals is consistent with that for reals.

Def. I.5.4.5 (Absolute value)**.** Let x be a real number. We define the *absolute value* $|x|$ of x to equal x if x is positive, $-x$ when x is negative, and 0 when x is zero.

Def. I.5.4.6 (Ordering of the real numbers)**.** Let x and y be real numbers. We say that x is *greater than* y, and write $x > y$, iff $x - y$ is a positive real number, and $x < y$ iff $x - y$ is a negative real number. We define $x \geq y$ iff $x > y$ or $x = y$, and similarly define $x \leq y$.

Note. Comparing Def. I.5.4.6 with the definition of order on the rationals from Def. I.4.2.8 we see that order on the reals is consistent with order on the rationals, i.e., if two rational numbers q, q' are such that q is less than q' in the rational number system, then q is still less than q' in the real number system, and similarly for "greater than." In the same way we see that the definition of absolute value given in Def. I.5.4.5 is consistent with that in Def. I.4.3.1.

Prop. I.5.4.7. All the claims in Prop. I.4.2.9 which held for rationals, continue to hold for real numbers.

Proof of Prop. I.5.4.7(a). By Prop. I.5.4.4 $x - y$ satisfy exactly one of the following three statements:

- $x - y = 0$. Then by Prop. I.5.3.11 we have $x = y$.
- $x - y$ is a positive rational number. Then by Def. I.5.4.6 we have $x > y$.
- $x - y$ is a negative rational number. Then by Def. I.5.4.6 we have $x < y$.

$\square$

Proof of Prop. I.5.4.7(b). We have

$$\begin{aligned}
& x < y \\
\iff & x - y \text{ is negative} && \text{(by Def. I.5.4.6)} \\
\iff & -(x - y) \text{ is positive} && \text{(by Prop. I.5.4.4)} \\
\iff & y - x \text{ is positive} && \text{(by Prop. I.5.3.11)} \\
\iff & y > x. && \text{(by Def. I.5.4.6)}
\end{aligned}$$

$\square$

Proof of Prop. I.5.4.7(c). We have

$$\begin{aligned}
& (x < y) \wedge (y < z) \\
\Longrightarrow & (x - y \text{ is negative}) \wedge (y - z \text{ is negative}) && \text{(by Def. I.5.4.6)} \\
\Longrightarrow & (-(x - y) \text{ is positive}) \wedge (-(y - z) \text{ is positive}) && \text{(by Prop. I.5.4.4)} \\
\Longrightarrow & (y - x \text{ is positive}) \wedge (z - y \text{ is positive}) && \text{(by Prop. I.5.3.11)} \\
\Longrightarrow & y - x + z - y \text{ is positive} && \text{(by Prop. I.5.4.4)} \\
\Longrightarrow & -(x - z) \text{ is positive} && \text{(by Prop. I.5.3.11)} \\
\Longrightarrow & x - z \text{ is negative} && \text{(by Prop. I.5.4.4)} \\
\Longrightarrow & x < z. && \text{(by Def. I.5.4.6)}
\end{aligned}$$

$\square$

Proof of Prop. I.5.4.7(d). We have

$$\begin{aligned}
& x < y \\
\Longrightarrow & x - y \text{ is negative} && \text{(by Def. I.5.4.6)} \\
\Longrightarrow & x + z - z - y \text{ is negative} && \text{(by Prop. I.5.3.11)} \\
\Longrightarrow & (x + z) - (y + z) \text{ is negative} && \text{(by Prop. I.5.3.11)} \\
\Longrightarrow & x + z < y + z. && \text{(by Def. I.5.4.6)}
\end{aligned}$$

$\square$

Proof of Prop. I.5.4.7(e). We have

$$\begin{aligned}
& x < y \\
\implies & y > x && \text{(by Prop. I.5.4.7(b))} \\
\implies & y - x \text{ is positive} && \text{(by Def. I.5.4.6)} \\
\implies & (y - x)z \text{ is positive} && \text{(by Prop. I.5.4.4)} \\
\implies & yz - xz \text{ is positive} && \text{(by Prop. I.5.3.11)} \\
\implies & yz > xz && \text{(by Def. I.5.4.6)} \\
\implies & xz < yz. && \text{(by Prop. I.5.4.7(b))}
\end{aligned}$$

□

Prop. I.5.4.8. Let x be a positive real number. Then x^{-1} is also positive. Also, if y is another positive number and $x > y$, then $x^{-1} < y^{-1}$.

Proof of Prop. I.5.4.8. Let x be positive. Since $xx^{-1} = 1$, the real number x^{-1} cannot be zero (since $x0 = 0 \neq 1$). Also, from Prop. I.5.4.4 it is easy to see that a positive number times a negative number is negative; this shows that x^{-1} cannot be negative, since this would imply that $xx^{-1} = 1$ is negative, a contradiction. Thus, by Prop. I.5.4.4, the only possibility left is that x^{-1} is positive.

Now let y be positive as well, so x^{-1} and y^{-1} are also positive. Suppose that $x > y$. If $x^{-1} \geq y^{-1}$, then by Prop. I.5.4.7 we have $xx^{-1} > yx^{-1} \geq yy^{-1}$, thus $1 > 1$, which is a contradiction. Thus, we must have $x^{-1} < y^{-1}$. □

Prop. I.5.4.9 (The non-negative reals are closed)**.** Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence of non-negative rational numbers. Then $\operatorname*{LIM}_{n\to\infty} a_n$ is a non-negative real number.

Proof of Prop. I.5.4.9. We argue by contradiction, and suppose that the real number $x := \operatorname*{LIM}_{n\to\infty} a_n$ is a negative number. Then by definition of negative real number, we have $x = \operatorname*{LIM}_{n\to\infty} b_n$ for some sequence $(b_n)_{n=1}^{\infty}$ which is negatively bounded away from 0, i.e., there is a negative rational $-c \in \mathbb{Q}^-$ such that $b_n \leq -c$ for all $n \in \mathbb{Z}^+$. On the other hand, we have $a_n \in \mathbb{Q}_{\geq 0}$ for all $n \in \mathbb{Z}^+$, by hypothesis. Thus, the numbers a_n and b_n are never $c/2$-close, since $c/2 < c$. Thus, the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are not eventually $c/2$-close. Since $c/2 \in \mathbb{Q}^+$, this implies that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are not equivalent. But this contradicts the fact that both these sequences have x as their formal limit. □

Note. Eventually, we will see a better explanation of Prop. I.5.4.9: the set of non-negative reals is *closed*, whereas the set of positive reals is *open*. See Sec. I.11.4.

Cor. I.5.4.10. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be Cauchy sequences of rationals such that $a_n \geq b_n$ for all $n \in \mathbb{Z}^+$. Then $\operatorname*{LIM}_{n\to\infty} a_n \geq \operatorname*{LIM}_{n\to\infty} b_n$.

Proof of Cor. I.5.4.10. Apply Prop. I.5.4.9 to the sequence $(a_n - b_n)_{n=1}^{\infty}$. □

Rmk. I.5.4.11. Note that Cor. I.5.4.10 does not work if the $\geq$ signs are replaced by $>$: for instance if $a_n := 1 + 1/n$ and $b_n := 1 - 1/n$, then a_n is always strictly greater than b_n, but the formal limit of a_n is not greater than the formal limit of b_n, instead they are equal.

A.Cor. I.5.4.1. We now define distance $d(x, y) := |x - y|$ just as we did for the rationals. In fact, Prop. I.4.3.3 and I.4.3.7 hold not only for the rationals, but for the reals; the proof is identical, since the real numbers obey all the laws of algebra and order that the rationals do.

Prop. I.5.4.12 (Bounding of reals by rationals)**.** Let x be a positive real number. Then there exists a positive rational number q such that $q \leq x$, and there exists a positive integer N such that $x \leq N$.

Proof of Prop. I.5.4.12. Since x is a positive real, it is the formal limit of a rational Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is positively bounded away from 0. Also, by Lem. I.5.1.15, this sequence is bounded. Thus, we have rationals $q, r \in \mathbb{Q}^+$ such that $q \leq a_n \leq r$ for all $n \in \mathbb{Z}^+$. But by Prop. I.4.4.1 we know that there is some integer N such that $r \leq N$; since q is positive and $q \leq r \leq N$, we see that N is positive. Thus, $q \leq a_n \leq N$ for all $n \in \mathbb{Z}^+$. Applying Cor. I.5.4.10 we obtain that $q \leq x \leq N$, as desired. ☐

Cor. I.5.4.13 (Archimedean property)**.** Let x and ε be any positive real numbers. Then there exists a positive integer M such that $M\varepsilon > x$.

Proof of Cor. I.5.4.13. The number x/ε is positive, and hence by Prop. I.5.4.12 there exists a positive integer N such that $x/\varepsilon \leq N$. If we set $M := N + 1$, then $x/\varepsilon < M$. Now multiply by ε. ☐

Note. This property (Cor. I.5.4.13) is quite important; it says that no matter how large x is and how small ε is, if one keeps adding ε to itself, one will eventually overtake x.

Prop. I.5.4.14. Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.

Proof of Prop. I.5.4.14. First, observe that

$$\begin{aligned}
& x < y \\
\implies & y > x && \text{(by Prop. I.5.4.7)} \\
\implies & y - x \text{ is positive} && \text{(by Def. I.5.4.6)} \\
\implies & \exists N \in \mathbb{Z}^+ : y - x > 1/N && \text{(by Ex. I.5.4.4)} \\
\implies & \exists N \in \mathbb{Z}^+ : y > x + 1/N. && \text{(by Prop. I.5.4.7)}
\end{aligned}$$

Fix one such N. Since x is a real number, by Prop. I.5.3.10 we know that Nx is also a real number. By Ex. I.5.4.3, there exists an $M \in \mathbb{Z}$ such that $M \leq Nx < M + 1$. So we have

$$M \leq Nx < M + 1$$

$$
\begin{aligned}
&\implies \frac{M}{N} \le x < \frac{M+1}{N} && \text{(by Prop. I.5.4.7(e))}\\
&\implies \left(\frac{M+1}{N} \le x + \frac{1}{N}\right) \wedge \left(x < \frac{M+1}{N}\right) && \text{(by Prop. I.5.4.7(d))}\\
&\implies x < \frac{M+1}{N} \le x + \frac{1}{N} < y. && \text{(by Prop. I.5.4.7(c))}
\end{aligned}
$$

Clearly, $(M+1)/N \in \mathbb{Q}$. So by setting $q = (M+1)/N$ we are done. □

Rmk. I.5.4.15. Up until now, we have not addressed the fact that real numbers can be expressed using the decimal system. For instance, the formal limit of

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

is more conventionally represented as the decimal $1.41421\ldots$. There are some subtleties in the decimal system, for instance $0.9999\ldots$ and $1.000\ldots$ are in fact the same real number.

A.Cor. I.5.4.2. Let X be an non-empty finite subset of $\mathbb{R}$. Then X has exactly one maximum $\max(X) \in X$ satisfying

$$\forall x \in X, x \le \max(X).$$

Similarly, X has exactly one minimum $\min(X) \in X$ satisfying

$$\forall x \in X, x \ge \min(X).$$

Proof of A.Cor. I.5.4.2. Let $n = \#(X)$. We induct on n to show that $\max(X) \in X$ and $\min(X) \in X$, and we start with $n = 1$. For $n = 1$, we have $X = \{x\}$ for some $x \in \mathbb{R}$. Then we have

$$
\begin{aligned}
&\forall y \in X, y = x && \text{(by Ax. I.3.3)}\\
\implies &(\forall y \in X, y \le x) \wedge (\forall y \in X, y \ge x) && \text{(by Def. I.5.4.6)}\\
\implies &\max(X) = \min(X) = x.
\end{aligned}
$$

Thus, the base case holds. Suppose inductively that for some $n \in \mathbb{Z}^+$ we have $\max(X) \in X$ and $\min(X) \in X$. Now let X be a finite subset of real number with $\#(X) = n+1$. Let $x \in X$ and let $X' = X \setminus \{x\}$. By Lem. I.3.6.9 we know that $\#(X') = n$. Since $n \in \mathbb{Z}^+$, we know that X' is non-empty. Then we have

$$
\begin{aligned}
&(\max(X') \in X') \wedge (\min(X') \in X') && \text{(by the induction hypothesis)}\\
\implies &(\max(X') \in X) \wedge (\min(X') \in X) && (X' \subseteq X)\\
\implies &\begin{cases} \forall y \in X', y \le \max(X') \le \max(\max(X'), x) \in X \\ \forall y \in X', y \ge \min(X') \ge \min(\min(X'), x) \in X \\ x \le \max(\max(X'), x) \\ x \ge \min(\min(X'), x) \end{cases} && \text{(by Prop. I.5.4.7(a))}
\end{aligned}
$$

$$\implies \begin{cases} \forall y \in X, y \leq \max(\max(X'), x) \\ \forall y \in X, y \geq \min(\min(X'), x) \end{cases}$$
$$\implies \begin{cases} \max(X) = \max(\max(X'), x) \\ \min(X) = \min(\min(X'), x) \end{cases}.$$

This closes the induction.

Now we show that both $\max(X), \min(X)$ are unique. Suppose there are two $x, x' \in X$ such that $y \leq x$ and $y \leq x'$ for all $y \in X$. But since $x, x' \in X$, we have $x \leq x'$ and $x' \leq x$. Thus, by Prop. I.5.4.7(a) we must have $x = x'$ and $\max(X)$ is unique. Using similarly arguments, we can show that $\min(X)$ is unique. $\square$

A.Cor. I.5.4.3. Let $x \in \mathbb{R}$. Then x is positive iff $x > 0$; x is negative iff $x < 0$.

Let $y \in \mathbb{R}$. We define the following eight subsets of $\mathbb{R}$:

$$\begin{array}{lll} \mathbb{R}_{\leq x} := \{r \in \mathbb{R} : r \leq x\}; & \mathbb{R}_{<x} := \{r \in \mathbb{R} : r < x\}; & \mathbb{R}^+ := \mathbb{R}_{>0}; \\ \mathbb{R}_{\geq x} := \{r \in \mathbb{R} : r \geq x\}; & \mathbb{R}_{>x} := \{r \in \mathbb{R} : r > x\}; & \mathbb{R}^- := \mathbb{R}_{<0}; \\ \mathbb{R}_{x \leq y} := \{r \in \mathbb{R} : x \leq r \leq y\}; & \mathbb{R}_{x<y} := \{r \in \mathbb{R} : x < r < y\}. & \end{array}$$

Proof of A.Cor. I.5.4.3. By Prop. I.5.3.11 we have $x = x - 0$. Thus, x is a positive rational number iff $x - 0$ is a positive rational number, iff $x > 0$ (Def. I.5.4.6). Similarly, x is a negative rational number iff $x - 0$ is a negative rational number, iff $x < 0$ (Def. I.5.4.6). $\square$

— Exercises —

Ex. I.5.4.1. Prove Prop. I.5.4.4.

Proof of Ex. I.5.4.1. See Prop. I.5.4.4. $\square$

Ex. I.5.4.2. Prove the remaining claims in Prop. I.5.4.7.

Proof of Ex. I.5.4.2. See Prop. I.5.4.7. $\square$

Ex. I.5.4.3. Show that for every real number x there is exactly one integer N such that $N \leq x < N + 1$. (This integer N is called the *integer part* of x, and is sometimes denoted $N = \lfloor x \rfloor$.)

Proof of Ex. I.5.4.3. We first prove the existence of the integer N. By Prop. I.5.4.4, exactly one of the following three statements is true:

- $x = 0$. Then we choose $N = 0$ so that $0 \leq 0 < 1$.
- x is positive. Then by Prop. I.5.4.12 there exist some $q \in \mathbb{Q}^+$ and $N_1' \in \mathbb{Z}^+$ such that $q \leq x \leq N_1'$. Let $N_1 = N_1' + 1$. Then we have $x < N_1$. By Prop. I.4.4.1 we

know that there exists an $N_2 \in \mathbb{Z}$ such that $N_2 \leq q$, thus by Prop. I.5.4.7(c) we have $N_2 \leq x < N_1$. Let X be the set

$$X = \{n \in \mathbb{Z} : N_2 \leq n \leq x < N_1\}.$$

We know that X is finite since $X \subseteq \mathbb{Z}_{N_2 \leq N_1}$ and $\mathbb{Z}_{N_2 \leq N_1}$ is finite (Prop. I.3.6.14(c)). We also know that X is non-empty since $N_2 \in X$. By A.Cor. I.5.4.2 we know that there exists an unique $\max(X) \in X$. Let $N = \max(X)$. By the definition of X we know that $N \leq x < N_1$. We must have $x < N + 1$, otherwise if $N + 1 \leq x$ then we would have $N + 1 \in X$, which means $N + 1 \leq \max(X) = N$, a contradiction. Thus, we have $N \leq x < N + 1$.

- x is negative. Then by Prop. I.5.4.4 we know that $-x$ is positive. By Cor. I.5.4.13 we know that there exists an $M \in \mathbb{Z}^+$ such that $-x < 1M = M$. Fix such M. By Prop. I.5.4.7(d) we know that $0 < x + M$. From the above case we know that there exists an $K \in \mathbb{Z}$ such that $K \leq x + M < K + 1$. So by setting $N = K - M$ we see that $N \leq x < N + 1$.

From all cases above, we conclude that for all $x \in \mathbb{R}$, there exists an $N \in \mathbb{Z}$ such that $N \leq x < N + 1$.

Now we prove the uniqueness of the integer N. Let $x \in \mathbb{R}$. Suppose that there exist some $N_1, N_2 \in \mathbb{Z}$ such that $N_1 \leq x < N_1 + 1$ and $N_2 \leq x < N_2 + 1$. Then we have

$$\begin{aligned}
& (N_1 \leq x < N_1 + 1) \wedge (N_2 \leq x < N_2 + 1) \\
\implies & (N_1 < N_2 + 1) \wedge (N_2 < N_1 + 1) && \text{(by Prop. I.5.4.7(c))} \\
\implies & (N_1 + 1 \leq N_2 + 1) \wedge (N_2 + 1 \leq N_1 + 1) && \text{(by Def. I.4.1.10)} \\
\implies & (N_1 \leq N_2) \wedge (N_2 \leq N_1) && \text{(by Def. I.4.1.10)} \\
\implies & N_1 = N_2. && \text{(by Lem. I.4.1.11)}
\end{aligned}$$

Thus, for all $x \in \mathbb{R}$, N is unique and $\lfloor x \rfloor$ is well-defined. □

Ex. I.5.4.4. Show that for any positive real number $x > 0$ there exists a positive integer N such that $x > 1/N > 0$.

Proof of Ex. I.5.4.4. We have

$$\begin{aligned}
& x > 0 \\
\implies & x^{-1} > 0 && \text{(by Prop. I.5.4.8)} \\
\implies & \exists N \in \mathbb{Z}^+ : N1 = N > x^{-1} && \text{(by Cor. I.5.4.13)} \\
\implies & N^{-1} = \frac{1}{N} < x. && \text{(by Prop. I.5.4.8)}
\end{aligned}$$

□

Ex. I.5.4.5. Prove Prop. I.5.4.14.

Proof of Ex. I.5.4.5. See Prop. I.5.4.14. □

Ex. I.5.4.6. Let $x, y \in \mathbb{R}$ and let $\varepsilon \in \mathbb{R}^+$. Show that $|x-y| < \varepsilon$ iff $y - \varepsilon < x < y + \varepsilon$, and that $|x-y| \leq \varepsilon$ iff $y - \varepsilon \leq x \leq y + \varepsilon$.

Proof of Ex. I.5.4.6. We first show that $|x-y| < \varepsilon \iff y-\varepsilon < x < y+\varepsilon$.

$$\begin{aligned}
& |x-y| < \varepsilon \\
\iff & (-(x-y) \leq x-y < \varepsilon) \vee (x-y \leq -(x-y) < \varepsilon) && \text{(by Def. I.5.4.5)} \\
\iff & (x-y < \varepsilon) \wedge (-(x-y) < \varepsilon) && \text{(by Prop. I.5.4.7(a))} \\
\iff & (x-y < \varepsilon) \wedge (y-x < \varepsilon) && \text{(by Prop. I.5.3.11)} \\
\iff & (x < y+\varepsilon) \wedge (y-\varepsilon < x) && \text{(by Prop. I.5.4.7(d))} \\
\iff & y-\varepsilon < x < y+\varepsilon. && \text{(by Prop. I.5.4.7(c))}
\end{aligned}$$

Now we show that $|x-y| \leq \varepsilon \iff y-\varepsilon \leq x \leq y+\varepsilon$. By replacing $<$ with $\leq$ in the above arguments, we are done. □

Ex. I.5.4.7. Let $x, y \in \mathbb{R}$. Show that $x \leq y + \varepsilon$ for all $\varepsilon \in \mathbb{R}^+$ iff $x \leq y$. Show that $|x-y| \leq \varepsilon$ for all $\varepsilon \in \mathbb{R}^+$ iff $x = y$.

Proof of Ex. I.5.4.7. We first show that $x \leq y+\varepsilon$ for all $\varepsilon \in \mathbb{R}^+$ iff $x \leq y$.

$$\begin{aligned}
& \forall \varepsilon \in \mathbb{R}^+, x \leq y+\varepsilon \\
\iff & \forall \varepsilon \in \mathbb{R}^+, x-y \leq \varepsilon && \text{(by Prop. I.5.4.7(d))} \\
\iff & \neg(x-y > 0) \\
\iff & x-y \leq 0 && \text{(by Prop. I.5.4.7(a))} \\
\iff & x \leq y. && \text{(by Prop. I.5.4.7(d))}
\end{aligned}$$

Now we show that $|x-y| \leq \varepsilon$ for all $\varepsilon \in \mathbb{R}^+$ iff $x = y$.

$$\begin{aligned}
& \forall \varepsilon \in \mathbb{R}^+, |x-y| \leq \varepsilon \\
\iff & \forall \varepsilon \in \mathbb{R}^+, y-\varepsilon \leq x \leq y+\varepsilon && \text{(by Ex. I.5.4.6)} \\
\iff & (x \leq y) \wedge (y \leq x) && \text{(from the proof above)} \\
\iff & x = y. && \text{(by Prop. I.5.4.7(a))}
\end{aligned}$$

□

Ex. I.5.4.8. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence of rationals, and let x be a real number. Show that if $a_n \leq x$ for all $n \in \mathbb{Z}^+$, then $\operatorname*{LIM}_{n\to\infty} a_n \leq x$. Similarly, show that if $a_n \geq x$ for all $n \in \mathbb{Z}^+$, then $\operatorname*{LIM}_{n\to\infty} a_n \geq x$.

Proof of Ex. I.5.4.8. We first show that if $a_n \leq x$ for all $n \in \mathbb{Z}^+$, then $\underset{n\to\infty}{\text{LIM}}\, a_n \leq x$. Let $a = \underset{n\to\infty}{\text{LIM}}\, a_n$. Suppose for the sake of contradiction that $a > x$. Then by Prop. I.5.4.14, there exists a $q \in \mathbb{Q}$ such that $a > q > x$. Since $q > x$, we have $a_n \leq x < q$ for all $n \in \mathbb{Z}^+$. But by Cor. I.5.4.10 we have $a = \underset{n\to\infty}{\text{LIM}}\, a_n \leq \underset{n\to\infty}{\text{LIM}}\, q = q$, which contradict to $a > q$. Thus, we must have $a \leq x$.

Now we show that if $a_n \geq x$ for all $n \in \mathbb{Z}^+$, then $\underset{n\to\infty}{\text{LIM}}\, a_n \geq x$. This is true since

$$\begin{aligned}
& a_n \geq x && \\
\implies & -a_n \leq -x && \text{(by Ex. I.4.2.6)} \\
\implies & \underset{n\to\infty}{\text{LIM}} -a_n \leq -x && \text{(from the proof above)} \\
\implies & -\underset{n\to\infty}{\text{LIM}}\, a_n \leq -x && \text{(by Prop. I.5.3.10)} \\
\implies & \underset{n\to\infty}{\text{LIM}}\, a_n \geq x. && \text{(by Ex. I.4.2.6)}
\end{aligned}$$

□

I.5.5 The least upper bound property

Def. I.5.5.1 (Upper bound)**.** Let E be a subset of $\mathbb{R}$, and let M be a real number. We say that M is an *upper bound* for E, iff we have $x \leq M$ for every element x in E.

E.g. I.5.5.3. Let $\mathbb{R}^+$ be the set of positive reals: $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$. Then $\mathbb{R}^+$ does not have any upper bounds at all. (More precisely, $\mathbb{R}^+$ has no upper bounds which are real numbers.)

Proof of E.g. I.5.5.3. Suppose for the sake of contradiction that there exists an $M \in \mathbb{R}$ such that M is an upper bound for $\mathbb{R}^+$. Then we have $x \leq M$ for all $x \in \mathbb{R}^+$. Since $x > 0$, by Prop. I.5.4.7 we have $M > 0$, thus $M \in \mathbb{R}^+$. But this means $M + 1 \in \mathbb{R}^+$, and we must have $M > M + 1$, a contradiction. Thus, such M does not exist and $\mathbb{R}^+$ has no upper bounds in $\mathbb{R}$. □

A.Cor. I.5.5.1. Let $x, y \in \mathbb{R}$. We define the following eight subsets of $\mathbb{R}$:

$$\begin{aligned}
&\mathbb{R}_{\leq x} := \{r \in \mathbb{R} : r \leq x\}; && \mathbb{R}_{<x} := \{r \in \mathbb{R} : r < x\}; && \mathbb{R}^+ := \mathbb{R}_{>0}; \\
&\mathbb{R}_{\geq x} := \{r \in \mathbb{R} : r \geq x\}; && \mathbb{R}_{>x} := \{r \in \mathbb{R} : r > x\}; && \mathbb{R}^- := \mathbb{R}_{<0}; \\
&\mathbb{R}_{x\leq y} := \{r \in \mathbb{R} : x \leq r \leq y\}; && \mathbb{R}_{x<y} := \{r \in \mathbb{R} : x < r < y\}.
\end{aligned}$$

E.g. I.5.5.4. Let $\emptyset$ be the empty set. Then every number M is an upper bound for $\emptyset$, because M is greater than every element of the empty set (this is a vacuously true statement, but still true).

Note. It is clear that if M is an upper bound of E, then any larger number $M' \geq M$ is also an upper bound of E. On the other hand, it is not so clear whether it is also possible for any number smaller than M to also be an upper bound of E. This motivates the Def. I.5.5.5.

Def. I.5.5.5 (Least upper bound)**.** Let E be a subset of $\mathbb{R}$, and M be a real number. We say that M is a *least upper bound* for E iff

(a) M is an upper bound for E, and also

(b) any other upper bound M' for E must be larger than or equal to M.

E.g. I.5.5.7. The empty set does not have a least upper bound.

Proof of E.g. I.5.5.7. Suppose for the sake of contradiction that there exists an $M \in \mathbb{R}$ such that M is a least upper bound of $\emptyset$. By Def. I.5.5.5 we know that $x \leq M$ for all $x \in \emptyset$. But by E.g. I.5.5.4 we know that $M - 1$ is also a upper bound of $\emptyset$, so by Def. I.5.5.5 we have $M < M - 1$, a contradiction. Thus, $\emptyset$ does not have a least upper bound. $\square$

Prop. I.5.5.8 (Uniqueness of least upper bound)**.** Let E be a subset of $\mathbb{R}$. Then E can have at most one least upper bound.

Proof of Prop. I.5.5.8. Let M_1 and M_2 be two least upper bounds of E. Since M_1 is a least upper bound and M_2 is an upper bound, then by definition of least upper bound we have $M_2 \geq M_1$. Since M_2 is a least upper bound and M_1 is an upper bound, we similarly have $M_1 \geq M_2$. Thus, $M_1 = M_2$. Thus, there is at most one least upper bound. $\square$

Thm. I.5.5.9 (Existence of least upper bound)**.** Let E be a non-empty subset of $\mathbb{R}$. If E has an upper bound (i.e., E has some upper bound M), then it must have exactly one least upper bound.

Proof of Thm. I.5.5.9. Let E be a non-empty subset of $\mathbb{R}$ with an upper bound M. By Prop. I.5.5.8, we know that E has at most one least upper bound; we have to show that E has at least one least upper bound. Since E is non-empty, we can choose some element x_0 in E.

Let $n \in \mathbb{Z}^+$. We know that E has an upper bound M. By the Archimedean property (Cor. I.5.4.13), we can find a $K \in \mathbb{Z}^+$ such that $K/n \geq M$, and hence K/n is also an upper bound for E. (Note that K is positive, and M can be either zero or negative, but K/n is positive, so we are fine.) By the Archimedean property again, there exists another $L \in \mathbb{Z}$ such that $L/n < x_0$. (Note that if $x_0 \geq 0$, then we can set $L = -1$; if $x_0 < 0$, then $-x_0$ is positive, so by Archimedean property we have some $-L \in \mathbb{Z}^+$ such that $-L/n > -x_0$.) Since x_0 lies in E, we see that L/n is not an upper bound for E. Since K/n is an upper bound but L/n is not, we see that $K > L$.

Since K/n is an upper bound for E and L/n is not, we can find an integer $L < m_n \leq K$ with the property that m_n/n is an upper bound for E, but $(m_n - 1)/n$ is not (see Ex. I.5.5.2). In fact, this integer m_n is unique (Ex. I.5.5.3). We subscript m_n by n to emphasize the fact that this integer m depends on the choice of n. This gives a well-defined (and unique) sequence $m_1, m_2, m_3, \ldots$ of integers, with each of the m_n/n being upper bounds and each of the $(m_n - 1)/n$ not being upper bounds.

Now let $N \in \mathbb{Z}^+$, and let $n, n' \in \mathbb{Z}_{\geq N}$. Since m_n/n is an upper bound for E and $(m_{n'} - 1)/n'$ is not, by Def. I.5.5.1 we must have $m_n/n > (m_{n'} - 1)/n'$. After a little algebra, this implies that

$$\frac{m_n}{n} - \frac{m_{n'}}{n'} > -\frac{1}{n'} \geq -\frac{1}{N}.$$

Similarly, since $m_{n'}/n'$ is an upper bound for E and $(m_n - 1)/n$ is not, we have $m_{n'}/n' > (m_n - 1)/n$, and hence

$$\frac{m_n}{n} - \frac{m_{n'}}{n'} < \frac{1}{n} \leq \frac{1}{N}.$$

Putting these two bounds together, we see that

$$\left|\frac{m_n}{n} - \frac{m_{n'}}{n'}\right| \leq \frac{1}{N} \text{ for all } n, n' \geq N \geq 1.$$

This implies that $\dfrac{m_n}{n}$ is a Cauchy sequence (Ex. I.5.5.4). Since the $\dfrac{m_n}{n}$ are rational numbers, we can now define the real number S as

$$S := \operatorname*{LIM}_{n\to\infty} \frac{m_n}{n}.$$

From Ex. I.5.3.5 we conclude that

$$S = \operatorname*{LIM}_{n\to\infty} \frac{m_n - 1}{n}.$$

To finish the proof of the theorem, we need to show that S is the least upper bound for E. First, we show that it is an upper bound. Let x be any element of E. Then, since m_n/n is an upper bound for E, we have $x \leq m_n/n$ for all $n \in \mathbb{Z}^+$. Applying Ex. I.5.4.8, we conclude that $x \leq \operatorname*{LIM}_{n\to\infty} m_n/n = S$. Thus, S is indeed an upper bound for E.

Now we show it is a least upper bound. Suppose y is an upper bound for E. Since $(m_n - 1)/n$ is not an upper bound, we conclude that $y \geq (m_n - 1)/n$ for all $n \in \mathbb{Z}^+$. Applying Ex. I.5.4.8, we conclude that $y \geq \operatorname*{LIM}_{n\to\infty}(m_n - 1)/n = S$. Thus, the upper bound S is less than or equal to every upper bound of E, and S is thus a least upper bound of E. □

Def. I.5.5.10 (Supremum)**.** Let E be a subset of the real numbers. If E is non-empty and has some upper bound, we define $\sup(E)$ to be the least upper bound of E (this is well-defined by Thm. I.5.5.9). We introduce two additional symbols, $+\infty$ and $-\infty$. If E is non-empty and has no upper bound, we set $\sup(E) := +\infty$; if E is empty, we set $\sup(E) := -\infty$. We refer to $\sup(E)$ as the *supremum* of E, and also denote it by $\sup E$.

Rmk. I.5.5.11. At present, $+\infty$ and $-\infty$ are meaningless symbols; we have no operations on them at present, and none of our results involving real numbers apply to $+\infty$ and $-\infty$, because these are not real numbers. In Sec. I.6.2 we add $+\infty$ and $-\infty$ to the reals to form the *extended real number system*, but this system is not as convenient to work with as the real number system, because many of the laws of algebra break down. For instance, it is not a good idea to try to define $+\infty + -\infty$; setting this equal to 0 causes some problems.

Prop. I.5.5.12. There exists a positive real number x such that $x^2 = 2$.

Proof of Prop. I.5.5.12. Let E be the set $\{y \in \mathbb{R} : y \geq 0 \text{ and } y^2 < 2\}$; thus E is the set of all non-negative real numbers whose square is less than 2. Observe that E has an upper bound of 2 (because if $y > 2$, then $y^2 > 4 > 2$ and hence $y \notin E$). Also, E is non-empty (for instance, 1 is an element of E). Thus, by the least upper bound property (Thm. I.5.5.9), we have a real number $x := \sup(E)$ which is the least upper bound of E. Then x is greater than or equal to 1 (since $1 \in E$) and less than or equal to 2 (since 2 is an upper bound for E). So x is positive. Now we show that $x^2 = 2$.

We argue this by contradiction. We show that both $x^2 < 2$ and $x^2 > 2$ lead to contradictions. First, suppose that $x^2 < 2$. Let $\varepsilon \in \mathbb{Q}_{0<1}$ be a small number; then we have

$$(x+\varepsilon)^2 = x^2 + 2\varepsilon x + \varepsilon^2 \leq x^2 + 4\varepsilon + \varepsilon = x^2 + 5\varepsilon$$

since $x \leq 2$ and $\varepsilon^2 \leq \varepsilon$. Since $x^2 < 2$, we see that we can use the Archimedean property (Cor. I.5.4.13) to choose an $\varepsilon \in \mathbb{Q}_{0<1}$ such that $x^2 + 5\varepsilon < 2$, thus $(x+\varepsilon)^2 < 2$. By construction of E, this means that $x + \varepsilon \in E$; but this contradicts the fact that x is an upper bound of E.

Now suppose that $x^2 > 2$. Let $\varepsilon \in \mathbb{Q}_{0<1}$ be a small number; then we have

$$(x-\varepsilon)^2 = x^2 - 2\varepsilon x + \varepsilon^2 \geq x^2 - 2\varepsilon x \geq x^2 - 4\varepsilon$$

since $x \leq 2$ and $\varepsilon^2 \geq 0$. Since $x^2 > 2$, we can choose $\varepsilon \in \mathbb{Q}_{0<1}$ such that $x^2 - 4\varepsilon > 2$, and thus $(x-\varepsilon)^2 > 2$. But then this implies that $x - \varepsilon \geq y$ for all $y \in E$. (Why? If $x - \varepsilon < y$ then $(x-\varepsilon)^2 < y^2 \leq 2$, a contradiction.) Thus, $x - \varepsilon$ is an upper bound for E, which contradicts the fact that x is the *least* upper bound of E. From these two contradictions we see that $x^2 = 2$, as desired. □

Rmk. I.5.5.13. Comparing Prop. I.5.5.12 with Prop. I.4.4.4, we see that certain numbers are real but not rational. The proof of Prop. I.5.5.12 also shows that the rationals $\mathbb{Q}$ do not obey the least upper bound property, otherwise one could use that property to construct a square root of 2, which by Prop. I.4.4.4 is not possible.

Rmk. I.5.5.14. In Ch. I.6 we will use the least upper bound property to develop the theory of limits, which allows us to do many more things than just take square roots.

Rmk. I.5.5.15. We can of course talk about lower bounds, and greatest lower bounds, of sets E; the greatest lower bound of a set E is also known as the *infimum* of E and is denoted $\inf(E)$ or $\inf E$. Everything we say about suprema has a counterpart for infima; A precise relationship between the two notions is given by Ex. I.5.5.1. See also Sec. I.6.2.

Note. Supremum means "highest" and infimum means "lowest," and the plurals are suprema and infima. Supremum is to superior, and infimum to inferior, as maximum is to major, and minimum to minor. The root words are "super," which means "above," and "infer," which means "below" (this usage only survives in a few rare English words such as "infernal," with the Latin prefix "sub" having mostly replaced "infer" in English).

— Exercises —

Ex. I.5.5.1. Let E be a subset of the real numbers $\mathbb{R}$, and suppose that E has a least upper bound M which is a real number, i.e., $M = \sup(E)$. Let $-E$ be the set

$$-E \coloneqq \{-x : x \in E\}.$$

Show that $-M$ is the greatest lower bound of $-E$, i.e., $-M = \inf(-E)$.

Proof of Ex. I.5.5.1. We first show that $-M$ is a lower bound for $-E$. This is true since

$$\begin{aligned}
&\forall x \in E, x \leq M && \text{(by Def. I.5.5.1)}\\
\implies &\forall x \in E, -x \geq -M && \text{(by Prop. I.5.4.7)}\\
\implies &\forall -x \in -E, -x \geq -M &&\\
\implies &-M \text{ is a lower bound of } -E. && \text{(by Rmk. I.5.5.15)}
\end{aligned}$$

Next we show that $-M$ is a greatest lower bound for $-E$. Let $L \in \mathbb{R}$ be any lower bound for $-E$. Then we have

$$\begin{aligned}
&\forall x \in E, L \leq -x &&\\
\implies &\forall x \in E, x \leq -L && \text{(by Prop. I.5.4.7)}\\
\implies &-L \text{ is an upper bound of } E && \text{(by Def. I.5.5.1)}\\
\implies &M \leq -L && \text{(by Def. I.5.5.5)}\\
\implies &-M \geq L && \text{(by Prop. I.5.4.7)}\\
\implies &-M \text{ is a greatest lower bound of } -E. && \text{(by Rmk. I.5.5.15)}
\end{aligned}$$

Now we show that the greatest lower bound is unique. Let M, M' be two greatest lower bounds of $-E$. Then we have $M \leq M'$ and $M \geq M'$, which means $M = M'$. So the greatest lower bound is unique. □

Ex. I.5.5.2. Let E be a non-empty subset of $\mathbb{R}$, let $n \in \mathbb{Z}^+$, and let $L < K$ be integers. Suppose that K/n is an upper bound for E, but that L/n is not an upper bound for E. Without using Thm. I.5.5.9, show that there exists an integer $L < m \leq K$ such that m/n is an upper bound for E, but that $(m-1)/n$ is not an upper bound for E.

Proof of Ex. I.5.5.2. Let $d = K - L$, so d is positive by Lem. I.4.1.11(a). Now we induct on d to show that for every $d \in \mathbb{Z}^+$, there exists an $m \in \mathbb{Z}_{>L} \cap \mathbb{Z}_{\leq K}$ such that m/n is an upper bound for E, but that $(m-1)/n$ is not. We start with $d = 1$. For $d = 1$, we have $K - 1 = L$. Then let $m = K$. So by hypothesis we have $m = K \in \mathbb{Z}_{>L} \cap \mathbb{Z}_{\leq K}$, $m/n = K/n$ is an upper bound for E, and $(m-1)/n = (K-1)/n = L/n$ is not an upper bound for E. Thus, the base case holds. Suppose inductively that the statement holds for some $d \in \mathbb{Z}^+$. Now we show that for $d+1$ the statement is also true. So suppose that for some $K, L \in \mathbb{Z}$, we have $K - L = d + 1$, K/n is an upper bound of E, but L/n is not. Since K/n is an upper bound for E, we can ask whether $(K-1)/n$ is an upper bound for E.

- If $(K-1)/n$ is not an upper bound for E, then we can choose $m = K$ and we are done.
- If $(K-1)/n$ is an upper bound for E, then by Def. I.5.5.1 we have $L < K - 1$. Since $K - 1 - L = d$, by the induction hypothesis we know that there exists an $m \in \mathbb{Z}_{>L} \cap \mathbb{Z}_{\leq K}$ such that m/n is an upper bound for E, but $(m-1)/n$ is not.

From all cases above, we found an $m \in \mathbb{Z}_{>L} \cap \mathbb{Z}_{\leq K}$ such that m/n is an upper bound for E, but $(m-1)/n$ is not. This closes the induction. $\square$

Ex. I.5.5.3. Let E be a non-empty subset of $\mathbb{R}$, let $n \in \mathbb{Z}^+$, and let $m, m' \in \mathbb{Z}$ with the properties that m/n and m'/n are upper bounds for E, but $(m-1)/n$ and $(m'-1)/n$ are not upper bounds for E. Show that $m = m'$. This shows that the integer m constructed in Ex. I.5.5.2 is unique.

Proof of Ex. I.5.5.3. Suppose for the sake of contradiction that $m \neq m'$. Then by Lem. I.4.1.11(f we have either $m < m'$ or $m > m'$.

- If $m < m'$, then we have $m \leq m' - 1$. By Prop. I.4.2.9(e) we have $m/n \leq (m'-1)/n$, which means $(m'-1)/n$ is an upper bound for E. But this contradicts the hypothesis.
- If $m > m'$, then by switching the row of m and m' in the previous case we can also derive contradiction.

From all cases above, we derive contradictions. Thus, we must have $m = m'$. $\square$

Ex. I.5.5.4. Let $(q_n)_{n=1}^{\infty}$ be a sequence of rational numbers with the property that $|q_n - q_{n'}| \leq \dfrac{1}{M}$ whenever $M \in \mathbb{Z}^+$ and $n, n' \in \mathbb{Z}_{\geq M}$. Show that $(q_n)_{n=1}^{\infty}$ is a Cauchy sequence. Furthermore, if $S := \operatorname*{LIM}_{n\to\infty} q_n$, show that $|q_M - S| \leq \dfrac{1}{M}$ for every $M \in \mathbb{Z}^+$.

Proof of Ex. I.5.5.4. We first show that $(q_n)_{n=1}^{\infty}$ is a Cauchy sequence. Let $\varepsilon \in \mathbb{Q}^+$. By Archimedean property (Cor. I.5.4.13) we know that there exists an $M \in \mathbb{Z}^+$ such that $M\varepsilon > 1$. By Prop. I.4.2.9(e) we have $\varepsilon > 1/M$. But by hypothesis we have $|q_n - q_{n'}| \leq \dfrac{1}{M} < \varepsilon$ for all $n, n' \in \mathbb{Z}_{\geq M}$. Since ε was arbitrary, by Def. I.5.1.8 this means $(q_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Now we show that if $S = \operatorname*{LIM}_{n\to\infty} q_n$, then $|q_M - S| \le \dfrac{1}{M}$ for every $M \in \mathbb{Z}^+$. From the proof above we know that $(q_n)_{n=1}^{\infty}$ is a Cauchy sequence, thus $S = \operatorname*{LIM}_{n\to\infty} q_n$ is well-defined. By hypothesis we have $|q_M - q_n| \le 1/M$ for all $M \in \mathbb{Z}^+$ and for all $n \ge M$. Then we have

$$
\begin{aligned}
&\forall M \in \mathbb{Z}^+, \forall n \in \mathbb{Z}_{\ge M}, |q_M - q_n| \le 1/M \\
\Longrightarrow\ &\forall M \in \mathbb{Z}^+, \forall n \in \mathbb{Z}_{\ge M}, |q_n - q_M| \le 1/M && \text{(by Def. I.4.3.1)} \\
\Longrightarrow\ &\forall M \in \mathbb{Z}^+, \forall n \in \mathbb{Z}_{\ge M}, -1/M \le q_n - q_M \le 1/M && \text{(by Prop. I.4.3.3(c))} \\
\Longrightarrow\ &\forall M \in \mathbb{Z}^+, \forall n \in \mathbb{Z}_{\ge M}, -1/M + q_M \le q_n \le 1/M + q_M && \text{(by Prop. I.4.2.9(d))} \\
\Longrightarrow\ &\forall M \in \mathbb{Z}^+, -1/M + q_M \le S \le 1/M + q_M && \text{(by Ex. I.5.4.8)} \\
\Longrightarrow\ &\forall M \in \mathbb{Z}^+, -1/M \le S - q_M \le 1/M && \text{(by Prop. I.5.4.7(d))} \\
\Longrightarrow\ &\forall M \in \mathbb{Z}^+, |S - q_M| \le 1/M && \text{(by Ex. I.5.4.6)} \\
\Longrightarrow\ &\forall M \in \mathbb{Z}^+, |q_M - S| \le 1/M. && \text{(by Def. I.5.4.5)}
\end{aligned}
$$

□

Ex. I.5.5.5. Establish an analogue of Prop. I.5.4.14, in which “rational” is replaced by “irrational.”

Proof of Ex. I.5.5.5. Let $x, y, z \in \mathbb{R}$ where $x < y$ and $z^2 = 2$. (z is well-defined thanks to Prop. I.5.5.12.) So by Prop. I.5.4.7 we have $x - z < y - z$. But by Prop. I.5.4.14 there exists a $q \in \mathbb{Q}$ such that $x - z < q < y - z$. So by Prop. I.5.4.7 again we have $x < q + z < y$. Because z is irrational, $q + z$ is also irrational (otherwised we have $a = q + z \in \mathbb{Q}$ and $z = a - q \in \mathbb{Q}$, contradicts to Prop. I.4.4.4). So we have an irrational number in between any two real numbers x, y where $x < y$. □

I.5.6 Real exponentiation, part I

Def. I.5.6.1 (Exponentiating a real by a natural number)**.** Let $x \in \mathbb{R}$. To raise x to the power 0, we define $x^0 := 1$. Now suppose recursively that x^n has been defined for some natural number n, then we define $x^{n+1} := x^n \times x$.

Def. I.5.6.2 (Exponentiating a real by an integer)**.** Let x be a non-zero real number. Then for any negative integer $-n$, we define $x^{-n} := 1/x^n$.

Prop. I.5.6.3. All the properties in Prop. I.4.3.10 and I.4.3.12 remain valid if x and y are assumed to be real numbers instead of rational numbers.

Meta-proof (Proof of Prop. I.5.6.3). If one inspects the proof of Prop. I.4.3.10 and I.4.3.12 we see that they rely on the laws of algebra and the laws of order for the rationals (Prop. I.4.2.4 and I.4.2.9). But by Prop. I.5.3.11 and I.5.4.7, and the identity $xx^{-1} = x^{-1}x = 1$ we know that all these laws of algebra and order continue to hold for real numbers as well as rationals. Thus, we can modify the proof of Prop. I.4.3.10 and I.4.3.12 to hold in the case when x and y are real.

Note. Instead of giving an actual proof of Prop. I.5.6.3, we shall give a meta-proof (an argument appealing to the nature of proofs, rather than the nature of real and rational numbers).

Def. I.5.6.4. Let $x \in \mathbb{R}_{\geq 0}$, and let $n \in \mathbb{Z}^+$. We define $x^{1/n}$, also known as the n^{th} *root of* x, by the formula

$$x^{1/n} := \sup\{y \in \mathbb{R} : y \geq 0 \text{ and } y^n \leq x\}.$$

We often write $\sqrt{x}$ for $x^{1/2}$.

Note. We do not define the n^{th} roots of a negative number. In fact, we will leave the n^{th} roots of negative numbers undefined for the rest of the text (one can define these n^{th} roots once one defines the complex numbers, but we shall refrain from doing so).

Lem. I.5.6.5 (Existence of n^{th} roots)**.** Let $x \in \mathbb{R}_{\geq 0}$, and let $n \in \mathbb{Z}^+$. Then the set $E := \{y \in \mathbb{R} : y \geq 0 \text{ and } y^n \leq x\}$ is non-empty and is also bounded above. In particular, $x^{1/n}$ is a real number.

Proof of Lem. I.5.6.5. The set E contains 0, so it is certainly not empty. Now we show it has an upper bound. We divide into two cases: $x \leq 1$ and $x > 1$. First, suppose that we are in the case where $x \leq 1$. Then we claim that the set E is bounded above by 1. To see this, suppose for the sake of contradiction that there was an element $y \in E$ for which $y > 1$. But then $y^n > 1$, and hence $y^n > x$, a contradiction. Thus, E has an upper bound. Now suppose that we are in the case where $x > 1$. Then we claim that the set E is bounded above by x. To see this, suppose for contradiction that there was an element $y \in E$ for which $y > x$. Since $x > 1$, we thus have $y > 1$. Since $y > x$ and $y > 1$, we have $y^n > x$, a contradiction. Thus, in both cases E has an upper bound, and so $x^{1/n}$ is finite. $\square$

Lem. I.5.6.6. Let $x, y \in \mathbb{R}_{\geq 0}$, and let $n, m \in \mathbb{Z}^+$.

(a) If $y = x^{1/n}$, then $y^n = x$.

(b) Conversely, if $y^n = x$, then $y = x^{1/n}$.

(c) $x^{1/n}$ is a non-negative real number, and is positive iff x is positive.

(d) We have $x > y$ iff $x^{1/n} > y^{1/n}$.

(e) Let $k, l \in \mathbb{Z}^+$. If $x > 1$, then $x^{1/k}$ is a decreasing (i.e., $x^{1/k} > x^{1/l}$ whenever $k < l$) function of k. If $0 < x < 1$, then $x^{1/k}$ is an increasing (i.e., $x^{1/k} < x^{1/l}$ whenever $k < l$) function of k. If $x = 1$, then $x^{1/k} = 1$ for all k.

(f) We have $(xy)^{1/n} = x^{1/n}y^{1/n}$.

(g) We have $(x^{1/n})^{1/m} = x^{1/nm}$.

Proof of Lem. I.5.6.6(a). Let $E = \{z \in \mathbb{R}_{\geq 0} : z^n \leq x\}$. By Lem. I.5.6.5 we have $y = x^{1/n} = \sup(E)$. Suppose for the sake of contradiction that $y^n \neq x$. Then by Prop. I.5.4.7(a) we have either $y^n < x$ or $y^n > x$.

- If $y^n < x$, then we can find a small number $\varepsilon \in \mathbb{Q}_{0<1}$ such that $(y+\varepsilon)^n < x$ (this can be proved by induction, and using the binomial formula, see Ex. I.7.1.4). But by the definition of E we see that $y + \varepsilon \in E$, so we must have $y + \varepsilon \leq y$, a contradiction.

- If $y^n > x$, then we can find a small number $\varepsilon \in \mathbb{Q}_{0<1}$ such that $(y-\varepsilon)^n > x$. Then by the definition of E we see that $(y-\varepsilon)^n > x$ implies $y - \varepsilon$ is an upper bound of E. But by Def. I.5.5.5 this means $y - \varepsilon \geq y$, a contradiction.

From all cases above, we derived contradictions. So we must have $y^n = x$. □

Proof of Lem. I.5.6.6(b). Let $E = \{z \in \mathbb{R}_{\geq 0} : z^n \leq x\}$. By Lem. I.5.6.5 we have $x^{1/n} = \sup(E)$. Since $y^n = x$, by the definition of E we know that $y \in E$. Suppose for the sake of contradiction that $y \neq x^{1/n}$. Then by Prop. I.5.4.7(a) exactly one of the following statements is true:

- $y < x^{1/n}$. But then we have

$$\begin{aligned} x &= y^n \\ &< (x^{1/n})^n && \text{(by Prop. I.5.6.3)} \\ &= x, && \text{(by Lem. I.5.6.6(a))} \end{aligned}$$

  a contradiction.

- $y > x^{1/n}$. But then we have

$$\begin{aligned} x &= y^n \\ &> (x^{1/n})^n && \text{(by Prop. I.5.6.3)} \\ &= x, && \text{(by Lem. I.5.6.6(a))} \end{aligned}$$

  a contradiction.

From all cases above, we derived contradictions. So we must have $y = x^{1/n}$. □

Proof of Lem. I.5.6.6(c). Let $E = \{z \in \mathbb{R}_{\geq 0} : z^n \leq x\}$. By Lem. I.5.6.5 we have $x^{1/n} = \sup(E)$. Since $0 \in E$, by Def. I.5.5.5 we know that $0 \leq x^{1/n}$, thus $x^{1/n}$ is a non-negative real number.

Next suppose that $x^{1/n} \in \mathbb{R}^+$. Then we have

$$\begin{aligned} & x^{1/n} \in \mathbb{R}^+ \\ \Longrightarrow\ & x = (x^{1/n})^n \in \mathbb{R}^+. && \text{(by Prop. I.5.6.3 and Lem. I.5.6.6(a))} \end{aligned}$$

Finally, suppose that $x \in \mathbb{R}^+$. Suppose for the sake of contradiction that $x^{1/n} \notin \mathbb{R}^+$. Then from the proof above we know that $x^{1/n} = 0$. But by Lem. I.5.6.6(a) we have

$$x = (x^{1/n})^n = 0^n = 0,$$

a contradiction. Thus, we must have $x^{1/n} \in \mathbb{R}^+$. We conclude that $x^{1/n} \in \mathbb{R}^+$ iff $x \in \mathbb{R}^+$. $\square$

Proof of Lem. I.5.6.6(d). We first show that $x^{1/n} > y^{1/n} \implies x > y$.

$$\begin{aligned}
&x^{1/n} > y^{1/n} \\
\implies &(x^{1/n})^n > (y^{1/n})^n && \text{(by Prop. I.5.6.3)} \\
\implies &x > y. && \text{(by Lem. I.5.6.6(a))}
\end{aligned}$$

Now we show that $x > y \implies x^{1/n} > y^{1/n}$. Suppose for the sake of contradiction that $x^{1/n} \leq y^{1/n}$. But then we have

$$\begin{aligned}
&x^{1/n} \leq y^{1/n} \\
\implies &(x^{1/n})^n \leq (y^{1/n})^n && \text{(by Prop. I.5.6.3)} \\
\implies &x \leq y, && \text{(by Lem. I.5.6.6(a))}
\end{aligned}$$

a contradiction. Thus, we must have $x^{1/n} > y^{1/n}$. We conclude that $x > y \iff x^{1/n} > y^{1/n}$. $\square$

Proof of Lem. I.5.6.6(e). If $x = 0$, then by Lem. I.5.6.6(c) we know that $x^{1/k} = 0$ for every $k \in \mathbb{Z}^+$. Thus, we only consider the case $x \in \mathbb{R}^+$.

We first show that if $x > 1$, then $x^{1/k}$ is a decreasing function of $k \in \mathbb{Z}^+$. Suppose for the sake of contradiction that there exists a $k \in \mathbb{Z}^+$ such that $x^{1/k} \leq x^{1/(k+1)}$. But then we have

$$\begin{aligned}
&x^{1/k} \leq x^{1/(k+1)} \\
\implies &(x^{1/k})^k \leq (x^{1/(k+1)})^k && \text{(by Prop. I.5.6.3)} \\
\implies &x \leq (x^{1/(k+1)})^k && \text{(by Lem. I.5.6.6(a))} \\
\implies &(x^{1/(k+1)})^{k+1} \leq (x^{1/(k+1)})^k && \text{(by Lem. I.5.6.6(a)(b))} \\
\implies &(x^{1/(k+1)})^{k+1} \cdot (x^{1/(k+1)})^{-1} \leq (x^{1/(k+1)})^k \cdot (x^{1/(k+1)})^{-1} && \text{(by Lem. I.5.6.6(c))} \\
\implies &(x^{1/(k+1)})^{k+1} \cdot (x^{1/(k+1)})^{-k} \leq (x^{1/(k+1)})^k \cdot (x^{1/(k+1)})^{-k} && \text{(by Prop. I.5.6.3)} \\
\implies &x^{1/(k+1)} \leq 1 && \text{(by Prop. I.5.6.3)} \\
\implies &(x^{1/(k+1)})^{k+1} \leq 1^{k+1} = 1 && \text{(by Prop. I.5.6.3)} \\
\implies &x \leq 1, && \text{(by Lem. I.5.6.6(a))}
\end{aligned}$$

a contradiction. Thus, such k does not exist, and $x^{1/k}$ is a decreasing function of k when $x > 1$.

Next we show that if $x < 1$, then $x^{1/k}$ is an increasing function of $k \in \mathbb{Z}^+$. Suppose for the sake of contradiction that there exists a $k \in \mathbb{Z}^+$ such that $x^{1/k} \geq x^{1/(k+1)}$. But then we have

$$\begin{aligned}
&x^{1/k} \geq x^{1/(k+1)} && \\
\implies &(x^{1/k})^k \geq (x^{1/(k+1)})^k && \text{(by Prop. I.5.6.3)}\\
\implies &x \geq (x^{1/(k+1)})^k && \text{(by Lem. I.5.6.6(a))}\\
\implies &(x^{1/(k+1)})^{k+1} \geq (x^{1/(k+1)})^k && \text{(by Lem. I.5.6.6(a)(b))}\\
\implies &(x^{1/(k+1)})^{k+1} \cdot (x^{1/(k+1)})^{-1} \geq (x^{1/(k+1)})^k \cdot (x^{1/(k+1)})^{-1} && \text{(by Lem. I.5.6.6(c))}\\
\implies &(x^{1/(k+1)})^{k+1} \cdot (x^{1/(k+1)})^{-k} \geq (x^{1/(k+1)})^k \cdot (x^{1/(k+1)})^{-k} && \text{(by Prop. I.5.6.3)}\\
\implies &x^{1/(k+1)} \geq 1 && \text{(by Prop. I.5.6.3)}\\
\implies &(x^{1/(k+1)})^{k+1} \geq 1^{k+1} = 1 && \text{(by Prop. I.5.6.3)}\\
\implies &x \geq 1, && \text{(by Lem. I.5.6.6(a))}
\end{aligned}$$

a contradiction. Thus, such k does not exist, and $x^{1/k}$ is a increasing function of k when $x < 1$.

Finally we show that if $x = 1$, then $x^{1/k} = 1$ for every $k \in \mathbb{Z}^+$. Suppose for the sake of contradiction that there exists a $k \in \mathbb{Z}^+$ such that $x^{1/k} \neq 1$. Then by Prop. I.5.4.7 exactly one of the following two statements is true:

- $x^{1/k} > 1$. But then we have

  $$\begin{aligned}
  &(x^{1/k})^k > 1^k = 1 && \text{(by Prop. I.5.6.3)}\\
  \implies &x > 1, && \text{(by Lem. I.5.6.6(a))}
  \end{aligned}$$

  a contradiction.

- $x^{1/k} < 1$. But then we have

  $$\begin{aligned}
  &(x^{1/k})^k < 1^k = 1 && \text{(by Prop. I.5.6.3)}\\
  \implies &x < 1, && \text{(by Lem. I.5.6.6(a))}
  \end{aligned}$$

  a contradiction.

From all cases above, we derived contradictions. Thus, we must have $x^{1/k} = 1$ for all $k \in \mathbb{Z}^+$. □

Proof of Lem. I.5.6.6(f). We have

$$\begin{aligned}
((xy)^{1/n})^n &= xy && \text{(by Lem. I.5.6.6(a))}\\
&= (x^{1/n})^n (y^{1/n})^n && \text{(by Lem. I.5.6.6(a)(b))}\\
&= (x^{1/n} y^{1/n})^n. && \text{(by Prop. I.5.6.3)}
\end{aligned}$$

Thus, by Lem. I.5.6.6(b) we have $(xy)^{1/n} = x^{1/n} y^{1/n}$. □

Proof of Lem. I.5.6.6(g). We have

$$\begin{aligned}
(x^{1/nm})^{nm} &= x && \text{(by Lem. I.5.6.6(a))}\\
&= (x^{1/n})^n && \text{(by Lem. I.5.6.6(a)(b))}\\
&= \left(\left((x^{1/n})^{1/m}\right)^m\right)^n && \text{(by Lem. I.5.6.6(a)(b))}\\
&= \left((x^{1/n})^{1/m}\right)^{nm}. && \text{(by Prop. I.5.6.3)}
\end{aligned}$$

Thus, by Lem. I.5.6.6(b) we have $x^{1/nm} = (x^{1/n})^{1/m}$. □

Note. The observant reader may note that this definition of $x^{1/n}$ might possibly be inconsistent with our previous notion of x^n when $n = 1$, but it is easy to check that $x^{1/1} = x = x^1$ by using Lem. I.5.6.6(e), so there is no inconsistency.

Note. One consequence of Lem. I.5.6.6(b) is another proof of the cancellation law from Prop. I.4.3.12(c) and Prop. I.5.6.3: if y and z are positive and $y^n = z^n$, then $y = z$. This only works when y and z are positive; for instance, $(-3)^2 = 3^2$, but we cannot conclude from this that $-3 = 3$.

Def. I.5.6.7. Let $x \in \mathbb{R}^+$, and let $q \in \mathbb{Q}$. To define x^q, we write $q = a/b$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, and define

$$x^q \coloneqq (x^{1/b})^a.$$

Note. Every rational q, whether positive, negative, or zero, can be written in the form a/b where a is an integer and b is positive. However, the rational number q can be expressed in the form a/b in more than one way, for instance $1/2$ can also be expressed as $2/4$ or $3/6$. So to ensure that Def. I.5.6.7 is well-defined, we need to check that different expressions a/b give the same formula for x^q.

Lem. I.5.6.8. Let a, a' be integers and b, b' be positive integers such that $a/b = a'/b'$, and let x be a positive real number. Then we have $(x^{1/b'})^{a'} = (x^{1/b})^a$.

Proof of Lem. I.5.6.8. There are three cases: $a = 0, a > 0, a < 0$. If $a = 0$, then we must have $a' = 0$ and so both $(x^{1/b'})^{a'}$ and $(x^{1/b})^a$ are equal to 1, so we are done.

Now suppose that $a > 0$. Then $a' > 0$, and $ab' = ba'$. Write $y \coloneqq x^{1/(ab')} = x^{1/(ba')}$. By Lem. I.5.6.6(g) we have $y = (x^{1/b'})^{1/a}$ and $y = (x^{1/b})^{1/a'}$; by Lem. I.5.6.6(a) we thus have $y^{a'} = x^{1/b}$ and $y^a = x^{1/b'}$. Thus, we have

$$(x^{1/b'})^{a'} = (y^a)^{a'} = y^{aa'} = (y^{a'})^a = (x^{1/b})^a$$

as desired.

Finally, suppose that $a < 0$. Then we have $(-a)/b = (-a')/b'$. But $-a$ is positive, so the previous case applies and we have $(x^{1/b'})^{-a'} = (x^{1/b})^{-a}$. Taking the reciprocal of both sides we obtain the result. □

Note. Thus, x^q is well-defined for every rational q. Def. I.5.6.7 is consistent with our old definition for $x^{1/n}$ (since $x^{1/n} = (x^{1/n})^1$) and is also consistent with our old definition for x^n (since $x^n = (x^{1/1})^n$).

Lem. I.5.6.9. Let $x, y \in \mathbb{R}^+$, and let $q, r \in \mathbb{Q}$.

(a) $x^q \in \mathbb{R}^+$.

(b) $x^{q+r} = x^q x^r$ and $(x^q)^r = x^{qr}$.

(c) $x^{-q} = 1/x^q$.

(d) If $q > 0$, then $x > y$ iff $x^q > y^q$.

(e) If $x > 1$, then $x^q > x^r$ iff $q > r$. If $x < 1$, then $x^q > x^r$ iff $q < r$.

(f) $(xy)^q = x^q y^q$.

Proof of Lem. I.5.6.9(a). Let $q = a/b$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. Then we have

$$\begin{aligned}
& x \in \mathbb{R}^+ \\
\implies & x^{1/b} \in \mathbb{R}^+ && \text{(by Lem. I.5.6.6(c))} \\
\implies & (x^{1/b})^a \in \mathbb{R}^+ && \text{(by Prop. I.5.6.3)} \\
\implies & x^q \in \mathbb{R}^+. && \text{(by Def. I.5.6.7)}
\end{aligned}$$

□

Proof of Lem. I.5.6.9(b). Let $q = a/b$ and $r = c/d$ where $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{Z}^+$. Then we have

$$\begin{aligned}
x^{q+r} &= x^{(ad+bc)/bd} && \text{(by Def. I.4.2.2)} \\
&= (x^{1/bd})^{(ad+bc)} && \text{(by Def. I.5.6.7)} \\
&= (x^{1/bd})^{ad}(x^{1/bd})^{bc} && \text{(by Prop. I.5.6.3)} \\
&= x^{ad/bd}x^{bc/bd} && \text{(by Def. I.5.6.7)} \\
&= x^{a/b}x^{c/d} && \text{(by Lem. I.5.6.8)} \\
&= x^q x^r
\end{aligned}$$

and

$$\begin{aligned}
x^{qr} &= x^{ac/bd} && \text{(by Def. I.4.2.2)} \\
&= (x^{1/bd})^{ac} && \text{(by Def. I.5.6.7)} \\
&= \left((x^{1/b})^{1/d}\right)^{ac} && \text{(by Lem. I.5.6.6(g))}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\left(\left((x^{1/b})^a\right)^{1/a}\right)^{1/d}\right)^{ac} && \text{(by Lem. I.5.6.6(a)(b))} \\
&= \left(\left((x^{a/b})^{1/a}\right)^{1/d}\right)^{ac} && \text{(by Def. I.5.6.7)} \\
&= \left((x^{a/b})^{1/ad}\right)^{ac} && \text{(by Lem. I.5.6.6(g))} \\
&= (x^{a/b})^{ac/ad} && \text{(by Def. I.5.6.7)} \\
&= (x^{a/b})^{c/d} && \text{(by Lem. I.5.6.8)} \\
&= (x^q)^r.
\end{aligned}
$$

□

Proof of Lem. I.5.6.9(c). Let $q = a/b$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. Then we have

$$
\begin{aligned}
x^{-q} &= x^{-a/b} \\
&= (x^{1/b})^{-a} && \text{(by Def. I.5.6.7)} \\
&= 1/(x^{1/b})^a && \text{(by Prop. I.5.6.3)} \\
&= 1/x^q. && \text{(by Def. I.5.6.7)}
\end{aligned}
$$

□

Proof of Lem. I.5.6.9(d). Let $q = a/b$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. Then we have

$$
\begin{aligned}
& x > y \\
\implies & x^{1/b} > y^{1/b} && \text{(by Lem. I.5.6.6(d))} \\
\implies & (x^{1/b})^a > (y^{1/b})^a && \text{(by Prop. I.5.6.3)} \\
\implies & x^q > y^q && \text{(by Def. I.5.6.7)}
\end{aligned}
$$

and

$$
\begin{aligned}
& x^q > y^q \\
\implies & (x^{1/b})^a > (y^{1/b})^a && \text{(by Def. I.5.6.7)} \\
\implies & \left((x^{1/b})^a\right)^{1/a} > \left((y^{1/b})^a\right)^{1/a} && \text{(by Lem. I.5.6.6(d))} \\
\implies & x^{1/b} > y^{1/b} && \text{(by Lem. I.5.6.6(a)(b))} \\
\implies & x > y. && \text{(by Lem. I.5.6.6(d))}
\end{aligned}
$$

Thus, we conclude that $x > y \iff x^q > y^q$ when $x, y \in \mathbb{R}^+$ and $q \in \mathbb{Q}^+$. □

Proof of Lem. I.5.6.9(e). Let $q = a/b$ and $r = c/d$ where $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{Z}^+$. First, suppose that $x > 1$ and $x^q > x^r$. Then we have

$$\begin{aligned}
& x^q > x^r \\
\implies & x^q x^{-r} > x^r x^{-r} && \text{(by Lem. I.5.6.9(a))} \\
\implies & x^{q-r} > x^{r-r} = x^0 = 1 && \text{(by Lem. I.5.6.9(b))} \\
\implies & x^{(ad-bc)/bd} > 1 && \text{(by Def. I.4.2.2)} \\
\implies & \left(x^{(ad-bc)/bd}\right)^{bd} > 1^{bd} = 1 && \text{(by Prop. I.5.6.3)} \\
\implies & x^{(ad-bc)} > 1 = 1^{(ad-bc)} && \text{(by Lem. I.5.6.9(b))} \\
\implies & ad - bc > 0 && \text{(by Prop. I.4.3.12(b))} \\
\implies & ad > bc && \text{(by Lem. I.4.1.11(b))} \\
\implies & a/b > c/d && \text{(by Prop. I.4.2.9(e))} \\
\implies & q > r.
\end{aligned}$$

Now suppose that $x > 1$ and $q > r$. Then we have

$$\begin{aligned}
& q > r \\
\implies & q - r > 0 && \text{(by Prop. I.4.2.9)} \\
\implies & x^{q-r} > 1^{q-r} && \text{(by Lem. I.5.6.9(d))} \\
\implies & x^{q-r} > 1^{(ad-bc)/bd} && \text{(by Def. I.4.2.2)} \\
\implies & x^{q-r} > (1^{1/bd})^{ad-bc} && \text{(by Def. I.5.6.7)} \\
\implies & x^{q-r} > 1^{ad-bc} = 1 && \text{(by Lem. I.5.6.6(e))} \\
\implies & x^{q-r} x^r > x^r && \text{(by Lem. I.5.6.9(a))} \\
\implies & x^q > x^r. && \text{(by Lem. I.5.6.9(b))}
\end{aligned}$$

Thus, we conclude that if $x > 1$, then $x^q > x^r \iff q > r$.

Next suppose that $x < 1$ and $x^q > x^r$. Then we have

$$\begin{aligned}
& x^q > x^r \\
\implies & x^q x^{-r} > x^r x^{-r} && \text{(by Lem. I.5.6.9(a))} \\
\implies & x^{q-r} > x^{r-r} = x^0 = 1 && \text{(by Lem. I.5.6.9(b))} \\
\implies & x^{(ad-bc)/bd} > 1 && \text{(by Def. I.4.2.2)} \\
\implies & \left(x^{(ad-bc)/bd}\right)^{bd} > 1^{bd} = 1 && \text{(by Prop. I.5.6.3)} \\
\implies & x^{(ad-bc)} > 1 = 1^{(ad-bc)} && \text{(by Lem. I.5.6.9(b))} \\
\implies & ad - bc < 0 && \text{(by Prop. I.4.3.12(b))} \\
\implies & ad < bc && \text{(by Lem. I.4.1.11(b))}
\end{aligned}$$

$$\begin{aligned}
&\implies a/b < c/d && \text{(by Prop. I.4.2.9(e))}\\
&\implies q < r.
\end{aligned}$$

Finally suppose that $x < 1$ and $q < r$. Then we have

$$\begin{aligned}
&q < r\\
\implies\ & r - q > 0 && \text{(by Prop. I.4.2.9)}\\
\implies\ & x^{r-q} < 1^{r-q} && \text{(by Lem. I.5.6.9(d))}\\
\implies\ & x^{r-q} < 1^{(bc-ad)/bd} && \text{(by Def. I.4.2.2)}\\
\implies\ & x^{r-q} < (1^{1/bd})^{bc-ad} && \text{(by Def. I.5.6.7)}\\
\implies\ & x^{r-q} < 1^{bc-ad} = 1 && \text{(by Lem. I.5.6.6(e))}\\
\implies\ & x^{r-q}x^q < x^q && \text{(by Lem. I.5.6.9(a))}\\
\implies\ & x^r < x^q. && \text{(by Lem. I.5.6.9(b))}
\end{aligned}$$

Thus, we conclude that if $x < 1$, then $x^q > x^r \iff q < r$. □

Proof of Lem. I.5.6.9(f). Let $q = a/b$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. Then we have

$$\begin{aligned}
(xy)^q &= \left((xy)^{1/b}\right)^a && \text{(by Def. I.5.6.7)}\\
&= (x^{1/b}y^{1/b})^a && \text{(by Lem. I.5.6.6(f))}\\
&= (x^{1/b})^a(y^{1/b})^a && \text{(by Prop. I.5.6.3)}\\
&= x^q y^q. && \text{(by Def. I.5.6.7)}
\end{aligned}$$

□

— Exercises —

Ex. I.5.6.1. Prove Lem. I.5.6.6.

Proof of Ex. I.5.6.1. See Lem. I.5.6.6. □

Ex. I.5.6.2. Prove Lem. I.5.6.9.

Proof of Ex. I.5.6.2. See Lem. I.5.6.9. □

Ex. I.5.6.3. If x is a real number, show that $|x| = (x^2)^{1/2}$.

Proof of Ex. I.5.6.3. By Prop. I.5.4.7(a) exactly one of the following three statements is true:

- $x > 0$. Then by Def. I.5.4.5 we have $|x| = x$ and by Lem. I.5.6.6(a)(b) we have $|x| = x = (x^2)^{1/2}$.

- $x = 0$. Then by Def. I.5.4.5 we have $|0| = 0$ and by Lem. I.5.6.6(c) we have $|0| = 0 = (0^2)^{1/2}$.

- $x < 0$. Then by Def. I.5.4.5 we have $|x| = -x > 0$. By Lem. I.5.6.6(b)(c) we have $-x = ((-x)^2)^{1/2}$. But by Prop. I.5.3.11 we know that $(-x)^2 = (-x)(-x) = x^2$. Thus, we have $|x| = -x = (x^2)^{1/2}$.

From all cases above, we conclude that $|x| = (x^2)^{1/2}$. $\square$

Chapter I.6

Limits of sequences

I.6.1 Convergence and limit laws

Def. I.6.1.1 (Distance between two real numbers)**.** Given two real numbers x and y, we define their distance $d(x, y)$ to be $d(x, y) \coloneqq |x - y|$.

Note. Clearly, Def. I.6.1.1 is consistent with Def. I.4.3.2. Further, Prop. I.4.3.3 works just as well for real numbers as it does for rationals, because the real numbers obey all the rules of algebra that the rationals do.

Def. I.6.1.2 (ε-close real numbers)**.** Let $\varepsilon \in \mathbb{R}_{\geq 0}$. We say that two real numbers x, y are *ε-close* iff we have $d(y, x) \leq \varepsilon$.

Note. Again, it is clear that Def. I.6.1.2 is consistent with Def. I.4.3.4.

Note. Now let $(a_n)_{n=m}^{\infty}$ be a sequence of *real* numbers; i.e., we assign a real number a_n for every integer $n \geq m$. The starting index m is some integer; usually this will be 1, but in some cases we will start from some index other than 1. (The choice of label used to index this sequence is unimportant; we could use for instance $(a_k)_{k=m}^{\infty}$ and this would represent exactly the same sequence as $(a_n)_{n=m}^{\infty}$.) We can define the notion of a Cauchy sequence in the same manner as before.

Def. I.6.1.3 (Cauchy sequences of reals)**.** Let $\varepsilon \in \mathbb{R}^+$. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers starting at some integer index N is said to be *ε-steady* iff a_j and a_k are ε-close for every $j, k \in \mathbb{Z}_{\geq N}$. A sequence $(a_n)_{n=m}^{\infty}$ starting at some integer index m is said to be *eventually ε-steady* iff there exists an $N \in \mathbb{Z}_{\geq m}$ such that $(a_n)_{n=N}^{\infty}$ is ε-steady. We say that $(a_n)_{n=m}^{\infty}$ is a *Cauchy sequence* iff it is eventually ε-steady for every $\varepsilon \in \mathbb{R}^+$.

Note. To put it another way, a sequence $(a_n)_{n=m}^{\infty}$ of real numbers is a Cauchy sequence if, for every $\varepsilon \in \mathbb{R}^+$, there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|a_n - a_{n'}| \leq \varepsilon$ for all $n, n' \in \mathbb{Z}_{\geq N}$. These definitions are consistent with the corresponding definitions for rational numbers (Def. I.5.1.3, I.5.1.6 and I.5.1.8), although verifying consistency for Cauchy sequences takes a little bit of care.

Prop. I.6.1.4. Let $(a_n)_{n=m}^{\infty}$ be a sequence of rational numbers starting at some integer index m. Then $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of Def. I.5.1.8 iff it is a Cauchy sequence in the sense of Def. I.6.1.3.

Proof of Prop. I.6.1.4. Suppose first that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of Def. I.6.1.3; then it is eventually ε-steady for every $\varepsilon \in \mathbb{R}^+$. In particular, it is eventually ε-steady for every $\varepsilon \in \mathbb{Q}^+$, which makes it a Cauchy sequence in the sense of Def. I.5.1.8.

Now suppose that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of Def. I.5.1.8; then it is eventually ε'-steady for every $\varepsilon' \in \mathbb{Q}^+$. If $\varepsilon \in \mathbb{R}^+$, then there exists an $\varepsilon' \in \mathbb{Q}^+$ which is smaller than ε, by Prop. I.5.4.12. Since ε' is rational, we know that $(a_n)_{n=m}^{\infty}$ is eventually ε'-steady; since $\varepsilon' < \varepsilon$, this implies that $(a_n)_{n=m}^{\infty}$ is eventually ε-steady. Since ε is an arbitrary positive real number, we thus see that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence in the sense of Def. I.6.1.3. □

Note. Because of Prop. I.6.1.4, we will no longer care about the distinction between Def. I.5.1.8 and Def. I.6.1.3, and view the concept of a Cauchy sequence as a single unified concept.

Def. I.6.1.5 (Convergence of sequences)**.** Let $\varepsilon \in \mathbb{R}^+$, and let $L \in \mathbb{R}$. A sequence $(a_n)_{n=N}^{\infty}$ of real numbers is said to be *ε-close to L* iff a_n is ε-close to L for every $n \in \mathbb{Z}_{\geq N}$, i.e., we have $|a_n - L| \leq \varepsilon$ for every $n \in \mathbb{Z}_{\geq N}$. We say that a sequence $(a_n)_{n=m}^{\infty}$ is *eventually ε-close to L* iff there exists an $N \in \mathbb{Z}_{\geq m}$ such that $(a_n)_{n=N}^{\infty}$ is ε-close to L. We say that a sequence $(a_n)_{n=m}^{\infty}$ *converges to L* iff it is eventually ε-close to L for every $\varepsilon \in \mathbb{R}^+$.

Prop. I.6.1.7 (Uniqueness of limits)**.** Let $(a_n)_{n=m}^{\infty}$ be a real sequence starting at some integer index m, and let $L \neq L'$ be two distinct real numbers. Then it is not possible for $(a_n)_{n=m}^{\infty}$ to converge to L while also converging to L'.

Proof of Prop. I.6.1.7. Suppose for the sake of contradiction that $(a_n)_{n=m}^{\infty}$ was converging to both L and L'. Let $\varepsilon = \left|L - L'\right|/3$. Note that ε is positive since $L \neq L'$. Since $(a_n)_{n=m}^{\infty}$ converges to L, we know that $(a_n)_{n=m}^{\infty}$ is eventually ε-close to L; thus there is an $N \in \mathbb{Z}_{\geq m}$ such that $d(a_n, L) \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$. Similarly, there is an $M \in \mathbb{Z}_{\geq m}$ such that $d(a_n, L') \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq M}$. In particular, if we set $n := \max(N, M)$, then we have $d(a_n, L) \leq \varepsilon$ and $d(a_n, L') \leq \varepsilon$, hence by the triangle inequality $d(L, L') \leq 2\varepsilon = 2\left|L - L'\right|/3$. But then we have $\left|L - L'\right| \leq 2\left|L - L'\right|/3$, which contradicts the fact that $\left|L - L'\right| > 0$. Thus, it is not possible to converge to both L and L'. □

Def. I.6.1.8 (Limits of sequences)**.** If a sequence $(a_n)_{n=m}^{\infty}$ converges to some real number L, we say that $(a_n)_{n=m}^{\infty}$ is *convergent* and that its *limit* is L; we write

$$L = \lim_{n \to \infty} a_n$$

to denote this fact. If a sequence $(a_n)_{n=m}^{\infty}$ is not converging to any real number L, we say that the sequence $(a_n)_{n=m}^{\infty}$ is *divergent* and we leave $\lim\limits_{n \to \infty} a_n$ undefined.

Note. Prop. I.6.1.7 ensures that a sequence can have at most one limit. Thus, if the limit exists, it is a single real number, otherwise it is undefined.

Rmk. I.6.1.9. The notation $\lim_{n\to\infty} a_n$ does not give any indication about the starting index m of the sequence, but the starting index is irrelevant (Ex. I.6.1.3). Thus, in the rest of this discussion we shall not be too careful as to where these sequences start, as we shall be mostly focused on their limits.

Note. We sometimes use the phrase "$a_n \to x$ as $n \to \infty$" as an alternate way of writing the statement "$(a_n)_{n=m}^\infty$ converges to x." Bear in mind, though, that the individual statements $a_n \to x$ and $n \to \infty$ do not have any rigorous meaning; this phrase is just a convention, though of course a very suggestive one.

Rmk. I.6.1.10. The exact choice of letter used to denote the index (in this case n) is irrelevant: the phrase $\lim_{n\to\infty} a_n$ has exactly the same meaning as $\lim_{k\to\infty} a_k$, for instance. Sometimes it will be convenient to change the label of the index to avoid conflicts of notation; for instance, we might want to change n to k because n is simultaneously being used for some other purpose, and we want to reduce confusion. See Ex. I.6.1.4.

Prop. I.6.1.11. We have $\lim_{n\to\infty} 1/n = 0$.

Proof of Prop. I.6.1.11. We have to show that the sequence $(a_n)_{n=1}^\infty$ converges to 0, where $a_n := 1/n$. In other words, for every $\varepsilon \in \mathbb{R}^+$, we need to show that the sequence $(a_n)_{n=1}^\infty$ is eventually ε-close to 0. So, let $\varepsilon \in \mathbb{R}^+$ be an arbitrary real number. We have to find an $N \in \mathbb{Z}^+$ such that $|a_n - 0| \le \varepsilon$ for every $n \in \mathbb{Z}_{\ge N}$. But if $n \in \mathbb{Z}_{\ge N}$, then

$$|a_n - 0| = |1/n - 0| = 1/n \le 1/N.$$

Thus, if we pick $N > 1/\varepsilon$ (which we can do by the Archimedean principle, Cor. I.5.4.13), then $1/N < \varepsilon$, and so $(a_n)_{n=N}^\infty$ is ε-close to 0. Thus, $(a_n)_{n=1}^\infty$ is eventually ε-close to 0. Since ε was arbitrary, $(a_n)_{n=1}^\infty$ converges to 0. □

Prop. I.6.1.12 (Convergent sequences are Cauchy)**.** Suppose that $(a_n)_{n=m}^\infty$ is a convergent sequence of real numbers. Then $(a_n)_{n=m}^\infty$ is also a Cauchy sequence.

Proof of Prop. I.6.1.12. Suppose that $(a_n)_{n=m}^\infty$ converges to $L \in \mathbb{R}$. Let $\varepsilon \in \mathbb{R}^+$. Since $\lim_{n\to\infty} a_n = L$, by Def. I.6.1.5 there exists an $N \in \mathbb{Z}_{\ge m}$ such that $(a_n)_{n=N}^\infty$ is $\varepsilon/2$-close to L. This means $|a_n - L| \le \varepsilon/2$ for all $n \in \mathbb{Z}_{\ge N}$. Then we have

$$\begin{aligned} \forall j, k \in \mathbb{Z}_{\ge N}, |a_j - a_k| &\le |a_j - L| + |a_k - L| && \text{(by A.Cor. I.5.4.1(f)(g))} \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. && \text{(by Prop. I.5.4.7(c)(d))} \end{aligned}$$

Thus, $(a_n)_{n=N}^\infty$ is ε-steady and $(a_n)_{n=m}^\infty$ is eventually ε-steady. Since ε was arbitrary, by Def. I.6.1.3 we see that $(a_n)_{n=m}^\infty$ is a Cauchy sequence of reals. □

Rmk. I.6.1.14. For a converse to Prop. I.6.1.12, see Thm. I.6.4.18 below.

Prop. I.6.1.15 (Formal limits are genuine limits)**.** Suppose that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence of rational numbers. Then $(a_n)_{n=m}^{\infty}$ converges to $\operatorname*{LIM}_{n\to\infty} a_n$, i.e.

$$\operatorname*{LIM}_{n\to\infty} a_n = \lim_{n\to\infty} a_n.$$

Proof of Prop. I.6.1.15. Let $L = \operatorname*{LIM}_{n\to\infty} a_n$. By Def. I.5.3.1 we know that $L \in \mathbb{R}$. Thus, by Prop. I.6.1.4 and Def. I.6.1.5 we can ask whether $(a_n)_{n=m}^{\infty}$ converges to L. Suppose for the sake of contradiction that $(a_n)_{n=m}^{\infty}$ does not converge to L. Then there must exist an $\varepsilon \in \mathbb{R}^+$ such that $(a_n)_{n=m}^{\infty}$ is not eventually ε-close to L. Fix such ε.

Since $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence of reals, we know that there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|a_j - a_k| \leq \frac{\varepsilon}{4}$ for all $j, k \in \mathbb{Z}_{\geq N}$. Fix such N. Since $(a_n)_{n=m}^{\infty}$ is not eventually ε-close to L, we know that $(a_n)_{n=m}^{\infty}$ is not eventually ε'-close to L for every $\varepsilon' \in \mathbb{R}_{0<\varepsilon}$. So $(a_n)_{n=m}^{\infty}$ is not eventually $\frac{2\varepsilon}{3}$-close to L, and we can find a $j \in \mathbb{Z}_{\geq N}$ such that $|a_j - L| > \frac{2\varepsilon}{3}$. Similarly, $(a_n)_{n=m}^{\infty}$ is not eventually $\frac{\varepsilon}{3}$-close to L, and we can find a $k \in \mathbb{Z}_{\geq N}$ such that $|a_k - L| > \frac{\varepsilon}{3}$. Fix such j and k. But then we have

$$\begin{aligned}
\frac{\varepsilon}{4} &\geq |a_j - a_k| && \text{(by Def. I.6.1.3)}\\
&\geq |a_j - L| - |a_k - L| && \text{(by Prop. I.4.3.3(f)(g))}\\
&\geq \frac{2\varepsilon}{3} - \frac{\varepsilon}{3} = \frac{\varepsilon}{3},
\end{aligned}$$

a contradiction. Thus, such ε does not exist. Therefore we must have $\lim_{n\to\infty} a_n = L = \operatorname*{LIM}_{n\to\infty} a_n$. □

Def. I.6.1.16 (Bounded sequences)**.** A sequence $(a_n)_{n=m}^{\infty}$ of reals is *bounded by* a real number $M \in \mathbb{R}_{\geq 0}$ iff we have $|a_n| \leq M$ for all $n \in \mathbb{Z}_{\geq m}$. We say that $(a_n)_{n=m}^{\infty}$ is bounded iff it is *bounded* by M for some real number $M \in \mathbb{R}_{\geq 0}$.

Note. Def. I.6.1.16 is consistent with Def. I.5.1.12. See Ex. I.6.1.7.

Cor. I.6.1.17. Every convergent sequence of real numbers is bounded.

Proof of Cor. I.6.1.17. Recall from Lem. I.5.1.15 that every Cauchy sequence of rationals is bounded. An inspection of the proof of Lem. I.5.1.15 shows that the same argument works for reals; every Cauchy sequence of reals is bounded. From Prop. I.6.1.12 we see that every convergent sequence of reals is a Cauchy sequence. Thus, every convergent sequence of reals is bounded. □

A.Cor. I.6.1.1. Let $x, y \in \mathbb{R}$. Then we have $\min(x, y) = -\max(-x, -y)$. Similarly, we have $\max(x, y) = -\min(-x, -y)$.

Proof of A.Cor. I.6.1.1. First, we show that $\min(x, y) = -\max(-x, -y)$. We split into two cases:

- If $x \le y$, then we have $-x \ge -y$ by Ex. I.4.2.6. Thus,

$$\begin{aligned}
\min(x, y) &= x \\
&= -(-x) && \text{(by A.Cor. I.5.3.3)} \\
&= -\max(-x, -y). && (-x \ge -y)
\end{aligned}$$

- If $x > y$, then we have $-x < -y$ by Ex. I.4.2.6. Thus,

$$\begin{aligned}
\min(x, y) &= y \\
&= -(-y) && \text{(by A.Cor. I.5.3.3)} \\
&= -\max(-x, -y). && (-x < -y)
\end{aligned}$$

From all cases above, we see that $\min(x, y) = -\max(-x, -y)$. Thus, we conclude that $\min(x, y) = -\max(-x, -y)$.

Now we show that $\max(x, y) = -\min(-x, -y)$. This is true since

$$\begin{aligned}
\max(x, y) &= -(-\max(-(-x), -(-y))) && \text{(by A.Cor. I.5.3.3)} \\
&= -\min(-x, -y). && \text{(from the proof above)}
\end{aligned}$$

□

Thm. I.6.1.19 (Limit Laws)**.** Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be convergent sequences of real numbers, and let x, y be the real numbers $x := \lim_{n\to\infty} a_n$ and $y := \lim_{n\to\infty} b_n$.

(a) The sequence $(a_n + b_n)_{n=m}^{\infty}$ converges to $x + y$; in other words,

$$\lim_{n\to\infty} (a_n + b_n) = \Big(\lim_{n\to\infty} a_n\Big) + \Big(\lim_{n\to\infty} b_n\Big).$$

(b) The sequence $(a_n b_n)_{n=m}^{\infty}$ converges to xy; in other words,

$$\lim_{n\to\infty} (a_n b_n) = \Big(\lim_{n\to\infty} a_n\Big)\Big(\lim_{n\to\infty} b_n\Big).$$

(c) For any real number c, the sequence $(ca_n)_{n=m}^{\infty}$ converges to cx; in other words,

$$\lim_{n\to\infty} (ca_n) = c\Big(\lim_{n\to\infty} a_n\Big).$$

(d) The sequence $(a_n - b_n)_{n=m}^{\infty}$ converges to $x - y$; in other words,

$$\lim_{n\to\infty} (a_n - b_n) = \Big(\lim_{n\to\infty} a_n\Big) - \Big(\lim_{n\to\infty} b_n\Big).$$

(e) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(b_n^{-1})_{n=m}^{\infty}$ converges to y^{-1}; in other words,

$$\lim_{n\to\infty} b_n^{-1} = \left(\lim_{n\to\infty} b_n\right)^{-1}.$$

(f) Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \in \mathbb{Z}_{\geq m}$. Then the sequence $(a_n/b_n)_{n=m}^{\infty}$ converges to x/y; in other words,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

(g) The sequence $(\max(a_n, b_n))_{n=m}^{\infty}$ converges to $\max(x, y)$; in other words,

$$\lim_{n\to\infty} \max(a_n, b_n) = \max\left(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right).$$

(h) The sequence $(\min(a_n, b_n))_{n=m}^{\infty}$ converges to $\min(x, y)$; in other words,

$$\lim_{n\to\infty} \min(a_n, b_n) = \min\left(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right).$$

Proof of Thm. I.6.1.19(a). Let $\varepsilon \in \mathbb{R}^+$. By Def. I.6.1.8, there exists an $N_a \in \mathbb{Z}_{\geq m}$ such that $|a_n - x| \leq \varepsilon/2$ for every $n \in \mathbb{Z}_{\geq N_a}$. Similarly, there exists an $N_b \in \mathbb{Z}_{\geq m}$ such that $|b_n - y| \leq \varepsilon/2$ for every $n \in \mathbb{Z}_{\geq N_b}$. Now we fix both N_a and N_b. Let $N = \max(N_a, N_b)$. Then we have

$$\begin{aligned} \forall n \in \mathbb{Z}_{\geq N}, |(a_n + b_n) - (x + y)| &= |(a_n - x) + (b_n - y)| && \text{(by A.Cor. I.5.3.3)} \\ &\leq |a_n - x| + |b_n - y| && \text{(by A.Cor. I.5.4.1)} \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon. && \text{(by Prop. I.5.4.7(c)(d))} \end{aligned}$$

Thus, by Def. I.6.1.5, $(a_n + b_n)_{n=N}^{\infty}$ is ε-close to $x + y$, and $(a_n + b_n)_{n=m}^{\infty}$ is eventually ε-close to $x + y$. Since ε was arbitrary, by Def. I.6.1.5 again we know that $(a_n + b_n)_{n=m}^{\infty}$ converges to $x + y$. □

Proof of Thm. I.6.1.19(b). Since $x = \lim_{n\to\infty} a_n$ and $y = \lim_{n\to\infty} b_n$, by Cor. I.6.1.17, there exist some $A, B \in \mathbb{R}_{\geq 0}$ such that $|a_n| \leq A$ and $|b_n| \leq B$ for all $n \in \mathbb{Z}_{\geq m}$. Fix both A and B. Clearly, we have $A < |x| + A + 1$ and $B < B + 1$, so we must have $|a_n| \leq |x| + A + 1$ and $|b_n| \leq B + 1$ for all $n \in \mathbb{Z}_{\geq m}$.

Let $\varepsilon \in \mathbb{R}^+$. Observe that $\dfrac{\varepsilon}{2(|x| + A + 1)} \in \mathbb{R}^+$ and $\dfrac{\varepsilon}{2(B + 1)} \in \mathbb{R}^+$. Since $x = \lim_{n\to\infty} a_n$, by Def. I.6.1.8, there exists an $N_a \in \mathbb{Z}_{\geq m}$ such that $|a_n - x| \leq \dfrac{\varepsilon}{2(B + 1)}$ for all $n \in \mathbb{Z}_{\geq N_a}$.

Similarly, since $y = \lim_{n\to\infty} b_n$, there exists an $N_b \in \mathbb{Z}_{\geq m}$ such that $|b_n - y| \leq \dfrac{\varepsilon}{2(|x| + A + 1)}$ for all $n \in \mathbb{Z}_{\geq N_b}$. Now we fix both N_a and N_b. Let $N = \max(N_a, N_b)$. Then we have

$$\begin{aligned}
\forall n \in \mathbb{Z}_{\geq N}, |a_n b_n - xy| &= |a_n b_n - xy + xb_n - xb_n| && \text{(by A.Cor. I.5.3.3)}\\
&= |b_n(a_n - x) + x(b_n - y)| && \text{(by A.Cor. I.5.3.3)}\\
&\leq |b_n(a_n - x)| + |x(b_n - y)| && \text{(by A.Cor. I.5.4.1)}\\
&= |b_n||a_n - x| + |x||b_n - y| && \text{(by A.Cor. I.5.4.1)}\\
&\leq (B+1) \times \frac{\varepsilon}{2(B+1)} + |x||b_n - y| && \text{(by Prop. I.5.4.7(c)(d)(e))}\\
&\leq \frac{\varepsilon}{2} + (|x| + A + 1) \times \frac{\varepsilon}{2(|x| + A + 1)} && \text{(by Prop. I.5.4.7(c)(d)(e))}\\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus, by Def. I.6.1.5, $(a_n b_n)_{n=N}^{\infty}$ is ε-close to xy, and $(a_n b_n)_{n=m}^{\infty}$ is eventually ε-close to xy. Since ε was arbitrary, by Def. I.6.1.5 again we know that $(a_n b_n)_{n=m}^{\infty}$ converges to xy. □

Proof of Thm. I.6.1.19(c). Let $(c_n)_{n=m}^{\infty}$ be a sequence of reals where $c_n = c$ for all $n \in \mathbb{Z}_{\geq m}$. Clearly, we have $\lim_{n\to\infty} c_n = \lim_{n\to\infty} c = c$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} (ca_n) &= \lim_{n\to\infty} (c_n a_n)\\
&= \Big(\lim_{n\to\infty} c_n\Big)\Big(\lim_{n\to\infty} a_n\Big) && \text{(by Thm. I.6.1.19(b))}\\
&= c\Big(\lim_{n\to\infty} a_n\Big).
\end{aligned}$$

□

Proof of Thm. I.6.1.19(d). We have

$$\begin{aligned}
\lim_{n\to\infty} (a_n - b_n) &= \lim_{n\to\infty} (a_n + (-1)(b_n)) && \text{(by A.Cor. I.5.3.2)}\\
&= \Big(\lim_{n\to\infty} a_n\Big) + \Big(\lim_{n\to\infty} ((-1)(b_n))\Big) && \text{(by Thm. I.6.1.19(a))}\\
&= \Big(\lim_{n\to\infty} a_n\Big) + \Big((-1)\Big(\lim_{n\to\infty} b_n\Big)\Big) && \text{(by Thm. I.6.1.19(c))}\\
&= \Big(\lim_{n\to\infty} a_n\Big) - \Big(\lim_{n\to\infty} b_n\Big). && \text{(by A.Cor. I.5.3.2)}
\end{aligned}$$

□

Proof of Thm. I.6.1.19(e). First, we show that $(b_n)_{n=m}^{\infty}$ is bounded away from zero. Since $y \neq 0$, we know that $|y| > 0$. Since $y = \lim_{n\to\infty} b_n$, we know that there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|b_n - y| \leq \dfrac{|y|}{2}$ for all $n \in \mathbb{Z}_{\geq N}$. Then we have

$$\forall n \in \mathbb{Z}_{\geq N}, \frac{-|y|}{2} \leq b_n - y \leq \frac{|y|}{2} \qquad \text{(by A.Cor. I.5.4.1)}$$

$$\implies \forall n \in \mathbb{Z}_{\geq N}, \begin{cases} \dfrac{y}{2} \leq b_n \leq \dfrac{3y}{2} & \text{if } y \in \mathbb{R}^+ \\ \dfrac{3y}{2} \leq b_n \leq \dfrac{y}{2} & \text{if } y \in \mathbb{R}^- \end{cases} \qquad \text{(by Prop. I.5.4.7(c)(d))}$$

$$\implies \forall n \in \mathbb{Z}_{\geq N}, \begin{cases} \dfrac{y}{2} \leq b_n & \text{if } y \in \mathbb{R}^+ \\ \dfrac{-y}{2} \leq -b_n & \text{if } y \in \mathbb{R}^- \end{cases} \qquad \text{(by Ex. I.4.2.6)}$$

$$\implies \forall n \in \mathbb{Z}_{\geq N}, \left|\frac{y}{2}\right| \leq |b_n|. \qquad \text{(by A.Cor. I.5.4.1)}$$

Since $y \neq 0$, we see that $\left|\frac{y}{2}\right| \in \mathbb{R}^+$. Thus, $(b_n)_{n=N}^{\infty}$ is bounded away from zero. Since $b_n \neq 0$ for all $n \in \mathbb{Z}_{\geq m}$, we see that $(b_n)_{n=m}^{N-1}$ is also bounded away from zero. Combining the results we see that $(b_n)_{n=m}^{\infty}$ is bounded away from zero.

Now we show that $\lim_{n\to\infty} b_n^{-1} = y^{-1}$. Let $\varepsilon \in \mathbb{R}^+$. Since $(b_n)_{n=m}^{\infty}$ is bounded away from zero, there exists an $M \in \mathbb{R}^+$ such that $|b_n| \geq M$ for all $n \in \mathbb{Z}_{\geq m}$. Fix such M. Clearly, we have $\varepsilon M|y| \in \mathbb{R}^+$ and $\frac{1}{|b_n|} \leq \frac{1}{M}$ for all $n \in \mathbb{Z}_{\geq m}$. Since $y = \lim_{n\to\infty} b_n \neq 0$, by Def. I.6.1.8, there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|b_n - y| \leq \varepsilon M|y|$ for all $n \in \mathbb{Z}_{\geq N}$. Fix such N. Then we have

$$\begin{aligned} \forall n \in \mathbb{Z}_{\geq N}, \left|b_n^{-1} - y^{-1}\right| &= \left|\frac{1}{b_n} - \frac{1}{y}\right| && \text{(by Def. I.5.6.2)} \\ &= \left|\frac{y - b_n}{b_n y}\right| && \text{(by A.Cor. I.5.3.3)} \\ &= |y - b_n| \frac{1}{|b_n||y|} && \text{(by A.Cor. I.5.4.1)} \\ &\leq |y - b_n| \frac{1}{M|y|} && \text{(by Prop. I.5.4.7(e))} \\ &\leq \varepsilon M|y| \frac{1}{M|y|} = \varepsilon. && \text{(by Prop. I.5.4.7(e))} \end{aligned}$$

Thus, by Def. I.6.1.5, $\left(b_n^{-1}\right)_{n=N}^{\infty}$ is ε-close to y^{-1}, and $\left(b_n^{-1}\right)_{n=m}^{\infty}$ is eventually ε-close to y^{-1}. Since ε was arbitrary, by Def. I.6.1.5 again we know that $\left(b_n^{-1}\right)_{n=m}^{\infty}$ converges to y^{-1}. □

Proof of Thm. I.6.1.19(f). We have

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left(a_n b_n^{-1}\right) && \text{(by A.Cor. I.5.3.5)} \\ &= \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n^{-1}\right) && \text{(by Thm. I.6.1.19(b))} \\ &= \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)^{-1} && \text{(by Thm. I.6.1.19(e))} \end{aligned}$$

$$= \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}. \qquad \text{(by A.Cor. I.5.3.5)}$$

□

Proof of Thm. I.6.1.19(g). First, suppose that $x = y$. Let $\varepsilon \in \mathbb{R}^+$. By Def. I.6.1.8, there exists an $N_a \in \mathbb{Z}_{\geq m}$ such that $|a_n - x| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N_a}$. Similarly, there exists an $N_b \in \mathbb{Z}_{\geq m}$ such that $|b_n - y| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N_b}$. Fix both N_a and N_b. Let $N = \max(N_a, N_b)$. Then we have $|a_n - x| \leq \varepsilon$ and $|b_n - y| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$. Since $x = y$, we have $\max(x, y) = x$. Thus,

$$\begin{aligned}
&\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} |a_n - \max(x,y)| = |a_n - x| \leq \varepsilon \\ |b_n - \max(x,y)| = |b_n - x| = |b_n - y| \leq \varepsilon \end{cases} \\
\implies &\forall n \in \mathbb{Z}_{\geq N}, |\max(a_n, b_n) - \max(x,y)| = \begin{cases} |a_n - x| \leq \varepsilon & \text{if } a_n \geq b_n \\ |b_n - x| \leq \varepsilon & \text{if } a_n < b_n \end{cases} \\
\implies &\forall n \in \mathbb{Z}_{\geq N}, |\max(a_n, b_n) - \max(x,y)| \leq \varepsilon.
\end{aligned}$$

By Def. I.6.1.5, $(\max(a_n, b_n))_{n=N}^{\infty}$ is ε-close to $\max(x, y)$, and $(\max(a_n, b_n))_{n=m}^{\infty}$ is eventually ε-close to $\max(x, y)$. Since ε was arbitrary, by Def. I.6.1.5 again we know that $(\max(a_n, b_n))_{n=m}^{\infty}$ converges to $\max(x, y)$.

Now suppose that $x \neq y$. We have either $x < y$ or $x > y$, so without the loss of generality, suppose that $x < y$. Then we have $\dfrac{y-x}{2} \in \mathbb{R}^+$. Let $\varepsilon \in \mathbb{R}^+$. Clearly, we have $\min\left(\varepsilon, \dfrac{y-x}{2}\right) \in \mathbb{R}^+$. By Def. I.6.1.8, there exists an $N_a \in \mathbb{Z}_{\geq m}$ such that $|a_n - x| \leq \min\left(\varepsilon, \dfrac{y-x}{2}\right)$ for all $n \in \mathbb{Z}_{\geq N_a}$. Similarly, there exists an $N_b \in \mathbb{Z}_{\geq m}$ such that $|b_n - y| \leq \min\left(\varepsilon, \dfrac{y-x}{2}\right)$ for all $n \in \mathbb{Z}_{\geq N_b}$. Fix both N_a and N_b. Let $N = \max(N_a, N_b)$. Then we have

$$\begin{aligned}
&\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} |a_n - x| \leq \min\left(\varepsilon, \dfrac{y-x}{2}\right) \leq \dfrac{y-x}{2} \\ |b_n - y| \leq \min\left(\varepsilon, \dfrac{y-x}{2}\right) \leq \dfrac{y-x}{2} \end{cases} && \text{(by Prop. I.5.4.7(c))} \\
\implies &\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} -\dfrac{y-x}{2} \leq a_n - x \leq \dfrac{y-x}{2} \\ -\dfrac{y-x}{2} \leq b_n - y \leq \dfrac{y-x}{2} \end{cases} && \text{(by A.Cor. I.5.4.1)} \\
\implies &\forall n \in \mathbb{Z}_{\geq N}, \begin{cases} a_n \leq \dfrac{y-x}{2} + x \\ y - \dfrac{y-x}{2} \leq b_n \end{cases} && \text{(by Prop. I.5.4.7(d))}
\end{aligned}$$

$$\implies \forall n \in \mathbb{Z}_{\geq N}, \begin{cases} a_n \leq \dfrac{x+y}{2} \\ \dfrac{x+y}{2} \leq b_n \end{cases} \qquad \text{(by A.Cor. I.5.3.3)}$$

$$\implies \forall n \in \mathbb{Z}_{\geq N}, a_n \leq \frac{x+y}{2} \leq b_n. \qquad \text{(by Prop. I.5.4.7(c))}$$

This means $\max(a_n, b_n) = b_n$ for all $n \in \mathbb{Z}_{\geq N}$. Thus,

$$\begin{aligned} \forall n \in \mathbb{Z}_{\geq N}, |\max(a_n, b_n) - \max(x, y)| &= |b_n - y| \\ &\leq \min\left(\varepsilon, \frac{y-x}{2}\right) \\ &\leq \varepsilon. && \text{(by Prop. I.5.4.7(c))} \end{aligned}$$

By Def. I.6.1.5, $(\max(a_n, b_n))_{n=N}^{\infty}$ is ε-close to $\max(x, y)$, and $(\max(a_n, b_n))_{n=m}^{\infty}$ is eventually ε-close to $\max(x, y)$. Since ε was arbitrary, by Def. I.6.1.5 again we know that $(\max(a_n, b_n))_{n=m}^{\infty}$ converges to $\max(x, y)$. □

Proof of Thm. I.6.1.19(h). We have

$$\begin{aligned} \lim_{n\to\infty} \min(a_n, b_n) &= \lim_{n\to\infty} -\max(-a_n, -b_n) && \text{(by A.Cor. I.6.1.1)} \\ &= -\left(\lim_{n\to\infty} \max(-a_n, -b_n)\right) && \text{(by Thm. I.6.1.19(c))} \\ &= -\max\left(\lim_{n\to\infty} -a_n, \lim_{n\to\infty} -b_n\right) && \text{(by Thm. I.6.1.19(g))} \\ &= -\max\left(-\left(\lim_{n\to\infty} a_n\right), -\left(\lim_{n\to\infty} b_n\right)\right) && \text{(by Thm. I.6.1.19(c))} \\ &= \min\left(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n\right). && \text{(by A.Cor. I.6.1.1)} \end{aligned}$$

□

— Exercises —

Ex. I.6.1.1. Let $(a_n)_{n=m}^{\infty}$ be a sequence of reals, such that $a_{n+1} > a_n$ for each $n \in \mathbb{Z}_{\geq m}$. Prove that whenever $j, k \in \mathbb{Z}_{\geq m}$ such that $j > k$, then we have $a_j > a_k$. (We refer to these sequences as *increasing* sequences.)

Proof of Ex. I.6.1.1. Let $E = \{z \in \mathbb{Z}_{\geq m} : j \leq z \leq k\}$. Then E is finite (since $\#(E) = k - j + 1$) and non-empty (since $j, k \in E$). So $(a_n)_{n=j}^{k}$ is a finite sequence, and the elements in $(a_n)_{n=j}^{k}$ are $\{a_j, a_{j+1}, \ldots, a_{k-1}, a_k\}$. By hypothesis, we have $a_{n+1} > a_n$ for each $n \in \mathbb{Z}_{\geq m}$. Thus, we have $a_j < a_{j+1} < \cdots < a_{k-1} < a_k$, and by Prop. I.5.4.7(c) we have $a_j < a_k$. □

Ex. I.6.1.2. Let $(a_n)_{n=m}^{\infty}$ be a sequence of reals, and let $L \in \mathbb{R}$. Show that $(a_n)_{n=m}^{\infty}$ converges to L iff, given any $\varepsilon \in \mathbb{R}^+$, one can find an $N \in \mathbb{Z}_{\geq m}$ such that $|a_n - L| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$.

Proof of Ex. I.6.1.2. We have

$$\begin{aligned} & (a_n)_{n=m}^{\infty} \text{ converges to } L \\ \iff & \forall \varepsilon \in \mathbb{R}^+, (a_n)_{n=m}^{\infty} \text{ is eventually } \varepsilon\text{-close to } L && \text{(by Def. I.6.1.5)} \\ \iff & \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}_{\geq m} : (a_n)_{n=N}^{\infty} \text{ is } \varepsilon\text{-close to } L && \text{(by Def. I.6.1.5)} \\ \iff & \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}_{\geq m} : \forall n \in \mathbb{Z}_{\geq N}, |a_n - L| \leq \varepsilon. && \text{(by Def. I.6.1.5)} \end{aligned}$$

□

Ex. I.6.1.3. Let $(a_n)_{n=m}^{\infty}$ be a sequence of reals, let $c \in \mathbb{R}$, and let $m' \in \mathbb{Z}_{\geq m}$. Show that $(a_n)_{n=m}^{\infty}$ converges to c iff $(a_n)_{n=m'}^{\infty}$ converges to c.

Proof of Ex. I.6.1.3. First, suppose that $(a_n)_{n=m}^{\infty}$ converges to c. Let $\varepsilon \in \mathbb{R}^+$. By Def. I.6.1.5, there exists an $N \in \mathbb{Z}_{\geq m}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$. Fix such N. Now we split into two cases:

- If $N \geq m'$, then we have found an $N \in \mathbb{Z}_{\geq m'}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$.
- If $N < m'$, then by setting $N' = m'$ we see that there exists an $N' \in \mathbb{Z}_{\geq m'}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N'}$.

From all cases above, we see that we can find an $M \in \mathbb{Z}_{\geq m'}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq M}$. Thus, by Def. I.6.1.5, we see that $(a_n)_{n=m'}^{\infty}$ is eventually ε-close to c. Since ε was arbitrary, by Def. I.6.1.5 again we see that $(a_n)_{n=m'}^{\infty}$ converges to c.

Now suppose that $(a_n)_{n=m'}^{\infty}$ converges to c. Let $\varepsilon \in \mathbb{R}^+$. By Def. I.6.1.5, there exists an $N \in \mathbb{Z}_{\geq m'}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$. Fix such N. Since $m \leq m'$, we have found an $N \in \mathbb{Z}_{\geq m}$ such that $|a_n - c| \leq \varepsilon$ for all $n \in \mathbb{Z}_{\geq N}$. Thus, by Def. I.6.1.5, we see that $(a_n)_{n=m}^{\infty}$ is eventually ε-close to c. Since ε was arbitrary, by Def. I.6.1.5 again we see that $(a_n)_{n=m}^{\infty}$ converges to c. □

Ex. I.6.1.4. Let $(a_n)_{n=m}^{\infty}$ be a sequence of reals, let $c \in \mathbb{R}$, and let $k \in \mathbb{Z}_{\geq 0}$. Show that $(a_n)_{n=m}^{\infty}$ converges to c iff $(a_{n+k})_{n=m}^{\infty}$ converges to c.

Proof of Ex. I.6.1.4. First, observe that $(a_{n+k})_{n=m}^{\infty} = (a_n)_{n=m+k}^{\infty}$. Since $m + k \geq m$, by Ex. I.6.1.3 we see that $(a_n)_{n=m}^{\infty}$ converges to c iff $(a_n)_{n=m+k}^{\infty}$ converges to c. Thus, $(a_n)_{n=m}^{\infty}$ converges to c iff $(a_{n+k})_{n=m}^{\infty}$ converges to c. □

Ex. I.6.1.5. Prove Prop. I.6.1.12.

Proof of Ex. I.6.1.5. See Prop. I.6.1.12. □

Ex. I.6.1.6. Prove Prop. I.6.1.15.

Proof of Ex. I.6.1.6. See Prop. I.6.1.15. □

Ex. I.6.1.7. Show that Def. I.6.1.16 is consistent with Def. I.5.1.12 (i.e., prove an analogue of Prop. I.6.1.4 for bounded sequences instead of Cauchy sequences).

Proof of Ex. I.6.1.7. First, suppose that $(a_n)_{n=m}^\infty$ is a sequence of reals which is bounded in the sense of Def. I.6.1.16. Then there exists an $M \in \mathbb{R}_{\geq 0}$ such that $|a_n| \leq M$ for all $n \in \mathbb{Z}_{\geq m}$. By Prop. I.5.4.12, there exists an $M' \in \mathbb{Z}^+$ such that $M \leq M'$. Clearly, $M' \in \mathbb{Z}^+$ implies $M' \in \mathbb{Q}_{\geq 0}$. Thus, by Prop. I.5.4.7(c), we have $|a_n| \leq M'$ for all $n \in \mathbb{Z}_{\geq m}$. This means that $(a_n)_{n=m}^\infty$ is a bounded sequence in the sense of Def. I.5.1.12.

Now suppose that $(a_n)_{n=m}^\infty$ is a sequence of reals which is bounded in the sense of Def. I.5.1.12. Then there exists an $M \in \mathbb{Q}_{\geq 0}$ such that $|a_n| \leq M$ for all $n \in \mathbb{Z}_{\geq m}$. Since M is also a real number, we see that $(a_n)_{n=m}^\infty$ is a bounded sequence in the sense of Def. I.6.1.16. □

Ex. I.6.1.8. Proof Thm. I.6.1.19.

Proof of Ex. I.6.1.8. See Thm. I.6.1.19. □

Ex. I.6.1.9. Explain why Thm. I.6.1.19(f) fails when the limit of the denominator is 0. (To repair that problem requires *L'Hôpital's rule*, see Sec. I.10.5.)

Proof of Ex. I.6.1.9. Suppose for the sake of contradiction that Thm. I.6.1.19(f) works when denominator is 0. Let $(a_n)_{n=1}^\infty = (1/n)_{n=1}^\infty$. Then we have

$$\lim_{n\to\infty} \frac{a_n}{a_n} = \lim_{n\to\infty} \frac{1/n}{1/n} = \lim_{n\to\infty} 1 = 1.$$

But by Prop. I.6.1.11 we also have

$$\frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} a_n} = \frac{0}{0}$$

which is undefined. Thus, Thm. I.6.1.19(f) fails when denominator is 0. □

Ex. I.6.1.10. Show that the concept of equivalent Cauchy sequence, as defined in Def. I.5.2.6, does not change if ε is required to be positive real instead of positive rational. More precisely, if $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are sequences of reals, show that $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are eventually ε-close for every $\varepsilon \in \mathbb{Q}^+$ iff they are eventually ε-close for every $\varepsilon \in \mathbb{R}^+$.

Proof of Ex. I.6.1.10. Suppose first that $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are eventually ε-close for all $\varepsilon \in \mathbb{Q}^+$. Let $\varepsilon' \in \mathbb{R}^+$. By Prop. I.5.4.12, there exists an $\varepsilon \in \mathbb{Q}^+$ such that $\varepsilon \leq \varepsilon'$. Fix such ε. Since $\varepsilon \in \mathbb{Q}^+$, by hypothesis we know that $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are eventually ε-close. This implies that $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are eventually ε'-close. Since ε' was arbitrary, we see that $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are eventually ε'-close for all $\varepsilon' \in \mathbb{R}^+$.

Now suppose that $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are eventually ε'-close for all $\varepsilon' \in \mathbb{R}^+$. This implies that $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are eventually ε-close for all $\varepsilon \in \mathbb{Q}^+$. Thus, we conclude that $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are eventually ε-close for all $\varepsilon \in \mathbb{Q}^+$ iff they are eventually ε'-close for all $\varepsilon' \in \mathbb{R}^+$. □

I.6.2 The Extended real number system

Def. I.6.2.1 (Extended real number system)**.** The *extended real number system* $\mathbb{R}^*$ is the real line $\mathbb{R}$ with two additional elements attached, called $+\infty$ and $-\infty$. These elements are distinct from each other and also distinct from every real number. An extended real number x is called *finite* iff it is a real number, and *infinite* iff it is equal to $+\infty$ or $-\infty$. (This definition is not directly related to the notion of finite and infinite sets in Sec. I.3.6, though it is of course similar in spirit.)

Def. I.6.2.2 (Negation of extended reals)**.** The operation of negation $x \to -x$ on $\mathbb{R}$, we now extend to $\mathbb{R}^*$ by defining $-(+\infty) := -\infty$ and $-(-\infty) := +\infty$.

Note. Thus, every extended real number x has a negation, and $-(-x)$ is always equal to x.

Def. I.6.2.3 (Ordering of extended reals)**.** Let x and y be extended real numbers. We say that $x \leq y$, i.e., x is less than or equal to y, iff one of the following three statements is true:

(a) x and y are real numbers, and $x \leq y$ as real numbers.

(b) $y = +\infty$.

(c) $x = -\infty$.

We say that $x < y$ if we have $x \leq y$ and $x \neq y$. We sometimes write $x < y$ as $y > x$, and $x \leq y$ as $y \geq x$.

Prop. I.6.2.5. Let x, y, z be extended real numbers. Then the following statements are true:

(a) (Reflexivity) We have $x \leq x$.

(b) (Trichotomy) Exactly one of the statements $x < y$, $x = y$, or $x > y$ is true.

(c) (Transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$.

(d) (Negation reverses order) If $x \leq y$, then $-y \leq -x$.

Proof. (a) By Prop. I.5.4.7 we already have $x \leq x$ when $x \in \mathbb{R}$. So we only need to consider the cases $x \in \{+\infty, -\infty\}$. By Def. I.6.2.3 we have $x \leq +\infty$ for every $x \in \mathbb{R}^*$. So we have $+\infty \leq +\infty$. Again by Def. I.6.2.3 we have $-\infty \leq x$ for every $x \in \mathbb{R}^*$. So we have $-\infty \leq -\infty$. Thus, we conclude that $x \leq x$ for every $x \in \mathbb{R}^*$. $\square$

Proof. (b) By Prop. I.5.4.7, we already have exactly one of the statements $x < y$, $x = y$, or $x > y$ is true when $x, y \in \mathbb{R}$. So we only need to consider the cases $x, y \in \{+\infty, -\infty\}$.

- If $x = +\infty$, then by Def. I.6.2.3 we have $x \geq y$ for every $y \in \mathbb{R}^*$.
  - If $y = +\infty$, then we have $x = y$.

- If $y \in \mathbb{R}$, then by Def. I.6.2.1 $x \neq y$. Thus, by Def. I.6.2.3 we have $x > y$.
- If $y = -\infty$, then by Def. I.6.2.1 $x \neq y$. Thus, by Def. I.6.2.3 we have $x > y$.

- If $x = -\infty$, then by Def. I.6.2.3 we have $x \leq y$ for every $y \in \mathbb{R}^*$.
  - If $y = +\infty$, then by Def. I.6.2.1 $x \neq y$. Thus, by Def. I.6.2.3 we have $x < y$.
  - If $y \in \mathbb{R}$, then by Def. I.6.2.1 $x \neq y$. Thus, by Def. I.6.2.3 we have $x < y$.
  - If $y = -\infty$, then we have $x = y$.

- If $y = +\infty$, then by Def. I.6.2.3 we have $x \leq y$ for every $x \in \mathbb{R}^*$.
  - If $x = +\infty$, then we have $x = y$.
  - If $x \in \mathbb{R}$, then by Def. I.6.2.1 $x \neq y$. Thus, by Def. I.6.2.3 we have $x < y$.
  - If $x = -\infty$, then by Def. I.6.2.1 $x \neq y$. Thus, by Def. I.6.2.3 we have $x < y$.

- If $y = -\infty$, then by Def. I.6.2.3 we have $x \geq y$ for every $x \in \mathbb{R}^*$.
  - If $x = +\infty$, then by Def. I.6.2.1 $x \neq y$. Thus, by Def. I.6.2.3 we have $x > y$.
  - If $x \in \mathbb{R}$, then by Def. I.6.2.1 $x \neq y$. Thus, by Def. I.6.2.3 we have $x > y$.
  - If $x = -\infty$, then we have $x = y$.

From all cases above, we conclude that exactly one of the statements $x < y$, $x = y$, or $x > y$ is true. $\square$

Proof. (c) By Prop. I.5.4.7, we already have $(x \leq y) \wedge (y \leq z) \implies x \leq z$ when $x, y, z \in \mathbb{R}$. So we only need to consider the cases $x, y, z \in \{+\infty, -\infty\}$.

- If $x = +\infty$, then by Def. I.6.2.3 $(x = +\infty) \wedge (x \leq y) \implies y = +\infty$. Similarly, $(y = +\infty) \wedge (y \leq z) \implies z = +\infty$. Thus, we have $x = +\infty = z$, and by Prop. I.6.2.5(a) we have $x \leq z$.

- If $x = -\infty$, then by Def. I.6.2.3 $x \leq z$ for every $z \in \mathbb{R}^*$.

- If $y = +\infty$, then by Def. I.6.2.3 $(y = +\infty) \wedge (y \leq z) \implies z = +\infty$. Again by Def. I.6.2.3 we have $x \leq +\infty = z$ for every $x \in \mathbb{R}^*$.

- If $y = -\infty$, then by Def. I.6.2.3 $(y = -\infty) \wedge (x \leq y) \implies x = -\infty$. Again by Def. I.6.2.3 we have $x = -\infty \leq z$ for every $z \in \mathbb{R}^*$.

- If $z = +\infty$, then by Def. I.6.2.3, we have $x \leq +\infty = z$ for every $x \in \mathbb{R}^*$.

- If $z = -\infty$, then by Def. I.6.2.3 $(z = -\infty) \wedge (y \leq z) \implies y = -\infty$. Similarly, $(y = -\infty) \wedge (x \leq y) \implies x = -\infty$. Thus, we have $x = -\infty = z$, and by Prop. I.6.2.5(a) we have $x \leq z$.

From all cases above, we conclude that $(x \le y) \wedge (y \le z) \implies x \le z$. □

Proof. (d) By Prop. I.5.4.7 we already have $x \le y \implies -y \le -x$ for every $x, y \in \mathbb{R}$. So we only need to consider the cases $x, y \in \{+\infty, -\infty\}$.

- If $x = +\infty$, then by Def. I.6.2.3 $(x = +\infty) \wedge (x \le y) \implies y = +\infty$. And by Def. I.6.2.2 we have $-x = -\infty = -y$. Thus, by Prop. I.6.2.5(a) we have $-y \le -x$.
- If $x = -\infty$, then by Def. I.6.2.2 $-x = +\infty$ and by Def. I.6.2.3 we have $-y \le -x$ for every $-y \in \mathbb{R}^*$.
- If $y = +\infty$, then by Def. I.6.2.2 $-y = -\infty$ and by Def. I.6.2.3 we have $-y \le -x$ for every $-x \in \mathbb{R}^*$.
- If $y = -\infty$, then by Def. I.6.2.3 $(y = -\infty) \wedge (x \le y) \implies x = -\infty$. And by Def. I.6.2.2 we have $-x = +\infty = -y$. Thus, by Prop. I.6.2.5(a) we have $-y \le -x$.

From all cases above, we conclude that $x \le y \implies -y \le -x$. □

Note. One could also introduce other operations on the extended real number system, such as addition, multiplication, etc. However, this is somewhat dangerous as these operations will almost certainly fail to obey the familiar rules of algebra. For instance, to define addition it seems reasonable (given one's intuitive notion of infinity) to set $+\infty + 5 = +\infty$ and $+\infty + 3 = +\infty$, but then this implies that $+\infty + 5 = +\infty + 3$, while $5 \ne 3$. So things like the cancellation law begin to break down once we try to operate involving infinity. To avoid these issues we shall simply not define any arithmetic operations on the extended real number system other than negation and order.

Def. I.6.2.6 (Supremum of sets of extended reals)**.** Let E be a subset of $\mathbb{R}^*$. Then we define the *supremum* $\sup(E)$ or *least upper bound* of E by the following rule.

(a) If E is contained in $\mathbb{R}$ (i.e., $+\infty$ and $-\infty$ are not elements of E), then we let $\sup(E)$ be as defined in Def. I.5.5.10.

(b) If E contains $+\infty$, then we set $\sup(E) := +\infty$.

(c) If E does not contain $+\infty$ but does contain $-\infty$, then we set $\sup(E) := \sup(E \setminus \{-\infty\})$ (which is a subset of $\mathbb{R}$ and thus falls under case (a)).

We also define the *infimum* $\inf(E)$ of E (also known as the *greatest lower bound* of E) by the formula

$$\inf(E) := -\sup(-E)$$

where $-E$ is the set $-E := \{-x : x \in E\}$.

E.g. I.6.2.10. Let E be the empty set. Then $\sup(E) = -\infty$ and $\inf(E) = +\infty$. This is the only case in which the supremum can be less than the infimum.

Proof. Since $+\infty \notin \emptyset$ and $-\infty \notin \emptyset$, by Def. I.6.2.6 we know that $\sup(\emptyset) = -\infty$. Since $-\emptyset$ is also empty, by Def. I.6.2.6 we know that $\sup(-\emptyset) = -\infty$, thus by Def. I.6.2.2 and I.6.2.6 we have $\inf(\emptyset) = -\sup(-\emptyset) = -(-\infty) = +\infty$.

Now we show that the only case in which the supremum can be less than the infimum is when $E = \emptyset$. Suppose for the sake of contradiction that there is a set E such that $E \neq \emptyset$ and $\sup(E) < \inf(E)$. Since $E \neq \emptyset$, let $x \in E$. Then we have $\sup(E) < \inf(E) \leq x \leq \sup(E)$, a contradiction. Thus, $E = \emptyset$. □

Note. One can intuitively think of the supremum of E as follows. Imagine the real line with $+\infty$ somehow on the far right, and $-\infty$ on the far left. Imagine a piston at $+\infty$ moving leftward until it is stopped by the presence of a set E; the location where it stops is the supremum of E. Similarly, if one imagines a piston at $-\infty$ moving rightward until it is stopped by the presence of E, the location where it stops is the infimum of E. In the case when E is the empty set, the pistons pass through each other, the supremum landing at $-\infty$ and the infimum landing at $+\infty$.

Thm. I.6.2.11. Let E be a subset of $\mathbb{R}^*$. Then the following statements are true.

(a) For every $x \in E$ we have $x \leq \sup(E)$ and $x \geq \inf(E)$.

(b) Suppose that $M \in \mathbb{R}^*$ is an upper bound for E, i.e., $x \leq M$ for all $x \in E$. Then we have $\sup(E) \leq M$.

(c) Suppose that $M \in \mathbb{R}^*$ is a lower bound for E, i.e., $x \geq M$ for all $x \in E$. Then we have $\inf(E) \geq M$.

Proof. (a) We first show that $x \leq \sup(E)$ for every $x \in E$. First, suppose that $E = \emptyset$. Then the statement "$x \leq \sup(\emptyset)$ for every $x \in \emptyset$" is vacuously true. Now suppose that $E \neq \emptyset$. We split into two cases:

- If $+\infty \notin E$, then we can further split into two cases:
  - If $-\infty \in E$, then by Def. I.6.2.6 we know that $\sup(E) = \sup(E \setminus \{-\infty\})$. Let $E' = E \setminus \{-\infty\}$. Since $E' \subseteq \mathbb{R}$, by Thm. I.5.5.9 we know that $x \leq \sup(E)$ for every $x \in E'$. By Def. I.6.2.3 we know that $-\infty \leq \sup(E)$, thus we have $x \leq \sup(E)$ for every $x \in E$.
  - If $-\infty \notin E$, then $E \subseteq \mathbb{R}$, thus by Thm. I.5.5.9 we know that $x \leq \sup(E)$ for every $x \in E$.
- If $+\infty \in E$, then by Def. I.6.2.6 we have $\sup(E) = +\infty$, and by Def. I.6.2.3 we have $x \leq \sup(E)$ for every $x \in E$.

From all cases above, we conclude that $x \leq \sup(E)$ for every $x \in E$.

Now we show that $x \geq \inf(E)$ for every $x \in E$. First, suppose that $E = \emptyset$. Then the statement "$x \geq \inf(E)$ for every $x \in E$" is vacuously true. Now suppose that $E \neq \emptyset$. From the proof above we know that $x \leq \sup(E)$ for every $x \in E$. Then we have

$$\begin{aligned}
& x \leq \sup(E) && \text{(by Thm. I.5.5.9)} \\
\implies & -x \geq -\sup(E) && \\
\implies & \sup(-E) \geq -x \geq -\sup(E) && \text{(by Thm. I.5.5.9)} \\
\implies & \inf(E) = -\sup(-E) \leq x \leq \sup(E). && \text{(by Def. I.6.2.6)}
\end{aligned}$$

Thus, we conclude that $x \geq \inf(E)$ for every $x \in E$. □

Proof. (b) First, suppose that $E = \emptyset$. Then by E.g. I.6.2.10 and Def. I.6.2.3 we have $\sup(E) = -\infty \leq M$. Now suppose that $E \neq \emptyset$. We split into two cases:

- If $+\infty \notin E$, then we can further split into two cases:
  - If $-\infty \in E$, then by Def. I.6.2.6 we know that $\sup(E) = \sup(E \setminus \{-\infty\})$. Let $E' = E \setminus \{-\infty\}$. Then by Thm. I.5.5.9 we know that $\sup(E) \leq M$.
  - If $-\infty \notin E$, then $E \subseteq \mathbb{R}$, thus by Thm. I.5.5.9 we know that $\sup(E) \leq M$.
- If $+\infty \in E$, then by hypothesis we have $+\infty \leq M$, and by Def. I.6.2.3 we have $M = +\infty$. Again by Def. I.6.2.3 we have $\sup(E) \leq M$.

From all cases above, we conclude that $\sup(E) \leq M$. □

Proof. (c) We have

$$\begin{aligned}
& \forall x \in E, x \geq M && \\
\implies & -x \leq -M && \\
\implies & \sup(-E) \leq -M && \text{(by Thm. I.6.2.11(b))} \\
\implies & -\sup(-E) \geq M && \\
\implies & \inf(E) \geq M. && \text{(by Def. I.6.2.6)}
\end{aligned}$$

□

— Exercises —

Ex. I.6.2.1. Prove Prop. I.6.2.5.

Proof. See Prop. I.6.2.5. □

Ex. I.6.2.2. Prove Thm. I.6.2.11.

Proof. See Thm. I.6.2.11. □

I.6.3 Suprema and Infima of sequences

Def. I.6.3.1 (Sup and inf of sequences)**.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Then we define $\sup(a_n)_{n=m}^{\infty}$ to be the supremum of the set $\{a_n : n \geq m\}$, and $\inf(a_n)_{n=m}^{\infty}$ to the infimum of the same set $\{a_n : n \geq m\}$.

Rmk. I.6.3.2. The quantities $\sup(a_n)_{n=m}^{\infty}$ and $\inf(a_n)_{n=m}^{\infty}$ are sometimes written as $\sup\limits_{n\geq m} a_n$ and $\inf\limits_{n\geq m} a_n$ respectively.

E.g. I.6.3.4. Let $a_n := 1/n$; thus $(a_n)_{n=1}^{\infty}$ is the sequence $1, 1/2, 1/3, \ldots$. Then the set $\{a_n : n \geq 1\}$ is the countable set $\{1, 1/2, 1/3, 1/4, \ldots\}$. Thus, $\sup(a_n)_{n=1}^{\infty} = 1$ and $\inf(a_n)_{n=1}^{\infty} = 0$.

Proof. We first show that $\sup(a_n)_{n=1}^{\infty} = 1$. By hypothesis $\forall n \in \mathbb{Z}^+$ we have $a_n \leq 1$. If $x \in \mathbb{R}$ and $x < 1$, then $x < a_1$, which means x is not an upper bound of $(a_n)_{n=1}^{\infty}$. Thus, $\sup(a_n)_{n=1}^{\infty} = 1$.

Now we show that $\inf(a_n)_{n=1}^{\infty} = 0$. $\forall n \in \mathbb{Z}^+$, we have $-a_n = -1/n \leq 0$. So 0 is an upper bound of $\{-a_n : n \geq 1\}$, and $\sup(\{-a_n : n \geq 1\}) \leq 0$. Then we have

$$\begin{aligned}
\inf(a_n)_{n=1}^{\infty} &= \inf(\{a_n : n \geq 1\}) && \text{(by Def. I.6.3.1)} \\
&= -\sup(-\{a_n : n \geq 1\}) && \text{(by Def. I.6.2.6)} \\
&= -\sup(\{-a_n : n \geq 1\}) && \text{(by Def. I.6.2.6)} \\
&\geq 0.
\end{aligned}$$

So 0 is a lower bound of $\{a_n : n \geq 1\}$, i.e., $0 \leq \inf(a_n)_{n=1}^{\infty}$. Suppose for the sake of contradiction that $\exists x \in \mathbb{R}^+$ such that $x = \inf(a_n)_{n=1}^{\infty}$. Then by Rmk. I.6.3.7 $\forall n \in \mathbb{Z}^+$ we must have $0 < x \leq a_n$. But by Prop. I.5.4.12 $\exists q \in \mathbb{Q}^+$ such that $q \leq x$. Let such $q = a/b$ where $a, b \in \mathbb{Z}^+$. Since $b \in \mathbb{Z}^+$, we have $1/(b+1) \in \{a_n : n \geq 1\}$. Since $1/(b+1) < 1/b \leq a/b$, we have $1/(b+1) < x$, which contradict to $x \leq a_n$ for every $n \in \mathbb{Z}^+$. Thus, $\nexists x \in \mathbb{R}^+$ such that $x = \inf(a_n)_{n=1}^{\infty}$, therefore $\inf(a_n)_{n=1}^{\infty} = 0$. □

Note. It is a little inaccurate to think of the supremum and infimum as the "largest element of the sequence" and "smallest element of the sequence" respectively.

Note. It is possible for the supremum or infimum of a sequence to be $+\infty$ or $-\infty$. However, if a sequence $(a_n)_{n=m}^{\infty}$ is bounded, say bounded by M, then all the elements a_n of the sequence lie between $-M$ and M, so that the set $\{a_n : n \geq m\}$ has M as an upper bound and $-M$ as a lower bound. Since this set is clearly non-empty, we can thus conclude that the supremum and infimum of a bounded sequence are real numbers (i.e., not $+\infty$ and $-\infty$).

Prop. I.6.3.6 (Least upper bound property)**.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be the extended real number $x := \sup(a_n)_{n=m}^{\infty}$. Then we have $a_n \leq x$ for all $n \geq m$. Also, whenever $M \in \mathbb{R}^*$ is an upper bound for a_n (i.e., $a_n \leq M$ for all $n \geq m$), we have $x \leq M$. Finally, for every extended real number y for which $y < x$, there exists at least one $n \geq m$ for which $y < a_n \leq x$.

Proof. We first show that $\forall n \geq m$ we have $a_n \leq x$. By Def. I.6.3.1 we have $\sup(a_n)_{n=m}^{\infty} = \sup(\{a_n : n \geq m\})$. Thus, by Thm. I.6.2.11(a), $\forall a_n \in \{a_n : n \geq m\}$ we have $a_n \leq x$.

Next we show that $M \in \mathbb{R}^*$ is an upper bound of $(a_n)_{n=m}^{\infty}$ implies $x \leq M$. By Def. I.6.3.1 we have $\sup(a_n)_{n=m}^{\infty} = \sup(\{a_n : n \geq m\})$. Thus, by Thm. I.6.2.11(b) we have $x \leq M$.

Finally we show that if $y \in \mathbb{R}^*$ and $y < x$, then $\exists n \geq m$ such that $y < a_n \leq x$. Suppose for the sake of contradition that such n does not exist. Then $\forall n \geq m$ we must have $a_n \leq y < x$. But then y is the least upper bound, a contradiction. Thus, $\exists n \geq m$ such that $y < a_n \leq x$. □

Rmk. I.6.3.7. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be the extended real number $x := \inf(a_n)_{n=m}^{\infty}$. Then we have $a_n \geq x$ for all $n \geq m$. Also, whenever $M \in \mathbb{R}^*$ is an lower bound for a_n (i.e., $a_n \geq M$ for all $n \geq m$), we have $x \geq M$. Finally, for every extended real number y for which $y > x$, there exists at least one $n \geq m$ for which $y > a_n \geq x$. This is the corresponding Proposition for infima, but with all the references to order reversed, e.g., all upper bounds should now be lower bounds, etc. The proof is exactly the same.

Note. In the previous section we saw that all convergent sequences are bounded. It is natural to ask whether the converse is true: are all bounded sequences convergent? The answer is no; for instance, the sequence $1, -1, 1, -1, \ldots$ is bounded, but not Cauchy and hence not convergent. However, if we make the sequence both bounded and *monotone* (i.e., increasing or decreasing), then it is true that it must converge.

Prop. I.6.3.8 (Monotone bounded sequences converge)**.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which has some finite upper bound $M \in \mathbb{R}$, and which is also increasing (i.e., $a_{n+1} \geq a_n$ for all $n \geq m$). Then $(a_n)_{n=m}^{\infty}$ is convergent, and in fact

$$\lim_{n\to\infty} a_n = \sup(a_n)_{n=m}^{\infty} \leq M.$$

Proof. Since $(a_n)_{n=m}^{\infty}$ have an upper bound M, by Def. I.6.3.1 the set $E = \{a_n : n \geq m\}$ have an upper bound M. By Thm. I.5.5.9 $\sup(E)$ must exist and $\sup(E) \leq M$. Now we want to show that $\lim_{n\to\infty} a_n = \sup(E)$. By Def. I.6.1.5 and I.6.1.8, we need to show that $\forall \varepsilon \in \mathbb{R}^+$, $\exists N \in \mathbb{N}$ and $N \geq m$ such that $|a_n - \sup(E)| \leq \varepsilon$ for every $n \geq N$. Since $\forall n \geq m$ we have $a_n \leq \sup(E)$, we must also have $|a_n - \sup(E)| = \sup(E) - a_n$. Thus

$$\begin{aligned}
& \forall \varepsilon \in \mathbb{R}^+, -\varepsilon < 0 \\
\implies & \sup(E) - \varepsilon < \sup(E) \\
\implies & \exists N \geq m : \sup(E) - \varepsilon < a_N \leq \sup(E) && \text{(by Prop. I.6.3.6)} \\
\implies & \forall n \geq N : \sup(E) - \varepsilon < a_N \leq a_n \leq \sup(E) && \text{(by hypothesis)} \\
\implies & \sup(E) - \varepsilon \leq a_n \\
\implies & \sup(E) - a_n \leq \varepsilon \\
\implies & |a_n - \sup(E)| \leq \varepsilon
\end{aligned}$$

and we conclude that $\lim_{n\to\infty} a_n = \sup(E) = \sup(a_n)_{n=m}^{\infty}$. □

A.Cor. I.6.3.1. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which has some finite lower bound $M \in \mathbb{R}$, and which is also decreasing (i.e., $a_{n+1} \leq a_n$ for all $n \geq m$). Then $(a_n)_{n=m}^{\infty}$ is convergent, and in fact

$$\lim_{n\to\infty} a_n = \inf(a_n)_{n=m}^{\infty} \geq M.$$

Proof. Since $(a_n)_{n=m}^{\infty}$ is decreasing, we know that $(-a_n)_{n=m}^{\infty}$ is increasing since

$$n \leq m \iff a_n \leq a_m \iff -a_n \geq -a_m.$$

Thus, we have

$$\begin{aligned}
& \forall n \geq m, a_n \geq M \\
\implies & -a_n \leq -M \\
\implies & \lim_{n\to\infty} -a_n = \sup(-a_n)_{n=m}^{\infty} && \text{(by Prop. I.6.3.8)} \\
\implies & -\lim_{n\to\infty} a_n = \sup(-a_n)_{n=m}^{\infty} && \text{(by Thm. I.6.1.19(c))} \\
\implies & \lim_{n\to\infty} a_n = -\sup(-a_n)_{n=m}^{\infty} \\
\implies & \lim_{n\to\infty} a_n = -\sup\{-a_n : n \geq m\} && \text{(by Def. I.6.3.1)} \\
\implies & \lim_{n\to\infty} a_n = \inf\{a_n : n \geq m\} && \text{(by Def. I.6.2.6)} \\
\implies & \lim_{n\to\infty} a_n = \inf(a_n)_{n=m}^{\infty}. && \text{(by Def. I.6.3.1)}
\end{aligned}$$

□

Note. A sequence is said to be *monotone* if it is either increasing or decreasing. From Prop. I.6.3.8 and Cor. I.6.1.17 we see that a monotone sequence converges iff it is bounded.

E.g. I.6.3.9. The sequence

$$3, 3.1, 3.14, 3.141, 3.1415, \ldots$$

is increasing, and is bounded above by 4. Hence by Prop. I.6.3.8 it must have a limit, which is a real number less than or equal to 4.

Note. Prop. I.6.3.8 asserts that the limit of a monotone sequence exists, but does not directly say what that limit is. Nevertheless, with a little extra work one can often find the limit once one is given that the limit does exist.

Prop. I.6.3.10. Let $0 < x < 1$. Then we have $\lim_{n\to\infty} x^n = 0$.

Proof. Since $0 < x < 1$, one can show that the sequence $(x^n)_{n=1}^{\infty}$ is decreasing (since $x^{n+1}/x^n = x < 1$, so $x^{n+1} < x^n$). On the other hand, the sequence $(x^n)_{n=1}^{\infty}$ has a lower bound of 0. Thus, by A.Cor. I.6.3.1 the sequence $(x^n)_{n=1}^{\infty}$ converges to some limit L. Since $x^{n+1} = x \times x^n$, we thus see from the limit laws (Thm. I.6.1.19) that $(x^{n+1})_{n=1}^{\infty}$ converges to

xL. But the sequence $(x^{n+1})_{n=1}^{\infty}$ is just the sequence $(x^n)_{n=2}^{\infty}$ shifted by one, and so they must have the same limits by Ex. I.6.1.3 and I.6.1.4. So $xL = L$. Since $x \neq 1$, we can solve for L to obtain $L = 0$. Thus, $(x^n)_{n=1}^{\infty}$ converges to 0. □

— Exercises —

Ex. I.6.3.1. Verify the claim in E.g. I.6.3.4.

Proof. See E.g. I.6.3.4. □

Ex. I.6.3.2. Prove Prop. I.6.3.6.

Proof. See Prop. I.6.3.6. □

Ex. I.6.3.3. Prove Prop. I.6.3.8.

Proof. See Prop. I.6.3.8. □

Ex. I.6.3.4. Explain why Prop. I.6.3.10 fails when $x > 1$. In fact, show that the sequence $(x_n)_{n=1}^{\infty}$ diverges when $x > 1$.

Proof. Since $x = x^{n+1}/x^n > 1$, we have $x^{n+1} > x^n$, which means $(x^n)_{n=1}^{\infty}$ is increasing. Suppose for the sake of contradiction that $(x^n)_{n=1}^{\infty}$ has an upper bound of M. Then by Prop. I.6.3.8 the sequence $(x^n)_{n=1}^{\infty}$ converges to some limit L. Since $(1/x)^n x^n = 1$, we thus see from the limit laws (Thm. I.6.1.19) that $((1/x)^n x^n)_{n=1}^{\infty}$ converges to 1. But $0 < (1/x)^n < 1$, by Prop. I.6.3.10 we have $((1/x)^n)_{n=1}^{\infty}$ converges to 0. So by Thm. I.6.1.19 we have $0L = 1$, a contradiction. Thus, $(x^n)_{n=1}^{\infty}$ does not have an upper bound. This means $(x^n)_{n=1}^{\infty}$ is diverge by Cor. I.6.1.17. □

I.6.4 Limsup, Liminf, and limit points

Def. I.6.4.1 (Limit points)**.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let x be a real number, and let $\varepsilon > 0$ be a real number. We say that x is *ε-adherent* to $(a_n)_{n=m}^{\infty}$ iff there exists an $n \geq m$ such that a_n is ε-close to x. We say that x is *continually ε-adherent* to $(a_n)_{n=m}^{\infty}$ iff it is ε-adherent to $(a_n)_{n=N}^{\infty}$ for every $N \geq m$. We say that x is a *limit point* or *adherent point* of $(a_n)_{n=m}^{\infty}$ iff it is continually ε-adherent to $(a_n)_{n=m}^{\infty}$ for every $\varepsilon > 0$.

Rmk. I.6.4.2. The verb "to adhere" means much the same as "to stick to"; hence the term "adhesive."

Note. Unwrapping all the definitions, we see that x is a limit point of $(a_n)_{n=m}^{\infty}$ if, for every $\varepsilon > 0$ and every $N \geq m$, there exists an $n \geq N$ such that $|a_n - x| \leq \varepsilon$. Note the difference between a sequence being ε-close to L (which means that *all* the elements of the sequence stay within a distance ε of L) and L being ε-adherent to the sequence (which only needs a *single* element of the sequence to stay within a distance ε of L). Also, for L to be continually

ε-adherent to $(a_n)_{n=m}^{\infty}$, it has to be ε-adherent to $(a_n)_{n=N}^{\infty}$ for *all* $N \geq m$, whereas for $(a_n)_{n=m}^{\infty}$ to be eventually ε-close to L, we only need $(a_n)_{n=m}^{\infty}$ to be ε-close to L for *some* $N \geq m$. Thus, there are some subtle differences in quantifiers between limits and limit points.

Prop. I.6.4.5 (Limits are limit points)**.** Let $(a_n)_{n=m}^{\infty}$ be a sequence which converges to a real number c. Then c is a limit point of $(a_n)_{n=m}^{\infty}$, and in fact it is the only limit point of $(a_n)_{n=m}^{\infty}$.

Proof. We first show that c is a limit point of $(a_n)_{n=m}^{\infty}$. Let $N_1, N_2 \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} a_n = c$, we have $\forall \varepsilon \in \mathbb{R}^+$, $\exists N_1 \geq m$ such that $|a_n - c| \leq \varepsilon$ for every $n \geq N_1$. This means $\forall N_2 \geq m$, $\exists n \geq \max(N_1, N_2)$ such that $|a_n - c| \leq \varepsilon$. Thus, by Def. I.6.4.1 c is a limit point of $(a_n)_{n=m}^{\infty}$.

Now we show that c is the only limit point of $(a_n)_{n=m}^{\infty}$. Let $N \in \mathbb{N}$. Suppose for the sake of contradiction that c' is also a limit point of $(a_n)_{n=m}^{\infty}$ and $c' \neq c$. Since $\lim\limits_{n\to\infty} a_n = c$, we have $\forall \varepsilon \in \mathbb{R}^+$, $\exists N \geq m$ such that $|a_n - c| \leq \varepsilon$ for every $n \geq N$. In particular, $|a_n - c| \leq \left|c - c'\right|/2$. So

$$\begin{aligned}
\left|c - c'\right| &= \left|a_n - a_n + c - c'\right| \\
&= \left|c - a_n + a_n - c'\right| \\
&\leq |c - a_n| + \left|a_n - c'\right| \\
&= |a_n - c| + \left|a_n - c'\right|.
\end{aligned}$$

This means $\left|a_n - c'\right| \geq \left|c - c'\right| - |a_n - c|$. Then $\forall n \geq N$,

$$\begin{aligned}
\left|a_n - c'\right| &\geq \left|c - c'\right| - |a_n - c| \\
&\geq \left|c - c'\right| - \left|c - c'\right|/2 \\
&= \left|c - c'\right|/2.
\end{aligned}$$

Since $\left|c - c'\right|/2 > 0$, this means we can not find an $n \geq N$ where $\left|a_n - c'\right| < \left|c - c'\right|/2$, which contradict to the fact that c' is limit point. Thus, $\lim\limits_{n\to\infty} a_n = c$ is the only limit point of $(a_n)_{n=m}^{\infty}$. □

Def. I.6.4.6 (Limit superior and limit inferior)**.** Suppose that $(a_n)_{n=m}^{\infty}$ is a sequence. We define a new sequence $(a_N^+)_{N=m}^{\infty}$ by the formula

$$a_N^+ := \sup(a_n)_{n=N}^{\infty}.$$

More informally, a_N^+ is the supremum of all the elements in the sequence from a_N onwards. We then define the *limit superior* of the sequence $(a_n)_{n=m}^{\infty}$ by the formula

$$\limsup_{n\to\infty} a_n := \inf(a_N^+)_{N=m}^{\infty}.$$

Similarly, we can define

$$a_N^- := \inf(a_n)_{n=N}^\infty$$

and define the *limit inferior* of the sequence $(a_n)_{n=m}^\infty$ by the formula

$$\liminf_{n\to\infty} a_n := \sup(a_N^-)_{N=m}^\infty.$$

Rmk. I.6.4.11. Some authors use the notation $\overline{\lim}_{n\to\infty} a_n$ instead of $\limsup_{n\to\infty} a_n$, and $\underline{\lim}_{n\to\infty} a_n$ instead of $\liminf_{n\to\infty} a_n$. Note that the starting index m of the sequence is irrelevant.

Note. Returning to the piston analogy, imagine a piston at $+\infty$ moving leftward until it is stopped by the presence of the sequence $a_1, a_2, \ldots$. The place it will stop is the supremum of $a_1, a_2, a_3, \ldots$, which in our new notation is a_1^+. Now let us remove the first element a_1 from the sequence; this may cause our piston to slip leftward, to a new point a_2^+ (though in many cases the piston will not move and a_2^+ will just be the same as a_1^+). Then we remove the second element a_2, causing the piston to slip a little more. If we keep doing this the piston will keep slipping, but there will be some point where it cannot go any further, and this is the limit superior of the sequence. A similar analogy can describe the limit inferior of the sequence.

Prop. I.6.4.12. Let $(a_n)_{n=m}^\infty$ be a sequence of real numbers, let L^+ be the limit superior of this sequence, and let L^- be the limit inferior of this sequence (thus both L^+ and L^- are extended real numbers).

(a) For every $x > L^+$, there exists an $N \geq m$ such that $a_n < x$ for all $n \geq N$. (In other words, for every $x > L^+$, the elements of the sequence $(a_n)_{n=m}^\infty$ are eventually less than x.) Similarly, for every $y < L^-$ there exists an $N \geq m$ such that $a_n > y$ for all $n \geq N$.

(b) For every $x < L^+$, and every $N \geq m$, there exists an $n \geq N$ such that $a_n > x$. (In other words, for every $x < L^+$, the elements of the sequence $(a_n)_{n=m}^\infty$ exceed x infinitely often.) Similarly, for every $y > L^-$ and every $N \geq m$, there exists an $n \geq N$ such that $a_n < y$.

(c) We have $\inf(a_n)_{n=m}^\infty \leq L^- \leq L^+ \leq \sup(a_n)_{n=m}^\infty$.

(d) If c is any limit point of $(a_n)_{n=m}^\infty$, then we have $L^- \leq c \leq L^+$.

(e) If L^+ is finite, then it is a limit point of $(a_n)_{n=m}^\infty$. Similarly, if L^- is finite, then it is a limit point of $(a_n)_{n=m}^\infty$.

(f) Let c be a real number. If $(a_n)_{n=m}^\infty$ converges to c, then we must have $L^+ = L^- = c$. Conversely, if $L^+ = L^- = c$, then $(a_n)_{n=m}^\infty$ converges to c.

Proof. (a) Suppose first that $x > L^+$. Then by definition of L^+, we have $x > \inf(a_N^+)_{N=m}^\infty$. By Rmk. I.6.3.7, there must then exist an integer $N \geq m$ such that $x > a_N^+$. By definition of

a_N^+, this means that $x > \sup(a_n)_{n=N}^\infty$. Thus, by Prop. I.6.3.6, we have $x > a_n$ for all $n \geq N$, as desired.

Now suppose that $y < L^-$. Then by definition of L^-, we have $y < \sup(a_N^-)_{N=m}^\infty$. By Prop. I.6.3.6, there must then exist an integer $N \geq m$ such that $y < a_N^-$. By definition of a_N^-, this means that $y < \inf(a_n)_{n=N}^\infty$. Thus, by Rmk. I.6.3.7, we have $y < a_n$ for all $n \geq N$, as desired. □

Proof. (b) Suppose that $x < L^+$. Then we have $x < \inf(a_N^+)_{N=m}^\infty$. If we fix any $N \geq m$, then by Rmk. I.6.3.7, we thus have $x < a_N^+$. By definition of a_N^+, this means that $x < \sup(a_n)_{n=N}^\infty$. By Prop. I.6.3.6, there must thus exist $n \geq N$ such that $a_n > x$, as desired.

Now suppose that $y > L^-$. Then we have $y > \sup(a_N^-)_{N=m}^\infty$. If we fix any $N \geq m$, then by Prop. I.6.3.6, we thus have $y > a_N^-$. By definition of a_N^-, this means that $y > \inf(a_n)_{n=N}^\infty$. By Rmk. I.6.3.7, there must thus exist $n \geq N$ such that $a_n < y$, as desired. □

Proof. (c) Let $N \in \mathbb{N}$. We first show that $\inf(a_n)_{n=m}^\infty \leq L^-$.

$$\begin{aligned}
L^- &= \liminf_{n\to\infty} a_n && \text{(by Def. I.6.4.6)} \\
&= \sup(a_N^-)_{N=m}^\infty && \text{(by Def. I.6.4.6)} \\
&\geq a_m^- && \text{(by Def. I.6.3.1)} \\
&= \inf(a_n)_{n=m}^\infty. && \text{(by Def. I.6.4.6)}
\end{aligned}$$

Next we show that $L^+ \leq \sup(a_n)_{n=m}^\infty$.

$$\begin{aligned}
L^+ &= \limsup_{n\to\infty} a_n && \text{(by Def. I.6.4.6)} \\
&= \inf(a_N^+)_{N=m}^\infty && \text{(by Def. I.6.4.6)} \\
&\leq a_m^+ && \text{(by Def. I.6.3.1)} \\
&= \sup(a_n)_{n=m}^\infty. && \text{(by Def. I.6.4.6)}
\end{aligned}$$

Finally we show that $L^- \leq L^+$. Let $N_1, N_2 \in \mathbb{N}$. Suppose for the sake of contradiction that $L^- > L^+$. Then we have

$$\begin{aligned}
& L^+ < L^- \\
\implies & \inf(a_N^+)_{N=m}^\infty < \sup(a_N^-)_{N=m}^\infty && \text{(by Def. I.6.4.6)} \\
\implies & \exists N_1 \geq m : a_{N_1}^+ < \sup(a_N^-)_{N=m}^\infty && \text{(by Rmk. I.6.3.7)} \\
\implies & \exists N_2 \geq m : a_{N_1}^+ < a_{N_2}^- && \text{(by Prop. I.6.3.6)} \\
\implies & \sup(a_n)_{n=N_1}^\infty < \inf(a_n)_{n=N_2}^\infty. && \text{(by Def. I.6.4.6)}
\end{aligned}$$

Let $N = \max(N_1, N_2)$. Then we have

$$\sup(a_n)_{n=N}^\infty \leq \sup(a_n)_{n=N_1}^\infty < \inf(a_n)_{n=N_2}^\infty \leq \inf(a_n)_{n=N}^\infty,$$

a contradiction. Thus, $L^- \leq L^+$, and we conclude that

$$\inf(a_n)_{n=m}^{\infty} \leq L^- \leq L^+ \leq \sup(a_n)_{n=m}^{\infty}.$$

□

Proof. (d) Let $N, n' \in \mathbb{N}$. Suppose that c is a limit point of $(a_n)_{n=m}^{\infty}$. We first show that $L^- \leq c$. Suppose for the sake of contradiction that $L^- > c$. Let $\varepsilon = (L^- - c)/2$. Since c is a limit point, by Def. I.6.4.1 $\forall N \geq m$, $\exists n \geq N$ such that $|a_n - c| \leq \varepsilon$. So we have $-\varepsilon \leq a_n - c \leq \varepsilon$ and thus $a_n \leq c + \varepsilon$. Since $L^- - \varepsilon < L^-$, by Prop. I.6.4.12(a) we have $\exists N \geq m$ such that $a_{n'} > L^- - \varepsilon$ for every $n' \geq N$. Combine with the above result we get $L^- - \varepsilon < c + \varepsilon$, but substituting ε with $(L^- - c)/2$ we get $(L^- + c)/2 < (L^- + c)/2$, a contradiction. Thus, $L^- \leq c$.

Now we show that $L^+ \geq c$. Suppose for the sake of contradiction that $L^+ < c$. Let $\varepsilon = (c - L^+)/2$. Since c is a limit point, by Def. I.6.4.1 $\forall N \geq m$, $\exists n \geq N$ such that $|a_n - c| \leq \varepsilon$. So we have $-\varepsilon \leq a_n - c \leq \varepsilon$ and thus $c - \varepsilon \leq a_n$. Since $L^+ < L^+ + \varepsilon$, by Prop. I.6.4.12(a) we have $\exists N \geq m$ such that $a_{n'} < L^+ + \varepsilon$ for every $n' \geq N$. Combine with the above result we get $c - \varepsilon < L^+ + \varepsilon$, but substituting ε with $(c - L^+)/2$ we get $(c + L^+)/2 < (c + L^+)/2$, a contradiction. Thus, $L^+ \geq c$. And we conclude that $L^- \leq c \leq L^+$. □

Proof. (e) Let $N, N' \in \mathbb{N}$. We first show that if L^+ is finite, then L^+ is a limit point of $(a_n)_{n=m}^{\infty}$. By Def. I.6.4.1 we need to show that $\forall \varepsilon \in \mathbb{R}^+$, $\forall N \geq m$, $\exists n \geq N$ such that $\left|a_n - L^+\right| \leq \varepsilon$. Since $L^+ < L^+ + \varepsilon$, by Prop. I.6.4.12(a) $\exists N' \geq m$ such that $a_n < L^+ + \varepsilon$ for every $n \geq N'$. But this also means $\forall N \geq m$, $\exists n \geq N$ such that $a_n < L^+ + \varepsilon$ as long as we always choose $n \geq \max(N', N)$. Since $L^+ - \varepsilon < L^+$, by Prop. I.6.4.12(b), $\forall N \geq m$, $\exists n \geq N$ such that $a_n > L^+ - \varepsilon$. Combine the two statements above we have $\forall N \geq m$, $\exists n \geq N$ such that $L^+ - \varepsilon < a_n < L^+ + \varepsilon$. Thus, $\left|a_n - L^+\right| < \varepsilon$ and L^+ is a limit point.

Now we show that if L^- is finite, then L^- is a limit point of $(a_n)_{n=m}^{\infty}$. By Def. I.6.4.1 we need to show that $\forall \varepsilon \in \mathbb{R}^+$, $\forall N \geq m$, $\exists n \geq N$ such that $\left|a_n - L^-\right| \leq \varepsilon$. Since $L^- - \varepsilon < L^-$, by Prop. I.6.4.12(a) $\exists N' \geq m$ such that $a_n > L^- - \varepsilon$ for every $n \geq N'$. But this also means $\forall N \geq m$, $\exists n \geq N$ such that $a_n > L^- - \varepsilon$ as long as we always choose $n \geq \max(N', N)$. Since $L^- < L^- + \varepsilon$, by Prop. I.6.4.12(b), $\forall N \geq m$, $\exists n \geq N$ such that $a_n < L^- + \varepsilon$. Combine the two statements above we have $\forall N \geq m$, $\exists n \geq N$ such that $L^- - \varepsilon < a_n < L^- + \varepsilon$. Thus, $\left|a_n - L^-\right| < \varepsilon$ and L^- is also a limit point. □

Proof. (f) Let $N, N_1, N_2, n_1, n_2 \in \mathbb{N}$. We first show that if $\lim_{n\to\infty} a_n = c$, then $L^+ = c$. By Prop. I.6.4.5, c is the only limit point of $(a_n)_{n=m}^{\infty}$. And by Prop. I.6.4.12(d), we have $L^- \leq c \leq L^+$. Suppose for the sake of contradiction that $c \neq L^+$. Then we must have $c < L^+$. Let $\varepsilon = (L^+ - c)/2$. Since $L^+ - \varepsilon < L^+$, by Prop. I.6.4.12(b), $\forall N_1 \geq m$, $\exists n_1 \geq N_1$ such that $a_{n_1} > L^+ - \varepsilon$. Since $\lim_{n\to\infty} a_n = c$, $\exists N_2 \geq m$ such that $|a_{n_2} - c| \leq \varepsilon$ for every $n_2 \geq N_2$, so $-\varepsilon \leq a_{n_2} - c \leq \varepsilon$ and $a_{n_2} \leq c + \varepsilon$. By setting $N = \max(N_1, N_2)$ we have

$L^+ - \varepsilon < a_n \leq c + \varepsilon$, but substituting ε with $(L^+ - c)/2$ we get $(L^+ + c)/2 < (L^+ + c)/2$, a contradiction. Thus, $c = L^+$.

Next we show that if $\lim_{n\to\infty} a_n = c$, then $L^- = c$. By Prop. I.6.4.5, c is the only limit point of $(a_n)_{n=m}^\infty$. And by Prop. I.6.4.12(d), we have $L^- \leq c \leq L^+$. Suppose for the sake of contradiction that $c \neq L^-$. Then we must have $L^- < c$. Let $\varepsilon = (c - L^-)/2$. Since $L^- < L^- + \varepsilon$, by Prop. I.6.4.12(b), $\forall N_1 \geq m$, $\exists n_1 \geq N_1$ such that $a_{n_1} < L^- + \varepsilon$. Since $\lim_{n\to\infty} a_n = c$, $\exists N_2 \geq m$ such that $|a_{n_2} - c| \leq \varepsilon$ for every $n_2 \geq N_2$, so $-\varepsilon \leq a_{n_2} - c \leq \varepsilon$ and $c - \varepsilon \leq a_{n_2}$. By setting $N = \max(N_1, N_2)$ we have $c - \varepsilon \leq a_n < L^- + \varepsilon$, but substituting ε with $(c - L^-)/2$ we get $(L^- + c)/2 < (L^- + c)/2$, a contradiction. Thus, $c = L^-$. We conclude that if $\lim_{n\to\infty} a_n = c$, then $L^+ = L^- = c$.

Finally we show that if $L^+ = L^- = c$, then $\lim_{n\to\infty} a_n = c$. Let $\varepsilon \in \mathbb{R}^+$. Then we have $c - \varepsilon < c < c + \varepsilon$. Since $c = L^+$, by Rmk. I.6.4.11(a) $\exists N_1 \geq m$ such that $a_{n_1} < c + \varepsilon$ for every $n_1 \geq N_1$. Similarly, since $c = L^-$, by Rmk. I.6.4.11(a) $\exists N_2 \geq m$ such that $c - \varepsilon < a_{n_2}$ for every $n_2 \geq N_2$. Let $N = \max(N_1, N_2)$. Then $\exists N \geq m$ such that $c - \varepsilon < a_n < c + \varepsilon$ for every $n \geq N$. But this means $|a_n - c| < \varepsilon$, and thus $\lim_{n\to\infty} a_n = c$. We conclude that $L^+ = L^- = c \iff \lim_{n\to\infty} a_n = c$. □

Note. Parts (d) and (e) of Prop. I.6.4.12 say, in particular, that L^+ is the largest limit point of $(a_n)_{n=m}^\infty$, and L^- is the smallest limit point (providing that L^+ and L^- are finite). Prop. I.6.4.12 (f) then says that if L^+ and L^- coincide (so there is only one limit point), then the sequence in fact converges. This gives a way to test if a sequence converges: compute its limit superior and limit inferior, and see if they are equal.

Lem. I.6.4.13 (Comparison principle)**.** Suppose that $(a_n)_{n=m}^\infty$ and $(b_n)_{n=m}^\infty$ are two sequences of real numbers such that $a_n \leq b_n$ for all $n \geq m$. Then we have the inequalities

$$\begin{aligned} \sup(a_n)_{n=m}^\infty &\leq \sup(b_n)_{n=m}^\infty \\ \inf(a_n)_{n=m}^\infty &\leq \inf(b_n)_{n=m}^\infty \\ \limsup_{n\to\infty} a_n &\leq \limsup_{n\to\infty} b_n \\ \liminf_{n\to\infty} a_n &\leq \liminf_{n\to\infty} b_n \end{aligned}$$

Proof. Let $n' \in \mathbb{N}$. We first show that $\sup(a_n)_{n=m}^\infty \leq \sup(b_n)_{n=m}^\infty$. Suppose for the sake of contradiction that $\sup(a_n)_{n=m}^\infty > \sup(b_n)_{n=m}^\infty$. Then by Prop. I.6.3.6 $\exists n' \geq m$ such that $a_{n'} > \sup(b_n)_{n=m}^\infty$. Also by Def. I.6.3.1 $\forall n \geq m$ we have $\sup(b_n)_{n=m}^\infty \geq b_n$. But this means $a_{n'} > b_n$, in particular, $a_{n'} > b_{n'}$, a contradiction. Thus, we must have $\sup(a_n)_{n=m}^\infty \leq \sup(b_n)_{n=m}^\infty$.

Next we show that $\inf(a_n)_{n=m}^\infty \leq \inf(b_n)_{n=m}^\infty$. Suppose for the sake of contradiction that $\inf(a_n)_{n=m}^\infty > \inf(b_n)_{n=m}^\infty$. Then by Rmk. I.6.3.7 $\exists n' \geq m$ such that $b_{n'} < \inf(a_n)_{n=m}^\infty$. Also by Def. I.6.3.1 $\forall n \geq m$ we have $\inf(a_n)_{n=m}^\infty \leq a_n$. But this means $b_{n'} < a_n$, in particular, $b_{n'} < a_{n'}$, a contradiction. Thus, we must have $\inf(a_n)_{n=m}^\infty \leq \inf(b_n)_{n=m}^\infty$.

Next we show that $\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n$. Suppose for the sake of contradiction that $\limsup_{n\to\infty} a_n > \limsup_{n\to\infty} b_n$. By Def. I.6.4.6 we have $\inf(a_n^+)_{n=m}^\infty > \inf(b_n^+)_{n=m}^\infty$. By Rmk. I.6.3.7 $\exists n' \ge m$ such that $b_{n'}^+ < \inf(a_n^+)_{n=m}^\infty$. By Def. I.6.3.1 $\forall n \ge m$ we have $\inf(a_n^+)_{n=m}^\infty \le a_n^+$. But this means $b_{n'}^+ < a_n^+$, in particular, $b_{n'}^+ < a_{n'}^+$. Again by Def. I.6.4.6 we have $\sup(b_n)_{n=n'}^\infty < \sup(a_n)_{n=n'}^\infty$. But this contradicts the proof above that $\sup(b_n)_{n=n'}^\infty \ge \sup(a_n)_{n=n'}^\infty$. Thus, we must have $\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n$.

Finally we show that $\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n$. Suppose for the sake of contradiction that $\liminf_{n\to\infty} a_n > \liminf_{n\to\infty} b_n$. By Def. I.6.4.6 we have $\sup(a_n^-)_{n=m}^\infty > \sup(b_n^-)_{n=m}^\infty$. By Prop. I.6.3.6 $\exists n' \ge m$ such that $a_{n'}^- > \sup(b_n^-)_{n=m}^\infty$. By Def. I.6.3.1 $\forall n \ge m$ we have $\sup(b_n^-)_{n=m}^\infty \ge b_n^-$. But this means $a_{n'}^- > b_n^-$, in particular, $a_{n'}^- > b_{n'}^-$. Again by Def. I.6.4.6 we have $\inf(a_n)_{n=n'}^\infty > \inf(b_n)_{n=n'}^\infty$. But this contradicts the proof above that $\inf(a_n)_{n=n'}^\infty \le \inf(b_n)_{n=n'}^\infty$. Thus, we must have $\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n$. □

Cor. I.6.4.14 (Squeeze test)**.** Let $(a_n)_{n=m}^\infty$, $(b_n)_{n=m}^\infty$, and $(c_n)_{n=m}^\infty$ be sequences of real numbers such that

$$a_n \le b_n \le c_n$$

for all $n \ge m$. Suppose also that $(a_n)_{n=m}^\infty$ and $(c_n)_{n=m}^\infty$ both converge to the same limit L. Then $(b_n)_{n=m}^\infty$ is also convergent to L.

Proof. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, by Prop. I.6.4.12(f) we have

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L = \limsup_{n\to\infty} c_n = \liminf_{n\to\infty} c_n.$$

Thus, we have

$$\begin{aligned} & a_n \le b_n \le c_n \\ \implies & \limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n \le \limsup_{n\to\infty} c_n && \text{(by Lem. I.6.4.13)} \\ \implies & L \le \limsup_{n\to\infty} b_n \le L && \text{(by Prop. I.6.4.12(f))} \\ \implies & \limsup_{n\to\infty} b_n = L \end{aligned}$$

and

$$\begin{aligned} & a_n \le b_n \le c_n \\ \implies & \liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n \le \liminf_{n\to\infty} c_n && \text{(by Lem. I.6.4.13)} \\ \implies & L \le \liminf_{n\to\infty} b_n \le L && \text{(by Prop. I.6.4.12(f))} \\ \implies & \liminf_{n\to\infty} b_n = L. \end{aligned}$$

Since
$$\limsup_{n\to\infty} b_n = \liminf_{n\to\infty} b_n = L,$$
by Prop. I.6.4.12(f) we have $\lim_{n\to\infty} b_n = L$. □

Rmk. I.6.4.16. The squeeze test, combined with the limit laws and the principle that monotone bounded sequences always have limits, allows one to compute a large number of limits.

Cor. I.6.4.17 (Zero test for sequences)**.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Then the limit $\lim_{n\to\infty} a_n$ exists and is equal to zero iff the limit $\lim_{n\to\infty} |a_n|$ exists and is equal to zero.

Proof. Let $N \in \mathbb{N}$. We first show that $\lim_{n\to\infty} a_n = 0$ implies $\lim_{n\to\infty} |a_n| = 0$.

$$\begin{aligned}
&\lim_{n\to\infty} a_n = 0\\
\implies& \forall\varepsilon \in \mathbb{R}^+, \exists N \geq m : \forall n \geq m, |a_n - 0| \leq \varepsilon\\
\implies& \forall\varepsilon \in \mathbb{R}^+, \exists N \geq m : \forall n \geq m, |a_n| \leq \varepsilon\\
\implies& \forall\varepsilon \in \mathbb{R}^+, \exists N \geq m : \forall n \geq m, ||a_n| - 0| \leq \varepsilon\\
\implies& \lim_{n\to\infty} |a_n| = 0.
\end{aligned}$$

Now we show that $\lim_{n\to\infty} |a_n| = 0$ implies $\lim_{n\to\infty} a_n = 0$. Since $-|a_n| \leq a_n \leq |a_n|$, we have

$$\begin{aligned}
&\lim_{n\to\infty} |a_n| = 0 && \\
\implies& \lim_{n\to\infty} -|a_n| = 0 && \text{(by Thm. I.6.1.19(c))}\\
\implies& \lim_{n\to\infty} a_n = 0. && \text{(by Cor. I.6.4.14)}
\end{aligned}$$

We conclude that $\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0$. □

Thm. I.6.4.18 (Completeness of the reals)**.** A sequence $(a_n)_{n=1}^{\infty}$ of real numbers is a Cauchy sequence iff it is convergent.

Proof. Prop. I.6.1.12 already tells us that every convergent sequence is Cauchy, so it suffices to show that every Cauchy sequence is convergent.

Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence. We know from Lem. I.5.1.15 (or more precisely, from the extension of this lemma to the real numbers, which is proven in exactly the same fashion) that the sequence $(a_n)_{n=1}^{\infty}$ is bounded; by Lem. I.6.4.13 (or Prop. I.6.4.12(c)) this implies that $L^- := \liminf_{n\to\infty} a_n$ and $L^+ := \limsup_{n\to\infty} a_n$ of the sequence are both finite. To show that the sequence converges, it will suffice by Prop. I.6.4.12(f) to show that $L^- = L^+$.

Now let $\varepsilon > 0$ be any real number. Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, it must be eventually ε-steady. So, in particular, there exists an $N \geq 1$ such that the sequence $(a_n)_{n=N}^{\infty}$

is ε-steady. In particular, we have $a_N - \varepsilon \leq a_n \leq a_N + \varepsilon$ for all $n \geq N$. By Prop. I.6.3.6 (or Lem. I.6.4.13) this implies that

$$a_N - \varepsilon \leq \inf(a_n)_{n=N}^{\infty} \leq \sup(a_n)_{n=N}^{\infty} \leq a_N + \varepsilon$$

and hence by the definition of L^- and L^+ (and Prop. I.6.3.6 again)

$$a_N - \varepsilon \leq L^- \leq L^+ \leq a_N + \varepsilon.$$

Thus, we have

$$0 \leq L^+ - L^- \leq 2\varepsilon.$$

But this is true for all $\varepsilon > 0$, and L^+ and L^- do not depend on ε; so we must therefore have $L^+ = L^-$. (If $L^+ > L^-$ then we could set $\varepsilon := (L^+ - L^-)/3$ and obtain a contradiction.) By Prop. I.6.4.12(f) we thus see that the sequence converges. □

Rmk. I.6.4.19. While Thm. I.6.4.18 is very similar in spirit to Prop. I.6.1.15, it is a bit more general, since Prop. I.6.1.15 refers to Cauchy sequences of rationals instead of real numbers.

Rmk. I.6.4.20. In the language of metric spaces, Thm. I.6.4.18 asserts that the real numbers are a *complete* metric space - that they do not contain "holes" the same way the rationals do. (Certainly the rationals have lots of Cauchy sequences which do not converge to other rationals; take for instance the sequence $1, 1.4, 1.41, 1.414, 1.4142, \ldots$ which converges to the irrational $\sqrt{2}$.) This property is closely related to the least upper bound property (Thm. I.5.5.9), and is one of the principal characteristics which make the real numbers superior to the rational numbers for the purposes of doing analysis (taking limits, taking derivatives and integrals, finding zeroes of functions, that kind of thing).

— Exercises —

Ex. I.6.4.1. Prove Prop. I.6.4.5.

Proof. See Prop. I.6.4.5. □

Ex. I.6.4.2. Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let c be a real number, let $m' \geq m$ be an integer, and let $k \geq 0$ be a non-negative integer. Show that

(a) c is a limit point of $(a_n)_{n=m}^{\infty}$ iff c is a limit point of $(a_n)_{n=m'}^{\infty}$.

(b) c is the limit superior of $(a_n)_{n=m}^{\infty}$ iff c is the limit superior of $(a_n)_{n=m'}^{\infty}$.

(c) c is the limit inferior of $(a_n)_{n=m}^{\infty}$ iff c is the limit inferior of $(a_n)_{n=m'}^{\infty}$.

(d) c is a limit point of $(a_n)_{n=m}^{\infty}$ iff c is a limit point of $(a_{n+k})_{n=m}^{\infty}$.

(e) c is the limit superior of $(a_n)_{n=m}^{\infty}$ iff c is the limit superior of $(a_{n+k})_{n=m}^{\infty}$.

(f) c is the limit inferior of $(a_n)_{n=m}^{\infty}$ iff c is the limit inferior of $(a_{n+k})_{n=m}^{\infty}$.

Proof. (a) Let $N, \in \mathbb{N}$. We first show that c is a limit point of $(a_n)_{n=m}^{\infty}$ implies c is a limit point of $(a_n)_{n=m'}^{\infty}$. Since c is a limit point of $(a_n)_{n=m}^{\infty}$, by Def. I.6.4.1 $\forall \varepsilon \in \mathbb{R}^+$, we have $\forall N \geq m$, $\exists n \geq N$ such that $|a_n - c| \leq \varepsilon$. Since $m' \geq m$, we must also have $\forall N \geq m'$, $\exists n \geq N$ such that $|a_n - c| \leq \varepsilon$. Thus, by Def. I.6.4.1 c is a limit point of $(a_n)_{n=m'}^{\infty}$.

Now we show that c is a limit point of $(a_n)_{n=m'}^{\infty}$ implies c is a limit point of $(a_n)_{n=m}^{\infty}$. Since c is a limit point of $(a_n)_{n=m'}^{\infty}$, by Def. I.6.4.1 $\forall \varepsilon \in \mathbb{R}^+$, we have $\forall N \geq m'$, $\exists n \geq N$ such that $|a_n - c| \leq \varepsilon$. Since $m' \geq m$, we only need to show that $\forall N' \in \mathbb{N}$ and $m \leq N' < m'$, $\exists n \geq N'$ such that $|a_n - c| \leq \varepsilon$. Since $N \geq m' > N'$, we can always find an $n \geq N > N'$ such that $|a_n - c| \leq \varepsilon$. Thus, by Def. I.6.4.1 c is a limit point of $(a_n)_{n=m}^{\infty}$. We conclude that c is a limit point of $(a_n)_{n=m}^{\infty}$ iff c is a limit point of $(a_n)_{n=m'}^{\infty}$. □

Proof. (b) We have

$$\begin{aligned}
& m \leq m' \\
\implies & \{a_n : n \geq m'\} \subseteq \{a_n : n \geq m\} \\
\implies & \sup(a_n)_{n=m}^{\infty} \geq \sup(a_n)_{n=m'}^{\infty} \\
\implies & a_m^+ \geq a_{m'}^+ && \text{(by Def. I.6.4.6)}
\end{aligned}$$

and thus

$$\begin{aligned}
& (m \leq m') \wedge (a_m^+ \geq a_{m'}^+) \\
\implies & (\{a_N^+ : N \geq m'\} \subseteq \{a_N^+ : N \geq m\}) \wedge (a_m^+ \geq a_{m'}^+) \\
\implies & (\inf(a_N^+)_{N=m}^{\infty} \leq \inf(a_N^+)_{N=m'}^{\infty}) \wedge (a_m^+ \geq a_{m'}^+) \\
\implies & \inf(a_N^+)_{N=m}^{\infty} = \inf(a_N^+)_{N=m'}^{\infty} \\
\implies & c = \limsup_{n\to\infty} a_n = \inf(a_N^+)_{N=m}^{\infty} = \inf(a_N^+)_{N=m'}^{\infty}. && \text{(by Def. I.6.4.6)}
\end{aligned}$$

□

Proof. (c) We have

$$\begin{aligned}
& m \leq m' \\
\implies & \{a_n : n \geq m'\} \subseteq \{a_n : n \geq m\} \\
\implies & \inf(a_n)_{n=m}^{\infty} \leq \inf(a_n)_{n=m'}^{\infty} \\
\implies & a_m^- \leq a_{m'}^- && \text{(by Def. I.6.4.6)}
\end{aligned}$$

and thus

$$\begin{aligned}
& (m \leq m') \wedge (a_m^- \leq a_{m'}^-) \\
\implies & (\{a_N^- : N \geq m'\} \subseteq \{a_N^- : N \geq m\}) \wedge (a_m^- \leq a_{m'}^-) \\
\implies & (\sup(a_N^-)_{N=m'}^{\infty} \leq \sup(a_N^-)_{N=m}^{\infty}) \wedge (a_m^- \leq a_{m'}^-)
\end{aligned}$$

$$\implies \sup(a_N^-)_{N=m}^{\infty} = \sup(a_N^-)_{N=m'}^{\infty}$$
$$\implies c = \liminf_{n\to\infty} a_n = \sup(a_N^-)_{N=m}^{\infty} = \sup(a_N^-)_{N=m'}^{\infty}. \qquad \text{(by Def. I.6.4.6)}$$

□

Proof. (d) Since $(a_{n+k})_{n=m}^{\infty} = (a_n)_{n=m+k}^{\infty}$, by Ex. I.6.4.2(a) we conclude that c is a limit point of $(a_n)_{n=m}^{\infty}$ iff c is a limit point of $(a_{n+k})_{n=m}^{\infty}$. □

Proof. (e) Since $(a_{n+k})_{n=m}^{\infty} = (a_n)_{n=m+k}^{\infty}$, by Ex. I.6.4.2(b) we conclude that c is the limit superior of $(a_n)_{n=m}^{\infty}$ iff c is the limit superior of $(a_{n+k})_{n=m}^{\infty}$. □

Proof. (f) Since $(a_{n+k})_{n=m}^{\infty} = (a_n)_{n=m+k}^{\infty}$, by Ex. I.6.4.2(c) we conclude that c is the limit inferior of $(a_n)_{n=m}^{\infty}$ iff c is the limit inferior of $(a_{n+k})_{n=m}^{\infty}$. □

Ex. I.6.4.3. Prove Prop. I.6.4.12.

Proof. See Prop. I.6.4.12. □

Ex. I.6.4.4. Prove Lem. I.6.4.13.

Proof. See Lem. I.6.4.13. □

Ex. I.6.4.5. Use Lem. I.6.4.13 to prove Cor. I.6.4.14.

Proof. See Cor. I.6.4.14. □

Ex. I.6.4.6. Give an example of two bounded sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $a_n < b_n$ for all $n \geq 1$, but that $\sup(a_n)_{n=1}^{\infty} \not< \sup(b_n)_{n=1}^{\infty}$. Explain why this does not contradict Lem. I.6.4.13.

Proof. Let $a_n = -1/(n+1)$ and $b_n = -1/(n+1)^2$. Then we have $a_n < b_n$ for every $n \geq 1$. But $\sup(a_n)_{n=1}^{\infty} = \sup(b_n)_{n=1}^{\infty} = 0$. This does not contradict to Lem. I.6.4.13 since $\sup(a_n)_{n=1}^{\infty} \leq \sup(b_n)_{n=1}^{\infty}$ does not enforce strictly order, i.e., $\sup(a_n)_{n=1}^{\infty} < \sup(b_n)_{n=1}^{\infty}$. □

Ex. I.6.4.7. Prove Cor. I.6.4.17. Is the corollary still true if we replace zero in the statement of this Corollary by some other number?

Proof. See Cor. I.6.4.17.

Now we show that Cor. I.6.4.17 is not true if we replace zero by some other number. Let $N \in \mathbb{N}$ and let $(a_n)_{n=m}^{\infty}$ be a sequence of reals. Suppose that $\lim_{n\to\infty} |a_n| = c$ for some $c \in \mathbb{R}^-$. Then we have

$$\lim_{n\to\infty} |a_n| = c$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \geq m : \forall n \geq N, ||a_n| - c| \leq \varepsilon$$
$$\implies -\varepsilon \leq |a_n| - c \leq \varepsilon$$

$$\implies -\varepsilon + c \leq |a_n| \leq \varepsilon + c.$$

But by setting $\varepsilon = -c/2$ we see that $|a_n| \leq c/2 < 0$, a contradiction.

Now suppose that $\lim_{n\to\infty} |a_n| = c$ for some $c \in \mathbb{R}^+$. By setting $a_n = (-1)^n c$ we see that

$$\forall n \geq m, |a_n| = c$$
$$\implies \lim_{n\to\infty} |a_n| = c.$$

But we know that $(a_n)_{n=m}^\infty$ is not eventually $2c$-close, thus $\lim_{n\to\infty} a_n$ does not exist. We conclude that Cor. I.6.4.17 is not true if we replace zero by some other number. □

Ex. I.6.4.8. Let us say that a sequence $(a_n)_{n=m}^\infty$ of real numbers has $+\infty$ as a limit point iff it has no finite upper bound, and that it has $-\infty$ as a limit point iff it has no finite lower bound. With this definition, show that $\limsup_{n\to\infty} a_n$ is a limit point of $(a_n)_{n=m}^\infty$, and furthermore that it is larger than all the other limit points of $(a_n)_{n=m}^\infty$; in other words, the limit superior is the largest limit point of a sequence. Similarly, show that the limit inferior is the smallest limit point of a sequence.

Proof. By Prop. I.6.4.12(e) we already know the cases where $(a_n)_{n=m}^\infty$ has finite upper bound or lower bound. So we only consider the cases where $(a_n)_{n=m}^\infty$ has no finite upper bound or lower bound. By Prop. I.6.4.12(d) we know that $L^- \leq c \leq L^+$ if c is a limit point of $(a_n)_{n=m}^\infty$. By definition $(a_n)_{n=m}^\infty$ has no finite upper bound iff $+\infty$ is a limit point. This means $+\infty \leq L^+$, and thus $L^+ = +\infty$ is the largest limit point. Similarly, by definition $(a_n)_{n=m}^\infty$ has no finite lower bound iff $-\infty$ is a limit point. This means $L^- \leq -\infty$, and thus $L^- = \infty$ is the smallest limit point. □

Ex. I.6.4.9. Using the definition in Ex. I.6.4.8, construct a sequence $(a_n)_{n=1}^\infty$ which has exactly three limit points, at $-\infty$, 0 and $+\infty$.

Proof. Let $(a_n)_{n=1}^\infty$ be the sequence $1, 1/2, -3, -1/4, 5, 1/6, -7, -1/8, \ldots$. Then $(a_n)_{n=1}^\infty$ has no finite upper bound and lower bound, and has 0 as limit point. □

Ex. I.6.4.10. Let $(a_n)_{n=N}^\infty$ be a sequence of real numbers, and let $(b_m)_{m=M}^\infty$ be another sequence of real numbers such that each b_m is a limit point of $(a_n)_{n=N}^\infty$. Let c be a limit point of $(b_m)_{m=M}^\infty$. Prove that c is also a limit point of $(a_n)_{n=N}^\infty$. (In other words, limit points of limit points are themselves limit points of the original sequence.)

Proof. Let $\varepsilon \in \mathbb{R}^+$, and let $i, j \in \mathbb{N}$. Since c is a limit point of $(b_m)_{m=M}^\infty$, by Def. I.6.4.1 we have $\forall i \geq M$, $\exists m \geq i$ such that $|b_m - c| \leq \varepsilon/2$. Now we fix such m. Since b_m is a limit point of $(a_n)_{n=N}^\infty$, by Def. I.6.4.1 we have $\forall j \geq N$, $\exists n \geq j$ such that $|a_n - b_m| \leq \varepsilon/2$. Then we have

$$|a_n - c| = |a_n - c + b_m - b_m|$$

$$\begin{aligned}&= |(a_n - b_m) + (b_m - c)|\\&\le |a_n - b_m| + |b_m - c|\\&\le \varepsilon/2 + \varepsilon/2\\&= \varepsilon\end{aligned}$$

This means $\forall i \ge N$, $\exists n \ge i$ such that $|a_n - c| \le \varepsilon$. Since ε was arbitrary, by Def. I.6.4.1 we know that c is a limit point of $(a_n)_{n=N}^{\infty}$. □

I.6.5 Some standard limits

A.Cor. I.6.5.1. We have

$$\lim_{n\to\infty} c = c$$

for any constant c.

Proof. Let $(a_n)_{n=1}^{\infty}$ be a constant sequence where $a_n = c$ for all $n \ge 1$, and let $N \in \mathbb{N}$. Then $\forall \varepsilon \in \mathbb{R}^+$, $\exists N \ge 1$ such that for every $n \ge N$,

$$|a_n - c| = |c - c| = 0 \le \varepsilon.$$

Thus, by Def. I.6.1.8 we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c = c$. □

Cor. I.6.5.1. We have $\lim_{n\to\infty} 1/n^{1/k} = 0$ for every integer $k \ge 1$.

Proof. From Lem. I.5.6.6 we know that $1/n^{1/k}$ is a decreasing function of n, while being bounded below by 0. By A.Cor. I.6.3.1 (for decreasing sequences instead of increasing sequences) we thus know that this sequence converges to some limit $L \ge 0$:

$$L = \lim_{n\to\infty} 1/n^{1/k}.$$

Raising this to the k^{th} power and using the limit laws (or more precisely, Thm. I.6.1.19(b) and induction), we obtain

$$L^k = \lim_{n\to\infty} 1/n.$$

By Prop. I.6.1.11 we thus have $L^k = 0$; but this means that L cannot be positive (else L^k would be positive), so $L = 0$, and we are done. □

Lem. I.6.5.2. Let x be a real number. Then the limit $\lim_{n\to\infty} x^n$ exists and is equal to zero when $|x| < 1$, exists and is equal to 1 when $x = 1$, and diverges when $x = -1$ or when $|x| > 1$.

Proof. We first show that if $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$. Since $0 \le |x| < 1$, we have

$$\begin{aligned} & 0 \le |x| < 1 \\ \implies & \lim_{n\to\infty} |x|^n = 0 && \text{(by Prop. I.6.3.10)} \\ \implies & (-1)\lim_{n\to\infty} |x|^n = (-1)\times 0 = 0 \\ \implies & \lim_{n\to\infty} -|x|^n = 0. && \text{(by Thm. I.6.1.19)} \end{aligned}$$

So $\lim_{n\to\infty} -|x|^n = \lim_{n\to\infty} |x|^n$. And because $-|x|^n \le x^n \le |x|^n$, by Squeeze test (Cor. I.6.4.14) we have $\lim_{n\to\infty} x^n = 0$.

Next we show that if $x = 1$, then $\lim_{n\to\infty} x^n = 1$. This is done by A.Cor. I.6.5.1.

Finally we show that if $x = -1$ or $|x| > 1$, then $\lim_{n\to\infty} x^n$ does not exist. If $x = -1$, then we have sequence $-1, 1, -1, 1, \ldots$, which is not eventually 1-steady for any $n \ge 1$ and thus does not converge to any value. If $|x| > 1$, then we can divide into two cases:

- If $x > 1$, then by Ex. I.6.3.4 $\lim_{n\to\infty} x^n$ does not exist.
- If $x < -1$, then $\forall n \ge 1$ we have

$$\begin{aligned} \left|x^{n+1} - x^n\right| &= |x^n(x-1)| \\ &= |x^n||x-1| \\ &> |x^n||-2| \\ &= 2|x^n| \\ &> 2. \end{aligned}$$

  This means x is not a Cauchy sequence, so by Thm. I.6.4.18 $\lim_{n\to\infty} x^n$ does not exist.

From all cases above, we conclude that if $|x| > 1$ then $\lim_{n\to\infty} x^n$ does not exist. ☐

Lem. I.6.5.3. For any $x > 0$, we have $\lim_{n\to\infty} x^{1/n} = 1$.

Proof. We first show that $\forall \varepsilon, M \in \mathbb{R}^+$, $\exists n \ge 1$ such that $M^{1/n} \le 1 + \varepsilon$. Since $1/(1+\varepsilon) < 1$, by Lem. I.6.5.2 we have $\lim_{n\to\infty} 1/(1+\varepsilon)^n = 0$. Let $a_n = 1/(1+\varepsilon)^n$. Then $\inf(a_n)_{n=1}^{\infty} = 0$. Since $1/M > 0$, we have $1/M > \inf(a_n)_{n=1}^{\infty}$. By Rmk. I.6.3.7 $\exists n \ge 1$ such that $a_n = 1/(1+\varepsilon)^n < 1/M$. Thus, we have $M < (1+\varepsilon)^n$, and by Lem. I.5.6.9(d) $M^{1/n} < 1 + \varepsilon$.

Now we show that $\lim_{n\to\infty} x^{1/n} = 1$. We split into two cases:

- If $x \ge 1$, then by the proof above we have $\forall \varepsilon \in \mathbb{R}^+$, $\exists n \ge 1$ such that $x^{1/n} < 1 + \varepsilon$. Thus

$$x \ge 1$$

$$\begin{aligned}
&\implies x^{1/n} \geq 1^{1/n} = 1 && \text{(by Lem. I.5.6.6(d)(e))} \\
&\implies \left|x^{1/n} - 1\right| = x^{1/n} - 1 < \varepsilon
\end{aligned}$$

and by Def. I.6.1.8 $\lim_{n\to\infty} x^{1/n} = 1$.

- If $x < 1$, then $1/x > 1$. So from the proof above we have $\lim_{n\to\infty} x^{-1/n} = 1$. By Thm. I.6.1.19(e) we have $\lim_{n\to\infty} x^{-1/n} = (\lim_{n\to\infty} x^{1/n})^{-1} = 1^{-1} = 1$.

From all cases above, we conclude that $\lim_{n\to\infty} x^{1/n} = 1$. □

— Exercises —

Ex. I.6.5.1. Show that $\lim_{n\to\infty} 1/n^q = 0$ for any rational $q > 0$. Conclude that the limit $\lim_{n\to\infty} n^q$ does not exist.

Proof. We firs show that $\lim_{n\to\infty} 1/n^q = 0$ for every $q \in \mathbb{Q}^+$. Let $q = a/b$ where $a, b \in \mathbb{Z}^+$. Then we have

$$\begin{aligned}
&\frac{1}{n^q} = \frac{1}{n^{a/b}} = \left(\frac{1}{n^{1/b}}\right)^a \\
&\implies \lim_{n\to\infty} 1/n^{1/b} = 0 && \text{(by Cor. I.6.5.1)} \\
&\implies \lim_{n\to\infty} (1/n^{1/b})^a = 0 && \text{(by Thm. I.6.1.19(b))} \\
&\implies \lim_{n\to\infty} 1/n^q = 0. && \text{(by Def. I.5.6.7)}
\end{aligned}$$

Now we show that $\lim_{n\to\infty} n^q$ does not exist. Suppose for the sake of contradiction that $\lim_{n\to\infty} n^q$ exists and equals to y. Since $n \geq 1$, we have $n^q \geq 1$, so $y \geq 1$. Then we have

$$\begin{aligned}
&(\lim_{n\to\infty} n^q)^{-1} = y^{-1} && \text{(by Thm. I.6.1.19(e))} \\
&\implies \lim_{n\to\infty} (n^q)^{-1} = y^{-1} && \text{(by Thm. I.6.1.19(e))} \\
&\implies \lim_{n\to\infty} \frac{1}{n^q} = y^{-1} \\
&\implies y^{-1} = 0. && \text{(by proof above)}
\end{aligned}$$

But this means $y = 1/0$, which means such y does not exist, a contradiction. Thus, $\lim_{n\to\infty} n^q$ does not exist. □

Ex. I.6.5.2. Prove Lem. I.6.5.2.

Proof. See Lem. I.6.5.2. □

Ex. I.6.5.3. Prove Lem. I.6.5.3.

Proof. See Lem. I.6.5.3. □

I.6.6 Subsequences

Def. I.6.6.1 (Subsequences)**.** Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of real numbers. We say that $(b_n)_{n=0}^{\infty}$ is a *subsequence* of $(a_n)_{n=0}^{\infty}$ iff there exists a function $f : \mathbb{N} \to \mathbb{N}$ which is strictly increasing (i.e., $f(n+1) > f(n)$ for all $n \in \mathbb{N}$) such that

$$b_n = a_{f(n)} \text{ for all } n \in \mathbb{N}.$$

Lem. I.6.6.4. Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of real numbers. Then $(a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$. Furthermore, if $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, and $(c_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$, then $(c_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$.

Proof. We first show that $(a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$. Let $f : \mathbb{N} \to \mathbb{N}$ be a function where $f(n) = n$. Since f is strictly increasing, by Def. I.6.6.1 we know that $(a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$.

Now we show that if $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, and $(c_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$, then $(c_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$. Since $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, by Def. I.6.6.1 $\exists f : \mathbb{N} \to \mathbb{N}$ such that f is strictly increasing and $b_n = a_{f(n)}$. Since $(c_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$, by Def. I.6.6.1 $\exists g : \mathbb{N} \to \mathbb{N}$ such that g is strictly increasing and $c_n = b_{g(n)}$. Let $h = g \circ f$. Since f is strictly increasing, $\forall n_1, n_2 \in \mathbb{N}$, we have $n_1 < n_2 \implies f(n_1) < f(n_2)$. Since g is strictly increasing, we have $f(n_1) < f(n_2) \implies g(f(n_1)) < g(f(n_2))$. Thus, h is also strictly increasing, and by Def. I.6.6.1 $(c_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$ where $c_n = a_{g(f(n))}$. □

Prop. I.6.6.5 (Subsequences related to limits)**.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number. Then the following two statements are logically equivalent (each one implies the other):

(a) The sequence $(a_n)_{n=0}^{\infty}$ converges to L.

(b) Every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L.

Proof. We first show that $\lim\limits_{n\to\infty} a_n = L$ implies every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L. Let $(b_n)_{n=0}^{\infty}$ be a subsequence of $(a_n)_{n=0}^{\infty}$, and let $f : \mathbb{N} \to \mathbb{N}$ be a function where $b_n = a_{f(n)}$. Since $(a_n)_{n=0}^{\infty}$ converges to L, we have $\forall \varepsilon \in \mathbb{R}^+$, $\exists N \in \mathbb{N}$ such that $|a_n - L| \leq \varepsilon$ for every $n \geq N$. Since $f(n) \in \mathbb{N}$, we know that $\forall f(n) \geq N$, $\left|a_{f(n)} - L\right| = |b_n - L| \leq \varepsilon$. Thus, $(b_n)_{n=0}^{\infty}$ also converges to L. Since $(b_n)_{n=0}^{\infty}$ was arbitrary, we conclude that $\lim\limits_{n\to\infty} a_n = L$ implies every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L.

Now we show that every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L implies $\lim\limits_{n\to\infty} a_n = L$. By Lem. I.6.6.4 we know that $(a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, thus $\lim\limits_{n\to\infty} a_n = L$. We conclude that every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L iff $\lim\limits_{n\to\infty} a_n = L$. □

Prop. I.6.6.6 (Subsequences related to limit points)**.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number. Then the following two statements are logically equivalent.

(a) L is a limit point of $(a_n)_{n=0}^{\infty}$.

(b) There exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges to L.

Proof. We first show that L is a limit point of $(a_n)_{n=0}^{\infty}$ implies there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges to L. Since L is a limit point of $(a_n)_{n=0}^{\infty}$, by Def. I.6.4.1 $\forall \varepsilon \in \mathbb{R}^+$, $\forall n_j > 0$, $\exists n \geq n_j$ such that $|a_n - L| \leq \varepsilon$. In particular, $|a_n - L| \leq 1/j$ for every $j \in \mathbb{Z}^+$. Let $f : \mathbb{N} \to \mathbb{N}$ be a function where $f(0) = 0$ and $f(n_j) = \min\{n > n_{j-1} : |a_n - L| \leq 1/j\}$. Thus, such f is well-defined and $(a_{f(n)})_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$. Now we show that $\lim_{n\to\infty} a_{f(n)} = L$. By the definition of f we know that $\forall j \geq 1$, $\exists N \in \mathbb{Z}^+$ such that $\left|a_{f(n)} - L\right| \leq 1/j$ for every $n \geq N$. By Archimedian property (Cor. I.5.4.13) we know that $\forall \varepsilon \in \mathbb{R}^+$, $\exists j \in \mathbb{Z}^+$ such that $j\varepsilon > 1$. Thus, $\forall \varepsilon \in \mathbb{R}^+$, $\exists N \in \mathbb{Z}^+$ such that $\left|a_{f(n)} - L\right| \leq 1/j < \varepsilon$ for every $n \geq N$. This means $\lim_{n\to\infty} a_{f(n)} = L$.

Now we show that a subsequence of $(a_n)_{n=0}^{\infty}$ converges to L implies L is a limit point of $(a_n)_{n=0}^{\infty}$. Let $(b_n)_{n=0}^{\infty}$ be a subsequence of $(a_n)_{n=0}^{\infty}$ and $\lim_{n\to\infty} b_n = L$. Let $f : \mathbb{N} \to \mathbb{N}$ be a function where $b_n = a_{f(n)}$. Since $\lim_{n\to\infty} b_n = L$, $\forall \varepsilon \in \mathbb{R}^+$, $\exists n \in \mathbb{N}$ such that $|b_n - L| \leq \varepsilon$. This means $\forall N \in \mathbb{N}$, $\exists n \geq N$ such that $|b_n - L| = \left|a_{f(n)} - L\right| \leq \varepsilon$. Thus, by Def. I.6.4.1 L is a limit point of $(a_n)_{n=0}^{\infty}$. $\square$

Rmk. I.6.6.7. Prop. I.6.6.5 and I.6.6.6 give a sharp contrast between the notion of a limit, and that of a limit point. When a sequence has a limit L, then *all* subsequences also converge to L. But when a sequence has L as a limit point, then only *some* subsequences converge to L.

Note. We can now prove an important theorem in real analysis, due to Bernard Bolzano (1781–1848) and Karl Weierstrass (1815–1897): every bounded sequence has a convergent subsequence.

Thm. I.6.6.8 (Bolzano-Weierstrass theorem)**.** Let $(a_n)_{n=0}^{\infty}$ be a bounded sequence (i.e., there exists a real number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$). Then there is at least one subsequence of $(a_n)_{n=0}^{\infty}$ which converges.

Proof. Let L be the limit superior of the sequence $(a_n)_{n=0}^{\infty}$. Since we have $-M \leq a_n \leq M$ for all natural numbers n, it follows from the comparison principle (Lem. I.6.4.13) that $-M \leq L \leq M$. In particular, L is a real number (not $+\infty$ or $-\infty$). By Prop. I.6.4.12(e), L is thus a limit point of $(a_n)_{n=0}^{\infty}$. Thus, by Prop. I.6.6.6, there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges (in fact, it converges to L). $\square$

Note. we could as well have used the limit inferior instead of the limit superior in the argument of Thm. I.6.6.8.

Rmk. I.6.6.9. The Bolzano-Weierstrass theorem says that if a sequence is bounded, then eventually it has no choice but to converge in some places; it has “no room” to spread out and stop itself from acquiring limit points. It is not true for unbounded sequences; In the language of topology, this means that the interval $\{x \in \mathbb{R} : -M \leq x \leq M\}$ is *compact*, whereas an unbounded set such as the real line $\mathbb{R}$ is not compact.

— Exercises —

Ex. I.6.6.1. Prove Lem. I.6.6.4.

Proof. See Lem. I.6.6.4. □

Ex. I.6.6.2. Can you find two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ which are not the same sequence, but such that each is a subsequence of the other?

Proof. Let $(a_n)_{n=0}^{\infty} = \{0, 1, 0, 1, \ldots\}$ and $(b_n)_{n=0}^{\infty} = \{1, 0, 1, 0, \ldots\}$. □

Ex. I.6.6.3. Let $(a_n)_{n=0}^{\infty}$ be a sequence which is not bounded. Show that there exists a subsequence $(b_n)_{n=0}^{\infty}$ of $(a_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} 1/b_n$ exists and is equal to zero.

Proof. Let $j \in \mathbb{N}$, and let $f(n) : \mathbb{N} \to \mathbb{N}$ be a function defined as follow:

$$f(0) = \min\{n \in \mathbb{N} : |a_n| \geq 0\}$$
$$f(n_j) = \min\{n \in \mathbb{N} : (|a_n| \geq j) \wedge (n > n_{j-1})\}$$

Since $(a_n)_{n=0}^{\infty}$ is not bounded, we know that $\exists n \in \mathbb{N}$ such that $|a_n| \geq j$ for every $j \in \mathbb{N}$. Thus, by Thm. I.5.5.9 such f is well-defined. Let $(b_n)_{n=1}^{\infty}$ be a subsequence of $(a_n)_{n=0}^{\infty}$ where $b_n = a_{f(n)}$. Then $0 \leq |1/b_n| \leq 1/n$. Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} 1/n = 0$, by squeeze test (Cor. I.6.4.14) we have $\lim_{n\to\infty} |1/b_n| = 0$. Since $\lim_{n\to\infty} |1/b_n| = 0$, by zero test (Cor. I.6.4.17) we have $\lim_{n\to\infty} 1/b_n = 0$. Thus, there exists a subsequence $(b_n)_{n=0}^{\infty}$ of $(a_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} 1/b_n = 0$ □

Ex. I.6.6.4. Prove Prop. I.6.6.5.

Proof. See Prop. I.6.6.5. □

Ex. I.6.6.5. Prove Prop. I.6.6.6.

Proof. See Prop. I.6.6.6. □

I.6.7 Real exponentiation, part II

Lem. I.6.7.1 (Continuity of exponentiation)**.** Let $x > 0$, and let α be a real number. Let $(q_n)_{n=1}^{\infty}$ be any sequence of rational numbers converging to α. Then $(x^{q_n})_{n=1}^{\infty}$ is also a convergent sequence. Furthermore, if $(q'_n)_{n=1}^{\infty}$ is any other sequence of rational numbers converging to α, then $(x^{q'_n})_{n=1}^{\infty}$ has the same limit as $(x^{q_n})_{n=1}^{\infty}$:

$$\lim_{n\to\infty} x^{q_n} = \lim_{n\to\infty} x^{q'_n}.$$

Proof. There are three cases: $x < 1$, $x = 1$, and $x > 1$. The case $x = 1$ is rather easy (because then $x^q = 1$ for all rational q).

We first show that if $x > 1$ then $(x^{q_n})_{n=1}^{\infty}$ converges. By Thm. I.6.4.18 it is enough to show that $(x^{q_n})_{n=1}^{\infty}$ is a Cauchy sequence.

To do this, we need to estimate the distance between x^{q_n} and x^{q_m}; let us say for the time being that $q_n \geq q_m$, so that $x^{q_n} \geq x^{q_m}$ (since $x > 1$). We have

$$d(x^{q_n}, x^{q_m}) = x^{q_n} - x^{q_m} = x^{q_m}(x^{q_n - q_m} - 1).$$

Since $(q_n)_{n=1}^{\infty}$ is a convergent sequence, it has some upper bound M; since $x > 1$, we have $x^{q_m} \leq x^M$. Thus

$$d(x^{q_n}, x^{q_m}) = |x^{q_n} - x^{q_m}| \leq x^M(x^{q_n - q_m} - 1).$$

Now let $\varepsilon > 0$. We know by Lem. I.6.5.3 that the sequence $(x^{1/k})_{k=1}^{\infty}$ is eventually εx^{-M}-close to 1. Thus, there exists some $K \geq 1$ such that

$$\left| x^{1/K} - 1 \right| \leq \varepsilon x^{-M}.$$

Now since $(q_n)_{n=1}^{\infty}$ is convergent, it is a Cauchy sequence, and so there is an $N \geq 1$ such that q_n and q_m are $1/K$-close for all $n, m \geq N$. Thus, we have

$$d(x^{q_n}, x^{q_m}) \leq x^M(x^{q_n - q_m} - 1) \leq x^M(x^{1/K} - 1) \leq x^M \varepsilon x^{-M} = \varepsilon.$$

for every $n, m \geq N$ such that $q_n \geq q_m$. By symmetry we also have this bound when $n, m \geq N$ and $q_n \leq q_m$. Thus, the sequence $(x^{q_n})_{n=1}^{\infty}$ is ε-steady. Thus, the sequence $(x^{q_n})_{n=1}^{\infty}$ is eventually ε-steady for every $\varepsilon > 0$, and is thus a Cauchy sequence as desired. This proves the convergence of $(x^{q_n})_{n=1}^{\infty}$ when $x > 1$.

Next we show that if $x < 1$ then $(x^{q_n})_{n=1}^{\infty}$ also converges. By Thm. I.6.4.18 it is enough to show that $(x^{q_n})_{n=1}^{\infty}$ is a Cauchy sequence.

To do this, we need to estimate the distance between x^{q_n} and x^{q_m}; let us say for the time being that $q_n \leq q_m$, so that $x^{q_n} \geq x^{q_m}$ (since $x < 1$). We have

$$d(x^{q_n}, x^{q_m}) = x^{q_n} - x^{q_m} = x^{q_m}(x^{q_n - q_m} - 1).$$

Since $(q_n)_{n=1}^{\infty}$ is a convergent sequence, it has some lower bound M; since $x < 1$, we have $x^{q_m} \leq x^M$. Thus

$$d(x^{q_n}, x^{q_m}) = |x^{q_n} - x^{q_m}| \leq x^M(x^{q_n - q_m} - 1).$$

Now let $\varepsilon > 0$. We know by Lem. I.6.5.3 that the sequence $(x^{1/k})_{k=1}^{\infty}$ is eventually εx^{-M}-close to 1. Thus, there exists some $K \geq 1$ such that

$$\left| x^{1/K} - 1 \right| \leq \varepsilon x^{-M}.$$

Now since $(q_n)_{n=1}^{\infty}$ is convergent, it is a Cauchy sequence, and so there is an $N \geq 1$ such that q_n and q_m are $1/K$-close for all $n, m \geq N$. Thus, we have

$$d(x^{q_n}, x^{q_m}) \leq x^M(x^{q_n - q_m} - 1) \leq x^M(x^{1/K} - 1) \leq x^M \varepsilon x^{-M} = \varepsilon.$$

for every $n, m \geq N$ such that $q_n \leq q_m$. By symmetry we also have this bound when $n, m \geq N$ and $q_n \geq q_m$. Thus, the sequence $(x^{q_n})_{n=1}^{\infty}$ is ε-steady. Thus, the sequence $(x^{q_n})_{n=1}^{\infty}$ is eventually ε-steady for every $\varepsilon > 0$, and is thus a Cauchy sequence as desired. This proves the convergence of $(x^{q_n})_{n=1}^{\infty}$ when $x < 1$.

Now we prove the second claim. It will suffice to show that

$$\lim_{n\to\infty} x^{q_n - q'_n} = 1,$$

since the claim would then follow from limit laws (since $x^{q_n} = x^{q_n - q'_n} x^{q'_n}$).

Write $r_n := q_n - q'_n$; by limit laws we know that $(r_n)_{n=1}^{\infty}$ converges to 0. We have to show that for every $\varepsilon > 0$, the sequence $(x^{r_n})_{n=1}^{\infty}$ is eventually ε-close to 1. But from Lem. I.6.5.3 we know that the sequence $(x^{1/k})_{k=1}^{\infty}$ is eventually ε-close to 1. Since $\lim_{k\to\infty} x^{-1/k}$ is also equal to 1 by Lem. I.6.5.3, we know that $(x^{-1/k})_{k=1}^{\infty}$ is also eventually ε-close to 1. Thus, we can find a K such that $x^{1/K}$ and $x^{-1/K}$ are both ε-close to 1. But since $(r_n)_{n=1}^{\infty}$ is convergent to 0, it is eventually $1/K$-close to 0, so that eventually $-1/K \leq r_n \leq 1/K$, and thus when $x > 1$ we have $x^{-1/K} \leq x^{r_n} \leq x^{1/K}$, when $x < 1$ we have $x^{1/K} \leq x^{r_n} \leq x^{-1/K}$. In particular, x^{r_n} is also eventually ε-close to 1 (see Prop. I.4.3.7(f)), as desired. □

Def. I.6.7.2 (Exponentiation to a real exponent)**.** Let $x > 0$ be real, and let α be a real number. We define the quantity x^α by the formula $x^\alpha = \lim_{n\to\infty} x^{q_n}$, where $(q_n)_{n=1}^{\infty}$ is any sequence of rational numbers converging to α.

Note. Let us check that Def. I.6.7.2 is well-defined. First, of all, given any real number α we always have at least one sequence $(q_n)_{n=1}^{\infty}$ of rational numbers converging to α, by the definition of real numbers (and Prop. I.6.1.15). Secondly, given any such sequence $(q_n)_{n=1}^{\infty}$, the limit $\lim_{n\to\infty} x^{q_n}$ exists by Lem. I.6.7.1. Finally, even though there can be multiple choices for the sequence $(q_n)_{n=1}^{\infty}$, they all give the same limit by Lem. I.6.7.1. Thus, Def. I.6.7.2 is well-defined.

Note. If α is not just real but rational, i.e., $\alpha = q$ for some rational q, then Def. I.6.7.2 could in principle be inconsistent with our earlier definition of exponentiation in Sec. I.5.6. But in this case α is clearly the limit of the sequence $(q)_{n=1}^{\infty}$, so by definition $x^\alpha = \lim_{n\to\infty} x^q = x^q$. Thus, the new definition of exponentiation is consistent with the old one.

Prop. I.6.7.3. All the results of Lem. I.5.6.9, which held for rational numbers q and r, continue to hold for real numbers q and r.

Proof. (a) Let r be a real number. Then we can write $r = \lim_{n\to\infty} r_n$ for some sequences $(r_n)_{n=1}^{\infty}$ of rationals, by the definition of real numbers (and Prop. I.6.1.15). Since $(r_n)_{n=1}^{\infty}$ is a Cauchy sequence, it is bounded by some $M \in \mathbb{Q}^+$, i.e, $-M \leq r_n \leq M$ for every $n \geq 1$. By Lem. I.5.6.9, both x^M and x^{-M} are positive real numbers. If $0 < x < 1$, then $x^M \leq x^{r_n} \leq x^{-M}$. If $x \geq 1$, then $x^{-M} \leq x^{r_n} \leq x^M$. By Thm. I.6.1.19(h) we have

$$\lim_{n\to\infty} \min(x^{-M}, x^M, x^{r_n})$$

$$\begin{aligned}&= \min(\lim_{n\to\infty} x^{-M}, \lim_{n\to\infty} x^{M}, \lim_{n\to\infty} x^{r_n})\\ &= \min(x^M, x^{-M}).\end{aligned}$$

Since $\min(x^M, x^{-M})$ is positive real number, we know that x^r must also be a positive real number. $\square$

Proof. (b) Let q and r be real numbers. Then we can write $q = \lim_{n\to\infty} q_n$ and $r = \lim_{n\to\infty} r_n$ for some sequences $(q_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ of rationals, by the definition of real numbers (and Prop. I.6.1.15). Then by the limit laws, $q + r$ is the limit of $(q_n + r_n)_{n=1}^{\infty}$. By definition of real exponentiation, we have

$$x^{q+r} = \lim_{n\to\infty} x^{q_n+r_n}; x^q = \lim_{n\to\infty} x^{q_n}; x^r = \lim_{n\to\infty} x^{r_n}.$$

But by Lem. I.5.6.9(b) (applied to *rational* exponents) we have $x^{q_n+r_n} = x^{q_n}x^{r_n}$. Thus, by limit laws we have $x^{q+r} = x^q x^r$, as desired.

Now we show that $(x^q)^r = \lim_{n\to\infty}(x^{q_n})^{r_n}$. By Thm. I.6.1.19(c) we know that $qr_n = \lim_{m\to\infty} q_m r_n$. Thus, we have

$$\begin{aligned}(x^q)^r &= \lim_{n\to\infty}(x^q)^{r_n} && \text{(by Def. I.6.7.2)}\\ &= \lim_{n\to\infty}(\lim_{m\to\infty} x^{q_m})^{r_n} && \text{(by Def. I.6.7.2)}\\ &= \lim_{n\to\infty}(\lim_{m\to\infty} x^{q_m r_n}) && \text{(by Lem. I.5.6.9(b))}\\ &= \lim_{n\to\infty} x^{q r_n} && \text{(by Def. I.6.7.2)}\\ &= x^{qr}. && \text{(by Def. I.6.7.2)}\end{aligned}$$

$\square$

Proof. (c) Let $r \in \mathbb{R}$ where $r = \lim_{n\to\infty} r_n$ for some sequences $(r_n)_{n=1}^{\infty}$ of rationals. By Prop. I.6.1.15 r is well-defined and by Thm. I.6.1.19(c) we know that $-r$ is the limit of $(-r_n)_{n=1}^{\infty}$. By Prop. I.6.7.3(a) we know that $x^{-r} > 0$ and by Lem. I.5.6.9(a) we know that $x^{r_n} > 0$ for every $n \in \mathbb{Z}^+$. Thus, we have

$$\begin{aligned}x^{-r} &= \lim_{n\to\infty} x^{-r_n} && \text{(by Def. I.6.7.2)}\\ &= 1/\lim_{n\to\infty} x^{r_n} && \text{(by Thm. I.6.1.19(e))}\\ &= 1/x^r. && \text{(by Def. I.6.7.2)}\end{aligned}$$

$\square$

Proof. (d) Let $x, y, r \in \mathbb{R}^+$ where $r = \lim_{n\to\infty} r_n$ for some sequences $(r_n)_{n=1}^{\infty}$ of rationals. By Prop. I.6.1.15 r is well-defined. Since $r \in \mathbb{R}^+$, by Def. I.5.4.3 we know that $\exists c \in \mathbb{R}^+$ such that $r_n \geq c$ for every $n \in \mathbb{Z}^+$.

We first show that $x > y \implies x^r > y^r$.

$$\begin{aligned}
& x > y \\
\implies & \forall n \in \mathbb{Z}^+, x^{r_n} > y^{r_n} && \text{(by Lem. I.5.6.9(d))} \\
\implies & \lim_{n\to\infty} x^{r_n} \geq \lim_{n\to\infty} y^{r_n} && \text{(by Lem. I.6.4.13)} \\
\implies & x^r \geq y^r. && \text{(by Def. I.6.7.2)}
\end{aligned}$$

Now we show that $x^r \neq y^r$. Suppose for the sake of contradiction that $x^r = y^r$. Then we have

$$\begin{aligned}
& x^r = y^r \\
\implies & \lim_{n\to\infty} x^{r_n} = \lim_{n\to\infty} y^{r_n} && \text{(by Def. I.6.7.2)} \\
\implies & \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, |x^{r_n} - y^{r_n}| \leq \varepsilon. && \text{(by Def. I.5.3.1)}
\end{aligned}$$

But by Lem. I.5.6.9(d) we know that $x > y \implies x^{r_n} > y^{r_n}$, thus by setting $\varepsilon = (x^{r_n} - y^{r_n})/2$ we get

$$|x^{r_n} - y^{r_n}| = x^{r_n} - y^{r_n} \leq \frac{x^{r_n} - y^{r_n}}{2},$$

a contradiction. Thus, we must have $x^r > y^r$.

Finally we show that $x^r > y^r \implies x > y$. Suppose for the sake of contradiction that $x \leq y$. Then from the proof above we know that $x^r \leq y^r$, a contradiction. Thus, we must have $x > y$. We conclude that if $r > 0$, then $x > y \iff x^r > y^r$. □

Proof. (e) Let $x, q, r \in \mathbb{R}^+$ where $q = \lim_{n\to\infty} q_n$ and $r = \lim_{n\to\infty} r_n$ for some sequences $(q_n)_{n=1}^{\infty}$, $(r_n)_{n=1}^{\infty}$ of rationals. By Prop. I.6.1.15 q, r are well-defined. Since $q, r \in \mathbb{R}^+$, by Def. I.5.4.3 we know that $\exists c_1, c_2 \in \mathbb{R}^+$ such that $q_n \geq c_1$ and $r_n \geq c_2$ for every $n \in \mathbb{Z}^+$. Thus, by Lem. I.5.6.9(a) we know that $x^{q_n}, x^{r_n} > 0$ for every $n \in \mathbb{Z}^+$.

We first show that if $x > 1$, then $x^q > x^r \implies q > r$. We have

$$\begin{aligned}
& x^q > x^r > 0 && \text{(by Prop. I.6.7.3(a))} \\
\implies & x^{q-r} > 1 && \text{(by Prop. I.6.7.3(b)(c))} \\
\implies & \lim_{n\to\infty} x^{q_n - r_n} > 1. && \text{(by Def. I.6.7.2)}
\end{aligned}$$

Now we show that $\exists N \in \mathbb{Z}^+$ such that $q_n - r_n > 0$ for every $n \geq N$. Suppose for the sake of contradiction that $\forall N \in \mathbb{Z}^+$, $\exists n \geq N$ such that $q_n - r_n \leq 0$. Then we have

$$\begin{aligned}
& q_n - r_n \leq 0 \\
\implies & r_n - q_n \geq 0 \\
\implies & x^{r_n - q_n} \geq 1^{r_n - q_n} = 1 && \text{(by Prop. I.6.7.3(d))}
\end{aligned}$$

$$\implies x^{q_n - r_n} \le 1 \qquad \text{(by Prop. I.6.7.3(c))}$$
$$\implies \lim_{n\to\infty} x^{q_n - r_n} \le 1, \qquad \text{(by Lem. I.6.4.13)}$$

which contradict to $\lim_{n\to\infty} x^{q_n - r_n} > 1$. Thus, $\exists N \in \mathbb{Z}^+$ such that $q_n - r_n > 0$ for every $n \ge N$. This means $q_n > r_n$ for every $n \ge N$, and by Lem. I.6.4.13 we know that $q = \lim_{n\to\infty} q_n \ge \lim_{n\to\infty} r_n = r$.

Next we show that if $x > 1$, then $q > r \implies x^q > x^r$.

$$\begin{aligned}
& q > r \\
\implies & q - r > 0 \\
\implies & x^{q-r} > 1^{q-r} && \text{(by Prop. I.6.7.3(d))} \\
\implies & x^{q-r} > \lim_{n\to\infty} 1^{q_n - r_n} = \lim_{n\to\infty} 1 = 1 && \text{(by Def. I.6.7.2)} \\
\implies & x^{q-r} x^r > x^r && \text{(by Prop. I.6.7.3(a))} \\
\implies & x^{q-r+r} > x^r && \text{(by Prop. I.6.7.3(b))} \\
\implies & x^q > x^r. && \text{(by Lem. I.5.6.8)}
\end{aligned}$$

Thus, we conclude that if $x > 1$, then $x^q > x^r \iff q > r$.

Finally we show that if $x < 1$, then $x^q > x^r \iff q < r$.

$$\begin{aligned}
& x < 1 \\
\implies & x^{-1} > 1 \\
\implies & \big((x^{-1})^q < (x^{-1})^r \iff q < r\big) && \text{(from the proof above)} \\
\implies & (x^{-q} < x^{-r} \iff q < r) && \text{(by Prop. I.6.7.3(b))} \\
\implies & (x^q > x^r \iff q < r). && \text{(by Prop. I.6.7.3(a)(c))}
\end{aligned}$$

□

Proof. (f) Let $x, y, r \in \mathbb{R}^+$ where $r = \lim_{n\to\infty} r_n$ for some sequences $(r_n)_{n=1}^\infty$ of rationals. By Prop. I.6.1.15 r is well-defined. Then we have

$$\begin{aligned}
(xy)^r &= \lim_{n\to\infty} (xy)^{r_n} && \text{(by Def. I.6.7.2)} \\
&= \lim_{n\to\infty} x^{r_n} y^{r_n} && \text{(by Lem. I.5.6.9(f))} \\
&= \big(\lim_{n\to\infty} x^{r_n}\big)\big(\lim_{n\to\infty} y^{r_n}\big) && \text{(by Thm. I.6.1.19(b))} \\
&= x^r y^r. && \text{(by Def. I.6.7.2)}
\end{aligned}$$

□

— Exercises —

Ex. I.6.7.1. Prove the remaining components of Prop. I.6.7.3.

Proof. See Prop. I.6.7.3. □

Chapter I.7

Series

I.7.1 Finite series

Def. I.7.1.1 (Finite series)**.** Let m, n be integers, and let $(a_i)_{i=m}^n$ be a finite sequence of real numbers, assigning a real number a_i to each integer i between m and n inclusive (i.e., $m \leq i \leq n$). Then we define the finite sum (or finite series) $\sum_{i=m}^{n} a_i$ by the recursive formula

$$\sum_{i=m}^{n} a_i := 0 \text{ whenever } n < m;$$
$$\sum_{i=m}^{n+1} a_i := \left(\sum_{i=m}^{n} a_i \right) + a_{n+1} \text{ whenever } n \geq m-1.$$

Note. we sometimes express $\sum_{i=m}^{n} a_i$ less formally as

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n.$$

Rmk. I.7.1.2. The difference between "sum" and "series" is a subtle linguistic one. Strictly speaking, a series is an *expression* of the form $\sum_{i=m}^{n} a_i$; this series is mathematically (but not semantically) equal to a real number, which is then the *sum* of that series. For instance, $1+2+3+4+5$ is a series, whose sum is 15; if one were to be very picky about semantics, one would not consider 15 a series and one would not consider $1+2+3+4+5$ a sum, despite the two expressions having the same value. However, we will not be very careful about this distinction as it is purely linguistic and has no bearing on the mathematics; the expressions $1+2+3+4+5$ and 15 are the same number, and thus *mathematically* interchangeable, in the sense of the axiom of substitution, even if they are not semantically interchangeable.

Rmk. I.7.1.3. Note that the variable i (sometimes called the *index of summation*) is a *bound variable* (sometimes called a *dummy variable*); the expression $\sum_{i=m}^{n} a_i$ does not actually depend on any quantity named i. In particular, one can replace the index of summation i with any other symbol, and obtain the same sum:

$$\sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j.$$

Lem. I.7.1.4. (a) Let $m \le n < p$ be integers, and let a_i be a real number assigned to each integer $m \le i \le p$. Then we have

$$\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i = \sum_{i=m}^{p} a_i.$$

(b) Let $m \le n$ be integers, k be another integer, and let a_i be a real number assigned to each integer $m \le i \le n$. Then we have

$$\sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k}.$$

(c) Let $m \le n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \le i \le n$. Then we have

$$\sum_{i=m}^{n} (a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right).$$

(d) Let $m \le n$ be integers, and let a_i be a real number assigned to each integer $m \le i \le n$, and let c be another real number. Then we have

$$\sum_{i=m}^{n} (ca_i) = c\left(\sum_{i=m}^{n} a_i\right).$$

(e) (Triangle inequality for finite series) Let $m \le n$ be integers, and let a_i be a real number assigned to each integer $m \le i \le n$. Then we have

$$\left|\sum_{i=m}^{n} a_i\right| \le \sum_{i=m}^{n} |a_i|.$$

(f) (Comparison test for finite series) Let $m \le n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \le i \le n$. Suppose that $a_i \le b_i$ for all $m \le i \le n$. Then we have

$$\sum_{i=m}^{n} a_i \le \sum_{i=m}^{n} b_i$$

Proof. (a) Let $k = p - m$. By hypothesis we know that $k > 0$. Now we induct on k to show that Lem. I.7.1.4(a) is true and we start with $k = 1$. For $k = 1$, we have $p = m + 1$ and by Def. I.7.1.1 we have

$$\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i = \sum_{i=m}^{m} a_i + \sum_{i=m+1}^{p} a_i = a_m + a_{m+1} = \sum_{i=m}^{p} a_i.$$

Thus, the base case holds. Suppose inductively that for some $k \geq 1$ Lem. I.7.1.4(a) is true. Then for $k + 1 = p - m$, we have $p - 1 = k + m$ and

$$\begin{aligned}
\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i &= \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=n+1}^{p-1} a_i\right) + a_p && \text{(by Def. I.7.1.1)} \\
&= \left(\sum_{i=m}^{p-1} a_i\right) + a_p && \text{(by the induction hypothesis)} \\
&= \sum_{i=m}^{p} a_i. && \text{(by Def. I.7.1.1)}
\end{aligned}$$

This closes the induction. □

Proof. (b) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that Lem. I.7.1.4(b) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\sum_{j=m+k}^{m+k} a_{j-k} &= \left(\sum_{j=m+k}^{m+k-1} a_{j-k}\right) + a_{m+k-k} && \text{(by Def. I.7.1.1)} \\
&= 0 + a_{m+k-k} && \text{(by Def. I.7.1.1)} \\
&= 0 + a_m && \\
&= \left(\sum_{i=m}^{m-1} a_i\right) + a_m && \text{(by Def. I.7.1.1)} \\
&= \sum_{i=m}^{m} a_i. && \text{(by Def. I.7.1.1)}
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ Lem. I.7.1.4(b) is true. Then for $p + 1 = n - m$, we have $p = n - m - 1$ and

$$\begin{aligned}
\sum_{j=m+k}^{n+k} a_{j-k} &= \left(\sum_{j=m+k}^{n+k-1} a_{j-k}\right) + a_{n+k-k} && \text{(by Def. I.7.1.1)} \\
&= \left(\sum_{j=m+k}^{n+k-1} a_{j-k}\right) + a_n
\end{aligned}$$

$$= \left(\sum_{i=m}^{n-1} a_i\right) + a_n \qquad \text{(by the induction hypothesis)}$$

$$= \sum_{i=m}^{n} a_i. \qquad \text{(by Def. I.7.1.1)}$$

This closes the induction. □

Proof. (c) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that Lem. I.7.1.4(c) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\sum_{i=m}^{m}(a_i + b_i) &= \left(\sum_{i=m}^{m-1}(a_i + b_i)\right) + a_m + b_m && \text{(by Def. I.7.1.1)}\\
&= 0 + a_m + b_m && \text{(by Def. I.7.1.1)}\\
&= \left(\sum_{i=m}^{m-1} a_i\right) + \left(\sum_{i=m}^{m-1} b_i\right) + a_m + b_m && \text{(by Def. I.7.1.1)}\\
&= \left(\sum_{i=m}^{m} a_i\right) + \left(\sum_{i=m}^{m} b_i\right). && \text{(by Def. I.7.1.1)}
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ Lem. I.7.1.4(c) is true. Then for $p + 1 = n - m$, we have $p = n - m - 1$ and

$$\begin{aligned}
\sum_{i=m}^{n}(a_i + b_i) &= \left(\sum_{i=m}^{n-1}(a_i + b_i)\right) + a_n + b_n && \text{(by Def. I.7.1.1)}\\
&= \left(\sum_{i=m}^{n-1} a_i\right) + \left(\sum_{i=m}^{n-1} b_i\right) + a_n + b_n && \text{(by the induction hypothesis)}\\
&= \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right). && \text{(by Def. I.7.1.1)}
\end{aligned}$$

This closes the induction. □

Proof. (d) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that Lem. I.7.1.4(d) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\sum_{i=m}^{m} ca_i &= \left(\sum_{i=m}^{m-1} ca_i\right) + ca_m && \text{(by Def. I.7.1.1)}\\
&= 0 + ca_m && \text{(by Def. I.7.1.1)}\\
&= c \times 0 + ca_m
\end{aligned}$$

$$
\begin{aligned}
&= c\left(\sum_{i=m}^{m-1} a_i\right) + ca_m && \text{(by Def. I.7.1.1)}\\
&= c\left(\left(\sum_{i=m}^{m-1} a_i\right) + a_m\right)\\
&= c\left(\sum_{i=m}^{m} a_i\right). && \text{(by Def. I.7.1.1)}
\end{aligned}
$$

So the base case holds. Suppose inductively that for some $p \geq 0$ Lem. I.7.1.4(d) is true. Then for $p+1 = n-m$, we have $p = n-m-1$ and

$$
\begin{aligned}
\sum_{i=m}^{n} ca_i &= \left(\sum_{i=m}^{n-1} ca_i\right) + ca_n && \text{(by Def. I.7.1.1)}\\
&= c\left(\sum_{i=m}^{n-1} a_i\right) + ca_n && \text{(by the induction hypothesis)}\\
&= c\left(\left(\sum_{i=m}^{n-1} a_i\right) + a_n\right)\\
&= c\left(\sum_{i=m}^{n} a_i\right). && \text{(by Def. I.7.1.1)}
\end{aligned}
$$

This closes the induction. $\square$

Proof. (e) Let $p = n-m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that Lem. I.7.1.4(e) is true. For $p = 0$, we have $n = m$ and

$$
\begin{aligned}
\left|\sum_{i=m}^{m} a_i\right| &= \left|\left(\sum_{i=m}^{m-1} a_i\right) + a_m\right| && \text{(by Def. I.7.1.1)}\\
&= |0 + a_m| && \text{(by Def. I.7.1.1)}\\
&= 0 + |a_m|\\
&= \left(\sum_{i=m}^{m-1} |a_i|\right) + |a_m| && \text{(by Def. I.7.1.1)}\\
&= \sum_{i=m}^{m} |a_i|. && \text{(by Def. I.7.1.1)}
\end{aligned}
$$

So the base case holds. Suppose inductively that for some $p \geq 0$ Lem. I.7.1.4(e) is true. Then for $p+1 = n-m$, we have $p = n-m-1$ and

$$
\left|\sum_{i=m}^{n} a_i\right| = \left|\left(\sum_{i=m}^{n-1} a_i\right) + a_n\right| \qquad \text{(by Def. I.7.1.1)}
$$

$$\begin{aligned}
&\leq \left|\sum_{i=m}^{n-1} a_i\right| + |a_n| \\
&\leq \sum_{i=m}^{n-1} |a_i| + |a_n| && \text{(by the induction hypothesis)} \\
&= \sum_{i=m}^{n} |a_i|. && \text{(by Def. I.7.1.1)}
\end{aligned}$$

This closes the induction. □

Proof. (f) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that Lem. I.7.1.4(f) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\sum_{i=m}^{m} a_i &= \left(\sum_{i=m}^{m-1} a_i\right) + a_m && \text{(by Def. I.7.1.1)} \\
&= 0 + a_m && \text{(by Def. I.7.1.1)} \\
&\leq 0 + b_m && \text{(by hypothesis)} \\
&= \left(\sum_{i=m}^{m-1} b_i\right) + b_m && \text{(by Def. I.7.1.1)} \\
&= \sum_{i=m}^{m} b_i. && \text{(by Def. I.7.1.1)}
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ Lem. I.7.1.4(f) is true. Then for $p + 1 = n - m$, we have $p = n - m - 1$ and

$$\begin{aligned}
\sum_{i=m}^{n} a_i &= \left(\sum_{i=m}^{n-1} a_i\right) + a_n && \text{(by Def. I.7.1.1)} \\
&\leq \left(\sum_{i=m}^{n-1} b_i\right) + a_n && \text{(by the induction hypothesis)} \\
&\leq \left(\sum_{i=m}^{n-1} b_i\right) + b_n && \text{(by hypothesis)} \\
&= \sum_{i=m}^{n} b_i. && \text{(by Def. I.7.1.1)}
\end{aligned}$$

This closes the induction. □

Rmk. I.7.1.5. In the future we may omit some of the parentheses in series expressions, for instance we may write $\sum_{i=m}^{n}(a_i + b_i)$ simply as $\sum_{i=m}^{n} a_i + b_i$. This is reasonably safe from being mis-interpreted, because the alternative interpretation $(\sum_{i=m}^{n} a_i) + b_i$ does not make any sense (the index i in b_i is meaningless outside of the summation, since i is only a dummy variable).

Def. I.7.1.6 (Summations over finite sets)**.** Let X be a finite set with n elements (where $n \in \mathbb{N}$), and let $f : X \to \mathbb{R}$ be a function from X to the real numbers (i.e., f assigns a real number $f(x)$ to each element x of X). Then we can define the finite sum $\sum_{x\in X} f(x)$ as follows. We first select any bijection g from $\{i \in \mathbb{N} : 1 \le i \le n\}$ to X; such a bijection exists since X is assumed to have n elements. We then define

$$\sum_{x\in X} f(x) := \sum_{i=1}^{n} f(g(i)).$$

In some cases we would like to define the sum $\sum_{x\in X} f(x)$ when $f : Y \to \mathbb{R}$ is defined on a larger set Y than X. In such cases we use exactly the same definition as is given above.

Prop. I.7.1.8 (Finite summations are well-defined)**.** Let X be a finite set with n elements (where $n \in \mathbb{N}$), let $f : X \to \mathbb{R}$ be a function, and let $g : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$ and $h : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$ be bijections. Then we have

$$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(h(i)).$$

Proof. We induct on n; more precisely, we let $P(n)$ be the assertion that "For any set X of n elements, any function $f : X \to \mathbb{R}$, and any two bijections g, h from $\{i \in \mathbb{N} : 1 \le i \le n\}$ to X, we have $\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(h(i))$." (More informally, $P(n)$ is the assertion that Prop. I.7.1.8 is true for that value of n.) We want to prove that $P(n)$ is true for all natural numbers n.

We first check the base case $P(0)$. In this case $\sum_{i=1}^{0} f(g(i))$ and $\sum_{i=1}^{0} f(h(i))$ both equal to 0, by definition of finite series (Def. I.7.1.1), so we are done.

Now suppose inductively that $P(n)$ is true; we now prove that $P(n+1)$ is true. Thus, let X be a set with $n+1$ elements, let $f : X \to \mathbb{R}$ be a function, and let g and h be bijections from $\{i \in N : 1 \le i \le n+1\}$ to X. We have to prove that

$$\sum_{i=1}^{n+1} f(g(i)) = \sum_{i=1}^{n+1} f(h(i)). \tag{i:7.1}$$

Let $x := g(n+1)$; thus x is an element of X. By definition of finite series (Def. I.7.1.1), we can expand the left-hand side of Eq. (i:7.1) as

$$\sum_{i=1}^{n+1} f(g(i)) = \left(\sum_{i=1}^{n} f(g(i))\right) + f(x).$$

Now let us look at the right-hand side of Eq. (i:7.1). Ideally we would like to have $h(n+1)$ also equal to x - this would allow us to use the inductive hypothesis $P(n)$ much more easily - but we cannot assume this. However, since h is a bijection, we do know that there is *some* index j, with $1 \leq j \leq n+1$, for which $h(j) = x$. We now use Lem. I.7.1.4 and the definition of finite series (Def. I.7.1.1) to write

$$\begin{aligned}
\sum_{i=1}^{n+1} f(h(i)) &= \left(\sum_{i=1}^{j} f(h(i))\right) + \left(\sum_{i=j+1}^{n+1} f(h(i))\right) \\
&= \left(\sum_{i=1}^{j-1} f(h(i))\right) + f(h(j)) + \left(\sum_{i=j+1}^{n+1} f(h(i))\right) \\
&= \left(\sum_{i=1}^{j-1} f(h(i))\right) + f(x) + \left(\sum_{i=j}^{n} f(h(i+1))\right).
\end{aligned}$$

We now define the function $\tilde{h} : \{i \in \mathbb{N} : 1 \leq i \leq n\} \to X - \{x\}$ by setting $\tilde{h}(i) := h(i)$ when $i < j$ and $\tilde{h}(i) := h(i+1)$ when $i \geq j$. We can thus write the right-hand side of Eq. (i:7.1) as

$$= \left(\sum_{i=1}^{j-1} f(\tilde{h}(i))\right) + f(x) + \left(\sum_{i=j}^{n} f(\tilde{h}(i))\right) = \left(\sum_{i=1}^{n} f(\tilde{h}(i))\right) + f(x)$$

where we have used Lem. I.7.1.4 once again. Thus, to finish the proof of Eq. (i:7.1) we have to show that

$$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(\tilde{h}(i)). \tag{i:7.2}$$

But the function g (when restricted to $\{i \in \mathbb{N} : 1 \leq i \leq n\}$) is a bijection from $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ $X - \{x\}$. The function $\tilde{h}$ is also a bijection from $\{i \in \mathbb{N} : 1 \leq i \leq n\} \to X - \{x\}$ (cf. Lem. I.3.6.9). Since $X - \{x\}$ has n elements (by Lem. I.3.6.9), the claim Eq. (i:7.2) then follows directly from the induction hypothesis $P(n)$. □

Rmk. I.7.1.9. The issue is somewhat more complicated when summing over infinite sets; See Sec. I.8.2.

Rmk. I.7.1.10. Suppose that X is a set, that $P(x)$ is a property pertaining to an element x of X, and $f : \{y \in X : P(y) \text{ is true}\} \to \mathbb{R}$ is a function. Then we will often abbreviate

$$\sum_{x \in \{y \in X : P(y) \text{ is true}\}} f(x)$$

as $\displaystyle\sum_{x \in X : P(x) \text{ is true}} f(x)$ or even as $\displaystyle\sum_{P(x) \text{ is true}} f(x)$ when there is no change of confusion.

Prop. I.7.1.11 (Basic properties of summation over finite sets)**.** (a) If X is empty, and $f : X \to \mathbb{R}$ is a function (i.e., f is the empty function), we have

$$\sum_{x \in X} f(x) = 0.$$

(b) If X consists of a single element, $X = \{x_0\}$, and $f : X \to \mathbb{R}$ is a function, we have

$$\sum_{x \in X} f(x) = f(x_0).$$

(c) (Substitution, part I) If X is a finite set, $f : X \to \mathbb{R}$ is a function, and $g : Y \to X$ is a bijection, then

$$\sum_{x \in X} f(x) = \sum_{y \in Y} f(g(y)).$$

(d) (Substitution, part II) Let $n \leq m$ be integers, and let X be the set $X := \{i \in \mathbb{Z} : n \leq i \leq m$ If a_i is a real number assigned to each integer $i \in X$, then we have

$$\sum_{i=n}^{m} a_i = \sum_{i \in X} a_i.$$

(e) Let X, Y be disjoint finite sets (so $X \cap Y = \emptyset$), and $f : X \cup Y \to \mathbb{R}$ is a function. Then we have

$$\sum_{z \in X \cup Y} f(z) = \left(\sum_{x \in X} f(x) \right) + \left(\sum_{y \in Y} f(y) \right).$$

(f) (Linearity, part I) Let X be a finite set, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions. Then

$$\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x).$$

(g) (Linearity, part II) Let X be a finite set, let $f : X \to \mathbb{R}$ be a function, and let c be a real number. Then

$$\sum_{x \in X} cf(x) = c \sum_{x \in X} f(x).$$

(h) (Monotonicity) Let X be a finite set, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions such that $f(x) \leq g(x)$ for all $x \in \mathbf{X}$. Then we have

$$\sum_{x \in X} f(x) \leq \sum_{x \in X} g(x).$$

(i) (Triangle inequality) Let X be a finite set, and let $f : X \to \mathbb{R}$ be a function, then

$$\left|\sum_{x\in X} f(x)\right| \leq \sum_{x\in X} |f(x)|.$$

Proof. (a) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq 0\} \to \emptyset$ be a function. Then g is a bijection and

$$\begin{aligned}\sum_{x\in X} f(x) &= \sum_{i=1}^{0} f(g(i)) && \text{(by Def. I.7.1.6)}\\ &= 0. && \text{(by Def. I.7.1.1)}\end{aligned}$$

□

Proof. (b) Let $g : \{1\} \to \{x_0\}$ be a function. Then g is a bijection and

$$\begin{aligned}\sum_{x\in X} f(x) &= \sum_{i=1}^{1} f(g(i)) && \text{(by Def. I.7.1.6)}\\ &= \left(\sum_{i=1}^{0} f(g(i))\right) + f(g(1)) && \text{(by Def. I.7.1.1)}\\ &= 0 + f(g(1)) && \text{(by Def. I.7.1.1)}\\ &= f(x_0).\end{aligned}$$

□

Proof. (c) Let $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(Y)\} \to Y$ be a bijection. Since X is finite and g is a bijection between X and Y, we know that Y is finite and thus such h is well-defined. Then we know that $g \circ h : \{i \in \mathbb{N} : 1 \leq i \leq \#(Y)\} \to X$ is also a bijection and

$$\begin{aligned}\sum_{x\in X} f(x) &= \sum_{i=1}^{\#(Y)} f((g\circ h)(i)) && \text{(by Def. I.7.1.6)}\\ &= \sum_{i=1}^{\#(Y)} f(g(h(i)))\\ &= \sum_{i=1}^{\#(Y)} (f\circ g)(h(i))\\ &= \sum_{y\in Y} (f\circ g)(y) && \text{(by Def. I.7.1.6)}\\ &= \sum_{y\in Y} f(g(y)).\end{aligned}$$

□

Proof. (d) Let $f : X \to \{a_i \in \mathbb{R} : n \leq i \leq m\}$ be a function where $f = i \mapsto a_i$. Let $g : \{i \in \mathbb{N} : 1 \leq i \leq m - n + 1\} \to X$ be a function where $g = i \mapsto i + n - 1$. Then g is a bijection and

$$\begin{aligned}
\sum_{i \in X} a_i &= \sum_{i \in X} f(i) \\
&= \sum_{i=1}^{m-n+1} f(g(i)) && \text{(by Def. I.7.1.6)} \\
&= \sum_{i=1}^{m-n+1} f(i + n - 1) \\
&= \sum_{i=1}^{m-n+1} a_{i+n-1} \\
&= \sum_{i=1+n-1}^{m-n+1+n-1} a_{i+n-1-n+1} && \text{(by Lem. I.7.1.4(b))} \\
&= \sum_{i=n}^{m} a_i.
\end{aligned}$$

□

Proof. (e) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ and $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(Y)\} \to Y$ be bijections. Since X, Y are finite, we know that g, h are well-defined and $X \cup Y$ is finite. Let $k : \{i \in \mathbb{N} : 1 \leq i \leq \#(X \cup Y)\} \to X \cup Y$ be a bijection where

$$k(i) = \begin{cases} g(i) & \text{if } 1 \leq i \leq \#(X) \\ h(i - \#(X)) & \text{if } \#(X) + 1 \leq i \leq \#(X) + \#(Y). \end{cases}$$

Since $X \cup Y$ is finite, we know that k is well-defined and $\#(X \cup Y) = \#(X) + \#(Y)$. Then we have

$$\begin{aligned}
\sum_{z \in X \cup Y} f(z) &= \sum_{i=1}^{\#(X \cup Y)} f(k(i)) && \text{(by Def. I.7.1.6)} \\
&= \sum_{i=1}^{\#(X)} f(k(i)) + \sum_{i=\#(X)+1}^{\#(X \cup Y)} f(k(i)) && \text{(by Lem. I.7.1.4(a))} \\
&= \sum_{i=1}^{\#(X)} f(g(i)) + \sum_{i=\#(X)+1}^{\#(X \cup Y)} f(h(i - \#(X))) \\
&= \sum_{i=1}^{\#(X)} f(g(i)) + \sum_{i=1}^{\#(Y)} f(h(i)) && \text{(by Lem. I.7.1.4(b))}
\end{aligned}$$

$$= \sum_{x\in X} f(x) + \sum_{y\in Y} f(y). \qquad \text{(by Def. I.7.1.6)}$$

□

Proof. (f) Let $h : \{i \in \mathbb{N} : 1 \le i \le \#(X)\} \to X$ be a bijection. Since X is finite, we know that h is well-defined and

$$\begin{aligned}
\sum_{x\in X}(f(x)+g(x)) &= \sum_{x\in X}(f+g)(x) \\
&= \sum_{i=1}^{\#(X)} (f+g)(h(i)) && \text{(by Def. I.7.1.6)} \\
&= \sum_{i=1}^{\#(X)} (f(h(i))+g(h(i))) \\
&= \sum_{i=1}^{\#(X)} f(h(i)) + \sum_{i=1}^{\#(X)} g(h(i)) && \text{(by Lem. I.7.1.4(c))} \\
&= \sum_{x\in X} f(x) + \sum_{x\in X} g(x). && \text{(by Def. I.7.1.6)}
\end{aligned}$$

□

Proof. (g) Let $g : \{i \in \mathbb{N} : 1 \le i \le \#(X)\} \to X$ be a bijection. Since X is finite, we know that g is well-defined and

$$\begin{aligned}
\sum_{x\in X} cf(x) &= \sum_{x\in X}(cf)(x) \\
&= \sum_{i=1}^{\#(X)} (cf)(g(i)) && \text{(by Def. I.7.1.6)} \\
&= \sum_{i=1}^{\#(X)} cf(g(i)) \\
&= c\sum_{i=1}^{\#(X)} f(g(i)) && \text{(by Lem. I.7.1.4(d))} \\
&= c\sum_{x\in X} f(x). && \text{(by Def. I.7.1.6)}
\end{aligned}$$

□

Proof. (h) Let $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that h is well-defined and

$$\begin{aligned}
\sum_{x\in X} f(x) &= \sum_{i=1}^{\#(X)} f(h(i)) && \text{(by Def. I.7.1.6)}\\
&\leq \sum_{i=1}^{\#(X)} g(h(i)) && \text{(by Lem. I.7.1.4(f))}\\
&= \sum_{x\in X} g(x). && \text{(by Def. I.7.1.6)}
\end{aligned}$$

□

Proof. (i) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that g is well-defined and

$$\begin{aligned}
\left|\sum_{x\in X} f(x)\right| &= \left|\sum_{i=1}^{\#(X)} f(g(i))\right| && \text{(by Def. I.7.1.6)}\\
&\leq \sum_{i=1}^{\#(X)} |f(g(i))| && \text{(by Lem. I.7.1.4(e))}\\
&= \sum_{x\in X} |f(x)|. && \text{(by Def. I.7.1.6)}
\end{aligned}$$

□

Rmk. I.7.1.12. The substitution rule in Prop. I.7.1.11(c) can be thought of as making the substitution $x := g(y)$ (hence the name). Note that the assumption that g is a bijection is essential. From Prop. I.7.1.11(c) and (d) we see that

$$\sum_{i=n}^{m} a_i = \sum_{i=n}^{m} a_{f(i)}$$

for any bijection f from the set $\{i \in \mathbb{Z} : n \leq i \leq m\}$ to itself. Informally, this means that we can rearrange the elements of a finite sequence at will and still obtain the same value.

Lem. I.7.1.13. Let X, Y be finite sets, and let $f : X \times Y \to \mathbb{R}$ be a function. Then

$$\sum_{x\in X}\left(\sum_{y\in Y} f(x,y)\right) = \sum_{(x,y)\in X\times Y} f(x,y).$$

Proof. Let n be the number of elements in X. We will use induction on n (cf. Prop. I.7.1.8); i.e., we let $P(n)$ be the assertion that Lem. I.7.1.13 is true for any set X with n elements, and any finite set Y and any function $f : X \times Y \to \mathbb{R}$. We wish to prove $P(n)$ for all natural numbers n.

The base case $P(0)$ is easy, following from Prop. I.7.1.11(a). Now suppose that $P(n)$ is true; we now show that $P(n+1)$ is true. Let X be a set with $n+1$ elements. In particular, by Lem. I.3.6.9, we can write $X = X' \cup \{x_0\}$, where x_0 is an element of X and $X' := X - \{x_0\}$ has n elements. Then by Prop. I.7.1.11(e) we have

$$\sum_{x \in X} \left(\sum_{y \in Y} f(x, y) \right) = \sum_{x \in X'} \left(\sum_{y \in Y} f(x, y) \right) + \left(\sum_{y \in Y} f(x_0, y) \right);$$

by the induction hypothesis this is equal to

$$\sum_{(x,y) \in X' \times Y} f(x, y) + \left(\sum_{y \in Y} f(x_0, y) \right).$$

By Prop. I.7.1.11(c) this is equal to

$$\sum_{(x,y) \in X' \times Y} f(x, y) + \left(\sum_{(x,y) \in \{x_0\} \times Y} f(x, y) \right).$$

By Prop. I.7.1.11(e) this is equal to

$$\sum_{(x,y) \in X \times Y} f(x, y)$$

as desired. $\square$

Cor. I.7.1.14 (Fubini's theorem for finite series)**.** Let X, Y be finite sets, and let $f : X \times Y \to \mathbb{R}$ be a function. Then

$$\begin{aligned} \sum_{x \in X} \left(\sum_{y \in Y} f(x, y) \right) &= \sum_{(x,y) \in X \times Y} f(x, y) \\ &= \sum_{(y,x) \in Y \times X} f(x, y) \\ &= \sum_{y \in Y} \left(\sum_{x \in X} f(x, y) \right). \end{aligned}$$

Proof. In light of Lem. I.7.1.13, it suffices to show that

$$\sum_{(x,y) \in X \times Y} f(x, y) = \sum_{(y,x) \in Y \times X} f(x, y).$$

But this follows from Prop. I.7.1.11(c) by applying the bijection $h : Y \times X \to X \times Y$ defined by $h(y, x) := (x, y)$. $\square$

Rmk. I.7.1.15. We anticipate something interesting to happen when we move from finite sums to infinite sums. However, see Thm. I.8.2.2.

A.Cor. I.7.1.1 (Products over finite sets)**.** Let m, n be integers, and let $(a_i)_{i=m}^n$ be a finite sequence of real numbers, assigning a real number a_i to each integer i between m and n inclusive (i.e., $m \leq i \leq n$). Then we define the finite product $\prod_{i=m}^{n} a_i$ by the recursive formula

$$\prod_{i=m}^{n} a_i := 1 \text{ whenever } n < m;$$

$$\prod_{i=m}^{n+1} a_i := \left(\prod_{i=m}^{n} a_i\right) \times a_{n+1} \text{ whenever } n \geq m-1.$$

A.Cor. I.7.1.2. (a) Let $m \leq n < p$ be integers, and let a_i be a real number assigned to each integer $m \leq i \leq p$. Then we have

$$\prod_{i=m}^{n} a_i \times \prod_{i=n+1}^{p} a_i = \prod_{i=m}^{p} a_i.$$

(b) Let $m \leq n$ be integers, k be another integer, and let a_i be a real number assigned to each integer $m \leq i \leq n$. Then we have

$$\prod_{i=m}^{n} a_i = \prod_{j=m+k}^{n+k} a_{j-k}.$$

(c) Let $m \leq n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \leq i \leq n$. Then we have

$$\prod_{i=m}^{n} (a_i \times b_i) = \left(\prod_{i=m}^{n} a_i\right) \times \left(\prod_{i=m}^{n} b_i\right).$$

(d) Let $m \leq n$ be integers, and let a_i be a real number assigned to each integer $m \leq i \leq n$, and let c be another real number. Then we have

$$\prod_{i=m}^{n} (ca_i) = c^{n-m+1}\left(\prod_{i=m}^{n} a_i\right).$$

(e) Let $m \leq n$ be integers, and let a_i be a real number assigned to each integer $m \leq i \leq n$. Then we have

$$\left|\prod_{i=m}^{n} a_i\right| = \prod_{i=m}^{n} |a_i|.$$

Proof. (a) Let $k = p - m$. By hypothesis we know that $k > 0$. Now we induct on k to show that A.Cor. I.7.1.2(a) is true and we start with $k = 1$. For $k = 1$, we have $p = m + 1$ and by A.Cor. I.7.1.1 we have

$$\prod_{i=m}^{n} a_i \times \prod_{i=n+1}^{p} a_i = \prod_{i=m}^{m} a_i \times \prod_{i=m+1}^{p} a_i = a_m \times a_{m+1} = \prod_{i=m}^{p} a_i.$$

Thus, the base case holds. Suppose inductively that for some $k \geq 1$ A.Cor. I.7.1.2(a) is true. Then for $k + 1 = p - m$, we have $p - 1 = k + m$ and

$$\begin{aligned}
&\prod_{i=m}^{n} a_i \times \prod_{i=n+1}^{p} a_i \\
&= \left(\prod_{i=m}^{n} a_i\right) \times \left(\prod_{i=n+1}^{p-1} a_i\right) \times a_p && \text{(by A.Cor. I.7.1.1)} \\
&= \left(\prod_{i=m}^{p-1} a_i\right) \times a_p && \text{(by the induction hypothesis)} \\
&= \prod_{i=m}^{p} a_i. && \text{(by A.Cor. I.7.1.1)}
\end{aligned}$$

This closes the induction. □

Proof. (b) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that A.Cor. I.7.1.2(b) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\prod_{j=m+k}^{m+k} a_{j-k} &= \left(\prod_{j=m+k}^{m+k-1} a_{j-k}\right) \times a_{m+k-k} && \text{(by A.Cor. I.7.1.1)} \\
&= 1 \times a_{m+k-k} && \text{(by A.Cor. I.7.1.1)} \\
&= 1 \times a_m \\
&= \left(\prod_{i=m}^{m-1} a_i\right) \times a_m && \text{(by A.Cor. I.7.1.1)} \\
&= \prod_{i=m}^{m} a_i. && \text{(by A.Cor. I.7.1.1)}
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ A.Cor. I.7.1.2(b) is true. Then for $p + 1 = n - m$, we have $p = n - m - 1$ and

$$\prod_{j=m+k}^{n+k} a_{j-k} = \left(\prod_{j=m+k}^{n+k-1} a_{j-k}\right) \times a_{n+k-k} \qquad \text{(by A.Cor. I.7.1.1)}$$

$$= \left(\prod_{j=m+k}^{n+k-1} a_{j-k}\right) \times a_n$$

$$= \left(\prod_{i=m}^{n-1} a_i\right) \times a_n \qquad \text{(by the induction hypothesis)}$$

$$= \prod_{i=m}^{n} a_i. \qquad \text{(by A.Cor. I.7.1.1)}$$

This closes the induction. □

Proof. (c) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that A.Cor. I.7.1.2(c) is true. For $p = 0$, we have $n = m$ and

$$\prod_{i=m}^{m}(a_i \times b_i) = \left(\prod_{i=m}^{m-1}(a_i \times b_i)\right) \times a_m \times b_m \qquad \text{(by A.Cor. I.7.1.1)}$$

$$= 1 \times a_m \times b_m \qquad \text{(by A.Cor. I.7.1.1)}$$

$$= \left(\prod_{i=m}^{m-1} a_i\right) \times \left(\prod_{i=m}^{m-1} b_i\right) \times a_m \times b_m \qquad \text{(by A.Cor. I.7.1.1)}$$

$$= \left(\prod_{i=m}^{m} a_i\right) \times \left(\prod_{i=m}^{m} b_i\right). \qquad \text{(by A.Cor. I.7.1.1)}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ A.Cor. I.7.1.2(c) is true. Then for $p + 1 = n - m$, we have $p = n - m - 1$ and

$$\prod_{i=m}^{n}(a_i \times b_i) = \left(\prod_{i=m}^{n-1}(a_i \times b_i)\right) \times a_n \times b_n \qquad \text{(by A.Cor. I.7.1.1)}$$

$$= \left(\prod_{i=m}^{n-1} a_i\right) \times \left(\prod_{i=m}^{n-1} b_i\right) \times a_n \times b_n \qquad \text{(by the induction hypothesis)}$$

$$= \left(\prod_{i=m}^{n} a_i\right) \times \left(\prod_{i=m}^{n} b_i\right). \qquad \text{(by A.Cor. I.7.1.1)}$$

This closes the induction. □

Proof. (d) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that A.Cor. I.7.1.2(d) is true. For $p = 0$, we have $n = m$ and

$$\prod_{i=m}^{m} ca_i = \left(\prod_{i=m}^{m-1} ca_i\right) \times ca_m \qquad \text{(by A.Cor. I.7.1.1)}$$

$$\begin{aligned}
&= 1 \times ca_m && \text{(by A.Cor. I.7.1.1)}\\
&= c \times a_m\\
&= c\left(\prod_{i=m}^{m} a_i\right) && \text{(by A.Cor. I.7.1.1)}\\
&= c^{m-m+1}\left(\prod_{i=m}^{m} a_i\right).
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ A.Cor. I.7.1.2(d) is true. Then for $p + 1 = n - m$, we have $p = n - m - 1$ and

$$\begin{aligned}
\prod_{i=m}^{n} ca_i &= \left(\prod_{i=m}^{n-1} ca_i\right) \times ca_n && \text{(by A.Cor. I.7.1.1)}\\
&= c^{n-1-m+1}\left(\prod_{i=m}^{n-1} a_i\right) \times ca_n && \text{(by the induction hypothesis)}\\
&= c^{n-m+1}\left(\left(\prod_{i=m}^{n-1} a_i\right) \times a_n\right)\\
&= c^{n-m+1}\left(\prod_{i=m}^{n} a_i\right). && \text{(by A.Cor. I.7.1.1)}
\end{aligned}$$

This closes the induction. □

Proof. (e) Let $p = n - m$. By hypothesis we know that $p \geq 0$. Now we induct on p to show that A.Cor. I.7.1.2(e) is true. For $p = 0$, we have $n = m$ and

$$\begin{aligned}
\left|\prod_{i=m}^{m} a_i\right| &= \left|\left(\prod_{i=m}^{m-1} a_i\right) \times a_m\right| && \text{(by A.Cor. I.7.1.1)}\\
&= |1a_m| && \text{(by A.Cor. I.7.1.1)}\\
&= |1||a_m|\\
&= \left(\prod_{i=m}^{m-1} |a_i|\right) \times |a_m| && \text{(by A.Cor. I.7.1.1)}\\
&= \prod_{i=m}^{m} |a_i|. && \text{(by A.Cor. I.7.1.1)}
\end{aligned}$$

So the base case holds. Suppose inductively that for some $p \geq 0$ A.Cor. I.7.1.2(e) is true. Then for $p + 1 = n - m$, we have $p = n - m - 1$ and

$$\left|\prod_{i=m}^{n} a_i\right| = \left|\left(\prod_{i=m}^{n-1} a_i\right) \times a_n\right| \qquad \text{(by A.Cor. I.7.1.1)}$$

$$
\begin{aligned}
&= \left|\prod_{i=m}^{n-1} a_i\right| \times |a_n| \\
&= \left(\prod_{i=m}^{n-1} |a_i|\right) \times |a_n| && \text{(by the induction hypothesis)} \\
&= \prod_{i=m}^{n} |a_i|. && \text{(by A.Cor. I.7.1.1)}
\end{aligned}
$$

This closes the induction. □

A.Cor. I.7.1.3. Let X be a finite set with n elements (where $n \in \mathbb{N}$), and let $f : X \to \mathbb{R}$ be a function from X to the real numbers (i.e., f assigns a real number $f(x)$ to each element x of X). Then we can define the finite product $\prod_{x\in X} f(x)$ as follows. We first select any bijection g from $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ to X; such a bijection exists since X is assumed to have n elements. We then define

$$\prod_{x\in X} f(x) := \prod_{i=1}^{n} f(g(i))$$

A.Cor. I.7.1.4 (Finite products are well-defined)**.** Let X be a finite set with n elements (where $n \in \mathbb{N}$), let $f : X \to \mathbb{R}$ be a function, and let $g : \{i \in \mathbb{N} : 1 \leq i \leq n\} \to X$ and $h : \{i \in \mathbb{N} : 1 \leq i \leq n\} \to X$ be bijections. Then we have

$$\prod_{i=1}^{n} f(g(i)) = \prod_{i=1}^{n} f(h(i)).$$

Proof. Let $P(n)$ be the assertion that "For any set X of n elements, any function $f : X \to \mathbb{R}$, and any two bijections g, h from $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ to X, we have $\prod_{i=1}^{n} f(g(i)) = \prod_{i=1}^{n} f(h(i))$." (More informally, $P(n)$ is the assertion that A.Cor. I.7.1.4 is true for that value of n.) We induct on n.

We first check the base case $P(0)$. In this case $\prod_{i=1}^{0} f(g(i))$ and $\prod_{i=1}^{0} f(h(i))$ both equal to 1, by A.Cor. I.7.1.1, so we are done.

Now suppose inductively that $P(n)$ is true; we now prove that $P(n+1)$ is true. Thus, let X be a set with $n+1$ elements, let $f : X \to \mathbb{R}$ be a function, and let g and h be bijections from $\{i \in N : 1 \leq i \leq n+1\}$ to X. We have to prove that

$$\prod_{i=1}^{n+1} f(g(i)) = \prod_{i=1}^{n+1} f(h(i)). \tag{i:ac:7.1}$$

Let $x := g(n+1)$; thus x is an element of X. By A.Cor. I.7.1.1, we can expand the left-hand side of Eq. (i:ac:7.1) as

$$\prod_{i=1}^{n+1} f(g(i)) = \left(\prod_{i=1}^{n} f(g(i))\right) \times f(x).$$

Now let us look at the right-hand side of Eq. (i:ac:7.1). Since h is a bijection, we do know that there is *some* index j, with $1 \le j \le n+1$, for which $h(j) = x$. We now use A.Cor. I.7.1.1 and I.7.1.2 to write

$$\begin{aligned}\prod_{i=1}^{n+1} f(h(i)) &= \left(\prod_{i=1}^{j} f(h(i))\right) \times \left(\prod_{i=j+1}^{n+1} f(h(i))\right)\\ &= \left(\prod_{i=1}^{j-1} f(h(i))\right) \times f(h(j)) \times \left(\prod_{i=j+1}^{n+1} f(h(i))\right)\\ &= \left(\prod_{i=1}^{j-1} f(h(i))\right) \times f(x) \times \left(\prod_{i=j}^{n} f(h(i+1))\right).\end{aligned}$$

We now define the function $\tilde{h} : \{i \in \mathbb{N} : 1 \le i \le n\} \to X - \{x\}$ by setting $\tilde{h}(i) := h(i)$ when $i < j$ and $\tilde{h}(i) := h(i+1)$ when $i \ge j$. We can thus write the right-hand side of Eq. (i:ac:7.1) as

$$= \left(\prod_{i=1}^{j-1} f(\tilde{h}(i))\right) \times f(x) \times \left(\prod_{i=j}^{n} f(\tilde{h}(i))\right) = \left(\prod_{i=1}^{n} f(\tilde{h}(i))\right) \times f(x)$$

where we have used A.Cor. I.7.1.2 once again. Thus, to finish the proof of Eq. (i:ac:7.1) we have to show that

$$\prod_{i=1}^{n} f(g(i)) = \prod_{i=1}^{n} f(\tilde{h}(i)). \tag{i:ac:7.2}$$

But the function g (when restricted to $\{i \in \mathbb{N} : 1 \le i \le n\}$) is a bijection from $\{i \in \mathbb{N} : 1 \le i \le n\}$ $X - \{x\}$. The function $\tilde{h}$ is also a bijection from $\{i \in \mathbb{N} : 1 \le i \le n\} \to X - \{x\}$ (cf. Lem. I.3.6.9). Since $X - \{x\}$ has n elements (by Lem. I.3.6.9), the claim Eq. (i:ac:7.2) then follows directly from the induction hypothesis $P(n)$. □

A.Cor. I.7.1.5 (Basic properties of product over finite sets)**.** (a) If X is empty, and $f : X \to \mathbb{R}$ is a function (i.e., f is the empty function), we have

$$\prod_{x \in X} f(x) = 1.$$

(b) If X consists of a single element, $X = \{x_0\}$, and $f : X \to \mathbb{R}$ is a function, we have

$$\prod_{x \in X} f(x) = f(x_0).$$

(c) (Substitution, part I) If X is a finite set, $f : X \to \mathbb{R}$ is a function, and $g : Y \to X$ is a bijection, then

$$\prod_{x \in X} f(x) = \prod_{y \in Y} f(g(y)).$$

(d) (Substitution, part II) Let $n \le m$ be integers, and let X be the set $X := \{i \in \mathbb{Z} : n \le i \le m$
If a_i is a real number assigned to each integer $i \in X$, then we have

$$\prod_{i=n}^{m} a_i = \prod_{i \in X} a_i.$$

(e) Let X, Y be disjoint finite sets (so $X \cap Y = \emptyset$), and $f : X \cup Y \to \mathbb{R}$ is a function. Then we have

$$\prod_{z \in X \cup Y} f(z) = \left(\prod_{x \in X} f(x)\right) \times \left(\prod_{y \in Y} f(y)\right).$$

(f) Let X be a finite set, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions. Then

$$\prod_{x \in X} (f(x) \times g(x)) = \prod_{x \in X} f(x) \times \prod_{x \in X} g(x).$$

(g) Let X be a finite set, let $f : X \to \mathbb{R}$ be a function, and let c be a real number. Then

$$\prod_{x \in X} cf(x) = c^{\#(X)} \prod_{x \in X} f(x).$$

(h) Let X be a finite set, and let $f : X \to \mathbb{R}$ be a function, then

$$\left|\prod_{x \in X} f(x)\right| = \prod_{x \in X} |f(x)|.$$

Proof. (a) Let $g : \{i \in \mathbb{N} : 1 \le i \le 0\} \to \emptyset$ be a function. Then g is a bijection and

$$\begin{aligned}\prod_{x \in X} f(x) &= \prod_{i=1}^{0} f(g(i)) && \text{(by A.Cor. I.7.1.3)} \\ &= 1. && \text{(by A.Cor. I.7.1.1)}\end{aligned}$$

□

Proof. (b) Let $g : \{1\} \to \{x_0\}$ be a function. Then g is a bijection and

$$\prod_{x \in X} f(x) = \prod_{i=1}^{1} f(g(i)) \qquad \text{(by A.Cor. I.7.1.3)}$$

$$
\begin{aligned}
&= \left(\prod_{i=1}^{0} f(g(i))\right) \times f(g(1)) && \text{(by A.Cor. I.7.1.1)} \\
&= 1 \times f(g(1)) && \text{(by A.Cor. I.7.1.1)} \\
&= f(x_0).
\end{aligned}
$$

□

Proof. (c) Let $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(Y)\} \to Y$ be a bijection. Since X is finite and g is a bijection between X and Y, we know that Y is finite and thus such h is well-defined. Then we know that $g \circ h : \{i \in \mathbb{N} : 1 \leq i \leq \#(Y)\} \to X$ is also a bijection and

$$
\begin{aligned}
\prod_{x \in X} f(x) &= \prod_{i=1}^{\#(Y)} f((g \circ h)(i)) && \text{(by A.Cor. I.7.1.3)} \\
&= \prod_{i=1}^{\#(Y)} f(g(h(i))) \\
&= \prod_{i=1}^{\#(Y)} (f \circ g)(h(i)) \\
&= \prod_{y \in Y} (f \circ g)(y) && \text{(by A.Cor. I.7.1.3)} \\
&= \prod_{y \in Y} f(g(y)).
\end{aligned}
$$

□

Proof. (d) Let $f : X \to \{a_i \in \mathbb{R} : n \leq i \leq m\}$ be a function where $f = i \mapsto a_i$. Let $g : \{i \in \mathbb{N} : 1 \leq i \leq m - n + 1\} \to X$ be a function where $g = i \mapsto i + n - 1$. Then g is a bijection and

$$
\begin{aligned}
\prod_{i \in X} a_i &= \prod_{i \in X} f(i) \\
&= \prod_{i=1}^{m-n+1} f(g(i)) && \text{(by A.Cor. I.7.1.3)} \\
&= \prod_{i=1}^{m-n+1} f(i + n - 1) \\
&= \prod_{i=1}^{m-n+1} a_{i+n-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{i=1+n-1}^{m-n+1+n-1} a_{i+n-1-n+1} && \text{(by A.Cor. I.7.1.2(b))} \\
&= \prod_{i=n}^{m} a_i.
\end{aligned}
$$

□

Proof. (e) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ and $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(Y)\} \to Y$ be bijections. Since X, Y are finite, we know that g, h are well-defined and $X \cup Y$ is finite. Let $k : \{i \in \mathbb{N} : 1 \leq i \leq \#(X \cup Y)\} \to X \cup Y$ be a bijection where

$$
k(i) = \begin{cases} g(i) & \text{if } 1 \leq i \leq \#(X) \\ h(i - \#(X)) & \text{if } \#(X) + 1 \leq i \leq \#(X) + \#(Y). \end{cases}
$$

Since $X \cup Y$ is finite, we know that k is well-defined and $\#(X \cup Y) = \#(X) + \#(Y)$. Then we have

$$
\begin{aligned}
&\prod_{z \in X \cup Y} f(z) \\
&= \prod_{i=1}^{\#(X \cup Y)} f(k(i)) && \text{(by A.Cor. I.7.1.3)} \\
&= \prod_{i=1}^{\#(X)} f(k(i)) \times \prod_{i=\#(X)+1}^{\#(X \cup Y)} f(k(i)) && \text{(by A.Cor. I.7.1.2(a))} \\
&= \prod_{i=1}^{\#(X)} f(g(i)) \times \prod_{i=\#(X)+1}^{\#(X \cup Y)} f(h(i - \#(X))) \\
&= \prod_{i=1}^{\#(X)} f(g(i)) \times \prod_{i=1}^{\#(Y)} f(h(i)) && \text{(by A.Cor. I.7.1.2(b))} \\
&= \prod_{x \in X} f(x) \times \prod_{y \in Y} f(y). && \text{(by A.Cor. I.7.1.3)}
\end{aligned}
$$

□

Proof. (f) Let $h : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that h is well-defined and

$$
\begin{aligned}
&\prod_{x \in X} (f(x) \times g(x)) \\
&= \prod_{x \in X} (f \times g)(x)
\end{aligned}
$$

$$= \prod_{i=1}^{\#(X)} (f \times g)(h(i)) \qquad \text{(by A.Cor. I.7.1.3)}$$

$$= \prod_{i=1}^{\#(X)} (f(h(i)) \times g(h(i)))$$

$$= \prod_{i=1}^{\#(X)} f(h(i)) \times \prod_{i=1}^{\#(X)} g(h(i)) \qquad \text{(by A.Cor. I.7.1.2(c))}$$

$$= \prod_{x \in X} f(x) \times \prod_{x \in X} g(x). \qquad \text{(by A.Cor. I.7.1.3)}$$

□

Proof. (g) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that g is well-defined and

$$\prod_{x \in X} cf(x) = \prod_{x \in X} (cf)(x)$$

$$= \prod_{i=1}^{\#(X)} (cf)(g(i)) \qquad \text{(by A.Cor. I.7.1.3)}$$

$$= \prod_{i=1}^{\#(X)} cf(g(i))$$

$$= c^{\#(X)} \prod_{i=1}^{\#(X)} f(g(i)) \qquad \text{(by A.Cor. I.7.1.2(d))}$$

$$= c^{\#(X)} \prod_{x \in X} f(x). \qquad \text{(by A.Cor. I.7.1.3)}$$

□

Proof. (h) Let $g : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ be a bijection. Since X is finite, we know that g is well-defined and

$$\left| \prod_{x \in X} f(x) \right| = \left| \prod_{i=1}^{\#(X)} f(g(i)) \right| \qquad \text{(by A.Cor. I.7.1.3)}$$

$$= \prod_{i=1}^{\#(X)} |f(g(i))| \qquad \text{(by A.Cor. I.7.1.2(e))}$$

$$= \prod_{x \in X} |f(x)|. \qquad \text{(by A.Cor. I.7.1.3)}$$

□

A.Cor. I.7.1.6. Let X, Y be finite sets, and let $f : X \times Y \to \mathbb{R}$ be a function. Then

$$\prod_{x\in X}\left(\prod_{y\in Y} f(x,y)\right) = \prod_{(x,y)\in X\times Y} f(x,y).$$

Proof. Let n be the number of elements in X. We will use induction on n (cf. A.Cor. I.7.1.4); i.e., we let $P(n)$ be the assertion that A.Cor. I.7.1.6 is true for any set X with n elements, and any finite set Y and any function $f : X \times Y \to \mathbb{R}$. We wish to prove $P(n)$ for all natural numbers n.

The base case $P(0)$ is easy, following from A.Cor. I.7.1.5(a). Now suppose that $P(n)$ is true; we now show that $P(n+1)$ is true. Let X be a set with $n+1$ elements. In particular, by Lem. I.3.6.9, we can write $X = X' \cup \{x_0\}$, where x_0 is an element of X and $X' := X - \{x_0\}$ has n elements. Then by Rmk. I.7.1.5(e) we have

$$\prod_{x\in X}\left(\prod_{y\in Y} f(x,y)\right) = \prod_{x\in X'}\left(\prod_{y\in Y} f(x,y)\right) \times \left(\prod_{y\in Y} f(x_0,y)\right);$$

by the induction hypothesis this is equal to

$$\prod_{(x,y)\in X'\times Y} f(x,y) \times \left(\prod_{y\in Y} f(x_0,y)\right).$$

By Prop. I.7.1.11(c) this is equal to

$$\prod_{(x,y)\in X'\times Y} f(x,y) \times \left(\prod_{(x,y)\in \{x_0\}\times Y} f(x,y)\right).$$

By Prop. I.7.1.11(e) this is equal to

$$\prod_{(x,y)\in X\times Y} f(x,y)$$

as desired. □

A.Cor. I.7.1.7. Let X, Y be finite sets, and let $f : X \times Y \to \mathbb{R}$ be a function. Then

$$\begin{aligned}\prod_{x\in X}\left(\prod_{y\in Y} f(x,y)\right) &= \prod_{(x,y)\in X\times Y} f(x,y)\\ &= \prod_{(y,x)\in Y\times X} f(x,y)\\ &= \prod_{y\in Y}\left(\prod_{x\in X} f(x,y)\right).\end{aligned}$$

Proof. In light of A.Cor. I.7.1.6, it suffices to show that

$$\prod_{(x,y)\in X\times Y} f(x,y) = \prod_{(y,x)\in Y\times X} f(x,y).$$

But this follows from A.Cor. I.7.1.5(c) by applying the bijection $h : Y \times X \to X \times Y$ defined by $h(y,x) := (x,y)$. ☐

— Exercises —

Ex. I.7.1.1. Prove Lem. I.7.1.4.

Proof. See Lem. I.7.1.4. ☐

Ex. I.7.1.2. Prove Prop. I.7.1.11.

Proof. See Prop. I.7.1.11. ☐

Ex. I.7.1.3. Form a definition for the finite products $\prod_{i=1}^{n} a_i$ and $\prod_{x\in X} f(x)$. Which of the above result for finite series have analoges for finite products?

Proof. See A.Cor. I.7.1.1 to I.7.1.7. ☐

Ex. I.7.1.4. Define the *factorial function* $n!$ for natural numbers n by the recursive definition $0! := 1$ and $(n+1)! := n! \times (n+1)$. If x and y are real numbers, prove the *binomial formula*

$$(x+y)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n-j}$$

for all natural numbers n.

Proof. We induct on n. For $n = 0$, we have

$$\begin{aligned}
(x+y)^0 &= 1 \\
&= \frac{0!}{0!(0-0)!} x^0 y^{0-0} && \text{(by definition)} \\
&= \sum_{j=0}^{-1} \frac{0!}{j!(0-j)!} x^j y^{0-j} + \frac{0!}{0!(0-0)!} x^0 y^{0-0} && \text{(by Def. I.7.1.1)} \\
&= \sum_{j=0}^{0} \frac{0!}{j!(0-j)!} x^j y^{0-j} && \text{(by Def. I.7.1.1)}
\end{aligned}$$

So the base case holds. Suppose inductively that for some $n \geq 0$ the statement holds. Then for $n+1$, we have

$$
\begin{aligned}
(x+y)^{n+1} &= (x+y)^n \times (x+y) \\
&= \left(\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n-j} \right) \times (x+y) && \text{(by the induction hypothesis)} \\
&= \left(\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^{j+1} y^{n-j} \right) \\
&\quad + \left(\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \right) \\
&= \left(\sum_{j=0}^{n-1} \frac{n!}{j!(n-j)!} x^{j+1} y^{n-j} \right) && \text{(by Def. I.7.1.1)} \\
&\quad + \left(\frac{n!}{n!0!} x^{n+1} y^0 \right) \\
&\quad + \left(\sum_{j=1}^{n} \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \right) \\
&\quad + \left(\frac{n!}{0!n!} x^0 y^{n+1} \right) \\
&= \left(\sum_{j=0}^{n-1} \frac{n!}{j!(n-j)!} x^{j+1} y^{n-j} \right) + x^{n+1} && \text{(by definition)} \\
&\quad + \left(\sum_{j=1}^{n} \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \right) + y^{n+1} \\
&= \left(\sum_{j=1}^{n} \frac{n!}{(j-1)!(n+1-j)!} x^j y^{n+1-j} \right) + x^{n+1} && \text{(by Lem. I.7.1.4(b))} \\
&\quad + \left(\sum_{j=1}^{n} \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \right) + y^{n+1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\sum_{j=1}^{n} \frac{n!}{(j-1)!(n+1-j)!} x^j y^{n+1-j} \right) \\
&\quad + \left(\sum_{j=1}^{n} \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \left(\frac{n!}{(j-1)!(n+1-j)!} x^j y^{n+1-j} + \frac{n!}{j!(n-j)!} x^j y^{n+1-j} \right) && \text{(by Lem. I.7.1.4(c))} \\
&= \sum_{j=1}^{n} \left(\frac{j \times n!}{j!(n+1-j)!} x^j y^{n+1-j} + \frac{(n+1-j) \times n!}{j!(n+1-j)!} x^j y^{n+1-j} \right) \\
&= \sum_{j=1}^{n} \left(\frac{j \times n! + (n+1-j) \times n!}{j!(n+1-j)!} x^j y^{n+1-j} \right) \\
&= \sum_{j=1}^{n} \left(\frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j} \right).
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\sum_{j=0}^{n+1} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j} \\
&\quad = \frac{(n+1)!}{(n+1)!0!} x^{n+1} y^0 + \left(\sum_{j=0}^{n} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j} \right) && \text{(by Def. I.7.1.1)} \\
&\quad = x^{n+1} + \left(\sum_{j=0}^{n} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j} \right) && \text{(by definition)} \\
&\quad = x^{n+1} + \left(\sum_{j=0}^{0} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j} \right) && \text{(by Lem. I.7.1.4(a))} \\
&\qquad + \left(\sum_{j=1}^{n} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j} \right) \\
&\quad = x^{n+1} + \frac{(n+1)!}{0!(n+1)!} x^0 y^{n+1} && \text{(by Def. I.7.1.1)} \\
&\qquad + \left(\sum_{j=1}^{n} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j} \right) \\
&\quad = x^{n+1} + y^{n+1} + \left(\sum_{j=1}^{n} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j} \right). && \text{(by definition)}
\end{aligned}
$$

Thus, we have

$$
(x+y)^{n+1} = \sum_{j=0}^{n+1} \frac{(n+1)!}{j!(n+1-j)!} x^j y^{n+1-j}.
$$

and this closes the induction. □

Ex. I.7.1.5. Let X be a finite set, let m be an integer, and for each $x \in X$ let $(a_n(x))_{n=m}^{\infty}$ be a convergent sequence of real numbers. Show that the sequence $(\sum_{x \in X} a_n(x))_{n=m}^{\infty}$ is convergent, and

$$\lim_{n\to\infty} \sum_{x \in X} a_n(x) = \sum_{x \in X} \lim_{n\to\infty} a_n(x).$$

Thus, we may always interchange finite sums with convergent limits. Things however get trickier with infinite sums.

Proof. Let $k = \#(X)$. We induct on k. For $k = 0$, we have $X = \emptyset$. So

$$\begin{aligned}
\lim_{n\to\infty} \sum_{x \in X} a_n(x) &= \lim_{n\to\infty} 0 && \text{(by Prop. I.7.1.11)} \\
&= 0 \\
&= \sum_{x \in X} \lim_{n\to\infty} a_n(x). && \text{(by Prop. I.7.1.11)}
\end{aligned}$$

Thus, the base case holds. Suppose inductively that for some $k \geq 0$ the statement is true. Then for $k+1$, we have to show that the statement is also true. Let $x_0 \in X$ and $X' = X \setminus \{x_0\}$. So $\#(X') = \#(X) - 1 = n$, and we have

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{x \in X} a_n(x) \\
&= \lim_{n\to\infty} \sum_{x \in \{x_0\} \cup X'} a_n(x) \\
&= \lim_{n\to\infty} \left(\sum_{x \in \{x_0\}} a_n(x) + \sum_{x \in X'} a_n(x) \right) && \text{(by Prop. I.7.1.11(e))} \\
&= \left(\lim_{n\to\infty} \sum_{x \in \{x_0\}} a_n(x) \right) + \left(\lim_{n\to\infty} \sum_{x \in X'} a_n(x) \right) && \text{(by Thm. I.6.1.19(a))} \\
&= \left(\lim_{n\to\infty} a_n(x_0) \right) + \left(\lim_{n\to\infty} \sum_{x \in X'} a_n(x) \right) && \text{(by Prop. I.7.1.11(b))} \\
&= \left(\sum_{x \in \{x_0\}} \lim_{n\to\infty} a_n(x) \right) + \left(\lim_{n\to\infty} \sum_{x \in X'} a_n(x) \right) && \text{(by Prop. I.7.1.11(b))} \\
&= \left(\sum_{x \in \{x_0\}} \lim_{n\to\infty} a_n(x) \right) + \left(\sum_{x \in X'} \lim_{n\to\infty} a_n(x) \right) && \text{(by the induction hypothesis)} \\
&= \left(\sum_{x \in \{x_0\} \cup X'} \lim_{n\to\infty} a_n(x) \right) && \text{(by Prop. I.7.1.11(e))} \\
&= \sum_{x \in X} \lim_{n\to\infty} a_n(x).
\end{aligned}$$

This closes the induction. □

I.7.2 Infinite series

Def. I.7.2.1 (Formal infinite series)**.** A (formal) infinite series is any expression of the form

$$\sum_{n=m}^{\infty} a_n,$$

where m is an integer, and a_n is a real number for any integer $n \geq m$.

Note. We sometimes write this series as

$$a_m + a_{m+1} + a_{m+2} + \dots.$$

Note. At present, this series is only defined *formally*; we have not set this sum equal to any real number; the notation $a_m + a_{m+1} + a_{m+2} + \dots$ is of course designed to look very suggestively like a sum, but is not actually a finite sum because of the "$\dots$" symbol. To rigorously define what the series actually sums to, we need another definition.

Def. I.7.2.2 (Convergence of series)**.** Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, we define the N^{th} *partial sum* S_N of this series to be $S_N := \sum_{n=m}^{N} a_n$; of course, S_N is a real number. If the sequence $(S_N)_{N=m}^{\infty}$ converges to some limit L as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is *convergent*, and *converges to* L; we also write $L = \sum_{n=m}^{\infty} a_n$, and say that L is the *sum* of the infinite series $\sum_{n=m}^{\infty} a_n$. If the partial sums S_N diverge, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is *divergent*, and we do not assign any real number value to that series.

Rmk. I.7.2.3. Note that Prop. I.6.1.7 shows that if a series converges, then it has a unique sum, so it is safe to talk about *the* sum $L = \sum_{n=m}^{\infty} a_n$ of a convergent series.

Prop. I.7.2.5. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. Then $\sum_{n=m}^{\infty} a_n$ converges iff, for every real number $\varepsilon > 0$, there exists an integer $N \geq m$ such that

$$\left| \sum_{n=p+1}^{q} a_n \right| \leq \varepsilon \text{ for all } p, q \geq N.$$

Proof. Let $N, p, q \in \mathbb{N}$. We first show that if $\sum_{n=m}^{\infty} a_n$ converges, then $\forall \varepsilon \in \mathbb{R}^+$, $\exists N \geq m$ such that $\left|\sum_{n=p+1}^{q} a_n\right| \leq \varepsilon$ for every $p, q \geq N$. Let $k \in \mathbb{N}$ and let $S_k = \sum_{n=m}^{k} a_n$ be the k^{th} partial sum of $(a_n)_{n=m}^{\infty}$. Since $(S_k)_{k=m}^{\infty}$ converges, by Thm. I.6.4.18 $(S_k)_{k=m}^{\infty}$ is a Cauchy sequence. Then we have $\forall \varepsilon \in \mathbb{R}^+$, $\exists N \geq m$ such that $|S_q - S_p| \leq \varepsilon$ for every $p, q \geq N$. We now split into two cases:

- If $p \geq q$, then $p + 1 > q$. By Def. I.7.1.1 we have $\left|\sum_{n=p+1}^{q} a_n\right| = |0| = 0 \leq \varepsilon$.

- If $p < q$, then

$$
\begin{aligned}
& |S_q - S_p| \leq \varepsilon \\
\Longrightarrow & \left|\left(\sum_{n=m}^{q} a_n\right) - \left(\sum_{n=m}^{p} a_n\right)\right| \leq \varepsilon \\
\Longrightarrow & \left|\left(\sum_{n=m}^{p} a_n\right) + \left(\sum_{n=p+1}^{q} a_n\right) - \left(\sum_{n=m}^{p} a_n\right)\right| \leq \varepsilon \qquad \text{(by Lem. I.7.1.4(a))} \\
\Longrightarrow & \left|\sum_{n=p+1}^{q} a_n\right| \leq \varepsilon.
\end{aligned}
$$

From all cases above, we conclude that $\left|\sum_{n=p+1}^{q} a_n\right| \leq \varepsilon$ for every $p, q \geq N$.

Now we show that if $\forall \varepsilon \in \mathbb{R}^+$, $\exists N \geq m$ such that $\left|\sum_{n=p+1}^{q} a_n\right| \leq \varepsilon$ for every $p, q \geq N$, then $\sum_{n=m}^{\infty} a_n$ converges. Since $\left|\sum_{n=p+1}^{q} a_n\right| \leq \varepsilon$ for every $p, q \geq N$, we can choose some $p \leq q$ such that

$$
\begin{aligned}
& \left|\sum_{n=p+1}^{q} a_n\right| \leq \varepsilon \\
\Longrightarrow & \left|\sum_{n=p+1}^{q} a_n + \sum_{n=m}^{p} a_n - \sum_{n=m}^{p} a_n\right| \leq \varepsilon
\end{aligned}
$$

$$\implies \left| \sum_{n=m}^{q} a_n - \sum_{n=m}^{p} a_n \right| \le \varepsilon \qquad \text{(by Lem. I.7.1.4)}$$
$$\implies |S_q - S_p| \le \varepsilon.$$

This means $(S_k)_{k=m}^{\infty}$ is a Cauchy Sequence, so by Thm. I.6.4.18 $(S_k)_{k=m}^{\infty}$ converges, and by Def. I.7.2.2 $\sum_{n=m}^{\infty} a_n$ converges. We conclude that $\sum_{n=m}^{\infty} a_n$ converges iff $\forall \varepsilon \in \mathbb{R}^+$, $\exists N \ge m$ such that $\left| \sum_{n=p+1}^{q} a_n \right| \le \varepsilon$ for every $p, q \ge N$. □

Cor. I.7.2.6 (Zero test). Let $\sum_{n=m}^{\infty} a_n$ be a convergent series of real numbers. Then we must have $\lim_{n\to\infty} a_n = 0$. To put this another way, if $\lim_{n\to\infty} a_n$ is non-zero or divergent, then the series $\sum_{n=m}^{\infty} a_n$ is divergent.

Proof. Let $N, p, q \in \mathbb{N}$. Then we have

$$\sum_{n=m}^{\infty} a_n \text{ converges}$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \ge m : \forall p, q \ge N, \left| \sum_{n=p+1}^{q} a_n \right| \le \varepsilon \qquad \text{(by Prop. I.7.2.5)}$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \ge m : \forall p \ge N, \left| \sum_{n=p+1}^{p+1} a_n \right| \le \varepsilon$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \ge m : \forall p \ge N, |a_{p+1}| \le \varepsilon \qquad \text{(by Def. I.7.1.1)}$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \ge m : \forall p \ge N, |a_{p+1} - 0| \le \varepsilon$$
$$\implies \lim_{n\to\infty} a_n = 0. \qquad \text{(by Def. I.6.1.8)}$$

□

Note. If a sequence $(a_n)_{n=m}^{\infty}$ *does* converge to zero, then the series $\sum_{n=m}^{\infty} a_n$ may or may not be convergent; it depends on the series.

Def. I.7.2.8 (Absolute convergence). Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. We say that this series is *absolutely convergent* iff the series $\sum_{n=m}^{\infty} |a_n|$ is convergent.

Note. In order to distinguish convergence from absolute convergence, we sometimes refer to the former as *conditional convergence.*

Prop. I.7.2.9 (Absolute convergence test)**.** Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is also conditionally convergent. Furthermore, in this case we have the triangle inequality

$$\left|\sum_{n=m}^{\infty} a_n\right| \le \sum_{n=m}^{\infty} |a_n|.$$

Proof. We first show that if $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, then it is also conditionally convergent. Let $N, p, q \in \mathbb{N}$. Then we have

$$\begin{aligned}
&\sum_{n=m}^{\infty} |a_n| \text{ converge} \\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \ge m : \forall p, q \ge N, \left|\sum_{n=p+1}^{q} |a_n|\right| \le \varepsilon, && \text{(by Prop. I.7.2.5)} \\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \ge m : \forall p, q \ge N, \sum_{n=p+1}^{q} |a_n| \le \varepsilon \\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \ge m : \forall p, q \ge N, \\
&\quad \left|\sum_{n=p+1}^{q} a_n\right| \le \sum_{n=p+1}^{q} |a_n| \le \varepsilon && \text{(by Lem. I.7.1.4(e))} \\
&\implies \sum_{n=m}^{\infty} a_n \text{ converge.} && \text{(by Prop. I.7.2.5)}
\end{aligned}$$

Now we show that the triangle inequality is true. From the proof above we know that $\sum_{n=m}^{\infty} a_n$ converges, thus $\left|\sum_{n=m}^{\infty} a_n\right|$ exists and

$$\begin{aligned}
&\forall N \ge m, \left|\sum_{n=m}^{N} a_n\right| \le \sum_{n=m}^{N} |a_n| && \text{(by Lem. I.7.1.4(e))} \\
&\implies \lim_{N\to\infty} \left|\sum_{n=m}^{N} a_n\right| \le \lim_{N\to\infty} \sum_{n=m}^{N} |a_n| && \text{(by Lem. I.6.4.13)} \\
&\implies \lim_{N\to\infty} \max\left(\sum_{n=m}^{N} a_n, -\sum_{n=m}^{N} a_n\right) \le \lim_{N\to\infty} \sum_{n=m}^{N} |a_n|
\end{aligned}$$

$$\implies \max\left(\lim_{N\to\infty}\sum_{n=m}^{N} a_n, \lim_{N\to\infty} -\sum_{n=m}^{N} a_n\right) \leq \lim_{N\to\infty}\sum_{n=m}^{N} |a_n| \qquad \text{(by Thm. I.6.1.19(g))}$$

$$\implies \left|\lim_{N\to\infty}\sum_{n=m}^{N} a_n\right| \leq \lim_{N\to\infty}\sum_{n=m}^{N} |a_n|$$

$$\implies \left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n|. \qquad \text{(by Def. I.7.2.2)}$$

□

Rmk. I.7.2.10. The converse to this proposition is not true; there exist series which are conditionally convergent but not absolutely convergent.

Rmk. I.7.2.11. We consider the class of conditionally convergent series to include the class of absolutely convergent series as a subclass. Thus, when we say a statement such as "$\sum_{n=m}^{\infty} a_n$ is conditionally convergent," this does not automatically mean that $\sum_{n=m}^{\infty} a_n$ is not absolutely convergent. If we wish to say that a series is conditionally convergent but not absolutely convergent, then we will instead use a phrasing such as "$\sum_{n=m}^{\infty} a_n$ is *only* conditionally convergent," or "$\sum_{n=m}^{\infty} a_n$ converges conditionally, but not absolutely." We caution however that in most other texts, the terminology "conditional convergence" is meant in this latter sense (that is, of a series that converges but does not converge absolutely).

Prop. I.7.2.12 (Alternating series test)**.** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which are non-negative and decreasing, thus $a_n \geq 0$ and $a_n \geq a_{n+1}$ for every $n \geq m$. Then the series $\sum_{n=m}^{\infty} (-1)^n a_n$ is convergent iff the sequence a_n converges to 0 as $n \to \infty$.

Proof. From the zero test (Cor. I.7.2.6), we know that if $\sum_{n=m}^{\infty} (-1)^n a_n$ is a convergent series, then the sequence $((-1)^n a_n)_{n=m}^{\infty}$ converges to 0, which implies that a_n also converges to 0, since $(-1)^n a_n$ and a_n have the same distance from 0.

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} \wedge N \geq m : \forall n \geq N,$$
$$|a_n - 0| = |a_n| = |(-1)^n a_n| = |(-1)^n a_n - 0| \leq \varepsilon$$

Now suppose conversely that a_n converges to 0. For each N, let S_N be the partial sum $S_N := \sum_{n=m}^{N} (-1)^n a_n$; our job is to show that S_N converges. Observe that

$$S_{N+2} = S_N + (-1)^{N+1} a_{N+1} + (-1)^{N+2} a_{N+2}$$

$$= S_N + (-1)^{N+1}(a_{N+1} - a_{N+2}).$$

But by hypothesis, $(a_{N+1} - a_{N+2})$ is non-negative. Thus, we have $S_{N+2} \geq S_N$ when N is odd and $S_{N+2} \leq S_N$ if N is even.

Now suppose that N is even. From the above discussion and induction we see that $S_{N+2k} \leq S_N$ for all natural numbers k. Also we have $S_{N+2k+1} \geq S_{N+1} = S_N - a_{N+1}$. Finally, we have $S_{N+2k+1} = S_{N+2k} - a_{N+2k+1} \leq S_{N+2k}$. Thus, we have

$$S_N - a_{N+1} \leq S_{N+2k+1} \leq S_{N+2k} \leq S_N$$

for all k. In particular, we have

$$S_N - a_{N+1} \leq S_n \leq S_N \text{ for all } n \geq N.$$

In particular, the sequence $(S_n)_{n=m}^{\infty}$ is eventually a_{N+1}-steady

$$S_p - S_n \leq S_N - S_n \leq a_{N+1} \text{ for all } p, n \geq N.$$

But the sequence $(a_N)_{N=m}^{\infty}$ converges to 0 as $N \to \infty$, thus this implies that $(S_n)_{n=m}^{\infty}$ is eventually ε-steady for every $\varepsilon > 0$. Thus, $(S_n)_{n=m}^{\infty}$ converges, and so the series $\sum_{n=m}^{\infty}(-1)^n a_n$ is convergent. □

Note. Lack of absolute convergence does not imply lack of conditional convergence, even though absolute convergence implies conditional convergence.

Prop. I.7.2.14 (Series law)**.** (a) If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and $\sum_{n=m}^{\infty} b_n$ is a series of real numbers converging to y, then $\sum_{n=m}^{\infty}(a_n + b_n)$ is also a convergent series, and converges to $x + y$. In particular, we have

$$\sum_{n=m}^{\infty}(a_n + b_n) = \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n.$$

(b) If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and c is a real number, then $\sum_{n=m}^{\infty}(ca_n)$ is also a convergent series, and converges to cx. In particular, we have

$$\sum_{n=m}^{\infty}(ca_n) = c\sum_{n=m}^{\infty} a_n.$$

(c) Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $k \geq 0$ be an integer. If one of the two series $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m+k}^{\infty} a_n$ are convergent, then the other one is also, and we have the identity

$$\sum_{n=m}^{\infty} a_n = \sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^{\infty} a_n.$$

(d) Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers converging to x, and let k be an integer. Then $\sum_{n=m+k}^{\infty} a_{n-k}$ also converges to x.

Proof. (a) We have

$$\begin{aligned} x + y &= \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n \\ &= \lim_{N\to\infty} \sum_{n=m}^{N} a_n + \lim_{N\to\infty} \sum_{n=m}^{N} b_n && \text{(by Def. I.7.2.2)} \\ &= \lim_{N\to\infty} \left(\sum_{n=m}^{N} a_n + \sum_{n=m}^{N} b_n \right) && \text{(by Thm. I.6.1.19(a))} \\ &= \lim_{N\to\infty} \sum_{n=m}^{N} (a_n + b_n) && \text{(by Lem. I.7.1.4(c))} \\ &= \sum_{n=m}^{\infty} (a_n + b_n). && \text{(by Def. I.7.2.2)} \end{aligned}$$

□

Proof. (b) We have

$$\begin{aligned} cx &= c \sum_{n=m}^{\infty} a_n \\ &= c \lim_{N\to\infty} \sum_{n=m}^{N} a_n && \text{(by Def. I.7.2.2)} \\ &= \lim_{N\to\infty} \left(c \sum_{n=m}^{N} a_n \right) && \text{(by Thm. I.6.1.19(c))} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \sum_{n=m}^{N} (ca_n) && \text{(by Lem. I.7.1.4(d))} \\
&= \sum_{n=m}^{\infty} (ca_n). && \text{(by Def. I.7.2.2)}
\end{aligned}
$$

□

Proof. (c) Let $M \in \mathbb{N}$. Then we have

$$
\begin{aligned}
& L = \sum_{n=m}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=m}^{N} a_n && \text{(by Def. I.7.2.2)} \\
\iff & \forall \varepsilon \in \mathbb{R}^+, \exists M \geq m : \forall N \geq M, \left| \sum_{n=m}^{N} a_N - L \right| \leq \varepsilon \\
\iff & \forall \varepsilon \in \mathbb{R}^+, \exists M \geq m+k : \forall N \geq M, \left| \sum_{n=m}^{N} a_N - L \right| \leq \varepsilon \\
\iff & \forall \varepsilon \in \mathbb{R}^+, \exists M \geq m+k : \forall N \geq M, \\
& \left| \sum_{n=m}^{m+k-1} a_N + \sum_{n=m+k}^{N} a_N - L \right| \leq \varepsilon && \text{(by Lem. I.7.1.4(a))} \\
\iff & L = \lim_{N\to\infty} \left(\sum_{n=m}^{m+k-1} a_N + \sum_{n=m+k}^{N} a_n \right) \\
\iff & L = \sum_{n=m}^{m+k-1} a_N + \lim_{N\to\infty} \sum_{n=m+k}^{N} a_n \\
\iff & L = \sum_{n=m}^{m+k-1} a_N + \sum_{n=m+k}^{\infty} a_n. && \text{(by Def. I.7.2.2)}
\end{aligned}
$$

□

Proof. (d) Let $M \in \mathbb{N}$. Then we have

$$
\begin{aligned}
& x = \sum_{n=m}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=m}^{N} a_n && \text{(by Def. I.7.2.2)} \\
\implies & \forall \varepsilon \in \mathbb{R}^+, \exists M \geq m : \forall N \geq M, \left| \sum_{n=m}^{N} a_n - x \right| \leq \varepsilon \\
\implies & \forall \varepsilon \in \mathbb{R}^+, \exists M \geq m : \forall N \geq M, \left| \sum_{n=m}^{N} a_n - x \right| \leq \varepsilon
\end{aligned}
$$

$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists M \geq m-k : \forall N \geq M, \left| \sum_{n=m}^{N} a_n - x \right| \leq \varepsilon$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists M \geq m-k : \forall N \geq M,$$
$$\left| \sum_{n=m+k}^{N+k} a_{n-k} - x \right| \leq \varepsilon. \qquad \text{(by Lem. I.7.1.4(b))}$$

Now let $M' = M + k$ and $N' = N + k$. Then we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists M \geq m-k : \forall N \geq M,$$
$$\left| \sum_{n=m+k}^{N+k} a_{n-k} - x \right| \leq \varepsilon$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists M+k \geq m : \forall N+k \geq M+k,$$
$$\left| \sum_{n=m+k}^{N+k} a_{n-k} - x \right| \leq \varepsilon$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists M' \geq m : \forall N' \geq M',$$
$$\left| \sum_{n=m+k}^{N'} a_{n-k} - x \right| \leq \varepsilon$$
$$\implies x = \lim_{N' \to \infty} \sum_{n=m+k}^{N'} a_{n-k} = \sum_{n=m+k}^{\infty} a_{n-k}. \qquad \text{(by Def. I.7.2.2)}$$

□

Note. From Prop. I.7.2.14(c) we see that the convergence of a series does not depend on the first few elements of the series (though of course those elements do influence which value the series converges to). Because of this, we will usually not pay much attention as to what the initial index m of the series is.

Lem. I.7.2.15 (Telescoping series)**.** Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers which converge to 0, i.e., $\lim_{n\to\infty} a_n = 0$. Then the series $\sum_{n=0}^{\infty}(a_n - a_{n+1})$ converges to a_0. If $\lim_{n\to\infty} a_n = L$, then the series $\sum_{n=0}^{\infty}(a_n - a_{n+1})$ converges to $a_0 + L$.

Proof. Let $S_N = \sum_{n=0}^{N}(a_n - a_{n+1})$ be the N^{th} partial sum of $\sum_{n=m}^{\infty}(a_n - a_{n+1})$. We first show that $S_N = a_0 - a_{N+1}$ for every $N \in \mathbb{N}$. We induct on N. For $N = 0$, by Def. I.7.1.1 we have

$$S_0 = \sum_{n=0}^{0} a_n - a_{n+1} = a_0 - a_1,$$

so the base case holds. Suppose inductively that for some $N \geq 0$ the statement holds. Then for $N + 1$, we have

$$\begin{aligned} S_{N+1} &= \sum_{n=0}^{N+1} (a_n - a_{n+1}) \\ &= \sum_{n=0}^{N} (a_n - a_{n+1}) + a_{N+1} - a_{N+2} && \text{(by Def. I.7.1.1)} \\ &= a_0 - a_{N+1} + a_{N+1} - a_{N+2} && \text{(by the induction hypothesis)} \\ &= a_0 - a_{N+2}. \end{aligned}$$

This closes the induction.

Now we show that if $\lim_{n\to\infty} a_n = L$, then $\sum_{n=0}^{\infty} (a_n - a_{n+1}) = a_0 + L$.

$$\begin{aligned} \sum_{n=0}^{\infty} (a_n - a_{n+1}) &= \lim_{N\to\infty} S_N && \text{(by Def. I.7.2.2)} \\ &= \lim_{N\to\infty} a_0 + a_N && \text{(from claim above)} \\ &= a_0 + \lim_{N\to\infty} a_N && \text{(by Thm. I.6.1.19(a))} \\ &= a_0 + L. \end{aligned}$$

In particular, we have $\sum_{n=0}^{\infty} (a_n - a_{n+1}) = a_0$ if $\lim_{n\to\infty} a_n = 0$. □

— Exercises —

Ex. I.7.2.1. Is the series $\sum_{n=1}^{\infty} (-1)^n$ convergent or divergent? Justify your answer.

Proof. Let $(a_n)_{n=1}^{\infty}$ be a sequence where $a_n = 1$ for every $n \geq 1$. Since $\lim_{n\to\infty} a_n = 1$, by alternating series test (Prop. I.7.2.12) we know that $\sum_{n=m}^{\infty} (-1)^n a_n$ diverges. □

Ex. I.7.2.2. Prove Prop. I.7.2.5.

Proof. See Prop. I.7.2.5. □

Ex. I.7.2.3. Use Prop. I.7.2.5 to prove Cor. I.7.2.6.

Proof. See Cor. I.7.2.6. □

Ex. I.7.2.4. Prove Prop. I.7.2.9.

Proof. See Prop. I.7.2.9. □

Ex. I.7.2.5. Prove Prop. I.7.2.14.

Proof. See Prop. I.7.2.14. □

Ex. I.7.2.6. Prove Lem. I.7.2.15. How does the proposition change if we assume that an does not converge to zero, but instead converges to some other real number L?

Proof. See Lem. I.7.2.15. □

I.7.3 Sums of non-negative numbers

Note. When all the terms in a series are non-negative, there is no distinction between conditional convergence and absolute convergence.

Prop. I.7.3.1. Let $\sum_{n=m}^{\infty} a_n$ be a formal series of non-negative real numbers. Then this series is convergent iff there is a real number M such that

$$\sum_{n=m}^{N} a_n \leq M \text{ for all integers } N \geq m.$$

Proof. Suppose $\sum_{n=m}^{\infty} a_n$ is a series of non-negative numbers. Then the partial sums $S_N := \sum_{n=m}^{N} a_n$ are increasing, i.e., $S_{N+1} \geq S_N$ for all $N \geq m$. From Prop. I.6.3.8 and Cor. I.6.1.17, we thus see that the sequence $(S_N)_{n=m}^{\infty}$ is convergent iff it has an upper bound M. □

Cor. I.7.3.2 (Comparison test)**.** Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two formal series of real numbers, and suppose that $|a_n| \leq b_n$ for all $n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, and in fact

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Proof. Let $N \in \mathbb{N}$. Then we have

$$\begin{aligned}
&\sum_{n=m}^{\infty} b_n \text{ converges} \\
\implies &\exists M \in \mathbb{R} : \forall N \geq m, \sum_{n=m}^{N} b_n \leq M && \text{(by Prop. I.7.3.1)} \\
\implies &\exists M \in \mathbb{R} : \forall N \geq m, \sum_{n=m}^{N} |a_n| \leq \sum_{n=m}^{N} b_n \leq M && \text{(by hypothesis)} \\
\implies &\sum_{n=m}^{\infty} |a_n| \text{ converges} && \text{(by Prop. I.7.3.1)} \\
\implies &\sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n && \text{(by Lem. I.6.4.13)} \\
\implies &\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n. && \text{(by Prop. I.7.2.9)}
\end{aligned}$$

□

Note. We can also run the comparison test in the contrapositive: if we have $|a_n| \leq b_n$ for all $n \geq m$, and $\sum_{n=m}^{\infty} a_n$ is not absolutely convergent, then $\sum_{n=m}^{\infty} b_n$ is not conditionally convergent.

Lem. I.7.3.3 (Geometric series)**.** Let x be a real number. If $|x| \geq 1$, then the series $\sum_{n=0}^{\infty} x^n$ is divergent. If however $|x| < 1$, then the series is absolutely convergent and

$$\sum_{n=0}^{\infty} x^n = 1/(1-x).$$

Proof. We first show that if $|x| \geq 1$, then $\sum_{n=0}^{\infty} x^n$ is divergent. We split into two cases:

(a) If $x = 1$, then $\lim_{n\to\infty} x^n = 1$. By zero test (Cor. I.7.2.6) $\sum_{n=0}^{\infty} x^n$ diverges.

(b) If $x = -1$ or $|x| > 1$, then by Lem. I.6.5.2 $\lim_{n\to\infty} x^n$ diverges. Thus, by zero test (Cor. I.7.2.6) $\sum_{n=0}^{\infty} x^n$ diverges.

From all cases above, we conclude that if $|x| \geq 1$, then $\displaystyle\sum_{n=0}^{\infty} x^n$ diverges.

Next we show that $\displaystyle\sum_{n=0}^{N} x^n = (1-x^{N+1})/(1-x)$. We induct on N. For $N=0$, by Def. I.7.1.1 we have

$$\sum_{n=0}^{0} x^n = x^0 = 1 = \frac{1-x}{1-x} = \frac{1-x^1}{1-x}.$$

So the base case holds. Suppose inductivly that for some $N \geq 0$ we have $\displaystyle\sum_{n=0}^{N} x^n = (1-x^{N+1})/(1-x)$. Then for $N+1$, we have

$$\begin{aligned}
\sum_{n=0}^{N+1} x^n &= \sum_{n=0}^{N} x^n + x^{N+1} && \text{(by Def. I.7.1.1)} \\
&= \frac{1-x^{N+1}}{1-x} + x^{N+1} && \text{(by the induction hypothesis)} \\
&= \frac{1-x^{N+1}}{1-x} + \frac{(1-x)x^{N+1}}{1-x} \\
&= \frac{1-x^{N+1}}{1-x} + \frac{x^{N+1}-x^{N+2}}{1-x} \\
&= \frac{1-x^{N+2}}{1-x}.
\end{aligned}$$

This closes the induction. Using similar arguments, we can show that

$$\sum_{n=0}^{N} |x^n| = \frac{1-|x^{N+1}|}{1-|x|}.$$

Now we use the induction result to show that if $|x| < 1$, then $\displaystyle\sum_{n=0}^{\infty} x^n$ is absolutely convergent and $\displaystyle\sum_{n=0}^{\infty} x^n = 1/(1-x)$.

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n &= \lim_{N\to\infty} \sum_{n=0}^{N} x^n && \text{(by Def. I.7.2.2)} \\
&= \lim_{N\to\infty} \frac{1-x^{N+1}}{1-x} && \text{(from claim above)} \\
&= \frac{\lim_{N\to\infty} 1-x^{N+1}}{1-x} && \text{(by Thm. I.6.1.19(f))}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lim_{N\to\infty} 1 - \lim_{N\to\infty} x^{N+1}}{1-x} && \text{(by Thm. I.6.1.19(d))}\\
&= \frac{1 - (\lim_{N\to\infty} x^{N+1})}{1-x} && \text{(by A.Cor. I.6.5.1)}\\
&= \frac{1-0}{1-x} && \text{(by Lem. I.6.5.2)}\\
&= \frac{1}{1-x}.
\end{aligned}
$$

Using similar arguments, we can show that $\sum_{n=0}^{\infty} |x^n| = 1/(1-|x|)$. Thus, we conclude that if $|x| < 1$, then $\sum_{n=0}^{\infty} x^n$ is absolutely convergent and $\sum_{n=0}^{\infty} x^n = 1/(1-x)$. □

Prop. I.7.3.4 (Cauchy criterion). Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of non-negative real numbers (so $a_n \geq 0$ and $a_{n+1} \leq a_n$ for all $n \geq 1$). Then the series $\sum_{n=1}^{\infty} a_n$ is convergent iff the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \dots$$

is convergent.

Proof. Let $S_N := \sum_{n=1}^{N} a_n$ be the partial sums of $\sum_{n=1}^{\infty} a_n$, and let $T_K := \sum_{k=0}^{K} 2^k a_{2^k}$ be the partial sums of $\sum_{k=0}^{\infty} 2^k a_{2^k}$. In light of Prop. I.7.3.1, our task is to show that the sequence $(S_N)_{N=1}^{\infty}$ is bounded iff the sequence $(T_K)_{K=0}^{\infty}$ is bounded. From Lem. I.7.3.6 we see that if $(S_N)_{N=1}^{\infty}$ is bounded, then $(S_{2^K})_{K=0}^{\infty}$ is bounded, and hence $(T_K)_{K=0}^{\infty}$ is bounded. Conversely, if $(T_K)_{K=0}^{\infty}$ is bounded, then Lem. I.7.3.6 implies that $(S_{2^{K+1}-1})_{K=0}^{\infty}$ is bounded, i.e., there is an M such that $S_{2^{K+1}-1} \leq M$ for all natural numbers K. But one can easily show (using induction) that $2^{K+1} - 1 \geq K + 1$, and hence that $S_{K+1} \leq M$ for all natural numbers K, hence $(S_N)_{N=1}^{\infty}$ is bounded. □

Rmk. I.7.3.5. An interesting feature of this criterion is that it only uses a small number of elements of the sequence a_n (namely, those elements whose index n is a power of 2, $n = 2^k$) in order to determine whether the whole series is convergent or not.

Lem. I.7.3.6. For any natural number K, we have $S_{2^{K+1}-1} \leq T_K \leq 2S_{2^K}$.

Proof. We induct on K. First, we prove the claim when $K = 0$, i.e.

$$S_1 \leq T_0 \leq 2S_1.$$

This becomes

$$a_1 \leq a_1 \leq 2a_1$$

which is clearly true, since a_1 is non-negative. Now suppose the claim has been proven for K, and now we try to prove it for $K+1$:

$$S_{2^{K+2}-1} \leq T_{K+1} \leq 2S_{2^{K+1}}.$$

Clearly, we have

$$T_{K+1} = T_K + 2^{K+1} a_{2^{K+1}}.$$

Also, we have (using Lem. I.7.1.4(a) and (f), and the hypothesis that the a_n are decreasing)

$$S_{2^{K+1}} = S_{2^K} + \sum_{n=2^K+1}^{2^{K+1}} a_n \geq S_{2^K} + \sum_{n=2^K+1}^{2^{K+1}} a_{2^{K+1}} = S_{2^K} + 2^K a_{2^{K+1}}$$

and hence

$$2S_{2^{K+1}} \geq 2S_{2^K} + 2^{K+1} a_{2^{K+1}}.$$

Similarly, we have

$$\begin{aligned} S_{2^{K+2}-1} &= S_{2^{K+1}-1} + \sum_{n=2^{K+1}}^{2^{K+2}-1} a_n \\ &\leq S_{2^{K+1}-1} + \sum_{n=2^{K+1}}^{2^{K+2}-1} a_{2^{K+1}} \\ &= S_{2^{K+1}-1} + 2^{K+1} a_{2^{K+1}}. \end{aligned}$$

Combining these inequalities with the induction hypothesis

$$S_{2^{K+1}-1} \leq T_K \leq 2S_{2^K}$$

we obtain

$$S_{2^{K+2}-1} \leq T_{K+1} \leq 2S_{2^{K+1}}$$

as desired. This proves the claim. □

Cor. I.7.3.7. Let $q > 0$ be a real number. Then the series $\sum_{n=1}^{\infty} 1/n^q$ is convergent when $q > 1$ and divergent when $q \leq 1$.

Proof. The sequence $(1/n^q)_{n=1}^{\infty}$ is non-negative and decreasing (by Prop. I.6.7.3), and so the Cauchy criterion (Prop. I.7.3.4) applies. Thus, this series is convergent iff

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^q}$$

is convergent. But by the laws of exponentiation (Prop. I.6.7.3) we can rewrite this as the geometric series

$$\sum_{k=0}^{\infty}(2^{1-q})^k.$$

By Lem. I.7.3.3, the geometric series $\sum_{k=0}^{\infty} x^k$ converges iff $|x| < 1$. Thus, the series $\sum_{n=1}^{\infty} 1/n^q$ will converge iff $\left|2^{1-q}\right| < 1$, which happens iff $q > 1$. $\square$

Note. In particular, the series $\sum_{n=1}^{\infty} 1/n$ (also known as the *harmonic series*) is divergent, as claimed earlier. However, the series is $\sum_{n=1}^{\infty} 1/n^2$ convergent.

Rmk. I.7.3.8. The quantity $\sum_{n=1}^{\infty} 1/n^q$, when it converges, is called $\zeta(q)$, the *Riemann-zeta function of q*. This function is very important in number theory, particularly in the distribution of the primes; there is a very famous unsolved problem regarding this function, called the *Riemann hypothesis*, but to discuss it further is far beyond the scope of this text. I will mention however that there is a US$ 1 million prize - and instant fame among all mathematicians - attached to the solution to this problem.

— Exercises —

Ex. I.7.3.1. Use Prop. I.7.3.1 to prove Cor. I.7.3.2.

Proof. See Cor. I.7.3.2. $\square$

Ex. I.7.3.2. Prove Lem. I.7.3.3.

Proof. See Lem. I.7.3.3. $\square$

Ex. I.7.3.3. Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers such that $\sum_{n=0}^{\infty}|a_n| = 0$. Show that $a_n = 0$ for every natural number n.

Proof. Let $N, k \in \mathbb{N}$. Then we have $\forall 0 \leq k \leq N$,

$$0 \leq |a_k| \leq \sum_{n=0}^{N}|a_n|$$

$$\implies \lim_{N\to\infty} 0 \le \lim_{N\to\infty} |a_k| \le \lim_{N\to\infty} \sum_{n=0}^{N} |a_n| \qquad \text{(by Lem. I.6.4.13)}$$

$$\implies 0 \le |a_k| \le \sum_{n=0}^{\infty} |a_n| = 0 \qquad \text{(by Def. I.7.2.2)}$$

$$\implies |a_k| = 0 = a_k.$$

Since N was arbitrary, we have $a_k = 0$ for every $k \ge 0$. □

I.7.4 Rearrangement of series

Note. One feature of finite sums is that no matter how one rearranges the terms in a sequence, the total sum is the same. A more rigorous statement of this, involving bijections, has already appeared earlier, see Rmk. I.7.1.12.

Prop. I.7.4.1. Let $\sum_{n=0}^{\infty} a_n$ be a convergent series of non-negative real numbers, and let $f : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{m=0}^{\infty} a_{f(m)}$ is also convergent, and has the same sum:

$$\sum_{n=0}^{\infty} a_n = \sum_{m=0}^{\infty} a_{f(m)}.$$

Proof. We introduce the partial sums $S_N := \sum_{n=0}^{N} a_n$ and $T_M := \sum_{m=0}^{M} a_{f(m)}$. We know that the sequences $(S_N)_{N=0}^{\infty}$ and $(T_M)_{M=0}^{\infty}$ are increasing. Write $L := \sup(S_N)_{N=0}^{\infty}$ and $L' := \sup(T_M)_{M=0}^{\infty}$. By Prop. I.6.3.8 we know that L is finite, and in fact $L = \sum_{n=0}^{\infty} a_n$; by Prop. I.6.3.8 again we see that we will thus be done as soon as we can show that $L' = L$.

Fix M, and let Y be the set $Y := \{m \in \mathbb{N} : m \le M\}$. Note that f is a bijection between Y and $f(Y)$. By Prop. I.7.1.11, we have

$$T_M = \sum_{m=0}^{M} a_{f(m)} = \sum_{m\in Y} a_{f(m)} = \sum_{n\in f(Y)} a_n.$$

The sequence $(f(m))_{m=0}^{M}$ is finite, hence bounded, i.e., there exists an N such that $f(m) \le N$ for all $m \le M$. In particular, $f(Y)$ is a subset of $\{n \in \mathbb{N} : n \le N\}$, and so by Prop. I.7.1.11 again (and the assumption that all the a_n are non-negative)

$$T_M = \sum_{n\in f(Y)} a_n \le \sum_{n\in\{n\in\mathbb{N}:n\le N\}} a_n = \sum_{n=0}^{N} a_n = S_N.$$

But since $(S_N)_{N=0}^{\infty}$ has a supremum of L, we thus see that $S_N \le L$, and hence that $T_M \le L$ for all M. Since L' is the least upper bound of $(T_M)_{M=0}^{\infty}$, this implies that $L' \le L$.

Now we fix N, and let X be the set $X := \{n \in \mathbb{N} : n \le N\}$. Note that f^{-1} is a bijection between X and $f^{-1}(X)$. By Prop. I.7.1.11, we have

$$S_N = \sum_{n=0}^{N} a_n = \sum_{n \in X} a_n = \sum_{m \in f^{-1}(X)} a_{f(m)}.$$

The sequence $(f^{-1}(n))_{n=0}^{N}$ is finite, hence bounded, i.e., there exists an M such that $f^{-1}(n) \le M$ for all $n \le N$. In particular, $f^{-1}(X)$ is a subset of $\{m \in \mathbb{N} : m \le M\}$, and so by Prop. I.7.1.11 again (and the assumption that all the a_n are non-negative)

$$S_N = \sum_{m \in f^{-1}(X)} a_{f(m)} \le \sum_{m \in \{m \in \mathbb{N} : m \le M\}} a_{f(m)} = \sum_{m=0}^{M} a_{f(m)} = T_M.$$

But since $(T_M)_{M=0}^{\infty}$ has a supremum of L', we thus see that $T_M \le L'$, and hence that $S_N \le L'$ for all N. Since L is the least upper bound of $(S_N)_{N=0}^{\infty}$, this implies that $L \le L'$.

Combining these two inequalities we obtain $L = L'$, as desired. □

Prop. I.7.4.3 (Rearrangement of series)**.** Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers, and let $f : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{m=0}^{\infty} a_{f(m)}$ is also absolutely convergent, and has the same sum:

$$\sum_{n=0}^{\infty} a_n = \sum_{m=0}^{\infty} a_{f(m)}.$$

Proof. We apply Prop. I.7.4.1 to the infinite series $\sum_{n=0}^{\infty} |a_n|$, which by hypothesis is a convergent series of non-negative numbers. If we write $L := \sum_{n=0}^{\infty} |a_n|$, then by Prop. I.7.4.1 we know that $\sum_{m=0}^{\infty} |a_{f(m)}|$ also converges to L.

Now write $L' := \sum_{n=0}^{\infty} a_n$. We have to show that $\sum_{m=0}^{\infty} a_{f(m)}$ also converges to L'. In other words, given any $\varepsilon > 0$, we have to find an M such that $\sum_{m=0}^{M'} a_{f(m)}$ is ε-close to L' for every $M' \ge M$.

Since $\sum_{n=0}^{\infty}|a_n|$ is convergent, we can use Prop. I.7.2.5 and find an N_1 such that $\sum_{n=p+1}^{q}|a_n| \leq \varepsilon/2$ for all $p, q \geq N_1$. Since $\sum_{n=0}^{\infty} a_n$ converges to L', the partial sums $\sum_{n=0}^{N} a_n$ also converge to L', and so there exists $N \geq N_1$ such that $\sum_{n=0}^{N} a_n$ is $\varepsilon/2$-close to L'.

Now the sequence $(f^{-1}(n))_{n=0}^{N}$ is finite, hence bounded, so there exists an M such that $f^{-1}(n) \leq M$ for all $0 \leq n \leq N$. In particular, for any $M' \geq M$, the set $\{f(m) : m \in \mathbb{N}; m \leq M'\}$ contains $\{n \in \mathbb{N} : n \leq N\}$. So by Prop. I.7.1.11, for any $M' \geq M$,

$$\sum_{m=0}^{M'} a_{f(m)} = \sum_{n \in \{f(m):m\in\mathbb{N};m\leq M'\}} a_n = \sum_{n=0}^{N} a_n + \sum_{n\in X} a_n$$

where X is the set

$$X = \{f(m) : m \in \mathbb{N}; m \leq M'\} \setminus \{n \in \mathbb{N} : n \leq N\}.$$

The set X is finite, and is therefore bounded by some natural number q; we must therefore have

$$X \subseteq \{n \in \mathbb{N} : N + 1 \leq n \leq q\}.$$

Thus

$$\left|\sum_{m=0}^{M'} a_{f(m)} - \sum_{n=0}^{N} a_n\right| = \left|\sum_{n\in X} a_n\right| \leq \sum_{n\in X}|a_n| \leq \sum_{n=N+1}^{q}|a_n| \leq \varepsilon/2$$

by our choice of N. Thus, $\sum_{m=0}^{M'} a_{f(m)}$ is $\varepsilon/2$-close to $\sum_{n=0}^{N} a_n$, which as mentioned before is $\varepsilon/2$-close to L'. Thus, $\sum_{m=0}^{M'} a_{f(m)}$ is ε-close to L' for all $M' \geq M$, as desired. □

Note. There is in fact a surprising result of Riemann, which shows that a series which is conditionally convergent but not absolutely convergent can in fact be rearranged to converge to *any* value (or rearranged to diverge).

Note. To summarize, rearranging series is safe when the series is absolutely convergent, but is somewhat dangerous otherwise. (This is not to say that rearranging a series that is not absolutely convergent necessarily gives you the wrong answer - for instance, in theoretical physics one often performs similar maneuvres, and one still (usually) obtains a correct answer at the end - but doing so is risky, unless it is backed by a rigorous result such as Prop. I.7.4.3.)

— Exercises —

Ex. I.7.4.1. Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers. Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function (i.e., $f(n+1) > f(n)$ for all $n \in \mathbb{N}$). Show that $\sum_{n=0}^{\infty} a_{f(n)}$ is also an absolutely convergent series. What happens if we assume f is merely one-to-one, rather than increasing?

Proof. Since f is bijective implies f is one-to-one, we only need to proof the case for f being one-to-one.

Let $S_N = \sum_{n=0}^{N} |a_n|$ and $T_N = \sum_{n=0}^{N} \left|a_{f(n)}\right|$. Since $\sum_{n=0}^{\infty} a_n$ is absolutely convergent and S_N is an increasing sequence, by Prop. I.6.3.8 we have $\lim_{N\to\infty} S_N = \sup(S_N)_{N=0}^{\infty}$. Since $(f(n))_{n=0}^{N}$ is a finite sequence with $N+1$ unique elements (f is assumed to be one-to-one), it is bounded by some $M \in \mathbb{N}$, thus $\{f(n) : n \in \mathbb{N} \wedge n \le N\} \subseteq \{n \in \mathbb{N} : n \le M\}$. Now we have

$$T_N = \sum_{n=0}^{N} \left|a_{f(n)}\right| \le \sum_{n \in \mathbb{N}: n \le M} |a_n| = S_M \le \sup(S_M)_{M=0}^{\infty} = \lim_{M\to\infty} S_M,$$

which means T_N is bounded. Since $(T_N)_{N=0}^{\infty}$ is an increasing sequence and is bounded, by Prop. I.6.3.8 $(T_N)_{N=0}^{\infty}$ converges, and thus $\sum_{n=0}^{\infty} a_{f(n)}$ is absolutely convergent. □

Ex. I.7.4.2. Obtain an alternate proof of Prop. I.7.4.3 using Prop. I.7.4.1, Prop. I.7.2.14, and expressing a_n as the difference of $a_n + |a_n|$ and $|a_n|$. (This proof is due to Will Ballard.)

Proof. From hypothesis we know that $\sum_{n=0}^{\infty} |a_n|$ converges, thus by Prop. I.7.2.9 we know that $\sum_{n=0}^{\infty} a_n$ converges. Since $\sum_{n=0}^{\infty} |a_n|$ converges, by Prop. I.7.2.14(b) we know that $\sum_{n=0}^{\infty} 2|a_n|$ converges and

$$\begin{aligned}
&\forall n \ge 0, -|a_n| \le a_n \le |a_n| \\
\implies &0 \le a_n + |a_n| \le 2|a_n| \\
\implies &\sum_{n=0}^{\infty} \left(a_n + |a_n|\right) \text{ converges.} \qquad \text{(by Cor. I.7.3.2)}
\end{aligned}$$

Now we write $a_n = a_n + |a_n| - |a_n|$. Since $0 \le a_n + |a_n|$, we have

$$\sum_{n=0}^{\infty} (a_n + |a_n|) \text{ and } \sum_{n=0}^{\infty} |a_n| \text{ converges}$$

$$\begin{aligned}
&\implies \left(\sum_{n=0}^{\infty}(a_n + |a_n|) = \sum_{m=0}^{\infty}\left(a_{f(m)} + \left|a_{f(m)}\right|\right)\right) \\
&\qquad \wedge \left(\sum_{n=0}^{\infty}|a_n| = \sum_{m=0}^{\infty}\left|a_{f(m)}\right|\right) && \text{(by Prop. I.7.4.1)} \\
&\implies \sum_{n=0}^{\infty}\left(a_n + |a_n|\right) - \sum_{n=0}^{\infty}|a_n| \\
&\qquad = \sum_{m=0}^{\infty}\left(a_{f(m)} + \left|a_{f(m)}\right|\right) - \sum_{m=0}^{\infty}\left|a_{f(m)}\right| \\
&\implies \sum_{n=0}^{\infty} a_n = \sum_{m=0}^{\infty} a_{f(m)}. && \text{(by Prop. I.7.2.14(a)(b))}
\end{aligned}$$

□

I.7.5 The root and ratio tests

Thm. I.7.5.1 (Root test)**.** Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$.

- If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent (and hence conditionally convergent).
- If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is not conditionally convergent (and hence cannot be absolutely convergent either).
- If $\alpha = 1$, we cannot assert any conclusion.

Proof. By Prop. I.7.2.14(c), we may assume without loss of generality that $m \geq 1$ (in particular, $|a_n|^{1/n}$ is well-defined for any $n \geq m$).

First, suppose that $\alpha < 1$. Note that we must have $\alpha \geq 0$, since by Lem. I.5.6.6(c) $|a_n|^{1/n} \geq 0$ for every n. Then we can find an $\varepsilon > 0$ such that $0 < \alpha + \varepsilon < 1$ (for instance, we can set $\varepsilon := (1-\alpha)/2$). By Prop. I.6.4.12(a), there exists an $N \geq m$ such that $|a_n|^{1/n} \leq \alpha + \varepsilon$ for all $n \geq N$. In other words, we have $|a_n| \leq (\alpha+\varepsilon)^n$ for all $n \geq N$. But from the geometric series (Lem. I.7.3.3) we have that $\sum_{n=N}^{\infty} (\alpha+\varepsilon)^n$ is absolutely convergent, since $0 < \alpha + \varepsilon < 1$ (note that the fact that we start from N is irrelevant by Prop. I.7.2.14(c)). Thus, by the comparison test (Cor. I.7.3.2), we see that $\sum_{n=N}^{\infty} a_n$ is absolutely convergent, and thus $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, by Prop. I.7.2.14(c) again.

Now suppose that $\alpha > 1$. Then by Prop. I.6.4.12(b), we see that for every $N \geq m$ there exists an $n \geq N$ such that $|a_n|^{1/n} \geq 1$, and hence that $|a_n| \geq 1$. In particular, $(a_n)_{n=N}^{\infty}$ is not 1-close to 0 for any N, and hence $(a_n)_{n=m}^{\infty}$ is not eventually 1-close to 0. In particular, $(a_n)_{n=m}^{\infty}$ does not converge to zero. Thus, by the zero test (Cor. I.7.2.6), $\sum_{n=m}^{\infty} a_n$ is not conditionally convergent.

For $\alpha = 1$, we show two sequences $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ where

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{n\to\infty} |b_n|^{1/n} = 1$$

and

$$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \limsup_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = 1$$

but $\sum_{n=m}^{\infty} a_n$ converges and $\sum_{n=m}^{\infty} b_n$ diverges. Let $a_n = 1/n$ and $b_n = (-1)^n/n$. Then by Cor. I.7.3.7 $\sum_{n=m}^{\infty} a_n$ diverges and by alternating series test (Prop. I.7.2.12) $\sum_{n=m}^{\infty} b_n$ converges.

Since

$$\begin{aligned}
\limsup_{n\to\infty} \left|\frac{1}{n}\right|^{1/n} &\leq \limsup_{n\to\infty} \frac{|n|}{|n+1|} && \text{(by Lem. I.7.5.2)} \\
&= \limsup_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \\
&= 1 && \text{(by Prop. I.6.1.11)}
\end{aligned}$$

and

$$\begin{aligned}
\liminf_{n\to\infty} \left|\frac{1}{n}\right|^{1/n} &\geq \liminf_{n\to\infty} \frac{|n|}{|n+1|} && \text{(by Lem. I.7.5.2)} \\
&= \liminf_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \\
&= 1, && \text{(by Prop. I.6.1.11)}
\end{aligned}$$

by Prop. I.6.4.12(f) we have

$$\limsup_{n\to\infty} \frac{|n|}{|n+1|} = \liminf_{n\to\infty} \frac{|n|}{|n+1|} = 1 = \lim_{n\to\infty} \frac{|n|}{|n+1|}$$

and
$$\limsup_{n\to\infty}\left|\frac{1}{n}\right|^{1/n}=\liminf_{n\to\infty}\left|\frac{1}{n}\right|^{1/n}=1=\lim_{n\to\infty}\left|\frac{1}{n}\right|^{1/n}.$$
Thus, we have $\limsup\limits_{n\to\infty}|a_n|^{1/n}=\limsup\limits_{n\to\infty}\dfrac{|a_{n+1}|}{|a_n|}=1$ but $\sum\limits_{n=m}^{\infty}a_n$ diverges.

Since
$$\begin{aligned}\limsup_{n\to\infty}\left|\frac{(-1)^n}{n}\right|^{1/n}&\le\limsup_{n\to\infty}\frac{\left|(-1)^{n+1}n\right|}{|(-1)^n(n+1)|} && \text{(by Lem. I.7.5.2)}\\&=\limsup_{n\to\infty}\frac{n}{n+1}\\&=1 && \text{(by Prop. I.6.1.11)}\end{aligned}$$
and
$$\begin{aligned}\liminf_{n\to\infty}\left|\frac{(-1)^n}{n}\right|^{1/n}&\ge\liminf_{n\to\infty}\frac{\left|(-1)^{n+1}n\right|}{|(-1)^n(n+1)|} && \text{(by Lem. I.7.5.2)}\\&=\liminf_{n\to\infty}\frac{n}{n+1}\\&=1, && \text{(by Prop. I.6.1.11)}\end{aligned}$$
By Prop. I.6.4.12(f) we have
$$\limsup_{n\to\infty}\frac{\left|(-1)^{n+1}n\right|}{|(-1)^n(n+1)|}=\liminf_{n\to\infty}\frac{\left|(-1)^{n+1}n\right|}{|(-1)^n(n+1)|}=1=\lim_{n\to\infty}\frac{\left|(-1)^{n+1}n\right|}{|(-1)^n(n+1)|}$$
and
$$\limsup_{n\to\infty}\left|\frac{(-1)^n}{n}\right|^{1/n}=\liminf_{n\to\infty}\left|\frac{(-1)^n}{n}\right|^{1/n}=\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right|^{1/n}=1.$$
Thus, we have $\limsup\limits_{n\to\infty}|b_n|^{1/n}=\limsup\limits_{n\to\infty}\dfrac{|b_{n+1}|}{|b_n|}=1$ but $\sum\limits_{n=m}^{\infty}b_n$ converges. We conclude that when $\limsup\limits_{n\to\infty}|a_n|^{1/n}=\limsup\limits_{n\to\infty}\dfrac{|a_{n+1}|}{|a_n|}=1$ we cannot assert any conclusion. □

Note. The root test is phrased using the limit superior, but of course if $\lim\limits_{n\to\infty}|a_n|^{1/n}$ converges then the limit is the same as the limit superior. Thus, one can phrase the root test using the limit instead of the limit superior, but *only when the limit exists.*

Lem. I.7.5.2. Let $(c_n)_{n=m}^{\infty}$ be a sequence of positive numbers. Then we have
$$\liminf_{n\to\infty}\frac{c_{n+1}}{c_n}\le\liminf_{n\to\infty}c_n^{1/n}\le\limsup_{n\to\infty}c_n^{1/n}\le\limsup_{n\to\infty}\frac{c_{n+1}}{c_n}.$$

Proof. There are three inequalities to prove here. The middle inequality follows from Prop. I.6.4.12(c).

Next we show that $\limsup_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$. Write $L := \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$. If $L = +\infty$ then there is nothing to prove (since $x \leq +\infty$ for every extended real number x), so we may assume that L is a finite real number. (Note that L cannot equal $-\infty$). Since $\frac{c_{n+1}}{c_n}$ is always positive, we know that $L \geq 0$.

Let $\varepsilon > 0$. By Prop. I.6.4.12(a), we know that there exists an $N \geq m$ such that $\frac{c_{n+1}}{c_n} \leq L + \varepsilon$ for all $n \geq N$ (without loss of generality we may assume that $N \geq 1$). This implies that $c_{n+1} \leq c_n(L+\varepsilon)$ for all $n \geq N$. By induction this implies that

$$c_n \leq c_N(L+\varepsilon)^{n-N} \text{ for all } n \geq N.$$

If we write $A := c_N(L+\varepsilon)^{-N}$, then we have

$$c_n \leq A(L+\varepsilon)^n$$

and thus

$$c_n^{1/n} \leq A^{1/n}(L+\varepsilon)$$

for all $n \geq N$. But we have

$$\lim_{n\to\infty} A^{1/n}(L+\varepsilon) = L+\varepsilon$$

by the limit laws (Thm. I.6.1.19) and Lem. I.6.5.3. Thus, by the comparison principle (Lem. I.6.4.13) we have

$$\limsup_{n\to\infty} c_n^{1/n} \leq L+\varepsilon.$$

But this is true for all $\varepsilon > 0$, so this must imply that

$$\limsup_{n\to\infty} c_n^{1/n} \leq L.$$

(If $\limsup_{n\to\infty} c_n^{1/n} > L$, then when $\varepsilon = (\limsup_{n\to\infty} c_n^{1/n} - L)/2$ we have

$$\limsup_{n\to\infty} c_n^{1/n} \leq \frac{\limsup_{n\to\infty} c_n^{1/n} + L}{2},$$

a contradiction.), as desired.

Finally we show that $\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} c_n^{1/n}$. Write $L := \liminf_{n\to\infty} \frac{c_{n+1}}{c_n}$. Since $\frac{c_{n+1}}{c_n}$ is always positive, we know that $L \geq 0$.

Let $\varepsilon > 0$. By Prop. I.6.4.12(a), we know that there exists an $N \geq m$ such that $\frac{c_{n+1}}{c_n} \geq L - \varepsilon$ for all $n \geq N$ (without loss of generality we may assume that $N \geq 1$). This implies that $c_{n+1} \geq c_n(L-\varepsilon)$ for all $n \geq N$. By induction this implies that

$$c_n \geq c_N(L-\varepsilon)^{n-N} \text{ for all } n \geq N.$$

If we write $A := c_N(L-\varepsilon)^{-N}$, then we have

$$c_n \geq A(L-\varepsilon)^n$$

and thus

$$c_n^{1/n} \geq A^{1/n}(L-\varepsilon)$$

for all $n \geq N$. But we have

$$\lim_{n\to\infty} A^{1/n}(L-\varepsilon) = L-\varepsilon$$

by the limit laws (Thm. I.6.1.19) and Lem. I.6.5.3. Thus, by the comparison principle (Lem. I.6.4.13) we have

$$\liminf_{n\to\infty} c_n^{1/n} \geq L-\varepsilon.$$

But this is true for all $\varepsilon > 0$, so this must imply that

$$\liminf_{n\to\infty} c_n^{1/n} \geq L.$$

(If $\liminf\limits_{n\to\infty} c_n^{1/n} < L$, then when $\varepsilon = (L - \liminf\limits_{n\to\infty} c_n^{1/n})/2$ we have

$$\liminf_{n\to\infty} c_n^{1/n} \geq \frac{\liminf_{n\to\infty} c_n^{1/n} + L}{2},$$

a contradiction.), as desired. □

Cor. I.7.5.3 (Ratio test)**.** Let $\sum\limits_{n=m}^{\infty} a_n$ be a series of non-zero numbers. (The non-zero hypothesis is required so that the ratios $|a_{n+1}|/|a_n|$ appearing below are well-defined.)

- If $\limsup\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum\limits_{n=m}^{\infty} a_n$ is absolutely convergent (hence conditionally convergent).
- If $\liminf\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} > 1$, then the series $\sum\limits_{n=m}^{\infty} a_n$ is not conditionally convergent (and thus cannot be absolutely convergent).
- In the remaining cases, we cannot assert any conclusion.

Proof. We first show that if $\limsup\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum\limits_{n=m}^{\infty} a_n$ is absolutely convergent.

$$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

$$\Longrightarrow \limsup_{n\to\infty} |a_n|^{1/n} \le \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1 \qquad \text{(by Lem. I.7.5.2)}$$
$$\Longrightarrow \sum_{n=m}^{\infty} a_n \text{ is absolutely convergent.} \qquad \text{(by Thm. I.7.5.1)}$$

Next we show that if $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is not conditionally convergent.

$$\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$$
$$\Longrightarrow \limsup_{n\to\infty} |a_n|^{1/n} \ge \liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1 \qquad \text{(by Lem. I.7.5.2)}$$
$$\Longrightarrow \sum_{n=m}^{\infty} a_n \text{ is not conditionally convergent.} \qquad \text{(by Thm. I.7.5.1)}$$

Finally we show that if $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \le 1$ or $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \ge 1$, then we cannot assert any conclusion. See Thm. I.7.5.1. $\square$

Prop. I.7.5.4. We have $\lim_{n\to\infty} n^{1/n} = 1$.

Proof. By Lem. I.7.5.2 we have

$$\limsup_{n\to\infty} n^{1/n} \le \limsup_{n\to\infty} (n+1)/n = \limsup_{n\to\infty} 1 + 1/n = 1$$

by Prop. I.6.1.11 and limit laws (Thm. I.6.1.19). Similarly, we have

$$\liminf_{n\to\infty} n^{1/n} \ge \liminf_{n\to\infty} (n+1)/n = \liminf_{n\to\infty} 1 + 1/n = 1.$$

The claim then follows from Prop. I.6.4.12(c) and (f). $\square$

Rmk. I.7.5.5. In addition to the ratio and root tests, another very useful convergence test is the *integral test*, which we will cover in Prop. I.11.6.4.

— Exercises —

Ex. I.7.5.1. Prove the first inequality in Lem. I.7.5.2.

Proof. See Lem. I.7.5.2. $\square$

Ex. I.7.5.2. Let x be a real number with $|x| < 1$, and q be a real number. Show that the series $\sum_{n=1}^{\infty} n^q x^n$ is absolutely convergent, and that $\lim_{n\to\infty} n^q x^n = 0$.

Proof. Let $N \in \mathbb{N}$. Since $q \in \mathbb{R}$, by Prop. I.5.4.12 $\exists N \geq q$. Then we have

$$\begin{aligned}
&\forall n \geq 1, |n^q x^n| \leq \left|n^N x^n\right| && \text{(by Lem. I.5.6.9)}\\
\implies &\forall n \geq 1, |n^q x^n|^{1/n} \leq \left|n^N x^n\right|^{1/n} && \text{(by Lem. I.5.6.9)}\\
\implies &\forall n \geq 1, n^{q/n}|x| \leq n^{N/n}|x| && \text{(by Prop. I.5.6.3)}\\
\implies &\limsup_{n\to\infty} n^{q/n}|x| \leq \limsup_{n\to\infty} n^{N/n}|x| && \text{(by Lem. I.6.4.13)}\\
\implies &\limsup_{n\to\infty} n^{q/n}|x| \leq (\limsup_{n\to\infty} n^{1/n})^N|x| && \text{(by Thm. I.6.1.19)}\\
\implies &\limsup_{n\to\infty} n^{q/n}|x| \leq 1^N|x| && \text{(by Prop. I.7.5.4)}\\
\implies &\limsup_{n\to\infty} n^{q/n}|x| \leq |x| < 1 && \text{(by hypothesis)}\\
\implies &\sum_{n=1}^{\infty} n^q x^n \text{ is absolutely convergent} && \text{(by Thm. I.7.5.1)}\\
\implies &\sum_{n=1}^{\infty} n^q x^n \text{ converges} && \text{(by Prop. I.7.2.9)}\\
\implies &\lim_{n\to\infty} n^q x^n = 0 && \text{(by Cor. I.7.2.6)}
\end{aligned}$$

□

Ex. I.7.5.3. Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ of positive numbers a_n such that $\lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty} a_n^{1/n} = 1$, and give an example of a convergent series $\sum_{n=1}^{\infty} b_n$ of positive numbers b_n such that $\lim_{n\to\infty} b_{n+1}/b_n = \lim_{n\to\infty} b_n^{1/n} = 1$. This shows that the ratio and root tests can be inconclusive even when the summands are positive and all the limits converge.

Proof. See Thm. I.7.5.1. □

Chapter I.8

Infinite sets

I.8.1 Countability

Note. From Thm. I.3.6.12 we know that the set $\mathbb{N}$ of natural numbers is infinite. The set $\mathbb{N} - \{0\}$ is also infinite, thanks to Prop. I.3.6.14(a), and is a proper subset of $\mathbb{N}$. However, the set $\mathbb{N} - \{0\}$, despite being "smaller" than $\mathbb{N}$, still has the same cardinality as $\mathbb{N}$, because the function $f : \mathbb{N} \to \mathbb{N} - \{0\}$ defined by $f(n) \coloneqq n + 1$, is a bijection from $\mathbb{N}$ to $\mathbb{N} - \{0\}$. This is one characteristic of infinite sets.

Def. I.8.1.1 (Countable sets)**.** A set X is said to be *countably infinite* (or just *countable*) iff it has equal cardinality with the natural numbers $\mathbb{N}$. A set X is said to be *at most countable* iff it is either countable or finite. We say that a set is *uncountable* if it is infinite but not countable.

Rmk. I.8.1.2. Countably infinite sets are also called *denumerable* sets.

E.g. I.8.1.3. The even natural numbers $\{2n : n \in \mathbb{N}\}$, since the function $f(n) \coloneqq 2n$ provides a bijection between $\mathbb{N}$ and the even natural numbers.

Note. Let X be a countable set. Then, by definition, we know that there exists a bijection $f : \mathbb{N} \to X$. Thus, every element of X can be written in the form $f(n)$ for exactly one natural number n. Informally, we thus have

$$X = \{f(0), f(1), f(2), f(3), \ldots\}.$$

Thus, a countable set can be arranged in a sequence, so that we have a zeroth element $f(0)$, followed by a first element $f(1)$, then a second element $f(2)$, and so forth, in such a way that all these elements $f(0), f(1), f(2), \ldots$ are all distinct, and together they fill out all of X. (This is why these sets are called *countable*; because we can literally count them one by one, starting from $f(0)$, then $f(1)$, and so forth.)

Prop. I.8.1.4 (Well ordering principle)**.** Let X be a non-empty subset of the natural numbers $\mathbb{N}$. Then there exists exactly one element $n \in X$ such that $n \leq m$ for all $m \in X$. In other words, every non-empty set of natural numbers has a minimum element.

Proof. Suppose for the sake of contradiction that X has no minimum element. Let $n \in \mathbb{N}$ and let $P(n)$ be the statement "$\forall m \in X$, we have $n \leq m$ and $n \notin X$." We now use induction to show that $P(n)$ is true $\forall n \in \mathbb{N}$. For $n = 0$, we have

$$\begin{aligned} & X \subseteq \mathbb{N} & \\ \implies & \forall m \in X, m \in \mathbb{N} & \text{(by Def. I.3.1.15)} \\ \implies & \forall m \in X, 0 \leq m & \text{(by Ax. I.2.3)} \\ \implies & 0 \notin X. & (X \text{ has no minimum element}) \end{aligned}$$

Thus, the base case holds. Suppose inductively that $P(n)$ is true for some $n \geq 0$. Then for $n+1$, we have

$$\begin{aligned} & \forall m \in X, n \leq m \wedge n \notin X & \text{(by the induction hypothesis)} \\ \implies & \forall m \in X, n < m & \text{(by Def. I.2.2.11)} \\ \implies & \forall m \in X, n+1 \leq m & \text{(by Prop. I.2.2.12(e))} \\ \implies & n+1 \notin X. & (X \text{ has no minimum element}) \end{aligned}$$

This closes the induction.

By hypothesis we know that $X \subseteq \mathbb{N}$ and $X \neq \emptyset$. So let $n \in X$. But $P(n)$ is true, we must have $n \notin X$, a contradiction. Thus, X must have a minimum element $\min(X) \in X$.

Now we show that such $\min(X)$ is unique. Suppose that $\exists n, n' \in X$ such that $\forall m \in X$, we have $n \leq m \wedge n' \leq m$. Since $n, n' \in X$, we have $n \leq n' \wedge n' \leq n$. Thus, $n = n'$. ☐

Note. We will refer to the element n given by the well-ordering principle as the *minimum* of X, and write it as $\min(X)$. This minimum is clearly the same as the infimum of X, as defined in Def. I.5.5.10.

Prop. I.8.1.5. Let X be an infinite subset of the natural numbers $\mathbb{N}$. Then there exists a unique bijection $f : \mathbb{N} \to X$ which is increasing, in the sense that $f(n+1) > f(n)$ for all $n \in N$. In particular, X has equal cardinality with $\mathbb{N}$ and is hence countable.

Proof. We now define a sequence $a_0, a_1, a_2, \ldots$ of natural numbers recursively by the formula

$$a_n := \min\{x \in X : x \neq a_m \text{ for all } m < n\}.$$

Intuitively speaking, a_0 is the smallest element of X; a_1 is the second smallest element of X, i.e., the smallest element of X once a_0 is removed; a_2 is the third smallest element of X; and so forth. Observe that in order to define a_n, one only needs to know the values of a_m for all $m < n$, so this definition is recursive. Also, since X is infinite, the set $\{x \in X : x \neq a_m \text{ for all } m < n\}$

is infinite, hence non-empty. (If it is finite, then its union with the set $\{a_0, \ldots, a_{n-1}\}$ is also finite, but the union is X, which contradict to X is infinite.) Thus, by the well-ordering principle (Prop. I.8.1.5), the minimum, $\min\{x \in X : x \neq a_m \text{ for all } m < n\}$ is always well-defined.

Since $a_{n+1} = \min\{x \in X : x \neq a_m \text{ for all } m < n+1\}$, we know that $a_n < a_{n+1}$. Since n was arbitrary, we see that a_n is an increasing sequence, i.e.

$$a_0 < a_1 < a_2 < \ldots,$$

and in particular, that $a_n \neq a_m$ for all $n \neq m$. Also, we have $a_n \in X$ for each natural number n (by Prop. I.8.1.4).

Now define the function $f : \mathbb{N} \to X$ by $f(n) := a_n$. From the previous paragraph we know that f is one-to-one. Now we show that f is onto. In other words, we claim that for every $y \in X$, there exists an n such that $a_n = y$.

Let $y \in X$. Suppose for the sake of contradiction that $a_n \neq y$ for every natural number n. Then this implies that y is an element of the set $\{x \in X : x \neq a_m \text{ for all } m < n\}$ for all n. By definition of a_n, this implies that $y > a_n$ for every natural number n. (If $y < a_n$, then $y = \min\{x \in X : a \neq a_m \text{ for all } m < n\}$ instead of a_n, a contradiction) However, since a_n is an increasing sequence, we have $a_n \geq n$, and hence $y \geq n$ for every natural number n. In particular, we have $y \geq y + 1$, which is a contradiction. Thus, we must have $a_n = y$ for some natural number n, and hence f is onto.

Since $f : \mathbb{N} \to X$ is both one-to-one and onto, it is a bijection. We have thus found at least one increasing bijection f from $\mathbb{N}$ to X. Now suppose for the sake of contradiction that there was at least one other increasing bijection g from $\mathbb{N}$ to X which was not equal to f. Then the set $\{n \in \mathbb{N} : g(n) \neq f(n)\}$ is non-empty, and define $m := \min\{n \in \mathbb{N} : g(n) \neq f(n)\}$, thus in particular, $g(m) \neq f(m) = a_m$, and $g(n) = f(n) = a_n$ for all $n < m$. But we then must have

$$g(m) = \min\{x \in X : x \neq a_t \text{ for all } t < m\} = a_m,$$

a contradiction. Thus, there is no other increasing bijection from $\mathbb{N}$ to X other than f. $\square$

Cor. I.8.1.6. All subsets of the natural numbers are at most countable.

Proof. Since finite sets are at most countable by definition, combine with Prop. I.8.1.5 we thus have all subsets of the natural numbers are at most countable. $\square$

Cor. I.8.1.7. If X is an at most countable set, and Y is a subset of X, then Y is at most countable.

Proof. If X is finite then this follows from Prop. I.3.6.14(c), so assume X is countable. Then there is a bijection $f : X \to \mathbb{N}$ between X and $\mathbb{N}$. Since Y is a subset of X, and f is a bijection from X and $\mathbb{N}$, then when we restrict f to Y, we obtain a bijection between Y and $f(Y)$. Thus, $f(Y)$ has equal cardinality with Y. But $f(Y)$ is a subset of $\mathbb{N}$, and hence at most countable by Cor. I.8.1.6. Hence Y is also at most countable. $\square$

Prop. I.8.1.8. Let Y be a set, and let $f : \mathbb{N} \to Y$ be a function. Then $f(\mathbb{N})$ is at most countable.

Proof. If $f(\mathbb{N})$ is finite then by Def. I.8.1.1 it is at most countable. So assume that $f(\mathbb{N})$ is infinite. Let A be the set

$$A = \{n \in \mathbb{N} : f(m) \neq f(n) \text{ for all } 0 \leq m < n\}.$$

So $A \subseteq \mathbb{N}$ and A is infinite. We now show that $f|_A : A \to f(A)$ is a bijection.

Let $p, q \in A$ and $p \neq q$. By the definition of A we know that $f|_A(p) \neq f|_A(q)$ and thus $f|_A$ is injective. By Def. I.3.4.1 we also know that $f|_A$ is surjective, thus $f|_A$ is bijective.

Now we show that $\forall y \in f(\mathbb{N})$, $\exists p \in A$ such that $f|_A(p) = y$. Suppose for the sake of contradiction that $\nexists p \in A$ such that $f|_A(p) = y$. Then we have $y \neq f|_A(p)$ for every $p \in A$. Since $y \in f(\mathbb{N})$, we know that $\exists q \in \mathbb{N}$ such that $f(q) = y$ and $q \notin A$. Since $q \notin A$, by the definition of A we know that $\exists 0 \leq m < q$ such that $f(m) = f(q) = y$. Now we let E be the set

$$E = \{m \in \mathbb{N} : f(m) = f(q) = y\}.$$

Since $E \subseteq \mathbb{N}$ and $E \neq \emptyset$, by well ordering principle (Prop. I.8.1.4) we know that $\min(E)$ exists. This means $\exists p \in E$ such that $\forall 0 \leq m < p$, we have $f(m) \neq f(p) = f(q)$. But then we must have $p \in A$, a contradiction. Thus, $\forall y \in f(\mathbb{N})$, $\exists p \in A$ such that $f|_A(p) = y$. This means $f(\mathbb{N}) \subseteq f(A)$, thus we have $f(\mathbb{N}) = f(A)$.

Since $A \subseteq \mathbb{N}$ and A is infinite, by Prop. I.8.1.5 $\exists g : \mathbb{N} \to A$ where g is bijective. This means $f|_A \circ g$ is bijective and we have

$$(f|_A \circ g)(\mathbb{N}) = f|_A\big(g(\mathbb{N})\big) = f|_A(A) = f(A) = f(\mathbb{N}).$$

Thus, by Def. I.8.1.1 $f(\mathbb{N})$ is countable, and thus at most countable. □

Cor. I.8.1.9. Let X be a countable set, and let $f : X \to Y$ be a function. Then $f(X)$ is at most countable.

Proof. By Def. I.8.1.1 $\exists g : \mathbb{N} \to X$ such that g is a bijection. Then we have $f \circ g : \mathbb{N} \to Y$ and by Prop. I.8.1.8 $(f \circ g)(\mathbb{N})$ is at most countable. But

$$(f \circ g)(\mathbb{N}) = f(g(\mathbb{N})) = f(X).$$

Thus, $f(X)$ is at most countable. □

Prop. I.8.1.10. Let X be a countable set, and let Y be a countable set. Then $X \cup Y$ is a countable set.

Proof. By Def. I.8.1.1 $\exists f : \mathbb{N} \to X$ and $g : \mathbb{N} \to Y$ such that f and g are bijections. Let $h : \mathbb{N} \to X \cup Y$ by setting $h(2n) = f(n)$ and $h(2n+1) = g(n)$ for every natural number n. We now show that $h(\mathbb{N}) = X \cup Y$.

$$z \in h(\mathbb{N})$$

$$
\begin{aligned}
&\iff \exists k \in \mathbb{N} : h(k) = z \\
&\iff (\exists k \in \mathbb{N} : h(k) = z) \\
&\qquad \wedge (\exists n \in \mathbb{N} : k = 2n \vee k = 2n+1) && \text{(by A.Cor. I.4.4.2)} \\
&\iff \exists n \in \mathbb{N} : z = h(2n) \vee z = h(2n+1) \\
&\iff z = f(n) \vee z = g(n) \\
&\iff z \in X \vee z \in Y \\
&\iff z \in X \cup Y.
\end{aligned}
$$

Then by Cor. I.8.1.9 we have $h(\mathbb{N}) = X \cup Y$ is at most countable. But since X and Y are infinite sets, $X \cup Y$ can not be finite, thus $X \cup Y$ is countable. □

Note. To summarize, any subset or image of a countable set is at most countable, and any finite union of countable sets is still countable.

A.Cor. I.8.1.1. Let X, Y be at most countable sets. Then $X \cup Y$ is at most countable.

Proof. We split into following three cases:

- X, Y are countable. Then by Prop. I.8.1.10 we know that $X \cup Y$ is countable, thus at most countable.
- X, Y are finite. Then by Prop. I.3.6.14(b) we know that $X \cup Y$ is finite, thus at most countable.
- X, Y consist of one finite set and one countable set. Without the loss of generality, suppose that X is finite and Y is countable. Since X is finite, there exists a function $f : \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} \to X$ such that f is bijective. Since Y is countable, by Def. I.8.1.1 there exists a function $g : \mathbb{N} \to Y$ such that g is bijective. Now we define a functiton $h : \mathbb{N} \to X \cup Y$ as follow:

$$
\forall n \in \mathbb{N}, h(n) = \begin{cases} f(n+1) & \text{if } n < \#(X) \\ g(n - \#(X)) & \text{if } n \geq \#(X) \end{cases}
$$

We need to show that $h(\mathbb{N}) = X \cup Y$. Since $h(\mathbb{N}) \subseteq X \cup Y$, it suffices to show that $X \cup Y \subseteq h(\mathbb{N})$.

$$
\begin{aligned}
&\quad \forall z \in X \cup Y \\
&\implies (z \in X) \vee (z \in Y) \\
&\implies \big(\exists n \in \{i \in \mathbb{N} : 1 \leq i \leq \#(X)\} : f(n) = z\big) \\
&\qquad \wedge \big(\exists n \in \mathbb{N} : g(n) = z\big) \\
&\implies \big(h(n-1) = f(n) = z\big) \wedge \big(h(n + \#(X)) = g(n) = z\big) \\
&\implies z \in h(\mathbb{N}),
\end{aligned}
$$

Thus, we have $X \cup Y \subseteq h(\mathbb{N})$. By Prop. I.8.1.8 $X \cup Y$ is at most countable.

From all cases above, we conclude that $X \cup Y$ is at most countable. $\square$

Cor. I.8.1.11. The integers $\mathbb{Z}$ are countable.

Proof. We already know that the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers are countable. The set $-\mathbb{N}$ defined by

$$-\mathbb{N} \coloneqq \{-n : n \in \mathbb{N}\} = \{0, -1, -2, -3, \dots\}$$

is also countable, since the map $f(n) \coloneqq -n$ is a bijection between $\mathbb{N}$ and this set. Since the integers are the union of $\mathbb{N}$ and $-\mathbb{N}$, the claim follows from Prop. I.8.1.10. $\square$

Note. To establish countability of the rationals, we need to relate countability with Cartesian products. In particular, we need to show that the set $\mathbb{N} \times \mathbb{N}$ is countable.

Lem. I.8.1.12. The set

$$A \coloneqq \{(n, m) \in \mathbb{N} \times \mathbb{N} : 0 \leq m \leq n\}$$

is countable.

Proof. Define the sequence $a_0, a_1, a_2, \dots$ recursively by setting $a_0 \coloneqq 0$, and $a_{n+1} \coloneqq a_n + n + 1$ for all natural numbers n. Thus

$$a_0 = 0; a_1 = 0 + 1; a_2 = 0 + 1 + 2; a_3 = 0 + 1 + 2 + 3; \dots$$

By induction one can show that a_n is increasing, i.e., that $a_n > a_m$ whenever $n > m$.

Now define the function $f : A \to \mathbb{N}$ by

$$f(n, m) \coloneqq a_n + m.$$

We claim that f is one-to-one. In other words, if (n, m) and (n', m') are any two distinct elements of A, then we claim that $f(n, m) \neq f(n', m')$.

To prove this claim, let (n, m) and (n', m') be two distinct elements of A. There are three cases: $n' = n$, $n' > n$, and $n' < n$. First, suppose that $n' = n$. Then we must have $m \neq m'$, otherwise (n, m) and (n', m') would not be distinct. Thus, $a_n + m \neq a_n + m'$, and hence $f(n, m) \neq f(n', m')$, as desired.

Now suppose that $n' > n$. Then $n' \geq n + 1$, and hence

$$f(n', m') = a_{n'} + m' \geq a_{n'} \geq a_{n+1} = a_n + n + 1.$$

But since $(n, m) \in A$, we have $m \leq n < n + 1$, and hence

$$f(n', m') \geq a_n + n + 1 > a_n + m = f(n, m),$$

and thus $f(n', m') \neq f(n, m)$.

The case $n' < n$ is proven similarly, by switching the roles of n and n' in the previous argument. Thus, we have shown that f is one-to-one. Thus, f is a bijection from A to $f(A)$, and so A has equal cardinality with $f(A)$. But $f(A)$ is a subset of $\mathbb{N}$, and hence by Cor. I.8.1.6 $f(A)$ is at most countable. Therefore A is at most countable. But, A is clearly not finite. (if A was finite, then every subset of A would be finite, and in particular, $\{(n, 0) : n \in \mathbb{N}\}$ would be finite, but this is clearly countably infinite, a contradiction.) Thus, A must be countable. $\square$

Cor. I.8.1.13. The set $\mathbb{N} \times \mathbb{N}$ is countable.

Proof. We already know that the set

$$A \coloneqq \{(n, m) \in \mathbb{N} \times \mathbb{N} : 0 \leq m \leq n\}$$

is countable. This implies that the set

$$B \coloneqq \{(n, m) \in \mathbb{N} \times \mathbb{N} : 0 \leq n \leq m\}$$

is also countable, since the map $f : A \to B$ given by $f(n, m) \coloneqq (m, n)$ is a bijection from A to B. We prove f is bijective by showing that f is both injective and surjective.

- To prove that f is injective, suppose that $(n, m), (n', m') \in A$ and $f(n, m) = f(n', m')$. Then we have

$$\begin{aligned} & f(n, m) = f(n', m') \\ \implies & (m, n) = (m', n') && \text{(by the definition of } f\text{)} \\ \implies & n = n' \wedge m = m' && \text{(by Def. I.3.5.1)} \\ \implies & (n, m) = (n', m'). && \text{(by Def. I.3.5.1)} \end{aligned}$$

  Thus, f is injective.

- Since $\forall (n, m) \in B$, we have $n \leq m$, thus $(m, n) \in A$ and $f(m, n) = (n, m)$. So f is surjective.

We now show that $\mathbb{N} \times \mathbb{N} = A \cup B$. By Prop. I.3.1.18 we need to show that $\mathbb{N} \times \mathbb{N} \subseteq A \cup B$ and $A \cup B \subseteq \mathbb{N} \times \mathbb{N}$. It is clearly that $A \cup B \subseteq \mathbb{N} \times \mathbb{N}$. So we only need to show that $\mathbb{N} \times \mathbb{N} \subseteq A \cup B$.

$$\begin{aligned} & \forall (a, b) \in \mathbb{N} \times \mathbb{N} \\ \implies & (a < b) \vee (a = b) \vee (a > b) && \text{(by Prop. I.2.2.13)} \\ \implies & (a, b) \in A \vee (a, b) \in B && \text{(by the definition of } A \text{ and } B\text{)} \\ \implies & (a, b) \in A \cup B. && \text{(by Ax. I.3.4)} \end{aligned}$$

Thus, by Def. I.3.1.15 we have $\mathbb{N} \times \mathbb{N} \subseteq A \cup B$.

Since $\mathbb{N} \times \mathbb{N}$ is the union of A and B, the claim then follows from Prop. I.8.1.10. $\square$

Cor. I.8.1.14. If X and Y are countable, then $X \times Y$ is countable.

Proof. By Def. I.8.1.1 $\exists f : \mathbb{N} \to X$ and $g : \mathbb{N} \to Y$ such that f and g are bijections. Let $h : \mathbb{N} \times \mathbb{N} \to X \times Y$ by setting $h(x, y) = (f(x), g(y))$. If $n, n', m, m' \in \mathbb{N}$ and $(n, m) \neq (n', m')$, then

$$h(n, m) = (f(n), g(m)) \neq (f(n'), g(m')) = h(n', m')$$

since f, g are bijections, so h is injective. Again since f, g are bijections, $\forall x \in X \wedge \forall y \in Y$, $\exists n, m \in \mathbb{N}$ such that $x = f(n) \wedge y = g(m)$. So h is surjective, and thus h is bijective.

Since h is bijective, $\mathbb{N} \times \mathbb{N}$ and $X \times Y$ has the same cardinality. But by Cor. I.8.1.13 we know that $\mathbb{N} \times \mathbb{N}$ is countable. Thus, by Def. I.8.1.1 $X \times Y$ is countable. □

Cor. I.8.1.15. The rationals $\mathbb{Q}$ are countable.

Proof. We already know that the integers $\mathbb{Z}$ are countable, which implies that the non-zero integers $\mathbb{Z} - \{0\}$ are countable. (since $\mathbb{Z} - \{0\} \subseteq \mathbb{Z}$, by Cor. I.8.1.7 we know that $\mathbb{Z} - \{0\}$ is at most countable, and clearly $\mathbb{Z} - \{0\}$ is not finite.) By Cor. I.8.1.14, the set

$$\mathbb{Z} \times (\mathbb{Z} - \{0\}) = \{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$$

is thus countable. If one lets $f : \mathbb{Z} \times (\mathbb{Z} - \{0\}) \to \mathbb{Q}$ be the function $f(a, b) \coloneqq a/b$ (note that f is well-defined since we prohibit b from being equal to 0), we see from Cor. I.8.1.9 that $f(\mathbb{Z} \times (\mathbb{Z} - \{0\}))$ is at most countable. But we have $f(\mathbb{Z} \times (\mathbb{Z} - \{0\})) = \mathbb{Q}$ (This is basically the definition of the rationals $\mathbb{Q}$). Thus, $\mathbb{Q}$ is at most countable. However, $\mathbb{Q}$ cannot be finite, since it contains the infinite set $\mathbb{N}$. Thus, $\mathbb{Q}$ is countable. □

Rmk. I.8.1.16. Because the rationals are countable, we know *in principle* that it is possible to arrange the rational numbers as a sequence:

$$\mathbb{Q} = \{a_0, a_1, a_2, a_3, \ldots\}$$

such that every element of the sequence is different from every other element, and that the elements of the sequence exhaust $\mathbb{Q}$ (i.e., every rational number turns up as one of the elements a_n of the sequence). However, it is quite difficult (though not impossible) to actually try and come up with an explicit sequence $a_0, a_1, \ldots$ which does this.

— Exercises —

Ex. I.8.1.1 (Dedekind-infinite set)**.** Let X be a set. Show that X is infinite iff there exists a proper subset $Y \subsetneq X$ of X which has the same cardinality as X.

Proof. We first show that X is infinite implies $\exists Y \subsetneq X$ such that Y has the same cardinality as X. Suppose that X is an infinite set. Then we have $X \neq \emptyset$ since by Ex. I.3.6.2 $\#(\emptyset) = 0$.

Let $n \in \mathbb{N}$ and let $P(n)$ be the statement "$\exists A_n \subseteq X$ such that $\#(A_n) = n$." We induct on n to show that $\forall n \in \mathbb{N}$, $P(n)$ is true. For $n = 0$, we have $\emptyset \subseteq X$ by E.g. I.3.1.17, Thus, the base case holds. Suppose inductively that $P(n)$ is true for some $n \geq 0$. Then we need to show that $P(n+1)$ is true. By the induction hypothesis, $\exists A_n \subseteq X : \#(A_n) = n$. Then by Prop. I.3.6.14(b) we know that $X \setminus A_n$ is infinite. Since $X \setminus A_n$ is infinite, we know that $X \setminus A_n \neq \emptyset$. Let $x \in X \setminus A_n$. Then we define $A_{n+1} = A_n \cup \{x\}$, and this closes the induction.

By axiom of choice (Ax. I.8.1) the set $\prod_{n \in \mathbb{Z}^+} A_n$ is non-empty since $\forall n \in \mathbb{Z}^+$, $P(n)$ is true. We can now choose an element $(x_n)_{n \in \mathbb{Z}^+}$ from $\prod_{n \in \mathbb{Z}^+} A_n$. In particular, we want to choose a $(x_n)_{n \in \mathbb{Z}^+}$ where $x_i \neq x_j$ for every $i, j \in \mathbb{Z}^+$ and $i \neq j$. This can be done since $\#(A_i) \neq \#(A_j)$ for every $i, j \in \mathbb{Z}^+$ and $i \neq j$. We collect x_i as a set $A = \left\{x_i : i \in \mathbb{Z}^+\right\}$. By axiom of choice (Ax. I.8.1) A can be construct as the image of $(x_n)_{n \in \mathbb{Z}^+}$. Now we define a function $f : X \to X \setminus \{x_1\}$ as follow:

$$f(x) = \begin{cases} x_{n+1} & \text{if } x = x_n \text{ for some } x_n \in A, \\ x & \text{if } x \notin A. \end{cases}$$

We show that such f is bijective. We start by showing f is injective. Let $x, x' \in X$ and $x \neq x'$. We split into four cases:

- If $x \in A \wedge x' \in A$, then $\exists n, n' \in \mathbb{Z}^+$ such that $x = x_n \wedge x' = x_{n'}$. By the definition of x_n and $x_{n'}$, we must have $x_n \neq x_{n'} \implies x_{n+1} \neq x_{n'+1}$. Thus, we have $x_{n+1} = f(x) \neq f(x') = x_{n'+1}$.
- If $x \in A \wedge x' \notin A$, then $f(x) \in A \wedge f(x') = x' \notin A$ and thus $f(x) \neq f(x')$.
- If $x \notin A \wedge x' \in A$, then $f(x) = x \notin A \wedge f(x') \in A$ and thus $f(x) \neq f(x')$.
- If $x \notin A \wedge x' \notin A$, then $f(x) = x \neq x' = f(x')$.

From all cases above, we conclude that $x \neq x' \implies f(x) \neq f(x')$, thus f is injective. Now we show that f is surjective. Let $x \in X \setminus \{x_1\}$. We split into two cases:

- If $x \in A$, then $x \neq x_1$ and $\exists n \in \mathbb{Z}^+ \setminus \{1\}$ such that $x = x_n$. Since $n \geq 2$, we have $n - 1 \geq 1$. Thus, by the definition of A we have $x_{n-1} \in A$ and $f(x_{n-1}) = x_n$.
- If $x \notin A$, then we have $f(x) = x$.

Since x was arbitrary, we know that f is surjective. Since f is both injective and surjective, we know that f is bijective, and by Def. I.3.6.1 X and $X \setminus \{x_1\}$ have the same cardinality. But $x_1 \in X \wedge x_1 \notin X \setminus \{x_1\}$, we have $X \neq X \setminus \{x_1\}$. Thus, by Def. I.3.1.15 $X \setminus \{x_1\} \subsetneq X$.

Now we show that if $\exists Y \subsetneq X$ where X and Y have the same cardinality, then X is infinite. We prove this by contradiction. Suppose for the sake of contradiction that X is finite. Then by Prop. I.3.6.14(c) we have $\#(Y) < \#(X)$, a contradiction. Thus, X is infinite. $\square$

Ex. I.8.1.2. Prove Prop. I.8.1.4.

Proof. See Prop. I.8.1.4. □

Ex. I.8.1.3. Fill in the gaps marked in Prop. I.8.1.5.

Proof. See Prop. I.8.1.5. □

Ex. I.8.1.4. Prove Prop. I.8.1.8.

Proof. See Prop. I.8.1.8. □

Ex. I.8.1.5. Use Prop. I.8.1.8 to prove Cor. I.8.1.9.

Proof. See Cor. I.8.1.9. □

Ex. I.8.1.6. Let A be a set. Show that A is at most countable iff there exists an injective map $f : A \to \mathbb{N}$ from A to $\mathbb{N}$.

Proof. We first show that if A is at most countable, then there exists an injective map $f : A \to \mathbb{N}$. Suppose that A is at most countable. By Def. I.8.1.1 A is either finite or countable.

- If A is finite, then by Def. I.3.6.10 $\exists g : A \to \{i \in \mathbb{N} : 1 \leq i \leq \#(A)\}$ such that g is bijective. Now let $f : A \to \mathbb{N}$ be the function $f(x) = g(x)$ for every $x \in A$. Since g is a bijection and $\{i \in \mathbb{N} : 1 \leq i \leq \#(A)\} \subseteq \mathbb{N}$, we have $f : A \to \mathbb{N}$ is injective.

- If A is countable, then by Def. I.8.1.1 $\exists f : A \to \mathbb{N}$ such that f is a bijection, and hence f is injective.

From all cases above, we conclude that if A is at most countable then there exists an injective map $f : A \to \mathbb{N}$.

Now we show that if there exists an injective map $f : A \to \mathbb{N}$, then A is at most countable. Suppose that $f : A \to \mathbb{N}$ is injective. Since $f(A) \subseteq \mathbb{N}$, by Cor. I.8.1.6 $f(A)$ is at most countable. Since f is bijective from A to $f(A)$, we know that A and $f(A)$ have equal cardinality, and thus A is at most countable. □

Ex. I.8.1.7. Prove Prop. I.8.1.10.

Proof. See Prop. I.8.1.10. □

Ex. I.8.1.8. Use Cor. I.8.1.13 to prove Cor. I.8.1.14.

Proof. See Cor. I.8.1.14. □

Ex. I.8.1.9. Suppose that I is an at most countable set, and for each $\alpha \in I$, let A_α be an at most countable set. Show that the set $\bigcup_{\alpha \in I} A_\alpha$ is also at most countable. In particular, countable unions of countable sets are countable.

Proof. Suppose that I be an at most countable set and $\forall \alpha \in I$ we have A_α is an at most countable set. By Def. I.8.1.1 I is either finite or countable.

We first show that if I is finite, then $\bigcup_{\alpha \in I} A_\alpha$ is at most countable. Since I is finite, by Def. I.3.6.5 $\exists n \in \mathbb{N}$ such that $\#(I) = n$. Let $P(n)$ be the statement "$\#(I) = n$ and $\bigcup_{\alpha \in I} A_\alpha$ is at most countable." We induct on n to show that $P(n)$ is true for every $n \in \mathbb{N}$. For $n = 0$, we have $\#(\emptyset) = 0$ and $\bigcup_{\alpha \in \emptyset} A_\alpha = \emptyset$. Thus, the base case holds. Suppose inductively that $P(n)$ is true for some $n \geq 0$. Then we need to show that $P(n+1)$ is also true. Since $\#(I) = n + 1 > 0$, we know that $I \neq \emptyset$. Let $i \in I$. Since $\#(I \setminus \{i\}) = n$, by the induction hypothesis we know that the set $\bigcup_{\alpha \in I \setminus \{i\}} A_\alpha$ is at most countable. By Ax. I.3.11 we have $\bigcup_{\alpha \in I} A_\alpha = (\bigcup_{\alpha \in I \setminus \{i\}} A_\alpha) \cup A_i$. Then by A.Cor. I.8.1.1 we know that $\bigcup_{\alpha \in I} A_\alpha$ is at most countable. This closes the induction. We conclude that finite union of at most countable sets is at most countable.

Now we show the case where I is countable. Let $J = \{\alpha \in I : A_\alpha \neq \emptyset\}$. Since $J \subseteq I$, by Cor. I.8.1.7 we know that J is at most countable. If J is finite (including the case where $J = \emptyset$), then we already show that finite union of at most countable sets is at most countable. So suppose that J is countable. Then we have

$$
\begin{aligned}
&\forall x \in \bigcup_{\alpha \in I} A_\alpha \\
&\iff \exists \alpha' \in I : x \in A'_{\alpha'} \\
&\iff A'_{\alpha'} \neq \emptyset \\
&\iff \alpha' \in J \\
&\iff \exists \alpha' \in J : x \in A'_{\alpha'} \\
&\iff x \in \bigcup_{\alpha \in J} A_\alpha.
\end{aligned}
$$

Thus, by Def. I.3.1.4 we have $\bigcup_{\alpha \in I} A_\alpha = \bigcup_{\alpha \in J} A_\alpha$. To show that $\bigcup_{\alpha \in I} A_\alpha$ is at most countable, it suffices to show that $\bigcup_{\alpha \in J} A_\alpha$ is at most countable.

Since $\forall \alpha \in J$, A_α is at most countable. We split into two cases:

- If A_α is finite, then by Def. I.3.6.5 $\exists f'_\alpha : \{n \in \mathbb{N} : 1 \leq n \leq \#(A_\alpha)\} \to A_\alpha$ such that f'_α is bijective. We now define a function $f_\alpha : \mathbb{N} \to A_\alpha$ as follow:

$$\forall n \in \mathbb{N} : f_\alpha(n) = \begin{cases} f'_\alpha(n) & \text{if } 1 \leq n \leq \#(A_\alpha), \\ f'_\alpha(1) & \text{if } n = 0 \vee n > \#(A_\alpha). \end{cases}$$

Thus, f_α is surjective. We can define F_α to be a set of functions

$$F_\alpha = \{f_\alpha : \mathbb{N} \to A_\alpha | f_\alpha \text{ follows the definition above}\}$$

and $F_\alpha \neq \emptyset$.

- If A_α is countable, then we define F_α to be a set of bijections

$$F_\alpha = \{f_\alpha : \mathbb{N} \to A_\alpha | f_\alpha \text{ is bijective}\}.$$

Since A_α is countable, we know that $F_\alpha \neq \emptyset$.

Since $\forall \alpha \in J$, $F_\alpha \neq \emptyset$, by axiom of choice (Ax. I.8.1) the set $\prod_{\alpha \in J} F_\alpha \neq \emptyset$. This means we can choose a function $(f_\alpha)_{\alpha \in J}$ from $\prod_{\alpha \in J} F_\alpha$ which maps $\alpha \in J$ to a function $f_\alpha : \mathbb{N} \to A_\alpha$.

We now use axiom of choice (Ax. I.8.1) to choose a function $(f_\alpha)_{\alpha \in J}$ and fix such function. Since J is countable, $\exists g : \mathbb{N} \to J$ such that g is bijective. We now define another function $h : \mathbb{N} \times \mathbb{N} \to \bigcup_{\alpha \in J} A_\alpha$ as follow:

$$\forall (n, m) \in \mathbb{N} \times \mathbb{N} : h(n, m) = f_{g(n)}(m).$$

By Cor. I.8.1.9 we now that $h(\mathbb{N} \times \mathbb{N})$ is at most countable. If we can show that h is surjective, then we can show that $\bigcup_{\alpha \in J} A_\alpha$ is at most countable. Let $x \in \bigcup_{\alpha \in J} A_\alpha$. We know that $\exists \beta \in J$ such that $x \in A_\beta$. By the definition of f_β we know that f_β is surjective. Since f_β is surjective, $\exists m \in \mathbb{N}$ such that $f_\beta(m) = x$. Since g is bijective, $\exists n \in \mathbb{N}$ such that $g(n) = \beta$. Then we have

$$(n, m) \in \mathbb{N} \times \mathbb{N} \implies h(n, m) = f_{g(n)}(m) = f_\beta(m) = x.$$

Since x was arbitrary, we thus know that h is surjective. We conclude that countable union of at most countable set is at most countable.

Finally we show that countable union of countable set is countable. Let I be a countable set and $\forall \alpha \in I$ let A_α be countable set. From the proof above we know that $\bigcup_{\alpha \in I} A_\alpha$ is at most countable. Suppose for the sake of contradiction that $\bigcup_{\alpha \in I} A_\alpha$ is finite. Let $\beta \in I$. By hypothesis we know that A_β is countable, and we have

$$A_\beta \subseteq \bigcup_{\alpha \in I} A_\alpha.$$

But $\bigcup_{\alpha \in I} A_\alpha$ is finite, thus by Prop. I.3.6.14(c) we know that A_β is finite, a contradiction. We conclude that countable union of countable set is countable. □

Ex. I.8.1.10. Find a bijection $f : \mathbb{N} \to \mathbb{Q}$ from the natural numbers to the rationals.

Proof. Helped needed. □

I.8.2 Summation on infinite sets

Def. I.8.2.1 (Series on countable sets)**.** Let X be a countable set, and let $f : X \to \mathbb{R}$ be a function. We say that the series $\sum_{x\in X} f(x)$ is absolutely convergent iff for some bijection $g : \mathbb{N} \to X$, the sum $\sum_{n=0}^{\infty} f(g(n))$ is absolutely convergent. We then define the sum of $\sum_{x\in X} f(x)$ by the formula

$$\sum_{x\in X} f(x) = \sum_{n=0}^{\infty} f(g(n)).$$

Note. From Prop. I.7.4.3, one can show that these definitions do not depend on the choice of g, and so are well defined.

Note. For finite sets X we adopt the convention that series $\sum_{x\in X} f(x)$ are automatically considered to be absolutely convergent.

A.Cor. I.8.2.1. Let X be an at most countable set, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions such that the series $\sum_{x\in X} f(x)$ and $\sum_{x\in X} g(x)$ are both absolutely convergent.

(a) The series $\sum_{x\in X} (f(x) + g(x))$ is absolutely convergent, and

$$\sum_{x\in X} (f(x) + g(x)) = \sum_{x\in X} f(x) + \sum_{x\in X} g(x).$$

(b) If c is a real number, then $\sum_{x\in X} cf(x)$ is absolutely convergent, and

$$\sum_{x\in X} cf(x) = c \sum_{x\in X} f(x).$$

(c) If $X = X_1 \cup X_2$ for some disjoint sets X_1 and X_2, then $\sum_{x\in X_1} f(x)$ and $\sum_{x\in X_2} f(x)$ are absolutely convergent, and

$$\sum_{x\in X_1\cup X_2} f(x) = \sum_{x\in X_1} f(x) + \sum_{x\in X_2} f(x).$$

Conversely, if $h : X \to \mathbb{R}$ is such that $\sum_{x\in X_1} h(x)$ and $\sum_{x\in X_2} h(x)$ are absolutely convergent, then $\sum_{x\in X_1\cup X_2} h(x)$ is also absolutely convergent, and

$$\sum_{x\in X_1\cup X_2} h(x) = \sum_{x\in X_1} h(x) + \sum_{x\in X_2} h(x).$$

(d) If Y is another set, and $\phi : Y \to X$ is a bijection, then $\sum_{y\in Y} f(\phi(y))$ is absolutely convergent, and

$$\sum_{y\in Y} f(\phi(y)) = \sum_{x\in X} f(x).$$

Proof. (a) Since X is at most countable, by Def. I.8.1.1 we know that X is either finite or countable. If X is finite, then the statement follows from Prop. I.7.1.11(f). So suppose that X is countable. By Def. I.8.2.1 we know that there exists a bijection $p : \mathbb{N} \to X$ such that $\sum_{n=0}^{\infty} f\big(p(n)\big)$ converges. Similarly, there exists a bijection $q : \mathbb{N} \to X$ such that $\sum_{n=0}^{\infty} g\big(q(n)\big)$ converges. Since p is bijective, by Prop. I.7.4.3 we know that

$$\sum_{x\in X} g(x) = \sum_{n=0}^{\infty} g(q(n)) = \sum_{n=0}^{\infty} g(p(n)).$$

Thus, we have

$$\begin{aligned}
&\sum_{x\in X}|f(x)| + \sum_{x\in X}|g(x)| \\
&= \sum_{n=0}^{\infty}\big|f\big(p(n)\big)\big| + \sum_{n=0}^{\infty}\big|g\big(p(n)\big)\big| && \text{(by Def. I.8.2.1)} \\
&= \sum_{n=0}^{\infty}\Big(\big|f\big(p(n)\big)\big| + \big|g\big(p(n)\big)\big|\Big) && \text{(by Prop. I.7.2.14(a))} \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\Big(\big|f\big(p(n)\big)\big| + \big|g\big(p(n)\big)\big|\Big) && \text{(by Def. I.7.2.2)} \\
&\geq \lim_{N\to\infty}\sum_{n=0}^{N}\big|f\big(p(n)\big) + g\big(p(n)\big)\big| && \text{(by Thm. I.6.1.19(h))} \\
&= \sum_{n=0}^{\infty}\big|f\big(p(n)\big) + g\big(p(n)\big)\big| && \text{(by Prop. I.6.3.8)} \\
&= \sum_{x\in X}|f(x) + g(x)| && \text{(by Def. I.8.2.1)}
\end{aligned}$$

and $\sum_{x\in X} f(x)+g(x)$ is absolutely convergent. This implies

$$\begin{aligned}
&\sum_{x\in X} f(x)+\sum_{x\in X} g(x) \\
&= \sum_{n=0}^{\infty} f\bigl(p(n)\bigr)+\sum_{n=0}^{\infty} g\bigl(p(n)\bigr) && \text{(by Def. I.8.2.1)} \\
&= \sum_{n=0}^{\infty} \Bigl(f\bigl(p(n)\bigr)+g\bigl(p(n)\bigr)\Bigr) && \text{(by Prop. I.7.2.14(a))} \\
&= \sum_{x\in X} \bigl(f(x)+g(x)\bigr). && \text{(by Def. I.8.2.1)}
\end{aligned}$$

□

Proof. (b) Since X is at most countable, by Def. I.8.1.1 we know that X is either finite or countable. If X is finite, then the statement follows from Prop. I.7.1.11(g). So suppose that X is countable. By Def. I.8.2.1 we know that there exists a bijection $p:\mathbb{N}\to X$ such that $\sum_{n=0}^{\infty} f\bigl(p(n)\bigr)$ converges. Then we have

$$\begin{aligned}
|c|\sum_{x\in X}|f(x)| &= |c|\sum_{n=0}^{\infty}\bigl|f\bigl(p(n)\bigr)\bigr| && \text{(by Def. I.8.2.1)} \\
&= \sum_{n=0}^{\infty}|c|\bigl|f\bigl(p(n)\bigr)\bigr| && \text{(by Prop. I.7.2.14(b))} \\
&= \sum_{n=0}^{\infty}\bigl|cf\bigl(p(n)\bigr)\bigr| \\
&= \sum_{x\in X}|cf(x)| && \text{(by Def. I.8.2.1)}
\end{aligned}$$

and thus $\sum_{x\in X}|cf(x)|$ is absolutely convergent. This implies

$$\begin{aligned}
c\sum_{x\in X} f(x) &= c\sum_{n=0}^{\infty} f\bigl(p(n)\bigr) && \text{(by Def. I.8.2.1)} \\
&= \sum_{n=0}^{\infty} cf\bigl(p(n)\bigr) && \text{(by Prop. I.7.2.14(b))} \\
&= \sum_{x\in X} f(x). && \text{(by Def. I.8.2.1)}
\end{aligned}$$

□

Proof. (c) We first show that if $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, then $\sum_{x \in X_1} f(x)$ and $\sum_{x \in X_2} f(x)$ is absolutely convergent. Since X is at most countable, by Def. I.8.1.1 we know that X is either finite or countable. If X is finite, then the statement follows from Prop. I.7.1.11(e). So suppose that X is countable. Since $X = X_1 \cup X_2$, we know that X_1 and X_2 cannot both be finite. Now we split into two cases:

- One of X_1, X_2 is finite and one is countable. Without the loss of generality suppose that X_1 is finite. Since X_1 is finite, we know that $\exists q_1 : \{i \in \mathbb{N} : 1 \leq i \leq \#(X_1)\} \to X_1$ such that q_1 is bijective. Since X_2 is countable, by Def. I.8.1.1 we know that $\exists q_2 : \mathbb{N} \to X_2$ such that q_2 is bijective. Then we define a function $q : \mathbb{N} \to X$ as follow:

$$\forall n \in \mathbb{N}, q(n) = \begin{cases} q_1(n+1) & \text{if } n < \#(X_1) \\ q_2(n - \#(X_1)) & \text{if } n \geq \#(X_1) \end{cases}$$

Such q is bijective since $X_1 \cap X_2 = \emptyset$ and q_1, q_2 are bijective. Then we have

$$\begin{aligned}
\sum_{x \in X} |f(x)| &= \sum_{n=0}^{\infty} \big|f\big(q(n)\big)\big| && \text{(by Def. I.8.2.1)} \\
&= \sum_{n=0}^{\#(X_1)-1} \big|f\big(q(n)\big)\big| + \sum_{n=\#(X_1)}^{\infty} \big|f\big(q(n)\big)\big| && \text{(by Prop. I.7.2.14(c))} \\
&= \sum_{n=0}^{\#(X_1)-1} \big|f\big(q_1(n+1)\big)\big| \\
&\quad + \sum_{n=\#(X_1)}^{\infty} \Big|f\Big(q_2\big(n - \#(X_1)\big)\Big)\Big| \\
&= \sum_{n=1}^{\#(X_1)} \big|f\big(q_1(n)\big)\big| && \text{(by Lem. I.7.1.4(b))} \\
&\quad + \sum_{n=0}^{\infty} \big|f\big(q_2(n)\big)\big| && \text{(by Prop. I.7.2.14(d))} \\
&= \sum_{x \in X_1} |f(x)| && \text{(by Def. I.7.1.6)} \\
&\quad + \sum_{x \in X_2} \big|f\big(q(n)\big)\big| && \text{(by Def. I.8.2.1)}
\end{aligned}$$

and thus both $\sum_{x \in X_1} f(x)$ and $\sum_{x \in X_2} f(x)$ are absolutely convergent. This implies

$$\sum_{x \in X} f(x) = \sum_{n=0}^{\infty} f(q(x)) \qquad \text{(by Def. I.8.2.1)}$$

$$\begin{aligned}
&= \sum_{n=0}^{\#(X_1)-1} f\big(q(n)\big) + \sum_{n=\#(X_1)}^{\infty} f\big(q(n)\big) && \text{(by Prop. I.7.2.14(c))} \\
&= \sum_{n=0}^{\#(X_1)-1} f\big(q_1(n+1)\big) \\
&\quad + \sum_{n=\#(X_1)}^{\infty} f\Big(q_2\big(n-\#(X_1)\big)\Big) \\
&= \sum_{n=1}^{\#(X_1)} f\big(q_1(n)\big) && \text{(by Lem. I.7.1.4(b))} \\
&\quad + \sum_{n=0}^{\infty} f\big(q_2(n)\big) && \text{(by Prop. I.7.2.14(c))} \\
&= \sum_{x\in X_1} f(x) && \text{(by Def. I.7.1.6)} \\
&\quad + \sum_{x\in X_2} f\big(q(n)\big). && \text{(by Def. I.8.2.1)}
\end{aligned}$$

- Both X_1, X_2 are countable. Since X_1 is countable, by Def. I.8.1.1 we know that $\exists q_1 : \mathbb{N} \to X_1$ such that q_1 is bijective. Similarly, $\exists q_2 : \mathbb{N} \to X_2$ such that q_2 is bijective. Then we define a function $q : \mathbb{N} \to X$ as follow:

$$\forall n \in \mathbb{N}, q(n) = \begin{cases} q_1(\dfrac{n}{2}) & \text{if } n \text{ is even} \\ q_2(\dfrac{n-1}{2}) & \text{if } n \text{ is odd} \end{cases}$$

Such q is bijective since $X_1 \cap X_2 = \emptyset$ and q_1, q_2 are bijective. Then we have

$$\begin{aligned}
&\sum_{x\in X} |f(x)| \\
&= \sum_{n=0}^{\infty} \big|f\big(q(n)\big)\big| && \text{(by Def. I.8.2.1)} \\
&= \lim_{N\to\infty} \sum_{n=0}^{2N} \big|f\big(q(n)\big)\big| && \text{(by Def. I.7.2.2)} \\
&= \lim_{N\to\infty} \sum_{n\le 2N} \big|f\big(q(n)\big)\big| && \text{(by Def. I.7.1.6)} \\
&= \lim_{N\to\infty} \Bigg(\sum_{n\le 2N \wedge n \text{ is even}} \big|f\big(q(n)\big)\big|
\end{aligned}$$

$$\begin{aligned}
&+ \sum_{n\leq 2N\wedge n \text{ is odd}} \left|f\big(q(n)\big)\right|\Bigg) && \text{(by Prop. I.7.1.11(e))}\\
&= \lim_{N\to\infty}\Bigg(\sum_{n\leq 2N\wedge n \text{ is even}} \left|f\Big(q_1\big(\frac{n}{2}\big)\Big)\right| \\
&+ \sum_{n\leq 2N\wedge n \text{ is odd}} \left|f\Big(q_2\big(\frac{n-1}{2}\big)\Big)\right|\Bigg)\\
&= \lim_{N\to\infty}\sum_{n\leq 2N\wedge n \text{ is even}} \left|f\Big(q_1\big(\frac{n}{2}\big)\Big)\right| \\
&+ \lim_{N\to\infty}\sum_{n\leq 2N\wedge n \text{ is odd}} \left|f\Big(q_2\big(\frac{n-1}{2}\big)\Big)\right| && \text{(by Thm. I.6.1.19(a))}\\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\left|f\big(q_1(n)\big)\right| + \lim_{N\to\infty}\sum_{n=0}^{N}\left|f\big(q_2(n)\big)\right| && \text{(by Def. I.7.1.6)}\\
&= \sum_{n=0}^{\infty}\left|f\big(q_1(n)\big)\right| + \sum_{n=0}^{\infty}\left|f\big(q_2(n)\big)\right| && \text{(by Def. I.7.2.2)}\\
&= \sum_{x\in X_1}|f(x)| + \sum_{x\in X_2}|f(x)| && \text{(by Def. I.8.2.1)}
\end{aligned}$$

and thus both $\sum_{x\in X_1} f(x)$ and $\sum_{x\in X_2} f(x)$ are absolutely convergent. This implies

$$\begin{aligned}
&\sum_{x\in X} f(x)\\
&= \sum_{n=0}^{\infty} f\big(q(n)\big) && \text{(by Def. I.8.2.1)}\\
&= \lim_{N\to\infty}\sum_{n=0}^{2N} f\big(q(n)\big) && \text{(by Def. I.7.2.2)}\\
&= \lim_{N\to\infty}\sum_{n\leq 2N} f\big(q(n)\big) && \text{(by Def. I.7.1.6)}\\
&= \lim_{N\to\infty}\Bigg(\sum_{n\leq 2N\wedge n \text{ is even}} f\big(q(n)\big)\\
&+ \sum_{n\leq 2N\wedge n \text{ is odd}} f\big(q(n)\big)\Bigg) && \text{(by Prop. I.7.1.11(e))}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty}\Bigg(\sum_{n\le 2N\wedge n\text{ is even}} f\big(q_1(\frac{n}{2})\big) \\
&\quad + \sum_{n\le 2N\wedge n\text{ is odd}} f\big(q_2(\frac{n-1}{2})\big)\Bigg) \\
&= \lim_{N\to\infty}\sum_{n\le 2N\wedge n\text{ is even}} f\big(q_1(\frac{n}{2})\big) \\
&\quad + \lim_{N\to\infty}\sum_{n\le 2N\wedge n\text{ is odd}} f\big(q_2(\frac{n-1}{2})\big) && \text{(by Thm. I.6.1.19(a))} \\
&= \lim_{N\to\infty}\sum_{n=0}^{N} f\big(q_1(n)\big) + \lim_{N\to\infty}\sum_{n=0}^{N} f\big(q_2(n)\big) && \text{(by Def. I.7.1.6)} \\
&= \sum_{n=0}^{\infty} f\big(q_1(n)\big) + \sum_{n=0}^{\infty} f\big(q_2(n)\big) && \text{(by Def. I.7.2.2)} \\
&= \sum_{x\in X_1} f(x) + \sum_{x\in X_2} f(x). && \text{(by Def. I.8.2.1)}
\end{aligned}
$$

From all cases above, we conclude that both $\sum_{x\in X_1} f(x)$ and $\sum_{x\in X_2} f(x)$ are absolutely convergent, and we have

$$\sum_{x\in X} f(x) = \sum_{x\in X_1} f(x) + \sum_{x\in X_2} f(x).$$

Now we show that if $X_1\cup X_2\subseteq X$, $X_1\cap X_2=\emptyset$, $\sum_{x\in X_1} h(x)$ and $\sum_{x\in X_2} h(x)$ are absolutely convergent, then $\sum_{x\in X_1\cup X_2} h(x)$ is absolutely convergent. Since X is at most countable, by Cor. I.8.1.7 we know that $X_1\cup X_2$ is at most countable. By Def. I.8.1.1 we know that $X_1\cup X_2$ is either finite or countable. If $X_1\cup X_2$ is finite, then the statement follows from Prop. I.7.1.11(e). So suppose that $X_1\cup X_2$ is countable. We know that X_1 and X_2 cannot both be finite. Now we split into two cases:

- One of X_1, X_2 is finite and one is countable. Without the loss of generality suppose that X_1 is finite. Since X_1 is finite, we know that $\exists q_1 : \{i\in\mathbb{N} : 1\le i\le \#(X_1)\}\to X_1$ such that q_1 is bijective. Since X_2 is countable, by Def. I.8.1.1 we know that $\exists q_2:\mathbb{N}\to X_2$ such that q_2 is bijective. Then we define a function $q:\mathbb{N}\to X_1\cup X_2$ as follow:

$$\forall n\in\mathbb{N}, q(n) = \begin{cases} q_1(n+1) & \text{if } n<\#(X_1) \\ q_2(n-\#(X_1)) & \text{if } n\ge \#(X_1)\end{cases}$$

Such q is bijective since $X_1 \cap X_2 = \emptyset$ and q_1, q_2 are bijective. Then we have

$$\begin{aligned}
&\sum_{x\in X_1} |h(x)| + \sum_{x\in X_2} |h(x)| \\
&= \sum_{n=1}^{\#(X_1)} \left|h\big(q_1(n)\big)\right| && \text{(by Def. I.7.1.6)} \\
&\quad + \sum_{n=0}^{\infty} \left|h\big(q_2(n)\big)\right| && \text{(by Def. I.8.2.1)} \\
&= \sum_{n=0}^{\#(X_1)-1} \left|h\big(q_1(n+1)\big)\right| && \text{(by Lem. I.7.1.4(b))} \\
&\quad + \sum_{n=\#(X_1)}^{\infty} \left|h\Big(q_2\big(n-\#(X_1)\big)\Big)\right| && \text{(by Prop. I.7.2.14(d))} \\
&= \sum_{n=0}^{\#(X_1)-1} \left|h\big(q(n)\big)\right| + \sum_{n=\#(X_1)}^{\infty} \left|h\big(q(n)\big)\right| \\
&= \sum_{n=0}^{\infty} \left|h\big(q(n)\big)\right| && \text{(by Prop. I.7.2.14(c))} \\
&= \sum_{x\in X_1\cup X_2} |h(x)| && \text{(by Def. I.8.2.1)}
\end{aligned}$$

and thus $\sum_{x\in X_1\cup X_2} h(x)$ is absolutely convergent.

- Both X_1, X_2 are countable. Since X_1 is countable, by Def. I.8.1.1 we know that $\exists q_1 : \mathbb{N} \to X_1$ such that q_1 is bijective. Similarly, $\exists q_2 : \mathbb{N} \to X_2$ such that q_2 is bijective. Then we define a function $q : \mathbb{N} \to X_1 \cup X_2$ as follow:

$$\forall n \in \mathbb{N}, q(n) = \begin{cases} q_1(\frac{n}{2}) & \text{if } n \text{ is even} \\ q_2(\frac{n-1}{2}) & \text{if } n \text{ is odd} \end{cases}$$

Such q is bijective since $X_1 \cap X_2 = \emptyset$ and q_1, q_2 are bijective. Then we have

$$\begin{aligned}
&\sum_{x\in X_1} |h(x)| + \sum_{x\in X_2} |h(x)| \\
&= \sum_{n=0}^{\infty} \left|h\big(q_1(n)\big)\right| + \sum_{n=0}^{\infty} \left|h\big(q_2(n)\big)\right| && \text{(by Def. I.8.2.1)} \\
&= \sum_{n=0}^{\infty} \Big(\left|h\big(q_1(n)\big)\right| + \left|h\big(q_2(n)\big)\right|\Big) && \text{(by Prop. I.7.2.14(a))}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty}\left(\sum_{n=0}^{N}\left|h\big(q_1(n)\big)\right| + \sum_{n=0}^{N}\left|h\big(q_2(n)\big)\right|\right) && \text{(by Def. I.7.2.2)}\\
&= \lim_{N\to\infty}\left(\sum_{n=0}^{N}\left|h\big(q(2n)\big)\right| + \sum_{n=0}^{N}\left|h\big(q(2n+1)\big)\right|\right) \\
&= \lim_{N\to\infty}\left(\sum_{n\le 2N: n \text{ is even}}\left|h\big(q(n)\big)\right|\right.\\
&\qquad \left. + \sum_{n\le 2N: n \text{ is odd}}\left|h\big(q(n)\big)\right|\right) && \text{(by Def. I.7.1.6)}\\
&= \lim_{N\to\infty}\sum_{n\le 2N}\left|h\big(q(n)\big)\right| && \text{(by Prop. I.7.1.11(e))}\\
&= \lim_{N\to\infty}\sum_{n=0}^{2N}\left|h\big(q(n)\big)\right| && \text{(by Def. I.7.1.6)}\\
&= \sum_{n=0}^{\infty}\left|h\big(q(n)\big)\right| && \text{(by Def. I.7.2.2)}\\
&= \sum_{x\in X_1\cup X_2}^{\infty}|h(x)| && \text{(by Def. I.8.2.1)}
\end{aligned}
$$

and thus $\sum_{x\in X_1\cup X_2} h(x)$ is absolutely convergent.

From all cases above, we conclude that $\sum_{x\in X_1\cup X_2} h(x)$ is absolutely convergent. Since $\sum_{x\in X_1\cup X_2} h(x)$ is absolutely convergent, from the proof above we have

$$\sum_{x\in X_1\cup X_2} h(x) = \sum_{x\in X_1} h(x) + \sum_{x\in X_2} h(x).$$

□

Proof. (d) Since X is at most countable, by Def. I.8.1.1 we know that X is either finite or countable. If X is finite, then the statement follows from Prop. I.7.1.11(c). So suppose that X is countable. By Def. I.8.2.1 we know that there exists a bijection $p : \mathbb{N} \to X$ such that $\sum_{n=0}^{\infty} f\big(p(n)\big)$ converges. Since ϕ is bijective, we know that Y is also countable and by Def. I.8.1.1 $\exists q : \mathbb{N} \to Y$ such that q is bijective. Then we have $\phi \circ q : \mathbb{N} \to X$ is bijective and

$$\sum_{x\in X} f(x) = \sum_{n=0}^{\infty} f\big(p(n)\big) \qquad \text{(by Def. I.8.2.1)}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} f\big((\phi \circ q)(n)\big) && \text{(by Prop. I.7.4.3)}\\
&= \sum_{n=0}^{\infty} f\Big(\phi\big(q(n)\big)\Big) &&\\
&= \sum_{y \in Y} f\big(\phi(y)\big). && \text{(by Def. I.8.2.1)}
\end{aligned}
$$

Thus, $\sum_{y \in Y} f\big(\phi(y)\big)$ is absolutely convergent. □

Thm. I.8.2.2 (Fubini's theorem for infinite sums)**.** Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a function such that $\sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(n,m)$ is absolutely convergent. Then we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} f(n,m) \right) &= \sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(n,m)\\
&= \sum_{(m,n) \in \mathbb{N} \times \mathbb{N}} f(n,m)\\
&= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} f(n,m) \right).
\end{aligned}
$$

In other words, we can switch the order of infinite sums *provided that the entire sum is absolutely convergent.*

Proof. The second equality follows easily from Prop. I.7.4.3 (and Prop. I.3.6.4).

Let us first consider the case when $f(n,m)$ is always non-negative (we will deal with the general case later). Write

$$L := \sum_{(n,m) \in \mathbb{N} \times \mathbb{N}} f(n,m);$$

our task is to show that the series $\sum_{n=0}^{\infty} (\sum_{m=0}^{\infty} f(n,m))$ converges to L.

One can easily show that $\sum_{(n,m) \in X} f(n,m) \leq L$ for all finite sets $X \subseteq \mathbb{N} \times \mathbb{N}$. (Use a bijection g between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$, and then use the fact that $g(X)$ is finite, hence bounded.) In particular, for every $n \in \mathbb{N}$ and $M \in \mathbb{N}$ we have $\sum_{m=0}^{M} f(n,m) \leq L$, which implies by Prop. I.6.3.8 that $\sum_{m=0}^{\infty} f(n,m)$ is convergent for each n. Similarly, for any $N \in \mathbb{N}$ and $M \in \mathbb{N}$

we have (by Cor. I.7.1.14)

$$\sum_{n=0}^{N} \sum_{m=0}^{M} f(n,m) = \sum_{(n,m)\in X} f(n,m) \leq L$$

where X is the set $\{(n,m) \in \mathbb{N} \times \mathbb{N} : n \leq N, m \leq M\}$ which is finite by Prop. I.3.6.14. Taking limits of this as $M \to \infty$ we have (by Ex. I.7.1.5 and either Prop. I.6.3.8 or Lem. I.6.4.13)

$$\sum_{n=0}^{N} \sum_{m=0}^{\infty} f(n,m) \leq L.$$

By Prop. I.6.3.8, this implies that $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n,m)$ converges, and

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n,m) \leq L.$$

To finish the proof, it will suffice to show that

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n,m) \geq L - \varepsilon$$

for every $\varepsilon > 0$.

$$\begin{aligned}
& L \geq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n,m) \geq L - \varepsilon \\
\implies & L + \varepsilon \geq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n,m) \geq L - \varepsilon \\
\implies & \varepsilon \geq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n,m) - L \geq -\varepsilon \\
\implies & \left| \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n,m) - L \right| \leq \varepsilon
\end{aligned}$$

So, let $\varepsilon > 0$. By definition of L, we can then find a finite set $X \subseteq \mathbb{N} \times \mathbb{N}$ such that $\sum_{(n,m)\in X} f(n,m) \geq L - \varepsilon$. (Since $\mathbb{N} \times \mathbb{N}$ is countable by Cor. I.8.1.13, we can find a bijection $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that $\sum_{i=0}^{\infty} f(g(i)) = L$, which means $\forall \varepsilon > 0$, $\exists H \in \mathbb{N}$ such that

$\left|\sum_{i=0}^{h} f(g(i)) - L\right| \leq \varepsilon$ for all $h \geq H$. Now we can choose $X = \{g(i) : 0 \leq i \leq H\})$ This set, being finite, must be contained in some set of the form $Y := \{(n, m) \in \mathbb{N} \times \mathbb{N} : n \leq N; m \leq M\}$. Thus, by Cor. I.7.1.14

$$\sum_{n=0}^{N} \sum_{m=0}^{M} f(n, m) = \sum_{(n,m)\in Y} f(n, m) \geq \sum_{(n,m)\in X} f(n, m) \geq L - \varepsilon$$

and hence

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n, m) \geq \sum_{n=0}^{N} \sum_{m=0}^{\infty} f(n, m) \geq \sum_{n=0}^{N} \sum_{m=0}^{M} f(n, m) \geq L - \varepsilon$$

as desired.

This proves the claim when the $f(n, m)$ are all non-negative. A similar argument works when the $f(n, m)$ are all non-positive (in fact, one can simply apply the result just obtained to the function $-f(n, m)$, and then use limit laws to remove the $-$. For the general case, note that any function $f(n, m)$ can be written as $f_+(n, m) + f_-(n, m)$, where $f_+(n, m)$ is the positive part of $f(n, m)$ (i.e., it equals $f(n, m)$ when $f(n, m)$ is positive, and 0 otherwise), and f_- is the negative part of $f(n, m)$ (it equals $f(n, m)$ when $f(n, m)$ is negative, and 0 otherwise). It is easy to show that if $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}} f(n, m)$ is absolutely convergent, then so are $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}} f_+(n, m)$ and $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}} f_-(n, m)$. (We can construct a bijection $g : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ and then since $\forall n \in \mathbb{N}$ we have $f_+(g(n)) \leq |f(g(n))|$ and $|f_-(g(n))| \leq |f(g(n))|$, we know that $(f_+(g(n)))_{n=0}^{\infty}$ and $(f_-(g(n)))_{n=0}^{\infty}$ are absolutely convergent by comparison test (Cor. I.7.3.2.)) So now one applies the results just obtained to f_+ and to f_- and adds them together using limit laws to obtain the result for a general f. □

Lem. I.8.2.3. Let X be a countable set, and let $f : X \to \mathbb{R}$ be a function. Then the series $\sum_{x\in X} f(x)$ is absolutely convergent iff

$$\sup\left\{\sum_{x\in A} |f(x)| : A \subseteq X, A \text{ finite}\right\} < \infty.$$

Proof. Let $P(X, f)$ be the statement

$$\sup\left\{\sum_{x\in A} |f(x)| : A \subseteq X, A \text{ finite}\right\} < \infty.$$

We first show that if $\sum_{x\in X} f(x)$ is absolutely convergent, then $P(X, f)$ is true. Let $L = \sum_{x\in X} f(x)$. Since $\sum_{x\in X} f(x)$ is absolutely convergent, by Def. I.8.2.1 $\exists g : \mathbb{N} \to X$ where g is a bijection such that

$$L = \sum_{x \in X} |f(x)| = \sum_{n=0}^{\infty} \left| f\big(g(n)\big) \right|.$$

Let $A \subseteq X$ be a finite set. Since g is a bijection, we have

$$\sum_{x \in A} |f(x)| = \sum_{n \in g^{-1}(A)} |f(g(n))|$$

Since A is finite, by Ex. I.3.6.3 $\exists M \in \mathbb{N}$ such that $g^{-1}(A)$ is bounded by M. So we have

$$\sum_{x \in A} |f(x)| = \sum_{n \in g^{-1}(A)} |f(g(n))| \leq \sum_{n=0}^{M} |f(g(n))| \leq L$$

This is true for any finite subset of X. Thus, by Thm. I.5.5.9 $P(X, f)$ is true.

Now we show that if $P(X, f)$ is true, then $\sum_{x \in X} f(x)$ is absolutely convergent. Let L be the supremum described by $P(X, f)$. Since X is countable, $\exists g : \mathbb{N} \to X$ where g is a bijection. So we have

$$\begin{aligned}
&\forall n \in \mathbb{N} : \sum_{x \in g(\{i \in \mathbb{N} : 0 \leq i \leq n\})} |f(x)| \leq L && (P(X, f) \text{ is true}) \\
\implies &\forall n \in \mathbb{N} : \sum_{i=0}^{n} |f(g(i))| \leq L && (\text{by Def. I.7.1.6}) \\
\implies &\sum_{i=0}^{\infty} |f(g(i))| \text{ converges} && (\text{by Prop. I.7.3.1}) \\
\implies &\sum_{x \in X} |f(x)| \text{ converges.} && (\text{by Def. I.8.2.1})
\end{aligned}$$

□

Note. Inspired by Lem. I.8.2.3, we may now define the concept of an absolutely convergent series even when the set X could be uncountable.

Def. I.8.2.4. Let X be a set (which could be uncountable), and let $f : X \to \mathbb{R}$ be a function. We say that the series $\sum_{x \in X} f(x)$ is absolutely convergent iff

$$\sup \left\{ \sum_{x \in A} |f(x)| : A \subseteq X, A \text{ finite} \right\} < \infty.$$

Lem. I.8.2.5. Let X be a set (which could be uncountable), and let $f : X \to \mathbb{R}$ be a function such that the series $\sum_{x\in X} f(x)$ is absolutely convergent. Then the set $\{x \in X : f(x) \neq 0\}$ is at most countable.

Proof. Suppose that X is a set and $f : X \to \mathbb{R}$ is a function such that $\sum_{x\in X} f(x)$ is absolutely convergent. Since $\sum_{x\in X} f(x)$ is absolutely convergent, by Def. I.8.2.4 we have

$$M = \sup\left\{\sum_{x\in A}|f(x)| : A \subseteq X, A \text{ finite}\right\} < \infty.$$

We first show that $\forall n \in \mathbb{Z}^+$, the set $S_n = \{x \in X : |f(x)| > 1/n\}$ is finite and $\#(S_n) \leq Mn$.

Suppose for the sake of contradiction that S_n is infinite. Then we can have a finite set $S \subseteq S_n$ where $\#(S) > (M + 1)n$. Since S is finite, we have $\sum_{x\in S}|f(x)| \leq M$. Since $S \subseteq S_n$, we have $|f(x)| > 1/n$ for every $x \in S$. But now we have

$$M \geq \sum_{x\in S}|f(x)| > \frac{(M+1)n}{n} = M + 1,$$

a contradiction. Thus, S_n must be finite.

Now suppose for the sake of contradiction $\#(S_n) > Mn$. Again we have

$$M \geq \sum_{x\in S_n} |f(x)| > \frac{Mn}{n} = M,$$

a contradiction. Thus, $\#(S_n) \leq Mn$.

Let $x \in X$ where $f(x) \neq 0$. If x does not exist, then we have $\{x \in X : f(x) \neq 0\} = \emptyset$ which is at most countable. So suppose that such x exists. Since $|f(x)| \in \mathbb{R}^+$, by Prop. I.5.4.12 we have

$$\begin{aligned}
&\exists N \in \mathbb{Z}^+ : \frac{1}{|f(x)|} < N \\
\implies &|f(x)| > \frac{1}{N} \\
\implies &x \in S_N && \text{(by the definition of } S_N\text{)} \\
\implies &x \in \bigcup_{n\in\mathbb{Z}^+} S_n && \text{(by Ax. I.3.11)} \\
\implies &\{x \in X : f(x) \neq 0\} \subseteq \bigcup_{n\in\mathbb{Z}^+} S_n. && \text{(by Def. I.3.1.15)}
\end{aligned}$$

Since $\forall n \in \mathbb{Z}^+$, S_n is finite, by Cor. I.8.1.9 we know that $\bigcup_{n\in\mathbb{Z}^+} S_n$ is at most countable. Since $\{x \in X : f(x) \neq 0\} \subseteq \bigcup_{n\in\mathbb{Z}^+} S_n$, by Cor. I.8.1.7 we know that $\{x \in X : f(x) \neq 0\}$ is at most countable. □

Note. Because of Lem. I.8.2.5, we can define the value of $\sum_{x\in X} f(x)$ for any absolutely convergent series on an uncountable set X by the formula

$$\sum_{x\in X} := \sum_{x\in X: f(x)\neq 0} f(x),$$

since we have replaced a sum on an uncountable set X by a sum on the at most countable set $\{x \in X : f(x) \neq 0\}$. (If the former sum is absolutely convergent, then the latter one is also.) Def. I.8.2.4 is consistent with the definitions we already have for series on countable sets (Def. I.8.2.1).

Prop. I.8.2.6 (Absolutely convergent series laws)**.** Let X be an arbitrary set (possibly uncountable), and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions such that the series $\sum_{x\in X} f(x)$ and $\sum_{x\in X} g(x)$ are both absolutely convergent.

(a) The series $\sum_{x\in X}(f(x) + g(x))$ is absolutely convergent, and

$$\sum_{x\in X}(f(x) + g(x)) = \sum_{x\in X} f(x) + \sum_{x\in X} g(x).$$

(b) If c is a real number, then $\sum_{x\in X} cf(x)$ is absolutely convergent, and

$$\sum_{x\in X} cf(x) = c\sum_{x\in X} f(x).$$

(c) If $X = X_1 \cup X_2$ for some disjoint sets X_1 and X_2, then $\sum_{x\in X_1} f(x)$ and $\sum_{x\in X_2} f(x)$ are absolutely convergent, and

$$\sum_{x\in X_1\cup X_2} f(x) = \sum_{x\in X_1} f(x) + \sum_{x\in X_2} f(x).$$

Conversely, if $h : X \to \mathbb{R}$ is such that $\sum_{x\in X_1} h(x)$ and $\sum_{x\in X_2} h(x)$ are absolutely convergent, then $\sum_{x\in X_1\cup X_2} h(x)$ is also absolutely convergent, and

$$\sum_{x\in X_1\cup X_2} h(x) = \sum_{x\in X_1} h(x) + \sum_{x\in X_2} h(x).$$

(d) If Y is another set, and $\phi : Y \to X$ is a bijection, then $\sum_{y\in Y} f(\phi(y))$ is absolutely convergent, and

$$\sum_{y\in Y} f(\phi(y)) = \sum_{x\in X} f(x).$$

Proof. (a) Suppose that X is a set and $f : X \to \mathbb{R}, g : X \to \mathbb{R}$ are functions such that $\sum_{x\in X} f(x)$ and $\sum_{x\in X} g(x)$ are both absolutely convergent. By Prop. I.7.1.11(f) we already show that the statement is true when X is finite. So suppose that X is infinite.

We first show that $\sum_{x\in X}(f(x)+g(x))$ is absolutely convergent. Since $\sum_{x\in X} f(x)$ and $\sum_{x\in X} g(x)$ are both absolutely convergent, by Def. I.8.2.4 $\exists N, M \in \mathbb{R}$ such that

$$N = \sup\left\{\sum_{x\in A}|f(x)| : A \subseteq X, A \text{ finite}\right\} < \infty$$

and

$$M = \sup\left\{\sum_{x\in A}|g(x)| : A \subseteq X, A \text{ finite}\right\} < \infty.$$

Let $A \subseteq X$ be a finite set. Then we have

$$\begin{aligned}\sum_{x\in A}|f(x)+g(x)| &\le \sum_{x\in A}(|f(x)|+|g(x)|)\\ &= \sum_{x\in A}|f(x)| + \sum_{x\in A}|g(x)| && \text{(by Prop. I.7.1.11(f))}\\ &\le N+M.\end{aligned}$$

Since A was arbitrary, we have

$$\sup\left\{\sum_{x\in A}|f(x)+g(x)| : A \subseteq X, A \text{ finite}\right\} \le N+M < \infty.$$

Thus, by Def. I.8.2.4 $\sum_{x\in X}\big(f(x)+g(x)\big)$ is absolutely convergent.

Now we show that $\sum_{x\in X}(f(x)+g(x)) = \sum_{x\in X} f(x) + \sum_{x\in X} g(x)$. If X is at most countable, then the statement follows by A.Cor. I.8.2.1(a). So suppose that X is uncountable. Let $X_f = \{x \in X : f(x) \neq 0\}$, $X_g = \{x \in X : g(x) \neq 0\}$ and $X_h = \{x \in X : f(x)+g(x) \neq 0\}$ be sets. Then by Lem. I.8.2.5 we know that X_f, X_g and X_h are at most countable. Since

$$\begin{aligned}
&\forall x \in X_f \setminus X_g \\
\implies& f(x) \neq 0 \wedge g(x) = 0 \\
\implies& f(x) + g(x) \neq 0 \\
\implies& x \in X_h,
\end{aligned}$$

we know that $X_f \setminus X_g \subseteq X_h$. Similarly, we have $X_g \setminus X_f \subseteq X_h$. Then we have

$$\begin{aligned}
&\forall x \in X_h \\
\implies& f(x) + g(x) \neq 0 \\
\implies& f(x) \neq -g(x) \\
\implies& \big(f(x) \neq 0 \wedge g(x) = 0\big) \vee \big(f(x) = 0 \wedge g(x) \neq 0\big) \\
&\vee \big(f(x) \neq -g(x) \wedge f(x) \neq 0 \wedge g(x) \neq 0\big) \\
\implies& (x \in X_f \setminus X_g) \vee (x \in X_g \setminus X_f) \vee (x \in X_f \cap X_g \wedge x \in X_h) \\
\implies& \big(x \in (X_f \setminus X_g) \cup (X_g \setminus X_f) \cup (X_f \cap X_g)\big) \\
&\wedge \big(x \in (X_f \setminus X_g) \cup (X_g \setminus X_f) \cup X_h\big) \\
\implies& (x \in X_f \cup X_g) \wedge (x \in X_h) \\
\implies& x \in X_f \cup X_g
\end{aligned}$$

and $X_h \subseteq X_f \cup X_g$. By A.Cor. I.8.1.1 we know that $X_f \cup X_g$ is at most countable. By Cor. I.8.1.7 we know that $X_f \cup X_g \setminus X_f$, $X_f \cup X_g \setminus X_g$, $X_f \cup X_g \setminus X_h$ are at most countable. Thus, we have

$$\begin{aligned}
&\sum_{x\in X} \big(f(x)+g(x)\big) \\
&= \sum_{x\in X_h} \big(f(x)+g(x)\big) && \text{(by Lem} \\
&= \sum_{x\in X_h} \big(f(x)+g(x)\big) \\
&\quad + \sum_{x\in (X_f\cup X_g)\setminus X_h} \big(f(x)+g(x)\big) && (x \notin X_h \iff f(x)+g(x) = 0) \\
&= \sum_{x\in X_f\cup X_g} \big(f(x)+g(x)\big) && \text{(by A.Co} \\
&= \sum_{x\in X_f\cup X_g} f(x) + \sum_{x\in X_f\cup X_g} g(x) && \text{(by A.Co}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{x\in X_f} f(x) + \sum_{x\in (X_f\cup X_g)\setminus X_f} f(x) && \text{(by A.Co} \\
&\quad + \sum_{x\in X_g} g(x) + \sum_{x\in (X_f\cup X_g)\setminus X_g} g(x) && \text{(by A.Co} \\
&= \sum_{x\in X_f} f(x) + \sum_{x\in X_g} g(x) && (x\notin X_f \iff f(x)=0, x\notin X_g \iff g(x)=0) \\
&= \sum_{x\in X} f(x) + \sum_{x\in X} g(x). && \text{(by Lem}
\end{aligned}
$$

□

Proof. (b) Suppose that $c\in\mathbb{R}$, X is a set and $f: X\to\mathbb{R}$ is a function such that $\sum_{x\in X} f(x)$ is absolutely convergent. Since $\sum_{x\in X} f(x)$ is absolutely convergent, by Def. I.8.2.4 $\exists N\in\mathbb{R}$ such that

$$N = \sup\left\{\sum_{x\in A}|f(x)| : A\subseteq X, A \text{ finite}\right\} < \infty.$$

Let $A\subseteq X$ be a finite set. Then we have

$$
\begin{aligned}
\sum_{x\in A}|cf(x)| &= \sum_{x\in A}|c||f(x)| \\
&= |c|\sum_{x\in A}|f(x)| && \text{(by Prop. I.7.1.11(g))} \\
&\le |c|N.
\end{aligned}
$$

Since A was arbitrary, we have

$$\sup\left\{\sum_{x\in A}|cf(x)| : A\subseteq X, A \text{ finite}\right\} \le |c|N < \infty.$$

Thus, by Def. I.8.2.4 $\sum_{x\in X} cf(x)$ is absolutely convergent.

Now we show that $\sum_{x\in X} cf(x) = c\sum_{x\in X} f(x)$. If X is at most countable, then the statement follows by A.Cor. I.8.2.1(b). So suppose that X is uncountable. Let $X_f = \{x\in X : f(x)\neq 0\}$ and $X_h = \{x\in X : cf(x)\neq 0\}$ be two sets. By Lem. I.8.2.5 we know that both X_f and X_h are at most countable.

- If $c=0$, then $X_h=\emptyset$ and we have

$$\sum_{x\in X} 0f(x) = \sum_{x\in X_h} 0f(x) \qquad \text{(by Lem. I.8.2.5)}$$

$$\begin{aligned} &= 0 && \text{(by Prop. I.7.1.11(a))} \\ &= 0 \sum_{x \in X} f(x). \end{aligned}$$

- If $c \neq 0$, then we have $X_f = X_h$ since

$$\forall x, x \in X_f \iff f(x) \neq 0 \iff cf(x) \neq 0 \iff x \in X_h.$$

Thus

$$\begin{aligned} \sum_{x \in X} cf(x) &= \sum_{x \in X_h} cf(x) && \text{(by Lem. I.8.2.5)} \\ &= c \sum_{x \in X_h} f(x) && \text{(by A.Cor. I.8.2.1(b))} \\ &= c \sum_{x \in X_f} f(x) \\ &= c \sum_{x \in X} f(x). && \text{(by Lem. I.8.2.5)} \end{aligned}$$

From all cases above, we conclude that $\sum_{x \in X} cf(x) = c \sum_{x \in X} f(x)$. □

Proof. (c) If X is at most countable, then the statements follow by A.Cor. I.8.2.1(c). So suppose that X is uncountable.

We first show that if $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, then $\sum_{x \in X_1} f(x)$ and $\sum_{x \in X_2} f(x)$ is absolutely convergent. Since $\sum_{x \in X} f(x)$ is absolutely convergent, by Def. I.8.2.4 $\exists N \in \mathbb{R}$ such that

$$N = \sup\left\{ \sum_{x \in A} |f(x)| : A \subseteq X, A \text{ finite} \right\} < \infty.$$

Let $A_1 \subseteq X_1, A_2 \subseteq X_2$ and both A_1, A_2 are finite. Then we have

$$\sum_{x \in A_1} |f(x)| \leq N,$$
$$\sum_{x \in A_2} |f(x)| \leq N.$$

Since A_1, A_2 were arbitrary, we have

$$\sup\left\{ \sum_{x \in A} |f(x)| : A \subseteq X_1, A \text{ finite} \right\} \leq N < \infty$$

and

$$\sup\left\{\sum_{x\in A}|f(x)| : A \subseteq X_2, A \text{ finite}\right\} \leq N < \infty.$$

Thus, by Def. I.8.2.4 both $\sum_{x\in X_1} f(x)$ and $\sum_{x\in X_2} f(x)$ are absolutely convergent.

Next we show that if $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, then

$$\sum_{x\in X_1\cup X_2} f(x) = \sum_{x\in X_1} f(x) + \sum_{x\in X_2} f(x).$$

Let $X_f = \{x \in X : f(x) \neq 0\}, X_{f_1} = \{x \in X_1 : f(x) \neq 0\}, X_{f_2} = \{x \in X_2 : f(x) \neq 0\}$ be sets. Clearly, we have $X_{f_1} \cup X_{f_2} = X_f$ and $X_{f_1} \cap X_{f_2} = \emptyset$. By Lem. I.8.2.5 we know that X_f, X_{f_1}, X_{f_2} are at most countable. Then we have

$$\begin{aligned}
\sum_{x\in X} f(x) &= \sum_{x\in X_f} f(x) && \text{(by Lem. I.8.2.5)}\\
&= \sum_{x\in X_{f_1}\cup X_{f_2}} f(x)\\
&= \sum_{x\in X_{f_1}} f(x) + \sum_{x\in X_{f_2}} f(x) && \text{(by A.Cor. I.8.2.1(c))}\\
&= \sum_{x\in X_1} f(x) + \sum_{x\in X_2} f(x). && \text{(by Lem. I.8.2.5)}
\end{aligned}$$

Finally we show that if $X_1 \cup X_2 \subseteq X$, $X_1 \cap X_2 = \emptyset$, $h : X \to \mathbb{R}$ is a function such that $\sum_{x\in X_1} h(x)$ and $\sum_{x\in X_2} h(x)$ are absolutely convergent, then $\sum_{x\in X_1\cup X_2} h(x)$ is also absolutely convergent. Since $\sum_{x\in X_1} h(x)$ and $\sum_{x\in X_2} h(x)$ are absolutely convergent, by Def. I.8.2.4 $\exists N, M \in \mathbb{R}$ such that

$$N = \sup\left\{\sum_{x\in A}|h(x)| : A \subseteq X_1, A \text{ finite}\right\} < \infty$$

and

$$M = \sup\left\{\sum_{x\in A}|h(x)| : A \subseteq X_2, A \text{ finite}\right\} < \infty.$$

Let $A \subseteq X_1 \cup X_2$ be a finite set. Let $A_1 = A \cap X_1$ and $A_2 = A \cap X_2$. Clearly, we have

$$\begin{aligned}
A_1 \cap A_2 &= \emptyset,\\
A_1 \cup A_2 &= A,
\end{aligned}$$

$$\begin{aligned}
&A_1 \subseteq X_1,\\
&A_2 \subseteq X_2,\\
&A_1 \text{ is finite},\\
&A_2 \text{ is finite}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\sum_{x\in A} |h(x)| &= \sum_{x\in A_1\cup A_2} |h(x)| \\
&= \sum_{x\in A_1} |h(x)| + \sum_{x\in A_2} |h(x)| \\
&\leq N + M.
\end{aligned}$$

Since A was arbitrary, we have

$$\sup\left\{\sum_{x\in A} |h(x)| : A \subseteq X_1 \cup X_2, A \text{ finite}\right\} \leq N + M < \infty$$

Thus, by Def. I.8.2.4 $\sum_{x\in X_1\cup X_2} h(x)$ is absolutely convergent. Since $\sum_{x\in X_1\cup X_2} h(x)$ is absolutely convergent, from the proof above we have

$$\sum_{x\in X_1\cup X_2} h(x) = \sum_{x\in X_1} h(x) + \sum_{x\in X_2} h(x).$$

□

Proof. (d) If X is at most countable, then the statements follow by A.Cor. I.8.2.1(d). So suppose that X is uncountable.

We first show that $\sum_{y\in Y} f\big(\phi(y)\big)$ is absolutely convergent. Since $\sum_{x\in X} f(x)$ is absolutely convergent, by Def. I.8.2.4, $\exists N \in \mathbb{R}$ such that

$$N = \sup\left\{\sum_{x\in A} |f(x)| : A \subseteq X, A \text{ finite}\right\} < \infty$$

Let $A \subseteq Y$ be a finite set. Then we have

$$\begin{aligned}
\sum_{y\in A} \big|f\big(\phi(y)\big)\big| &= \sum_{x\in \phi(A)} |f(x)| && \text{(by Prop. I.7.1.11(c))} \\
&\leq N. && (\phi(A) \text{ is finite})
\end{aligned}$$

Since A was arbitrary, we have

$$\sup\left\{\sum_{y\in A}\left|f\big(\phi(x)\big)\right| : A\subseteq Y, A \text{ finite}\right\}\leq N<\infty.$$

Thus, by Def. I.8.2.4 $\sum\limits_{y\in Y} f\big(\phi(y)\big)$ is absolutely convergent.

Now we show that $\sum\limits_{y\in Y} f\big(\phi(y)\big) = \sum\limits_{x\in X} f(x)$. Let $X_f = \{x\in X : f(x)\neq 0\}$ and $Y_f = \{y\in Y : f\big(\phi(y)\big)\neq 0\}$ be sets. By Lem. I.8.2.5 we know that X_f and Y_f are at most countable. Clearly, ϕ is a bijective between X_f and Y_f. Thus, we have

$$\begin{aligned}
\sum_{y\in Y} f\big(\phi(y)\big) &= \sum_{y\in Y_f} f\big(\phi(y)\big) && \text{(by Lem. I.8.2.5)}\\
&= \sum_{x\in X_f} f(x) && \text{(by A.Cor. I.8.2.1(d))}\\
&= \sum_{x\in X} f(x). && \text{(by Lem. I.8.2.5)}
\end{aligned}$$

□

Lem. I.8.2.7. Let $\sum\limits_{n=0}^{\infty} a_n$ be a series of real numbers which is conditionally convergent, but not absolutely convergent. Define the sets $A_+ := \{n\in\mathbb{N} : a_n\geq 0\}$ and $A_- := \{n\in\mathbb{N} : a_n<0\}$, thus $A_+\cup A_- = \mathbb{N}$ and $A_+\cap A_- = \emptyset$. Then both of the series $\sum\limits_{n\in A_+} a_n$ and $\sum\limits_{n\in A_-} a_n$ are not absolutely convergent.

Proof. Suppose for the sake of contradiction that at least one of the series $\sum\limits_{n\in A_+} a_n$ and $\sum\limits_{n\in A_-} a_n$ is absolutely convergent. Let $b_n = \max(a_n, 0)$ and $c_n = -\min(a_n, 0)$. Then we have $a_n = b_n - c_n$ and

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n &= \sum_{n=0}^{\infty} b_n - c_n\\
&= \sum_{n=0}^{\infty} b_n - \sum_{n=0}^{\infty} c_n && \text{(by Prop. I.7.2.14(a))}\\
&= \sum_{n=0}^{\infty} \max(a_n, 0) + \sum_{n=0}^{\infty} \min(a_n, 0)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{n\in A} \max(a_n, 0) + \sum_{n\in A} \min(a_n, 0) && \text{(by Def. I.8.2.1)}\\
&= \sum_{n\in A_+} a_n + \sum_{n\in A_-} a_n. && \text{(by Lem. I.8.2.5)}
\end{aligned}$$

Thus, $\sum_{n\in A_+} a_n$ and $\sum_{n\in A_-} a_n$ converges. Since $\sum_{n\in A_+} a_n$ converges and

$$\sum_{n\in A_+} |a_n| = \sum_{n\in A_+} a_n,$$

we know that $\sum_{n\in A_+} a_n$ is absolutely converges. Since

$$\begin{aligned}
\sum_{n\in A_-} |a_n| &= \sum_{n=0}^{\infty} |\min(a_n, 0)| && \text{(by Lem. I.8.2.5)}\\
&= \sum_{n=0}^{\infty} -\min(a_n, 0)\\
&= -\sum_{n=0}^{\infty} \min(a_n, 0) && \text{(by Prop. I.7.2.14(b))}\\
&= -\sum_{n\in A_-} a_n, && \text{(by Lem. I.8.2.5)}
\end{aligned}$$

we know that $\sum_{n\in A_-} |a_n|$ is absolutely convergent. But by Prop. I.8.2.6(c) we have

$$\sum_{n\in A_+} a_n + \sum_{n\in A_-} a_n = \sum_{n\in A} a_n$$

and $\sum_{n\in A} a_n$ is absolutely convergent, a contradiction. Thus, both $\sum_{n\in A_+} a_n$ and $\sum_{n\in A_-} a_n$ are not absolutely convergent. □

Note. Thm. I.8.2.8 is done by Georg Riemann (1826–1866), which asserts that a series which converges conditionally but not absolutely can be rearranged to converge to any value one pleases!

Thm. I.8.2.8. Let $\sum_{n=0}^{\infty} a_n$ be a series which is conditionally convergent, but not absolutely convergent, and let L be any real number. Then there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $\sum_{m=0}^{\infty} a_{f(m)}$ converges conditionally to L.

Proof. Let A_+ and A_- be the sets in Lem. I.8.2.7; from Lem. I.8.2.7 we know that $\sum_{n \in A_+} a_n$ and $\sum_{n \in A_-} a_n$ both fail to be absolutely convergent. In particular, A_+ and A_- are infinite. (If A_- is finite, then $\sum_{n \in A_-} a_n$ is absolutely convergent. If A_+ is finite, then $\sum_{n \in A_+} a_n$ is also absolutely convergent.) By Prop. I.8.1.5 we can then find increasing bijections $f_+ : \mathbb{N} \to A_+$ and $f_- : \mathbb{N} \to A_-$. Thus, the sums $\sum_{m=0}^{\infty} a_{f_+(m)}$ and $\sum_{m=0}^{\infty} a_{f_-(m)}$ both fail to be absolutely convergent (first by Def. I.8.2.1 then by Lem. I.8.2.7). The plan shall be to select terms from the divergent series $\sum_{m=0}^{\infty} a_{f_+(m)}$ and $\sum_{m=0}^{\infty} a_{f_-(m)}$ in a well-chosen order in order to keep their difference converging towards L.

We define the sequence $n_0, n_1, n_2, \dots$ of natural numbers recursively as follows. Suppose that j is a natural number, and that n_i has already been defined for all $i < j$ (this is vacuously true if $j = 0$). We then define n_j by the following rule:

(I) If $\sum_{0 \le i < j} a_{n_i} < L$, then we set

$$n_j := \min\{n \in A_+ : n \neq n_i \text{ for all } i < j\}.$$

(II) If instead $\sum_{0 \le i < j} a_{n_i} \ge L$, then we set

$$n_j := \min\{n \in A_- : n \neq n_i \text{ for all } i < j\}.$$

Note that this recursive definition is well-defined because A_+ and A_- are infinite, and so the sets $\{n \in A_+ : n \neq n_i \text{ for all } i < j\}$ and $\{n \in A_- : n \neq n_i \text{ for all } i < j\}$ are never empty. (Intuitively, we add a non-negative number to the series whenever the partial sum is too low, and add a negative number when the sum is too high.) One can then verify the following claims:

- The map $j \mapsto n_j$ is injective. This is true since

$$\begin{aligned} &\forall j_1, j_2 \in \mathbb{N}, j_1 \neq j_2 \\ \implies &j_1 < j_2 \vee j_1 > j_2 \\ \implies &n_{j_1} \neq n_{j_2}. \end{aligned}$$

- Case I occurs an infinite number of times, and Case II also occurs an infinite number of times. We prove this by contradiction. Suppose for the sake of contradiction that case I occurs only finite number of times. Then we have

$$\left(\sum_{0\le i<j} a_{n_i} < L\right) \wedge \left(\sum_{0\le i<j} a_{n_i} + a_{n_j} + \sum_{i>j} a_{n_i} \ge L\right)$$

where j is the last time case I occurs. Since $\sum_{0\le i<j} a_{n_i} + a_{n_j}$ is finite, $\sum_{i>j} a_{n_i}$ have a lower bound. Since all $i > j$ are cases II, $\sum_{i>j} a_{n_i}$ is decreasing. Since $\sum_{i>j} a_{n_i}$ is decreasing and has lower bound, by A.Cor. I.6.3.1 $\sum_{i>j} a_{n_i}$ is convergent. But this means $\sum_{n\in A_-} a_n$ is absolutely convergent, a contradiction. Thus, case I occurs infinite number of times. Similar proof show that case II also occurs infinite number of times.

- The map $j \mapsto n_j$ is surjective. We know that $\forall n \in \mathbb{N}$, either $n \in A_+$ or $n \in A_-$. If $n \in A_+$ and there is no $j \mapsto n$, then $\forall n' > n$ there must also have no $j \mapsto n'$, otherwise by definition we must have $n = \min\{n \in A_+ : n \ne n_i \text{ for all } i < j\}$. But then case I only occur finite number of times, a contradiction. Thus, $\exists j \mapsto n$. Similar arguments show that if $n \in A_-$, then $\exists j \mapsto n$. Thus, $j \mapsto n_j$ is surjective.

- We have $\lim_{j\to\infty} a_{n_j} = 0$. By Cor. I.7.2.6 we have $\lim_{j\to\infty} a_j = 0$. This means

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall j \ge N, |a_j - 0| \le \varepsilon.$$

Thus, the set $E = \{j \in \mathbb{N} : a_j - 0 > \varepsilon\}$ is finite. Since $j \to n_j$ is bijective, we know that the set $E' = \{n_j \in \mathbb{N} : j \in E\}$ is also finite. Let $M = \max(E')$. Then we have

$$\forall n_j \ge M, \left|a_{n_j} - 0\right| \le \varepsilon.$$

Thus, $\lim_{j\to\infty} a_{n_j} = 0$.

- We have $\lim_{j\to\infty} \sum_{0\le i\le j} a_{n_i} = L$. Since $\lim_{j\to\infty} a_{n_j} = 0$, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall j \ge N, \left|a_{n_j} - 0\right| \le \varepsilon.$$

Let K be the set

$$K = \left\{k \in \mathbb{N} : (k \ge j) \wedge \left(\sum_{i=0}^{k} a_{n_i} < L\right) \wedge \left(\sum_{i=0}^{k+1} a_{n_i} \ge L\right)\right\}.$$

We know that $K \ne \emptyset$ since case II occurs infinite number of times. Let $k = \min(K)$. Such k is well-defined by well ordering principle (Prop. I.8.1.4). Now we show that for every $p \in \mathbb{N}$, we have

$$L - \varepsilon \le \sum_{i=0}^{k+p} a_{n_i} \le L + \varepsilon.$$

we induct on p. For $p = 0$, we have

$$\begin{aligned}
&\sum_{i=0}^{k+p} a_{n_i} = \sum_{i=0}^{k} a_{n_i} \\
\implies &\sum_{i=0}^{k+1} a_{n_i} \geq L && \text{(by the definition of } k\text{)} \\
\implies &\sum_{i=0}^{k} a_{n_i} + a_{n_{k+1}} \geq L \\
\implies &\sum_{i=0}^{k} a_{n_i} \geq L - a_{n_{k+1}} \geq L - \varepsilon \qquad (\left|a_{n_{k+1}}\right| \leq \varepsilon) \\
\implies &L + \varepsilon \geq L > \sum_{i=0}^{k} a_{n_i} \geq L - \varepsilon. && \text{(by the definition of } k\text{)}
\end{aligned}$$

Thus, the base case holds. Suppose inductively that for some $p \geq 0$ the statement is true. Then we need to show that for $p + 1$ the statement is also true. We split into two cases:

- If $a_{n_{p+1}} \geq 0$, then this means case I happened. Thus, we have

$$\begin{aligned}
&(0 \leq a_{n_{p+1}} \leq \varepsilon) \wedge \left(\sum_{i=0}^{k+p} a_{n_i} < L\right) && \text{(case I)} \\
\implies &(0 \leq a_{n_{p+1}} \leq \varepsilon) \wedge \left(L - \varepsilon \leq \sum_{i=0}^{k+p} a_{n_i} < L\right) && \text{(by the induction hypothesis)} \\
\implies &L - \varepsilon \leq \sum_{i=0}^{k+p} a_{n_i} + a_{n_{p+1}} \leq L + \varepsilon \\
\implies &L - \varepsilon \leq \sum_{i=0}^{k+p+1} a_{n_i} \leq L + \varepsilon.
\end{aligned}$$

- If $a_{n_{p+1}} < 0$, then this means case II happened. Thus, we have

$$\begin{aligned}
&(-\varepsilon \leq a_{n_{p+1}} < 0) \wedge \left(L \leq \sum_{i=0}^{k+p} a_{n_i}\right) && \text{(case II)} \\
\implies &(-\varepsilon \leq a_{n_{p+1}} < 0) \\
&\wedge \left(L \leq \sum_{i=0}^{k+p} a_{n_i} \leq L + \varepsilon\right) && \text{(by the induction hypothesis)}
\end{aligned}$$

$$\implies L - \varepsilon \leq \sum_{i=0}^{k+p} a_{n_i} + a_{n_{p+1}} \leq L + \varepsilon$$

$$\implies L - \varepsilon \leq \sum_{i=0}^{k+p+1} a_{n_i} \leq L + \varepsilon.$$

From all cases above, we conclude that $L - \varepsilon \leq \sum_{i=0}^{k+p+1} a_{n_i} \leq L + \varepsilon$. This closes the induction. This means $\forall p \geq 0$, we have

$$L - \varepsilon \leq \sum_{i=0}^{k+p} \leq L + \varepsilon \iff \left| \sum_{i=0}^{k+p} - L \right| \leq \varepsilon.$$

Rewritting with $q = k + p$, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists k \geq 0 : \forall q \geq k, \left| \sum_{i=0}^{q} - L \right| \leq \varepsilon.$$

Thus, $\lim_{q \to \infty} \sum_{i=0}^{q} a_{n_i} = L$.

The claim then follows by setting $f(i) \coloneqq n_i$ for all $i \in \mathbb{N}$. □

— Exercises —

Ex. I.8.2.1. Prove Lem. I.8.2.3.

Proof. See Lem. I.8.2.3. □

Ex. I.8.2.2. Prove Lem. I.8.2.5.

Proof. See Lem. I.8.2.5. □

Ex. I.8.2.3. Prove Prop. I.8.2.6.

Proof. See Prop. I.8.2.6. □

Ex. I.8.2.4. Prove Lem. I.8.2.7.

Proof. See Lem. I.8.2.7. □

Ex. I.8.2.5. Explain the gaps marked (why?) in the proof of Thm. I.8.2.8.

Proof. See Thm. I.8.2.8. □

Ex. I.8.2.6. Let $\sum_{n=0}^{\infty} a_n$ be a series which is conditionally convergent, but not absolutely convergent. Show that there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $\sum_{m=0}^{\infty} a_{f(m)}$ diverges to $+\infty$, or more precisely that

$$\liminf_{N\to\infty} \sum_{m=0}^{N} a_{f(m)} = \limsup_{N\to\infty} \sum_{m=0}^{N} a_{f(m)} = +\infty.$$

(Of course, a similar statement holds with $+\infty$ replaced by $-\infty$.)

Proof. Let A_+ and A_- defined as Lem. I.8.2.7. In Thm. I.8.2.8 we know that both A_+ and A_- are countable, and there exist two increasing bijections $f_+ : \mathbb{N} \to A_+$ and $f_- : \mathbb{N} \to A_-$. We know that both $\sum_{m=0}^{\infty} a_{f_+(m)}$ and $\sum_{m=0}^{\infty} a_{f_-(m)}$ fail to be absolutely convergent.

We first show that there exists a bijection $f : \mathbb{N} \to \mathbb{N}$ such that $\sum_{m=0}^{\infty} a_{f(m)}$ diverges to $+\infty$. Let $L_0 = 0$. Suppose that $j \in \mathbb{N}$, and n_i has been defined for all $i < j$ (this is vacuously true if $j = 0$). We define n_j by the following rule:

(I) If $\sum_{0\le i<j} a_{n_i} < L_j$, then we set

$$\begin{aligned} n_j &= \min\{n \in A_+ : n \neq n_i \text{ for all } i < j\}; \\ L_{j+1} &= L_j. \end{aligned}$$

(II) If $\sum_{0\le i<j} a_{n_i} \ge L_j$, then we set

$$\begin{aligned} n_j &= \min\{n \in A_- : n \neq n_i \text{ for all } i < j\}; \\ L_{j+1} &= L_j + 1. \end{aligned}$$

Now we verify the following claims:

- The map $j \mapsto n_j$ is injective. Suppose that $i, j \in \mathbb{N}$ and $i \neq j$. Then by the definition of n_i, n_j we know that $n_i \neq n_j$. Thus, $j \mapsto n_j$ is injective.

- Both Case I and II occur infinite number of times. Obviously, at least one case must occur infinite number of times. Suppose for the sake of contradiction that Case I only occurs finite number of times. Let j be the largest number such that Case I occurs, i.e.,

$$\left(\sum_{0\le i<j} a_{n_i} < L_j\right) \wedge \left(\sum_{0\le i\le j} a_{n_i} \ge L_j\right).$$

Then $\forall k \in \mathbb{N}$ and $k > j$, Case II occurs, i.e.,

$$S_k = \sum_{i=0}^{k} a_{n_i} \geq L_k.$$

Since Case II occurs, we know that S_k is decreasing and L_k is increasing. Thus

$$\begin{aligned}
& S_k > S_{k+1} \geq L_{k+1} > L_k \geq 0 \\
\implies & (\lim_{k\to\infty} S_k \text{ converges}) && \text{(by A.Cor. I.6.3.1)} \\
& \wedge (S_k - L_k \geq S_{k+1} - L_{k+1} \geq 0) \\
\implies & \lim_{k\to\infty} S_k - L_k \text{ converges} && \text{(by A.Cor. I.6.3.1)} \\
\implies & \lim_{k\to\infty} L_k \text{ converges.} && \text{(by Thm. I.6.1.19(a))}
\end{aligned}$$

But since Case II occurs infinite number of times, we know that $\lim_{k\to\infty} L_k$ diverges to ∞, a contradiction. Thus, case I must occurs infinite number of times.

Now Suppose for the sake of contradiction that Case II only occurs finite number of times. Let j be the largest number such that Case II occurs, i.e.,

$$\left(\sum_{0\leq i<j} a_{n_i} \geq L_j\right) \wedge \left(\sum_{0\leq i\leq j} a_{n_i} < L_j\right).$$

Then $\forall k \in \mathbb{N}$ and $k > j$, Case I occurs, i.e.,

$$S_k = \sum_{i=0}^{k} a_{n_i} < L_k.$$

Since Case I occurs, we know that S_k is increasing. Thus

$$\begin{aligned}
& S_k < S_{k+1} < L_{k+1} = L_k \\
\implies & \lim_{k\to\infty} S_k \text{ converges} && \text{(by Prop. I.6.3.8)} \\
\implies & \sum_{k=j+1}^{\infty} a_{n_k} \text{ converges} && \text{(by Def. I.7.2.2)} \\
\implies & \sum_{k=j+1}^{\infty} |a_{n_k}| \text{ converges} && (\forall k > j, a_{n_k} \geq 0) \\
\implies & \sum_{k\in A_+} |a_k| \text{ converges} && \text{(by Prop. I.8.2.6(c))}
\end{aligned}$$

But we know that $\sum_{k\in A_+} |a_k|$ is not absolutely convergent, a contradiction. Thus, case II must occurs infinite number of times. We conclude that both Case I and II occur infinite number of times.

- The map $j \mapsto n_j$ is surjective. We know that $\forall n \in \mathbb{N}$, either $n \in A_+$ or $n \in A_-$. If $n \in A_+$ and there is no $j \mapsto n$, then $\forall n' > n$ there must also have no $j \mapsto n'$, otherwise by definition we must have $n = \min\{n \in A_+ : n \neq n_i \text{ for all } i < j\}$. But then case I only occur finite number of times, a contradiction. Thus, $\exists j \mapsto n$. Similar arguments show that if $n \in A_-$, then $\exists j \mapsto n$. Thus, $j \mapsto n_j$ is surjective.

- We have $\limsup_{j\to\infty} \sum_{i=0}^{j} a_{n_j}$ diverges to ∞. Since Case II occurs infinite number of times, we know that for every $M \in \mathbb{R}^+$, $\exists j \geq 0$ such that $M \leq L_i$. This means

$$\exists k \in \mathbb{N} \wedge k > j : M \leq L_i \leq \sum_{i=0}^{k} a_{n_i}.$$

Since this is true for every $M \in \mathbb{R}^+$, by Ex. I.6.4.8 we thus have

$$\limsup_{j\to\infty} \sum_{i=0}^{j} a_{n_j} = \infty.$$

The claim then follows by setting $f(i) \coloneqq n_i$ for all $i \in \mathbb{N}$. Similar proof can be used to show the case $-\infty$. $\square$

I.8.3 Uncountable sets

Note. It was great shock when Georg Cantor (1845–1918) showed in 1873 that certain sets - including the real numbers $\mathbb{R}$ are in fact uncountable - no matter how hard you try, you cannot arrange the real numbers $\mathbb{R}$ as a sequence $a_0, a_1, a_2, \ldots$. (Of course, the real numbers $\mathbb{R}$ can contain many infinite sequences, e.g., the sequence $0, 1, 2, 3, 4, \ldots$. However, what Cantor proved is that no such sequence can ever exhaust the real numbers; no matter what sequence of real numbers you choose, there will always be some real numbers that are not covered by that sequence.)

Thm. I.8.3.1 (Cantor's theorem)**.** Let X be an arbitrary set (finite or infinite). Then the sets X and 2^X cannot have equal cardinality.

Proof. Suppose for the sake of contradiction that the sets X and 2^X had equal cardinality. Then there exists a bijection $f : X \to 2^X$ between X and the power set of X. Now consider the set

$$A \coloneqq \{x \in X : x \notin f(x)\}.$$

Note that this set is well-defined since $f(x)$ is an element of 2^X and is hence a subset of X. Clearly, A is a subset of X, hence is an element of 2^X. Since f is a bijection, there must therefore exist $x \in X$ such that $f(x) = A$. There are now two cases, depending on whether $x \in A$ or $x \notin A$. If $x \in A$, then by definition of A we have $x \notin f(x)$, hence $x \notin A$, a contradiction. But if $x \notin A$, then $x \notin f(x)$, hence by definition of A we have $x \in A$, a contradiction. Thus, in either case we have a contradiction. □

Rmk. I.8.3.2. The reader should compare the proof of Cantor's theorem with the statement of Russell's paradox (Sec. I.3.2). The point is that a bijection between X and 2^X would come dangerously close to the concept of a set X "containing itself."

Cor. I.8.3.3. $2^{\mathbb{N}}$ is uncountable.

Proof. By Thm. I.8.3.1, $2^{\mathbb{N}}$ cannot have equal cardinality with $\mathbb{N}$, hence is either uncountable or finite. However, $2^{\mathbb{N}}$ contains as a subset the set of singletons $\{\{n\} : n \in \mathbb{N}\}$, which is clearly bijective to $\mathbb{N}$ and hence countably infinite. Thus, $2^{\mathbb{N}}$ cannot be finite (by Prop. I.3.6.14), and is hence uncountable. □

Cor. I.8.3.4. $\mathbb{R}$ is uncountable.

Proof. Let us define the map $f : 2^{\mathbb{N}} \to \mathbb{R}$ by the formula

$$f(A) := \sum_{n\in A} 10^{-n}.$$

Observe that since $\sum_{n=0}^{\infty} 10^{-n}$ is an absolutely convergent series (by Lem. I.7.3.3), the series $\sum_{n\in A} 10^{-n}$ is also absolutely convergent (by Prop. I.8.2.6(c)). Thus, the map f is well defined. We now claim that f is injective. Suppose for the sake of contradiction that there were two distinct sets $A, B \in 2^{\mathbb{N}}$ such that $f(A) = f(B)$. Since $A \neq B$, the set $(A \setminus B) \cup (B \setminus A)$ is a non-empty subset of $\mathbb{N}$. By the well-ordering principle (Prop. I.8.1.4), we can then define the minimum of this set, say $n_0 := \min(A \setminus B) \cup (B \setminus A)$. Thus, n_0 either lies in $A \setminus B$ or $B \setminus A$. By symmetry we may assume it lies in $A \setminus B$. Then $n_0 \in A$, $n_0 \notin B$, and for all $n < n_0$ we either have $n \in A, B$ or $n \notin A, B$. Thus

$$\begin{aligned}
0 &= f(A) - f(B) \\
&= \sum_{n\in A} 10^{-n} - \sum_{n\in B} 10^{-n} \\
&= \left(\sum_{n<n_0 : n\in A} 10^{-n} + 10^{-n_0} + \sum_{n>n_0 : n\in A} 10^{-n}\right) \\
&\quad - \left(\sum_{n<n_0 : n\in B} 10^{-n} + \sum_{n>n_0 : n\in B} 10^{-n}\right) \\
&= 10^{-n_0} + \sum_{n>n_0 : n\in A} 10^{-n} - \sum_{n>n_0 : n\in B} 10^{-n}
\end{aligned}$$

$$\geq 10^{-n_0} + 0 - \sum_{n>n_0} 10^{-n}$$
$$\geq 10^{-n_0} - \frac{1}{9}10^{-n_0}$$
$$> 0,$$

a contradiction, where we have used the geometric series lemma (Lem. I.7.3.3) to sum

$$\sum_{n>n_0} 10^{-n} = \sum_{m=0}^{\infty} 10^{-(n_0+1+m)} = 10^{-n_0-1}\sum_{m=0}^{\infty} 10^{-m} = \frac{1}{9}10^{-n_0}.$$

Thus, f is injective, which means that $f(2^{\mathbb{N}})$ has the same cardinality as $2^{\mathbb{N}}$ and is thus uncountable. Since $f(2^{\mathbb{N}})$ is a subset of $\mathbb{R}$, this forces $\mathbb{R}$ to be uncountable also (otherwise this would contradict Cor. I.8.1.7), and we are done. □

Rmk. I.8.3.6. Cor. I.8.3.4 shows that the reals have strictly larger cardinality than the natural numbers (in the sense of Ex. I.3.6.7). One could ask whether there exist any sets which have strictly larger cardinality than the natural numbers, but strictly smaller cardinality than the reals. The *Continuum Hypothesis* asserts that no such sets exist. Interestingly, it was shown in separate works of Kurt Gödel (1906–1978) and Paul Cohen (1934–2007) that this hypothesis is independent of the other axioms of set theory; it can neither be proved nor disproved in that set of axioms (unless those axioms are inconsistent, which is highly unlikely).

— Exercises —

Ex. I.8.3.1. Let X be a finite set of cardinality n. Show that 2^X is a finite set of cardinality 2^n.

Proof. We induct on n. For $n = 0$, $X = \emptyset$ and $2^\emptyset = \{\emptyset\}$. Clearly, $\#(2^\emptyset) = 1$. So the base case holds. Suppose inductively that $\#(2^X) = 2^n$ for some $n \geq 0$. We need to show that for $n+1$, $\#(2^X) = 2^{n+1}$. Since $\#(X) = n+1$, we have $X \neq \emptyset$. Let $x \in X$. Then we have

$$\begin{aligned}
\#(2^X) &= \#(2^{(X\setminus\{x\})\cup\{x\}}) \\
&= \#(2^{X\setminus\{x\}}) \times \#(2^{\{x\}}) && \text{(by Ex. I.3.6.6)} \\
&= 2^n \times \#(2^{\{x\}}) && \text{(by the induction hypothesis)} \\
&= 2^n \times \#(\{\emptyset, \{x\}\}) \\
&= 2^n \times 2 \\
&= 2^{n+1}.
\end{aligned}$$

This closes the induction. □

Ex. I.8.3.2. Let A, B, C be sets such that $A \subseteq B \subseteq C$, and suppose that there is a injection $f : C \to A$. Define the sets $D_0, D_1, D_2, \ldots$ recursively by setting $D_0 := B \setminus A$, and then $D_{n+1} := f(D_n)$ for all natural numbers n. Prove that the sets $D_0, D_1, \ldots$ are all disjoint from each other (i.e., $D_n \cap D_m = \emptyset$ whenever $n \neq m$). Also show that if $g : A \to B$ is the function defined by setting $g(x) := f^{-1}(x)$ when $x \in \bigcup_{n=1}^{\infty} D_n$, and $g(x) := x$ when $x \notin \bigcup_{n=1}^{\infty} D_n$, then g does indeed map A to B and is a bijection between the two. In particular, A and B have the same cardinality.

Proof. We first show that $\forall n, m \in \mathbb{N}, n \neq m \implies D_n \cap D_m = \emptyset$. Let $P(n)$ be the statement "$\forall k \in \mathbb{Z}^+ : D_n \cap D_{n+k} = \emptyset$." We induct on n to show that $P(n)$ is true. For $n = 0$, we have $D_0 = B \setminus A$ and $D_k = f(D_{k-1}) \subseteq A$. Thus, we have $D_0 \cap D_k = \emptyset$ and the base case holds. Suppose inductively that $P(n)$ is true for some $n \geq 0$. Then for $n + 1$, we need to show that $P(n + 1)$ is also true. Suppose for the sake of contradiction that $D_{n+1} \cap D_{n+k+1} \neq \emptyset$. Then we have

$$\begin{aligned}
&\exists x : (x \in D_{n+1} \cap D_{n+k+1}) \\
\implies &\exists x : \big(x \in f(D_n) \cap f(D_{n+k})\big) \\
\implies &\exists z, z' : \big(z \in D_n \wedge z' \in D_{n+k} \wedge f(z) = f(z')\big) \\
\implies &z = z'. && (f \text{ is injective})
\end{aligned}$$

But by the induction hypothesis we know that $D_n \cap D_{n+k} = \emptyset$, a contradiction. Thus, $D_{n+1} \cap D_{n+k+1} = \emptyset$. This closes the induction.

Now we use $P(n)$ to show that $\forall n, m \in \mathbb{N}, n \neq m \implies D_n \cap D_m = \emptyset$. Since $m \neq n$, we have

$$\begin{cases} n = m + k & \text{if } m < n, \\ m = n + k & \text{if } n < m, \end{cases}$$

where $k \in \mathbb{Z}^+$. Using $P(n)$ we can thus have $D_n \cap D_m = \emptyset$.

Finally we show that if $g : A \to B$ is a function where

$$\forall x \in A : g(x) = \begin{cases} f^{-1}(x) & \text{if } x \in \bigcup_{n=1}^{\infty} D_n, \\ x & \text{otherwise,} \end{cases}$$

then g is bijective. Since f is injective, f is thus bijective from $\bigcup_{n=0}^{\infty} D_n$ to $f(\bigcup_{n=0}^{\infty} D_n)$. By definition we have $f(\bigcup_{n=0}^{\infty} D_n) = \bigcup_{n=1}^{\infty} D_n$. Thus, g is well-defined.

We now show that g is bijective. We start by showing that g is injective. Let $x, x' \in A$ such that $g(x) = g(x')$. We split into four cases:

- If $x, x' \in \bigcup_{n=1}^{\infty} D_n$, then since f is bijective from $\bigcup_{n=0}^{\infty} D_n$ to $\bigcup_{n=1}^{\infty} D_n$, we have $f^{-1}(x) = f^{-1}(x') \implies x = x'$.

- If $x, x' \in A \setminus \bigcup_{n=1}^{\infty} D_n$, then we have $g(x) = x = x' = g(x')$.

- One of x, x' is in $A \setminus \bigcup_{n=1}^{\infty} D_n$ and the other is in $\bigcup_{n=1}^{\infty} D_n$. Without the loss of generality suppose that $x \in \bigcup_{n=1}^{\infty} D_n$ and $x' \in A \setminus \bigcup_{n=1}^{\infty} D_n$. Then we show that this case is impossible. Since $x \in \bigcup_{n=1}^{\infty} D_n$, we have

$$f^{-1}(x) \in \bigcup_{n=0}^{\infty} D_n = D_0 \cup (\bigcup_{n=1}^{\infty} D_n) = (B \setminus A) \cup (\bigcup_{n=1}^{\infty} D_n).$$

  But $g(x) = f^{-1}(x) = x' = g(x')$ implies $x' \in (B \setminus A) \cup (\bigcup_{n=1}^{\infty} D_n)$, a contradiction. Thus, this case is impossible.

From all possible cases above we conclude that $x = x'$. Thus, g is injective.

Next we show that g is surjective. Let $y \in B$. We split into two cases:

- If $y \in \bigcup_{n=0}^{\infty} D_n$, then $\exists x \in \bigcup_{n=1}^{\infty} D_n$ such that $f^{-1}(x) = y$. This is true since f is bijective from $\bigcup_{n=0}^{\infty} D_n$ to $\bigcup_{n=1}^{\infty} D_n$.

- If $y \in B \setminus (\bigcup_{n=0}^{\infty} D_n)$, then we know that $y \in A$ since

$$\begin{aligned}
B \setminus \bigcup_{n=0}^{\infty} D_n &= B \setminus \left((B \setminus A) \cup (\bigcup_{n=1}^{\infty} D_n) \right) \\
&= \Big(B \setminus (B \setminus A) \Big) \cap \left(B \setminus (\bigcup_{n=1}^{\infty} D_n) \right) && \text{(by Prop. I.3.1.28)} \\
&= (A \cap B) \cap \left(B \setminus (\bigcup_{n=1}^{\infty} D_n) \right)
\end{aligned}$$

$$\subseteq A.$$

Since $y \notin \bigcup_{n=0}^{\infty} D_n$, we have $g(y) = y$.

From all cases above, we can find a $x \in A$ such that $g(x) = y$. Thus, g is surjective. Since g is both injective and surjective, we know that g is bijective. $\square$

Ex. I.8.3.3. Recall from Ex. I.3.6.7 that a set A is said to have lesser or equal cardinality than a set B iff there is an injective map $f : A \to B$ from A to B. Using Ex. I.8.3.2, show that if A, B are sets such that A has lesser or equal cardinality to B and B has lesser or equal cardinality to A, then A and B have equal cardinality. (This is known as the *Schröder-Bernstein theorem*, after Ernst Schröder (1841–1902) and Felix Bernstein (1878–1956).)

Proof. Suppose that A, B are sets, A has lesser or equal cardinality to B and B has lesser or equal cardinality to A. Then by Ex. I.3.6.7 we know that $\exists f : A \to B, g : B \to A$ such that both f, g are injective. By Ex. I.3.3.2 we have $f \circ g : B \to B$ is injective. By Def. I.3.4.1 we know that $f \circ g : B \to f\big(g(B)\big)$ is bijective and

$$f\big(g(B)\big) \subseteq f(A) \subseteq B.$$

Define the sets D_n recursively by setting

$$D_0 = f(A) \setminus f\big(g(B)\big),$$
$$D_{n+1} = f\big(g(D_n)\big),$$

where $n \in \mathbb{N}$. By Ex. I.8.3.2 we know that $\forall i, j \in \mathbb{N}$, $i \neq j \implies D_i \cap D_j = \emptyset$. Now let $h : f\big(g(B)\big) \to f(A)$ be a function as follow:

$$\forall x \in f\big(g(B)\big), h(x) = \begin{cases} (g^{-1} \circ f^{-1})(x) & \text{if } x \in \bigcup_{n=1}^{\infty} D_n, \\ x & \text{if } x \notin \bigcup_{n=1}^{\infty} D_n. \end{cases}$$

By Ex. I.8.3.2 we know that h is bijective. Thus, by Def. I.3.6.1 we know $f(A)$ and $f\big(g(B)\big)$ have the same cardinality. Since $f \circ g$ is bijective, we know that $B, f\big(g(B)\big)$ have the same cardinality, thus by Prop. I.3.6.4 $f(A), B$ have the same cardinality. But since f is injective, we know that f is bijective from A to $f(A)$, which means $A, f(A)$ have the same cardinality. Thus, by Prop. I.3.6.4 A, B have the same cardinality. $\square$

Ex. I.8.3.4. Let us say that a set A has *strictly lesser cardinality* than a set B if A has lesser than or equal cardinality to B (in the sense of Ex. I.3.6.7) but A does not have equal cardinality to B. Show that for any set X, that X has strictly lesser cardinality than 2^X. Also, show that if A has strictly lesser cardinality than B, and B has strictly lesser cardinality than C, then A has strictly lesser cardinality than C.

Proof. We first show that X has strictly lesser cardinality than 2^X. Suppose that X is a set. Since 2^X has a subset $S = \{\{x\} : x \in X\}$, we have a bijection $f : X \to S$ which maps $x \mapsto \{x\}$ for every $x \in X$. Now we define $g : X \to 2^X$ where $\forall x \in X : g(x) = f(x)$. By Thm. I.8.3.1 we know that g is not bijective. Since f is bijective, we know that g is injective. Thus, by definition X has strictly lesser cardinality than 2^X.

Now we show that if A has strictly lesser cardinality than B, and B has strictly lesser cardinality than C, then A has strictly lesser cardinality than C. By Ex. I.3.6.7 $\exists f : A \to B, g : B \to C$ such that both f, g are injective. By Ex. I.3.3.2 we know that $g \circ f : A \to C$ is injective. Thus, to show that A has strictly lesser cardinality than B, by definition it suffices to show that A does not have equal cardinality to C.

Suppose for the sake of contradiction that A, C have the same cardinality. Then by Def. I.3.6.1 $\exists h : C \to A$ such that h is bijective. Since g is injective, by Ex. I.3.3.2 we know that $h \circ g : B \to A$ is injective. But f is also injective, by Ex. I.8.3.3 we know that A, B have the same cardinality, a contradiction. Thus, A, C does not have the same cardinality, therefore A has strictly lesser cardinality than C. □

Ex. I.8.3.5. Show that no power set (i.e., a set of the form 2^X for some set X) can be countably infinite.

Proof. Suppose for the sake of contradiction that there exists a set X such that 2^X is countable. By Ex. I.8.3.4 X has strictly lesser cardinality than 2^X. Since 2^X is countable, by Def. I.8.1.1 we know that X has strictly lesser cardinality than $\mathbb{N}$ and X can not be countable. Since X has strictly lesser cardinality than $\mathbb{N}$, by Ex. I.3.6.7, $\exists f : X \to \mathbb{N}$ such that f is injective. Since f is injective, we know that f is bijective from X to $f(X)$. Since $f(X) \subseteq \mathbb{N}$, by Cor. I.8.1.6 $f(X)$ is at most countable. Since $X, f(X)$ have the same cardinality and X is not countable, X must be finite. But by Ex. I.8.3.1 we know that X is finite implies 2^X is finite, a contradiction. Thus, such X does not exists. □

I.8.4 The axiom of choice

Note. We now discuss the final axiom of the standard Zermelo-Fraenkel-Choice system of set theory, namely the *axiom of choice*. We have delayed introducing this axiom for a while now, to demonstrate that a large portion of the foundations of analysis can be constructed without appealing to this axiom. However, in many further developments of the theory, it is very convenient (and in some cases even essential) to employ this powerful axiom. On the other hand, the axiom of choice can lead to a number of unintuitive consequences (for instance the *Banach-Tarski paradox*), and can lead to proofs that are philosophically somewhat unsatisfying. Nevertheless, the axiom is almost universally accepted by mathematicians. One reason for this confidence is a theorem due to the great logician Kurt Gödel, who showed that a result proven using the axiom of choice will never contradict a result proven without the axiom of choice (unless all the other axioms of set theory are themselves inconsistent, which is highly unlikely). More precisely, Gödel demonstrated that the axiom of choice is *undecidable*; it can neither be proved nor disproved from the other axioms of set theory, so long as those axioms are themselves consistent. (From a set of inconsistent axioms one can prove that every statement is both true and false.) In practice, this means that any "real-life" application of analysis (more precisely, any application involving only "decidable" questions) which can be rigorously supported using the axiom of choice, can also be rigorously supported without the axiom of choice, though in many cases it would take a much more complicated and lengthier argument to do so if one were not allowed to use the axiom of choice. Thus, one can view the axiom of choice as a convenient and safe labour-saving device in analysis. In other disciplines of mathematics, notably in set theory in which many of the questions are not decidable, the issue of whether to accept the axiom of choice is more open to debate, and involves some philosophical concerns as well as mathematical and logical ones.

Def. I.8.4.1 (Infinite Cartesian products)**.** Let I be a set (possibly infinite), and for each $\alpha \in I$ let X_α be a set. We then define the Cartesian product $\prod_{\alpha\in I} X_\alpha$ to be the set

$$\prod_{\alpha\in I} X_\alpha = \left\{ (x_\alpha)_{\alpha\in I} \in (\bigcup_{\beta\in I} X_\beta)^I : x_\alpha \in X_\alpha \ \forall\alpha \in I \right\},$$

where we recall (from Ax. I.3.10) that $(\bigcup_{\alpha\in I} X_\alpha)^I$ is the set of all functions $(x_\alpha)_{\alpha\in I}$ which assign an element $x_\alpha \in \bigcup_{\beta\in I} X_\beta$ to each $\alpha \in I$. Thus, $\prod_{\alpha\in I} X_\alpha$ is a subset of that set of functions, consisting instead of those functions $(x_\alpha)_{\alpha\in I}$ which assign an element $x_\alpha \in X_\alpha$ to each $\alpha \in I$.

E.g. I.8.4.2. For any sets I and X, we have $\prod_{\alpha\in I} X = X^I$. If I is a set of the form $I :=$ $\{i \in \mathbb{N} : 1 \leq i \leq n\}$, then $\prod_{\alpha\in I} X_\alpha$ is the same set as the set $\prod_{1\leq i\leq N} X_i$ defined in Def. I.3.5.7.

Note. Recall from Lem. I.3.5.12 that if $X_1, \ldots, X_n$ were any finite collection of non-empty sets, then the finite Cartesian product $\prod_{1\leq i\leq n} X_i$ was also non-empty. The axiom of choice asserts that this statement is also true fo infinite Cartesian products.

Ax. I.8.1 (Choice)**.** Let I be a set, and for each $\alpha \in I$, let X_α be a non-empty set. Then $\prod_{\alpha\in I} X_\alpha$ is also non-empty. In other words, there exists a function $(x_\alpha)_{\alpha\in I}$ which assigns to each $\alpha \in I$ an element $x_\alpha \in X_\alpha$.

Rmk. I.8.4.3. The intuition behind this axiom is that given a (possibly infinite) collection of non-empty sets X_α, one should be able to choose a single element x_α from each one, and then form the possibly infinite tuple $(x_\alpha)_{\alpha\in I}$ from all the choices one has made. On one hand, this is a very intuitively appealing axiom; in some sense one is just applying Lem. I.3.1.6 over and over again. On the other hand, the fact that one is making an infinite number of arbitrary choices, with no explicit rule as to *how* to make these choices, is a little disconcerting. Indeed, there are many theorems proven using the axiom of choice which assert the abstract existence of some object x with certain properties, without saying at all *what* that object is, or how to construct it. Thus, the axiom of choice can lead to proofs which are *non-constructive* - demonstrating existence of an object without actually constructing the object explicitly. This problem is not unique to the axiom of choice - it already appears for instance in Lem. I.3.1.6 - but the objects shown to exist using the axiom of choice tend to be rather extreme in their level of non-constructiveness. However, as long as one is aware of the distinction between a non-constructive existence statement, and a constructive existence statement (with the latter being preferable, but not strictly necessary in many cases), there is no difficulty here, except perhaps on a philosophical level.

Rmk. I.8.4.4. There are many equivalent formulations of the axiom of choice.

Note. In analysis one often does not need the full power of the axiom of choice. Instead, one often only needs the *axiom of countable choice*, which is the same as the axiom of choice but with the index set I restricted to be at most countable.

Lem. I.8.4.5. Let E be a non-empty subset of the real line with $\sup(E) < \infty$ (i.e., E is bounded from above). Then there exists a sequence $(a_n)_{n=1}^{\infty}$ whose elements a_n all lie in E, such that $\lim_{n\to\infty} a_n = \sup(E)$.

Proof. For each positive natural number n, let X_n denote the set

$$X_n := \{x \in E : \sup(E) - 1/n \leq x \leq \sup(E)\}.$$

Since $\sup(E)$ is the least upper bound for E, then $\sup(E) - 1/n$ cannot be an upper bound for E, and so X_n is non-empty for each n. Using the axiom of choice (Ax. I.8.1, or the axiom of countable choice), we can then find a sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \in X_n$ for all $n \geq 1$. In particular, $a_n \in E$ for all n, and $\sup(E) - 1/n \leq a_n \leq \sup(E)$ for all n. But then we have $\lim_{n\to\infty} a_n = \sup(E)$ by the squeeze test (Cor. I.6.4.14). □

Rmk. I.8.4.6. In many special cases, one can obtain the conclusion of Lem. I.8.4.5 without using the axiom of choice. For instance, if E is a closed set then one can define a_n without choice by the formula $a_n := \inf(X_n)$; the extra hypothesis that E is closed will ensure that a_n lies in E.

Prop. I.8.4.7. Let X and Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$ such that for every $x \in X$ there is at least one $y \in Y$ such that $P(x, y)$ is true. Then there exists a function $f : X \to Y$ such that $P(x, f(x))$ is true for all $x \in X$.

Proof. We first show that axiom of choice (Ax. I.8.1) implies $\exists f : X \to Y$ such that $P(x, f(x))$ is true for all $x \in X$. Define

$$Y_x := \{y \in Y : P(x, y) \text{ is true}\}$$

for each $x \in X$. Such a set exist by Ax. I.3.5 and is non-empty by the hypothesis. By Ax. I.3.10, we know that the set

$$\prod_{x\in X} Y_x = \left\{(y_x)_{x\in X} \in (\bigcup_{x\in X} Y_x)^X : y_x \in Y_x \text{ for all } x \in X\right\}$$

exists. By axiom of choice (Ax. I.8.1) we also know that the set $\prod\limits_{x\in X} Y_x$ is non-empty. Now we can choose an element $f \in \prod\limits_{x\in X} Y_x$. We know that f is a function with domain X and codomain Y. Also, $\forall x \in X$, we have an unique $f(x) \in Y_x$. By the definition of Y_x we know that $P(x, f(x))$ is true. Thus, axiom of choice (Ax. I.8.1) implies $\exists f : X \to Y$ such that $P(x, f(x))$ is true for all $x \in X$.

Now we show that if $\forall x \in X$, $\exists y \in Y$ such that $P(x, y)$ is true and $\exists f : X \to Y$ such that $P(x, f(x))$ is true for all $x \in X$, then axiom of choice (Ax. I.8.1) is true. Using the definition of Y_x again we know that $f(x) \in Y_x$, so $Y_x \neq \emptyset$. By Ax. I.3.10 we can have a set $\prod\limits_{x\in X} Y_x$. We want to show that $\prod\limits_{x\in X} Y_x \neq \emptyset$. But this is true since $f \in \prod\limits_{x\in X} Y_x$. Thus, Prop. I.8.4.7 implies axiom of choice (Ax. I.8.1). □

— Exercises —

Ex. I.8.4.1. Show that the axiom of choice implies Prop. I.8.4.7. Conversely, show that if Prop. I.8.4.7 is true, then the axiom of choice is also true.

Proof. See Prop. I.8.4.7. □

Ex. I.8.4.2. Let I be a set, and for each $\alpha \in I$ let X_α be a non-empty set. Suppose that all the sets X_α are disjoint from each other, i.e., $X_\alpha \cap X_\beta = \emptyset$ for all distinct $\alpha, \beta \in I$. Using the axiom of choice, show that there exists a set Y such that $\#(Y \cap X_\alpha) = 1$ for all $\alpha \in I$ (i.e., Y intersects each X_α in exactly one element). Conversely, show that if the above statement was true for an arbitrary choice of sets I and non-empty disjoint sets X_α, then the axiom of choice is true.

Proof. We first show that axiom of choice (Ax. I.8.1) implies there exists a set Y such that $\#(Y \cap X_\alpha) = 1$ for all $\alpha \in I$. By Ax. I.8.1, the set $\prod\limits_{\alpha\in I} X_\alpha$ is non-empty. Let $f \in \prod\limits_{\alpha\in I} X_\alpha$ and let $Y = f(I)$. Then we have

$$\forall \alpha \in I, Y \cap X_\alpha$$

$$
\begin{aligned}
&= f(I) \cap X_\alpha \\
&= \left(\bigcup_{\beta \in I} \{f(\beta)\} \right) \cap X_\alpha \\
&= \left(\{f(\alpha)\} \cup \left(\bigcup_{\beta \in I : \beta \neq \alpha} \{f(\beta)\} \right) \right) \cap X_\alpha \\
&= \{f(\alpha)\} \cap X_\alpha && \text{(by hypothesis)} \\
&= \{f(\alpha)\}.
\end{aligned}
$$

Thus, $\#(Y \cap X_\alpha) = 1$ for every $\alpha \in I$. We conclude that axiom of choice (Ax. I.8.1) implies there exists a set Y such that $\#(Y \cap X_\alpha) = 1$ for all $\alpha \in I$.

Now we show that if there exists a set Y such that $\#(Y \cap X_\alpha) = 1$ for all $\alpha \in I$, then axiom of choice (Ax. I.8.1) is true. Since we know that $\exists Y : \#(Y \cap X_\alpha) = 1$ for all $\alpha \in I$, we can let $Y \cap X_\alpha = \{x_\alpha\}$ for some $x_\alpha \in X_\alpha$. Thus, by Ax. I.3.6 we have a function $f : I \to \bigcup_{\alpha \in I} X_\alpha$ such that $f(\alpha) = x_\alpha \in X_\alpha$ for all $\alpha \in I$. But this means $f \in \prod_{\alpha \in I} X_\alpha$, so $\prod_{\alpha \in I} X_\alpha \neq \emptyset$, and axiom of choice (Ax. I.8.1) is true. Thus, if there exists a set Y such that $\#(Y \cap X_\alpha) = 1$ for all $\alpha \in I$, then axiom of choice (Ax. I.8.1) is true. $\square$

Ex. I.8.4.3. Let A and B be sets such that there exists a surjection $g : B \to A$. Using the axiom of choice, show that there exists an injection $f : A \to B$ with $g \circ f : A \to A$ the identity map; in particular, A has lesser or equal cardinality to B in the sense of Ex. I.3.6.7. Compare this with Ex. I.3.6.8. Conversely, show that if the above statement is true for arbitrary sets A, B and surjections $g : B \to A$, then the axiom of choice is true.

Proof. We first show that if $g : B \to A$ is a surjection, then there exists an injection $f : A \to B$. Let $g^{-1}(A)$ be the inverse image of A. Since g is surjective, we know that $g^{-1}(A) = \bigcup_{a \in A} g^{-1}(\{a\}) = B$. By axiom of choice (Ax. I.8.1), we know that the set $\prod_{a \in A} g^{-1}(\{a\}) \neq \emptyset$. Let $f \in \prod_{a \in A} g^{-1}(\{a\})$. We know that f has domain A and codomain $\bigcup_{a \in A} g^{-1}(\{a\}) = B$. We now show that f is injective. $\forall a_1, a_2 \in A$, if $a_1 \neq a_2$, then we must have $g^{-1}(\{a_1\}) \cap g^{-1}(\{a_2\}) = \emptyset$. Otherwise by Def. I.3.4.4 $\exists b \in B$ such that $g(b) = a_1 \wedge g(b) = a_2$, a contradiction. By the definition of f, we know that $f(a_1) \in g^{-1}(\{a_1\}) \wedge f(a_2) \in g^{-1}(\{a_2\})$. Since $g^{-1}(\{a_1\}) \cap g^{-1}(\{a_2\}) = \emptyset$, we know that $f(a_1) \neq f(a_2)$, thus f is injective.

Next we show that $g \circ f : A \to A$ is an identity map.

$$
\begin{aligned}
&\forall a \in A, f(a) \in g^{-1}(\{a\}) \\
\implies &g(f(a)) = a. && \text{(by Def. I.3.4.4)}
\end{aligned}
$$

Thus, $g \circ f : A \to A$ is an identity map.

Now we show that if $g : B \to A$ is a surjection and there exists an injection $f : A \to B$ where $g \circ f : A \to A$ is an identity map, then the axiom of choice is true. Let $A_a = \{a \in A : g(f(a)) = a\}$. Since $g \circ f$ is an identity map, $\forall a_1, a_2 \in A$, we have $a_1 \neq a_2 \implies g(f(a_1)) \neq g(f(a_2))$. Thus, $\#(A_a) = 1$ and $\forall a_1, a_2 \in A$, $a_1 \neq a_2 \implies A_{a_1} \cap A_{a_2} = \emptyset$. Also we have $A \cap A_a = A_a$ for every $a \in A$, and thus $\#(A \cap A_a) = 1$. So by Ex. I.8.4.2 we know that axiom of choice (Ax. I.8.1) is true. $\square$

I.8.5 Ordered sets

Def. I.8.5.1 (Partially ordered sets)**.** A *partially ordered set* (or *poset*) is a set X, together with a relation $\leq_X$ on X (thus for any two objects $x, y \in X$, the statement $x \leq_X y$ is either a true statement or a false statement). Furthermore, this relation is assumed to obey the following three properties:

- (Reflexivity) For any $x \in X$, we have $x \leq_X x$.
- (Anti-symmetry) If $x, y \in X$ are such that $x \leq_X y$ and $y \leq_X x$, then $x = y$.
- (Transitivity) If $x, y, z \in X$ are such that $x \leq_X y$ and $y \leq_X z$, then $x \leq_X z$.

We refer to $\leq_X$ as the *ordering relation*. In most situations it is understood what the set X is from context, and in those cases we shall simply write $\leq$ instead of $\leq_X$. We write $x <_X y$ (or $x < y$ for short) if $x \leq_X y$ and $x \neq y$.

Note. Strictly speaking, a partially ordered set is not a set X, but rather a pair $(X, \leq_X)$. But in many cases the ordering $\leq_X$ will be clear from context, and so we shall refer to X itself as the partially ordered set even though this is technically incorrect.

E.g. I.8.5.2. The natural numbers $\mathbb{N}$ together with the usual less-than-or-equal-to relation $\leq$ (as defined in Def. I.2.2.11) forms a partially ordered set, by Prop. I.2.2.12. Similar arguments (using the appropriate definitions and propositions) show that the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, the reals $\mathbb{R}$, and the extended reals $\mathbb{R}^*$ are also partially ordered sets. Meanwhile, if X is any collection of sets, and one uses the relation of is-a-subset-of $\subseteq$ (as defined in Def. I.3.1.15) for the ordering relation $\leq_X$, then X is also partially ordered (Prop. I.3.1.18). Note that it is certainly possible to give these sets a different partial ordering than the standard one.

Def. I.8.5.3 (Totally ordered set)**.** Let X be a partially ordered set with some order relation $\leq_X$. A subset Y of X is said to be *totally ordered* if, given any two $y, y' \in Y$, we either have $y \leq_X y'$ or $y' \leq_X y$ (or both). If X itself is totally ordered, we say that X is a *totally ordered set* (or *chain*) with order relation $\leq_X$.

E.g. I.8.5.4. The natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, reals $\mathbb{R}$, and the extended reals $\mathbb{R}^*$, all with the usual ordering relation $\leq$, are totally ordered (by Prop. I.2.2.13, Lem. I.4.1.11, Prop. I.4.2.9, Prop. I.5.4.7, and Prop. I.6.2.5 respectively). Also, any subset of a totally ordered set is again totally ordered. On the other hand, a collection of sets with the $\subseteq$ relation is usually not totally ordered.

Def. I.8.5.5 (Maximal and minimal elements)**.** Let X be a partially ordered set, and let Y be a subset of X. We say that y is a *minimal element* of Y if $y \in Y$ and there is no element $y' \in Y$ such that $y' < y$. We say that y is a *maximal element* of Y if $y \in Y$ and there is no element $y' \in Y$ such that $y < y'$.

E.g. I.8.5.7. The natural numbers $\mathbb{N}$ (ordered by $\leq$) has a minimal element, namely 0, but no maximal element. The set of integers $\mathbb{Z}$ has no maximal and no minimal element.

Def. I.8.5.8 (Well-ordered sets)**.** Let X be a partially ordered set, and let Y be a totally ordered subset of X. We say that Y is *well-ordered* if every non-empty subset Z of Y has a minimal element $\min(Z)$.

E.g. I.8.5.9. The natural numbers $\mathbb{N}$ are well-ordered by Prop. I.8.1.4. However, the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, and the real numbers $\mathbb{R}$ are not (see Ex. I.8.1.2). Every subset of a well-ordered set is again well-ordered.

Prop. I.8.5.10 (Principle of strong induction)**.** Let X be a well-ordered set with an ordering relation $\leq_X$, and let $P(n)$ be a property pertaining to an element $n \in X$ (i.e., for each $n \in X$, $P(n)$ is either a true statement or a false statement). Suppose that for every $n \in X$, we have the following implication: if $P(m)$ is true for all $m \in X$ with $m <_X n$, then $P(n)$ is also true. Then $P(n)$ is true for all $n \in X$.

Proof. Since $(X, \leq_X)$ is well-ordered, by Def. I.8.5.8 we know that $X \neq \emptyset$ and $\min\big((X, \leq_X)\big)$ exists. By Def. I.8.5.5 we know that the statement "$\forall m \in X,\ m <_X \min\big((X, \leq_X)\big) \implies P(m)$ is true" is vacuously true since there is no $m <_X n$. Thus, by hypothesis we know that $P\Big(\min\big((X, \leq_X)\big)\Big)$ is vacuously true.

Now let Y be the set

$$Y = \{m \in X : P(m) \text{ is false}\}.$$

Suppose for the sake of contradiction that $Y \neq \emptyset$. Since X is well-ordered and $Y \subseteq X$, by Def. I.8.5.8 we know that $\min\big((Y, \leq_X)\big)$ exists. From previous claim we know that $\min\big((Y, \leq_X)\big) \neq \min\big((X, \leq_X)\big)$, thus the set

$$Y' = \big\{m \in X : m <_X \min\big((Y, \leq_X)\big)\big\}$$

is not empty. Also by Def. I.8.5.5 we know that $\forall m \in Y'$, $P(m)$ is true, otherwise contradict to the definition of $\min\big((Y, \leq_X)\big)$. But this means

$$\forall m \in X, m <_X \min\big((Y, \leq_X)\big) \implies P\Big(\min\big((Y, \leq_X)\big)\Big)$$

is false, which contradict to the hypothesis. Thus, we must have $P(n)$ is true for all $n \in X$. $\square$

Rmk. I.8.5.11. It may seem strange that there is no "base" case in strong induction, corresponding to the hypothesis $P(0)$ in Ax. I.2.5. However, such a base case is automatically included in the strong induction hypothesis. Indeed, if 0 is the minimal element of X, then by specializing the hypothesis "if $P(m)$ is true for all $m \in X$ with $m <_X n$, then $P(n)$ is also true" to the $n = 0$ case, we automatically obtain that $P(0)$ is true. (Since there is no element $m <_X n$, such statement "if $P(m)$ is true for all $m \in X$ with $m <_X n$" is false, thus the implication holds vacuously.)

Def. I.8.5.12 (Upper bounds and strict upper bounds)**.** Let X be a partially ordered set with ordering relation $\leq$, and let Y be a subset of X. If $x \in X$, we say that x is an *upper bound* for Y iff $y \leq x$ for all $y \in Y$. If in addition $x \notin Y$, we say that x is a *strict upper bound* for Y. Equivalently, x is a strict upper bound for Y iff $y < x$ for all $y \in Y$.

Lem. I.8.5.14. Let X be a partially ordered set with ordering relation $\leq$, and let x_0 be an element of X. Then there is a well-ordered subset Y of X which has x_0 as its minimal element, and which has no strict upper bound.

Proof. The intuition behind this lemma is that one is trying to perform the following algorithm: we initialize $Y := \{x_0\}$. If Y has no strict upper bound, then we are done; otherwise, we choose a strict upper bound and add it to Y. Then we look again to see if Y has a strict upper bound or not. If not, we are done; otherwise we choose another strict upper bound and add it to Y. We continue this algorithm "infinitely often" until we exhaust all the strict upper bounds; the axiom of choice comes in because infinitely many choices are involved. This is however not a rigorous proof because it is quite difficult to precisely pin down what it means to perform an algorithm "infinitely often." Instead, what we will do is that we will isolate a collection of "partially completed" sets Y, which we shall call *good sets*, and then take the union of all these good sets to obtain a "completed" object Y_∞ which will indeed have no strict upper bound.

We now begin the rigorous proof. Suppose for the sake of contradiction that every well-ordered subset Y of X which has x_0 as its minimal element has at least one strict upper bound. Using the axiom of choice (in the form of Prop. I.8.4.7), we can thus assign a strict upper bound $s(Y) \in X$ to each well-ordered subset Y of X which has x_0 as its minimal element.

Henceforth we fix a single such strict upper bound function s. Let us define a special class of subsets Y of X. We say that a subset Y of X is *good* iff it is well-ordered, contains x_0 as its minimal element, and obeys the property that

$$x = s(\{y \in Y : y < x\}) \text{ for all } x \in Y \setminus \{x_0\}.$$

Note that if $x \in Y \setminus \{x_0\}$ then the set $\{y \in Y : y < x\}$ is a subset of X which is well-ordered and contains x_0 as its minimal element. Let $\Omega := \{Y \subseteq X : Y \text{ is good}\}$ be the collection of all good subsets of X. This collection is not empty, since the subset $\{x_0\}$ of X is clearly good (which is vacuously true).

We make the following important observation: if Y and Y' are two good subsets of X, then every element of $Y' \setminus Y$ is a strict upper bound for Y, and every element of $Y \setminus Y'$ is a strict upper bound for Y'. (Ex. I.8.5.13). In particular, given any two good sets Y and Y', at least one of $Y' \setminus Y$ and $Y \setminus Y'$ must be empty (since they are both strict upper bounds of each other). In other words, Ω is totally ordered by set inclusion: given any two good sets Y and Y', either $Y \subseteq Y'$ or $Y' \subseteq Y$.

Let $Y_\infty = \bigcup \Omega$, i.e., Y_∞ is the set of all elements of X which belong to at least one good subset of X. Clearly, $x_0 \in Y_\infty$. Also, since each good subset of X has x_0 as its minimal element, the set Y_∞ also has x_0 as its minimal element.

Next, we show that Y_∞ is totally ordered. Let x, x' be two elements of Y_∞. By definition of Y_∞, we know that x lies in some good set Y and x' lies in some good set Y'. But since Ω is totally ordered, one of these good sets contains the other. Thus, x, x' are contained in a single good set (either Y or Y'); since good sets are totally ordered, we thus see that either $x \leq x'$ or $x' \leq x$ as desired.

Next, we show that Y_∞ is well-ordered. Let A be any non-empty subset of Y_∞. Then we can pick an element $a \in A$, which then lies in Y_∞. Therefore there is a good set Y such that $a \in Y$. Then $A \cap Y$ is a non-empty subset of Y; since Y is well-ordered, the set $A \cap Y$ thus has a minimal element, call it b. Now recall that for any other good set Y', every element of $Y' \setminus Y$ is a strict upper bound for Y, and in particular, is larger than b. Since b is a minimal element of $A \cap Y$, this implies that b is also a minimal element of $A \cap Y'$ for any good set Y' with $A \cap Y' \neq \emptyset$. This is true since

$$\begin{aligned}
&\forall y_1 \in Y, y_1 < \min(Y' \setminus Y) \\
\Longrightarrow\ & y_1 \in Y \cap Y' \\
\Longrightarrow\ & \forall y_2 \in A \cap Y, \big(y_2 < \min(Y' \setminus Y)\big) \wedge \big(y_2 \in Y \cap Y'\big) \\
\Longrightarrow\ & y_2 \in A \cap Y' \\
\Longrightarrow\ & b = \min(A \cap Y) = \min(A \cap Y').
\end{aligned}$$

Since every element of A belongs to Y_∞ and hence belongs to at least one good set Y', we thus see that b is a minimal element of A. Thus, Y_∞ is well-ordered as claimed.

Since Y_∞ is well-ordered with x_0 as its minimal element, it has a strict upper bound $s(Y_\infty)$. But then $Y_\infty \cup \{s(Y_\infty)\}$ is well-ordered (by Ex. I.8.5.11) and has x_0 as its minimal element. We now claim that $Y_\infty \cup \{s(Y_\infty)\}$ is good. By the preceding discussion, it suffices to show that $x = s\big(\{y \in Y_\infty \cup \{s(Y_\infty)\} : y < x\}\big)$ when $x \in \big(Y_\infty \cup \{s(Y_\infty)\}\big) \setminus \{x_0\}$. If $x = s(Y_\infty)$ this is clear since $\{y \in Y_\infty \cup \{s(Y_\infty)\} : y < x\} = Y_\infty$ in this case. If instead $x \in Y_\infty$, then $x \in Y$ for some good Y. Then the set $\{y \in Y_\infty \cup \{s(Y_\infty)\} : y < x\}$ is equal to $\{y \in Y : y < x\}$ (why? use the previous observation that every element of $Y' \setminus Y$ is an upper bound for x for every good Y'), and the claim then follows since Y is good. By definition of Y_∞, we conclude that the good set $Y_\infty \cup \{s(Y_\infty)\}$ is contained in Y_∞. But this is a contradiction since $s(Y_\infty)$ is a strict upper bound for Y_∞. Thus, we have constructed a set with no strict upper bound, as desired. $\square$

Lem. I.8.5.15 (Zorn's lemma)**.** Let X be a non-empty partially ordered set, with the property that every totally ordered subset Y of X has an upper bound. Then X contains at least one maximal element.

Proof. Let $(X, \leq)$ be partially ordered such that $X \neq \emptyset$. Suppose for the sake of contradiction that X has no maximal element. We show that any subset $Y \subseteq X$ which has an upper bound also has a strict upper bound. Let s be an upper bound of Y. By Def. I.8.5.12, we know that $s \in X$ and $\forall y \in Y \implies y \leq s$. Since X has no maximal element, we know $\exists s' \in X$ such that $s < s'$, otherwise by Def. I.8.5.5 we have $s = \max((X, \leq))$, a contradiction. Since $\forall y \in Y$, $y < s'$, we know that $s' \notin Y$, and thus by Def. I.8.5.12 s' is a strict upper bound of Y.

Since $X \neq \emptyset$, let $x_0 \in X$. By Lem. I.8.5.14, $\exists Y \subseteq X$ such that $(Y, \leq)$ is well-ordered, $\min\big((Y, \leq)\big) = x_0$ and Y has no strict upper bound. But by hypothesis we know that Y has an upper bound and thus has a strict upper bound, a contradiction. Thus, X must has at least one maximal element. $\square$

Note. Zorn's lemma is also called the *principle of transfinite induction.*

— Exercises —

Ex. I.8.5.1. Consider the empty set $\emptyset$ with the empty order relation $\leq_\emptyset$ (this relation is vacuous because the empty set has no elements). Is this set partially ordered? totally ordered? well-ordered? Explain.

Proof. Since

$$\forall x \in \emptyset, x \leq_\emptyset x$$

is vacuously true, we know that $(\emptyset, \leq_\emptyset)$ is reflexive. Since

$$\forall x, y \in \emptyset, (x \leq_\emptyset y) \wedge (y \leq_\emptyset x) \implies x = y$$

is vacuously true, we know that $(\emptyset, \leq_\emptyset)$ is anti-symmetric. Since

$$\forall x, y, z \in \emptyset, (x \leq_\emptyset y) \wedge (y \leq_\emptyset z) \implies x \leq_\emptyset z$$

is vacuously true, we know that $(\emptyset, \leq_\emptyset)$ is transitive. Since $(\emptyset, \leq_\emptyset)$ is reflexive, anti-symmetric and transitive, by Def. I.8.5.1 $(\emptyset, \leq_\emptyset)$ is partially ordered. Since

$$\forall x, y \in \emptyset, (x \leq_\emptyset y) \vee (y \leq_\emptyset x)$$

is vacuously true, by Def. I.8.5.3 we know that $(\emptyset, \leq_\emptyset)$ is totally ordered. Since

$$\forall X \subseteq \emptyset, X \neq \emptyset \implies \exists \min\big((X, \leq_\emptyset)\big) \in X$$

is vacuously true, by Def. I.8.5.8 we know that $(\emptyset, \leq_\emptyset)$ is well-ordered. $\square$

Ex. I.8.5.2. Give examples of a set X and a relation $\leq_X$ such that

(a) The relation $\leq_X$ is reflexive and anti-symmetric, but not transitive;

(b) The relation $\leq_X$ is reflexive and transitive, but not anti-symmetric;

(c) The relation $\leq_X$ is anti-symmetric and transitive, but not reflexive.

Proof. (a) Let $X = \{1, 2, 4\}$ be a set and let $\leq_X$ be the relation

$$\forall a, b \in X, a \leq_X b \iff (a = b) \vee (2a = b).$$

Since

$$\forall a \in X, a = a \implies a \leq_X a,$$

we know that $(X, \leq_X)$ is reflexive. Since

$$(1 \leq_X 1) \wedge (1 \leq_X 2) \wedge (2 \leq_X 2) \wedge (2 \leq_X 4) \wedge (4 \leq_X 4),$$

we know that when pairs of $a, b \in X$ satisfying $a \leq_X b \wedge b \leq_X a$ we must have $a = b$, thus $(X, \leq_X)$ is anti-symmetric. Since we have $(1 \leq_X 2) \wedge (2 \leq_X 4)$ but not $1 \leq_X 4$, we know that $(X, \leq_X)$ is not transitive.

(b) Let $X = \mathbb{Z}$ be a set and let $\leq_X$ be the relation

$$\forall a, b \in \mathbb{Z}, a \leq_X b \iff |a| \leq |b|.$$

Since

$$\forall a \in \mathbb{Z}, |a| \leq |a| \implies a \leq_X a,$$

we know that $(\mathbb{Z}, \leq_X)$ is reflexive. Since

$$(|1| \leq |-1|) \wedge (|-1| \leq |1|) \implies (1 \leq_X -1) \wedge (-1 \leq_X 1)$$

but $1 \neq -1$, we know that $(\mathbb{Z}, \leq_X)$ is not anti-symmetric. Since

$$\begin{aligned} &\forall a, b, c \in \mathbb{Z}, (a \leq_X b) \wedge (b \leq_X c) \\ \implies &(|a| \leq |b|) \wedge (|b| \leq |c|) \\ \implies &|a| \leq |c| \\ \implies &a \leq_X c, \end{aligned}$$

we know that $(\mathbb{Z}, \leq_X)$ is transitive.

(c) Let $X = \{0\}$ be a set and let $\leq_X$ be the relation

$$\forall a, b \in X, a \leq_X b \iff a \leq a + 1.$$

Since

$$\forall a \in X, a + 1 \not\leq a,$$

we know that $(X, \leq_X)$ is not reflexive. Since

$$\forall a, b \in X, (a \leq_X b) \wedge (b \leq_X a) \implies a = b$$

is vacuously true, we know that $(X, \leq_X)$ is anti-symmetric. Since

$$\forall a, b \in X, (a \leq_X b) \wedge (b \leq_X c) \implies a \leq_X b$$

is vacuously true, we know that $(X, \leq_X)$ is transitive.

□

Ex. I.8.5.3. Given two positive integers $n, m \in \mathbb{N} \setminus \{0\}$, we say that n *divides* m, and write $n|m$, if there exists a positive integer a such that $m = na$. Show that the set $\mathbb{N} \setminus \{0\}$ with the ordering relation $|$ is a partially ordered set but not a totally ordered one. Note that this is a different ordering relation from the usual $\leq$ ordering of $\mathbb{N} \setminus \{0\}$.

Proof. Since

$$\forall n \in \mathbb{N} \setminus \{0\}, n = 1n \implies n|n,$$

we know that $(X, |)$ is reflexive. Since

$$\begin{aligned}
&\forall n, m \in \mathbb{N} \setminus \{0\}, (n|m) \wedge (m|n) \\
\implies &\exists a, b \in \mathbb{N} \setminus \{0\}, (m = na) \wedge (n = bm) \\
\implies &n = abn \\
\implies &ab = 1 \\
\implies &(a = 1) \wedge (b = 1) \\
\implies &n = m,
\end{aligned}$$

we know that $(X, |)$ is anti-symmetric. Since

$$\begin{aligned}
&\forall n, m, p \in \mathbb{N} \setminus \{0\}, (n|m) \wedge (m|p) \\
\implies &\exists a, b \in \mathbb{N} \setminus \{0\}, (m = na) \wedge (p = bm) \\
\implies &(p = abn) \wedge (ab > 0) \\
\implies &n|p,
\end{aligned}$$

we know that $(X, |)$ is transitive. Since $(X, |)$ is reflexive, anti-symmetric and transitive, by Def. I.8.5.1 $(X, |)$ is partially ordered. Since $(2|3) \vee (3|2)$ is false, by Def. I.8.5.3 $(X, |)$ is not totally ordered. □

Ex. I.8.5.4. Show that the set of positive reals $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ have no minimal element.

Proof. Suppose for the sake of contradiction that $\exists x \in \mathbb{R}^+$ such that $x = \min(\mathbb{R}^+)$. Since $x > 0$, we know that $x/2 > 0$ and $x/2 \in \mathbb{R}^+$. But $x = \min(\mathbb{R}^+)$ implies we have $x < x/2$, a contradiction. Thus, $\nexists x \in \mathbb{R}^+$ such that $x = \min(\mathbb{R}^+)$. $\square$

Ex. I.8.5.5. Let $f : X \to Y$ be a function from one set X to another set Y. Suppose that Y is partially ordered with some ordering relation $\leq_Y$. Define a relation $\leq_X$ on X by defining $x \leq_X x'$ iff $f(x) <_Y f(x')$ or $x = x'$. Show that this relation $\leq_X$ turns X into a partially ordered set. If we know in addition that the relation $\leq_Y$ makes Y totally ordered, does this mean that the relation $\leq_X$ makes X totally ordered also? If not, what additional assumption needs to be made on f in order to ensure that $\leq_X$ makes X totally ordered?

Proof. We first show that $(X, \leq_X)$ is partially ordered. Since

$$\forall x \in X, x = x \implies x \leq_X x,$$

we know that $(X, \leq_X)$ is reflexive. Since

$$\begin{aligned}
&\forall x, x' \in X, (x \leq_X x') \wedge (x' \leq_X x) \\
\implies &\Big(\big(f(x) <_Y f(x')\big) \vee \big(x = x'\big)\Big) \wedge \Big(\big(f(x') <_Y f(x)\big) \vee \big(x = x'\big)\Big) \\
\implies &x = x',
\end{aligned}$$

we know that $(X, \leq_X)$ is anti-symmetric. Since

$$\begin{aligned}
&\forall x_1, x_2, x_3 \in X, (x_1 \leq_X x_2) \wedge (x_2 \leq_X x_3) \\
\implies &\Big(\big(f(x_1) <_Y f(x_2)\big) \vee \big(x_1 = x_2\big)\Big) \wedge \Big(\big(f(x_2) <_Y f(x_3)\big) \vee \big(x_2 = x_3\big)\Big) \\
\implies &\bigg(\Big(\big(f(x_1) <_Y f(x_2)\big) \vee \big(x_1 = x_2\big)\Big) \wedge \big(f(x_2) <_Y f(x_3)\big)\bigg) \\
&\vee \bigg(\Big(\big(f(x_1) <_Y f(x_2)\big) \vee \big(x_1 = x_2\big)\Big) \wedge \big(x_2 = x_3\big)\bigg) \\
\implies &\big(f(x_1) <_Y f(x_2) <_Y f(x_3)\big) \\
&\vee \Big(\big(x_1 = x_2\big) \wedge \big(f(x_1) <_Y f(x_3)\big)\Big) \\
&\vee \Big(\big(f(x_1) <_Y f(x_2)\big) \wedge \big(x_2 = x_3\big)\Big) \\
&\vee \big(x_1 = x_2 = x_3\big) \\
\implies &\big(f(x_1) <_Y f(x_3)\big) \\
&\vee \big(f(x_1) <_Y f(x_3)\big) \\
&\vee \big(f(x_1) <_Y f(x_3)\big) \\
&\vee \big(x_1 = x_3\big) \\
\implies &x_1 \leq_X x_3,
\end{aligned}$$

we know that $(X, \leq_X)$ is transitive. Since $(X, \leq_X)$ is reflexive, anti-symmetric and transitive, by Def. I.8.5.1 $(X, \leq_X)$ is partially ordered.

Next we show that $(X, \leq_X)$ may not be totally ordered when $(Y, \leq_Y)$ is totally ordered. If $\exists x, x' \in X : x \neq x' \implies f(x) = f(x')$, then we do not have the ordering relation $x \leq_X x'$ and $x' \leq_X x$. Thus, by Def. I.8.5.3 $(X, \leq_X)$ is not totally ordered.

Next we show that if $f : X \to Y$ is injective and $(Y, \leq_Y)$ is totally ordered, then $(X, \leq_X)$ is totally ordered. Since f is injective, we know that $\forall x, x' \in X, x \neq x' \implies f(x) \neq f(x')$. Since Y is totally ordered, we know that $\big(f(x) \leq_Y f(x')\big) \vee \big(f(x') \leq_Y f(x)\big)$ is true. Since $f(x) \neq f(x')$, we know that exactly one of $\big(f(x) <_Y f(x')\big) \vee \big(f(x') <_Y f(x)\big)$ is true. In either case, we get exactly one of $(x \leq_X x') \vee (x' \leq_X x)$ is true. Thus, by Def. I.8.5.3 $(X, \leq_X)$ is totally ordered. □

Ex. I.8.5.6. Let X be a partially ordered set. For any x in X, define the *order ideal* $(x) \subseteq X$ to be the set $(x) \coloneqq \{y \in X : y \leq_X x\}$. Let $(X) \coloneqq \{(x) : x \in X\}$ be the set of all order ideals, and let $f : X \to (X)$ be the map $f(x) \coloneqq (x)$ that sends every element of x to its order ideal. Show that f is a bijection, and that given any $x, y \in X$, that $x \leq_X y$ iff $f(x) \subseteq f(y)$. This exercise shows that any partially ordered set can be *represented* by a collection of sets whose ordering relation is given by set inclusion.

Proof. We first show that f is bijective. We start by showing f is injective. Since $(X, \leq_X)$ is partially ordered, by Def. I.8.5.1 we know that

$$\begin{aligned}
&\forall x, x' \in X, f(x) = f(x') \\
\implies &(x) = (x') \\
\implies &\big(x \in (x')\big) \wedge \big(x' \in (x)\big) && ((X, \leq_X) \text{ is reflexive}) \\
\implies &(x \leq_X x') \wedge (x' \leq_X x') \\
\implies &x = x'. && ((X, \leq_X) \text{ is anti-symmetric})
\end{aligned}$$

Thus, f is injective. Next we show that f is surjective.

$$\begin{aligned}
&\forall (x) \in (X), x \in X \\
\implies &f(x) = (x).
\end{aligned}$$

Thus, f is surjective. Since f is both injective and surjective, we know that f is bijective.

Finally we show that $\forall x, y \in X, x \leq_X y \iff f(x) \subseteq f(y)$.

$$\begin{aligned}
&\forall x, y \in X, x \leq_X y \\
\iff &\forall z \in X, (z \leq_X x \implies z \leq_X y) && ((X, \leq_X) \text{ is transitive}) \\
\iff &(x) \subseteq (y) \\
\iff &f(x) \subseteq f(y).
\end{aligned}$$

□

Ex. I.8.5.7. Let X be a partially ordered set with ordering relation $\leq_X$, and let Y be a totally ordered subset of X. Show that Y can have at most one maximum and at most one minimum.

Proof. Suppose for the sake of contradiction that $\exists y, y' \in Y$ such that $y \neq y'$ and both y, y' are maximum of Y. Then by Def. I.8.5.5 we have $(y \leq_X y') \wedge (y' \leq_X y)$, so $y = y'$, a contradiction. Thus, Y can have at most one maximum. Similar arguments show that Y can have at most one minimum. □

Ex. I.8.5.8. Show that every finite non-empty subset of a totally ordered set has a minimum and a maximum. Conclude in particular that every finite totally ordered set is well-ordered.

Proof. Let $(X, \leq_X)$ be totally ordered such that $X \neq \emptyset$, let $Y \subseteq X$ be a finite set where $Y \neq \emptyset$ and let $n = \#(Y)$. We induct on n to show that $\min\big((Y, \leq_X)\big)$ and $\max\big((Y, \leq_X)\big)$ exist. For $n = 1$, let $y \in Y$. Then we have

$$\begin{aligned}
&\forall y' \in Y, y' = y \\
\implies& y' = y \leq_X y = y' && ((Y, \leq_X) \text{ is totally ordered}) \\
\implies& y = \min\big((Y, \leq_X)\big) = \max\big((Y, \leq_X)\big) && (\text{by Def. I.8.5.5})
\end{aligned}$$

and the base case holds. Suppose inductively that for some $n \geq 1$ we know that $\min\big((Y, \leq_X)\big)$ and $\max\big((Y, \leq_X)\big)$ exist. Then for $n+1$, let $y \in Y$. By the induction hypothesis we know that $y_{\min} = \min\big((Y \setminus \{y\}, \leq_X)\big)$ and $\max\big((Y \setminus \{y\}, \leq_X)\big)$ exist. Since $y \notin Y \setminus \{y\}$, we know that $y \neq y_{\min}$. Since $Y \subseteq X$, we know that Y is totally ordered, and thus we have $(y <_X y_{\min}) \vee (y_{\min} <_X y)$. Now we split into two cases:

- If $y <_X y_{\min}$, then $y = \min\big((Y, \leq_X)\big)$ since

$$\begin{aligned}
&\forall y' \in Y \\
\implies& (y' = y) \vee (y' \in Y \setminus \{y\}) \\
\implies& (y' = y) \vee (y_{\min} <_X y') && (\text{by Def. I.8.5.5}) \\
\implies& (y' = y) \vee (y <_X y') && ((X, \leq_X) \text{ is transitive}) \\
\implies& y \leq_X y'.
\end{aligned}$$

- If $y_{\min} <_X y$, then $y_{\min} = \min\big((Y, \leq_X)\big)$ since $\forall y' \in Y, y_{\min} \leq_X y'$.

From all cases above, we conclude that $\min\big((Y, \leq_X)\big)$ exists. Similar arguments show that $\max\big((Y, \leq_X)\big)$ exists. This closes the induction. □

Ex. I.8.5.9. Let X be a totally ordered set such that every non-empty subset of X has both a minimum and a maximum. Show that X is finite.

Proof. Let $(X, \leq_X)$ be totally ordered such that $\forall Y \subseteq X$, $Y \neq \emptyset$ implies $\min\big((Y, \leq_X)\big)$ and $\max\big((Y, \leq_X)\big)$ exist. Suppose for the sake of contradiction that X is infinite. Since $X \subseteq X$, by hypothesis $\min\big((X, \leq_X)\big)$ and $\max\big((X, \leq_X)\big)$ exist. Define x_n recursively as follow

$$\forall n \in \mathbb{N}, x_n = \begin{cases} \min\big((X, \leq_X)\big) & \text{if } n = 0 \\ \min\big((X \setminus \bigcup_{m=0}^{n-1} \{x_m\}, \leq_X)\big) & \text{if } n > 0 \end{cases}$$

Then we have an strictly increasing sequence $(x_n)_{n=0}^{\infty}$, i.e., $x_0 \leq_X x_1 \leq_X x_2 \ldots$. Let $X_n = \{x_n : n \in \mathbb{N}\}$ be the set of all elements in sequence $(x_n)_{n=0}^{\infty}$. We know that $X_n \subseteq X$ and $X_n \neq \emptyset$. By hypothesis we know that $\exists \max\big((X_n, \leq_X)\big) \in X_n$. Let $m \in \mathbb{N}$ be the index of the maximum in X_n, i.e., $x_m = \min\big((X_n, \leq_X)\big)$. But $(x_n)_{n=0}^{\infty}$ is an strictly increasing sequence, so we have $x_m <_X x_{m+1}$ and x_m is not a maximum, a contradiction. Thus, X must be finite. □

Ex. I.8.5.10. Prove Prop. I.8.5.10, without using the axiom of choice.

Proof. See Prop. I.8.5.10. □

Ex. I.8.5.11. Let X be a partially ordered set, and let Y and Y' be well-ordered subsets of X. Show that $Y \cup Y'$ is well-ordered iff it is totally ordered.

Proof. Suppose that $(X, \leq_X)$ is partially ordered. Let $Y \subseteq X$, $Y' \subseteq X$ and both $(Y, \leq_X), (Y', \leq_X)$ are well-ordered. By Def. I.8.5.8 we know that if $(Y \cup Y', \leq_X)$ is well-ordered, then $(Y \cup Y', \leq_X)$ is totally ordered. So we only need to show that if $(Y \cup Y', \leq_X)$ is totally ordered, then $(Y \cup Y', \leq_X)$ is well-ordered. Suppose that $(Y \cup Y', \leq_X)$ is totally ordered. Let $Z \subseteq Y \cup Y'$ and $Z \neq \emptyset$. Let $Z_Y = Z \cap Y$ and let $Z_{Y'} = Z \cap Y'$. We know that

$$\begin{aligned} & (Z_Y = \emptyset) \vee (Z_{Y'} = \emptyset) \\ \implies & (Z \subseteq Y') \vee (Z \subseteq Y) \\ \implies & Z \text{ is well-ordered.} && \text{(by Def. I.8.5.8)} \end{aligned}$$

So suppose that $Z_Y \neq \emptyset \wedge Z_{Y'} \neq \emptyset$. Since $Z_Y \subseteq Y$ and $Z_{Y'} \subseteq Y'$, by Def. I.8.5.8 $z_Y = \min\big((Z_Y, \leq_X)\big)$ and $z_{Y'} = \min\big((Z_{Y'}, \leq_X)\big)$ exist. Since $(Y \cup Y', \leq_X)$ is totally order, we know that $(z_Y \leq_X z_{Y'}) \vee (z_{Y'} \leq_X z_Y)$. Without the loss of generality suppose that $z_Y \leq_X z_{Y'}$. Then we have

$$\begin{aligned} & \forall z \in Z \\ \implies & z \in Y \cup Y' \\ \implies & z \in Y \vee z \in Y' \\ \implies & (z_Y \leq_X z) \vee (z_{Y'} \leq_X z) && \text{(by Def. I.8.5.5)} \\ \implies & (z_Y \leq_X z) \vee (z_Y \leq_X z_{Y'} \leq_X z) && \text{(by Def. I.8.5.1)} \end{aligned}$$

$$
\begin{aligned}
&\Longrightarrow z_Y \leq_X z \\
&\Longrightarrow z_Y = \min\big((Z, \leq_X)\big). && \text{(by Def. I.8.5.5)}
\end{aligned}
$$

This means for any $Z \subseteq Y \cup Y'$, $\min\big((Z, \leq_X)\big)$ must exists. Thus, by Def. I.8.5.8 $(Y \cup Y', \leq_X)$ is well-ordered. $\square$

Ex. I.8.5.12. Let X and Y be partially ordered sets with ordering relations $\leq_X$ and $\leq_Y$ respectively. Define a relation $\leq_{X\times Y}$ on the Cartesian product $X \times Y$ by defining $(x, y) \leq_{X\times Y} (x', y')$ if $x <_X x'$, or if $x = x'$ and $y \leq_Y y'$. (This is called the *lexicographical ordering* on $X \times Y$, and is similar to the alphabetical ordering of words; a word w appears earlier in a dictionary than another word w' if the first letter of w is earlier in the alphabet than the first letter of w', or if the first letters match and the second letter of w is earlier than the second letter of w', and so forth.) Show that $\leq_{X\times Y}$ defines a partial ordering on $X \times Y$. Furthermore, show that if X and Y are totally ordered, then so is $X \times Y$, and if X and Y are well-ordered, then so is $X \times Y$.

Proof. We first show that $(X \times Y, \leq_{X\times Y})$ is partially ordered. If $X = \emptyset \vee Y = \emptyset$, then by Ex. I.3.5.8 $X \times Y = \emptyset$ and by Ex. I.8.5.1 we know that $\emptyset$ is partially ordered. So suppose that $X \neq \emptyset \wedge Y \neq \emptyset$. Since

$$\forall (x, y) \in X \times Y, x = x \implies (x, y) \leq_{X\times Y} (x, y),$$

we know that $(X \times Y, \leq_{X\times Y})$ is reflexive. Since

$$
\begin{aligned}
&\forall (x, y), (x', y') \in X \times Y, \\
&\big((x, y) \leq_{X\times Y} (x', y')\big) \wedge \big((x', y') \leq_{X\times Y} (x, y)\big) \\
\Longrightarrow &\Big((x <_X x') \vee \big((x = x') \wedge (y \leq_Y y')\big)\Big) \\
&\wedge \Big((x' <_X x) \vee \big((x = x') \wedge (y' \leq_Y y)\big)\Big) \\
\Longrightarrow &\Big((x \leq_X x') \wedge \big((x <_X x') \vee (y \leq_Y y')\big)\Big) \\
&\wedge \Big((x' \leq_X x) \wedge \big((x' <_X x) \vee (y' \leq_Y y)\big)\Big) \\
\Longrightarrow &(x = x') \wedge (y \leq_Y y') \wedge (y' \leq_Y y) \\
\Longrightarrow &(x = x') \wedge (y = y') && \text{(by Def. I.8.5.1)} \\
\Longrightarrow &(x, y) = (x', y'), && \text{(by Def. I.3.5.1)}
\end{aligned}
$$

we know that $(X \times Y, \leq_{X\times Y})$ is anti-symmetric. Since

$$
\begin{aligned}
&\forall (x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y : \\
&\big((x_1, y_1) \leq_{X\times Y} (x_2, y_2)\big) \wedge \big((x_2, y_2) \leq_{X\times Y} (x_3, y_3)\big) \\
\Longrightarrow &\Big((x_1 <_X x_2) \vee \big((x_1 = x_2) \wedge (y_1 \leq_Y y_2)\big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\wedge \Big((x_2 <_X x_3) \vee \big((x_2 = x_3) \wedge (y_2 \leq_Y y_3) \big) \Big) \\
\Longrightarrow\ & (x_1 <_X x_2 <_X x_3) && \text{(by Def. I.8.5.1)} \\
&\vee \big((x_1 = x_2 <_X x_3) \wedge (y_1 \leq_Y y_2) \big) \\
&\vee \big((x_1 <_X x_2 = x_3) \wedge (y_2 \leq_Y y_3) \big) \\
&\vee \big((x_1 = x_2 = x_3) \wedge (y_1 \leq_Y y_2 \leq_Y y_3) \big) && \text{(by Def. I.8.5.1)} \\
\Longrightarrow\ & (x_1, y_1) \leq_{X\times Y} (x_3, y_3),
\end{aligned}
$$

we know that $(X \times Y, \leq_{X\times Y})$ is transitive. Since $(X \times Y, \leq_{X\times Y})$ is reflexive, anti-symmetric and transitive, by Def. I.8.5.1 $(X \times Y, \leq_{X\times Y})$ is partially ordered.

Now we show that if $(X, \leq_X), (Y, \leq_Y)$ are totally ordered, then $(X \times Y, \leq_{X\times Y})$ is totally ordered. If $X = \emptyset \vee Y = \emptyset$, then by Ex. I.3.5.8 $X \times Y = \emptyset$ and by Ex. I.8.5.1 we know that $\emptyset$ is totally ordered. So suppose that $X \neq \emptyset \wedge Y \neq \emptyset$. From the proof above we know that $(X \times Y, \leq_{X\times Y})$ is partially ordered. Since

$$
\begin{aligned}
&\forall (x, y), (x', y') \in X \times Y, (x, y) \neq (x', y') \\
\Longrightarrow\ & (x \neq x') \vee \big((x = x') \wedge (y \neq y') \big), && \text{(by Def. I.3.5.1)}
\end{aligned}
$$

we split into two cases:

- If $x \neq x'$, then $(x <_X x') \vee (x' <_X x)$ since $(X, \leq_X)$ is totally ordered. Thus, we have $\big((x, y) \leq_{X\times Y} (x', y')\big) \vee \big((x, y) \leq_{X\times Y} (x', y')\big)$.
- If $(x = x') \wedge (y \neq y')$, then $(y <_Y y') \vee (y' <_Y y)$ since $(Y, \leq_Y)$ is totally ordered. Thus, we have $\big((x, y) \leq_{X\times Y} (x', y')\big) \vee \big((x, y) \leq_{X\times Y} (x', y')\big)$.

From all cases above, we conclude that $\big((x, y) \leq_{X\times Y} (x', y')\big) \vee \big((x, y) \leq_{X\times Y} (x', y')\big)$. Thus, $(X \times Y, \leq_{X\times Y})$ is totally ordered.

Now we show that if $(X, \leq_X), (Y, \leq_Y)$ are well-ordered, then $(X \times Y, \leq_{X\times Y})$ is well-ordered. If $X = \emptyset \vee Y = \emptyset$, then by Ex. I.3.5.8 $X \times Y = \emptyset$ and by Ex. I.8.5.1 we know that $(\emptyset, \leq_{X\times Y})$ is well-ordered. So suppose that $X \neq \emptyset \wedge Y \neq \emptyset$. From the proof above we know that $(X \times Y, \leq_{X\times Y})$ is totally ordered. Let $Z \subseteq X \times Y$ and $Z \neq \emptyset$. We know that the set $Z_X = \{x \in X | \exists y \in Y : (x, y) \in Z\} \neq \emptyset$. Since $(X, \leq_X)$ is well-ordered and $Z_X \subseteq X$, by Def. I.8.5.8 we know that $z_x = \min\big((Z_X, \leq_X)\big)$ exists. Since $z_x \in Z_X$, we know that the set $Z_Y = \{y \in Y : (z_x, y) \in Z\} \neq \emptyset$. Since $(Y, \leq_Y)$ is well-ordered and $Z_Y \subseteq Y$, by Def. I.8.5.8 we know that $z_y = \min\big((Z_Y, \leq_Y)\big)$ exists. Then we have

$$
\begin{aligned}
&\forall (x, y) \in Z \\
\Longrightarrow\ & (z_x <_X x) \vee (x = z_x) \\
\Longrightarrow\ & \big((z_x, z_y) \leq_{X\times Y} (x, y)\big) \vee (x = z_x) \\
\Longrightarrow\ & \big((z_x, z_y) \leq_{X\times Y} (x, y)\big) \vee \big((x = z_x) \wedge (y \in Z_y)\big) \\
\Longrightarrow\ & \big((z_x, z_y) \leq_{X\times Y} (x, y)\big) \vee \big((x = z_x) \wedge (z_y \leq_Y y)\big)
\end{aligned}
$$

$$\implies (z_x, z_y) \leq_{X\times Y} (x, y).$$

Thus, $\min\big((Z, \leq_{X\times Y})\big) = (z_x, z_y)$. Since Z was arbitrary, by Def. I.8.5.8 we know that $(X \times Y, \leq_{X\times Y})$ is well-ordered. $\square$

Ex. I.8.5.13. Prove the claim in the proof of Lem. I.8.5.14, namely that every element of $Y' \setminus Y$ is an strict upper bound for Y and vice versa.

Proof. Since Y, Y' are good, we know that $x_0 \in Y \cap Y'$ and thus $Y \cap Y' \neq \emptyset$. Let $n \in Y \cap Y'$ and let $P(n)$ be the statement define as follow:

$$\{y \in Y : y \leq n\} = \big\{y \in Y' : y \leq n\big\} = \big\{y \in Y \cap Y' : y \leq n\big\}.$$

Let $Q(n)$ be the statement "if $P(m)$ is true for every $m \in Y \cap Y'$ and $m < n$, then $P(n)$ is true." Note that if we can show that $Q(n)$ is true for every $n \in Y \cap Y'$, then by principle of strong induction (Prop. I.8.5.10) we know that $P(n)$ is true for every $n \in Y \cap Y'$.

Since $\min(Y) = \min(Y') = x_0$, we know that $\min(Y \cap Y') = x_0$ and

$$\{y \in Y : y \leq x_0\} = \big\{y \in Y' : y \leq x_0\big\} = \big\{y \in Y \cap Y' : y \leq x_0\big\} = \{x_0\}.$$

Thus, $Q(x_0)$ is vacuously true (there is no element $z \in Y \cap Y'$ which satisfy $z < x_0$). Suppose inductively that $Q(m)$ is true for some $m \geq x_0$. We need to show that $Q(n)$ is also true for the next smallest item $n \in Y \cap Y'$. By smallest we mean that

$$n = \min\Big(\big\{n \in Y \cap Y' : Q(m) \text{ is true and } x_0 \leq m < n \text{ for every } m \in Y \cap Y'\big\}\Big).$$

Such n is well-defined since in the proof of Lem. I.8.5.14 we suppose for the sake of contradiction that every well-ordered subset Y of X which has x_0 as its minimal element has at least one strict upper bound. We know that the set $\{y \in Y : y \leq n\}$ is not empty since it contains x_0, so let $x \in \{y \in Y : y \leq n\}$. Now we split into two cases:

- If $x = n$, then by the definition of n we know that $x \in \big\{y \in Y \cap Y' : y \leq n\big\}$.
- If $x < n$, then by the definition of n we know that by the induction hypothesis $Q(x)$ is true. Then we have

$$\begin{aligned} &P(x) \text{ is true} \\ \implies &\{y \in Y : y \leq x\} = \big\{y \in Y \cap Y' : y \leq x\big\} \\ \implies &x \in \big\{y \in Y \cap Y' : y \leq x < n\big\}. \end{aligned}$$

From all cases above, we conclude that $\{y \in Y : y \leq n\} \subseteq \big\{y \in Y \cap Y' : y \leq n\big\}$. We also have $\big\{y \in Y \cap Y' : y \leq n\big\} \subseteq \{y \in Y : y \leq n\}$ since

$$\begin{aligned} &\forall x \in \big\{y \in Y \cap Y' : y \leq n\big\} \\ \implies &(x \in Y \cap Y') \wedge (x \leq n) \end{aligned}$$

$$\begin{aligned}&\Longrightarrow (x \in Y) \wedge (x \le n)\\&\Longrightarrow x \in \{y \in Y : y \le n\}.\end{aligned}$$

Thus, we have $\{y \in Y : y \le n\} = \big\{y \in Y \cap Y' : y \le n\big\}$. Using similar arguments above, we have $\big\{y \in Y' : y \le n\big\} = \big\{y \in Y \cap Y' : y \le n\big\}$. Thus, we conclude that $P(n)$ is also true, i.e.,

$$\{y \in Y : y \le n\} = \big\{y \in Y' : y \le n\big\} = \big\{y \in Y \cap Y' : y \le n\big\}.$$

By the induction hypothesis we know that $Q(m)$ is true for every $m \in Y \cap Y'$ and $m < n$. Thus, we conclude that $Q(n)$ is also true, this closes the induction.

By principle of strong induction (Prop. I.8.5.10) we know that $P(n)$ is true for all $n \in Y \cap Y'$. Thus, we have

$$\begin{aligned}&\forall x \in (Y \cap Y') \setminus \{x_0\}, P(x) \text{ is true}\\
\Longrightarrow\ & \big\{y \in Y \cap Y' : y \le x\big\} = \{y \in Y : y \le x\}\\
\Longrightarrow\ & \big\{y \in Y \cap Y' : y \le x\big\} \setminus \{x\} = \{y \in Y : y \le x\} \setminus \{x\}\\
\Longrightarrow\ & \big\{y \in Y \cap Y' : y < x\big\} = \{y \in Y : y < x\}\\
\Longrightarrow\ & s(\big\{y \in Y \cap Y' : y < x\big\}) = s(\{y \in Y : y < x\}) = x \qquad (Y \text{ is good})\end{aligned}$$

and $Y \cap Y'$ is good.

Now we show that if $Y \setminus Y' \neq \emptyset$, then $s(Y \cap Y') = \min(Y \setminus Y')$. Since $Y \cap Y'$ is good, $s(Y \cap Y')$ is well-defined. Since Y is well-ordered and $Y \setminus Y' \subseteq Y$, $Y \setminus Y'$ is also well-ordered and thus $\min(Y \setminus Y')$ is well-defined. We know that $\forall y_1 \in Y$, $y_1 < \min(Y \setminus Y') \implies y_1 \notin Y \setminus Y'$, otherwise $y_1 = \min(Y \setminus Y')$, a contradiction. Thus, $y_1 \in Y \cap Y'$ and $\big\{y \in Y : y < \min(Y \setminus Y')\big\} \subseteq Y \cap Y$. Similarly, we know that $\forall y_2 \in Y \cap Y'$, $y_2 \in Y$ and $y_2 < \min(Y \setminus Y')$, so $y_2 \in \big\{y \in Y : y < \min(Y \setminus Y')\big\}$ and $Y \cap Y' \subseteq \big\{y \in Y : y < \min(Y \setminus Y')\big\}$. We now conclude that $\big\{y \in Y : y < \min(Y \setminus Y')\big\} = Y \cap Y'$ and

$$\begin{aligned}&\min(Y \setminus Y') \in Y \setminus \{x_0\}\\
\Longrightarrow\ & \min(Y \setminus Y') = s\big(\big\{y \in Y : y < \min(Y \setminus Y')\big\}\big) \qquad (Y \text{ is good})\\
\Longrightarrow\ & \min(Y \setminus Y') = s(Y \cap Y').\end{aligned}$$

Similar arguments show that if $Y' \setminus Y \neq \emptyset$, then $s(Y \cap Y') = \min(Y' \setminus Y)$.

Finally we show that every element of $Y' \setminus Y$ is an upper bound for Y and vice versa. Since $(Y \setminus Y') \cap (Y' \setminus Y) = \emptyset$, we know that at least one of $Y \setminus Y'$ or $Y' \setminus Y$ is empty. Otherwise we have $s(Y \cap Y') = \min(Y \setminus Y') = \min(Y' \setminus Y)$, which means $\big(s(Y \cap Y') \in Y \setminus Y'\big) \wedge \big(s(Y \cap Y') \in Y' \setminus Y\big)$, a contradiction. Now we split into two cases:

- If $Y \setminus Y' = \emptyset$, then it is vacuously true that every element of $Y \setminus Y'$ is an strict upper bound of Y'.
- If $Y \setminus Y' \neq \emptyset$, then $Y' \setminus Y = \emptyset$ and $Y' \subseteq Y$. Since $Y \cap Y' = Y'$ and $s(Y \cap Y') = s(Y') = \min(Y \setminus Y')$, by Def. I.8.5.5 we know that every element of $Y \setminus Y'$ is an strict upper bound of Y'.

From all cases above, we conclude that every element of $Y \setminus Y'$ is an strict upper bound of Y'. Using similar arguments, we can show that every element of $Y' \setminus Y$ is an strict upper bound of Y. $\square$

Ex. I.8.5.14. Use Lem. I.8.5.14 to prove Lem. I.8.5.15.

Proof. See Lem. I.8.5.15 $\square$

Ex. I.8.5.15. Let A and B be two non-empty sets such that A does not have lesser or equal cardinality to B. Using Zorn's lemma, prove that B has lesser or equal cardinality to A. This exercise (combined with Ex. I.8.3.3) shows that the cardinality of any two sets is comparable, as long as one assumes the axiom of choice.

Proof. For every subset $X \subseteq B$, let $P(X)$ denote the property that there exists an injective map from $X \to A$. Let $S = \{X \subseteq B : P(X)\}$ be a set. Since $B \neq \emptyset$, let $b \in B$ and let $f_{\{b\}} : \{b\} \to A$. Clearly, $f_{\{b\}}$ is injective, so we know that $S \neq \emptyset$. Let $\leq_S$ be a ordering relation of S by setting

$$\begin{aligned}
&\forall X, Y \in S, X \leq_S Y \\
\iff &(X \subseteq Y) \\
&\wedge \big(\exists f_X : X \to A \text{ where } f_X \text{ is injective}\big) \\
&\wedge \big(\exists f_Y : Y \to A \text{ where } f_Y \text{ is injective}\big) \\
&\wedge \big(\forall x \in X, f_X(x) = f_Y(x)\big).
\end{aligned}$$

Since

$$\forall X \in S, (X \subseteq X) \wedge \big(\forall x \in X, f_X(x) = f_X(x)\big) \implies X \leq_S X,$$

we know that $(S, \leq_S)$ is reflexive. Since

$$\begin{aligned}
&\forall X, Y \in S, (X \leq_S Y) \wedge (Y \leq_S X) \\
\implies &(X \subseteq Y) \wedge (Y \subseteq X) \\
\implies &X = Y, && \text{(by Prop. I.3.1.18)}
\end{aligned}$$

we know that $(S, \leq_S)$ is anti-symmetric. Since

$$\begin{aligned}
&\forall X, Y, Z \in S, (X \leq_S Y) \wedge (Y \leq_S Z) \\
\implies &(X \subseteq Y) \wedge (Y \subseteq Z) \wedge \big(\forall x \in X, f_X(x) = f_Y(x)\big) \wedge \big(\forall y \in Y, f_Y(y) = f_Z(y)\big) \\
\implies &(X \subseteq Z) \wedge \big(\forall x \in X, f_X(x) = f_Z(x)\big) \\
\implies &X \leq_S Z,
\end{aligned}$$

we know that $(S, \leq_S)$ is transitive. Since $(S, \leq_S)$ is reflexive, anti-symmetric and transitive, by Def. I.8.5.1 we know that $(S, \leq_S)$ is partially ordered.

Next we show that $\forall S' \subseteq S$, if $(S', \leq_S)$ is totally ordered, then $\bigcup S' \in S$. Clearly, we have $\bigcup S' \subseteq B$. So we only need to show that there exists a function $f : \bigcup S' \to A$ such that f is injective. For each $X \in S'$, let $F_X = \{f_X : X \to A \text{ is injective}\}$ be a set. Since $P(X)$ is true, we know that $F_X \neq \emptyset$. By axiom of choice (Ax. I.8.1) we know that $\prod\limits_{X\in S'} F_X \neq \emptyset$. Let $(f_X)_{X\in S'} \in \prod\limits_{X\in S'} F_X$. Fix such $(f_X)_{X\in S'}$. We define a function $f : \bigcup S' \to A$ as follow:

$$\forall x_0 \in \bigcup S', f(x_0) = f_Y(x_0)$$

for some $Y \in S'$, $x_0 \in Y$ and $f_Y = (f_X)_{X\in S'}(Y)$. To show that such f is well-defined, we need to show that Y is unique for each $x_0 \in \bigcup S'$. Since $(S', \leq_S)$ is totally ordered, we know that $\forall Z \in S'$, we have $Z \leq_S Y$ or $Y \leq_S Z$. Let $f_Z = (f_X)_{X\in S'}(Z)$. We split into two cases:

- If $Z \leq_S Y$, then by the definition of $\leq_S$ we have $Z \subseteq Y$ and $\forall z \in Z$, $f_Z(z) = f_Y(z)$. Now we further split into two cases:
  - If $x_0 \in Z$, then $f_Z(x_0) = f_Y(x_0)$.
  - If $x_0 \notin Z$, then $\forall z \in Z$, we have $f_Z(z) = f_Y(z) \neq f_Y(x_0)$ since f_Y, f_Z are injective.
- If $Y \leq_S Z$, then by the definition of $\leq_S$ we have $Y \subseteq Z$ and $\forall y \in Y$, $f_Y(y) = f_Z(y)$. Thus, $x_0 \in Z$ and $f_Y(x_0) = f_Z(x_0)$.

From all cases above, we conclude that f is well-defined. Now we show that f is injective. Let $y, z \in \bigcup S'$ and $y \neq z$. Then $\exists Y, Z \in S'$ such that $y \in Y$ and $z \in Z$. Again we have either $Y \leq_S Z$ or $Z \leq_S Y$. Without the loss of generality suppose that $Y \leq_S Z$. Then we have

$$f_Y(y) = f_Z(y) \neq f_Z(z)$$

where $f_Y = (f_X)_{X\in S'}(Y)$ and $f_Z = (f_X)_{X\in S'}(Z)$. Thus, f is injective and $\bigcup S' \in S$.

Now we show that $\forall S' \subseteq S$, if $(S', \leq_S)$ is totally ordered, then there exists an upper bound of S'. Let $S' \subseteq S$ such that $(S', \leq_S)$ is totally ordered. Since

$$\begin{aligned}
&\forall Y \in S' \\
\implies &\left(Y \subseteq \bigcup S' \in S\right) && \text{(from claim above)} \\
&\wedge \left(\forall y \in Y, f_Y(x) = \big((f_X)_{X\in S'}(Y)\big)(x) = f(y)\right) \\
\implies &Y \leq_S \bigcup S',
\end{aligned}$$

we know that $\bigcup S'$ is an upper bound of S'. Since S' was arbitrary, we conclude that every totally ordered subset of S with ordering relation $\leq_S$ has an upper bound.

By Zorn's lemma (Lem. I.8.5.15) we know that there exists at least one maximal element of S. Suppose for the sake of contradiction that $B \neq \max\big((S, \leq_S)\big)$. Let $X = \max\big((S, \leq_S)\big)$. So $B \setminus X \neq \emptyset$. Then we know that $P(X)$ is true, i.e., $\exists f_X : X \to A$ such that f_X is injective. We must have $f_X(X) = A$, otherwise $f_X(X) \subsetneq A$, we know that $\exists a \in A \setminus f_X(X)$, and we can let $b \in B \setminus X$ map to a, i.e., $f_X(b) = a$. This cause $(X \subsetneq X \cup \{b\}) \wedge \big(\forall x \in X, f_X(x) = f_{X\cup\{b\}}(x)\big)$, which means $X \leq_S X \cup \{b\}$ and contradicts to $X = \max\big((S, \leq_S)\big)$. So we have $f_X(X) = A$. But this also means f_X is a bijection from X to A. So we can set $g : A \to B$ as $g(a) = f_X^{-1}(a)$, which is injective. By hypothesis we know that we cannot have a injection from A to B (this is the definition of A having lesser or equal cardinality than B, see Ex. I.3.6.7). Thus, we derived a contradiction. So $B = \max\big((S, \leq_S)\big) \in S$ and $\exists f_B : B \to A$ such that f_B is injective. By Ex. I.8.3.3 B has lesser or equal cardinality than A. $\square$

Ex. I.8.5.16. Let X be a set, and let P be the set of all partial orderings of X. (For instance, if $X \coloneqq \mathbb{N} \setminus \{0\}$, then both the usual partial ordering $\leq$, and the partial ordering in Ex. I.8.5.3, are elements of P.) We say that one partial ordering $\leq \in P$ is *coarser* than another partial ordering $\leq' \in P$ if for any $x, y \in X$, we have the implication $(x \leq y) \implies (x \leq' y)$. Thus, for instance the partial ordering in Ex. I.8.5.3 is coarser than the usual ordering $\leq$. Let us write $\leq \preceq \leq'$ if $\leq$ is coarser than $\leq'$. Show that $\preceq$ turns P into a partially ordered set; thus the set of partial orderings on X is itself partially ordered. There is exactly one minimal element of P; what is it? Show that the maximal elements of P are precisely the total orderings of P. Using Zorn's lemma (Lem. I.8.5.15), show that given any partial ordering $\leq$ of X there exists a total ordering $\leq'$ such that $\leq$ is coarser than $\leq'$.

Proof. Since

$$\begin{aligned} &\forall (X, \leq) \in P \\ \implies &(\forall x, y \in X, x \leq y \implies x \leq y) \\ \implies &\leq \preceq \leq, \end{aligned}$$

we know that $(P, \preceq)$ is reflexive. Since

$$\begin{aligned} &\forall (X, \leq_1), (X, \leq_2) \in P, (\leq_1 \preceq \leq_2) \wedge (\leq_2 \preceq \leq_1) \\ \implies &\big(\forall x, y \in X, (x \leq_1 y \implies x \leq_1 y') \wedge (x \leq_2 y \implies x \leq_1 y)\big) \\ \implies &\big(\forall x, y \in X, (x \leq_1 y \iff x \leq_2 y)\big) \\ \implies &\leq_1 = \leq_2, \end{aligned}$$

we know that $(P, \preceq)$ is anti-symmetric. Since

$$\begin{aligned} &\forall (X, \leq_1), (X, \leq_2), (X, \leq_3) \in P, (\leq_1 \preceq \leq_2) \wedge (\leq_2 \preceq \leq_3) \\ \implies &\big(\forall x, y \in X, (x \leq_1 y \implies x \leq_2 y) \wedge (x \leq_2 y \implies x \leq_3 y)\big) \\ \implies &\big(\forall x, y \in X, (x \leq_1 y \implies x \leq_3 y)\big) \\ \implies &\leq_1 \preceq \leq_3, \end{aligned}$$

we know that $(P, \preceq)$ is transitive. Since $(P, \preceq)$ is reflexive, anti-symmetric and transitive, by Def. I.8.5.1 we know that $(P, \preceq)$ is partially ordered.

Next we show that $(P, \preceq)$ has exactly one minimal element. We claim that the equivalent relation on X, denote as $=_X$, is the minimal element of $(P, \preceq)$. To show that $(X, =_X)$ is partially ordered, since equivalent relation is reflexive and transitive, we only need to show that $(X, =_X)$ is anti-symmetric. Since

$$\forall x, y \in X, (x =_X y) \wedge (y =_X x) \implies x =_X y,$$

we know that $(X, =_X)$ is anti-symmetric. Thus, by Def. I.8.5.1 $(X, =_X)$ is partially ordered and $(X, =_X) \in P$. Since

$$\begin{aligned}
&\forall (X, \leq) \in P \\
\implies &(\forall x, y \in X, x =_X y \implies x \leq y) && \text{(by Def. I.8.5.1)} \\
\implies &=_X \preceq \leq,
\end{aligned}$$

by Def. I.8.5.5 we have $(X, =_X) = \min\big((P, \preceq)\big)$.

Next we show that if $S \subseteq P$, $S \neq \emptyset$ and $(S, \preceq)$ is totally ordered, then the relation $(X, \leq_S)$ defined as follow is a element of P:

$$\forall x, y \in X, x \leq_S y \iff \exists \leq \in S : x \leq y.$$

Since

$$\begin{aligned}
&\forall x \in X, \forall \leq \in S, x \leq x && ((X, \leq) \text{ is partially ordered}) \\
\implies &x \leq_S x,
\end{aligned}$$

we know that $(X, \leq_S)$ is reflexive. Since

$$\begin{aligned}
&\forall x, y \in X, (x \leq_S y) \wedge (y \leq_S x) \\
\implies &\exists \leq_1, \leq_2 \in S : (x \leq_1 y) \wedge (y \leq_2 x) && ((X, \leq) \text{ is partially ordered}) \\
\implies &(\leq_2 \preceq \leq_1) \vee (\leq_1 \preceq \leq_2) && ((S, \preceq) \text{ is totally ordered}) \\
\implies &\big((x \leq_1 y) \wedge (y \leq_1 x)\big) \vee \big((x \leq_2 y) \wedge (y \leq_2 x)\big) \\
\implies &x =_X y,
\end{aligned}$$

we know that $(X, \leq_S)$ is anti-symmetric. Since

$$\begin{aligned}
&\forall x, y, z \in X, (x \leq_S y) \wedge (y \leq_S z) \\
\implies &\exists \leq_1, \leq_2 \in S : (x \leq_1 y) \wedge (y \leq_2 z) && ((X, \leq) \text{ is partially ordered}) \\
\implies &(\leq_2 \preceq \leq_1) \vee (\leq_1 \preceq \leq_2) && ((S, \preceq) \text{ is totally ordered}) \\
\implies &\big((x \leq_1 y) \wedge (y \leq_1 z)\big) \vee \big((x \leq_2 y) \wedge (y \leq_2 z)\big) \\
\implies &(x \leq_1 z) \vee (x \leq_2 z)
\end{aligned}$$

$$\implies x \leq_S z,$$

we know that $(X, \leq_S)$ is transitive. Since $(X, \leq_S)$ is reflexive, anti-symmetric and transitive, by Def. I.8.5.1 we know that $(X, \leq_S)$ is partially ordered. Thus, $(X, \leq_S) \in P$.

Next we show that every totally ordered subset of P has an upper bound. Let $S \subseteq P$ such that $(S, \preceq)$ is totally ordered. If $S = \emptyset$, by Ex. I.8.5.1 we know that $(\emptyset, \preceq)$ is totally ordered, and the statement

$$\forall (X, \leq) \in \emptyset, \leq \preceq \leq'$$

is vacuously true for every $(X, \leq') \in P$. Thus, $(\emptyset, \preceq)$ has an upper bound in P. So suppose that $S \neq \emptyset$. Then the relation $(S, \leq_S)$ defined as above is an upper bound of S since for every $(X, \leq) \in S$, we have

$$\begin{aligned} &\forall x, y \in X, x \leq y \\ \implies & x \leq_S y && \text{(by the definition of } (X, \leq_S)) \\ \implies & \leq \preceq \leq_S . \end{aligned}$$

Thus, every totally ordered subset of P has an upper bound.

By Zorn's lemma (Lem. I.8.5.15) we know that $(X, \leq_{\max}) = \max\big((P, \preceq)\big)$ exists. We show that $(X, \leq_{\max})$ is totally ordered. Suppose for the sake of contradiction that $(X, \leq_{\max})$ is not totally ordered. Then $\exists a, b \in X$ such that $a \not\leq_{\max} b$ and $b \not\leq_{\max} a$. Note that we must have $a \neq_X b$ since $\leq_{\max}$ is reflexive. Now we define a relation $(X, \leq_p)$ as follow:

$$\forall x, y \in X, x \leq_p y \iff (x, y) = (a, b) \vee (x =_X y)$$

We show that $(X, \leq_p)$ is partially ordered. Since

$$\forall x \in X, x =_X x \implies x \leq_p x,$$

we know that $(X, \leq_p)$ is reflexive. Since

$$\begin{aligned} &\forall x, y \in X, (x \leq_p y) \wedge (y \leq_p x) \\ \iff & \begin{cases} \big((x, y) = (a, b)\big) \wedge \big((y, x) = (a, b)\big) & \iff \text{false} \\ \big((x, y) = (a, b)\big) \wedge (y =_X x) & \iff \text{false} \\ (x =_X y) \wedge \big((y, x) = (a, b)\big) & \iff \text{false} \\ (x =_X y) \wedge (y =_X x) & \iff (x =_X y) \end{cases} \\ \iff & (\text{false}) \vee (x =_X y) \\ \iff & (x =_X y), && \text{(vacuously true)} \end{aligned}$$

we know that $(X, \leq_p)$ is anti-symmetric. Since

$$\forall x, y, z \in X, (x \leq_p y) \wedge (y \leq_p z)$$

$$\begin{aligned}
&\iff \begin{cases} \big((x,y)=(a,b)\big)\wedge\big((y,z)=(a,b)\big) & \iff \text{false} \\ \big((x,y)=(a,b)\big)\wedge(y=_X z) & \iff (x,z)=(a,b) \\ (x=_X y)\wedge\big((y,z)=(a,b)\big) & \iff (x,z)=(a,b) \\ (x=_X y)\wedge(y=_X z) & \iff x=_X y=_X z \end{cases} \\
&\iff (\text{false})\vee\big((x,z)=(a,b)\big)\vee(x=_X z) \\
&\iff x\le_p z, && \text{(vacuously true)}
\end{aligned}$$

we know that $(X,\le_p)$ is transitive. Since $(X,\le_p)$ is reflexive, anti-symmetric and transitive, by Def. I.8.5.1 we know that $(X,\le_p)$ is partially ordered. Thus, $(X,\le_p)\in P$. Then we must have $\le_p\preceq\le_{\max}$. But we know that $a\le_p b$ does not implies $a\le_{\max} b$, a contradiction. Thus, $(X,\le_{\max})$ is totally ordered.

Finally we show that any partial order $(X,\le)$, there exists a total order $(X,\le')$ such that $\le\preceq\le'$. Since $\le\preceq\le_{\max}$ and $(X,\le_{\max})$ is totally ordered, the claim follows. $\square$

Ex. I.8.5.17. Use Zorn's lemma (Lem. I.8.5.15) to give another proof of the claim in Ex. I.8.4.2. Deduce that Zorn's lemma (Lem. I.8.5.15) and the axiom of choice (Ax. I.8.1) are in fact logically equivalent (i.e., they can be deduced from each other).

Proof. Let I be a set, and for each $\alpha\in I$ let X_α be a non-empty set. Suppose that all the sets X_α are disjoint from each other, i.e., $X_\alpha\cap X_\beta=\emptyset$ for all distinct $\alpha,\beta\in I$. We want to show that there exists a set Y such that $\#(Y\cap X_\alpha)=1$ for all $\alpha\in I$.

We define a set Ω as follow:

$$\Omega=\left\{Y\subseteq\bigcup_{\alpha\in I}X_\alpha : \#(Y\cap X_\alpha)\le 1 \text{ for all } \alpha\in I\right\}.$$

Clearly, $\emptyset\in\Omega$, thus $\Omega\neq\emptyset$. By E.g. I.8.5.2 we know that $(\Omega,\subseteq)$ is partially ordered.

Let $S\subseteq\Omega$ such that $(S,\subseteq)$ is totally ordered. We show that $\bigcup S\in\Omega$. If $S=\emptyset$ or $S=\{\emptyset\}$, then $\bigcup S=\emptyset$ and $\emptyset\in\Omega$. So suppose that $S\neq\emptyset$ and $S\neq\{\emptyset\}$. Since

$$\begin{aligned}
&\forall\beta\in I,\forall x_1,x_2\in\Big(X_\beta\cap\bigcup S\Big) \\
\Longrightarrow\ &\exists Y_1,Y_2\in S:(x_1\in X_\beta\cap Y_1)\wedge(x_2\in X_\beta\cap Y_2) \\
&\wedge(Y_1\subseteq Y_2\vee Y_2\subseteq Y_1) && ((S,\subseteq)\text{ is totally ordered}) \\
\Longrightarrow\ &(x_1,x_2\in X_\beta\cap Y_1)\vee(x_1,x_2\in X_\beta\cap Y_2) \\
\Longrightarrow\ &\big(\#(X_\beta\cap Y_1)=\#(X_\beta\cap Y_2)=1\big)\wedge(x_1=x_2) \\
\Longrightarrow\ &\#\Big(X_\beta\cap\bigcup S\Big)=1,
\end{aligned}$$

we know that $\bigcup S\in\Omega$. Thus, we conclude that if $S\subseteq\Omega$ and $(S,\subseteq)$ is totally ordered, then $\bigcup S\in\Omega$.

Now we show that every totally ordered subset of Ω has an upper bound. Let $S \subseteq \Omega$ such that $(S, \subseteq)$ is totally ordered. If $S = \emptyset$, then $\emptyset \subseteq \emptyset$ and the claim follows vacuously. So suppose that $S \neq \emptyset$. Since $\bigcup S \in \Omega$ and

$$\forall Y \in S, Y \subseteq \bigcup S,$$

by Def. I.8.5.12 we know that $\bigcup S$ is an upper bound of S (under the relation $\subseteq$). Thus, we conclude that every totally ordered subset of Ω has an upper bound. By Zorn's lemma (Lem. I.8.5.15) we know that $M = \max\big((\Omega, \subseteq)\big)$ exists. We claim that $\#(M \cap X_\alpha) = 1$ for every $\alpha \in I$. Suppose for the sake of contradiction that $\exists \beta \in I$ such that $\#(M \cap X_\beta) = \#(\emptyset) = 0$. Let $x_\beta \in X_\beta$. Now we define a set $Y_\beta = \{x_\beta\}$. Clearly, $\#(Y_\beta \cap X_\alpha) = \#(\emptyset) = 0$ for every $\alpha \in I \setminus \{\beta\}$, and $\#(Y_\beta \cap X_\beta) = \#(\{x_\beta\}) = 1$. Thus, we have $Y_\beta \in \Omega$. But then we have

$$\begin{aligned} & Y_\beta \subseteq M \\ \Longrightarrow\ & x_\beta \in M \\ \Longrightarrow\ & M \cap X_\beta \neq \emptyset \\ \Longrightarrow\ & \#(M \cap X_\beta) \neq 0, \end{aligned}$$

a contradiction. Thus, we have $\#(M \cap X_\alpha) = 1$ for every $\alpha \in I$. By Ex. I.8.4.2 this means Zorn's lemma and the axiom of choice are in fact logically equivalent. $\square$

Ex. I.8.5.18. Using Zorn's lemma (Lem. I.8.5.15), prove *Hausdorff's maximality principle*: if X is a partially ordered set, then there exists a totally ordered subset Y of X which is maximal with respect to set inclusion (i.e. there is no other totally ordered subset Y' of X which contains Y). Conversely, show that if Hausdorff's maximality principle is true, then Zorn's lemma (Lem. I.8.5.15) is true. Thus, by Ex. I.8.5.17, these two statements are logically equivalent to the axiom of choice.

Proof. We first show that Zorn's lemma (Lem. I.8.5.15) implies Hausdorff's maximality principle. Suppose that $(X, \leq)$ is partially ordered. Let $X_T = \{Y \subseteq X : (Y, \leq) \text{ is totally ordered}\}$. By Ex. I.8.5.1 we know that $(\emptyset, \leq)$ is totally ordered, thus $\emptyset \in X_T$ and $X_T \neq \emptyset$. By E.g. I.8.5.2 we know that $(X_T, \subseteq)$ is partially ordered.

Let $S \subseteq X_T$ such that $(S, \subseteq)$ is totally ordered. We claim that $\bigcup S \in X_T$. Clearly, $\bigcup S \subseteq X$. If $S = \emptyset$, then $\bigcup S = \emptyset$ and $\emptyset \in X_T$. So suppose that $S \neq \emptyset$. Since

$$\forall x \in \bigcup S, \exists Y \in S : x \in Y \implies x \leq x,$$

we know that $(\bigcup S, \leq)$ is reflexive. Since

$$\forall x_1, x_2 \in \bigcup S, (x_1 \leq x_2) \wedge (x_2 \leq x_1)$$

$$
\begin{aligned}
&\implies \exists Y_1, Y_2 \in S : (x_1 \in Y_1) \wedge (x_2 \in Y_2) \\
&\qquad \wedge (Y_1 \subseteq Y_2 \vee Y_2 \subseteq Y_1) && ((S, \subseteq) \text{ is totally ordered}) \\
&\implies (x_1, x_2 \in Y_1) \vee (x_1, x_2 \in Y_2) \\
&\implies x_1 = x_2,
\end{aligned}
$$

(the last implication follows since $(Y_1, \leq)$, $(Y_2, \leq)$ are partially ordered) we know that $(\bigcup S, \leq)$ is anti-symmetric. Since

$$
\begin{aligned}
&\forall x_1, x_2, x_3 \in \bigcup S, (x_1 \leq x_2) \wedge (x_2 \leq x_3) \\
\implies &\exists Y_1, Y_2, Y_3 \in S : \\
&(x_1 \in Y_1) \wedge (x_2 \in Y_2) \wedge (x_3 \in Y_3) \\
&\wedge (Y_1 \subseteq Y_2 \vee Y_2 \subseteq Y_1) && ((S, \subseteq) \text{ is totally ordered}) \\
&\wedge (Y_1 \subseteq Y_3 \vee Y_3 \subseteq Y_1) \\
&\wedge (Y_2 \subseteq Y_3 \vee Y_3 \subseteq Y_2) \\
\implies &(x_1, x_2, x_3 \in Y_1) \vee (x_1, x_2, x_3 \in Y_2) \vee (x_1, x_2, x_3 \in Y_3) \\
\implies &x_1 \leq x_3,
\end{aligned}
$$

(the last implication follows since $(Y_1, \leq)$, $(Y_2, \leq)$, $(Y_3, \leq)$ are partially ordered) we know that $(\bigcup S, \leq)$ is transitive. Since $(\bigcup S, \leq)$ is reflexive, anti-symmetric and transitive, by Def. I.8.5.1 we know that $(\bigcup S, \leq)$ is partially ordered. Since

$$
\begin{aligned}
&\forall x_1, x_2 \in \bigcup S, \exists Y_1, Y_2 \in S : \\
&(x_1 \in Y_1) \wedge (x_2 \in Y_2) \\
&\wedge (Y_1 \subseteq Y_2 \vee Y_2 \subseteq Y_1) && ((S, \subseteq) \text{ is totally ordered}) \\
\implies &(x_1, x_2 \in Y_1) \vee (x_1, x_2 \in Y_2) \\
\implies &(x_1 \leq x_2) \vee (x_2 \leq x_1), && ((Y_1, \leq), (Y_2, \leq) \text{ are totally ordered})
\end{aligned}
$$

by Def. I.8.5.3 we know that $(\bigcup S, \leq)$ is totally ordered. Thus, $\bigcup S \in X_T$.

Next we show that every totally ordered subset of X_T has an upper bound. Let $S \subseteq X_T$ such that $(S, \subseteq)$ is totally ordered. Since $\bigcup S \in X_T$ and

$$\forall Y \in S, Y \subseteq \bigcup S,$$

by Def. I.8.5.12 we know that $\bigcup S$ is an upper bound of S (under the relation $\subseteq$). Thus, the claim follows.

By Zorn's lemma (Lem. I.8.5.15) we know that $\max\big((X_T, \subseteq)\big)$ exists. By Def. I.8.5.5 we know that

$$\forall Y \in X_T, Y \subseteq \max\big((X_T, \subseteq)\big).$$

Thus, we conclude that Zorn's lemma implies Hausdorff's maximality principle.

Finally we show that Hausdorff's maximality principle implies Zorn's lemma. Suppose that $(X, \leq)$ is partially ordered and every totally ordered subset of X has an upper bound. Let $X_T = \{Y \subseteq X : (Y, \leq) \text{ is totally ordered}\}$. By hypothesis we know that $Y_{\max} = \max\big((X_T, \subseteq)\big)$ exists. We want to show that $\max\big((X, \leq)\big)$ exists.

Suppose for the sake of contradiction that $\max\big((X, \leq)\big)$ does not exists. Then every totally ordered subset of X must have a strictly upper bound. To be precise, let $Y \subseteq X$ such that $(Y, \leq)$ is totally ordered. Let $u(Y)$ be the upper bound of Y. Then $\exists s(Y) \in X$ such that $s(Y)$ is a strict upper bound of Y, i.e., $u(Y) < s(Y)$. If such strict upper bound $s(Y)$ does not exists, then by Def. I.8.5.5 we must have $u(Y) = \max\big((X, \leq)\big)$, a contradiction. Since $(Y_{\max}, \leq)$ is totally ordered, we know that $\exists s(Y_{\max}) \in X$ such that $u(Y_{\max}) < s(Y_{\max})$. But then $\big(Y_{\max} \cup \{s(Y_{\max})\}, \leq\big)$ is totally ordered and $Y_{\max} \subsetneq Y_{\max} \cup \{s(Y_{\max})\}$, a contradiction. Thus, $\max\big((X, \leq)\big)$ must exists. We conclude that Zorn's lemma, Hausdorff's maximality principle and axiom of choice are logically equivalent. □

Ex. I.8.5.19. Let X be a set, and let Ω be the space of all pairs $(Y, \leq)$, where Y is a subset of X and $\leq$ is a well-ordering of Y. If $(Y, \leq)$ and $(Y', \leq')$ are elements of Ω, we say that $(Y, \leq)$ is an *initial segment* of $(Y', \leq')$ if there exists an $x \in Y'$ such that $Y := \big\{y \in Y' : y <' x\big\}$ (so, in particular, $Y \subsetneq Y'$), and for any $y, y' \in Y$, $y \leq y'$ iff $y \leq' y'$. Define a relation $\preceq$ on Ω by defining $(Y, \leq) \preceq (Y', \leq')$ if either $(Y, \leq) = (Y', \leq')$, or if $(Y, \leq)$ is an initial segment of $(Y', \leq')$. Show that $\preceq$ is a partial ordering of Ω. There is exactly one minimal element of Ω; what is it? Show that the maximal elements of Ω are precisely the well-orderings $(X, \leq)$ of X. Using Zorn's lemma (Lem. I.8.5.15), conclude the *well-ordering principle*: every set X has at least one well-ordering. Conversely, use the well-ordering principle to prove the axiom of choice, Ax. I.8.1. We thus see that the axiom of choice, Zorn's lemma (Lem. I.8.5.15), and the well-ordering principle are all logically equivalent to each other.

Proof. We first show that $(\Omega, \preceq)$ is partially ordered. Since

$$\forall (Y, \leq), \in \Omega, (Y, \leq) = (Y, \leq) \implies (Y, \leq) \preceq (Y, \leq),$$

we know that $(\Omega, \preceq)$ is reflexive. Since

$$\begin{aligned}
&\forall (Y_1, \leq_1), (Y_2, \leq_2) \in \Omega, \big((Y_1, \leq_1) \preceq (Y_2, \leq_2)\big) \wedge \big((Y_2, \leq_2) \preceq (Y_1, \leq_1)\big) \\
\iff &\begin{cases}
\big((Y_1, \leq_1) = (Y_2, \leq_2)\big) \wedge \big((Y_2, \leq_2) = (Y_1, \leq_1)\big) \\
\big((Y_1, \leq_1) = (Y_2, \leq_2)\big) \wedge (Y_2 \text{ is an initial segment of } Y_1) \\
(Y_1 \text{ is an initial segment of } Y_2) \wedge \big((Y_2, \leq_2) = (Y_1, \leq_1)\big) \\
(Y_1 \text{ is an initial segment of } Y_2) \wedge (Y_2 \text{ is an initial segment of } Y_1)
\end{cases} \\
\iff &\begin{cases}
(Y_1, \leq_1) = (Y_2, \leq_2) \\
(Y_1 = Y_2) \wedge (Y_2 \subsetneq Y_1) \\
(Y_1 \subsetneq Y_2) \wedge (Y_1 = Y_2) \\
(Y_1 \subsetneq Y_2) \wedge (Y_2 \subsetneq Y_1)
\end{cases}
\end{aligned}$$

$$\begin{aligned}&\iff \big((Y_1,\leq_1)=(Y_2,\leq_2)\big)\vee(\text{false})\\&\iff (Y_1,\leq_1)=(Y_2,\leq_2),\end{aligned}$$

we know that $(\Omega,\preceq)$ is anti-symmetric. Since

$$\forall (Y_1,\leq_1),(Y_2,\leq_2),(Y_3,\leq_3)\in\Omega, \big((Y_1,\leq_1)\preceq(Y_2,\leq_2)\big)\wedge\big((Y_2,\leq_2)\preceq(Y_3,\leq_3)\big)$$

$$\iff \begin{cases}\big((Y_1,\leq_1)=(Y_2,\leq_2)\big)\wedge\big((Y_2,\leq_2)=(Y_3,\leq_3)\big)\\ \big((Y_1,\leq_1)=(Y_2,\leq_2)\big)\wedge(Y_2\text{ is an initial segment of }Y_3)\\ (Y_1\text{ is an initial segment of }Y_2)\wedge\big((Y_2,\leq_2)=(Y_3,\leq_3)\big)\\ (Y_1\text{ is an initial segment of }Y_2)\wedge(Y_2\text{ is an initial segment of }Y_3)\end{cases}$$

$$\iff \begin{cases}(Y_1,\leq_1)=(Y_3,\leq_3)\\ Y_1\text{ is an initial segment of }Y_3\\ Y_1\text{ is an initial segment of }Y_3\\ \big((\exists y_2\in Y_2: Y_1=\{y\in Y_2: y<_2 y_2\})\\ \quad\wedge(\forall y_1,y_1'\in Y_1, y_1\leq_1 y_1' \iff y_1\leq_2 y_1')\\ \quad\wedge(\exists y_3\in Y_3: Y_2=\{y\in Y_3: y<_3 y_3\})\\ \quad\wedge(\forall y_2,y_2'\in Y_2, y_2\leq_2 y_2' \iff y_2\leq_3 y_2')\big)\end{cases}$$

$$\iff \begin{cases}(Y_1,\leq_1)=(Y_3,\leq_3)\\ Y_1\text{ is an initial segment of }Y_3\\ Y_1\text{ is an initial segment of }Y_3\\ \big((\exists y_3\in Y_3: Y_1=\{y\in Y_3: y<_2 y_3\})\\ \quad\wedge(\forall y_1,y_1'\in Y_1, y_1\leq_1 y_1' \iff y_1\leq_3 y_1')\big)\end{cases}$$

$$\begin{aligned}&\iff \big((Y_1,\leq_1)=(Y_3,\leq_3)\big)\vee(Y_1\text{ is an initial segment of }Y_3)\\&\iff (Y_1,\leq_1)\preceq(Y_3,\leq_3),\end{aligned}$$

we know that $(\Omega,\preceq)$ is transitive. Since $(\Omega,\preceq)$ is reflexive, anti-symmetric and transitive, by Def. I.8.5.1 we know that $(\Omega,\preceq)$ is partially ordered.

Next we show that $\min\big((\Omega,\preceq)\big)$ exists. We claim that $(\emptyset,\leq_\emptyset)=\min\big((\Omega,\preceq)\big)$. By Ex. I.8.5.1 we know that $(\emptyset,\leq_\emptyset)$ is well-ordered. Let $(Y,\leq)\in\Omega$. Now we split into two cases:

- If $Y=\emptyset$, then we know that

$$\forall y_1,y_2\in\emptyset, y_1\leq_\emptyset y_2 \iff y_1\leq y_2$$

  is vacuously true. Thus, we have $(\emptyset,\leq_\emptyset)=(\emptyset,\leq)$ and $(\emptyset,\leq_\emptyset)\preceq(\emptyset,\leq)$.

- If $Y\neq\emptyset$, then by Def. I.8.5.8 we know that $y_{\min}=\min\big((Y,\leq)\big)$ exists. By Def. I.8.5.5 we know that

$$\emptyset=\{y\in Y: y<y_{\min}\}.$$

Since

$$\forall y_1, y_2 \in \emptyset, y_1 \leq_\emptyset y_2 \iff y_1 \leq y_2$$

is vacuously true, we know that $\emptyset$ is an initial segment of $(Y, \leq)$. Thus, $(\emptyset, \leq_\emptyset) \preceq (\emptyset, \leq)$.

From all cases above, we conclude that $(\emptyset, \leq_\emptyset) \preceq (\emptyset, \leq)$. Since $(Y, \leq)$ was arbitrary, we know that $(\emptyset, \leq_\emptyset) = \min\big((\Omega, \preceq)\big)$.

Next we show that if $S \subseteq \Omega$ and $(S, \preceq)$ is well-ordered, then there exists a well-ordered relation $\leq_S$ on $\bigcup S$ and $(\bigcup S, \leq_S) \in \Omega$. If $S = \emptyset \vee S = \{\emptyset\}$, then $\bigcup S = \emptyset$. By Ex. I.8.5.1 we know that $(\emptyset, \leq_\emptyset)$ is well-ordered, thus $(\emptyset, \leq_\emptyset) \in \Omega$. So suppose that $S \neq \emptyset \wedge S \neq \{\emptyset\}$. Observe that

$$\begin{aligned}
&\forall y_1, y_2 \in \bigcup S \\
&\implies \exists (Y_1, \leq_1), (Y_2, \leq_2) \in S : (y_1 \in Y_1) \wedge (y_2 \in Y_2) \\
&\implies \big((Y_1, \leq_1) \preceq (Y_2, \leq_2)\big) \vee \big((Y_2, \leq_2) \preceq (Y_1, \leq_1)\big) && ((S, \preceq) \text{ is well-ordered}) \\
&\implies \begin{cases} (Y_1, \leq_1) = (Y_2, \leq_2) \\ Y_1 \text{ is an initial segment of } Y_2 \\ Y_2 \text{ is an initial segment of } Y_1 \end{cases} \\
&\implies (Y_1 \subseteq Y_2) \vee (Y_2 \subseteq Y_1) \\
&\implies (y_1 \leq_1 y_2) \vee (y_1 \leq_2 y_2).
\end{aligned}$$

We know that $\exists (Y_S, \leq_S) \in S$ such that $\forall y \in \bigcup S \implies y \in Y_S$. If not, then we would have some $y_1, y_2 \in \bigcup S$ such that

$$(y_1 \in Y_S) \wedge (y_2 \in Y_S') \wedge (y_1 \notin Y_S' \vee y_2 \notin Y_S) \text{ for some } (Y_S, \leq_S), (Y_S', \leq_S') \in S.$$

But this contradicts $Y_S \subseteq Y_S' \vee Y_S' \subseteq Y_S$. So such $(Y_S, \leq_S)$ must exists. Since $(Y_S, \leq_S)$ is well-ordered and $\bigcup S \subseteq Y_S$, by Def. I.8.5.8 $(\bigcup S, \leq_S) \in \Omega$.

Next we show that if $S \subseteq \Omega$ and $(S, \preceq)$ is totally ordered, then there exists an upper bound of $(S, \preceq)$ in Ω. Since $(\bigcup S, \leq_S) \in \Omega$ and

$$\begin{aligned}
&\forall (Y, \leq) \in S, Y \subseteq \bigcup S \\
&\implies \begin{cases} Y = \bigcup S \\ Y \subsetneq \bigcup S \end{cases} \\
&\implies \begin{cases} (Y, \leq) = (\bigcup S, \leq_S) \\ Y \text{ is an initial segment of } \bigcup S \end{cases} \\
&\implies (Y, \leq) \preceq (\bigcup S, \leq_S),
\end{aligned}$$

we know that $(\bigcup S, \leq_S)$ is an upper bound of $(S, \preceq)$. Since S was arbitrary, we conclude that every totally ordered subset of Ω has an upper bound.

By Zorn's lemma (Lem. I.8.5.15) we know that $(Y_{\max}, \leq_{\max}) = \max\big((\Omega, \preceq)\big)$ exists. By the definition of Ω we know that $(Y_{\max}, \leq_{\max})$ is well-ordered. Now we show that $X = Y_{\max}$. By the definition of Ω we know that $Y_{\max} \subseteq X$. So we only need to show that $X \subseteq Y_{\max}$. Suppose for the sake of contradiction that $\exists y \in X$ such that $y \notin Y_{\max}$. We claim that $(\{y\}, =)$ is well-ordered. Since

$$\forall y_1, y_2 \in \{y\}, (y_1 = y_2) \wedge (y_2 = y_1) \implies y_1 = y_2 = y,$$

we know that $(\{y\}, =)$ is anti-symmetric. Since $(\{y\}, =)$ is reflexive, anti-symmetric and transitive, by Def. I.8.5.1 we know that $(\{y\}, =)$ is partially ordered. Since

$$\forall y_1, y_2 \in \{y\}, y_1 = y_2 = y,$$

by Def. I.8.5.3 we know that $(\{y\}, =)$ is totally ordered. Since every non-emptyset subset of $\{y\} = \{y\}$, by Def. I.8.5.5 $\min\big((\{y\}, =)\big) = y$ and by Def. I.8.5.8 $(\{y\}, =)$ is well-ordered. Then we have $(\{y\}, =) \in \Omega$ and $(\{y\}, =) \preceq (Y_{\max}, \leq_{\max})$, which means $y \in Y_{\max}$, a contradiction. Thus, $X \subseteq Y_{\max}$, $X = Y_{\max}$ and $(X, \leq_{\max})$ is well-ordered. We conclude that the well-ordering principle is true, i.e., every X has at least one well-ordering $\max\big((\Omega, \preceq)\big)$.

Finally we use well-ordering principle to prove axiom of choice (Ax. I.8.1). Suppose that every set has at least one well-ordering. Let I be a set and let every $\alpha \in I$ map to a set X_α. Since $\bigcup\limits_{\alpha \in I} X_\alpha$ is a set, we know that there exists a well-ordering $\leq$ on $\bigcup\limits_{\alpha \in I} X_\alpha$. Since $X_\beta \subseteq \bigcup\limits_{\alpha \in I} X_\alpha$ for every $\beta \in I$, by Def. I.8.5.8 we know that $\min\big((X_\beta, \leq)\big)$ exists. Now we define $f : I \to \bigcup\limits_{\alpha \in I} X_\alpha$ as follow:

$$\forall \alpha \in I, f(\alpha) = \min\big((X_\alpha, \leq)\big)$$

Then we know that $f \in \prod\limits_{\alpha \in I} X_\alpha$, so $\prod\limits_{\alpha \in I} X_\alpha \neq \emptyset$ and axiom of choice is true. □

Ex. I.8.5.20. Let X be a set, and let $\Omega \subseteq 2^X$ be a collection of subsets of X. Assume that Ω does not contain the empty set $\emptyset$. Using Zorn's lemma, show that there is a subcollection $\Omega' \subseteq \Omega$ such that all the elements of Ω' are disjoint from each other (i.e., $A \cap B = \emptyset$ whenever A, B are distinct elements of Ω'), but that all the elements of Ω intersect at least one element of Ω' (i.e., for all $C \in \Omega$ there exists $A \in \Omega'$ such that $C \cap A \neq \emptyset$). Conversely, if the above claim is true, show that it implies the claim in Ex. I.8.4.2, and thus this is yet another claim which is logically equivalent to the axiom of choice.

Proof. Let S be the set

$$S = \{A \subseteq \Omega : \forall A_1, A_2 \in A, A_1 \neq A_2 \implies A_1 \cap A_2 = \emptyset\}.$$

By E.g. I.8.5.2 we know that $(S, \subseteq)$ is partially ordered. Let $S_T \subseteq S$ such that $(S_T, \subseteq)$ is totally ordered. We claim that $\bigcup S_T \in S$ and $\bigcup S_T$ is an upper bound of $(S_T, \subseteq)$. Clearly, $\bigcup S_T \subseteq \Omega$. Since

$$\begin{aligned}
&\forall A_1, A_2 \in \bigcup S_T, A_1 \neq A_2 \\
\implies &\exists S_1, S_2 \in S_T : (A_1 \in S_1) \wedge (A_2 \in S_2) \\
\implies &(S_1 \subseteq S_2) \vee (S_2 \subseteq S_1) && ((S_T, \subseteq) \text{ is totally ordered}) \\
\implies &(A_1, A_2 \in S_1) \vee (A_1, A_2 \in S_2) \\
\implies &A_1 \cap A_2 = \emptyset,
\end{aligned}$$

we know that $\bigcup S_T \in S$. Since $\bigcup S_T \in S$ and

$$\forall A \in S_T, A \subseteq \bigcup S_T,$$

we know that $\bigcup S_T$ is an upper bound of $(S_T, \subseteq)$. Thus, we conclude that every totally ordered subset of S has an upper bounded.

By Zorn's lemma (Lem. I.8.5.15) we know that $\Omega' = \max\big((S, \subseteq)\big)$ exists. Clearly, $\Omega' \subseteq \Omega$. Since $\Omega' \in S$, we know that all the elements of Ω' are disjoint from each other. Now we show that $\forall C \in \Omega$, $\exists A \in \Omega'$ such that $C \cap A \neq \emptyset$. Suppose for the sake of contradiction that $\exists C \in \Omega$ such that $\forall A \in \Omega'$, $C \cap A = \emptyset$. By hypothesis we know that $C \neq \emptyset$. Then we have $\{C\} \cup \Omega' \in S$ and $\Omega' \subsetneq \{C\} \cup \Omega'$, a contradiction. Thus, the claim is true.

Now we show that if for any set X, $\Omega \subseteq 2^X$ and $\exists \Omega' \subseteq \Omega$ such that

$$\forall A_1, A_2 \in \Omega', A_1 \neq A_2 \implies A_1 \cap A_2$$

and

$$\forall C \in \Omega, \exists A \in \Omega' : C \cap A \neq \emptyset,$$

then Ex. I.8.4.2 is true. Suppose that I is a set and for each $\alpha \in I$, X_α is a set and $X_\alpha \neq \emptyset$. Suppose that $\forall \alpha, \beta \in I$, $X_\alpha \cap X_\beta = \emptyset$. We want to show that $\exists Y$ such that $\#(Y \cap X_\alpha) = 1$ for all $\alpha \in I$. Let X be the set

$$X = (\{0\} \times I) \cup (\{1\} \times \bigcup_{\alpha \in I} X_\alpha),$$

let $\Omega \subseteq 2^X$ be the set

$$\Omega = \{\{(0, \alpha), (1, x_\alpha)\} : (\alpha \in I) \wedge (x_\alpha \in X_\alpha)\}.$$

We know that $\exists \Omega' \subseteq \Omega$ satisfying the hypothesis. Let Y be the set

$$Y = \left\{ x_\beta \in \bigcup_{\alpha \in I} X_\alpha | \exists \beta \in I : \{(0, \beta), (1, x_\beta)\} \in \Omega' \right\}.$$

We claim that $\#(Y \cap X_\alpha) = 1$ for all $\alpha \in I$. Since

$$
\begin{aligned}
& \forall \alpha \in I, X_\alpha \neq \emptyset \\
\implies & \exists x_\alpha \in X_\alpha \\
\implies & \exists \{(0,\alpha),(1,x_\alpha)\} \in \Omega \\
\implies & \exists \{(0,\beta),(1,x_\beta)\} \in \Omega' : \{(0,\alpha),(1,x_\alpha)\} \cap \{(0,\beta),(1,x_\beta)\} \neq \emptyset && \text{(by hypothesis)} \\
\implies & (\alpha \neq \beta \implies X_\alpha \cap X_\beta = \emptyset \implies x_\alpha \neq x_\beta) && \text{(by hypothesis)} \\
\implies & (\alpha = \beta) \wedge (x_\alpha, x_\beta \in X_\alpha = X_\beta) \\
\implies & (x_\alpha \in Y) \wedge (Y \cap X_\alpha \neq \emptyset)
\end{aligned}
$$

and

$$
\begin{aligned}
& \forall x_\alpha, x_\beta \in Y \\
\implies & \exists \alpha, \beta \in I : \{(0,\alpha),(1,x_\alpha)\}, \{(0,\beta),(1,x_\beta)\} \in \Omega' \\
\implies & \{(0,\alpha),(1,x_\alpha)\} \cap \{(0,\beta),(1,x_\beta)\} = \emptyset && \text{(by hypothesis)} \\
\implies & (\alpha \neq \beta) \wedge (x_\alpha \neq x_\beta) \\
\implies & \#(Y \cap X_\alpha) = \#(Y \cap X_\beta) = 1,
\end{aligned}
$$

we conclude that Ex. I.8.4.2 is true. □

Chapter I.9

Continuous functions on R

Note. Roughly speaking a set is discrete if each element is separated from the rest of the set by some non-zero distance, whereas a set is a *continuum* if it is connected and contains no "holes."

I.9.1 Subsets of the real line

Def. I.9.1.1 (Intervals)**.** Let $a, b \in \mathbb{R}^*$ be extended real numbers. We define the *closed interval* $[a, b]$ by

$$[a, b] := \{x \in \mathbb{R}^* : a \leq x \leq b\},$$

the *half-open intervals* $[a, b)$ and $(a, b]$ by

$$[a, b) := \{x \in \mathbb{R}^* : a \leq x < b\}; (a, b] := \{x \in \mathbb{R}^* : a < x \leq b\},$$

and the *open interval* (a, b) by

$$(a, b) := \{x \in \mathbb{R}^* : a < x < b\}.$$

We call a the *left endpoint* of these intervals, and b the *right endpoint.*

Rmk. I.9.1.2. Once again, we are overloading the parenthesis notation; for instance, we are now using $(2, 3)$ to denote both an open interval from 2 to 3, as well as an ordered pair in the Cartesian plane $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$. This can cause some genuine ambiguity, but the reader should still be able to resolve which meaning of the parentheses is intended from context. In some texts, this issue is resolved by using reversed brackets instead of parenthesis, thus for instance $[a, b)$ would now be $[a, b[$, $(a, b]$ would be $]a, b]$, and (a, b) would be $]a, b[$.

E.g. I.9.1.3. The positive real axis $\{x \in \mathbb{R} : x > 0\}$ is the open interval $(0, +\infty)$, while the non-negative real axis $\{x \in \mathbb{R} : x \geq 0\}$ is the half-open interval $[0, +\infty)$. Similarly, the negative real axis $\{x \in \mathbb{R} : x < 0\}$ is $(-\infty, 0)$, and the non-positive real axis $\{x \in \mathbb{R} : x \leq 0\}$ is $(-\infty, 0]$. Finally, the real line $\mathbb{R}$ itself is the open interval $(-\infty, +\infty)$, while the extended real line $\mathbb{R}^*$ is the closed interval $[-\infty, +\infty]$. We sometimes refer to an interval in which one endpoint is infinite (either $+\infty$ or $-\infty$) as *half-infinite* intervals, and intervals in which both endpoints are infinite as *doubly-infinite* intervals; all other intervals are *bounded intervals*. Thus, the positive and negative real axes are half-infinite intervals, and $\mathbb{R}$ and $\mathbb{R}^*$ are infinite intervals.

E.g. I.9.1.4. If $a > b$ then all four of the intervals $[a, b], [a, b), (a, b]$, and (a, b) are the empty set (by trichotomy, see Prop. I.5.4.7(a)). If $a = b$, then the three intervals $[a, b), (a, b]$, and (a, b) are the empty set, while $[a, b]$ is just the singleton set $\{a\}$. Because of this, we call these intervals *degenerate*; most (but not all) of our analysis will be restricted to non-degenerate intervals.

Def. I.9.1.5 (ε-adherent points)**.** Let X be a subset of $\mathbb{R}$, let $\varepsilon > 0$, and let $x \in \mathbb{R}$. We say that x is *ε-adherent to X* iff there exists a $y \in X$ which is ε-close to x (i.e., $|x - y| \le \varepsilon$).

Rmk. I.9.1.6. The terminology "ε-adherent" is not standard in the literature. However, we shall shortly use it to define the notion of an adherent point, which is standard.

Def. I.9.1.8 (Adherent points)**.** Let X be a subset of $\mathbb{R}$, and let $x \in \mathbb{R}$. We say that x is an *adherent point* of X iff it is ε-adherent to X for every $\varepsilon > 0$.

Def. I.9.1.10 (Closure)**.** Let X be a subset of $\mathbb{R}$. The *closure* of X, sometimes denoted $\overline{X}$ is defined to be the set of all the adherent points of X.

Lem. I.9.1.11 (Elementary properties of closures)**.** Let X and Y be arbitrary subsets of $\mathbb{R}$. Then $X \subseteq \overline{X}$, $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$, and $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$. If $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$.

Proof. We first show that $X \subseteq \overline{X}$. Since

$$\begin{aligned}
&\forall x \in X, |x - x| = 0 \\
\Longrightarrow\ &\forall \varepsilon \in \mathbb{R}^+, |x - x| \le \varepsilon \\
\Longrightarrow\ &x \in \overline{X}, && \text{(by Def. I.9.1.10)}
\end{aligned}$$

by Def. I.3.1.15 we have $X \subseteq \overline{X}$.

Next we show that $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$. Since

$$\begin{aligned}
&\forall x \in \overline{X} \cup \overline{Y} \\
\Longrightarrow\ &x \in \overline{X} \vee x \in \overline{Y} && \text{(by Ax. I.3.4)} \\
\Longrightarrow\ &(\exists a \in X : |x - a| \le \varepsilon) \vee (\exists a \in Y : |x - a| \le \varepsilon) && \text{(by Def. I.9.1.10)} \\
\Longrightarrow\ &\exists a \in X \cup Y : |x - a| \le \varepsilon && \text{(by Ax. I.3.4)} \\
\Longrightarrow\ &x \in \overline{X \cup Y} && \text{(by Def. I.9.1.10)}
\end{aligned}$$

and

$$\forall \varepsilon \in \mathbb{R}^+, \forall x \in \overline{X \cup Y}, \exists a \in X \cup Y : |x - a| \le \varepsilon \qquad \text{(by Def. I.9.1.10)}$$

$$
\begin{aligned}
&\implies \begin{cases} \exists \varepsilon' \in \mathbb{R}^+ : \forall b \in X, |x-b| > \varepsilon' & \text{if } x \notin \overline{X} \\ \exists \varepsilon' \in \mathbb{R}^+ : \forall b \in Y, |x-b| > \varepsilon' & \text{if } x \notin \overline{Y} \end{cases} && \text{(by Def. I.9.1.8)} \\
&\implies \begin{cases} (a \in Y) \wedge (|x-a| < \varepsilon') & \text{if } x \notin \overline{X} \\ (a \in X) \wedge (|x-a| < \varepsilon') & \text{if } x \notin \overline{Y} \end{cases} \\
&\implies \begin{cases} x \in \overline{Y} & \text{if } x \notin \overline{X} \\ x \in \overline{X} & \text{if } x \notin \overline{Y} \end{cases} && \text{(by Def. I.9.1.10)} \\
&\implies x \in \overline{X} \cup \overline{Y}, && \text{(by Ax. I.3.4)}
\end{aligned}
$$

by Prop. I.3.1.18 we have $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$.

Next we show that $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$. Since

$$
\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \forall x \in \overline{X \cap Y}, \exists y \in X \cap Y : |x-y| \le \varepsilon && \text{(by Def. I.9.1.10)} \\
\implies &(y \in X) \wedge (y \in Y) && \text{(by Def. I.3.1.23)} \\
\implies &(x \in \overline{X}) \wedge (x \in \overline{Y}) && \text{(by Def. I.9.1.10)} \\
\implies &x \in \overline{X} \cap \overline{Y}, && \text{(by Def. I.3.1.23)}
\end{aligned}
$$

by Def. I.3.1.15 we have $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$.

Finally we show that $X \subseteq Y \implies \overline{X} \subseteq \overline{Y}$. Suppose that $X \subseteq Y$. Then we have

$$
\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \forall x \in \overline{X}, \exists y \in X : |x-y| \le \varepsilon && \text{(by Def. I.9.1.10)} \\
\implies &y \in Y && (X \subseteq Y) \\
\implies &x \in \overline{Y}. && \text{(by Def. I.9.1.10)}
\end{aligned}
$$

Thus, by Def. I.3.1.15 we have $\overline{X} \subseteq \overline{Y}$. □

Lem. I.9.1.12 (Closures of intervals)**.** Let $a < b$ be real numbers, and let I be any one of the four intervals (a, b), $(a, b]$, $[a, b)$, or $[a, b]$. Then the closure of I is $[a, b]$. Similarly, the closure of (a, ∞) or $[a, \infty)$ is $[a, \infty)$, while the closure of $(-\infty, a)$ or $(-\infty, a]$ is $(-\infty, a]$. Finally, the closure of $(-\infty, \infty)$ is $(-\infty, \infty)$.

Proof. First, let us show that every element of $[a, b]$ is adherent to (a, b). Let $x \in [a, b]$. If $x \in (a, b)$ then it is definitely adherent to (a, b). This is true since $\forall \varepsilon \in \mathbb{R}^+$ we have $|x - x| \le \varepsilon$. If $x = b$ then x is also adherent to (a, b). Otherwise $\exists \varepsilon \in \mathbb{R}^+$ such that

$$\forall y \in (a, b), |b - y| > \varepsilon.$$

But this means

$$
\begin{aligned}
&|b - y| = b - y > \varepsilon && (y \in (a, b)) \\
\implies &b - \varepsilon > y
\end{aligned}
$$

$$
\begin{aligned}
&\implies b > b - \varepsilon > y > a && (\varepsilon \in \mathbb{R}^+ \wedge y \in (a,b)) \\
&\implies b - \varepsilon \in (a,b) && \text{(by Def. I.9.1.1)} \\
&\implies \varepsilon < |b - (b - \varepsilon)| = \varepsilon,
\end{aligned}
$$

a contradiction. Thus, b is adherent to (a,b). Similarly, when $x = a$. Thus, every point in $[a,b]$ is adherent to (a,b).

Now we show that every point x that is adherent to (a,b) lies in $[a,b]$. Suppose for the sake of contradiction that x does not lie in $[a,b]$, then either $x > b$ or $x < a$. If $x > b$ then x is not $(x-b)$-adherent to (a,b), and is hence not an adherent point to (a,b) (by setting $\varepsilon = x - b$ we have $\forall y \in (a,b)$, $|x-y| = x - y > x - b = \varepsilon$). Similarly, if $x < a$, then x is not $(a-x)$-adherent to (a,b), and is hence not an adherent point to (a,b). This contradiction shows that x is in fact in $[a,b]$ as claimed. Using similar arguments, we can show that $\overline{(a,b]} = \overline{[a,b)} = \overline{[a,b]} = [a,b]$.

Now we show that $\overline{(a,\infty)} = [a,\infty)$. By Lem. I.9.1.11 we know that $(a,\infty) \subseteq \overline{(a,\infty)}$. We also know that a is an adherent point of (a,∞). If not, then $\exists \varepsilon \in \mathbb{R}^+$ such that

$$\forall y \in (a,\infty), |a - y| > \varepsilon.$$

But this means

$$
\begin{aligned}
&|a - y| = y - a > \varepsilon && (y \in (a,\infty)) && \text{(I.9.1)} \\
&\implies y > a + \varepsilon && && \text{(I.9.2)} \\
&\implies b > y > a + \varepsilon > a && (\varepsilon \in \mathbb{R}^+ \wedge y \in (a,b)) && \text{(I.9.3)} \\
&\implies a + \varepsilon \in (a,b) && \text{(by Def. I.9.1.1)} && \text{(I.9.4)} \\
&\implies \varepsilon < |a - (a - \varepsilon)| = \varepsilon, && && \text{(I.9.5)}
\end{aligned}
$$

a contradiction. Thus, a is an adherent point of (a,∞) and $[a,\infty) \subseteq \overline{(a,\infty)}$. Suppose for the sake of contradiction that $\exists x \in \overline{(a,\infty)}$ such that $x \notin [a,\infty)$. Then $x < a$ (by Def. I.9.1.8 $x \in \mathbb{R}$ so $x \neq \infty$). But we know x is not $(a-x)$-adherent to (a,∞), and is hence not an adherent point to (a,b), a contradiction. Thus, $\overline{(a,\infty)} = [a,\infty)$. Using similar arguments, we can show that $\overline{[a,\infty)} = [a,\infty)$ and $\overline{(-\infty,b)} = \overline{(-\infty,b]} = (-\infty,b]$.

Finally we show that $\overline{(-\infty,\infty)} = (-\infty,\infty)$. By Lem. I.9.1.11 we know that $(-\infty,\infty) \subseteq \overline{(-\infty,\infty)}$. By Def. I.9.1.8 we know that $\overline{(-\infty,\infty)} \subseteq \mathbb{R} = (-\infty,\infty)$. Thus, $\overline{(-\infty,\infty)} = (-\infty,\infty)$. □

Lem. I.9.1.13. The closure of $\mathbb{N}$ is $\mathbb{N}$. The closure of $\mathbb{Z}$ is $\mathbb{Z}$. The closure of $\mathbb{Q}$ is $\mathbb{R}$, and the closure of $\mathbb{R}$ is $\mathbb{R}$. The closure of the empty set $\emptyset$ is $\emptyset$.

Proof. We first show that $\overline{\mathbb{N}} = \mathbb{N}$. Let $\overline{\mathbb{N}}$ be the closure of $\mathbb{N}$. By Def. I.9.1.8 we have $\overline{\mathbb{N}} \subseteq \mathbb{R}$. By Lem. I.9.1.11 we have $\mathbb{N} \subseteq \overline{\mathbb{N}}$. Now we show that $\overline{\mathbb{N}} \subseteq \mathbb{N}$. Suppose for the sake of contradiction that $\exists x \in \overline{\mathbb{N}}$ such that $x \notin \mathbb{N}$. Since

$$\mathbb{N} \subseteq [0,\infty) \qquad \text{(by Def. I.9.1.1)}$$

$$
\begin{aligned}
&\implies \overline{\mathbb{N}} \subseteq \overline{[0,\infty)} && \text{(by Lem. I.9.1.11)}\\
&\implies \overline{\mathbb{N}} \subseteq [0,\infty), && \text{(by Lem. I.9.1.12)}
\end{aligned}
$$

we know that $x > 0$. By Prop. I.5.4.12 $\exists n \in \mathbb{N}$ such that $n < x < n+1$. Let $\varepsilon = \min(x-n, n+1-x)/2$. By Def. I.9.1.10, $\exists m \in \mathbb{N}$ such that $|x-m| \le \varepsilon$. We split into two cases:

- If $m \le n$, then we have $x - m \ge x - n \ge \min(x-n, n+1-x) > \varepsilon$, a contradiction.
- If $m > n$, then we have $m \ge n+1$ and $m - x \ge n+1-x \ge \min(x-n, n+1-x) > \varepsilon$, a contradiction.

From all cases above, we derived contradictions. Thus, such m does not exists and by Def. I.9.1.10 $x \notin \overline{\mathbb{N}}$. So we have $\overline{\mathbb{N}} \subseteq \mathbb{N}$. Since $\mathbb{N} \subseteq \overline{\mathbb{N}} \wedge \overline{\mathbb{N}} \subseteq \mathbb{N}$, by Prop. I.3.1.18 we have $\mathbb{N} = \overline{\mathbb{N}}$.

Next we show that $\overline{\mathbb{Z}} = \mathbb{Z}$. Let $\overline{\mathbb{Z}}$ be the closure of $\mathbb{Z}$ and let $\mathbb{Z}^- = \{z \in \mathbb{Z} : z < 0\}$. Then we have

$$
\begin{aligned}
\overline{\mathbb{Z}} &= \overline{\mathbb{N} \cup \mathbb{Z}^-} \\
&= \overline{\mathbb{N}} \cup \overline{\mathbb{Z}^-} && \text{(by Lem. I.9.1.11)}\\
&= \mathbb{N} \cup \overline{\mathbb{Z}^-}. && \text{(from the proof above)}
\end{aligned}
$$

Thus, to show that $\mathbb{Z} = \overline{\mathbb{Z}}$, it suffices to show that $\mathbb{Z}^- = \overline{\mathbb{Z}^-}$. By Lem. I.9.1.11 we have $\mathbb{Z}^- \subseteq \overline{\mathbb{Z}^-}$. We need to show that $\overline{\mathbb{Z}^-} \subseteq \mathbb{Z}^-$. Suppose for the sake of contradiction that $\exists x \in \overline{\mathbb{Z}^-}$ such that $x \notin \mathbb{Z}^-$. Since

$$
\begin{aligned}
&\mathbb{Z}^- \subseteq (-\infty, 0) && \text{(by Def. I.9.1.1)}\\
\implies &\overline{\mathbb{Z}^-} \subseteq \overline{(-\infty, 0)} && \text{(by Lem. I.9.1.11)}\\
\implies &\overline{\mathbb{Z}^-} \subseteq (-\infty, 0], && \text{(by Lem. I.9.1.12)}
\end{aligned}
$$

we know that $x < 0$. By Prop. I.5.4.12 $\exists n \in \mathbb{Z}^-$ such that $n < x < n+1$. Let $\varepsilon = \min(x-n, n+1-x)/2$. By Def. I.9.1.10, $\exists m \in \mathbb{Z}^-$ such that $|x-m| \le \varepsilon$. We split into two cases:

- If $m \le n$, then we have $x - m \ge x - n \ge \min(x-n, n+1-x) > \varepsilon$, a contradiction.
- If $m > n$, then we have $m \ge n+1$ and $m - x \ge n+1-x \ge \min(x-n, n+1-x) > \varepsilon$, a contradiction.

From all cases above, we derived contradictions. Thus, such m does not exists and by Def. I.9.1.10 $x \notin \overline{\mathbb{Z}^-}$. So we have $\overline{\mathbb{Z}^-} \subseteq \mathbb{Z}^-$. Since $\mathbb{Z}^- \subseteq \overline{\mathbb{Z}^-} \wedge \overline{\mathbb{Z}^-} \subseteq \mathbb{Z}^-$, by Prop. I.3.1.18 we have $\mathbb{Z}^- = \overline{\mathbb{Z}^-}$, and thus $\mathbb{Z} = \overline{\mathbb{Z}}$.

Next we show that $\overline{\mathbb{Q}} = \mathbb{R}$. Let $\overline{\mathbb{Q}}$ be the closure of $\mathbb{Q}$. We have

$$
\mathbb{Q} \subseteq \mathbb{R} = (-\infty, \infty) \qquad \text{(by Def. I.9.1.1)}
$$

$$\begin{aligned} &\implies \overline{\mathbb{Q}} \subseteq \overline{(-\infty,\infty)} && \text{(by Lem. I.9.1.11)} \\ &\implies \overline{\mathbb{Q}} \subseteq (-\infty,\infty) = \mathbb{R}. && \text{(by Lem. I.9.1.12)} \end{aligned}$$

Since

$$\begin{aligned} &\forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, x - \varepsilon < x < x + \varepsilon \\ \implies &\exists q \in \mathbb{Q} : x - \varepsilon < q < x + \varepsilon && \text{(by Prop. I.5.4.14)} \\ \implies &|x - q| < \varepsilon \\ \implies &x \in \overline{\mathbb{Q}}, && \text{(by Def. I.9.1.10)} \end{aligned}$$

by Def. I.3.1.15 we have $\mathbb{R} \subseteq \overline{\mathbb{Q}}$. Since $\mathbb{R} \subseteq \overline{\mathbb{Q}} \wedge \overline{\mathbb{Q}} \subseteq \mathbb{R}$, by Prop. I.3.1.18 we have $\mathbb{R} = \overline{\mathbb{Q}}$.

Next we show that $\overline{\mathbb{R}} = \mathbb{R}$. Since

$$\begin{aligned} &\mathbb{R} = (-\infty,\infty) && \text{(by Def. I.9.1.1)} \\ \iff &\overline{\mathbb{R}} = \overline{(-\infty,\infty)} = (-\infty,\infty), && \text{(by Lem. I.9.1.12)} \end{aligned}$$

we know that $\overline{\mathbb{R}} = \mathbb{R}$.

Finally we show that $\overline{\emptyset} = \emptyset$. Suppose for the sake of contradiction that $\overline{\emptyset} \neq \emptyset$. Let $x \in \overline{\emptyset}$ Then by Def. I.9.1.10 $\forall \varepsilon \in \mathbb{R}^+$, $\exists y \in \emptyset$ such that $|x - y| \leq \varepsilon$, a contradiction. Thus, $\overline{\emptyset} = \emptyset$. $\square$

Lem. I.9.1.14. Let X be a subset of $\mathbb{R}$, and let $x \in \mathbb{R}$. Then x is an adherent point of X iff there exists a sequence $(a_n)_{n=0}^{\infty}$, consisting entirely of elements in X, which converges to x.

Proof. We first show that if x is an adherent point of X, then there exists a sequence $(a_n)_{n=0}^{\infty}$ such that $\forall n \in \mathbb{N}$, $a_n \in X$ and $\lim_{n\to\infty} a_n = x$. For each $n \in \mathbb{N}$ let A_n be the set

$$A_n = \left\{ y \in X : |x - y| \leq \frac{1}{n} \right\}.$$

We know by Def. I.9.1.10 that $A_n \neq \emptyset$. By axiom of choice (Ax. I.8.1) we know $\prod_{n\in\mathbb{N}} A_n \neq \emptyset$. Let $f \in \prod_{n\in\mathbb{N}} A_n$. We can define a sequence $(a_n)_{n=0}^{\infty}$ by setting $a_n = f(n)$. Then we have

$$\begin{aligned} &\forall n \in \mathbb{N}, a_n \in A_n \\ \implies &0 \leq |x - a_n| \leq \frac{1}{n} \\ \implies &\lim_{n\to\infty} |x - a_n| = 0 && \text{(by Cor. I.6.4.14)} \\ \implies &\lim_{n\to\infty} x - a_n = 0 && \text{(by Cor. I.6.4.17)} \\ \implies &x = \lim_{n\to\infty} a_n. && \text{(by Thm. I.6.1.19)} \end{aligned}$$

Now we show that if there exists a sequence $(a_n)_{n=0}^{\infty}$ such that $\forall n \in \mathbb{N}$, $a_n \in X$ and $\lim_{n\to\infty} a_n = x$, then x is an adherent point of X. Since $\lim_{n\to\infty} a_n = x$, by Prop. I.6.4.5 x is the only limit point of $(a_n)_{n=m}^{\infty}$. So we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists n \in \mathbb{N} : |x - a_n| \le \varepsilon && \text{(by Def. I.6.4.1)}\\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists a_n \in X : |x - a_n| \le \varepsilon\\
\implies &x \in \overline{X}. && \text{(by Def. I.9.1.10)}
\end{aligned}$$

We conclude that x is an adherent point of X iff there exists a sequence $(a_n)_{n=0}^{\infty}$ such that $\forall n \in \mathbb{N}$, $a_n \in X$ and $\lim_{n\to\infty} a_n = x$. $\square$

Def. I.9.1.15. A subset $E \subseteq \mathbb{R}$ is said to be *closed* if $\overline{E} = E$, or in other words that E contains all of its adherent points.

E.g. I.9.1.16. From Lem. I.9.1.12 we see that if $a < b$ are real numbers, then $[a, b]$, $[a, +\infty)$, $(-\infty, a]$, and $(-\infty, +\infty)$ are closed, while (a, b), $(a, b]$, $[a, b)$, $(a, +\infty)$, and $(-\infty, a)$ are not. From Lem. I.9.1.13 we see that $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\emptyset$ are closed, while $\mathbb{Q}$ is not.

Cor. I.9.1.17. Let X be a subset of $\mathbb{R}$. If X is closed, and $(a_n)_{n=0}^{\infty}$ is a convergent sequence consisting of elements in X, then $\lim_{n\to\infty} a_n$ also lies in X. Conversely, if it is true that every convergent sequence $(a_n)_{n=0}^{\infty}$ of elements in X has its limit in X as well, then X is necessarily closed.

Proof. We first show that if X is closed, and $(a_n)_{n=0}^{\infty}$ is a convergent sequence consisting of elements in X, then $\lim_{n\to\infty} a_n$ also lies in X. Let $x = \lim_{n\to\infty} a_n$. Then we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists n \in \mathbb{N} : \forall n' \in \mathbb{N} \wedge n' \ge n, |x - a_{n'}| \le \varepsilon\\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists a_n \in X : |x - a_n| \le \varepsilon\\
\implies &x \in \overline{X} && \text{(by Def. I.9.1.10)}\\
\implies &x \in X. && \text{(by Def. I.9.1.15)}
\end{aligned}$$

Now we show that if every convergent sequence $(a_n)_{n=0}^{\infty}$ of elements in X has its limit in X as well, then X is closed. By Lem. I.9.1.11 we have $X \subseteq \overline{X}$. Since

$$\begin{aligned}
&\forall x \in \overline{X}, \exists (a_n)_{n=0}^{\infty} : (\forall n \in \mathbb{N}, a_n \in X) \wedge (\lim_{n\to\infty} a_n = x) && \text{(by Lem. I.9.1.14)}\\
\implies &x \in X, && \text{(by hypothesis)}
\end{aligned}$$

by Def. I.3.1.15 we have $\overline{X} \subseteq X$. Since $X \subseteq \overline{X} \wedge \overline{X} \subseteq X$, by Rmk. I.3.1.8 we have $X = \overline{X}$, and thus by Def. I.9.1.15 X is closed. $\square$

Def. I.9.1.18 (Limit points)**.** Let X be a subset of the real line. We say that x is a *limit point* (or a *cluster point*) of X iff it is an adherent point of $X \setminus \{x\}$. We say that x is an *isolated point* of X if $x \in X$ and there exists some $\varepsilon > 0$ such that $|x - y| > \varepsilon$ for all $y \in X \setminus \{x\}$.

Rmk. I.9.1.20. From Lem. I.9.1.14 we see that x is a limit point of X iff there exists a sequence $(a_n)_{n=0}^{\infty}$, consisting entirely of elements in X that are distinct from x, and such that $(a_n)_{n=0}^{\infty}$ converges to x. It turns out that the set of adherent points splits into the set of limit points and the set of isolated points.

Lem. I.9.1.21. Let I be an interval (possibly infinite), i.e., I is a set of the form (a, b), $(a, b]$, $[a, b)$, $[a, b]$, $(a, +\infty)$, $[a, +\infty)$, $(-\infty, a)$, or $(-\infty, a]$, with $a < b$ in the first four cases. Then every element of I is a limit point of I.

Proof. We show this for the case $I = [a, b]$; the other cases are similar. Let $x \in I$; we have to show that x is a limit point of I. There are three cases: $x = a$, $a < x < b$, and $x = b$. If $x = a$, then consider the sequence $(x + \dfrac{1}{n})_{n=N}^{\infty}$. This sequence converges to x, and will lie inside $I \setminus \{a\} = (a, b]$ if N is chosen large enough (by Prop. I.5.4.14). Thus, by Rmk. I.9.1.20 we see that $x = a$ is a limit point of $[a, b]$. A similar argument works when $a < x < b$. When $x = b$ one has to use the sequence $(x - \dfrac{1}{n})_{n=N}^{\infty}$ instead. This sequence converges to x, and will lie inside $I \setminus \{b\} = [a, b)$ if N is chosen large enough (by Prop. I.5.4.14). Thus, by Rmk. I.9.1.20 we see that $x = b$ is a limit point of $[a, b]$. $\square$

Def. I.9.1.22 (Bounded sets)**.** A subset X of the real line is said to be *bounded* if we have $X \subseteq [-M, M]$ for some real number $M > 0$.

E.g. I.9.1.23. For any real numbers a, b, the interval $[a, b]$ is bounded, because it is contained inside $[-M, M]$, where $M := \max(|a|, |b|)$. However, the half-infinite interval $[0, +\infty)$ is unbounded. In fact, no half-infinite interval or doubly infinite interval can be bounded. The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ are all unbounded.

Thm. I.9.1.24 (Heine-Borel theorem for the line)**.** Let X be a subset of $\mathbb{R}$. Then the following two statements are equivalent:

(a) X is closed and bounded.

(b) Given any sequence $(a_n)_{n=0}^{\infty}$ of real numbers which takes values in X (i.e., $a_n \in X$ for all n), there exists a subsequence $(a_{n_j})_{j=0}^{\infty}$ of the original sequence, which converges to some number L in X.

Proof. We first show that statement (a) implies statement (b). Suppose that X is a set such that X is closed and bounded. Let $(a_n)_{n=0}^{\infty}$ be a sequence where $\forall n \in \mathbb{N}$, $a_n \in X$. Since X is bounded, by Def. I.9.1.22 $\exists M \in \mathbb{R}^+$ such that $X \subseteq [-M, M]$, thus $(a_n)_{n=0}^{\infty}$ is also bounded by M, i.e., $\forall n \in \mathbb{N}$, $|a_n| \leq M$. By Bolzano-Weierstrass theorem (Thm. I.6.6.8) we know that there exists a subsequence $(a_{n_j})_{j=0}^{\infty}$ of $(a_n)_{n=0}^{\infty}$ such that $(a_{n_j})_{j=0}^{\infty}$ converges. Since X is closed, by Cor. I.9.1.17 we know that $\lim_{j \to \infty} a_{n_j} \in X$.

Now we show that statement (b) implies statement (a). Since given any sequence $(a_n)_{n=0}^{\infty}$ we can always find a subsequence $(a_{n_j})_{j=0}^{\infty}$ such that $\lim_{j \to \infty} a_{n_j} \in X$, we know that if $(a_n)_{n=0}^{\infty}$ converges then $\lim_{n\to\infty} a_n \in X$. Thus, every convergent sequence $(a_n)_{n=0}^{\infty}$ have its limit in X, and by Cor. I.9.1.17 we know that X is closed. Suppose for the sake of contradiction that X is unbounded. Then $\nexists M \in \mathbb{R}^+$ such that $X \subseteq [-M, M]$. Now we define $X_n = \{x \in X : |x| > n\}$ for every $n \in \mathbb{N}$. We know that $X_n \neq \emptyset$ since X is unbounded. By axiom of choice (Ax. I.8.1) we know that $\prod_{n\in\mathbb{N}} X_n \neq \emptyset$. Let $f \in \prod_{n\in\mathbb{N}} X_n$. We can define a sequence $(a_n)_{n=0}^{\infty}$ by setting $a_n = f(n)$. By hypothesis we know that there exists a subsequence $(a_{n_j})_{j=0}^{\infty}$ such that $L = \lim_{j\to\infty} a_{n_j} \in X$. We know that $(a_{n_j})_{j=0}^{\infty}$ is unbounded since $\left|a_{n_j}\right| > n_j$ for every $n_j \in \mathbb{N}$. But by Thm. I.6.4.18 $(a_{n_j})_{j=0}^{\infty}$ is Cauchy sequence and by Cor. I.6.1.17 $(a_{n_j})_{j=0}^{\infty}$ is bounded, a contradiction. Thus, X is closed and bounded. ☐

Rmk. I.9.1.25. This theorem shall play a key role in subsequent sections of Ch. I.9. In the language of metric space topology, it asserts that every subset of the real line which is closed and bounded, is also compact A more general version of this theorem, due to Eduard Heine (1821–1881) and Emile Borel (1871–1956), can be found is Analysis II, Theorem 1.5.7.

— Exercises —

Ex. I.9.1.1. Let X be any subset of the real line, and let Y be a set such that $X \subseteq Y \subseteq \overline{X}$. Show that $\overline{Y} = \overline{X}$.

Proof. By Lem. I.9.1.11 we have $X \subseteq Y \implies \overline{X} \subseteq \overline{Y}$. Since

$$\begin{aligned}
&\forall y \in \overline{Y}, \forall \varepsilon \in \mathbb{R}^+, \exists x \in Y : |y - x| \leq \frac{\varepsilon}{2} && \text{(by Def. I.9.1.10)}\\
\implies &\exists x \in \overline{X} : |y - x| \leq \frac{\varepsilon}{2} && (Y \subseteq \overline{X})\\
\implies &\exists z \in X : |x - z| \leq \frac{\varepsilon}{2} && \text{(by Def. I.9.1.10)}\\
\implies &|y - x| + |x - z| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\
\implies &|y - x + x - z| \leq |y - x| + |x - z| \leq \varepsilon\\
\implies &|y - z| \leq \varepsilon\\
\implies &y \in \overline{X}, && \text{(by Def. I.9.1.10)}
\end{aligned}$$

by Def. I.3.1.15 we know that $\overline{Y} \subseteq \overline{X}$. Since $\overline{Y} \subseteq \overline{X} \wedge \overline{X} \subseteq \overline{Y}$, by Prop. I.3.1.18 we have $\overline{Y} = \overline{X}$. ☐

Ex. I.9.1.2. Prove Lem. I.9.1.11.

Proof. See Lem. I.9.1.11. ☐

Ex. I.9.1.3. Prove Lem. I.9.1.13.

Proof. See Lem. I.9.1.13. ☐

Ex. I.9.1.4. Give an example of two subsets X, Y of the real line such that $\overline{X \cap Y} \neq \overline{X} \cap \overline{Y}$.

Proof. Let $X = [0, 0.5)$ and $Y = (0.5, 1]$. By Lem. I.9.1.12 we have $\overline{X} = [0, 0.5]$ and $\overline{Y} = [0.5, 1]$, so $\overline{X} \cap \overline{Y} = \{0.5\}$. By Lem. I.9.1.13 we have $\overline{X \cap Y} = \overline{\emptyset} = \emptyset$. Thus, $\overline{X \cap Y} \neq \overline{X} \cap \overline{Y}$. □

Ex. I.9.1.5. Prove Lem. I.9.1.14.

Proof. See Lem. I.9.1.14. □

Ex. I.9.1.6. Let X be a subset of $\mathbb{R}$. Show that $\overline{X}$ is closed (i.e., $\overline{\overline{X}} = \overline{X}$). Furthermore, show that if Y is any closed set that contains X, then Y also contains $\overline{X}$. Thus, the closure $\overline{X}$ of X is the smallest closed set which contains X.

Proof. We first show that $X \subseteq \mathbb{R} \implies \overline{\overline{X}} = \overline{X}$. We have

$$\begin{aligned}
& X \subseteq \mathbb{R} \\
\implies & \overline{X} \subseteq \overline{\mathbb{R}} && \text{(by Lem. I.9.1.11)} \\
\implies & \overline{X} \subseteq \mathbb{R} && \text{(by Lem. I.9.1.13)} \\
\implies & \overline{X} \subseteq \overline{\overline{X}}. && \text{(by Lem. I.9.1.11)}
\end{aligned}$$

Since

$$\begin{aligned}
& \forall x \in \overline{\overline{X}}, \forall \varepsilon \in \mathbb{R}^+, \exists y \in \overline{X} : |x - y| \leq \frac{\varepsilon}{2} && \text{(by Def. I.9.1.10)} \\
\implies & \exists z \in X : |y - z| \leq \frac{\varepsilon}{2} && \text{(by Def. I.9.1.10)} \\
\implies & |x - y| + |y - z| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
\implies & |x - y + y - z| \leq |x - y| + |y - z| \leq \varepsilon \\
\implies & |x - z| \leq \varepsilon \\
\implies & x \in \overline{X}, && \text{(by Def. I.9.1.10)}
\end{aligned}$$

by Def. I.3.1.15 we know that $\overline{\overline{X}} \subseteq \overline{X}$. Since $\overline{\overline{X}} \subseteq \overline{X} \wedge \overline{X} \subseteq \overline{\overline{X}}$, by Prop. I.3.1.18 we have $\overline{\overline{X}} = \overline{X}$. By Def. I.9.1.15 $\overline{X}$ is closed.

Now we show that if Y is any closed set that contains X, then Y also contains $\overline{X}$.

$$\begin{aligned}
& (X \subseteq Y) \wedge (Y = \overline{Y}) && \text{(by Def. I.9.1.15)} \\
\implies & (\overline{X} \subseteq \overline{Y}) \wedge (Y = \overline{Y}) && \text{(by Lem. I.9.1.11)} \\
\implies & \overline{X} \subseteq Y.
\end{aligned}$$

□

Ex. I.9.1.7. Let $n \geq 1$ be a positive integer, and let $X_1, \ldots, X_n$ be closed subsets of $\mathbb{R}$. Show that $X_1 \cup X_2 \cup \cdots \cup X_n$ is also closed.

Proof. Suppose that $\forall m \in \mathbb{N}$ we have X_m is a closed subset of $\mathbb{R}$. We induct on n to show that $X_1 \cup \cdots \cup X_n$ is closed and we start with $n = 1$. For $n = 1$, by the given hypothesis we have X_1 is closed. So the base case holds. Suppose inductively that for some $n \geq 1$ we have $X_1 \cup \cdots \cup X_n$ is closed. Then for $n + 1$, we have

$$\begin{aligned}
&\overline{X_1 \cup \cdots \cup X_n \cup X_{n+1}} \\
=&\overline{(X_1 \cup \cdots \cup X_n) \cup X_{n+1}} && \text{(by Prop. I.3.1.28(e))} \\
=&\overline{(X_1 \cup \cdots \cup X_n)} \cup \overline{X_{n+1}} && \text{(by Lem. I.9.1.11)} \\
=&(X_1 \cup \cdots \cup X_n) \cup \overline{X_{n+1}} && \text{(by the induction hypothesis)} \\
=&(X_1 \cup \cdots \cup X_n) \cup X_{n+1} && \text{(by hypothesis)} \\
=&X_1 \cup \cdots \cup X_n \cup X_{n+1}. && \text{(by Prop. I.3.1.28(e))}
\end{aligned}$$

This closes the induction. Thus, $\forall n \in \mathbb{N}$, if $X_1, \ldots, X_n$ are closed subset of $\mathbb{R}$, then $X_1 \cup \cdots \cup X_n$ is also closed. □

Ex. I.9.1.8. Let I be a set (possibly infinite), and for each $\alpha \in I$ let X_α be a closed subset of $\mathbb{R}$. Show that the intersection $\bigcap_{\alpha \in I} X_\alpha$ is also closed.

Proof. By Lem. I.9.1.11 we have

$$\bigcap_{\alpha \in I} X_\alpha \subseteq \mathbb{R} \implies \bigcap_{\alpha \in I} X_\alpha \subseteq \overline{\bigcap_{\alpha \in I} X_\alpha}.$$

Since

$$\begin{aligned}
&\forall x \in \overline{\bigcap_{\alpha \in I} X_\alpha}, \forall \varepsilon \in \mathbb{R}^+, \exists y \in \bigcap_{\alpha \in I} X_\alpha : |x - y| \leq \varepsilon && \text{(by Def. I.9.1.10)} \\
\implies& \forall \alpha \in I, y \in X_\alpha \\
\implies& \forall \alpha \in I, x \in \overline{X_\alpha} && \text{(by Def. I.9.1.10)} \\
\implies& \forall \alpha \in I, x \in X_\alpha && \text{(by hypothesis)} \\
\implies& x \in \bigcap_{\alpha \in I} X_\alpha,
\end{aligned}$$

by Def. I.3.1.15 we have $\overline{\bigcap_{\alpha \in I} X_\alpha} \subseteq \bigcap_{\alpha \in I} X_\alpha$. By Prop. I.3.1.18 we have

$$\left(\overline{\bigcap_{\alpha \in I} X_\alpha} \subseteq \bigcap_{\alpha \in I} X_\alpha\right) \wedge \left(\bigcap_{\alpha \in I} X_\alpha \subseteq \overline{\bigcap_{\alpha \in I} X_\alpha}\right) \iff \overline{\bigcap_{\alpha \in I} X_\alpha} = \bigcap_{\alpha \in I} X_\alpha,$$

and thus by Def. I.9.1.15 $\bigcap_{\alpha \in I} X_\alpha$ is closed. □

Ex. I.9.1.9. Let X be a subset of the real line. Show that every adherent point of X is either a limit point or an isolated point of X, but cannot be both. Conversely, show that every limit point and every isolated point of X is an adherent point of X.

Proof. Let $X \subseteq \mathbb{R}$. We first show that every adherent point of X is either a limit point or an isolated point of X, but cannot be not both. Let $x \in \overline{X}$. Observe that

$$\begin{aligned}
& x \in \overline{X} \\
\Longrightarrow\ & x \in \overline{(X \setminus \{x\}) \cup \{x\}} \\
\Longrightarrow\ & x \in \overline{X \setminus \{x\}} \cup \overline{\{x\}} && \text{(by Lem. I.9.1.11)} \\
\Longrightarrow\ & (x \in \overline{X \setminus \{x\}}) \vee (x \in \overline{\{x\}}).
\end{aligned}$$

By Def. I.9.1.10 we know that $x \in \overline{\{x\}}$. Thus, we have either $x \in \overline{X \setminus \{x\}}$ or $x \notin \overline{X \setminus \{x\}}$.

- If $x \in \overline{X \setminus \{x\}}$, then by Def. I.9.1.18 x is a limit point of X.
- If $x \notin \overline{X \setminus \{x\}}$, then by Def. I.9.1.10 $\exists \varepsilon \in \mathbb{R}^+$ such that $\forall y \in X \setminus \{x\}$, $|x - y| > \varepsilon$. Since $x \in \overline{X}$, we must have $x \in X$, otherwise $\nexists y \in X \setminus \{x\}$ such that $|x - y| \leq \varepsilon$. Thus, by Def. I.9.1.18 x is a isolated point of X.

From all cases above, we conclude that x is either a limit point or an isolated point of X. Since $x \in \overline{X \setminus \{x\}}$ and $x \notin \overline{X \setminus \{x\}}$ cannot be true at the same time, we know that x is either a limit point or an isolated point of X, but cannot be both.

Now we show that every limit point and every isolated point of X is an adherent point of X. Suppose that x is a limit point of X. Then we have

$$\begin{aligned}
& x \in \overline{X \setminus \{x\}} && \text{(by Def. I.9.1.18)} \\
\Longrightarrow\ & \forall \varepsilon \in \mathbb{R}^+, \exists y \in X \setminus \{x\} : |x - y| \leq \varepsilon && \text{(by Def. I.9.1.10)} \\
\Longrightarrow\ & \forall \varepsilon \in \mathbb{R}^+, \exists y \in X : |x - y| \leq \varepsilon \\
\Longrightarrow\ & x \in \overline{X}. && \text{(by Def. I.9.1.10)}
\end{aligned}$$

Now suppose that x is a isolated point of X. Then we have

$$\begin{aligned}
& (x \in X) \wedge (X \subseteq \overline{X}) && \text{(by Def. I.9.1.18)} \\
\Longrightarrow\ & x \in \overline{X}. && \text{(by Def. I.9.1.10)}
\end{aligned}$$

Since x was arbitrary limit point or isolated point of X, we conclude that every limit point and every isolated point of X is an adherent point of X. $\square$

Ex. I.9.1.10. If X is a non-empty subset of $\mathbb{R}$, show that X is bounded iff $\inf(X)$ and $\sup(X)$ are finite.

Proof. Suppose that X is a set, $X \subseteq \mathbb{R}$ and $X \neq \emptyset$. We first show that if X is bounded then $\inf(X)$ and $\sup(X)$ are finite. Since

$$\begin{aligned}
&\exists M \in \mathbb{R}^+ : X \subseteq [-M, M] && \text{(by Def. I.9.1.22)}\\
\implies &\forall x \in X : -M \leq x \leq M && \text{(by Def. I.9.1.1)}\\
\implies &\forall x \in X : -M \leq \inf(X) \leq x \leq \sup(X) \leq M, && \text{(by Thm. I.6.2.11)}
\end{aligned}$$

we know that $\inf(X)$ and $\sup(X)$ are finite.

Now we show that if $\inf(X)$ and $\sup(X)$ are finite then X is bounded. Let $M = \max\big(|\inf(X)|, |\sup(X)|\big)$. Then we have

$$\begin{aligned}
&M = \max\big(|\inf(X)|, |\sup(X)|\big)\\
\implies &\big(|\inf(X)| \leq M\big) \wedge \big(\sup(X) \leq |\sup(X)| \leq M\big)\\
\implies &\big(-M \leq \inf(X) \leq M\big) \wedge \big(\sup(X) \leq M\big)\\
\implies &-M \leq \inf(X) \leq \sup(X) \leq M && \text{(by Thm. I.6.2.11)}\\
\implies &\forall x \in X, -M \leq \inf(X) \leq x \leq \sup(X) \leq M && \text{(by Thm. I.6.2.11)}\\
\implies &X \subseteq [-M, M]. && \text{(by Def. I.9.1.1)}
\end{aligned}$$

Thus, by Def. I.9.1.22 X is bounded. □

Ex. I.9.1.11. Show that if X is a bounded subset of $\mathbb{R}$, then the closure X is also bounded.

Proof. Since X is bounded, by Def. I.9.1.22 we know that $\exists M \in \mathbb{R}^+ : X \subseteq [-M, M]$. Let $\varepsilon \in \mathbb{R}^+$. Then we have

$$\begin{aligned}
&\forall x \in \overline{X}\\
\implies &\exists y \in X : |x - y| \leq \varepsilon && \text{(by Def. I.9.1.10)}\\
\implies &-\varepsilon \leq x - y \leq \varepsilon\\
\implies &y - \varepsilon \leq x \leq y + \varepsilon\\
\implies &-M - \varepsilon \leq y - \varepsilon \leq x \leq y + \varepsilon \leq M + \varepsilon \qquad (-M \leq y \leq M)\\
\implies &x \in [-M - \varepsilon, M + \varepsilon]. && \text{(by Def. I.9.1.1)}
\end{aligned}$$

By Def. I.3.1.15 we have $\overline{X} \subseteq [-M - \varepsilon, M + \varepsilon]$, and thus by Def. I.9.1.22 $\overline{X}$ is bounded. □

Ex. I.9.1.12. Show that the union of any finite collection of bounded subsets of $\mathbb{R}$ is still a bounded set. Is this conclusion still true if one takes an infinite collection of bounded subsets of $\mathbb{R}$?

Proof. Let $n \in \mathbb{N}$, let $I_n = \{i \in \mathbb{N} : 1 \leq i \leq n\}$ and let $X_1, \ldots, X_n$ be bounded subsets of $\mathbb{R}$. We want to show that $\bigcup_{i \in I_n} X_i$ is bounded. If $n = 0$, then $I_n = \emptyset \implies \bigcup_{i \in I_n} X_i = \emptyset$ and by Lem. I.5.1.14 $\emptyset$ is bounded. So suppose that $n \neq 0$. Since $X_1, \ldots, X_n$ are bounded,

$\exists M_1, \dots, M_n \in \mathbb{R}^+$ such that $\forall i \in I_n$, $X_i \subseteq [-M_i, M_i]$. Let $S = \{M_i : i \in I_n\}$. Clearly, S is finite. By Lem. I.5.1.14 we know that S is bounded by some $M \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\forall x \in \bigcup_{i \in I_n} X_i, \exists i \in I_n : x \in X_i \\
\implies &-M_i \leq x \leq M_i && (X_i \subseteq [-M_i, M_i]) \\
\implies &-M \leq -M_i \leq x \leq M_i \leq M && \text{(by Lem. I.5.1.14)} \\
\implies &x \in [-M, M].
\end{aligned}$$

By Def. I.3.1.15 we have $\bigcup_{i \in I_n} X_i \subseteq [-M, M]$, thus by Def. I.9.1.22 $\bigcup_{i \in I_n} X_i$ is bounded.

Now we show that the union of an infinite collection of bounded subsets of $\mathbb{R}$ may not be bounded. Let $n \in \mathbb{N}$ and let $X_n = \{n\}$. Clearly, $\forall n \in \mathbb{N}$, $X_n \subseteq [-n, n]$ and thus by Def. I.9.1.22 X_n is bounded. Then we have $\bigcup_{n \in \mathbb{N}} X_n = \mathbb{N}$ and by Thm. I.3.6.12 $\mathbb{N}$ is unbounded. □

Ex. I.9.1.13. Prove Thm. I.9.1.24.

Proof. See Thm. I.9.1.24. □

Ex. I.9.1.14. Show that any finite subset of $\mathbb{R}$ is closed and bounded.

Proof. Let $n \in \mathbb{N}$ and let $x_i \in \mathbb{R}$ for every $i \in \mathbb{N} \wedge i \leq n$. Given any sequence $(a_m)_{m=0}^{\infty}$ of real numbers which taks values in the singleton set $\{x_i\}$, we know that $\forall m \in \mathbb{N}$, $a_m = x_i$. Thus, by Thm. I.6.1.19 we have $\lim_{m \to \infty} a_m = x_i$. Since $x_i \in \{x_i\}$, by Heine-Borel theorem (Thm. I.9.1.24) we know that $\{x_i\}$ is closed and bounded. Let $S = \bigcup_{i \in \mathbb{N} : i \leq n} \{x_i\}$. By Ex. I.9.1.7 we know that S is closed, and by Ex. I.9.1.12 we know that S is bounded. Since n was arbitrary natural number, we conclude that every finite subset of $\mathbb{R}$ is closed and bounded. □

Ex. I.9.1.15. Let E be a non-empty bounded subset of $\mathbb{R}$, and let $S \coloneqq \sup(E)$ be the least upper bound of E. (Note from the least upper bound principle, Thm. I.5.5.9, that S is a real number.) Show that S is an adherent point of E, and is also an adherent point of $\mathbb{R} \setminus E$.

Proof. We first show that S is an adherent point of E. $\forall \varepsilon \in \mathbb{R}^+$, we can always find $x \in E$ such that $S - \varepsilon < x \leq S$. If not, then $\exists \varepsilon \in \mathbb{R}^+$ such that $S - \varepsilon < S$ and

$$\forall x \in E \implies x \leq S - \varepsilon < S \implies \sup(E) \neq S,$$

a contradiction. Then we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists x \in E : S - \varepsilon < x \leq S \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists x \in E : S - \varepsilon < x < S + \varepsilon
\end{aligned}$$

$$\begin{aligned}
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists x \in E : -\varepsilon < x - S < \varepsilon \\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists x \in E : |x - S| < \varepsilon \\
&\implies S \in \overline{E}. && \text{(by Def. I.9.1.10)}
\end{aligned}$$

Now we show that S is an adherent point of $\mathbb{R} \setminus E$. Let $\varepsilon \in \mathbb{R}^+$. Then we have

$$\begin{aligned}
&\forall x \in E, x \leq S && \text{(by Def. I.5.5.5)} \\
\implies& x \leq S < S + \varepsilon \\
\implies& x \notin (S, S + \varepsilon) && \text{(by Def. I.9.1.1)} \\
\implies& E \cap (S, S + \varepsilon) = \emptyset \\
\implies& (S, S + \varepsilon) \subseteq \mathbb{R} \setminus E \\
\implies& \overline{(S, S + \varepsilon)} \subseteq \overline{\mathbb{R} \setminus E} && \text{(by Lem. I.9.1.11)} \\
\implies& [S, S + \varepsilon] \subseteq \overline{\mathbb{R} \setminus E} && \text{(by Lem. I.9.1.12)} \\
\implies& S \in \overline{\mathbb{R} \setminus E}. && \text{(by Def. I.9.1.1)}
\end{aligned}$$

□

I.9.2 The algebra of real-valued functions

Note. We can take any one of the previous functions $f : \mathbb{R} \to \mathbb{R}$ defined on all of $\mathbb{R}$, and restrict the domain to a smaller set $X \subseteq \mathbb{R}$, creating a new function, sometimes called $f|_X$, from X to $\mathbb{R}$. This is the same function as the original function f, but is only defined on a smaller domain. (Thus, $f|_X(x) \coloneqq f(x)$ when $x \in X$, and $f|_X(x)$ is undefined when $x \notin X$.)

Note. If X is a subset of $\mathbb{R}$, and $f : X \to \mathbb{R}$ is a function, we can form the *graph* $\{(x, f(x)) : x \in X\}$ of the function f; this is a subset of $X \times \mathbb{R}$, and hence a subset of the Euclidean plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. One can certainly study a function through its graph, by using the geometry of the plane $\mathbb{R}^2$ (e.g., employing such concepts as tangent lines, area, and so forth). We however will pursue a more "analytic" approach, in which we rely instead on the properties of the real numbers to analyze these functions. The two approaches are complementary; the geometric approach offers more visual intuition, while the analytic approach offers rigour and precision. Both the geometric intuition and the analytic formalism become useful when extending analysis of functions of one variable to functions of many variables (or possibly even infinitely many variables).

Def. I.9.2.1 (Arithmetic operations on functions)**.** Given two functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$, we can define their sum $f + g : X \to \mathbb{R}$ by the formula

$$(f + g)(x) \coloneqq f(x) + g(x),$$

their difference $f - g : X \to \mathbb{R}$ by the formula

$$(f - g)(x) \coloneqq f(x) - g(x),$$

their maximum $\max(f,g) : X \to \mathbb{R}$ by

$$\max(f,g)(x) \coloneqq \max(f(x), g(x)),$$

their minimum $\min(f,g) : X \to \mathbb{R}$ by

$$\min(f,g)(x) \coloneqq \min(f(x), g(x)),$$

their product $fg : X \to \mathbb{R}$ (or $f \cdot g : X \to \mathbb{R}$) by the formula

$$(fg)(x) \coloneqq f(x)g(x),$$

and (provided that $g(x) = 0$ for all $x \in X$) the quotient $f/g : X \to \mathbb{R}$ by the formula

$$(f/g)(x) \coloneqq f(x)/g(x).$$

Finally, if c is a real number, we can define the function $cf : X \to \mathbb{R}$ (or $c \cdot f : X \to \mathbb{R}$) by the formula

$$(cf)(x) \coloneqq c \times f(x).$$

— Exercises —

Ex. I.9.2.1. Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$. Which of the following identities are true, and which ones are false? In the former case, give a proof; in the latter case, give a counterexample.

$$\begin{gathered}
(f+g) \circ h = (f \circ h) + (g \circ h) \\
f \circ (g+h) = (f \circ g) + (f \circ h) \\
(f+g) \cdot h = (f \cdot h) + (g \cdot h) \\
f \cdot (g+h) = (f \cdot g) + (f \cdot h)
\end{gathered}$$

Proof. We first show that $(f+g) \circ h = (f \circ h) + (g \circ h)$. Since

$$\begin{aligned}
& f+g \text{ has domain } \mathbb{R} \text{ and codomain } \mathbb{R} && \text{(by Def. I.9.2.1)} \\
\implies & (f+g) \circ h, f \circ h, g \circ h \text{ have domain } \mathbb{R} \text{ and codomain } \mathbb{R} && \text{(by Def. I.3.3.10)} \\
\implies & (f \circ h) + (g \circ h) \text{ has domain } \mathbb{R} \text{ and codomain } \mathbb{R} && \text{(by Def. I.9.2.1)} \\
\implies & (f+g) \circ h \text{ and} \\
& (f \circ h) + (g \circ h) \text{ have same domain and codomain}
\end{aligned}$$

and

$$\begin{aligned}
\forall x \in \mathbb{R}, \big((f+g) \circ h\big)(x) &= (f+g)\big(h(x)\big) && \text{(by Def. I.3.3.10)} \\
&= f\big(h(x)\big) + g\big(h(x)\big) && \text{(by Def. I.9.2.1)}
\end{aligned}$$

$$\begin{aligned} &= (f \circ h)(x) + (g \circ h)(x) && \text{(by Def. I.3.3.10)} \\ &= \big((f \circ h) + (g \circ h)\big)(x), && \text{(by Def. I.9.2.1)} \end{aligned}$$

by Def. I.3.3.7 we have $(f+g) \circ h = (f \circ h) + (g \circ h)$.

Next we show that $f \circ (g+h) = (f \circ g) + (f \circ h)$ may not be true. Let $f(x) = 2^x$, $g(x) = h(x) = x$. Then we have

$$\begin{aligned} \big(f \circ (g+h)\big)(x) &= f\big((g+h)(x)\big) && \text{(by Def. I.3.3.10)} \\ &= f\big(g(x) + h(x)\big) && \text{(by Def. I.9.2.1)} \\ &= f(x+x) \\ &= f(2x) \\ &= 2^{2x} \end{aligned}$$

and

$$\begin{aligned} \big((f \circ g) + (f \circ h)\big)(x) &= (f \circ g)(x) + (f \circ h)(x) && \text{(by Def. I.9.2.1)} \\ &= f\big(g(x)\big) + f\big(h(x)\big) && \text{(by Def. I.3.3.10)} \\ &= f(x) + f(x) \\ &= 2^x + 2^x \\ &= 2 \cdot 2^x. \end{aligned}$$

When $x = 2$ we have $2^{2x} = 16$ and $2 \cdot 2^x = 8$. Thus, $f \circ (g+h) = (f \circ g) + (f \circ h)$ may not be true.

Next we show that $(f+g) \cdot h = (f \cdot h) + (g \cdot h)$. Since

$$\begin{aligned} & f+g \text{ has domain } \mathbb{R} \text{ and codomain } \mathbb{R} && \text{(by Def. I.9.2.1)} \\ \Longrightarrow\ & (f+g) \cdot h, f \cdot h, g \cdot h \text{ have domain } \mathbb{R} \text{ and codomain } \mathbb{R} && \text{(by Def. I.3.3.10)} \\ \Longrightarrow\ & (f \cdot h) + (g \cdot h) \text{ has domain } \mathbb{R} \text{ and codomain } \mathbb{R} && \text{(by Def. I.9.2.1)} \\ \Longrightarrow\ & (f+g) \cdot h \text{ and} \\ & (f \cdot h) + (g \cdot h) \text{ have same domain and codomain} \end{aligned}$$

and

$$\begin{aligned} \forall x \in \mathbb{R}, \big((f+g) \cdot h\big)(x) &= (f+g)(x) \cdot h(x) && \text{(by Def. I.9.2.1)} \\ &= \big(f(x) + g(x)\big) \cdot h(x) && \text{(by Def. I.9.2.1)} \\ &= f(x)h(x) + g(x)h(x) \\ &= (f \cdot h)(x) + (g \cdot h)(x) && \text{(by Def. I.9.2.1)} \\ &= \big((f \cdot h) + (g \cdot h)\big)(x), && \text{(by Def. I.9.2.1)} \end{aligned}$$

by Def. I.3.3.7 we have $(f+g) \cdot h = (f \cdot h) + (g \cdot h)$.

Finally we show that $f \cdot (g+h) = (f \cdot g) + (f \cdot h)$. Since

$$
\begin{aligned}
& g+h \text{ has domain } \mathbb{R} \text{ and codomain } \mathbb{R} && \text{(by Def. I.9.2.1)}\\
\Longrightarrow\ & f \cdot (g+h), f \cdot g, f \cdot h \text{ have domain } \mathbb{R} \text{ and codomain } \mathbb{R} && \text{(by Def. I.3.3.10)}\\
\Longrightarrow\ & (f \cdot g) + (f \cdot h) \text{ has domain } \mathbb{R} \text{ and codomain } \mathbb{R} && \text{(by Def. I.9.2.1)}\\
\Longrightarrow\ & f \cdot (g+h) \text{ and}\\
& (f \cdot g) + (f \cdot h) \text{ have same domain and codomain}
\end{aligned}
$$

and

$$
\begin{aligned}
\forall x \in \mathbb{R}, \big(f \cdot (g+h)\big)(x) &= f(x) \cdot (g+h)(x) && \text{(by Def. I.9.2.1)}\\
&= f(x) \cdot \big(g(x) + h(x)\big) && \text{(by Def. I.9.2.1)}\\
&= f(x)g(x) + f(x)h(x)\\
&= (f \cdot g)(x) + (f \cdot h)(x) && \text{(by Def. I.9.2.1)}\\
&= \big((f \cdot g) + (f \cdot h)\big)(x), && \text{(by Def. I.9.2.1)}
\end{aligned}
$$

by Def. I.3.3.7 we have $f \cdot (g+h) = (f \cdot g) + (f \cdot h)$. □

I.9.3 Limiting values of functions

Def. I.9.3.1 (ε-closeness)**.** Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, let L be a real number, and let $\varepsilon > 0$ be a real number. We say that the function f is *ε-close to L* iff $f(x)$ is ε-close to L for every $x \in X$.

Def. I.9.3.3 (Local ε-closeness)**.** Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, let L be a real number, x_0 be an adherent point of X, and $\varepsilon > 0$ be a real number. We say that f is *ε-close to L near x_0* iff there exists a $\delta > 0$ such that f becomes ε-close to L when restricted to the set $\{x \in X : |x - x_0| < \delta\}$.

Def. I.9.3.6 (Convergence of functions at a point)**.** Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, let E be a subset of X, x_0 be an adherent point of E, and let L be a real number. We say that f *converges to L at x_0 in E*, and write $\lim_{x \to x_0; x \in E} f(x) = L$, iff f, after restricting to E, is ε-close to L near x_0 for every $\varepsilon > 0$. If f does not converge to any number L at x_0, we say that f *diverges at x_0*, and leave $\lim_{x \to x_0; x \in E} f(x)$ undefined.

Note. In other words, we have $\lim_{x \to x_0; x \in E} f(x) = L$ iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| \leq \varepsilon$ for all $x \in E$ such that $|x - x_0| < \delta$.

Rmk. I.9.3.7. In many cases we will omit the set E from the above notation (i.e., we will just say that f converges to L at x_0, or that $\lim_{x \to x_0} f(x) = L$), although this is slightly dangerous. For instance, it sometimes makes a difference whether E actually contains x_0

or not. To give an example, if $f : \mathbb{R} \to \mathbb{R}$ is the function defined by setting $f(x) = 1$ when $x = 0$ and $f(x) = 0$ when $x \neq 0$, then one has $\lim\limits_{x\to 0; x\in\mathbb{R}\setminus\{0\}} f(x) = 0$, but $\lim\limits_{x\to 0; x\in\mathbb{R}} f(x)$ is undefined. Some authors only define the limit $\lim\limits_{x\to x_0; x\in E} f(x)$ when E does not contain x_0 (so that x_0 is now a limit point of E rather than an adherent point), or would use $\lim\limits_{x\to x_0; x\in E} f(x)$ to denote what we would call $\lim\limits_{x\to x_0; x\in E\setminus\{x_0\}} f(x)$, but we have chosen a slightly more general notation, which allows the possibility that E contains x_0.

Prop. I.9.3.9. Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, and let L be a real number. Then the following two statements are logically equivalent:

(a) f converges to L at x_0 in E.

(b) For every sequence $(a_n)_{n=0}^{\infty}$ which consists entirely of elements of E and converges to x_0, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to L.

Proof. We first show that statement (a) implies statement (b). Since f converges to L at x_0 in E, by Def. I.9.3.6 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in E, |x - x_0| < \delta \implies |f(x) - L| \le \varepsilon\big).$$

Now we fix ε, and we have some δ satisfying the statement above, we also fix such δ. Let $(a_n)_{n=0}^{\infty}$ be a sequence which consists entirely of elements of E and $\lim\limits_{n\to\infty} a_n = x_0$. Such sequence exists since Lem. I.9.1.14. By Def. I.6.1.5 we have

$$\forall \varepsilon' \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \ge N, |a_n - x_0| \le \varepsilon'.$$

In particular, we have

$$\exists N \in \mathbb{N} : \forall n \ge N, |a_n - x_0| \le \frac{\delta}{2} < \delta.$$

Since $(a_n)_{n=0}^{\infty}$ consists entirely of elements of E, we have

$$|a_n - x_0| < \delta \implies |f(a_n) - L| \le \varepsilon.$$

Since ε was arbitrary, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \ge N, |f(a_n) - L| \le \varepsilon$$

and by Def. I.6.1.5 we have $\lim\limits_{n\to\infty} f(a_n) = L$.

Now we show that statement (b) implies statement (a). Suppose for the sake of contradiction that f does not converge to L at x_0 in E. Then by Def. I.9.3.6 we have

$$\exists \varepsilon \in \mathbb{R}^+ : \forall \delta \in \mathbb{R}^+, \Big(\forall x \in E, (|x - x_0| < \delta) \wedge \big(|f(x) - L| > \varepsilon\big)\Big).$$

Let $(a_n)_{n=0}^{\infty}$ be a sequence which consists entirely of elements of E and $\lim_{n\to\infty} a_n = x_0$. By hypothesis we have $\lim_{n\to\infty} f(a_n) = L$. By Def. I.6.1.5 the following two statements are true:

$$\exists N_1 \in \mathbb{N} : \forall n_1 \geq N_1, |a_{n_1} - x_0| \leq \frac{\delta}{2} < \delta$$
$$\exists N_2 \in \mathbb{N} : \forall n_2 \geq N_2, |f(a_{n_2}) - L| \leq \varepsilon$$

Let $N = \max(N_1, N_2)$. Then we have

$$\forall n \geq N, (|a_n - x_0| < \delta) \wedge \big(|f(a_n) - L| \leq \varepsilon\big).$$

But $a_n \in E$, so this contradicts $|f(a_n) - L| > \varepsilon$. Thus, $\lim_{x\to x_0; x\in E} f(x) = L$. □

Note. In view of Prop. I.9.3.9, we will sometimes write "$f(x) \to L$ as $x \to x_0$ in E" or "f has a limit L at x_0 in E" instead of "f converges to L at x_0," or "$\lim_{x\to x_0} f(x) = L$."

Rmk. I.9.3.10. With the notation of Prop. I.9.3.9, we have the following corollary: if $\lim_{x\to x_0; x\in E} f(x) = L$, and $\lim_{n\to\infty} a_n = x_0$, then $\lim_{n\to\infty} f(a_n) = L$.

Rmk. I.9.3.11. We only consider limits of a function f at x_0 in the case when x_0 is an adherent point of E. When x_0 is not an adherent point then it is not worth it to define the concept of a limit.

Rmk. I.9.3.12. The variable x used to denote a limit is a dummy variable; we could replace it by any other variable and obtain exactly the same limit. For instance, if $\lim_{x\to x_0; x\in E} f(x) = L$, then $\lim_{y\to x_0; y\in E} f(y) = L$, and conversely. (Since $x \in \mathbb{R}$.)

Cor. I.9.3.13. Let X be a subset of $\mathbb{R}$, let E be a subset of X, let x_0 be an adherent point of E, and let $f : X \to \mathbb{R}$ be a function. Then f can have at most one limit at x_0 in E.

Proof. Suppose for the sake of contradiction that there are two distinct numbers L and L' such that f has a limit L at x_0 in E, and such that f also has a limit L' at x_0 in E. Since x_0 is an adherent point of E, we know by Lem. I.9.1.14 that there is a sequence $(a_n)_{n=0}^{\infty}$ consisting of elements in E which converges to x_0. Since f has a limit L at x_0 in E, we thus see by Prop. I.9.3.9, that $(f(a_n))_{n=0}^{\infty}$ converges to L. But since f also has a limit L' at x_0 in E, we see that $(f(a_n))_{n=0}^{\infty}$ also converges to L'. But this contradicts the uniqueness of limits of sequences (Prop. I.6.1.7). □

Prop. I.9.3.14 (Limit laws for functions)**.** Let X be a subset of R, let E be a subset of X, let x_0 be an adherent point of E, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions. Suppose that f has a limit L at x_0 in E, and g has a limit M at x_0 in E. Then $f + g$ has a limit $L + M$ at x_0 in E, $f - g$ has a limit $L - M$ at x_0 in E, $\max(f, g)$ has a limit $\max(L, M)$ at x_0 in E, $\min(f, g)$ has a limit $\min(L, M)$ at x_0 in E and fg has a limit LM at x_0 in E. If c is a real number, then cf has a limit cL at x_0 in E. Finally, if g is non-zero on E (i.e., $g(x) \neq 0$ for all $x \in E$) and M is non-zero, then f/g has a limit L/M at x_0 in E.

Proof. Since x_0 is an adherent point of E, we know by Lem. I.9.1.14 that there is a sequence $(a_n)_{n=0}^{\infty}$ consisting of elements in E, which converges to x_0. Since f has a limit L at x_0 in E, we thus see by Prop. I.9.3.9, that $(f(a_n))_{n=0}^{\infty}$ converges to L. Similarly, $(g(a_n))_{n=0}^{\infty}$ converges to M.

By the limit laws for sequences (Thm. I.6.1.19) we conclude that

$$
\begin{aligned}
&(\lim_{n\to\infty} f(a_n) = L) \wedge (\lim_{n\to\infty} g(a_n) = M) \\
\Longrightarrow &\begin{cases}
\lim_{n\to\infty} f(a_n) + g(a_n) = L + M \\
\lim_{n\to\infty} f(a_n) - g(a_n) = L - M \\
\lim_{n\to\infty} \max\big(f(a_n), g(a_n)\big) = \max(L, M) \\
\lim_{n\to\infty} \min\big(f(a_n), g(a_n)\big) = \min(L, M) \\
\lim_{n\to\infty} f(a_n)g(a_n) = LM \\
\lim_{n\to\infty} cf(a_n) = cL
\end{cases} \\
\Longrightarrow &\begin{cases}
\lim_{n\to\infty} (f+g)(a_n) = L + M \\
\lim_{n\to\infty} (f-g)(a_n) = L - M \\
\lim_{n\to\infty} \max(f, g)(a_n) = \max(L, M) \\
\lim_{n\to\infty} \min(f, g)(a_n) = \min(L, M) \\
\lim_{n\to\infty} (fg)(a_n) = LM \\
\lim_{n\to\infty} (cf)(a_n) = cL
\end{cases} \qquad \text{(by Def. I.9.2.1)}
\end{aligned}
$$

If $\forall x \in E, g(x) \neq 0$ and $M \neq 0$, then by the limit laws for sequences (Thm. I.6.1.19) and Def. I.9.2.1 we have

$$\lim_{n\to\infty} \big(f(a_n)/g(a_n)\big) = \lim_{n\to\infty} \big((f/g)(a_n)\big)_{n=0}^{\infty} = L/M.$$

By Prop. I.9.3.9 again, this implies

$$\begin{cases} \lim_{x\to x_0; x\in E} (f+g)(x) = L+M \\ \lim_{x\to x_0; x\in E} (f-g)(x) = L-M \\ \lim_{x\to x_0; x\in E} \max(f,g)(x) = \max(L,M) \\ \lim_{x\to x_0; x\in E} \min(f,g)(x) = \min(L,M) \\ \lim_{x\to x_0; x\in E} (fg)(x) = LM \\ \lim_{x\to x_0; x\in E} (cf)(x) = cL \\ \lim_{x\to x_0; x\in E} (f/g)(x) = L/M & \text{if } \forall x\in E, g(x)\neq 0 \text{ and } M\neq 0 \end{cases}$$

(since $(a_n)_{n=0}^{\infty}$ was an arbitrary sequence in E converging to x_0). $\square$

Rmk. I.9.3.15. One can phrase Prop. I.9.3.14 more informally as saying that

$$\begin{aligned} \lim_{x\to x_0} (f\pm g)(x) &= \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) \\ \lim_{x\to x_0} \max(f,g)(x) &= \max\left(\lim_{x\to x_0} f(x), \lim_{x\to x_0} g(x)\right) \\ \lim_{x\to x_0} \min(f,g)(x) &= \min\left(\lim_{x\to x_0} f(x), \lim_{x\to x_0} g(x)\right) \\ \lim_{x\to x_0} (fg)(x) &= \lim_{x\to x_0} f(x) \lim_{x\to x_0} g(x) \\ \lim_{x\to x_0} (f/g)(x) &= \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)} \end{aligned}$$

(where we have dropped the restriction $x\in E$ for brevity) but bear in mind that these identities are only true when the right-hand side makes sense, and furthermore for the final identity we need g to be non-zero, and also $\lim_{x\to x_0} g(x)$ to be non-zero.

Note. If f converges to L at x_0 in X, and Y is any subset of X such that x_0 is still an adherent point of Y, then f will also converge to L at x_0 in Y (since $Y\subseteq X$ and $x\in\overline{Y}$). Thus, convergence on a large set implies convergence on a smaller set. The converse, however, is not true.

E.g. I.9.3.16. Consider the *signum function* $\operatorname{sgn}:\mathbb{R}\to\mathbb{R}$, defined by

$$\operatorname{sgn}(x) := \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -1 & \text{if } x<0 \end{cases}$$

Then $\lim_{x\to 0;x\in(0,\infty)} \operatorname{sgn}(x) = 1$, whereas $\lim_{x\to 0;x\in(-\infty,0)} = -1$ and $\lim_{x\to 0;x\in\mathbb{R}} \operatorname{sgn}(x)$ is undefined. Thus, it is sometimes dangerous to drop the set E from the notation of limit. However, in many cases it is safe to do so.

E.g. I.9.3.17. Let $f(x)$ be the function

$$f(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then $\lim_{x\to 0;x\in\mathbb{R}\setminus\{0\}} f(x) = 0$, but $\lim_{x\to 0;x\in\mathbb{R}} f(x)$ is undefined. (When this happens, we say that f has a "removable singularity" or "removable discontinuity" at 0. Because of such singularities, it is sometimes the convention when writing $\lim_{x\to x_0} f(x)$ to automatically exclude x_0 from the set; for instance, in some textbook, $\lim_{x\to x_0} f(x)$ is used as shorthand for $\lim_{x\to x_0;x\in X\setminus\{x_0\}} f(x)$.)

Note. On the other hand, the limit at x_0 should only depend on the values of the function near x_0; the values away from x_0 are not relevant.

Prop. I.9.3.18 (Limits are local)**.** Let X be a subset of $\mathbb{R}$, let E be a subset of X, let x_0 be an adherent point of E, let $f : X \to \mathbb{R}$ be a function, and let L be a real number. Let $\delta > 0$. Then we have

$$\lim_{x\to x_0;x\in E} f(x) = L$$

iff

$$\lim_{x\to x_0;x\in E\cap(x_0-\delta,x_0+\delta)} f(x) = L.$$

Proof. We know that $\lim_{x\to x_0;x\in E} f(x) = L \implies \lim_{x\to x_0;x\in E\cap(x_0-\delta,x_0+\delta)} f(x) = L$ since $E \cap (x_0 - \delta, x_0 + \delta) \subseteq E$. We also know that $\lim_{x\to x_0;x\in E\cap(x_0-\delta,x_0+\delta)} f(x) = L \implies \lim_{x\to x_0;x\in E} f(x) = L$ since

$$\begin{aligned}
&\lim_{x\to x_0;x\in E\cap(x_0-\delta,x_0+\delta)} f(x) = L \\
\implies & \forall\varepsilon \in \mathbb{R}^+, \exists\delta' \in \mathbb{R}^+ : \\
& \quad \big(\forall x \in E \cap (x_0 - \delta, x_0 + \delta), |x - x_0| < \delta' \implies |f(x) - L| \le \varepsilon\big) \\
\implies & \forall\varepsilon \in \mathbb{R}^+, \exists\delta' \in \mathbb{R}^+ : \\
& \quad \big(\forall x \in E, (x_0 - \delta < x < x_0 + \delta) \wedge (|x - x_0| < \delta') \implies |f(x) - L| \le \varepsilon\big) \\
\implies & \forall\varepsilon \in \mathbb{R}^+, \exists\delta' \in \mathbb{R}^+ : \\
& \quad \big(\forall x \in E, (-\delta < x - x_0 < \delta) \wedge (|x - x_0| < \delta') \implies |f(x) - L| \le \varepsilon\big) \\
\implies & \forall\varepsilon \in \mathbb{R}^+, \exists\delta' \in \mathbb{R}^+ : \\
& \quad \big(\forall x \in E, (|x - x_0| < \delta) \wedge (|x - x_0| < \delta') \implies |f(x) - L| \le \varepsilon\big)
\end{aligned}$$

$$
\begin{aligned}
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists \delta' \in \mathbb{R}^+ : \\
&\quad \big(\forall x \in E, |x - x_0| < \min(\delta, \delta') \implies |f(x) - L| \le \varepsilon\big) \\
&\implies \lim_{x \to x_0; x \in E} f(x) = L.
\end{aligned}
$$

Thus, we conclude that $\displaystyle\lim_{x \to x_0; x \in E} f(x) = \lim_{x \to x_0; x \in E \cap (x_0 - \delta, x_0 + \delta)} f(x) = L.$ □

Note. Informally, Prop. I.9.3.18 asserts that

$$\lim_{x \to x_0; x \in E} f(x) = \lim_{x \to x_0; x \in E \cap (x_0 - \delta, x_0 + \delta)} f(x).$$

Thus, the limit of a function at x_0, if it exists, only depends on the values of f near x_0; the values far away do not actually influence the limit.

A.Cor. I.9.3.1 (Limit superior and limi inferior)**.** Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, let E be a subset of X, and let x_0 be an adherent point of E. We define *limit superior at x_0 in E* as

$$\limsup_{x \to x_0; x \in E} f(x) = \inf\big\{\sup\{f(x) : x \in E \wedge |x - x_0| < \delta\} : \delta \in \mathbb{R}^+\big\}$$

and define *limit inferior at x_0 in E* as

$$\liminf_{x \to x_0; x \in E} f(x) = \sup\big\{\inf\{f(x) : x \in E \wedge |x - x_0| < \delta\} : \delta \in \mathbb{R}^+\big\}.$$

A.Cor. I.9.3.2. Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, let E be a subset of X, and let x_0 be an adherent point of E. Let $L \in \mathbb{R}$. We claim that the following statements are equivalent:

(a) $\displaystyle\limsup_{x \to x_0; x \in E} f(x) = L$

(b) For every sequence $(a_n)_{n=1}^{\infty}$ which consists entirely of elements of E and converges to x_0, the sequence $(f(a_n))_{n=1}^{\infty}$ has limit superior $\displaystyle\limsup_{n \to \infty} f(a_n) \le L$. There exists a sequence $(b_n)_{n=1}^{\infty}$ which consists entirely of elements of E and converges to x_0, and $\displaystyle\limsup_{n \to \infty} f(b_n) = L$.

Similarly, the following statements are equivalent:

(a) $\displaystyle\liminf_{x \to x_0; x \in E} f(x) = L$.

(b) For every sequence $(a_n)_{n=1}^{\infty}$ which consists entirely of elements of E and converges to x_0, the sequence $(f(a_n))_{n=1}^{\infty}$ has limit inferior $\displaystyle\liminf_{n \to \infty} f(a_n) \ge L$. There exists a sequence $(b_n)_{n=1}^{\infty}$ which consists entirely of elements of E and converges to x_0, and $\displaystyle\liminf_{n \to \infty} f(b_n) = L$.

Proof. We only prove for limit superior. The proof for limit inferior are similar. For each $\delta \in \mathbb{R}^+$, define X_δ as follow:

$$X_\delta = \{x \in E : |x - x_0| \leq \delta\}.$$

We know $X_\delta \neq \emptyset$ for each $\delta \in \mathbb{R}^+$ since $\delta \in \mathbb{R}^+$ and x_0 is an adherent point. By axiom of choice (Ax. I.8.1) we know that $\prod_{n\in\mathbb{N}} X_{\frac{1}{n}} \neq \emptyset$. If we choose some $g \in \prod_{n\in\mathbb{N}} X_{\frac{1}{n}}$ and define $(b_n)_{n=0}^{\infty}$ by setting $b_n = g(n)$ for each $n \in \mathbb{N}$, then we have

$$\begin{aligned}
&\forall n \in \mathbb{N}, b_n \in X_{\frac{1}{n}} \\
\implies &\forall n \in \mathbb{N}, 0 \leq |b_n - x_0| \leq \frac{1}{n} \\
\implies &\lim_{n\to\infty} |b_n - x_0| = 0 && \text{(by Cor. I.6.4.14)} \\
\implies &\lim_{n\to\infty} b_n - x_0 = 0 && \text{(by Cor. I.6.4.17)} \\
\implies &\lim_{n\to\infty} b_n = x_0. && \text{(by Thm. I.6.1.19)}
\end{aligned}$$

Let $S_\delta = \{f(x) : x \in X_\delta\}$. We can rewrite limit superior as follow:

$$\limsup_{x\to x_0; x\in E} f(x) = \inf\big\{\sup(S_\delta) : \delta \in \mathbb{R}^+\big\}.$$

We will use X_δ and S_δ throughout the proof of this corollary.

We first show that statement (a) implies statement (b). Suppose that

$$L = \limsup_{x\to x_0; x\in E} f(x) = \inf\big\{\sup(S_\delta) : \delta \in \mathbb{R}^+\big\}.$$

Let $(a_n)_{n=1}^{\infty}$ be a sequence which consists entirely of elements of E and $\lim_{n\to\infty} a_n = x_0$. Such sequence exists since Lem. I.9.1.14. Then we have

$$\begin{aligned}
&\lim_{n\to\infty} a_n = x_0 \\
\implies &\forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
&\forall n \geq N, |a_n - x_0| \leq \delta \\
\implies &\forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
&\forall n \geq N, \big(f(a_n) \in S_\delta\big) \land \big(f(a_n) \leq \sup(S_\delta)\big) && \text{(by Def. I.5.5.5)} \\
\implies &\forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
&\sup\big(f(a_n)\big)_{n=N}^{\infty} \leq \sup(S_\delta) && \text{(by Def. I.5.5.5)} \\
\implies &\forall \delta \in \mathbb{R}^+, \limsup_{n\to\infty} f(a_n) \leq \sup(S_\delta) && \text{(by Prop. I.6.4.12(c))}
\end{aligned}$$

$$\implies \limsup_{n\to\infty} f(a_n) \leq \inf\{\sup(S_\delta) : \delta \in \mathbb{R}^+\} = L. \qquad \text{(by Rmk. I.5.5.15)}$$

For each $n \in \mathbb{Z}^+$, define Y_n as follow:

$$Y_n = \left\{ x \in X_{\frac{1}{n}} : L - \frac{1}{n} < f(x) \leq L \right\}.$$

We must have $Y_n \neq \emptyset$ for each $n \in \mathbb{N}$, otherwise

$$\begin{aligned}
&Y_n = \emptyset \\
\implies &\forall x \in X_{\frac{1}{n}}, f(x) \leq L - \frac{1}{n} \\
\implies &\sup(S_{\frac{1}{n}}) \leq L - \frac{1}{n} \\
\implies &L = \inf\{\sup(S_\delta) : \delta \in \mathbb{R}^+\} \leq \sup(S_{\frac{1}{n}}) \leq L - \frac{1}{n},
\end{aligned}$$

a contradiction. By axiom of choice (Ax. I.8.1) we know that $\prod_{n\in\mathbb{Z}^+} Y_n \neq \emptyset$. Let $g \in \prod_{n\in\mathbb{Z}^+} Y_n$ and define $(b_n)_{n=1}^\infty$ by setting $b_n = g(n)$ for each $n \in \mathbb{Z}^+$. Then we have $\lim_{n\to\infty} b_n = x_0$ and $L - \frac{1}{n} < f(b_n) \leq L$ for each $n \in \mathbb{Z}^+$. By squeeze test (Cor. I.6.4.14) we have $\lim_{n\to\infty} f(b_n) = L$. By Prop. I.6.4.12(f) we have $\limsup_{n\to\infty} f(b_n) = \lim_{n\to\infty} f(b_n) = L$.

Now we show that statement (b) implies statement (a). Suppose that for every sequence $(a_n)_{n=1}^\infty$ in E, $\lim_{n\to\infty} a_n = x_0 \implies \limsup_{n\to\infty} f(a_n) \leq L$. Suppose also that there exists a sequence $(b_n)_{n=1}^\infty$ in E such that $\lim_{n\to\infty} b_n = x_0 \implies \limsup_{n\to\infty} f(b_n) = L$. Observe that

$$\begin{aligned}
&\lim_{n\to\infty} b_n = x_0 \\
\implies &\forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
&\forall n \geq N, |b_n - x_0| \leq \delta \\
\implies &\forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
&\forall n \geq N, \big(f(b_n) \in S_\delta\big) \wedge \big(f(b_n) \leq \sup(S_\delta)\big) && \text{(by Def. I.5.5.5)} \\
\implies &\forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
&\sup \big(f(b_n)\big)_{n=N}^\infty \leq \sup(S_\delta) && \text{(by Def. I.5.5.5)} \\
\implies &\forall \delta \in \mathbb{R}^+, L = \limsup_{n\to\infty} f(b_n) \leq \sup(S_\delta) && \text{(by Prop. I.6.4.12(c))} \\
\implies &L \leq \inf\{\sup(S_\delta) : \delta \in \mathbb{R}^+\}. && \text{(by Rmk. I.5.5.15)}
\end{aligned}$$

Suppose for the sake of contradiction that $L < \inf\{\sup(S_\delta) : \delta \in \mathbb{R}^+\}$. Then we know that

$$\inf\{\sup(S_\delta) : \delta \in \mathbb{R}^+\} = L + \varepsilon$$

for some $\varepsilon \in \mathbb{R}^+$. Since statement (a) implies statement (b), we know that there exist a sequence $(c_n)_{n=1}^\infty$ in E such that $\lim_{n\to\infty} c_n = x_0$ and $\lim_{n\to\infty} f(c_n) = L + \varepsilon$. But this contradicts the hypothesis that $\lim_{n\to\infty} f(c_n) \le L$. Thus, we must have $L = \inf\{\sup(S_\delta) : \delta \in \mathbb{R}^+\}$. □

— Exercises —

Ex. I.9.3.1. Prove Prop. I.9.3.9.

Proof. See Prop. I.9.3.9. □

Ex. I.9.3.2. Prove the remaining claims in Prop. I.9.3.14.

Proof. See Prop. I.9.3.14. □

Ex. I.9.3.3. Prove Prop. I.9.3.18.

Proof. See Prop. I.9.3.18. □

Ex. I.9.3.4. Propose a definition for limit superior $\limsup_{x\to x_0; x\in E} f(x)$ and limit inferior $\liminf_{x\to x_0; x\in E} f(x)$, and then propose an analogue of Prop. I.9.3.9 for your definition. (For an additional challenge: prove that analogue.)

Proof. See Addtional A.Cor. I.9.3.1 and Addtional A.Cor. I.9.3.2. □

Ex. I.9.3.5 (Continuous version of squeeze test)**.** Let X be a subset of $\mathbb{R}$, let E be a subset of X, let x_0 be an adherent point of E, and let $f : X \to \mathbb{R}$, $g : X \to \mathbb{R}$, $h : X \to \mathbb{R}$ be functions such that $f(x) \le g(x) \le h(x)$ for all $x \in E$. If we have $\lim_{x\to x_0; x\in E} f(x) = \lim_{x\to x_0; x\in E} h(x) = L$ for some real number L, show that $\lim_{x\to x_0; x\in E} g(x) = L$.

Proof. Since $\lim_{x\to x_0; x\in E} f(x) = L$, by Def. I.9.3.6 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta_1 \in \mathbb{R}^+ : \big(\forall x \in E, |x - x_0| < \delta_1 \implies |f(x) - L| \le \varepsilon\big).$$

Similarly, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta_2 \in \mathbb{R}^+ : \big(\forall x \in E, |x - x_0| < \delta_2 \implies |h(x) - L| \le \varepsilon\big).$$

Let $\delta = \min(\delta_1, \delta_2)$. Then we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ :$$

$$\forall x \in E, |x - x_0| < \delta \implies \big(|f(x) - L| \leq \varepsilon\big) \wedge \big(|h(x) - L| \leq \varepsilon\big).$$

Since $f(x) \leq g(x) \leq h(x)$, we have

$$\begin{aligned}
&\big(x \in E\big) \wedge \big(|x - x_0| < \delta\big) \\
\implies &\big(f(x) \leq g(x) \leq h(x)\big) \wedge \big(|f(x) - L| < \varepsilon\big) \wedge \big(|h(x) - L| < \varepsilon\big) \\
\implies &-\varepsilon \leq f(x) - L \leq g(x) - L \leq h(x) - L \leq \varepsilon \\
\implies &|g(x) - L| \leq \varepsilon.
\end{aligned}$$

But this means

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in E, |x - x_0| < \delta \implies |g(x) - L| \leq \varepsilon\big)$$

and thus by Def. I.9.3.6 $\lim_{x \to x_0; x \in E} g(x) = L$. □

I.9.4 Continuous functions

Def. I.9.4.1 (Continuity)**.** Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function. Let x_0 be an element of X. We say that f is *continuous at* x_0 iff we have

$$\lim_{x \to x_0; x \in X} f(x) = f(x_0);$$

in other words, the limit of $f(x)$ as x converges to x_0 in X exists and is equal to $f(x_0)$. We say that f is *continuous on* X (or simply *continuous*) iff f is continuous at x_0 for every $x_0 \in X$. We say that f is *discontinuous at* x_0 iff it is not continuous at x_0. We also extend these notions to functions $f : X \to Y$ that take values in a subset Y of $\mathbb{R}$, by identifying such functions (by abuse of notation) with the function $\tilde{f} : X \to \mathbb{R}$ that agrees everywhere with f (so $f(\tilde{x}) = f(x)$ for all $x \in X$) but where the codomain has been enlarged from Y to $\mathbb{R}$.

Note. Restricting the domain of a function can make a discontinuous function continuous again.

Prop. I.9.4.7 (Equivalent formulations of continuity)**.** Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a function, and let x_0 be an element of X. Then the following four statements are logically equivalent:

(a) f is continuous at x_0.

(b) For every sequence $(a_n)_{n=0}^{\infty}$ consisting of elements of X with $\lim_{n \to \infty} a_n = x_0$, we have $\lim_{n \to \infty} f(a_n) = f(x_0)$.

(c) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in X$ with $|x - x_0| < \delta$.

(d) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(x_0)| \leq \varepsilon$ for all $x \in X$ with $|x - x_0| \leq \delta$.

Proof. We first show that the statement (a) and the statement (b) are equivalent. By Def. I.9.4.1, f is continuous at x_0 iff f converges to $f(x_0)$ at x_0 in X. Thus, by Prop. I.9.3.9 we know that the statement (a) and the statement (b) are equivalent.

Next we show that the statement (a) and the statement (c) are equivalent. By Def. I.9.4.1, f is continuous at x_0 iff f converges to $f(x_0)$ at x_0 in X. By Def. I.9.3.6 this is equivalent to the statement

$$\forall\varepsilon \in \mathbb{R}^+, \exists\delta \in \mathbb{R}^+ : \big(\forall x \in X, |x - x_0| < \delta \implies |f(x) - f(x_0)| \leq \frac{\varepsilon}{2} < \varepsilon\big).$$

Thus, the statement (a) and the statement (c) are equivalent.

Finally we show that the statement (a) and the statement (d) are equivalent. By Def. I.9.4.1, f is continuous at x_0 iff f converges to $f(x_0)$ at x_0 in X. By Def. I.9.3.6 this is equivalent to the statement

$$\forall\varepsilon \in \mathbb{R}^+, \exists\delta' \in \mathbb{R}^+ : \big(\forall x \in X, |x - x_0| < \delta' \implies |f(x) - f(x_0)| \leq \varepsilon\big).$$

Let $\delta \in \mathbb{R}^+$ and $|x - x_0| \leq \delta < \delta'$. By Prop. I.5.4.14 we know such δ exists. Then we have

$$\forall\varepsilon \in \mathbb{R}^+, \exists\delta \in \mathbb{R}^+ : \big(\forall x \in X, |x - x_0| \leq \delta \implies |f(x) - f(x_0)| \leq \varepsilon\big).$$

Thus, the statement (a) and the statement (d) are equivalent. □

Rmk. I.9.4.8. A particularly useful consequence of Prop. I.9.4.7 is the following: if f is continuous at x_0, and $a_n \to x_0$ as $n \to \infty$, then $f(a_n) \to f(x_0)$ as $n \to \infty$ (provided that all the elements of the sequence $(a_n)_{n=0}^{\infty}$ lie in the domain of f, of course). Thus, continuous functions are very useful in computing limits.

Prop. I.9.4.9 (Arithmetic preserves continuity)**.** Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions. Let $x_0 \in X$. Then if f and g are both continuous at x_0, then the functions $f + g$, $f - g$, $\max(f, g)$, $\min(f, g)$ and fg are also continuous at x_0. If g is non-zero on X, then f/g is also continuous at x_0.

Proof. By Prop. I.9.3.14, we have

$$\begin{aligned}
\lim_{x \to x_0; x \in X} f(x) + g(x) &= \lim_{x \to x_0; x \in X} f(x) + \lim_{x \to x_0; x \in X} g(x) \\
&= f(x_0) + g(x_0); && \text{(by Def. I.9.4.1)} \\
\lim_{x \to x_0; x \in X} f(x) - g(x) &= \lim_{x \to x_0; x \in X} f(x) - \lim_{x \to x_0; x \in X} g(x) \\
&= f(x_0) - g(x_0); && \text{(by Def. I.9.4.1)} \\
\lim_{x \to x_0; x \in X} \max(f(x), g(x)) &= \max(\lim_{x \to x_0; x \in X} f(x), \lim_{x \to x_0; x \in X} g(x))
\end{aligned}$$

$$\begin{aligned}
&= \max(f(x_0), g(x_0)); && \text{(by Def. I.9.4.1)}\\
\lim_{x\to x_0; x\in X} \min(f(x), g(x)) &= \min(\lim_{x\to x_0; x\in X} f(x), \lim_{x\to x_0; x\in X} g(x))\\
&= \min(f(x_0), g(x_0)); && \text{(by Def. I.9.4.1)}\\
\lim_{x\to x_0; x\in X} f(x)g(x) &= \left(\lim_{x\to x_0; x\in X} f(x)\right)\left(\lim_{x\to x_0; x\in X} g(x)\right)\\
&= f(x_0)g(x_0); && \text{(by Def. I.9.4.1)}\\
\lim_{x\to x_0; x\in X} f(x)/g(x) &= \lim_{x\to x_0; x\in X} f(x)/\lim_{x\to x_0; x\in X} g(x) && \text{(g is non-zero on X)}\\
&= f(x_0)/g(x_0). && \text{(by Def. I.9.4.1)}
\end{aligned}$$

□

Prop. I.9.4.10 (Exponentiation is continuous, I)**.** Let $a > 0$ be a positive real number. Then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := a^x$ is continuous.

Proof. Let $x_0 \in \mathbb{R}$. By Lem. I.9.1.13 x_0 is an adherent point. By Prop. I.6.7.3 we know that $a^{x_0} > 0$. By Lem. I.6.5.3 we know that $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$, thus

$$\begin{aligned}
a^{x_0} &= a^{x_0} \lim_{n\to\infty} a^{\pm\frac{1}{n}} && \text{(by Prop. I.6.7.3)}\\
&= \lim_{n\to\infty} (a^{x_0} a^{\pm\frac{1}{n}}) && \text{(by Thm. I.6.1.19)}\\
&= \lim_{n\to\infty} a^{x_0 \pm \frac{1}{n}}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\lim_{n\to\infty} a^{x_0\pm\frac{1}{n}} = a^{x_0}\\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \geq N, \left| a^{x_0\pm\frac{1}{n}} - a^{x_0} \right| \leq \varepsilon\\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \left| a^{x_0\pm\frac{1}{N}} - a^{x_0} \right| \leq \varepsilon\\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : -\varepsilon \leq a^{x_0\pm\frac{1}{N}} - a^{x_0} \leq \varepsilon. && \text{(by Prop. I.6.7.3)}
\end{aligned}$$

Now fix N for each $\varepsilon \in \mathbb{R}^+$. By Prop. I.6.7.3 we have

$$
\begin{aligned}
&\forall x \in \mathbb{R}, |x - x_0| < \frac{1}{N} \\
\implies & \frac{-1}{N} < x - x_0 < \frac{1}{N} \\
\implies & x_0 - \frac{1}{N} < x < x_0 + \frac{1}{N} \\
\implies & \begin{cases} a^{x_0 - \frac{1}{N}} < a^x < a^{x_0 + \frac{1}{N}} & \text{if } a \geq 1 \\ a^{x_0 + \frac{1}{N}} < a^x < a^{x_0 - \frac{1}{N}} & \text{if } a < 1 \end{cases} \\
\implies & \begin{cases} -\varepsilon < a^{x_0 - \frac{1}{N}} - a^{x_0} < a^x - a^{x_0} < a^{x_0 + \frac{1}{N}} - a^{x_0} < \varepsilon & \text{if } a \geq 1 \\ -\varepsilon < a^{x_0 + \frac{1}{N}} - a^{x_0} < a^x - a^{x_0} < a^{x_0 - \frac{1}{N}} - a^{x_0} < \varepsilon & \text{if } a < 1 \end{cases} \\
\implies & |a^x - a^{x_0}| < \varepsilon.
\end{aligned}
$$

By setting $\delta = \dfrac{1}{N}$ we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in \mathbb{R}, |x - x_0| < \delta \implies |a^x - a^{x_0}| < \varepsilon\big).$$

Thus, by Def. I.9.3.6 we have $\lim_{x \to x_0; x \in \mathbb{R}} a^x = a^{x_0}$. Since x_0 was arbitrary, by Def. I.9.4.1 a^x is continuous on $\mathbb{R}$. □

Prop. I.9.4.11 (Exponentiation is continuous, II)**.** Let p be a real number. Then the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) := x^p$ is continuous.

Proof. By Prop. I.9.3.14 we know that

$$\forall n \in \mathbb{N}, \lim_{x \to 1; x \in (0,\infty)} x^n = 1.$$

Again by Prop. I.9.3.14 we know that

$$\forall n \in \mathbb{N}, \lim_{x \to 1; x \in (0,\infty)} x^{-n} = \lim_{x \to 1; x \in (0,\infty)} 1/x^n = 1.$$

By Ex. I.5.4.3 we have

$$\forall p \in \mathbb{R}, \exists N \in \mathbb{Z} : N \leq p < N + 1.$$

By Prop. I.6.7.3 we have

$$\forall x \in (0, \infty), \begin{cases} x^N \leq x^p < x^{N+1} & \text{if } x \in (0, 1) \\ x^{N+1} < x^p \leq x^N & \text{if } x \in [1, \infty) \end{cases}$$

Thus, by squeeze test (Ex. I.9.3.5) we have $\lim_{x\to 1; x\in(0,\infty)} x^p = 1$.

Let $x_0 \in (0,\infty)$. By Lem. I.9.1.12 x_0 is an adherent point. By Prop. I.6.7.3 $x_0^p > 0$. Since $\lim_{x\to 1; x\in(0,\infty)} x^p = 1$, by Def. I.9.3.6 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in (0,\infty), |x-1| < \delta \implies |x^p - 1| \le \varepsilon\big).$$

Let $y = x \cdot x_0$. Then $x = y/x_0$ and

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall y \in (0,\infty), |y/x_0 - 1| < \delta \implies \left|(\frac{y}{x_0})^p - 1\right| \le \varepsilon\big).$$

Since $\delta/x_0 < \delta$, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall y \in (0,\infty), |y/x_0 - 1| < \delta/x_0 \implies \left|(\frac{y}{x_0})^p - 1\right| \le \varepsilon\big).$$

This means

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall y \in (0,\infty), |y - x_0| < \delta \implies \left|(\frac{y}{x_0})^p - 1\right| \le \varepsilon\big).$$

In particular, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall y \in (0,\infty), |y - x_0| < \delta \implies \left|(\frac{y}{x_0})^p - 1\right| \le \frac{\varepsilon}{x_0^p}\big).$$

This means

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall y \in (0,\infty), |y - x_0| < \delta \implies |y^p - x_0^p| \le \varepsilon\big).$$

Thus, by Def. I.9.3.6 we have $\lim_{y\to x_0; y\in(0,\infty)} y^p = x_0^p$. Since x_0 was arbitrary, by Def. I.9.4.1 x^p is continuous on $(0,\infty)$. □

Prop. I.9.4.12 (Absolute value is continuous)**.** The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := |x|$ is continuous.

Proof. This follows since $|x| = \max(x, -x)$ and the functions $x, -x$ are already continuous. □

Prop. I.9.4.13 (Composition preserves continuity)**.** Let X and Y be subsets of $\mathbb{R}$, and let $f : X \to Y$ and $g : Y \to \mathbb{R}$ be functions. Let x_0 be a point in X. If f is continuous at x_0, and g is continuous at $f(x_0)$, then the composition $g \circ f : X \to \mathbb{R}$ is continuous at x_0.

Proof. Since $\lim_{y\to f(x_0);y\in Y} g(x) = g\big(f(x_0)\big)$, by Def. I.9.3.6 we have

$$\forall\varepsilon\in\mathbb{R}^+, \exists\delta'\in\mathbb{R}^+ : \Big(\forall y\in Y, |y-f(x_0)|<\delta' \implies \big|g(y)-g\big(f(x_0)\big)\big|<\varepsilon\Big).$$

Since $\lim_{x\to x_0;x\in X} f(x) = f(x_0)$, by Def. I.9.3.6 we have

$$\forall\varepsilon'\in\mathbb{R}^+, \exists\delta\in\mathbb{R}^+ : \Big(\forall x\in X, |x-x_0|<\delta \implies |f(x)-f(x_0)|<\varepsilon'\Big).$$

In particular, we have

$$\exists\delta\in\mathbb{R}^+ : \Big(\forall x\in X, |x-x_0|<\delta \implies |f(x)-f(x_0)|<\delta'\Big).$$

Since $f(x)\in Y$ and $|f(x)-f(x_0)|<\delta'$ implies $\big|g\big(f(x)\big)-g\big(f(x_0)\big)\big|<\varepsilon$, we have

$$\forall\varepsilon\in\mathbb{R}^+, \exists\delta\in\mathbb{R}^+ : \Big(\forall x\in X, |x-x_0|<\delta \implies \big|g\big(f(x)\big)-g\big(f(x_0)\big)\big|<\varepsilon\Big).$$

Thus, by Def. I.9.3.6 we have $\lim_{x\to x_0;x\in X} g\big(f(x)\big) = g\big(f(x_0)\big)$. By Def. I.9.4.1 $g\circ f$ is continuous at x_0. □

— Exercises —

Ex. I.9.4.1. Prove Prop. I.9.4.7.

Proof. See Prop. I.9.4.7 □

Ex. I.9.4.2. Let X be a subset of $\mathbb{R}$, and let $c\in\mathbb{R}$. Show that the constant function $f : X\to\mathbb{R}$ defined by $f(x) := c$ is continuous, and show that the identity function $g : X\to\mathbb{R}$ defined by $g(x) := x$ is also continuous.

Proof. We first show that the constant functions f is continuous. Let $x_0\in X$. By Lem. I.9.1.11 we know that $X\subseteq\overline{X}$, thus x_0 is an adherent point of X. Since

$$\forall\varepsilon\in\mathbb{R}^+, \forall\delta\in\mathbb{R}^+, |x-x_0|<\delta \implies |f(x)-c| = |c-c| = 0<\varepsilon,$$

we know that $\lim_{x\to x_0;x\in X} f(x) = c$. Since $f(x_0)=c$ and x_0 was arbitrary, by Def. I.9.4.1 we know that the constant functions f is continuous on X.

Now we show that the identity function g is continuous. Let $x_0\in X$. By Lem. I.9.1.11 we know that $X\subseteq\overline{X}$, thus x_0 is an adherent point of X. Since

$$\forall\varepsilon\in\mathbb{R}^+, |x-x_0|<\varepsilon \implies |g(x)-x_0| = |x-x_0|<\varepsilon,$$

we know that $\lim_{x\to x_0;x\in X} g(x) = x_0$. Since $g(x_0)=x_0$ and x_0 was arbitrary, by Def. I.9.4.1 we know that the constant functions g is continuous on X. □

Ex. I.9.4.3. Prove Prop. I.9.4.10.

Proof. See Prop. I.9.4.10. $\square$

Ex. I.9.4.4. Prove Prop. I.9.4.11.

Proof. See Prop. I.9.4.11. $\square$

Ex. I.9.4.5. Prove Prop. I.9.4.13.

Proof. See Prop. I.9.4.13. $\square$

Ex. I.9.4.6. Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a continuous function. If Y is a subset of X, show that the restriction $f|_Y : Y \to \mathbb{R}$ of f to Y is also a continuous function.

Proof. By Def. I.9.4.1, $\forall x_0 \in X$, we have $\lim_{x \to x_0; x \in X} f(x) = f(x_0)$. Since $Y \subseteq X$, we have $\forall y \in Y \implies y \in X$. Thus, $\forall y_0 \in Y$ we have $\lim_{y \to y_0; y \in Y} f(y) = f(y_0)$. By Def. I.9.4.1, $f|_Y$ is continuous on Y. $\square$

Ex. I.9.4.7. Let $n \geq 0$ be an integer, and for each $0 \leq i \leq n$ let c_i be a real number. Let $P : \mathbb{R} \to \mathbb{R}$ be the function

$$P(x) \coloneqq \sum_{i=0}^{n} c_i x^i;$$

Such a function is known as a *polynomial of one variable*; Show that P is continuous.

Proof. Let $F_n = \{$all polynomial function with the highest order being $n\}$. Let $Q(n)$ be the statement "all function $f \in F_n$ are continuous." We induct on n to show that $\forall n \in \mathbb{N}$, $Q(n)$ is true. For $n = 0$, we have

$$\sum_{n=0}^{0} c_i x^i = c_0 x^0 = c_0.$$

By Ex. I.9.4.2 we know that constant functions are continuous, and Thus, the base case holds. Suppose inductively that $Q(n)$ is true for some $n \geq 0$. To show that $Q(n+1)$ is true, observe that every function $f \in F_{n+1}$ are in the form

$$\begin{aligned} f(x) &= \sum_{i=0}^{n+1} c_i x^i \\ &= \sum_{i=0}^{n} c_i x^i + c_{n+1} x^{n+1} \\ &= \sum_{i=0}^{n} c_i x^i + c_{n+1}(x^n \cdot x). \end{aligned}$$

By the induction hypothesis we know that $\sum_{i=0}^{n} c_i x^i$ and x^n are continuous. By Ex. I.9.4.2 we know that c_{n+1} and x are continuous. Thus, by Prop. I.9.4.9 we know that f is continuous. This closes the induction. $\square$

I.9.5 Left and right limits

Def. I.9.5.1 (Left and right limits)**.** Let X be a subset of $\mathbb{R}$, $f : X \to \mathbb{R}$ be a function, and let x_0 be a real number. If x_0 is an adherent point of $X \cap (x_0, \infty)$, then we define the *right limit* $f(x_0+)$ of f at x_0 by the formula

$$f(x_0+) \coloneqq \lim_{x \to x_0; x \in X \cap (x_0, \infty)} f(x),$$

provided of course that this limit exists. Similarly, if x_0 is an adherent point of $X \cap (-\infty, x_0)$, then we define the *left limit* $f(x_0-)$ of f at x_0 by the formula

$$f(x_0-) \coloneqq \lim_{x \to x_0; x \in X \cap (-\infty, x_0)} f(x),$$

again provided that the limit exists. (Thus, in many cases $f(x_0+)$ and $f(x_0-)$ will not be defined.) Sometimes we use the shorthand notations

$$\begin{aligned} \lim_{x \to x_0+} f(x) &\coloneqq \lim_{x \to x_0; x \in X \cap (x_0, \infty)} f(x); \\ \lim_{x \to x_0-} f(x) &\coloneqq \lim_{x \to x_0; x \in X \cap (-\infty, x_0)} f(x) \end{aligned}$$

when the domain X of f is clear from context.

Note. From Prop. I.9.3.9 we see that if the right limit $f(x_0+)$ exists, and $(a_n)_{n=0}^{\infty}$ is a sequence in X converging to x_0 from the right (i.e., $a_n > x_0$ for all $n \in \mathbb{N}$), then $\lim_{n \to \infty} f(a_n) = f(x_0+)$. Similarly, if $(b_n)_{n=0}^{\infty}$ is a sequence converging to x_0 from the left (i.e., $a_n < x_0$ for all $n \in \mathbb{N}$) then $\lim_{n \to \infty} f(a_n) = f(x_0-)$.

A.Cor. I.9.5.1. Let x_0 be an adherent point of both $X \cap (x_0, \infty)$ and $X \cap (-\infty, x_0)$. If f is continuous at x_0, then $f(x_0+)$ and $f(x_0-)$ both exists and are equal to $f(x_0)$.

Proof. Since f is continuous at x_0, by Def. I.9.4.1 we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in X, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon\big).$$

Since $X \cap (-\infty, x_0) \subseteq X$, we must have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in X \cap (-\infty, x_0), |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon\big)$$

and by Def. I.9.5.1 we have $f(x_0+) = f(x_0)$. Similarly, we have $f(x_0-) = f(x_0)$. $\square$

Prop. I.9.5.3. Let X be a subset of $\mathbb{R}$ containing a real number x_0, and suppose that x_0 is an adherent point of both $X \cap (x_0, \infty)$ and $X \cap (-\infty, x_0)$. Let $f : X \to \mathbb{R}$ be a function. If $f(x_0+)$ and $f(x_0-)$ both exist and are both equal to $f(x_0)$, then f is continuous at x_0.

Proof. Let us write $L := f(x_0)$. Then by hypothesis we have

$$\lim_{x \to x_0; x \in X \cap (x_0, \infty)} f(x) = L$$

and

$$\lim_{x \to x_0; x \in X \cap (-\infty, x_0)} f(x) = L.$$

Let $\varepsilon > 0$ be given. From the first statement above and Prop. I.9.4.7 (applied to the restriction of f to $X \cap (x_0, +\infty)$), we know that there exists a $\delta_+ > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in X \cap (x_0, \infty)$ for which $|x - x_0| < \delta_+$. From the second statement above we similarly know that there exists a $\delta_- > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in X \cap (-\infty, x_0)$ for which $|x - x_0| < \delta_-$. Now let $\delta := \min(\delta_-, \delta_+)$; then $\delta > 0$, and suppose that $x \in X$ is such that $|x - x_0| < \delta$. Then there are three cases: $x > x_0$, $x = x_0$, and $x < x_0$, but in all three cases we know that $|f(x) - L| < \varepsilon$ since

- If $x > x_0$, then $x \in X \cap (x_0, \infty)$ and $|x - x_0| < \delta \leq \delta_+ \implies |f(x) - L| < \varepsilon$.
- If $x < x_0$, then $x \in X \cap (-\infty, x_0)$ and $|x - x_0| < \delta \leq \delta_- \implies |f(x) - L| < \varepsilon$.
- If $x = x_0$, then we have $|x_0 - x_0| = 0 < \delta$ and $|f(x_0) - f(x_0)| = 0 < \varepsilon$.

By Prop. I.9.4.7 we thus have that f is continuous at x_0, as desired. □

Note. When both $f(x_0+), f(x_0-)$ exist and $f(x_0+) \neq f(x_0-)$, we say that f has a *jump discontinuity* at x_0. When both $f(x_0+), f(x_0-)$ exist and $f(x_0+) = f(x_0-) \neq f(x_0)$, we say that f has a *removable discontinuity* (or *removable singularity*) at x_0.

Rmk. I.9.5.4. Jump discontinuities and removable discontinuities are not the only way a function can be discontinuous. Another way is for a function to go to infinity at the discontinuity: for instance, the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) := 1/x$ has a discontinuity at 0 which is neither a jump discontinuity or a removable singularity; informally, $f(x)$ converges to $+\infty$ when x approaches 0 from the right, and converges to $-\infty$ when x approaches 0 from the left. These types of singularities are sometimes known as *asymptotic discontinuities*. There are also *oscillatory discontinuities*, where the function remains bounded but still does not have a limit near x_0. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

has an oscillatory discontinuity at 0 (and in fact at any other real number also). This is because the function does not have left or right limits at 0, despite the fact that the function is bounded.

Note. The study of discontinuities is also called *singularities.*

A.Cor. I.9.5.2. We define $\lim_{x\to x_0;x\in E} f(x) = +\infty$ iff

$$\forall\varepsilon\in\mathbb{R}^+, \exists\delta\in\mathbb{R}^+ : \big(\forall x\in E, |x-x_0|<\delta \implies f(x)>\varepsilon\big).$$

And define $\lim_{x\to x_0;x\in E} f(x) = -\infty$ iff

$$\forall\varepsilon\in\mathbb{R}^+, \exists\delta\in\mathbb{R}^+ : \big(\forall x\in E, |x-x_0|<\delta \implies f(x)<-\varepsilon\big).$$

Show that $\lim_{x\to 0;x\in\mathbb{R}\cap(0,\infty)} 1/x = +\infty$ and $\lim_{x\to 0;x\in\mathbb{R}\cap(-\infty,0)} 1/x = -\infty$.

Proof. We first show that $\lim_{x\to 0;x\in\mathbb{R}\cap(0,+\infty)} 1/x = +\infty$ and $\lim_{x\to 0;x\in\mathbb{R}\cap(-\infty,0)} 1/x = -\infty$. Let $\varepsilon\in\mathbb{R}^+$. $\forall x\in\mathbb{R}\cap(0,\infty)$, we have

$$x = |x| = |x-0| < 1/\varepsilon \implies 1/x < \varepsilon.$$

By letting $\delta = 1/\varepsilon$ we have

$$\forall\varepsilon\in\mathbb{R}^+, \exists\delta\in\mathbb{R}^+ : \big(\forall x\in\mathbb{R}\cap(0,\infty), |x-0|<\delta \implies 1/x>\varepsilon\big).$$

Thus, by definition we have $\lim_{x\to 0;x\in\mathbb{R}\cap(0,+\infty)} 1/x = +\infty$. Similarly, $\forall x\in\mathbb{R}\cap(-\infty,0)$, we have

$$-x = |x| = |x-0| < 1/\varepsilon \implies 1/x < -\varepsilon.$$

By letting $\delta = 1/\varepsilon$ we have

$$\forall\varepsilon\in\mathbb{R}^+, \exists\delta\in\mathbb{R}^+ : \big(\forall x\in\mathbb{R}\cap(-\infty,0), |x-0|<\delta \implies 1/x<-\varepsilon\big).$$

Thus, by definition we have $\lim_{x\to 0;x\in\mathbb{R}\cap(-\infty,0)} 1/x = -\infty$. □

A.Cor. I.9.5.3. Let $X\subseteq\mathbb{R}$, let $E\subseteq X$, let $x_0\in\overline{E}$. Show that the following two statements are equivalent:

(a) $\lim_{x\to x_0;x\in E} f(x) = +\infty$.

(b) For every sequence $(a_n)_{n=0}^{\infty}$ which consists entirely of elements of E and converges to x_0, the sequence $(f(a_n))_{n=0}^{\infty}$ diverges to $+\infty$.

Proof. We first show that statement (a) implies statement (b). By A.Cor. I.9.5.2 we have

$$\begin{aligned}&\lim_{x\to x_0;x\in E} f(x) = +\infty\\ \iff &\forall\varepsilon\in\mathbb{R}^+, \exists\delta\in\mathbb{R}^+ : \big(\forall x\in E, |x-x_0|<\delta \implies f(x)>\varepsilon\big).\end{aligned}$$

Let $(a_n)_{n=0}^{\infty}$ be a sequence which consists entirely of elements of E and $\lim\limits_{n\to\infty} a_n = x_0$. Such sequence exists since Lem. I.9.1.14. Then we have

$$\begin{aligned}
&\lim_{n\to\infty} a_n = x_0 \\
\implies &\exists N \in \mathbb{N} : \forall n \geq N, |a_n - x_0| \leq \frac{\delta}{2} < \delta \\
\implies &\exists N \in \mathbb{N} : \forall n \geq N, f(a_n) > \varepsilon && (a_n \in E) \\
\implies &\lim_{n\to\infty} f(a_n) = +\infty.
\end{aligned}$$

Since $(a_n)_{n=0}^{\infty}$ was arbitrary, we conclude that statement (a) implies statement (b).

Now we show that statement (b) implies statement (a). Suppose for the sake of contradiction that $\lim\limits_{x\to x_0; x\in E} f(x) \neq +\infty$. Then we must have

$$\exists \varepsilon \in \mathbb{R}^+ : \forall \delta \in \mathbb{R}^+, (|x - x_0| < \delta) \wedge \big(f(x) \leq \varepsilon\big).$$

Let $(a_n)_{n=0}^{\infty}$ be a sequence in E such that $\lim\limits_{n\to\infty} a_n = x_0$. Such sequence exists since Lem. I.9.1.14. Then we have

$$\begin{aligned}
&\big(\lim_{n\to\infty} a_n = x_0\big) \wedge \big(\lim_{n\to\infty} f(a_n) = +\infty\big) && \text{(by hypothesis)} \\
\implies &\begin{cases} \forall \delta \in \mathbb{R}^+, \exists N_1 \in \mathbb{N} : \forall n \geq N_1, |a_n - x_0| \leq \dfrac{\delta}{2} < \delta \\ \forall \varepsilon \in \mathbb{R}^+, \exists N_2 \in \mathbb{N} : \forall n \geq N_2, f(a_n) > \varepsilon \end{cases} \\
\implies &\exists N = \max(N_1, N_2) : \forall n \geq N, (|a_n - x_0| < \delta) \wedge \big(f(a_n) > \varepsilon\big).
\end{aligned}$$

But this contradicts $(|x - x_0| < \delta) \wedge \big(f(x) \leq \varepsilon\big)$. Thus, $\lim\limits_{x\to x_0; x\in E} f(x) = +\infty$. ☐

A.Cor. I.9.5.4. Let $X \subseteq \mathbb{R}$, let $E \subseteq X$, let $x_0 \in \overline{E}$. Show that the following two statements are equivalent:

(a) $\lim\limits_{x\to x_0; x\in E} f(x) = -\infty$.

(b) For every sequence $(a_n)_{n=0}^{\infty}$ which consists entirely of elements of E and converges to x_0, the sequence $(f(a_n))_{n=0}^{\infty}$ diverges to $-\infty$.

Proof. We first show that statement (a) implies statement (b). By A.Cor. I.9.5.2 we have

$$\begin{aligned}
&\lim_{x\to x_0; x\in E} f(x) = -\infty \\
\iff &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in E, |x - x_0| < \delta \implies f(x) < -\varepsilon\big).
\end{aligned}$$

Let $(a_n)_{n=0}^{\infty}$ be a sequence which consists entirely of elements of E and $\lim\limits_{n\to\infty} a_n = x_0$. Such sequence exists since Lem. I.9.1.14. Then we have

$$\lim_{n\to\infty} a_n = x_0$$

$$\begin{aligned}
&\implies \exists N \in \mathbb{N} : \forall n \geq N, |a_n - x_0| \leq \frac{\delta}{2} < \delta \\
&\implies \exists N \in \mathbb{N} : \forall n \geq N, f(a_n) < -\varepsilon && (a_n \in E) \\
&\implies \lim_{n\to\infty} f(a_n) = -\infty.
\end{aligned}$$

Since $(a_n)_{n=0}^{\infty}$ was arbitrary, we conclude that statement (a) implies statement (b).

Now we show that statement (b) implies statement (a). Suppose for the sake of contradiction that $\lim_{x\to x_0; x\in E} f(x) \neq -\infty$. Then we must have

$$\exists \varepsilon \in \mathbb{R}^+ : \forall \delta \in \mathbb{R}^+, (|x - x_0| < \delta) \land \big(f(x) \geq -\varepsilon\big).$$

Let $(a_n)_{n=0}^{\infty}$ be a sequence in E such that $\lim_{n\to\infty} a_n = x_0$. Such sequence exists since Lem. I.9.1.14. Then we have

$$\begin{aligned}
&\big(\lim_{n\to\infty} a_n = x_0\big) \land \big(\lim_{n\to\infty} f(a_n) = -\infty\big) && \text{(by hypothesis)} \\
\implies &\begin{cases} \forall \delta \in \mathbb{R}^+, \exists N_1 \in \mathbb{N} : \forall n \geq N_1, |a_n - x_0| \leq \dfrac{\delta}{2} < \delta \\ \forall \varepsilon \in \mathbb{R}^+, \exists N_2 \in \mathbb{N} : \forall n \geq N_2, f(a_n) < -\varepsilon \end{cases} \\
\implies &\exists N = \max(N_1, N_2) : \forall n \geq N, (|a_n - x_0| < \delta) \land \big(f(a_n) < -\varepsilon\big).
\end{aligned}$$

But this contradicts $(|x - x_0| < \delta) \land \big(f(x) \geq -\varepsilon\big)$. Thus, $\lim_{x\to x_0; x\in E} f(x) = -\infty$. □

— Exercises —

Ex. I.9.5.1. Let E be a subset of $\mathbb{R}$, let $f : E \to \mathbb{R}$ be a function, and let x_0 be an adherent point of E. Write down a definition of what it would mean for the limit $\lim_{x\to x_0; x\in E} f(x)$ to exist and equal $+\infty$ or $-\infty$. If $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the function $f(x) := 1/x$, use your definition to conclude $f(0+) = +\infty$ and $f(0-) = -\infty$. Also, state and prove some analogue of Prop. I.9.3.9 when $L = +\infty$ or $L = -\infty$.

Proof. See A.Cor. I.9.5.2, A.Cor. I.9.5.3 and A.Cor. I.9.5.4. □

I.9.6 The maximum principle

Def. I.9.6.1. Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function. We say that f is *bounded from above* iff there exists a real number M such that $f(x) \leq M$ for all $x \in X$. We say that f is *bounded from below* iff there exists a real number M such that $f(x) \geq -M$ for all $x \in X$. We say that f is *bounded* iff there exists a real number M such that $|f(x)| \leq M$ for all $x \in X$.

Rmk. I.9.6.2. A function is bounded iff it is bounded both from above and below. Also, a function $f : X \to \mathbb{R}$ is bounded iff its image $f(X)$ is a bounded set in the sense of Def. I.9.1.22.

Lem. I.9.6.3. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$. Then f is a bounded function.

Proof. Suppose for the sake of contradiction that f is not bounded. Thus, for every real number M there exists an element $x \in [a, b]$ such that $|f(x)| \geq M$.

In particular, for every natural number n, the set $\{x \in [a, b] : |f(x)| \geq n\}$ is non-empty. We can thus choose a sequence $(x_n)_{n=0}^{\infty}$ in $[a, b]$ such that $|f(x_n)| \geq n$ for all n. This sequence lies in $[a, b]$, and so by Thm. I.9.1.24 there exists a subsequence $(x_{n_j})_{j=0}^{\infty}$ which converges to some limit $L \in [a, b]$, where $n_0 < n_1 < n_2 < \dots$ is an increasing sequence of natural numbers. In particular, we see that $n_j \geq j$ for all $j \in \mathbb{N}$ (use induction).

Since f is continuous on $[a, b]$, it is continuous at L, and in particular, we see that

$$\lim_{j \to \infty} f(x_{n_j}) = f(L).$$

Thus, the sequence $(f(x_{n_j}))_{j=0}^{\infty}$ is convergent, and hence it is bounded. On the other hand, we know from the construction that $\left|f(x_{n_j})\right| \geq n_j \geq j$ for all j, and hence the sequence $(f(x_{n_j}))_{j=0}^{\infty}$ is not bounded, a contradiction. ☐

Rmk. I.9.6.4. There are two things about the proof of Lem. I.9.6.3 that are worth noting. Firstly, it shows how useful the Heine-Borel theorem (Thm. I.9.1.24) is. Secondly, it is an indirect proof; it doesn't say *how* to find the bound for f, but it shows that having f unbounded leads to a contradiction.

Def. I.9.6.5 (Maxima and minima)**.** Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function, and let $x_0 \in X$. We say that f *attains its maximum at* x_0 if we have $f(x_0) \geq f(x)$ for all $x \in X$ (i.e., the value of f at the point x_0 is larger than or equal to the value of f at any other point in X). We say that f *attains its minimum at* x_0 if we have $f(x_0) \leq f(x)$ for all $x \in X$.

Rmk. I.9.6.6. If a function attains its maximum somewhere, then it must be bounded from above. Similarly, if it attains its minimum somewhere, then it must be bounded from below. These notions of maxima and minima are *global*.

Prop. I.9.6.7 (Maximum principle)**.** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function continuous on $[a, b]$. Then f attains its maximum at some point $x_{\text{max}} \in [a, b]$, and also attains its minimum at some point $x_{\text{min}} \in [a, b]$.

Proof. From Lem. I.9.6.3 we know that f is bounded, thus there exists an M such that $-M \leq f(x) \leq M$ for each $x \in [a, b]$. Now let E denote the set

$$E \coloneqq \{f(x) : x \in [a, b]\}.$$

(In other words, $E \coloneqq f([a, b])$.) By what we just said, this set is a subset of $[-M, M]$. It is also non-empty, since it contains for instance the point $f(a)$. Hence by the least upper bound principle, it has a supremum $\sup(E)$ which is a real number.

Write $m := \sup(E)$. By definition of supremum, we know that $y \leq m$ for all $y \in E$; by definition of E, this means that $f(x) \leq m$ for all $x \in [a, b]$. Thus, to show that f attains its maximum somewhere, it will suffice to find an $x_{\max} \in [a, b]$ such that $f(x_{\max}) = m$.

Let $n \geq 1$ be any integer. Then $m - \dfrac{1}{n} < m = \sup(E)$. As $\sup(E)$ is the least upper bound for E, $m - \dfrac{1}{n}$ cannot be an upper bound for E, thus there exists a $y \in E$ such that $m - \dfrac{1}{n} < y$. By definition of E, this implies that there exists an $x \in [a, b]$ such that $m - \dfrac{1}{n} < f(x)$.

We now choose a sequence $(x_n)_{n=1}^{\infty}$ by choosing, for each n, x_n to be an element of $[a, b]$ such that $m - \dfrac{1}{n} < f(x_n)$. (Again, this requires the axiom of choice; however it is possible to prove this principle without the axiom of choice. For instance, you will see a better proof of this proposition using the notion of *compactness*) This is a sequence in $[a, b]$; by the Heine-Borel theorem (Thm. I.9.1.24), we can thus find a subsequence $(x_{n_j})_{j=1}^{\infty}$, where $n_1 < n_2 < \ldots$, which converges to some limit $x_{\max} \in [a, b]$. Since $(x_{n_j})_{j=1}^{\infty}$ converges to $x_{\max}$, and f is continuous at $x_{\max}$, we have as before that

$$\lim_{j \to \infty} f(x_{n_j}) = f(x_{\max})$$

On the other hand, by construction we know that

$$f(x_{n_j}) > m - \frac{1}{n_j} \geq m - \frac{1}{j},$$

and so by taking limits of both sides we see that

$$f(x_{\max}) = \lim_{j \to \infty} f(x_{n_j}) \geq \lim_{j \to \infty} m - \frac{1}{j} = m.$$

On the other hand, we know that $f(x) \leq m$ for all $x \in [a, b]$, so, in particular, $f(x_{\max}) \leq m$. Combining these two inequalities we see that $f(x_{\max}) = m$ as desired.

By the least upper bound principle again, E has a infimum $\inf(E)$ which is a real number. Write $m := \inf(E)$. By definition of infimum, we know that $y \geq m$ for all $y \in E$; by definition of E, this means that $f(x) \geq m$ for all $x \in [a, b]$. Thus, to show that f attains its minimum somewhere, it will suffice to find an $x_{\min} \in [a, b]$ such that $f(x_{\min}) = m$.

Let $n \geq 1$ be any integer. Then $m + \dfrac{1}{n} > m = \inf(E)$. As $\inf(E)$ is the greatest lower bound for E, $m + \dfrac{1}{n}$ cannot be an lower bound for E, thus there exists a $y \in E$ such that $m + \dfrac{1}{n} > y$. By definition of E, this implies that there exists an $x \in [a, b]$ such that $m + \dfrac{1}{n} > f(x)$.

We now choose a sequence $(x_n)_{n=1}^{\infty}$ by choosing, for each n, x_n to be an element of $[a,b]$ such that $m + \frac{1}{n} > f(x_n)$. (Again, this requires the axiom of choice) This is a sequence in $[a,b]$; by the Heine-Borel theorem (Thm. I.9.1.24), we can thus find a subsequence $(x_{n_j})_{j=1}^{\infty}$, where $n_1 < n_2 < \ldots$, which converges to some limit $x_{\min} \in [a,b]$. Since $(x_{n_j})_{j=1}^{\infty}$ converges to $x_{\min}$, and f is continuous at $x_{\min}$, we have as before that

$$\lim_{j\to\infty} f(x_{n_j}) = f(x_{\min})$$

On the other hand, by construction we know that

$$f(x_{n_j}) < m + \frac{1}{n_j} \leq m + \frac{1}{j},$$

and so by taking limits of both sides we see that

$$f(x_{\min}) = \lim_{j\to\infty} f(x_{n_j}) \leq \lim_{j\to\infty} m + \frac{1}{j} = m.$$

On the other hand, we know that $f(x) \geq m$ for all $x \in [a,b]$, so, in particular, $f(x_{\min}) \geq m$. Combining these two inequalities we see that $f(x_{\min}) = m$ as desired. $\square$

Rmk. I.9.6.8. Strictly speaking, "maximum principle" is a misnomer, since the principle also concerns the minimum. Perhaps a more precise name would have been "extremum principle"; the word "extremum" is used to denote either a maximum or a minimum.

Note. The maximum principle (Prop. I.9.6.7) does not prevent a function from attaining its maximum or minimum at more than one point.

Note. Let us write $\sup_{x\in[a,b]} f(x)$ as short-hand for $\sup\{f(x) : x \in [a,b]\}$, and similarly define $\inf_{x\in[a,b]} f(x)$. The maximum principle (Prop. I.9.6.7) thus asserts that $m := \sup_{x\in[a,b]} f(x)$ is a real number and is the maximum value of f on $[a,b]$, i.e., there is at least one point $x_{\max}$ in $[a,b]$ for which $f(x_{\max}) = m$, and for every other $x \in [a,b]$, $f(x)$ is less than or equal to m. Similarly, $\inf_{x\in[a,b]} f(x)$ is the minimum value of f on $[a,b]$.

Rmk. I.9.6.9. You may encounter a rather different "maximum principle" in complex analysis or partial differential equations, involving analytic functions and harmonic functions respectively, instead of continuous functions. Those maximum principles are not directly related to this one (though they are also concerned with whether maxima exist, and where the maxima are located).

— Exercises —

Ex. I.9.6.1. Give example of

(a) a function $f : (1,2) \to \mathbb{R}$ which is continuous and bounded, attains its minimum somewhere, but does not attain its maximum anywhere;

(b) a function $f : [0,\infty) \to \mathbb{R}$ which is continuous, bounded, attains its maximum somewhere, but does not attain its minimum anywhere;

(c) a function $f : [-1,1] \to \mathbb{R}$ which is bounded but does not attain its minimum anywhere or its maximum anywhere.

(d) a function $f : [-1,1] \to \mathbb{R}$ which has no upper bound and no lower bound.

Explain why none of the examples you construct violate the maximum principle.

Proof. Let $f_a : (1,2) \to \mathbb{R}$ be a function where $f_a(x) = |x - 1.5|$. Then by Prop. I.9.4.12 and Prop. I.9.4.13 f_a is continuous. Since $f_a\big((1,2)\big) = [0, 0.5)$, by Def. I.9.6.1 f_a is bounded. Since $f_a\big((1,2)\big) = [0, 0.5)$, by Def. I.9.6.5 f_a attains its minimum at $x = 1.5$. Since $0.5 \notin f_a\big((1,2)\big)$, by Def. I.9.6.5 f_a does not attain its maximum anywhere. Since the domain of f_a is not closed, f_a does not violate the maximum principle (Prop. I.9.6.7).

Let $f_b : [0,\infty) \to \mathbb{R}$ be a function where $f_b(x) = 0.5^x$. Then by Prop. I.9.4.10 f_b is continuous. Since $f_b\big([0,\infty)\big) = (0,1]$, by Def. I.9.6.1 f_b is bounded. Since $f_b\big([0,\infty)\big) = (0,1]$, by Def. I.9.6.5 f_b attains its maximum at $x = 0$. Since $0 \notin f_b\big([0,\infty)\big)$, by Def. I.9.6.5 f_b does not attain its minimum anywhere. Since the domain of f_b is not closed, f_b does not violate the maximum principle (Prop. I.9.6.7).

Let $f_c : [-1,1] \to \mathbb{R}$ be a function where

$$f(x) = \begin{cases} 0 & \text{if } x = -1; \\ x & \text{if } x \in (-1,1); \\ 0 & \text{if } x = 1. \end{cases}$$

Since $f_c\big([-1,1]\big) = (-1,1)$, by Def. I.9.6.1 f_c is bounded. Since $-1 \notin f_c\big([-1,1]\big)$, by Def. I.9.6.5 f_c does not attain its minimum anywhere. Since $1 \notin f_c\big([-1,1]\big)$, by Def. I.9.6.5 f_c does not attain its maximum anywhere. Since f_c is not continuous on its domain, f_c does not violate the maximum principle (Prop. I.9.6.7).

Let $f_d : [-1,1] \to \mathbb{R}$ be a function where

$$f(x) = \begin{cases} 1/x & \text{if } x \in [-1,0) \cap (0,1]; \\ 0 & \text{if } x = 0. \end{cases}$$

Since $f_d\big([-1,1]\big) = (-\infty,-1) \cup \{0\} \cup (1,\infty)$, by Def. I.9.6.1 f_d is unbounded. Since f_d is not bounded, f_d does not violate the maximum principle (Prop. I.9.6.7). $\square$

Ex. I.9.6.2. Let $X \subseteq \mathbb{R}$. If $f, g : X \to \mathbb{R}$ are bounded functions, show that $f + g$, $f - g$, and $f \cdot g$ are also bounded functions. If we furthermore assume that $g(x) \neq 0$ for all $x \in X$, is it true that f/g is bounded? Prove this or give a counterexample.

Proof. Suppose that f is bounded by $M \in \mathbb{R}^+$ and g is bounded by $N \in \mathbb{R}^+$. Then we have

$$
\begin{aligned}
&\forall x \in X, \big(|f(x)| \le M\big) \wedge \big(|g(x)| \le N\big) && \text{(by Def. I.9.6.1)}\\
\implies &\big(-M \le f(x) \le M\big) \wedge \big(-N \le g(x) \le N\big)\\
\implies &\big(-M \le f(x) \le M\big) \wedge \big(N \ge -g(x) \ge -N\big)\\
\implies &\begin{cases} -(M+N) \le f(x)+g(x) \le M+N \\ -(M+N) \le f(x)-g(x) \le M+N \\ -MN \le f(x)g(x) \le MN \end{cases}\\
\implies &\begin{cases} |f(x)+g(x)| \le M+N \\ |f(x)-g(x)| \le M+N \\ |f(x)g(x)| \le MN \end{cases}
\end{aligned}
$$

Thus, by Def. I.9.6.1 $f+g$, $f-g$, $f \cdot g$ are bounded.

Now we show an counterexample when $g(x) \neq 0$ for all $x \in X$. Let $X = (0, \infty)$, let $f(x) = 1$ and let $g(x) = 1/x$. Then $g(x) \neq 0$ for all $x \in X$ and is bounded by 1, but $(f/g)(X) = (0, \infty)$ is unbounded. □

I.9.7 The intermediate value theorem

Thm. I.9.7.1 (Intermediate value theorem)**.** Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Let y be a real number between $f(a)$ and $f(b)$, i.e., either $f(a) \le y \le f(b)$ or $f(a) \ge y \ge f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = y$.

Proof. We have two cases: $f(a) \le y \le f(b)$ or $f(a) \ge y \ge f(b)$. We will assume the former, that $f(a) \le y \le f(b)$; the latter is proven similarly.

If $y = f(a)$ or $y = f(b)$ then the claim is easy, as one can simply set $c = a$ or $c = b$, so we will assume that $f(a) < y < f(b)$. Let E denote the set

$$E \coloneqq \{x \in [a, b] : f(x) < y\}.$$

Clearly, E is a subset of $[a, b]$, and is hence bounded. Also, since $f(a) < y$, we see that a is an element of E, so E is non-empty. By the least upper bound principle, the supremum

$$c \coloneqq \sup(E)$$

is thus finite. Since E is bounded by b, we know that $c \le b$; since E contains a, we know that $c \ge a$. Thus, we have $c \in [a, b]$. To complete the proof we now show that $f(c) = y$. The idea is to work from the left of c to show that $f(c) \le y$, and to work from the right of c to show that $f(c) \ge y$.

Let $n \ge 1$ be an integer. The number $c - \dfrac{1}{n}$ is less than $c = \sup(E)$ and hence cannot be an upper bound for E. Thus, there exists a point, call it x_n, which lies in E and which is greater than $c - \frac{1}{n}$. Also $x_n \le c$ since c is an upper bound for E. Thus

$$c - \frac{1}{n} \le x_n \le c.$$

By the squeeze test (Cor. I.6.4.14) we thus have $\lim_{n\to\infty} x_n = c$. Since f is continuous at c, this implies that $\lim_{n\to\infty} f(x_n) = f(c)$. But since x_n lies in E for every n, we have $f(x_n) < y$ for every n. By the comparison principle (Lem. I.6.4.13) we thus have $f(c) \le y$. Since $f(b) > f(c)$, we conclude $c \ne b$.

Since $c \ne b$ and $c \in [a, b]$, we must have $c < b$. In particular, there is an $N > 0$ such that $c + \frac{1}{n} < b$ for all $n > N$ (since $c + \frac{1}{n}$ converges to c as $n \to \infty$). Since c is the supremum of E and $c + \frac{1}{n} > c$, we thus have $c + \frac{1}{n} \notin E$ for all $n > N$. Since $c + \frac{1}{n} \in [a, b]$, we thus have $f(c + \frac{1}{n}) \ge y$ for all $n \ge N$. But $c + \frac{1}{n}$ converges to c, and f is continuous at c, thus $f(c) \ge y$. But we already knew that $f(c) \le y$, thus $f(c) = y$, as desired. □

Note. The intermediate value theorem says that if f takes the values $f(a)$ and $f(b)$, then it must also take all the values in between. If f is not assumed to be continuous, then the intermediate value theorem no longer applies. If a function is discontinuous, it can "jump" past intermediate values; however continuous functions cannot do so.

Rmk. I.9.7.2. A continuous function may take an intermediate value multiple times.

Rmk. I.9.7.3. The intermediate value theorem gives another way to show that one can take n^{th} roots of a number. For instance, to construct the square root of 2, consider the function $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = x^2$. This function is continuous, with $f(0) = 0$ and $f(2) = 4$. Thus, there exists a $c \in [0, 2]$ such that $f(c) = 2$, i.e., $c^2 = 2$. (This argument does not show that there is just one square root of 2, but it does prove that there is *at least* one square root of 2.)

Cor. I.9.7.4 (Images of continuous functions)**.** Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Let $M := \sup_{x\in[a,b]} f(x)$ be the maximum value of f, and let $m := \inf_{x\in[a,b]} f(x)$ be the minimum value. Let y be a real number between m and M (i.e., $m \le y \le M$). Then there exists a $c \in [a, b]$ such that $f(c) = y$. Furthermore, we have $f([a, b]) = [m, M]$.

Proof. We first show that $\exists c \in [a, b]$ such that $f(c) = y$. By maximum principle (Prop. I.9.6.7) we know that $\exists x_M, x_m \in [a, b]$ such that $f(x_M) = M$ and $f(x_m) = m$. We have either $x_m \le x_M$ or $x_m \ge x_M$. Without the loss of generality suppose that $x_m \le x_M$. Then we have $[x_m, x_M] \subseteq [a, b]$ and by Ex. I.9.4.6 we know that f is continuous on $[x_m, x_M]$. Since $m \le y \le M$, by Thm. I.9.7.1 $\exists c \in [x_m, x_M]$ such that $f(c) = y$. Since $[x_m, x_M] \subseteq [a, b]$, we have $c \in [a, b]$.

Now we show that $f([a,b]) = [m, M]$. Since

$$\begin{aligned}
&\big(M = \sup_{x\in[a,b]} f(x)\big) \wedge \big(m = \inf_{x\in[a,b]} f(x)\big) \\
\implies& \forall y \in f([a,b], m \le y \le M && \text{(by Def. I.9.6.5)} \\
\implies& \forall y \in f([a,b], y \in [m, M] && \text{(by Def. I.9.1.1)} \\
\implies& f([a,b]) \subseteq [m, M] && \text{(by Def. I.3.1.15)}
\end{aligned}$$

and

$$\begin{aligned}
&\forall y \in [m, M], \exists c \in [a,b] : f(c) = y && \text{(by proof above)} \\
\implies& \forall y \in [m, M], y \in f([a,b]) \\
\implies& [m, M] \subseteq f([a,b]), && \text{(by Def. I.3.1.15)}
\end{aligned}$$

by Prop. I.3.1.18 we know that $f([a,b]) = [m, M]$. □

— Exercises —

Ex. I.9.7.1. Prove Cor. I.9.7.4.

Proof. See Cor. I.9.7.4. □

Ex. I.9.7.2. Let $f : [0,1] \to [0,1]$ be a continuous function. Show that there exists a real number x in $[0,1]$ such that $f(x) = x$. This point x is known as a *fixed point* of f, and this result is a basic example of a *fixed point theorem*, which play an important role in certain types of analysis.

Proof. Let $F : [0,1] \to \mathbb{R}$ be a function where $F(x) = f(x) - x$. Since $x \mapsto x$ is continuous on $[0,1]$, by Prop. I.9.4.9 F is continuous on $[0,1]$. Since $f([0,1]) \subseteq [0,1]$, we know that $0 \le f(0)$ and $f(1) \le 1$. Thus

$$\begin{aligned}
&\big(0 \le f(0)\big) \wedge \big(f(1) \le 1\big) \\
\implies& \big(0 \le F(0) = f(0) - 0\big) \wedge \big(F(1) = f(1) - 1 \le 0\big) \\
\implies& F(1) \le 0 \le F(0) \\
\implies& \exists x \in [0,1] : F(x) = 0 && \text{(by Thm. I.9.7.1)} \\
\implies& \exists x \in [0,1] : f(x) - x = 0 \\
\implies& \exists x \in [0,1] : f(x) = x.
\end{aligned}$$

□

I.9.8 Monotonic functions

Def. I.9.8.1 (Monotonic functions)**.** Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function. We say that f is *monotone increasing* iff $f(y) \geq f(x)$ whenever $x, y \in X$ and $y > x$. We say that f is *strictly monotone increasing* iff $f(y) > f(x)$ whenever $x, y \in X$ and $y > x$. Similarly, we say f is *monotone decreasing* iff $f(y) \leq f(x)$ whenever $x, y \in X$ and $y > x$, and *strictly monotone decreasing* iff $f(y) < f(x)$ whenever $x, y \in X$ and $y > x$. We say that f is *monotone* if it is monotone increasing or monotone decreasing, and *strictly monotone* if it is strictly monotone increasing or strictly monotone decreasing.

Note. If a function is strictly monotone on a domain X, it is automatically monotone as well on the same domain X. Constant functions, when restricted to an arbitrary domain $X \subseteq \mathbb{R}$, are both monotone increasing and monotone decreasing, but is not strictly monotone (unless X consists of at most one point).

Note. Continuous functions are not necessarily monotone, and monotone functions are not necessarily continuous.

Prop. I.9.8.3. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is both continuous and strictly monotone increasing. Then f is a bijection from $[a, b]$ to $[f(a), f(b)]$, and the inverse $f^{-1} : [f(a), f(b)] \to [a, b]$ is also continuous and strictly monotone increasing.

Proof. We first show that f is bijective from $[a, b]$ to $[f(a), f(b)]$. Since $a < b$ and f is strictly monotone increasing, by Def. I.9.8.1 we know that $f(a) < f(b)$. In particular, $\forall c \in (a, b)$, we have $a < c < b$ and $f(a) < f(c) < f(b)$. By Def. I.9.6.5 this means f attains its minimum at a and attains its maximum at b. By Cor. I.9.7.4 we know that $f([a, b]) = [f(a), f(b)]$, thus f is surjective from $[a, b]$ to $[f(a), f(b)]$. Since f is strictly monotone increasing, by Def. I.9.8.1 $x \neq y \implies f(x) \neq f(y)$, thus f is injective from $[a, b]$ to $[f(a), f(b)]$. Since f is both injective and surjective from $[a, b]$ to $[f(a), f(b)]$, we know that f is bijective from $[a, b]$ to $[f(a), f(b)]$.

Next we show that f^{-1} is continuous. Let $y_0 \in [f(a), f(b)]$. By Lem. I.9.1.12 we know that $[f(a), f(b)]$ is closed, so y_0 is an adherent point of $[f(a), f(b)]$. Since f is bijective from $[a, b]$ to $[f(a), f(b)]$, we know that $\exists!\ x_0 \in [a, b]$ such that $f(x_0) = y_0$ and thus $f^{-1}(y_0) = x_0$. Again by Lem. I.9.1.12 we know that $[a, b]$ is closed, so x_0 is an adherent point of $[a, b]$. To show that f^{-1} is continuous at y_0, by Def. I.9.4.1 we need to show that

$$\lim_{y \to y_0; y \in [f(a), f(b)]} f^{-1}(y) = f^{-1}(y_0) = x_0.$$

By Def. I.9.3.6, it suffice to show that

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall y \in [f(a), f(b)], |y - y_0| < \delta \implies \left|f^{-1}(y) - x_0\right| \leq \varepsilon\big).$$

Now fix ε. Let $x_L = \max(x_0 - \varepsilon, a)$ and $x_H = \min(x_0 + \varepsilon, b)$. Then $x_L, x_H \in [a, b]$ and $f(x_L), f(x_H)$ are well-defined. Since $x_L \leq x_0 \leq x_H$ and f is strictly monotone increasing, we have $f(x_L) < y_0 < f(x_H)$ and thus $f(x_L) - y_0 < 0 < f(x_H) - y_0$. Let $\delta = \min\big(y_0 - f(x_L), f(x_H) - y_0\big)$. Then we have

$$\begin{aligned}
&\forall y \in [f(a), f(b)], |y - y_0| < \delta \\
\implies& -\delta < y - y_0 < \delta \\
\implies& f(x_L) - y_0 \leq -\delta < y - y_0 < \delta \leq f(x_H) - y_0 \\
\implies& f(x_L) \leq y \leq f(x_H) \\
\implies& \exists x \in [x_L, x_H] : \big(f(x) = y\big) \wedge (x_L \leq x \leq x_H) && \text{(by Thm. I.9.7.1)} \\
\implies& \exists x \in [x_L, x_H] : \big(f(x) = y\big) \\
&\wedge (x_0 - \varepsilon \leq x_L \leq x \leq x_H \leq x_0 + \varepsilon) \\
\implies& \exists x \in [x_L, x_H] : \big(f(x) = y\big) \wedge (-\varepsilon \leq x - x_0 \leq \varepsilon) \\
\implies& \exists x \in [x_L, x_H] : \big(f(x) = y\big) \wedge (|x - x_0| \leq \varepsilon) \\
\implies& \exists x \in [x_L, x_H] : \big(f(x) = y\big) \wedge (\left|f^{-1}(y) - x_0\right| \leq \varepsilon).
\end{aligned}$$

Since ε was arbitrary, f^{-1} is continuous at y_0. Since y_0 was arbitrary, f^{-1} is continuous on $[f(a), f(b)]$.

Next we show that f^{-1} is strictly monotone increasing. Let $y_1, y_2 \in [f(a), f(b)]$ and $y_1 < y_2$. We want to show that $f^{-1}(y_1) < f^{-1}(y_2)$. Suppose for the sake of contradiction that $f^{-1}(y_1) \geq f^{-1}(y_2)$. Since f is strictly monotone increasing, we know that

$$\begin{aligned}
&f^{-1}(y_1) \geq f^{-1}(y_2) \\
\implies& f\big(f^{-1}(y_1)\big) \geq f\big(f^{-1}(y_2)\big) && \text{(by Def. I.9.8.1)} \\
\implies& y_1 \geq y_2.
\end{aligned}$$

But this contradicts $y_1 < y_2$. Thus, $f^{-1}(y_1) < f^{-1}(y_2)$.

Next we show that if the continuity assumption is dropped, then the proposition is false. Let $f : [0, 1] \to \mathbb{R}$ be a function

$$f(x) = \begin{cases} x & \text{if } x \in [0, 0.5) \\ x + 1 & \text{if } x \in [0.5, 1]. \end{cases}$$

Then f is not continuous and $f([0, 1]) = [0, 0.5) \cup [1.5, 2] \neq [0, 2]$.

Finally we show that if strict monotonicity is replaced just by monotonicity, then the proposition is false. Let $f : [0, 1] \to \mathbb{R}$ be a function where $f(x) = 1$. Then f is monotone but not bijective. $\square$

A.Cor. I.9.8.1. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is both continuous and strictly monotone decreasing. Then f is a bijection from $[a, b]$ to $[f(b), f(a)]$, and the inverse $f^{-1} : [f(b), f(a)] \to [a, b]$ is also continuous and strictly monotone decreasing.

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be the function $h(x) = -x$. We define a function $g : [a, b] \to \mathbb{R}$ by setting $g = -f$. Then we have $g = h \circ f$ and $f = h \circ g$. By Prop. I.9.4.9 we know that h is continuous on $\mathbb{R}$, thus by Prop. I.9.4.13 we know that g is continuous. Since f is strictly monotone decreasing, we have

$$\begin{aligned}
&\forall x_1, x_2 \in [a, b], x_1 < x_2 \\
\implies & f(x_1) > f(x_2) && \text{(by Def. I.9.8.1)} \\
\implies & -f(x_1) < -f(x_2) \\
\implies & g(x_1) < g(x_2)
\end{aligned}$$

and g is strictly monotone increasing. Since g is monotone increasing and continuous, by Prop. I.9.8.3 we know that $g : [a, b] \to [g(a), g(b)]$ is bijective and $g^{-1} : [g(a), g(b)] \to [a, b]$ is continuous and strictly monotone increasing. Since h is bijective, by Ex. I.3.3.7 we know that $f = h \circ g$ is bijective from $[a, b]$ to $[-g(b), -g(a)] = [f(b), f(a)]$. Again by Ex. I.3.3.7 we know that

$$f^{-1} = g^{-1} \circ h^{-1}|_{[f(a),f(b)]}.$$

Since $h^{-1} = h$ is continuous, by Prop. I.9.4.13 we know that f^{-1} is continuous. Since g^{-1} is strictly monotone increasing, we have

$$\begin{aligned}
&\forall y_1, y_2 \in [f(b), f(a)], y_1 < y_2 \\
\implies & \exists x_1, x_2 \in [a, b] : \\
& \big(x_1 = f^{-1}(y_1)\big) \wedge \big(x_2 = f^{-1}(y_2)\big) \wedge \big(y_1 < y_2\big) \\
\implies & \exists x_1, x_2 \in [a, b] : \\
& \big(g(x_1) = -y_1\big) \wedge \big(g(x_2) = -y_2\big) \wedge \big(y_1 < y_2\big) \\
\implies & \exists x_1, x_2 \in [a, b] : \\
& \big(g(x_1) = -y_1\big) \wedge \big(g(x_2) = -y_2\big) \wedge \big(g(x_1) > g(x_2)\big) \\
\implies & \exists x_1, x_2 \in [a, b] : \\
& \big(g(x_1) = -y_1\big) \wedge \big(g(x_2) = -y_2\big) \\
& \wedge \Big(x_1 = g^{-1}\big(g(x_1)\big) > g^{-1}\big(g(x_2)\big) = x_2\Big) && \text{(by Def. I.9.8.1)} \\
\implies & f^{-1}(y_1) > f^{-1}(y_2)
\end{aligned}$$

and thus by Def. I.9.8.1 f^{-1} is strictly monotone decreasing. □

E.g. I.9.8.4. Let n be a positive integer and $R > 0$. Since the function $f(x) := x^n$ is strictly increasing on the interval $[0, R]$, we see from Prop. I.9.8.3 that this function is a bijection from $[0, R]$ to $[0, R^n]$, and hence there is an inverse from $[0, R^n]$ to $[0, R]$. This can be used to give an alternate means to construct the n^{th} root $x^{1/n}$ of a number $x \in [0, R]$ than what was done in Lem. I.5.6.5.

— Exercises —

Ex. I.9.8.1. Explain why the maximum principle remains true if the hypothesis that f is continuous is replaced with f being monotone, or with f being strictly monotone.

Proof. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function. Suppose that f is monotone. Then we have

$$\begin{aligned}
&\forall c \in [a, b] \\
\implies & a \leq c \leq b && \text{(by Def. I.9.1.1)} \\
\implies & \big(f(a) \leq f(c) \leq f(b)\big) \vee \big(f(a) \geq f(c) \geq f(b)\big) && \text{(by Def. I.9.8.1)}
\end{aligned}$$

Thus, f attains its maximum at $f(b)$ and attains its minimum at $f(a)$ when f is monotone increasing; f attains its maximum at $f(a)$ and attains its minimum at $f(b)$ when f is monotone decreasing. The same argument holds when f is strictly monotone. □

Ex. I.9.8.2. Give an example to show that the intermediate value theorem becomes false if the hypothesis that f is continuous is replaced with f being monotone, or with f being strictly monotone.

Proof. Let $f : [0, 1] \to \mathbb{R}$ be a function where

$$f(x) = \begin{cases} x & \text{if } x \in [0, 0.5); \\ x + 1 & \text{if } x \in [0.5, 1]. \end{cases}$$

Then f is strictly monotone and thus monotone. Since $\forall y \in [0.5, 1.5)$, $\nexists x \in \mathbb{N}$ such that $f(x) = y$, the intermediate value theorem (Thm. I.9.7.1) does not hold. □

Ex. I.9.8.3. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is both continuous and one-to-one. Show that f is strictly monotone.

Proof. Since $a < b$ and f is injective, we cannot have $f(a) = f(b)$. So we have two cases:

- $f(a) < f(b)$. Suppose for the sake of contradiction that f is not strictly monotone. Then $\exists c \in (a, b)$ such that

  $$\big(f(c) \leq f(a) < f(b)\big) \vee \big(f(a) < f(b) \leq f(c)\big).$$

  Since f is injective, we have $\big(f(c) \neq f(a)\big) \wedge \big(f(c) \neq f(b)\big)$. But then we have

  $$\begin{aligned}
  &\begin{cases} f(c) < f(a) < f(b) \\ f(a) < f(b) < f(c) \end{cases} \\
  \implies &\begin{cases} \exists x_1 \in (c, b) : f(x_1) = f(a) \\ \exists x_2 \in (a, c) : f(x_2) = f(b) \end{cases} && \text{(by Thm. I.9.7.1)}
  \end{aligned}$$

$$\implies \begin{cases} x_1 = a \\ x_2 = b \end{cases}, \qquad (f \text{ is injective})$$

a contradiction. Thus, f is strictly monotone. Since $f(a) < f(b)$, by Def. I.9.8.1 f is strictly monotone increasing.

- $f(a) > f(b)$. Suppose for the sake of contradiction that f is not strictly monotone. Then $\exists c \in (a, b)$ such that

$$\big(f(c) \leq f(b) < f(a)\big) \vee \big(f(b) < f(a) \leq f(c)\big).$$

Since f is injective, we have $\big(f(c) \neq f(a)\big) \wedge \big(f(c) \neq f(b)\big)$. But then we have

$$\begin{aligned} &\begin{cases} f(c) < f(b) < f(a) \\ f(b) < f(a) < f(c) \end{cases} \\ \implies &\begin{cases} \exists x_1 \in (a, c) : f(x_1) = f(b) \\ \exists x_2 \in (c, b) : f(x_2) = f(a) \end{cases} && (\text{by Thm. I.9.7.1}) \\ \implies &\begin{cases} x_1 = b \\ x_2 = a \end{cases}, && (f \text{ is injective}) \end{aligned}$$

a contradiction. Thus, f is strictly monotone. Since $f(a) > f(b)$, by Def. I.9.8.1 f is strictly monotone decreasing.

From all cases above, we conclude that f is strictly monotone. □

Ex. I.9.8.4. Prove Prop. I.9.8.3. Is the proposition still true if the continuity assumption is dropped, or if strict monotonicity is replaced just by monotonicity? How should one modify the proposition to deal with strictly monotone decreasing functions instead of strictly monotone increasing functions?

Proof. See Prop. I.9.8.3 and A.Cor. I.9.8.1. □

Ex. I.9.8.5. In this exercise we give an example of a function which has a discontinuity at every rational point, but is continuous at every irrational. Since the rationals are countable, we can write them as $\mathbb{Q} = \{q(0), q(1), q(2), \ldots\}$, where $q : \mathbb{N} \to \mathbb{Q}$ is a bijection from $\mathbb{N}$ to $\mathbb{Q}$. Now define a function $g : \mathbb{Q} \to \mathbb{R}$ by setting $g(q(n)) := 2^{-n}$ for each natural number n; thus g maps $q(0)$ to 1, $q(1)$ to 2^{-1}, etc. Since $\sum_{n=0}^{\infty} 2^{-n}$ is absolutely convergent, we see that $\sum_{r \in \mathbb{Q}} g(r)$ is also absolutely convergent. Now define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \sum_{r \in \mathbb{Q} : r < x} g(r).$$

Since $\sum_{r\in\mathbb{Q}} g(r)$ is absolutely convergent, we know that $f(x)$ is well-defined for every real number x.

(a) Show that f is strictly monotone increasing.

(b) Show that for every rational number r, f is discontinuous at r.

(c) Show that for every irrational number x, f is continuous at x.

Proof. (a) Let $a, b \in \mathbb{R}$ and $a < b$. By Prop. I.5.4.14, $\exists c \in \mathbb{Q}$ such that $a < c < b$. Then we have

$$\begin{aligned}
f(b) &= \sum_{r\in\mathbb{Q}:r<b} g(r) \\
&= \sum_{r\in\mathbb{Q}:r<a} g(r) + \sum_{r\in\mathbb{Q}:a\le r<c} g(r) + \sum_{r\in\mathbb{Q}:c\le r<b} g(r) && \text{(by Prop. I.8.2.6(c))} \\
&= f(a) + \sum_{r\in\mathbb{Q}:a\le r<c} g(r) + g(c) + \sum_{r\in\mathbb{Q}:c<r<b} g(r) && \text{(by Prop. I.8.2.6(c))} \\
&> f(a).
\end{aligned}$$

and by Def. I.9.8.1 f is strictly monotone increasing. □

Proof. (b) Let $\gamma \in \mathbb{Q}$. Since q is bijective, $\exists!\ n \in \mathbb{N}$ such that $q(n) = \gamma$. Then $\forall x \in (\gamma, \infty)$, we have

$$\begin{aligned}
f(x) &= \sum_{r\in\mathbb{Q}:r<x} g(r) \\
&= \sum_{r\in\mathbb{Q}:r<\gamma} g(r) + g(\gamma) + \sum_{r\in\mathbb{Q}:\gamma<r<x} g(r) && \text{(by Prop. I.8.2.6(c))} \\
&= f(\gamma) + g(\gamma) + \sum_{r\in\mathbb{Q}:\gamma<r<x} g(r) \\
&> f(\gamma) + g(\gamma) \\
&= f(\gamma) + 2^{-n}.
\end{aligned}$$

But this means

$$\begin{aligned}
& \forall x \in (\gamma, \infty), f(x) - f(\gamma) > 2^{-n} \\
\Longrightarrow\ & \forall x \in (\gamma, \infty), |f(x) - f(\gamma)| > 2^{-n} \\
\Longrightarrow\ & f(\gamma^+) \text{ does not exist} && \text{(by Def. I.9.5.1)} \\
\Longrightarrow\ & f \text{ is not continuous at } \gamma. && \text{(by A.Cor. I.9.5.1)}
\end{aligned}$$

Since γ was arbitrary, we conclude that f is discontinuous at each $\gamma \in \mathbb{Q}$. □

Proof. (c) Let $n \in \mathbb{N}$, let E_n be a set where

$$E_n = \{r \in \mathbb{Q} : g(r) \geq 2^{-n}\},$$

and let $f_n : \mathbb{R} \to \mathbb{R}$ be a function

$$f_n(x) = \sum_{r \in E_n : r < x} g(r) = \sum_{r \in \mathbb{Q} : r < x, g(r) \geq 2^{-n}} g(r).$$

Since q is bijective, there are at most $n+1$ rationals satisfying $(r \in \mathbb{Q}) \wedge \big(g(r) \geq 2^{-n}\big)$, thus E_n is finite and non-empty. Since $\mathbb{R} \setminus \mathbb{Q} \subseteq \mathbb{R}$, by Lem. I.9.1.11 $\forall x_0 \in \mathbb{R} \setminus \mathbb{Q}$, x_0 is an adherent point of $\mathbb{R} \setminus \mathbb{Q}$. Let $\varepsilon \in \mathbb{R}^+$ and let $\delta = \min\{|r - x_0| : r \in E_n\}$. Since $x_0 \notin \mathbb{Q}$, we have $\delta > 0$. Then we have

$$\begin{aligned}
& \forall x \in \mathbb{R} \setminus \mathbb{Q}, |x - x_0| < \delta \\
\Longrightarrow\ & \forall r \in E_n, |x - x_0| < \delta \leq |r - x_0| \\
\Longrightarrow\ & \forall r \in E_n, |x - x_0| < |r - x_0| \\
\Longrightarrow\ & \forall r \in E_n, -|r - x_0| < x - x_0 < |r - x_0| \\
\Longrightarrow\ & \forall r \in E_n, \begin{cases} r - x_0 < x - x_0 < x_0 - r & \text{if } r < x_0 \\ x_0 - r < x - x_0 < r - x_0 & \text{if } r > x_0 \end{cases} \\
\Longrightarrow\ & \forall r \in E_n, \begin{cases} r < x & \text{if } r < x_0 \\ r > x & \text{if } r > x_0 \end{cases} \\
\Longrightarrow\ & \{r \in E_n : r < x\} = \{r \in E_n : r < x_0\} \\
\Longrightarrow\ & f_n(x) = f_n(x_0) \\
\Longrightarrow\ & 0 = |f_n(x) - f_n(x_0)| < \varepsilon.
\end{aligned}$$

Since ε was arbitrary, by Def. I.9.3.6 we have $\lim_{x \to x_0; x \in \mathbb{R} \setminus \mathbb{Q}} f_n(x) = f_n(x_0)$, and by Def. I.9.4.1 f_n is continuous at x_0.

Now we show that f is continuous at x_0. We have

$$\begin{aligned}
& \forall x \in \mathbb{R} \setminus \mathbb{Q}, |f(x) - f_n(x)| \\
&= \left| \sum_{r \in \mathbb{Q} : r < x} g(r) - \sum_{r \in E_n : r < x} g(r) \right| \\
&= \left| \sum_{r \in \mathbb{Q} : r < x} g(r) - \sum_{r \in \mathbb{Q} : r < x, g(r) \geq 2^{-n}} g(r) \right| \\
&= \left| \sum_{r \in \mathbb{Q} : r < x, g(r) < 2^{-n}} g(r) \right| && \text{(by Prop. I.8.2.6(c))}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{r\in\mathbb{Q}: r<x, g(r)<2^{-n}} g(r) \\
&= \sum_{r\in\mathbb{Q}: r<x, g(r)\le 2^{-(n+1)}} g(r) \\
&\le \sum_{r\in\mathbb{Q}: g(r)\le 2^{-(n+1)}} g(r) \\
&\le \sum_{k=n+1}^{\infty} 2^{-k} \\
&= 2^{-(n+1)}\left(\sum_{k=0}^{\infty} 2^{-k}\right) \\
&= 2^{-n}. && \text{(by Lem. I.7.3.3)}
\end{aligned}
$$

By Prop. I.5.4.14, $\forall\varepsilon \in \mathbb{R}^+$, $\exists n \in \mathbb{N}$ such that $2^{-n} < \varepsilon/2$. From the proof above we also have

$$\forall x \in \mathbb{R}\setminus\mathbb{Q}, |x - x_0| < \delta \implies f_n(x) = f_n(x_0)$$

Combine the results above we have

$$
\begin{aligned}
&|f(x) - f(x_0)| \\
&= |f(x) - f_n(x) + f_n(x) - f(x_0)| \\
&\le |f(x) - f_n(x)| + |f_n(x) - f(x_0)| \\
&= |f(x) - f_n(x)| + |f_n(x_0) - f(x_0)| \\
&\le 2^{-n} + 2^{-n} \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}
$$

Since ε was arbitrary, by Def. I.9.3.6 we have $\lim_{x\to x_0; x\in\mathbb{R}\setminus\mathbb{Q}} f(x) = f(x_0)$, and by Def. I.9.4.1 f is continuous at x_0. □

I.9.9 Uniform continuity

Def. I.9.9.2 (Uniform continuity)**.** Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function. We say that f is *uniformly continuous* on X if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ and $f(x_0)$ are ε-close whenever $x, x_0 \in X$ are two points in X which are δ-close.

Rmk. I.9.9.3. This definition should be compared with the notion of continuity. From Prop. I.9.4.7(c), we know that a function f is *continuous* if for every $\varepsilon > 0$, and every $x_0 \in X$, there is a $\delta > 0$ such that $f(x)$ and $f(x_0)$ are ε-close whenever $x \in X$ is δ-close to x_0. The difference between uniform continuity and continuity is that in uniform continuity one can take a single δ which works for all $x_0 \in X$; for ordinary continuity, each $x_0 \in X$ might use a different δ. Thus, every uniformly continuous function is continuous, but not conversely.

Def. I.9.9.5 (Equivalent sequences)**.** Let m be an integer, let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be two sequences of real numbers, and let $\varepsilon > 0$ be given. We say that $(a_n)_{n=m}^{\infty}$ is *ε-close* to $(b_n)_{n=m}^{\infty}$ iff a_n is ε-close to b_n for each $n \geq m$. We say that $(a_n)_{n=m}^{\infty}$ is *eventually ε-close* to $(b_n)_{n=m}^{\infty}$ iff there exists an $N \geq m$ such that the sequences $(a_n)_{n=N}^{\infty}$ and $(b_n)_{n=N}^{\infty}$ are ε-close. Two sequences $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are *equivalent* iff for each $\varepsilon > 0$, the sequences $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are eventually ε-close.

Rmk. I.9.9.6. One could debate whether ε should be assumed to be rational or real, but a minor modification of Prop. I.6.1.4 shows that this does not make any difference to the above definitions.

Lem. I.9.9.7. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers (not necessarily bounded or convergent). Then $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent iff $\lim_{n\to\infty}(a_n - b_n) = 0$.

Proof.

$$\begin{aligned}
&(a_n)_{n=1}^{\infty} = (b_n)_{n=1}^{\infty} \\
\iff& \forall\varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, |a_n - b_n| \leq \varepsilon && \text{(by Def. I.9.9.5)} \\
\iff& \forall\varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, |(a_n - b_n) - 0| \leq \varepsilon \\
\iff& \lim_{n\to\infty}(a_n - b_n) = 0. && \text{(by Def. I.6.1.5)}
\end{aligned}$$

□

Prop. I.9.9.8. Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function. Then the following two statements are logically equivalent:

(a) f is uniformly continuous on X.

(b) Whenever $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ are two equivalent sequences consisting of elements of X, the sequences $(f(x_n))_{n=0}^{\infty}$ and $(f(y_n))_{n=0}^{\infty}$ are also equivalent.

Proof. We first show that statement (a) implies statement (b). By Def. I.9.9.2, f is uniformly continuous on X iff

$$\forall\varepsilon \in \mathbb{R}^+, \exists\delta \in \mathbb{R}^+ : \big(\forall x, y \in X, |x - y| < \delta \implies |f(x) - f(y)| \leq \varepsilon\big).$$

By Def. I.9.9.5, $(x_n)_{n=0}^{\infty} = (y_n)_{n=0}^{\infty}$ iff

$$\forall\varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \geq N, |x_n - y_n| \leq \varepsilon.$$

In particular, we have

$$\exists N \in \mathbb{N} : \forall n \geq N, |x_n - y_n| \leq \delta/2 < \delta.$$

Since $\forall n \geq 0$ we have $x_n, y_n \in X$, by replacing x, y with x_n, y_n we see that

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x_n, y_n \in X, |x_n - y_n| < \delta \implies |f(x_n) - f(y_n)| \leq \varepsilon\big).$$

Thus, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \geq N, |f(x_n) - f(y_n)| \leq \varepsilon$$

and by Def. I.9.9.5 $\big(f(x_n)\big)_{n=0}^{\infty} = \big(f(y_n)\big)_{n=0}^{\infty}$.

Now we show that statement (b) implies statement (a). By hypothesis we know that if $(x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}$ in X, then $(x_n)_{n=0}^{\infty} = (y_n)_{n=0}^{\infty} \implies \big(f(x_n)\big)_{n=0}^{\infty} = \big(f(y_n)\big)_{n=0}^{\infty}$. Suppose for the sake of contradiction that f is not uniformly continuous on X. Then we have

$$\exists \varepsilon \in \mathbb{R}^+ : \forall \delta \in \mathbb{R}^+, \Big(\forall x, y \in X, (|x - y| < \delta) \wedge \big(|f(x) - f(y)| > \varepsilon\big)\Big).$$

By replacing x, y with x_n, y_n we have

$$\exists \varepsilon \in \mathbb{R}^+ : \forall \delta \in \mathbb{R}^+, \Big(\forall x_n, y_n \in X, (|x_n - y_n| < \delta) \wedge \big(|f(x_n) - f(y_n)| > \varepsilon\big)\Big).$$

Since $(x_n)_{n=0}^{\infty} = (y_n)_{n=0}^{\infty}$, we have

$$\exists N_1 \in \mathbb{N} : \forall n \geq N_1, |x_n - y_n| \leq \delta/2 < \delta.$$

Since $\big(f(x_n)\big)_{n=0}^{\infty} = \big(f(y_n)\big)_{n=0}^{\infty}$, we have

$$\exists N_2 \in \mathbb{N} : \forall n \geq N_2, |f(x_n) - f(y_n)| \leq \varepsilon.$$

Let $N = \max(N_1, N_2)$. Then we have

$$\forall n \in \mathbb{N} \wedge n \geq N, (|x_n - y_n| < \delta) \wedge \big(|f(x_n) - f(y_n)| \leq \varepsilon\big),$$

which contradict to $(|x_n - y_n| < \delta) \wedge \big(|f(x_n) - f(y_n)| > \varepsilon\big)$. Thus, f is uniformly continuous on X. □

Rmk. I.9.9.9. The reader should compare Prop. I.9.9.8 with Prop. I.9.3.9. Prop. I.9.3.9 asserted that if f was continuous, then f maps convergent sequences to convergent sequences. In contrast, Prop. I.9.9.8 asserts that if f is *uniformly* continuous, then f maps *equivalent* pairs of sequences to equivalent pairs of sequences. To see how the two Propositions are connected, observe from Lem. I.9.9.7 that $(x_n)_{n=0}^{\infty}$ will converge to x_* iff the sequences $(x_n)_{n=0}^{\infty}$ and $(x_*)_{n=0}^{\infty}$ are equivalent.

Prop. I.9.9.12. Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a uniformly continuous function. Let $(x_n)_{n=0}^{\infty}$ be a Cauchy sequence consisting entirely of elements in X. Then $(f(x_n))_{n=0}^{\infty}$ is also a Cauchy sequence.

Proof. Since f is uniformly continuous, by Def. I.9.9.2 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x, y \in X, |x-y| < \delta \implies |f(x) - f(y)| \leq \varepsilon\big).$$

Let $i, j \in \mathbb{N}$. Since $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence, by Def. I.6.1.3 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall i, j \geq N, |x_i - x_j| \leq \varepsilon.$$

In particular, we have

$$\exists N \in \mathbb{N} : \forall i, j \geq N, |x_i - x_j| \leq \delta/2 < \delta.$$

Since $x_i, x_j \in X$, we have

$$|x_i - x_j| < \delta \implies |f(x_i) - f(x_j)| \leq \varepsilon.$$

Thus, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall i, j \geq N, |f(x_i) - f(x_j)| \leq \varepsilon.$$

and by Def. I.6.1.3 $\big(f(x_n)\big)_{n=0}^{\infty}$ is a Cauchy sequence. □

Cor. I.9.9.14. Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ be a uniformly continuous function, and let x_0 be an adherent point of X. Then the limit $\lim_{x \to x_0; x \in X} f(x)$ exists (in particular, it is a real number).

Proof. Since x_0 is an adherent point of X, by Lem. I.9.1.14 there exists a sequence $(a_n)_{n=0}^{\infty}$, consisting entirely of elements in X, which converges to x_0. Since $\lim_{n \to \infty} a_n = x_0$, by Prop. I.6.1.12 $(a_n)_{n=0}^{\infty}$ is a Cauchy sequence. Since f is uniformly continuous, by Prop. I.9.9.12 $\big(f(a_n)\big)_{n=0}^{\infty}$ is also a Cauchy sequence. Let $(b_n)_{n=0}^{\infty}$ in X such that $\lim_{n \to \infty} b_n = x_0$. Since

$$\begin{aligned}
&\lim_{n \to \infty} b_n = x_0 \\
\implies &\lim_{n \to \infty} a_n - b_n = 0 && \text{(by Thm. I.6.1.19)} \\
\implies &(a_n)_{n=0}^{\infty} = (b_n)_{n=0}^{\infty} && \text{(by Lem. I.9.9.7)} \\
\implies &\big(f(a_n)\big)_{n=0}^{\infty} = \big(f(b_n)\big)_{n=0}^{\infty} && \text{(by Prop. I.9.9.8)} \\
\implies &\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} f(b_n) && \text{(by Prop. I.9.9.12)}
\end{aligned}$$

and $(b_n)_{n=0}^{\infty}$ was arbitrary, by Prop. I.9.3.9 we know that $\lim_{x \to x_0; x \in X} f(x)$ exists.

Now we show that $f : (0, 2) \to \mathbb{R}$ defined by $f(x) = 1/x$ is not uniformly continuous. Suppose for the sake of contradiction that f is uniformly continuous. Since 0 is an adherent point of $(0, 2)$, we know that $\lim_{x \to 0; x \in (0,2)} f(x)$ exists. Since f is continuous, $f(0) = 1/0$ must exist, a contradiction. Thus, f is not uniformly continuous. □

Prop. I.9.9.15. Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a uniformly continuous function. Suppose that E is a bounded subset of X. Then $f(E)$ is also bounded.

Proof. Suppose for the sake of contradiction that $f(E)$ is unbounded. Thus, for every real number M there exists an element $x \in E$ such that $|f(x)| \geq M$.

In particular, for every natural number n, the set $\{x \in E : |f(x)| \geq n\}$ is non-empty. We can thus choose a sequence $(x_n)_{n=0}^{\infty}$ in E such that $|f(x_n)| \geq n$ for all n. Since $(x_n)_{n=0}^{\infty}$ in E, by Bolzano-Weierstrass theorem (Thm. I.6.6.8) there exists a subsequence $(x_{n_j})_{j=0}^{\infty}$ which converges, where $n_0 < n_1 < n_2 < \dots$ is an increasing sequence of natural numbers. In particular, we see that $n_j \geq j$ for all $j \in \mathbb{N}$ (use induction).

Let $\lim_{n\to\infty} x_n = x_*$. Since $(x_n)_{n=0}^{\infty}$ in E, by Lem. I.9.1.14 we know that x_* is an adherent point of E. Since f is continuous on X, by Ex. I.9.4.6 f is continuous on E. In particular, f is uniformly continuous on E. Thus, by Cor. I.9.9.14 we know that $\lim_{x\to x_*;x\in E} f(x)$ exists. By Prop. I.9.3.9 we see that

$$\lim_{j\to\infty} f(x_{n_j}) = \lim_{x\to x_*;x\in E} f(x).$$

Thus, the sequence $\big(f(x_{n_j})\big)_{j=0}^{\infty}$ is convergent, and hence it is bounded. On the other hand, we know from the construction that $\left|f(x_{n_j})\right| \geq n_j \geq j$ for all j, and hence the sequence $\big(f(x_{n_j})\big)_{j=0}^{\infty}$ is unbounded, a contradiction. $\square$

Thm. I.9.9.16. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$. Then f is also uniformly continuous.

Proof. Suppose for the sake of contradiction that f is not uniformly continuous. By Prop. I.9.9.8, there must therefore exist two equivalent sequences $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ in $[a, b]$ such that the sequences $\big(f(x_n)\big)_{n=0}^{\infty}$ and $\big(f(y_n)\big)_{n=0}^{\infty}$ are not equivalent. In particular, we can find an $\varepsilon > 0$ such that $\big(f(x_n)\big)_{n=0}^{\infty}$ and $\big(f(y_n)\big)_{n=0}^{\infty}$ are not eventually ε-close.

Fix this value of ε, and let E be the set

$$E := \{n \in \mathbb{N} : f(x_n) \text{ and } f(y_n) \text{ are not } \varepsilon\text{-close}\}.$$

We must have E infinite, since if E were finite then $\big(f(x_n)\big)_{n=0}^{\infty}$ and $\big(f(y_n)\big)_{n=0}^{\infty}$ would be eventually ε-close. By Prop. I.8.1.5, E is countable; in fact from the proof of that proposition we see that we can find an infinite sequence

$$n_0 < n_1 < n_2 < \dots$$

consisting entirely of elements in E. In particular, we have

$$\left|f(x_{n_j}) - f(y_{n_j})\right| > \varepsilon \text{ for all } j \in \mathbb{N}. \tag{i:9.3}$$

On the other hand, the sequence $(x_{n_j})_{j=0}^{\infty}$ is a sequence in $[a, b]$, and so by the Heine-Borel theorem (Thm. I.9.1.24) there must be a subsequence $(x_{n_{j_k}})_{k=0}^{\infty}$ which converges to some limit L in $[a, b]$. In particular, f is continuous at L, and so by Prop. I.9.4.7,

$$\lim_{k\to\infty} f(x_{n_{j_k}}) = f(L). \tag{i:9.4}$$

Note that $(x_{n_{j_k}})_{k=0}^{\infty}$ is a subsequence of $(x_n)_{n=0}^{\infty}$, and $(y_{n_{j_k}})_{k=0}^{\infty}$ is a subsequence of $(y_n)_{n=0}^{\infty}$, by Lem. I.6.6.4. On the other hand, from Lem. I.9.9.7 we have

$$\lim_{n\to\infty}(x_n - y_n) = 0.$$

By Prop. I.6.6.5, we thus have

$$\lim_{k\to\infty}(x_{n_{j_k}} - y_{n_{j_k}}) = 0.$$

Since $x_{n_{j_k}}$ converges to L as $k \to \infty$, we thus have by limit laws

$$\lim_{k\to\infty} y_{n_{j_k}} = L.$$

and hence by continuity of f at L

$$\lim_{k\to\infty} f(y_{n_{j_k}}) = f(L).$$

Subtracting this from (9.4) using limit laws, we obtain

$$\lim_{k\to\infty}(f(x_{n_{j_k}}) - f(y_{n_{j_k}})) = 0.$$

But this contradicts (9.3) (by Lem. I.9.9.7). From this contradiction we conclude that f is in fact uniformly continuous. □

Rmk. I.9.9.17. One should compare Lem. I.9.6.3, Prop. I.9.9.15, and Thm. I.9.9.16 with each other. Note, in particular, that Lem. I.9.6.3 follows from combining Prop. I.9.9.15 and Thm. I.9.9.16.

— Exercises —

Ex. I.9.9.1. Prove Lem. I.9.9.7.

Proof. See Lem. I.9.9.7. □

Ex. I.9.9.2. Prove Prop. I.9.9.8.

Proof. See Prop. I.9.9.8. □

Ex. I.9.9.3. Prove Prop. I.9.9.12.

Proof. See Prop. I.9.9.12. □

Ex. I.9.9.4. Use Prop. I.9.9.12 to prove Cor. I.9.9.14. Use this corollary to give an alternate demonstration of the results in Example 9.9.10.

Proof. See Cor. I.9.9.14. □

Ex. I.9.9.5. Prove Prop. I.9.9.15.

Proof. See Prop. I.9.9.15. □

Ex. I.9.9.6. Let X, Y, Z be subsets of $\mathbb{R}$. Let $f : X \to Y$ be a function which is uniformly continuous on X, and let $g : Y \to Z$ be a function which is uniformly continuous on Y. Show that the function $g \circ f : X \to Z$ is uniformly continuous on X.

Proof. Since f is continuous on X and g is continuous on Y, by Prop. I.9.4.13 we know that $g \circ f$ is continuous on X. Since g is uniformly continuous, by Def. I.9.9.2 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta' \in \mathbb{R}^+ : \big(\forall y_1, y_2 \in Y, |y_1 - y_2| < \delta' \implies |g(y_1) - g(y_2)| \le \varepsilon\big).$$

Similarly, since f is uniformly continuous, by Def. I.9.9.2 we have

$$\forall \varepsilon' \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x_1, x_2 \in X, |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| \le \varepsilon'\big).$$

In particular, we have

$$\exists \delta \in \mathbb{R}^+ : \big(\forall x_1, x_2 \in X, |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| \le \delta'/2 < \delta'\big).$$

Since $f(x_1), f(x_2) \in Y$, we have

$$|f(x_1) - f(x_2)| < \delta' \implies \big|g\big(f(x_1)\big) - g\big(f(x_2)\big)\big| \le \varepsilon.$$

Thus, we have showed that

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \Big(\forall x_1, x_2 \in X, |x_1 - x_2| < \delta \implies \big|g\big(f(x_1)\big) - g\big(f(x_2)\big)\big| \le \varepsilon\Big).$$

and by Def. I.9.9.2 $g \circ f$ is uniformly continuous on X. □

I.9.10 Limits at infinity

Def. I.9.10.1 (Infinite adherent points)**.** Let X be a subset of $\mathbb{R}$. We say that $+\infty$ is *adherent* to X iff for every $M \in \mathbb{R}$ there exists an $x \in X$ such that $x > M$; we say that $-\infty$ is *adherent* to X iff for every $M \in \mathbb{R}$ there exists an $x \in X$ such that $x < M$.

Note. In other words, $+\infty$ is adherent to X iff X has no upper bound, or equivalently iff $\sup(X) = +\infty$. Similarly, $-\infty$ is adherent to X iff X has no lower bound, or iff $\inf(X) = -\infty$. Thus, a set is bounded iff $+\infty$ and $-\infty$ are not adherent points.

Rmk. I.9.10.2. Def. I.9.10.1 may seem rather different from Def. I.9.1.8, but can be unified using the topological structure of the extended real line $\mathbb{R}^*$.

Def. I.9.10.3 (Limits at infinity)**.** Let X be a subset of $\mathbb{R}$ with $+\infty$ as an adherent point, and let $f : X \to \mathbb{R}$ be a function. We say that $f(x)$ *converges to* L as $x \to +\infty$ in X, and write $\lim_{x \to +\infty; x \in X} f(x) = L$, iff for every $\varepsilon > 0$ there exists an M such that f is ε-close to L on $X \cap (M, +\infty)$ (i.e., $|f(x) - L| \leq \varepsilon$ for all $x \in X$ such that $x > M$). Similarly, we say that $f(x)$ *converges to* L as $x \to -\infty$ iff for every $\varepsilon > 0$ there exists an M such that f is ε-close to L on $X \cap (-\infty, M)$.

Note. One can do many of the same things with these limits at infinity as we have been doing with limits at other points x_0; for instance, it turns out that all of the limit laws continue to hold. However, as we will not be using these limits much in this text, we will not devote much attention to these matters. We will note though that this definition is consistent with the notion of a limit $\lim_{n \to \infty} a_n$ of a sequence.

— Exercises —

Ex. I.9.10.1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, then a_n can also be thought of as a function from $\mathbb{N}$ to $\mathbb{R}$, which takes each natural number n to a real number a_n. Show that

$$\lim_{n \to +\infty; n \in \mathbb{N}} a_n = \lim_{n \to \infty} a_n$$

where the left-hand limit is defined by Def. I.9.10.3 and the right-hand limit is defined by Def. I.6.1.8. More precisely, show that if one of the above two limits exists then so does the other, and then they both have the same value. Thus, the two notions of limit here are compatible.

Proof. We first show that $\lim_{n \to +\infty; n \in \mathbb{N}} a_n = L$ implies $\lim_{n \to \infty} a_n = L$. By Def. I.9.10.3, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists M \in \mathbb{R} : \big(\forall n \in \mathbb{N}, n > M \implies |a_n - L| \leq \varepsilon\big).$$

Since $M \in \mathbb{R}$, by Prop. I.5.4.12 $\exists N \in \mathbb{N}$ such that $M \leq N$. Then we have

$$\forall n \in \mathbb{N}, n > \mathbb{N} \implies n > M \implies |a_n - L| \leq \varepsilon.$$

Thus, by Def. I.6.1.5 we have $\lim_{n \to \infty} a_n = L$.

Now we show that $\lim_{n \to \infty} a_n = L$ implies $\lim_{n \to +\infty; n \in \mathbb{N}} a_n = L$. By Def. I.6.1.5 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - L| \leq \varepsilon.$$

Since $\mathbb{N} \subseteq \mathbb{R}$, we have $N \in \mathbb{R}$ and thus by Def. I.9.10.3 we have $\lim_{n \to +\infty, n \in \mathbb{N}} a_n = L$. $\square$

Chapter I.10

Differentiation of functions

I.10.1 Basic definitions

Note. We can now define derivatives analyti cally, using limits, in contrast to the geometric definition of derivatives, which uses tangents. The advantage of working analytically is that (a) we do not need to know the axioms of geometry, and (b) these definitions can be modified to handle functions of several variables, or functions whose values are vectors instead of scalar. Furthermore, one's geometric intuition becomes difficult to rely on once one has more than three dimensions in play. (Conversely, one can use one's experience in analytic rigour to extend one's geometric intuition to such abstract settings; as mentioned earlier, the two viewpoints complement rather than oppose each other.)

Def. I.10.1.1 (Differentiability at a point)**.** Let X be a subset of $\mathbb{R}$, and let $x_0 \in X$ be an element of X which is also a limit point of X. Let $f : X \to \mathbb{R}$ be a function. If the limit

$$\lim_{x \to x_0; x \in X \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$$

converges to some real number L, then we say that f is *differentiable at* x_0 *on* X *with derivative* L, and write $f'(x_0) := L$. If the limit does not exist, or if x_0 is not an element of X or not a limit point of X, we leave $f'(x_0)$ undefined, and say that f is *not differentiable at* x_0 *on* X.

Rmk. I.10.1.2. Note that we need x_0 to be a limit point in order for x_0 to be adherent to $X \setminus \{x_0\}$, otherwise the limit

$$\lim_{x \to x_0; x \in X \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$$

would automatically be undefined. In particular, we do not define the derivative of a function at an isolated point; In practice, the domain X will almost always be an interval, and so by Lem. I.9.1.21 all elements x_0 of X will automatically be limit points and we will not have to care much about these issues.

Rmk. I.10.1.4. This point is trivial, but it is worth mentioning: if $f : X \to \mathbb{R}$ is differentiable at x_0, and $g : X \to \mathbb{R}$ is equal to f (i.e., $g(x) = f(x)$ for all $x \in X$), then g is also differentiable at x_0 and $g'(x_0) = f'(x_0)$. However, if two functions f and g merely have the same value at x_0, i.e., $g(x_0) = f(x_0)$, this does not imply that $g'(x_0) = f'(x_0)$. Thus, there is a big difference between two functions being equal on their whole domain, and merely being equal at one point.

Rmk. I.10.1.5. One sometimes writes $\dfrac{df}{dx}$ instead of f'. This notation is of course very familiar and convenient, but one has to be a little careful, because it is only safe to use as long as x is the only variable used to represent the input for f; otherwise one can get into all sorts of trouble. Because of this possible source of confusion, we will refrain from using the notation $\dfrac{df}{dx}$ whenever it could possibly lead to confusion. (This confusion becomes even worse in the calculus of several variables, and the standard notation of $\dfrac{\partial f}{\partial x}$ can lead to some serious ambiguities. There are ways to resolve these ambiguities, most notably by introducing the notion of differentiation along vector fields, but this is beyond the scope of this text.)

E.g. I.10.1.6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) := |x|$, and let $x_0 = 0$. To see whether f is differentiable at 0 on $\mathbb{R}$, we compute the limit

$$\lim_{x \to 0; x \in \mathbb{R} \setminus \{0\}} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0; x \in \mathbb{R} \setminus \{0\}} \frac{|x|}{x}.$$

Now we take left limits and right limits. The right limit is

$$\lim_{x \to 0; x \in (0, \infty)} \frac{|x|}{x} = \lim_{x \to 0; x \in (0, \infty)} \frac{x}{x} = \lim_{x \to 0; x \in (0, \infty)} 1 = 1,$$

while the left limit is

$$\lim_{x \to 0; x \in (-\infty, 0)} \frac{|x|}{x} = \lim_{x \to 0; x \in (-\infty, 0)} \frac{-x}{x} = \lim_{x \to 0; x \in (-\infty, 0)} -1 = -1,$$

and these limits do not match. Thus, $\displaystyle\lim_{x \to 0; x \in (0, \infty)} \frac{|x|}{x}$ does not exist, and f is not differentiable at 0 on $\mathbb{R}$. However, if one restricts f to $[0, \infty)$, then the restricted function $f|_{[0,\infty)}$ *is* differentiable at 0 on $[0, \infty)$, with derivative 1:

$$\lim_{x \to 0; x \in [0, \infty) \setminus \{0\}} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0; x \in (0, \infty)} \frac{|x|}{x} = 1.$$

Similarly, when one restricts f to $(-\infty, 0]$, the restricted function $f|_{(-\infty,0]}$ is differentiable at 0 on $(-\infty, 0]$, with derivative -1. Thus, even when a function is not differentiable, it is sometimes possible to restore the differentiability by restricting the domain of the function.

Prop. I.10.1.7 (Newton's approximation)**.** Let X be a subset of $\mathbb{R}$, let $x_0 \in X$ be a limit point of X, let $f : X \to \mathbb{R}$ be a function, and let L be a real number. Then the following statements are logically equivalent:

(a) f is differentiable at x_0 on X with derivative L.

(b) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ is $\varepsilon|x - x_0|$-close to $f(x_0)+L(x-x_0)$ whenever $x \in X$ is δ-close to x_0, i.e., we have

$$|f(x) - (f(x_0) + L(x - x_0))| \leq \varepsilon|x - x_0|$$

whenever $x \in X$ and $|x - x_0| \leq \delta$.

Proof. We first show that the first statement implies the second statement. Since f is differentiable at x_0 on X with derivative L, by Def. I.10.1.1 we have

$$\lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = L.$$

By Def. I.9.3.6 this means

$$\forall\varepsilon \in \mathbb{R}^+, \exists\delta \in \mathbb{R}^+ : \left(\forall x \in X \setminus \{x_0\}, |x - x_0| < \delta \implies \left|\frac{f(x) - f(x_0)}{x - x_0} - L\right| \leq \varepsilon\right).$$

Thus, we have

$$\begin{aligned}
&\forall x \in X \setminus \{x_0\}, |x - x_0| \leq \delta/2 < \delta \\
\implies &\left|\frac{f(x) - f(x_0)}{x - x_0} - L\right| \leq \varepsilon \\
\implies &\left|\frac{f(x) - f(x_0)}{x - x_0} - L\right||x - x_0| \leq \varepsilon|x - x_0| \\
\implies &|(f(x) - f(x_0)) - L(x - x_0)| \leq \varepsilon|x - x_0| \\
\implies &\big|f(x) - \big(f(x_0) + L(x - x_0)\big)\big| \leq \varepsilon|x - x_0|.
\end{aligned}$$

If $x = x_0$, then we have

$$\begin{aligned}
&0 = |x_0 - x_0| \leq \delta/2 < \delta \\
\implies &0 = \big|f(x_0) - \big(f(x_0) + L(x_0 - x_0)\big)\big| \leq \varepsilon|x_0 - x_0| = 0.
\end{aligned}$$

Thus, we have $\forall\varepsilon \in \mathbb{R}^+$, $\exists\delta \in \mathbb{R}^+$ such that

$$\forall x \in X, |x - x_0| \leq \delta/2 \implies \big|f(x) - \big(f(x_0) + L(x - x_0)\big)\big| \leq \varepsilon|x - x_0|.$$

Now we show that the second statement implies the first statement. By hypothesis we have $\forall\varepsilon \in \mathbb{R}^+$, $\exists\delta \in \mathbb{R}^+$ such that

$$\forall x \in X, |x - x_0| \leq \delta \implies \big|f(x) - \big(f(x_0) + L(x - x_0)\big)\big| \leq \varepsilon|x - x_0|.$$

In particular, we have

$$\forall x \in X \setminus \{x_0\}, |x - x_0| \le \delta \implies \big|f(x) - \big(f(x_0) + L(x - x_0)\big)\big| \le \varepsilon |x - x_0|.$$

Thus, we have

$$\begin{aligned}
&\forall x \in X \setminus \{x_0\}, |x - x_0| \le \delta \\
\implies &\big|f(x) - \big(f(x_0) + L(x - x_0)\big)\big| \le \varepsilon |x - x_0| \\
\implies &\frac{\big|f(x) - \big(f(x_0) + L(x - x_0)\big)\big|}{|x - x_0|} \le \varepsilon \\
\implies &\left|\frac{f(x) - f(x_0)}{x - x_0} - L\right| \le \varepsilon.
\end{aligned}$$

By Def. I.9.3.6 this means

$$\lim_{x \to x_0; x \in X \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = L$$

and by Def. I.10.1.1 we know that f is differentiable at x_0 on X with derivative L. □

Rmk. I.10.1.8. Newton's approximation is of course named after the great scientist and mathematician Isaac Newton (1642–1727), one of the founders of differential and integral calculus.

Rmk. I.10.1.9. We can phrase Prop. I.10.1.7 in a more informal way: if f is differentiable at x_0, then one has the approximation $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$, and conversely.

Prop. I.10.1.10 (Differentiability implies continuity)**.** Let X be a subset of $\mathbb{R}$, let $x_0 \in X$ be a limit point of X, and let $f : X \to \mathbb{R}$ be a function. If f is differentiable at x_0, then f is also continuous at x_0.

Proof. Since f is differentiable at x_0, by Def. I.10.1.1 we have

$$L = \lim_{x \to x_0; x \in X \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$$

for some $L \in \mathbb{R}$. By Prop. I.10.1.7, we have $\forall \varepsilon \in \mathbb{R}^+$, $\exists \delta \in \mathbb{R}^+$ such that

$$\begin{aligned}
&\forall x \in X, |x - x_0| \le \delta \\
\implies &\big|f(x) - \big(f(x_0) + L(x - x_0)\big)\big| \le \varepsilon |x - x_0| \\
\implies &\big|f(x) - \big(f(x_0) + L(x - x_0)\big)\big| + |L(x - x_0)| \le \varepsilon |x - x_0| + |L(x - x_0)| \\
\implies &\big|f(x) - \big(f(x_0) + L(x - x_0)\big)\big| + |L(x - x_0)| \le (\varepsilon + |L|)|x - x_0| \\
\implies &\big|f(x) - \big(f(x_0) + L(x - x_0)\big) + L(x - x_0)\big| \\
&\quad \le \big|f(x) - \big(f(x_0) + L(x - x_0)\big)\big| + |L(x - x_0)|
\end{aligned}$$

$$\begin{aligned}
&\leq (\varepsilon + |L|)|x - x_0| \\
\implies &|f(x) - f(x_0)| \leq (\varepsilon + |L|)|x - x_0|.
\end{aligned}$$

Let $\delta' = \min(\delta, \varepsilon/(\varepsilon + |L|))$. Then we have

$$\begin{aligned}
&\forall x \in X, |x - x_0| < \delta' \leq \delta \\
\implies &|f(x) - f(x_0)| \leq (\varepsilon + |L|)|x - x_0| \\
\implies &|f(x) - f(x_0)| \leq (\varepsilon + |L|)|x - x_0| \leq (\varepsilon + |L|)\delta' \leq (\varepsilon + |L|)\frac{\varepsilon}{\varepsilon + |L|} \\
\implies &|f(x) - f(x_0)| \leq \varepsilon.
\end{aligned}$$

Thus, by Def. I.9.3.6 we have $\lim_{x \to x_0; x \in X} f(x) = f(x_0)$, and by Def. I.9.4.1 f is continuous at x_0. □

Def. I.10.1.11 (Differentiability on a domain)**.** Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function. We say that f is *differentiable on* X if, for every limit point $x_0 \in X$, the function f is differentiable at x_0 on X.

Cor. I.10.1.12. Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function which is differentiable on X. Then f is also continuous on X.

Proof. By Lem. I.9.1.11 we know that $\forall x_0 \in X$, x_0 is an adherent point. By Ex. I.9.1.9 we know that x_0 is either a limit point or an isolated point. By Prop. I.10.1.10 and Def. I.10.1.11 we know that if x_0 is a limit point then f is continuous at x_0. So we only need to show that if x_0 is an isolated point, then f is also continuous at x_0. Suppose that x_0 is an isolated point of X. By Def. I.9.1.18 we know that $\exists \varepsilon' \in \mathbb{R}^+$ such that $|x - x_0| > \varepsilon'$ for all $x \in X \setminus \{x_0\}$. To show that f is continuous at x_0, by Def. I.9.4.1 and Def. I.9.3.6 we need to show that

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in X, |x - x_0| < \delta \implies |f(x) - f(x_0)| \leq \varepsilon\big).$$

Let $\delta = \varepsilon'$. Since x_0 is an isolated point, the only $x \in X$ satisfying $|x - x_0| < \varepsilon'$ is x_0. Thus, we have $0 = |f(x_0) - f(x_0)| \leq \varepsilon$ and $\lim_{x \to x_0; x \in X} f(x) = f(x_0)$. □

Thm. I.10.1.13 (Differential calculus)**.** Let X be a subset of $\mathbb{R}$, let $x_0 \in X$ be a limit point of X, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions.

(a) If f is a constant function, i.e., there exists a real number c such that $f(x) = c$ for all $x \in X$, then f is differentiable at x_0 and $f'(x_0) = 0$.

(b) If f is the identity function, i.e., $f(x) = x$ for all $x \in X$, then f is differentiable at x_0 and $f'(x_0) = 1$.

(c) (Sum rule) If f and g are differentiable at x_0, then $f + g$ is also differentiable at x_0, and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.

(d) (Product rule) If f and g are differentiable at x_0, then fg is also differentiable at x_0, and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

(e) If f is differentiable at x_0 and c is a real number, then cf is also differentiable at x_0, and $(cf)'(x_0) = cf'(x_0)$.

(f) (Difference rule) If f and g are differentiable at x_0, then $f - g$ is also differentiable at x_0, and $(f - g)'(x_0) = f'(x_0) - g'(x_0)$.

(g) If g is differentiable at x_0, and g is non-zero on X (i.e., $g(x) \neq 0$ for all $x \in X$), then $1/g$ is also differentiable at x_0, and $(\frac{1}{g})'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}$.

(h) (Quotient rule) If f and g are differentiable at x_0, and g is non-zero on X, then f/g is also differentiable at x_0, and

$$(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Proof. (a) We have $\forall \varepsilon \in \mathbb{R}^+$, $\forall \delta \in \mathbb{R}^+$ such that

$$\forall x \in X \setminus \{x_0\}, |x - x_0| < \delta \implies \left|\frac{f(x) - f(x_0)}{x - x_0} - 0\right| = \left|\frac{c - c}{x - x_0} - 0\right| = 0 \leq \varepsilon.$$

Thus, by Def. I.9.3.6 we have

$$\lim_{x \to x_0; x \in X \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = 0$$

and by Def. I.10.1.1 we have $f'(x_0) = 0$. □

Proof. (b) We have $\forall \varepsilon \in \mathbb{R}^+$, $\forall \delta \in \mathbb{R}^+$ such that

$$\forall x \in X \setminus \{x_0\}, |x - x_0| < \delta \implies \left|\frac{f(x) - f(x_0)}{x - x_0} - 1\right| = \left|\frac{x - x_0}{x - x_0} - 1\right| = 0 \leq \varepsilon.$$

Thus, by Def. I.9.3.6 we have

$$\lim_{x \to x_0; x \in X \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = 1$$

and by Def. I.10.1.1 we have $f'(x_0) = 1$. □

Proof. (c) By Def. I.10.1.1 and Prop. I.9.3.14 we have

$$f'(x_0) + g'(x_0)$$

$$
\begin{aligned}
&= \lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x)-f(x_0)}{x-x_0} + \lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{g(x)-g(x_0)}{x-x_0} \\
&= \lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)-g(x_0)}{x-x_0} \\
&= \lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x)-f(x_0)+g(x)-g(x_0)}{x-x_0} \\
&= \lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x)+g(x)-(f(x_0)+g(x_0))}{x-x_0} \\
&= \lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{(f+g)(x)-(f+g)(x_0)}{x-x_0} \\
&=(f+g)'(x_0).
\end{aligned}
$$

Thus, $f+g$ is differentiable at x_0 and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$. □

Proof. (d) By Def. I.10.1.1 and Prop. I.9.3.14 we have

$$
\begin{aligned}
& f'(x_0)g(x_0) + f(x_0)g'(x_0) \\
=& \left(\lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x)-f(x_0)}{x-x_0} \right) \left(\lim_{x\to x_0; x\in X\setminus\{x_0\}} g(x) \right) \\
& + f(x_0) \left(\lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{g(x)-g(x_0)}{x-x_0} \right) \\
=& \left(\lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{\big(f(x)-f(x_0)\big)g(x)}{x-x_0} \right) + \left(\lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x_0)\big(g(x)-g(x_0)\big)}{x-x_0} \right) \\
=& \left(\lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x)g(x)-f(x_0)g(x)}{x-x_0} \right) + \left(\lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0} \right) \\
=& \lim_{x\to x_0; x\in X\setminus\{x_0\}} \left(\frac{f(x)g(x)-f(x_0)g(x)}{x-x_0} + \frac{f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0} \right) \\
=& \lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0} \\
=& \lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} \\
=& \lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{(fg)(x)-(fg)(x_0)}{x-x_0} \\
=&(fg)'(x_0).
\end{aligned}
$$

Thus, fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$. □

Proof. (e) By Def. I.10.1.1 and Prop. I.9.3.14 we have

$$cf'(x_0)$$

$$\begin{aligned}
&=c\left(\lim_{x\to x_0;x\in X\setminus\{x_0\}}\frac{f(x)-f(x_0)}{x-x_0}\right)\\
&=\lim_{x\to x_0;x\in X\setminus\{x_0\}}\left(c\frac{f(x)-f(x_0)}{x-x_0}\right)\\
&=\lim_{x\to x_0;x\in X\setminus\{x_0\}}\frac{cf(x)-cf(x_0)}{x-x_0}\\
&=\lim_{x\to x_0;x\in X\setminus\{x_0\}}\frac{(cf)(x)-(cf)(x_0)}{x-x_0}\\
&=(cf)'(x_0).
\end{aligned}$$

Thus, cf is differentiable at x_0 and $(cf)'(x_0)=cf'(x_0)$. □

Proof. (f)

$$\begin{aligned}
&f'(x_0)-g'(x_0)\\
=&f'(x_0)+\big(-g'(x_0)\big)\\
=&f'(x_0)+\big((-g)'(x_0)\big) && \text{(by Thm. I.10.1.13(e))}\\
=&\big(f+(-g)\big)'(x_0) && \text{(by Thm. I.10.1.13(c))}\\
=&(f-g)'(x_0). && \text{(by Def. I.9.2.1)}
\end{aligned}$$

Thus, $f-g$ is differentiable at x_0 and $(f-g)'(x_0)=f'(x_0)-g'(x_0)$. □

Proof. (g) By Def. I.10.1.1 and Prop. I.9.3.14 we have

$$\begin{aligned}
&-\frac{g'(x_0)}{g(x_0)^2}\\
=&\left(\lim_{x\to x_0;x\in X\setminus\{x_0\}}\frac{g(x)-g(x_0)}{x-x_0}\right)\left(\frac{-1}{g(x_0)^2}\right)\\
=&\left(\lim_{x\to x_0;x\in X\setminus\{x_0\}}\frac{g(x)-g(x_0)}{x-x_0}\right)\left(\frac{-g(x_0)}{g(x_0)g(x_0)^2}\right)\\
=&\left(\lim_{x\to x_0;x\in X\setminus\{x_0\}}\frac{g(x)-g(x_0)}{x-x_0}\right)\left(\lim_{x\to x_0;x\in X\setminus\{x_0\}}\frac{-g(x_0)}{g(x)g(x_0)^2}\right)\\
=&\lim_{x\to x_0;x\in X\setminus\{x_0\}}\left(\left(\frac{g(x)-g(x_0)}{x-x_0}\right)\left(\frac{-g(x_0)}{g(x)g(x_0)^2}\right)\right)\\
=&\lim_{x\to x_0;x\in X\setminus\{x_0\}}\frac{\dfrac{g(x_0)(g(x_0)-g(x))}{g(x)g(x_0)^2}}{x-x_0}\\
=&\lim_{x\to x_0;x\in X\setminus\{x_0\}}\frac{\dfrac{g(x_0)-g(x)}{g(x)g(x_0)}}{x-x_0}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x\to x_0; x\in X\setminus\{x_0\}} \frac{\dfrac{1}{g(x)} - \dfrac{1}{g(x_0)}}{x - x_0} \\
&= (\frac{1}{g})'(x_0).
\end{aligned}$$

Thus, $1/g$ is differentiable at x_0 and $(1/g)'(x_0) = -\dfrac{g'(x_0)}{g(x_0)^2}$. □

Proof. (h)

$$\begin{aligned}
&(\frac{f}{g})'(x_0) \\
=&(f \cdot \frac{1}{g})'(x_0) && \text{(by Def. I.9.2.1)} \\
=&f'(x_0)\frac{1}{g}(x_0) + f(x_0)(\frac{1}{g})'(x_0) && \text{(by Thm. I.10.1.13(d))} \\
=&\frac{f'(x_0)}{g(x_0)} + f(x_0)(\frac{1}{g})'(x_0) && \text{(by Def. I.9.2.1)} \\
=&\frac{f'(x_0)}{g(x_0)} + f(x_0)\frac{-g'(x_0)}{g(x_0)^2} && \text{(by Thm. I.10.1.13(g))} \\
=&\frac{f'(x_0)g(x_0)}{g(x_0)^2} - \frac{f(x_0)g'(x_0)}{g(x_0)^2} \\
=&\frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.
\end{aligned}$$

Thus, f/g is differentiable at x_0 and $(f/g)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$. □

Rmk. I.10.1.14. The product rule is also known as the *Leibniz rule*, after Gottfried Leibniz (1646–1716), who was the other founder of differential and integral calculus besides Newton.

Note. The trick of adding and subtracting an intermediate term is sometimes known as the "middle-man trick" and is very useful in analysis.

Thm. I.10.1.15 (Chain rule)**.** Let X, Y be subsets of $\mathbb{R}$, let $x_0 \in X$ be a limit point of X, and let $y_0 \in Y$ be a limit point of Y. Let $f : X \to Y$ be a function such that $f(x_0) = y_0$, and such that f is differentiable at x_0. Suppose that $g : Y \to \mathbb{R}$ is a function which is differentiable at y_0. Then the function $g \circ f : X \to \mathbb{R}$ is differentiable at x_0, and

$$(g \circ f)'(x_0) = g'(y_0)f'(x_0)$$

Proof. By Prop. I.10.1.7 we want to show that

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in X, |x - x_0| \leq \delta \\ \implies &\big|g\big(f(x)\big) - g(y_0) - g'(y_0)f'(x_0)(x - x_0)\big| \leq \varepsilon |x - x_0|. \end{aligned}$$

Since $g'(y_0)$ exists, by Prop. I.10.1.7 we have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists \delta_g \in \mathbb{R}^+ : \forall y \in Y, |y - y_0| \leq \delta_g \\ \implies &\big|g(y) - g(y_0) - g'(y_0)(y - y_0)\big| \leq \varepsilon_g |y - y_0| \end{aligned}$$

where

$$\varepsilon_g = \frac{-\big(|f'(x_0)| + |g'(y_0)|\big) + \sqrt{\big(|f'(x_0)| + |g'(y_0)|\big)^2 + 4\varepsilon}}{2}.$$

Note that $\varepsilon_g \in \mathbb{R}^+$ since

$$\sqrt{\big(|f'(x_0)| + |g'(y_0)|\big)^2 + 4\varepsilon} > \sqrt{\big(|f'(x_0)| + |g'(y_0)|\big)^2} = \big|f'(x_0)\big| + \big|g'(y_0)\big|.$$

Now fix such ε (and ε_g) and δ_g. Since $f'(x_0)$ exists, by Cor. I.10.1.12 f is continuous at x_0 and by Def. I.9.4.1 $\lim\limits_{x \to x_0; x \in X} f(x) = f(x_0)$. This means

$$\begin{aligned} &\exists \delta \in \mathbb{R}^+ : \forall x \in X, |x - x_0| \leq \delta \\ \implies &\begin{cases} |f(x) - y_0| \leq \delta_g \\ \big|f(x) - y_0 - f'(x_0)(x - x_0)\big| \leq \varepsilon_g |x - x_0| \end{cases} \qquad \text{(by Prop. I.9.4.7(d) and Prop. I.10.1.7)} \\ \implies &\begin{cases} |f(x) - y_0| \leq \delta_g \\ \big|f(x) - y_0 - f'(x_0)(x - x_0)\big| \leq \varepsilon_g |x - x_0| \\ |f(x) - y_0| \leq \varepsilon_g |x - x_0| + \big|f'(x_0)(x - x_0)\big| \end{cases} \\ \implies &\begin{cases} |f(x) - y_0| \leq \delta_g \\ \big|f(x) - y_0 - f'(x_0)(x - x_0)\big| \leq \varepsilon_g |x - x_0| \\ |f(x) - y_0| \leq \big(\varepsilon_g + \big|f'(x_0)\big|\big)|x - x_0| \end{cases} \end{aligned}$$

We know that

$$\begin{aligned} &\exists \delta \in \mathbb{R}^+ : \forall x \in X, |x - x_0| \leq \delta \\ \implies &|f(x) - y_0| \leq \delta_g \\ \implies &\big|g\big(f(x)\big) - g(y_0) - g'(y_0)\big(f(x_0) - y_0\big)\big| \leq \varepsilon_g |f(x) - y_0| \\ \implies &\big|g\big(f(x)\big) - g(y_0) - g'(y_0)f'(x_0)(x - x_0) - g'(y_0)\big(f(x) - y_0 - f'(x_0)(x - x_0)\big)\big| \\ &\leq \varepsilon_g |f(x) - y_0| \\ \implies &\big|g\big(f(x)\big) - g(y_0) - g'(y_0)f'(x_0)(x - x_0)\big| \end{aligned}$$

$$
\begin{aligned}
&\le \varepsilon_g|f(x)-y_0| + \big|g'(y_0)\big(f(x)-y_0-f'(x_0)(x-x_0)\big)\big| \\
\implies &\big|g\big(f(x)\big)-g(y_0)-g'(y_0)f'(x_0)(x-x_0)\big| \\
&\le \varepsilon_g|f(x)-y_0| + \big|g'(y_0)\big|\big|f(x_0)-y_0-f'(x_0)(x-x_0)\big| \\
\implies &\big|g\big(f(x)\big)-g(y_0)-g'(y_0)f'(x_0)(x-x_0)\big| \\
&\le \varepsilon_g|f(x)-y_0| + \varepsilon_g\big|g'(y_0)\big||x-x_0| \\
\implies &\big|g\big(f(x)\big)-g(y_0)-g'(y_0)f'(x_0)(x-x_0)\big| \\
&\le \varepsilon_g\big(\varepsilon_g+\big|f'(x_0)\big|\big)|x-x_0| + \varepsilon_g\big|g'(y_0)\big||x-x_0| \\
\implies &\big|g\big(f(x)\big)-g(y_0)-g'(y_0)f'(x_0)(x-x_0)\big| \\
&\le \varepsilon_g\big(\varepsilon_g+\big|f'(x_0)\big|+\big|g'(y_0)\big|\big)|x-x_0|.
\end{aligned}
$$

Expanding ε_g we have

$$
\begin{aligned}
&\varepsilon_g\big(\varepsilon_g+\big|f'(x_0)\big|+\big|g'(y_0)\big|\big) \\
&= \frac{-\big(|f'(x_0)|+|g'(y_0)|\big)+\sqrt{\big(|f'(x_0)|+|g'(y_0)|\big)^2+4\varepsilon}}{2} \\
&\quad\times \frac{\big(|f'(x_0)|+|g'(y_0)|\big)+\sqrt{\big(|f'(x_0)|+|g'(y_0)|\big)^2+4\varepsilon}}{2} \\
&= \frac{\big(|f'(x_0)|+|g'(y_0)|\big)^2+4\varepsilon-\big(|f'(x_0)|+|g'(y_0)|\big)^2}{4} \\
&= \varepsilon.
\end{aligned}
$$

Thus, we conclude that

$$
\begin{aligned}
&\forall\varepsilon\in\mathbb{R}^+, \exists\delta\in\mathbb{R}^+ : \forall x\in X, |x-x_0|\le\delta \\
\implies &\big|g\big(f(x)\big)-g(y_0)-g'(y_0)f'(x_0)(x-x_0)\big| \le \varepsilon|x-x_0|
\end{aligned}
$$

and by Prop. I.10.1.7 we have $(g\circ f)'(x_0) = g'(y_0)f'(x_0)$. □

Rmk. I.10.1.17. If one writes y for $f(x)$, and z for $g(y)$, then the chain rule can be written in the more visually appealing manner $\dfrac{dz}{dx} = \dfrac{dz}{dy}\dfrac{dy}{dx}$. However, this notation can be misleading (for instance it blurs the distinction between dependent variable and independent variable, especially for y), and leads one to believe that the quantities dz, dy, dx can be manipulated like real numbers. However, these quantities are not real numbers (in fact, we have not assigned any meaning to them at all), and treating them as such can lead to problems in the future. For instance, if f depends on x_1 and x_2, which depend on t, then chain rule for several variables asserts that $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x_1}\dfrac{dx_1}{dt} + \dfrac{\partial f}{\partial x_2}\dfrac{dx_2}{dt}$, but this rule might seem suspect if one treated df, dt, etc. as real numbers. It is possible to think of dy, dx, etc. as "infinitesimal real numbers" if one knows what one is doing, but for those just starting out in analysis, I would not recommend this approach, especially if one wishes to work rigorously. (There is a way to make all of this rigorous, even for the calculus of several variables, but it requires the notion of a tangent vector, and the derivative map, both of which are beyond the scope of this text.)

— Exercises —

Ex. I.10.1.1. Suppose that X is a subset of $\mathbb{R}$, x_0 is a limit point of X, and $f : X \to \mathbb{R}$ is a function which is differentiable at x_0. Let $Y \subseteq X$ be such that $x_0 \in Y$, and x_0 is also a limit point of Y. Prove that the restricted function $f|_Y : Y \to \mathbb{R}$ is also differentiable at x_0, and has the same derivative as f at x_0. Explain why this does not contradict the discussion in Rmk. I.10.1.2.

Proof. Since f is differentiable at x_0, by Newton's approximation (Prop. I.10.1.7) we have $\forall \varepsilon \in \mathbb{R}^+$, $\exists \delta \in \mathbb{R}^+$ such that

$$\forall x \in X, |x - x_0| \leq \delta \implies \left|f(x) - (f(x_0) + f'(x_0)(x - x_0))\right| \leq \varepsilon |x - x_0|.$$

Since $Y \subseteq X$, we have

$$\begin{aligned} &\forall x \in Y, |x - x_0| \leq \delta \\ \implies &(x \in X) \wedge (|x - x_0| \leq \delta) \\ \implies &\left|f(x) - \big(f(x_0) + f'(x_0)(x - x_0)\big)\right| \leq \varepsilon |x - x_0| \\ \implies &\left|f|_Y(x) - \big(f|_Y(x_0) + f'(x_0)(x - x_0)\big)\right| \leq \varepsilon |x - x_0|. \end{aligned}$$

Thus, by Newton's approximation (Prop. I.10.1.7) we know that $f|_Y'(x_0) = f'(x_0)$. This does not contradict to Rmk. I.10.1.2 since x_0 is a limit point of Y implies x_0 is also a limit point of X. □

Ex. I.10.1.2. Prove Prop. I.10.1.7.

Proof. See Prop. I.10.1.7. □

Ex. I.10.1.3. Prove Prop. I.10.1.10.

Proof. See Prop. I.10.1.10. □

Ex. I.10.1.4. Prove Thm. I.10.1.13.

Proof. See Thm. I.10.1.13. □

Ex. I.10.1.5. Let n be a natural number, and let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) := x^n$. Show that f is differentiable on $\mathbb{R}$ and $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$ with the convention that $nx^{n-1} = 0$ when $n = 0$.

Proof. We induct on n to show that $\forall n \in \mathbb{N}$, $f_n(x) = x^n$ is differentiable on $\mathbb{R}$ and $f_n'(x) = nx^{n-1}$. For $n = 0$, we have $f_0(x) = x^0 = 1$. By Thm. I.10.1.13(a) we know that f_0 is differentiable on $\mathbb{R}$ and $f_0'(x) = 0$ for every $x \in X$. Thus, (by convention) the base case holds. Suppose inductively that for some $n \geq 0$ we have $f_n(x) = x^n$ is differentiable on $\mathbb{R}$ and $f_n'(x) = nx^{n-1}$. Then for $n+1$, we have $f_{n+1}(x) = x^{n+1} = x^n \cdot x^1 = f_n(x)f_1(x) = (f_n \cdot f_1)(x)$ and

$$\begin{aligned}
&(f_n \cdot f_1)'(x) \\
=&f_n'(x)f_1(x) + f_n(x)f_1'(x) && \text{(by Thm. I.10.1.13(d))} \\
=&(nx^{n-1})(x^1) + f_n(x)f_1'(x) && \text{(by the induction hypothesis)} \\
=&(nx^{n-1})(x^1) + (x^n)(1x^0) && \text{(by Thm. I.10.1.13(b))} \\
=&nx^n + x^n \\
=&(n+1)x^n.
\end{aligned}$$

This closes the induction. □

Ex. I.10.1.6. Let n be a *negative* integer, and let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the function $f(x) \coloneqq x^n$. Show that f is differentiable on $\mathbb{R} \setminus \{0\}$ and $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R} \setminus \{0\}$.

Proof. Let $x \in \mathbb{R} \setminus \{0\}$. Since $n \in \mathbb{Z}^-$, $-n \in \mathbb{Z}^+$. Then we have $(1/f)(x) = 1/x^n = x^{-n}$ and $f(x) = \big(1/(1/f)\big)(x)$. Thus, f is differentiable at x and

$$\begin{aligned}
f'(x) &= \big(1/(1/f)\big)'(x) \\
&= -\frac{(1/f)'(x)}{\big((1/f)(x)\big)^2} && \text{(by Thm. I.10.1.13(g))} \\
&= -\frac{(-n)x^{-n-1}}{x^{-2n}} && \text{(by Ex. I.10.1.5)} \\
&= nx^{n-1}.
\end{aligned}$$

□

Ex. I.10.1.7. Prove Thm. I.10.1.15.

Proof. See Thm. I.10.1.15. □

I.10.2 Local maxima, local minima, and derivatives

Def. I.10.2.1 (Local maxima and minima)**.** Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function, and let $x_0 \in X$. We say that f attains a *local maximum* at x_0 iff there exists a $\delta > 0$ such that the restriction $f|_{X \cap (x_0 - \delta, x_0 + \delta)}$ of f to $X \cap (x_0 - \delta, x_0 + \delta)$ attains a maximum at x_0. We say that f attains a *local minimum* at x_0 iff there exists a $\delta > 0$ such that the restriction $f|_{X \cap (x_0 - \delta, x_0 + \delta)}$ of f to $X \cap (x_0 - \delta, x_0 + \delta)$ attains a minimum at x_0.

Rmk. I.10.2.2. If f attains a maximum at x_0, we sometimes say that f attains a *global* maximum at x_0, in order to distinguish it from the local maxima defined in Def. I.10.2.1. Note that if f attains a global maximum at x_0, then it certainly also attains a local maximum at this x_0, and similarly for minima.

Rmk. I.10.2.5. If $f : X \to \mathbb{R}$ attains a local maximum at a point x_0 in X, and $Y \subseteq X$ is a subset of X which contains x_0, then the restriction $f|_Y : Y \to \mathbb{R}$ also attains a local maximum at x_0. Similarly, for minima.

Prop. I.10.2.6 (Local extrema are stationary)**.** Let $a < b$ be real numbers, and let $f : (a, b) \to \mathbb{R}$ be a function. If $x_0 \in (a, b)$, f is differentiable at x_0, and f attains either a local maximum or local minimum at x_0, then $f'(x_0) = 0$.

Proof. Suppose f attains local maximum at x_0. Then by Def. I.10.2.1 we know that

$$\exists \delta \in \mathbb{R}^+ : \forall x \in (a, b) \cap (x_0 - \delta, x_0 + \delta), f(x) \leq f(x_0).$$

Since f is differentiable at x_0, by Def. I.10.1.1 we know that

$$\lim_{x \to x_0; x \in (a,b) \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

By Def. I.9.3.6 we know that $\forall \varepsilon \in \mathbb{R}^+$, $\exists \delta' \in \mathbb{R}^+$ such that

$$\forall x \in (a, b) \setminus \{x_0\}, |x - x_0| < \delta' \implies \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \varepsilon.$$

In particular, we have

$$\forall x \in (a, b) \cap (x_0, x_0 + \delta), |x - x_0| < \delta' \implies \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \varepsilon$$

and

$$\forall x \in (a, b) \cap (x_0 - \delta, x_0), |x - x_0| < \delta' \implies \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| \leq \varepsilon.$$

Thus, by Def. I.9.3.6 we must have

$$\lim_{x \to x_0; x \in (a,b) \cap (x_0, x_0 + \delta)} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0; x \in (a,b) \cap (x_0 - \delta, x_0)} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

Since

$$\begin{aligned} &\forall x \in (a, b) \cap (x_0, x_0 + \delta) \\ \implies &f(x) \leq f(x_0) \\ \implies &f(x) - f(x_0) \leq 0 \end{aligned}$$

$$
\begin{aligned}
&\implies \frac{f(x)-f(x_0)}{x-x_0} \leq 0 \\
&\implies \lim_{x\to x_0; x\in(a,b)\cap(x_0,x_0+\delta)} \frac{f(x)-f(x_0)}{x-x_0} \leq 0 && \text{(by Prop. I.9.3.14)} \\
&\implies f'(x_0) \leq 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\quad \forall x \in (a,b)\cap(x_0-\delta,x_0) \\
&\implies f(x) \leq f(x_0) \\
&\implies f(x)-f(x_0) \leq 0 \\
&\implies \frac{f(x)-f(x_0)}{x-x_0} \geq 0 \\
&\implies \lim_{x\to x_0; x\in(a,b)\cap(x_0-\delta,x_0)} \frac{f(x)-f(x_0)}{x-x_0} \geq 0, && \text{(by Prop. I.9.3.14)} \\
&\implies f'(x_0) \geq 0,
\end{aligned}
$$

we must have $f'(x_0)=0$. Similar arguments work for the case f attains local minimum at x_0. □

Note. f must be differentiable for Prop. I.10.2.6 to work. Also, Prop. I.10.2.6 does not work if the open interval (a,b) is replaced by a closed interval $[a,b]$. For instance, the function $f:[1,2]\to\mathbb{R}$ defined by $f(x):=x$ has a local maximum at $x_0=2$ and a local minimum $x_0=1$ (in fact, these local extrema are global extrema), but at both points the derivative is $f'(x_0)=1$, not $f'(x_0)=0$. Thus, the endpoints of an interval can be local maxima or minima even if the derivative is not zero there. Finally, the converse of this proposition is false.

Thm. I.10.2.7 (Rolle's theorem)**.** Let $a<b$ be real numbers, and let $g:[a,b]\to\mathbb{R}$ be a continuous function which is differentiable on (a,b). Suppose also that $g(a)=g(b)$. Then there exists an $x\in(a,b)$ such that $g'(x)=0$.

Proof. Since g is continuous on $[a,b]$, by Prop. I.9.6.7 g attains its maximum at some point $x_{\max}\in[a,b]$, and also attains its minimum at some point $x_{\min}\in[a,b]$. If $(x_{\min}\in\{a,b\})\wedge(x_{\max}\in\{a,b\})$ is true, then by Def. I.9.6.5 we have $g(x)=g(a)=g(b)$ for every $x\in[a,b]$, and by Thm. I.10.1.13(a) we know that $g'(x)=0$. So suppose that at least one of $x_{\min},x_{\max}\notin\{a,b\}$, i.e., $\big(x_{\min}\in(a,b)\big)\vee\big(x_{\max}\in(a,b)\big)$ is true. If $x_{\min}\in(a,b)$, then by Prop. I.10.2.6 we know that $f'(x_{\min})=0$. Similarly, if $x_{\max}\in(a,b)$, then by Prop. I.10.2.6 we know that $f'(x_{\max})=0$. Thus, there exists an $x\in(a,b)$ such that $g'(x)=0$. □

Rmk. I.10.2.8. We only assume f is differentiable on the open interval (a,b), though of course Thm. I.10.2.7 also holds if we assume f is differentiable on the closed interval $[a,b]$, since this is larger than (a,b).

Cor. I.10.2.9 (Mean value theorem)**.** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b). Then there exists an $x \in (a, b)$ such that $f'(x) = \dfrac{f(b) - f(a)}{b - a}$.

Proof. Let $g : [a, b] \to \mathbb{R}$ be a function where $g(x) = f(x) - \dfrac{f(a) - f(b)}{a - b}x$. Since $a < b$, we know that g is well-defined. Since f is differentiable on (a, b), we know that

$$\begin{aligned}
& x \text{ is differentiable on } (a,b) && \text{(by Thm. I.10.1.13(b))}\\
\Longrightarrow\ & \frac{f(a)-f(b)}{a-b}x \text{ is differentiable on } (a,b) && \text{(by Thm. I.10.1.13(e))}\\
\Longrightarrow\ & f(x) - \frac{f(a)-f(b)}{a-b}x \text{ is differentiable on } (a,b) && \text{(by Thm. I.10.1.13(f))}\\
\Longrightarrow\ & g(x) \text{ is differentiable on } (a,b)\\
\Longrightarrow\ & g'(x) = f'(x) - \frac{f(a)-f(b)}{a-b}.
\end{aligned}$$

Since

$$g(a) = f(a) - \frac{f(a)-f(b)}{a-b}a = \frac{af(a) - bf(a) - af(a) + af(b)}{a-b} = \frac{af(b) - bf(a)}{a-b}$$

and

$$g(b) = f(b) - \frac{f(a)-f(b)}{a-b}b = \frac{af(b) - bf(b) - bf(a) + bf(b)}{a-b} = \frac{af(b) - bf(a)}{a-b},$$

we have $g(a) = g(b)$ and by Thm. I.10.2.7 $\exists x_0 \in (a, b)$ such that $g'(x_0) = 0$. Thus

$$\begin{aligned}
& g'(x_0) = 0\\
\Longrightarrow\ & f'(x_0) - \frac{f(a)-f(b)}{a-b} = 0\\
\Longrightarrow\ & f'(x_0) = \frac{f(a)-f(b)}{a-b}.
\end{aligned}$$

□

— Exercises —

Ex. I.10.2.1. Prove Prop. I.10.2.6.

Proof. See Prop. I.10.2.6. □

Ex. I.10.2.2. Give an example of a function $f : (-1, 1) \to \mathbb{R}$ which is continuous and attains a global maximum at 0, but which is not differentiable at 0. Explain why this does not contradict Prop. I.10.2.6.

Proof. Let $f(x) = -|x|$. Then f is continuous and attains a global maximum at 0, but which is not differentiable at 0. The fact that f is not differentiable at 0 does not contradict to Prop. I.10.2.6. □

Ex. I.10.2.3. Give an example of a function $f : (-1, 1) \to \mathbb{R}$ which is differentiable, and whose derivative equals 0 at 0, but such that 0 is neither a local minimum nor a local maximum. Explain why this does not contradict Prop. I.10.2.6.

Proof. Let $f(x) = x^3$. Then by Ex. I.10.1.5 we know that $f'(x) = 3x^2$. Then we have $f(0) = 0$ and $f'(0) = 0$. But $f(x) < f(0)$ for every $x \in (-1, 0)$ and $f(x) > f(0)$ for every $x \in (0, 1)$. Thus, 0 is neither a local minimum nor a local maximum. This does not contradict to Prop. I.10.2.6 since 0 is not given to be a local minimum or local maximum. □

Ex. I.10.2.4. Prove Thm. I.10.2.7.

Proof. See Thm. I.10.2.7. □

Ex. I.10.2.5. Use Thm. I.10.2.7 to prove Cor. I.10.2.9.

Proof. See Cor. I.10.2.9. □

Ex. I.10.2.6. Let $M > 0$, and let $f : [a, b] \to \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b), and such that $\left|f'(x)\right| \leq M$ for all $x \in (a, b)$ (i.e., the derivative of f is bounded). Show that for any $x, y \in [a, b]$ we have the inequality $|f(x) - f(y)| \leq M|x - y|$. Functions which obey the bound $|f(x) - f(y)| \leq M|x - y|$ are known as *Lipschitz continuous functions* with *Lipschitz constant* M; thus this exercise shows that functions with bounded derivative are Lipschitz continuous.

Proof. Let $x, y \in [a, b]$. If $x = y$, then we have $0 = |f(x) - f(y)| \leq M|x - y| = 0$. So suppose that $x \neq y$. We have either $x < y$ or $x > y$. Without the loss of generality suppose that $x < y$. Then we have $[x, y] \subseteq [a, b]$ and $(x, y) \subseteq (a, b)$. By Ex. I.9.4.6 we know that $f|_{[x,y]}$ is continuous on $[x, y]$. By Ex. I.10.1.1 we know that $f|_{[x,y]}$ is differentiable on (x, y). By mean value theorem (Cor. I.10.2.9) we know that $\exists c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}.$$

Since $c \in (x, y)$, we have $c \in (a, b)$. By hypothesis we have

$$\begin{aligned} \left|f'(c)\right| &= \left|\frac{f(y) - f(x)}{y - x}\right| \leq M \\ \implies &|f(y) - f(x)| \leq M|y - x| \\ \implies &|f(x) - f(y)| \leq M|x - y|. \end{aligned}$$

Thus, we conclude that $\forall x, y \in [a, b]$, we have $|f(x) - f(y)| \leq M|x - y|$. □

Ex. I.10.2.7. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is bounded. Show that f is uniformly continuous.

Proof. Since f' is bounded, by Def. I.9.6.1 we know that $\exists M \in \mathbb{R}^+$ such that $\left|f'(x)\right| \leq M$ for every $x \in \mathbb{R}$. By Ex. I.10.2.6 we know that $|f(x) - f(y)| \leq M|x - y|$ for every $x, y \in \mathbb{R}$. Then we have

$$\begin{gathered}\forall \varepsilon \in \mathbb{R}^+, \exists \delta = \varepsilon/M : \forall x, y \in \mathbb{R}, |x - y| \leq \delta \\ \implies |f(x) - f(y)| \leq M|x - y| \leq M\delta = M\frac{\varepsilon}{M} = \varepsilon\end{gathered}$$

and by Def. I.9.9.2 f is uniformly continuous. ☐

I.10.3 Monotone functions and derivatives

Prop. I.10.3.1. Let X be a subset of $\mathbb{R}$, let $x_0 \in X$ be a limit point of X, and let $f : X \to \mathbb{R}$ be a function. If f is monotone increasing and f is differentiable at x_0, then $f'(x_0) \geq 0$. If f is monotone decreasing and f is differentiable at x_0, then $f'(x_0) \leq 0$.

Proof. First, suppose that f is monotone increasing. Since

$$\begin{aligned}&\forall x \in X \setminus \{x_0\}, x \neq x_0 \\ \implies &\begin{cases} x < x_0 \\ x > x_0 \end{cases} \\ \implies &\begin{cases} (x - x_0 < 0) \wedge \big(f(x) - f(x_0) \leq 0\big) \\ (x - x_0 > 0) \wedge \big(f(x) - f(x_0) \geq 0\big) \end{cases} && \text{(by Def. I.9.8.1)} \\ \implies &\frac{f(x) - f(x_0)}{x - x_0} \geq 0,\end{aligned}$$

by Prop. I.9.3.14 we have $f'(x_0) \geq 0$.

Now suppose that f is monotone decreasing. Since

$$\begin{aligned}&\forall x \in X \setminus \{x_0\}, x \neq x_0 \\ \implies &\begin{cases} x < x_0 \\ x > x_0 \end{cases} \\ \implies &\begin{cases} (x - x_0 < 0) \wedge \big(f(x) - f(x_0) \geq 0\big) \\ (x - x_0 > 0) \wedge \big(f(x) - f(x_0) \leq 0\big) \end{cases} && \text{(by Def. I.9.8.1)} \\ \implies &\frac{f(x) - f(x_0)}{x - x_0} \leq 0,\end{aligned}$$

by Prop. I.9.3.14 we have $f'(x_0) \leq 0$. ☐

Rmk. I.10.3.2. We have to assume that f is differentiable at x_0; There exist monotone functions which are not always differentiable, and of course if f is not differentiable at x_0 we cannot possibly conclude that $f'(x_0) \geq 0$ or $f'(x_0) \leq 0$.

Note. One might naively guess that if f were strictly monotone increasing, and f was differentiable at x_0, then the derivative $f'(x_0)$ would be strictly positive instead of merely non-negative. Unfortunately, this is not always the case.

Prop. I.10.3.3. Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If $f'(x) > 0$ for all $x \in [a, b]$, then f is strictly monotone increasing. If $f'(x) < 0$ for all $x \in [a, b]$, then f is strictly monotone decreasing. If $f'(x) = 0$ for all $x \in [a, b]$, then f is a constant function.

Proof. We first show that if $f'(x) > 0$ for all $x \in [a, b]$, then f is strictly monotone increasing. Let $x_1, x_2 \in [a, b]$ and $x_1 < x_2$. Since f is differentiable on $[a, b]$, by Ex. I.10.1.1 we know that f is differentiable on (x_1, x_2), and by Cor. I.10.1.12 f is continuous on $[x_1, x_2]$. By mean value theorem (Cor. I.10.2.9) $\exists c \in (x_1, x_2)$ such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$. Since $c \in (x_1, x_2)$, we have $c \in [a, b]$. Now we split into three cases:

- If $f'(x) > 0$ for all $x \in [a, b]$, then we have

$$\begin{aligned} & f'(c) > 0 \\ \Longrightarrow\ & \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \\ \Longrightarrow\ & f(x_2) - f(x_1) > 0 && (x_2 > x_1) \\ \Longrightarrow\ & f(x_2) > f(x_1). \end{aligned}$$

  Since x_1, x_2 was arbitrary, by Def. I.9.8.1 we conclude that f is strictly monotone increasing.

- If $f'(x) < 0$ for all $x \in [a, b]$, then we have

$$\begin{aligned} & f'(c) < 0 \\ \Longrightarrow\ & \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0 \\ \Longrightarrow\ & f(x_2) - f(x_1) < 0 && (x_2 > x_1) \\ \Longrightarrow\ & f(x_2) < f(x_1). \end{aligned}$$

  Since x_1, x_2 was arbitrary, by Def. I.9.8.1 we conclude that f is strictly monotone decreasing.

- If $f'(x) = 0$ for all $x \in [a, b]$, then we have

$$f'(c) = 0$$

$$\begin{aligned}
&\Longrightarrow \frac{f(x_2)-f(x_1)}{x_2-x_1}=0 \\
&\Longrightarrow f(x_2)-f(x_1)=0 && (x_2>x_1) \\
&\Longrightarrow f(x_2)=f(x_1).
\end{aligned}$$

Since x_1, x_2 was arbitrary, we conclude that f is a constant function.

□

— Exercises —

Ex. I.10.3.1. Prove Prop. I.10.3.1.

Proof. See Prop. I.10.3.1. □

Ex. I.10.3.2. Give an example of a function $f:(-1,1)\to\mathbb{R}$ which is continuous and monotone increasing, but which is not differentiable at 0. Explain why this does not contradict Prop. I.10.3.1.

Proof. Define f as follow

$$\forall x\in(-1,1), f(x)=\begin{cases} x & \text{if } x\in(-1,0), \\ 2x & \text{if } x\in[0,1). \end{cases}$$

Then f is monotone increasing, $f(0+)\geq 0$ and $f(0-)<0$. Since $f(0+)\neq f(0-)$, by A.Cor. I.9.5.1 f is not continuous at 0, and by Prop. I.10.1.10 f is not differentiable at 0. This does not contradict to Prop. I.10.3.1 since 0 is not given to be differentiable. □

Ex. I.10.3.3. Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ which is strictly monotone increasing and differentiable, but whose derivative at 0 is zero. Explain why this does not contradict Prop. I.10.3.1 or Prop. I.10.3.3.

Proof. Let $f(x)=x^3$. By Ex. I.10.1.5 f is differentiable on $\mathbb{R}$ and $f'(x)=3x^2$, thus $f'(0)=0$. If $x,y\in\mathbb{R}$ and $x<y$, then $x^3<y^3$, thus by Def. I.9.8.1 f is strictly monotone increasing. This does not contradict to Prop. I.10.3.1 since $f'(0)=0\geq 0$. This does not contradict to Prop. I.10.3.3 since $\forall x\in\mathbb{R}$, $3x^2\geq 0$. □

Ex. I.10.3.4. Prove Prop. I.10.3.3.

Proof. See Prop. I.10.3.3. □

Ex. I.10.3.5. Give an example of a subset $X\subseteq\mathbb{R}$ and a function $f:X\to\mathbb{R}$ which is differentiable on X, is such that $f'(x)>0$ for all $x\in X$, but f is not strictly monotone increasing.

Proof. Let $X = [0, 0.5] \cup [1, 2]$ and let $f : X \to \mathbb{R}$ be the following function

$$\forall x \in X, f(x) = \begin{cases} 2x & \text{if } x \in [0, 0.5], \\ x & \text{if } x \in [1, 2]. \end{cases}$$

By Ex. I.10.1.5 we know that x and $2x$ are differentiable and

$$\forall x \in X, f'(x) = \begin{cases} 2 & \text{if } x \in [0, 0.5], \\ 1 & \text{if } x \in [1, 2]. \end{cases}$$

So we have $f'(x) > 0$ for every $x \in X$. Since $0.5 < 1$ and $f(0.5) = f(1) = 1$, by Def. I.9.8.1 f is not strictly monotone increasing. □

I.10.4 Inverse functions and derivatives

Lem. I.10.4.1. Let $f : X \to Y$ be an invertible function, with inverse $f^{-1} : Y \to X$. Suppose that $x_0 \in X$ and $y_0 \in Y$ are such that $y_0 = f(x_0)$ (which also implies that $x_0 = f^{-1}(y_0)$). If f is differentiable at x_0, and f^{-1} is differentiable at y_0, then

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. From the chain rule (Thm. I.10.1.15) we have

$$(f^{-1} \circ f)'(x_0) = (f^{-1})'(y_0) f'(x_0).$$

But $f^{-1} \circ f$ is the identity function on X, and hence by Thm. I.10.1.13(b) $(f^{-1} \circ f)'(x_0) = 1$. The claim follows. □

Note. As a particular corollary of Lem. I.10.4.1, we see that if f is differentiable at x_0 with $f'(x_0) = 0$, then f^{-1} cannot be differentiable at $y_0 = f(x_0)$, since $1/f'(x_0)$ is undefined in that case.

Note. If one writes $y = f(x)$, so that $x = f^{-1}(y)$, then one can write the conclusion of Lem. I.10.4.1 in the more appealing form $dx/dy = 1/(dy/dx)$. However, as mentioned before, this way of writing things, while very convenient and easy to remember, can be misleading and cause errors if applied too carelessly (especially when one begins to work in the calculus of several variables).

Note. Lem. I.10.4.1 seems to answer the question of how to differentiate the inverse of a function, however it has one significant drawback: the lemma only works if one assumes a *priori* that f^{-1} is differentiable. Thus, if one does not already know that f^{-1} is differentiable, one cannot use Lem. I.10.4.1 to compute the derivative of f^{-1}.

Thm. I.10.4.2 (Inverse function theorem)**.** Let $f : X \to Y$ be an invertible function, with inverse $f^{-1} : Y \to X$. Suppose that $x_0 \in X$ and $y_0 \in Y$ are such that $f(x_0) = y_0$. If f is differentiable at x_0, f^{-1} is continuous at y_0, and $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0 and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. We have to show that

$$\lim_{y \to y_0; y \in Y \setminus \{y_0\}} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$

By Prop. I.9.3.9, it suffices to show that

$$\lim_{n \to \infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{1}{f'(x_0)}$$

for any sequence $(y_n)_{n=1}^{\infty}$ of elements in $Y \setminus \{y_0\}$ which converge to y_0.

To prove this, we set $x_n := f^{-1}(y_n)$. Then $(x_n)_{n=1}^{\infty}$ is a sequence of elements in $X \setminus \{x_0\}$. (Note that f^{-1} is a bijection) Since f^{-1} is continuous by assumption, we know that $x_n = f^{-1}(y_n)$ converges to $f^{-1}(y_0) = x_0$ as $n \to \infty$. Thus, since f is differentiable at x_0, we have (by Prop. I.9.3.9 again)

$$\lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} = f'(x_0).$$

But since $x_n \neq x_0$ and f is a bijection, the fraction $\dfrac{f(x_n) - f(x_0)}{x_n - x_0}$ is non-zero. Also, by hypothesis $f(x_0)$ is non-zero. So by limit laws

$$\lim_{n \to \infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{f'(x_0)}.$$

But since $x_n = f^{-1}(y_n)$ and $x_0 = f^{-1}(y_0)$, we thus have

$$\lim_{n \to \infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{1}{f'(x_0)}.$$

as desired. □

— Exercises —

Ex. I.10.4.1. Let $n \geq 1$ be a natural number, and let $g : (0, \infty) \to (0, \infty)$ be the function $g(x) := x^{1/n}$.

(a) Show that g is continuous on $(0, \infty)$.

(b) Show that g is differentiable on $(0, \infty)$, and that $g'(x) = \dfrac{1}{n} x^{\frac{1}{n} - 1}$ for all $x \in (0, \infty)$.

Proof. We first show that g is continuous on $(0,\infty)$. Let $f:(0,\infty)\to(0,\infty)$ be a function where $f(x)=x^n$. Then $g\circ f:(0,\infty)\to(0,\infty)=x$ and thus $g=f^{-1}$. By Ex. I.10.1.5 we know that x^n is differentiable on $(0,\infty)$, thus by Cor. I.10.1.12 f is continuous on $(0,\infty)$. By E.g. I.9.8.4 f is strictly monotone increasing. Then by Prop. I.9.8.3 we know that g is also continuous and strictly monotone increasing.

Now we show that g is differentiable on $(0,\infty)$, and that $g'(x)=\frac{1}{n}x^{\frac{1}{n}-1}$ for all $x\in(0,\infty)$. Since g is continuous, $g=f^{-1}$, and $f(x)\neq 0$ for every $x\in(0,\infty)$, we have

$$
\begin{aligned}
& g(x)=f(x^{1/n}) \\
\implies & g'(x)=\frac{1}{f'(x^{1/n})} && \text{(by Thm. I.10.4.2)} \\
\implies & g'(x)=\frac{1}{n(x^{1/n})^{n-1}}=\frac{1}{n}x^{\frac{1}{n}-1}. && \text{(by Ex. I.10.1.5)}
\end{aligned}
$$

□

Ex. I.10.4.2. Let q be a rational number, and let $f:(0,\infty)\to\mathbb{R}$ be the function $f(x)=x^q$.

(a) Show that f is differentiable on $(0,\infty)$ and that $f'(x)=qx^{q-1}$.

(b) Show that $\displaystyle\lim_{x\to 1;x\in(0,\infty)\setminus\{1\}}\frac{x^q-1}{x-1}=q$ for every rational number q.

Proof. We first show that f is differentiable on $(0,\infty)$ and that $f'(x)=qx^{q-1}$. Let $q=a/b$ where $a,b\in\mathbb{Z}$ and $b>0$. Then we have

$$x^q=x^{a/b}=(x^a)^{1/b}.$$

By Ex. I.10.1.5 and Ex. I.10.1.6 we know that x^a is differentiable on $(0,\infty)$. By Ex. I.10.4.1(b) we know that $x^{1/b}$ is differentiable on $(0,\infty)$. Thus, by chain rule (Thm. I.10.1.15) we know that $x^q=x^{a/b}$ is differentiable and

$$
\begin{aligned}
(x^q)' &= (x^{a/b})' \\
&= \Big(\frac{1}{b}(x^a)^{\frac{1}{b}-1}\Big)(ax^{a-1}) && \text{(by Thm. I.10.1.15)} \\
&= \frac{a}{b}x^{\frac{a}{b}-1} \\
&= qx^{q-1}.
\end{aligned}
$$

Now we show that $\displaystyle\lim_{x\to 1;x\in(0,\infty)\setminus\{1\}}\frac{x^q-1}{x-1}=q$ for every $q\in\mathbb{Q}$. Since x^q is differentiable on $(0,\infty)$, we know that x^q is differentiable at 1. Thus, by Def. I.10.1.1 we have

$$\lim_{x\to 1;x\in(0,\infty)\setminus\{1\}}\frac{x^q-1}{x-1}=f'(1)=q\cdot 1^{q-1}=q.$$

$\square$

Ex. I.10.4.3. Let α be a real number, and let $f : (0, \infty) \to \mathbb{R}$ be the function $f(x) = x^\alpha$.

(a) Show that $\lim\limits_{x \to 1; x \in (0,\infty)\setminus\{1\}} \dfrac{f(x) - f(1)}{x - 1} = \alpha$.

(b) Show that f is differentiable on $(0, \infty)$ and that $f'(x) = \alpha x^{\alpha - 1}$.

Proof. We first show that $\lim\limits_{x \to 1; x \in (0,\infty)\setminus\{1\}} \dfrac{f(x) - f(1)}{x - 1} = \alpha$. By Def. I.6.7.2 we know that $1^\alpha = 1$. Then we have

$$
\begin{aligned}
& \forall \varepsilon \in \mathbb{R}^+, \alpha - \varepsilon < \alpha < \alpha + \varepsilon \\
\implies & \exists q_1, q_2 \in \mathbb{Q} : \alpha - \varepsilon < q_1 < \alpha < q_2 < \alpha + \varepsilon && \text{(by Prop. I.5.4.14)} \\
\implies & \forall x \in (0, \infty) \setminus \{1\}, \\
& \frac{x^{q_1} - 1}{x - 1} < \frac{x^\alpha - 1}{x - 1} < \frac{x^{q_2} - 1}{x - 1} && \text{(by Prop. I.6.7.3)} \\
\implies & q_1 = \lim_{x \to 1; x \in (0,\infty)\setminus\{1\}} \frac{x^{q_1} - 1}{x - 1} && \text{(by Ex. I.10.4.2(b))} \\
& \leq \lim_{x \to 1; x \in (0,\infty)\setminus\{1\}} \frac{x^\alpha - 1}{x - 1} && \text{(by Prop. I.9.3.14)} \\
& \leq \lim_{x \to 1; x \in (0,\infty)\setminus\{1\}} \frac{x^{q_2} - 1}{x - 1} = q_2 && \text{(by Ex. I.10.4.2(b))} \\
\implies & \alpha - \varepsilon < q_1 \leq \lim_{x \to 1; x \in (0,\infty)\setminus\{1\}} \frac{x^\alpha - 1}{x - 1} \leq q_2 < \alpha + \varepsilon \\
\implies & \left| \lim_{x \to 1; x \in (0,\infty)\setminus\{1\}} \frac{x^\alpha - 1}{x - 1} - \alpha \right| \leq \varepsilon \\
\implies & \lim_{x \to 1; x \in (0,\infty)\setminus\{1\}} \frac{x^\alpha - 1}{x - 1} = \alpha. && \text{(Since } \varepsilon \text{ was arbitrary)}
\end{aligned}
$$

Now we show that f is differentiable on $(0, \infty)$ and that $f'(x) = \alpha x^{\alpha-1}$. From the proof above we know that f is differentiable at 1. Let $x_0 \in (0, \infty)$. By Thm. I.10.1.13(b)(e) we know that $g(x) = x/x_0$ is differentiable at x_0. Then by chain rule (Thm. I.10.1.15) we know that $f \circ g$ is differentiable at x_0. Thus, we have

$$
\begin{aligned}
(f \circ g)'(x_0) &= f'(1) g'(x_0) && \text{(by Thm. I.10.1.15)} \\
&= \frac{\alpha}{x_0} && \text{(by Thm. I.10.1.13(b)(e))} \\
&= \lim_{x \to x_0; x \in (0,\infty)\setminus\{x_0\}} \frac{(f \circ g)(x) - (f \circ g)(x_0)}{x - x_0} && \text{(by Def. I.10.1.1)} \\
&= \lim_{x \to x_0; x \in (0,\infty)\setminus\{x_0\}} \frac{(x/x_0)^\alpha - (x_0/x_0)^\alpha}{x - x_0}
\end{aligned}
$$

$$= \frac{1}{x_0^\alpha}\left(\lim_{x\to x_0; x\in(0,\infty)\setminus\{x_0\}} \frac{x^\alpha - x_0^\alpha}{x - x_0}\right) \qquad \text{(by Thm. I.10.1.13(e))}$$

and

$$\lim_{x\to x_0; x\in(0,\infty)\setminus\{x_0\}} \frac{x^\alpha - x_0^\alpha}{x - x_0} = \alpha x_0^{\alpha-1}.$$

By Def. I.10.1.1 f is differentiable at x_0. Since x_0 was arbitrary, we know that f is differentiable on $(0, \infty)$. □

I.10.5 L'Hôpital's rule

Prop. I.10.5.1 (L'Hôpital's rule I)**.** Let X be a subset of $\mathbb{R}$, let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions, and let $x_0 \in X$ be a limit point of X. Suppose that $f(x_0) = g(x_0) = 0$, that f and g are both differentiable at x_0, but $g'(x_0) \neq 0$. Then there exists a $\delta > 0$ such that $g(x) \neq 0$ for all $x \in (X \cap (x_0 - \delta, x_0 + \delta)) \setminus \{x_0\}$, and

$$\lim_{x\to x_0; x\in(X\cap(x_0-\delta,x_0+\delta))\setminus\{x_0\}} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$

Proof. Since g is differentiable at x_0, by Newton's approximation (Prop. I.10.1.7) we have $\forall \varepsilon \in \mathbb{R}^+$, $\exists \delta \in \mathbb{R}^+$ such that

$$\begin{aligned}
&\forall x \in X, |x - x_0| \le \frac{\delta}{2} < \delta \\
\Longrightarrow\ & \big|g(x) - g(x_0) - g'(x_0)(x - x_0)\big| \le \varepsilon|x - x_0| \\
\Longrightarrow\ & \big|g(x) - g'(x_0)(x - x_0)\big| \le \varepsilon|x - x_0| && \big(g(x_0) = 0\big) \\
\Longrightarrow\ & \big|g'(x_0)(x - x_0) - g(x)\big| \le \varepsilon|x - x_0| \\
\Longrightarrow\ & \big|g'(x_0)(x - x_0)\big| \\
&\quad \le \big|g'(x_0)(x - x_0) - g(x)\big| + |g(x)| \\
&\quad \le \varepsilon|x - x_0| + |g(x)| \\
\Longrightarrow\ & \big|g'(x_0)(x - x_0)\big| - \varepsilon|x - x_0| \le |g(x)| \\
\Longrightarrow\ & \big(\big|g'(x_0)\big| - \varepsilon\big)|x - x_0| \le |g(x)|.
\end{aligned}$$

Since $g'(x_0) \neq 0$, we know $\dfrac{|g'(x_0)|}{2} > 0$. By setting $\varepsilon = \dfrac{|g'(x_0)|}{2}$ we know $\exists \delta \in \mathbb{R}^+$ such that

$$\begin{aligned}
&\forall x \in X, |x - x_0| < \delta \\
\Longrightarrow\ & \left(\big|g'(x_0)\big| - \frac{|g'(x_0)|}{2}\right)|x - x_0| \le |g(x)| \\
\Longrightarrow\ & 0 \le \frac{|g'(x_0)|}{2}|x - x_0| \le |g(x)|.
\end{aligned}$$

Observe that

$$(x \in X) \wedge (|x - x_0| < \delta) \iff x \in X \cap (x_0 - \delta, x_0 + \delta).$$

Then we have

$$\exists \delta \in \mathbb{R}^+ : \forall x \in X \cap (x_0 - \delta, x_0 + \delta), 0 \leq \frac{|g'(x_0)|}{2}|x - x_0| \leq |g(x)|.$$

In particular, we have

$$\exists \delta \in \mathbb{R}^+ : \forall x \in \big(X \cap (x_0 - \delta, x_0 + \delta)\big) \setminus \{x_0\}, 0 < \frac{|g'(x_0)|}{2}|x - x_0| \leq |g(x)|.$$

This means $g(x) \neq 0$ for every $x \in \big(X \cap (x_0 - \delta, x_0 + \delta)\big) \setminus \{x_0\}$. Thus

$$\begin{aligned}
&\lim_{x \to x_0; x \in \left(X \cap (x_0 - \delta, x_0 + \delta)\right) \setminus \{x_0\}} \frac{f(x)}{g(x)} \\
=&\lim_{x \to x_0; x \in \left(X \cap (x_0 - \delta, x_0 + \delta)\right) \setminus \{x_0\}} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} && (f(x_0) = g(x_0) = 0) \\
=&\lim_{x \to x_0; x \in \left(X \cap (x_0 - \delta, x_0 + \delta)\right) \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} \frac{x - x_0}{g(x) - g(x_0)} && (x \neq x_0) \\
=&\frac{\lim_{x \to x_0; x \in \left(X \cap (x_0 - \delta, x_0 + \delta)\right) \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}}{\lim_{x \to x_0; x \in \left(X \cap (x_0 - \delta, x_0 + \delta)\right) \setminus \{x_0\}} \frac{g(x) - g(x_0)}{x - x_0}} && (\text{by Prop. I.9.3.1}) \\
=&\frac{f'(x_0)}{g'(x_0)}. && (\text{by Def. I.10.1.1})
\end{aligned}$$

□

Note. The presence of the δ here may seem somewhat strange, but is needed because $g(x)$ might vanish at some points other than x_0, which would imply that quotient $\dfrac{f(x)}{g(x)}$ is not necessarily defined at all points in $X \setminus \{x_0\}$.

Prop. I.10.5.2 (L'Hôpital's rule II)**.** Let $a < b$ be real numbers, let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be functions which are differentiable on $[a, b]$. Suppose that $f(a) = g(a) = 0$, that g' is non-zero on $[a, b]$ (i.e., $g'(x) \neq 0$ for all $x \in [a, b]$), and $\displaystyle\lim_{x \to a; x \in (a,b]} \frac{f'(x)}{g'(x)}$ exists and equals to L. Then $g(x) \neq 0$ for all $x \in (a, b]$, and $\displaystyle\lim_{x \to a; x \in (a,b]} \frac{f(x)}{g(x)}$ exists and equals to L.

Proof. We first show that $g(x) \neq 0$ for all $x \in (a, b]$. Suppose for the sake of contradiction that $g(x) = 0$ for some $x \in (a, b]$. But since $g(a)$ is also zero, we can apply Rolle's theorem

(Thm. I.10.2.7) to obtain $g'(y) = 0$ for some $a < y < x$, but this contradicts the hypothesis that g' is non-zero on $[a, b]$.

Now we show that $\lim_{x \to a; x \in (a,b]} \frac{f(x)}{g(x)} = L$. By Prop. I.9.3.9, it will suffice to show that

$$\lim_{n \to \infty} \frac{f(x_n)}{g(x_n)} = L$$

for any sequence $(x_n)_{n=0}^{\infty}$ taking values in $(a, b]$ which converges to a.

Consider a single x_n, and consider the function $h_n : [a, x_n] \to \mathbb{R}$ defined by

$$h_n(x) \coloneqq f(x)g(x_n) - g(x)f(x_n).$$

Observe that h_n is continuous on $[a, x_n]$ and equals 0 at both a and x_n, and is differentiable on (a, x_n) with derivative $h_n'(x) = f'(x)g(x_n) - g'(x)f(x_n)$. (Note that $f(x_n)$ and $g(x_n)$ are constants with respect to x.) By Rolle's theorem (Thm. I.10.2.7), we can thus find $y_n \in (a, x_n)$ such that $h_n'(y_n) = 0$, which implies that

$$\frac{f(x_n)}{g(x_n)} = \frac{f'(y_n)}{g'(y_n)}.$$

Since $y_n \in (a, x_n)$ for all n, and x_n converges to a as $n \to \infty$, we see from the squeeze test (Cor. I.6.4.14) that y_n also converges to a as $n \to \infty$. Thus, $\frac{f'(y_n)}{g'(y_n)}$ converges to L, and thus $\frac{f(x_n)}{g(x_n)}$ also converges to L, as desired. □

Note. In Prop. I.10.5.2, the hypothesis that f, g be differentiable on $[a, b]$ may be weakened to being continuous on $[a, b]$ and differentiable on $(a, b]$, with g' only assumed to be non-zero on $(a, b]$ rather than $[a, b]$.

Rmk. I.10.5.3. This proposition only considers limits to the right of a, but one can easily state and prove a similar proposition for limits to the left of a, or around both sides of a. Speaking very informally, Prop. I.10.5.2 states that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

though one has to ensure all of the conditions of the proposition hold (in particular, that $f(a) = g(a) = 0$, and that the right-hand limit exists), before one can apply L'Hôpital's rule.

— Exercises —

Ex. I.10.5.1. Prove Prop. I.10.5.1.

Proof. See Prop. I.10.5.1. □

Ex. I.10.5.2. Explain why Exercise 1.2.12 does not contradict either of the propositions in this section.

Proof. In first example we have $g(0) \neq 0$. In second example the limit

$$\lim_{x \to 0; x \in (0,\infty)} \frac{f'(x)}{g'(x)}$$

doest not exists. Thus, Exercise 1.2.12 does not contradict either of the propositions in this section. $\square$

Chapter I.11

The Riemann integral

Note. In Ch. I.10 we reviewed *differentiation* - one of the two pillars of single variable calculus. The other pillar is, of course, *integration*, which is the focus of the current chapter. More precisely, we will turn to the *definite integral*, the integral of a function on a fixed interval, as opposed to the *indefinite integral*, otherwise known as the *antiderivative*. These two are of course linked by the *Fundamental theorem of calculus*.

Note. To actually *define* this integral $\int_I f$ is somewhat delicate (especially if one does not want to assume any axioms concerning geometric notions such as area), and not all functions f are integrable. It turns out that there are at least two ways to define this integral: the *Riemann integral*, named after Georg Riemann (1826–1866), which suffices for most applications, and the *Lebesgue integral*, named after Henri Lebesgue (1875–1941), which supercedes the Riemann integral and works for a much larger class of functions. There is also the *Riemann-Stieltjes integral* $\int_I f(x)d\alpha(x)$, a generalization of the Riemann integral due to Thomas Stieltjes (1856–1894).

I.11.1 Partitions

Def. I.11.1.1. Let X be a subset of $\mathbb{R}$. We say that X is *connected* iff X is nonempty and the following property is true: whenever x, y are elements in X such that $x < y$, the bounded interval $[x, y]$ is a subset of X (i.e., every number between x and y is also in X).

Lem. I.11.1.4. Let X be a subset of the real line. Then the following two statements are logically equivalent:

(a) X is bounded and either connected or empty.

(b) X is a bounded interval.

Proof. Both statements are logically equivalent when $X = \emptyset$ (which is vacuously true). So suppose that $X \neq \emptyset$.

We first show that X is bounded and connected implies X is a bounded interval. Since X is bounded, by Thm. I.5.5.9 we know that $\inf(X), \sup(X) \in \mathbb{R}$. Thus, $X \subseteq [\inf(X), \sup(X)]$. Now we split into four cases:

- If $\sup(X) \in X$ and $\inf(X) \in X$, then by Def. I.11.1.1 X is connected implies $[\inf(X), \sup(X)] \subseteq X$. Thus, by Prop. I.3.1.18 we have $X = [\inf(X), \sup(X)]$.

- If $\sup(X) \in X$ and $\inf(X) \notin X$, then we claim that $\big(\inf(X), \sup(X)\big] \subseteq X$. This is true since X is connected and by Def. I.11.1.1 we have $\big(a, \sup(X)\big] \subseteq X$ for every $a \in X$.

- If $\sup(X) \notin X$ and $\inf(X) \in X$, then we claim that $\big[\inf(X), \sup(X)\big) \subseteq X$. This is true since X is connected and by Def. I.11.1.1 we have $\big[\inf(X), b\big) \subseteq X$ for every $b \in X$.

- If $\sup(X) \notin X$ and $\inf(X) \notin X$, then we claim that $\big(\inf(X), \sup(X)\big) \subseteq X$. This is true since X is connected and by Def. I.11.1.1 we have $(a, b) \subseteq X$ for every $a, b \in X$ and $a < b$.

From all cases above, we conclude that X is a bounded interval.

Now we show that X is a bounded interval implies X is bounded and connected. Obviously X is bounded. Let $a, b \in \mathbb{R}$. Then X can be one of $(a, b), [a, b], (a, b], [a, b)$, and by Def. I.11.1.1 all of which are connected. $\square$

Rmk. I.11.1.5. Recall that intervals are allowed to be singleton points, or even the empty set.

Cor. I.11.1.6. If I and J are bounded intervals, then the intersection $I \cap J$ is also a bounded interval.

Proof. If $I \cap J = \emptyset$, then $I \cap J$ is bounded interval. So suppose that $I \cap J \neq \emptyset$. Since I, J are bounded intervals, by Lem. I.11.1.4 we know that I, J are bounded and connected. Since I, J are bounded, $\exists M_1, M_2 \in \mathbb{R}$ such that $I \subseteq [-M_1, M_1]$ and $J \subseteq [-M_2, M_2]$. Let $M = \min(M_1, M_2)$. Then we have $I \cap J \subseteq [-M, M]$ and thus $I \cap J$ is bounded. Let $x, y \in I \cap J$ and $x < y$. Since I is connected and $I \cap J \subseteq I$, we have $[x, y] \subseteq I$. Similarly, since J is connected and $I \cap J \subseteq J$, we have $[x, y] \subseteq J$. Thus, $[x, y] \subseteq I \cap J$ and by Def. I.11.1.1 $I \cap J$ is connected. Since $I \cap J$ is bounded and connected, by Lem. I.11.1.4 $I \cap J$ is bounded interval. $\square$

Def. I.11.1.8 (Length of intervals)**.** If I is a bounded interval, we define the *length* of I, denoted $|I|$ as follows. If I is one of the intervals $[a, b]$, (a, b), $[a, b)$, or $(a, b]$ for some real numbers $a < b$, then we define $|I| \coloneqq b - a$. Otherwise, if I is a point or the empty set, we define $|I| = 0$.

Def. I.11.1.10 (Partitions)**.** Let I be a bounded interval. A *partition* of I is a finite set $\mathbf{P}$ of bounded intervals contained in I, such that every x in I lies in exactly one of the bounded intervals J in $\mathbf{P}$.

Rmk. I.11.1.11. Note that a partition is a set of intervals, while each interval is itself a set of real numbers. Thus, a partition is a set consisting of other sets.

Thm. I.11.1.13 (Length is finitely additive)**.** Let I be a bounded interval, n be a natural number, and let $\mathbf{P}$ be a partition of I of cardinality n. Then

$$|I| = \sum_{J \in \mathbf{P}} |J|.$$

Proof. We prove this by induction on n. More precisely, we let $P(n)$ be the property that whenever I is a bounded interval, and whenever $\mathbf{P}$ is a partition of I with cardinality n, that $|I| = \sum_{J \in \mathbf{P}} |J|$.

The base case $P(0)$ is trivial; the only way that I can be partitioned into an empty partition is if I is itself empty, at which point the claim is easy. The case $P(1)$ is also very easy; the only way that I can be partitioned into a singleton set $\{J\}$ is if $J = I$, at which point the claim is again very easy.

Now suppose inductively that $P(n)$ is true for some $n \geq 1$, and now we prove $P(n+1)$. Let I be a bounded interval, and let $\mathbf{P}$ be a partition of I of cardinality $n+1$.

If I is the empty set or a point, then all the intervals in $\mathbf{P}$ must also be either the empty set or a point, and so every interval has length zero and the claim is trivial. Thus, we will assume that I is an interval of the form (a, b), $(a, b]$, $[a, b)$, or $[a, b]$.

Let us first suppose that $b \in I$, i.e., I is either $(a, b]$ or $[a, b]$. Since $b \in I$, we know that one of the intervals K in $\mathbf{P}$ contains b. Since K is contained in I, it must therefore be of the form $(c, b]$, $[c, b]$, or $\{b\}$ for some real number c, with $a \leq c \leq b$ (in the latter case of $K = \{b\}$, we set $c := b$). In particular, this means that the set $I \setminus K$ is also an interval of the form $[a, c]$, (a, c), $(a, c]$, $[a, c)$ when $c > a$, or a point or empty set when $a = c$. Either way, we easily see that

$$|I| = |K| + |I \setminus K|.$$

On the other hand, since $\mathbf{P}$ forms a partition of I, we see that $\mathbf{P} \setminus \{K\}$ forms a partition of $I \setminus K$. By the induction hypothesis, we thus have

$$|I \setminus K| = \sum_{J \in \mathbf{P} \setminus \{K\}} |J|.$$

Combining these two identities (and using the laws of addition for finite sets, see Prop. I.7.1.11(e)) we obtain

$$|I| = \sum_{J \in \mathbf{P}} |J|$$

as desired.

Now suppose that $b \notin I$, i.e., I is either (a,b) or $[a,b)$. Then one of the intervals K also is of the form (c,b) or $[c,b)$ (see Ex. I.11.1.3). In particular, this means that the set $I \setminus K$ is also an interval of the form $[a,c]$, (a,c), $(a,c]$, $[a,c)$ when $c > a$, or a point or empty set when $a = c$. The rest of the argument then proceeds as above. $\square$

Def. I.11.1.14 (Finer and coarser partitions)**.** Let I be a bounded interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. We say that $\mathbf{P}'$ is *finer* than $\mathbf{P}$ (or equivalently, that $\mathbf{P}$ is *coarser* than $\mathbf{P}'$) if for every J in $\mathbf{P}'$, there exists a K in $\mathbf{P}$ such that $J \subseteq K$.

Note. There is no such thing as a "finest" partition of some interval I. (recall all partitions are assumed to be finite.) We do not compare partitions of different intervals.

Def. I.11.1.16 (Common refinement)**.** Let I be a bounded interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. We define the *common refinement* $\mathbf{P}\#\mathbf{P}'$ of $\mathbf{P}$ and $\mathbf{P}'$ to be the set

$$\mathbf{P}\#\mathbf{P}' := \left\{K \cap J : K \in \mathbf{P} \text{ and } J \in \mathbf{P}'\right\}.$$

A.Cor. I.11.1.1. Let I be a bounded interval, and let $\mathbf{P}, \mathbf{P}'$ be two partitions of I. Then we have $I = \bigcup(\mathbf{P}\#\mathbf{P}')$.

Proof. Let $x \in I$. By Def. I.11.1.10 we know that $\exists!\ K \in \mathbf{P}$ such that $x \in K$. Similarly, $\exists!\ K' \in \mathbf{P}'$ such that $x \in K'$, thus $x \in K \cap K'$. By Def. I.11.1.16 we know that $K \cap K' \in \mathbf{P}\#\mathbf{P}'$, thus $x \in \bigcup(\mathbf{P}\#\mathbf{P}')$. Since x was arbitrary, we have

$$I \subseteq \bigcup \left(\mathbf{P}\#\mathbf{P}'\right).$$

Let $S \in \mathbf{P}\#\mathbf{P}'$. By Def. I.11.1.16 we know that $\exists J \in \mathbf{P}$ and $\exists J' \in \mathbf{P}'$ such that $S = J \cap J'$. Since $S = J \cap J'$, we have $S \subseteq I$. Since S was arbitrary, we have

$$\bigcup \left(\mathbf{P}\#\mathbf{P}'\right) \subseteq I.$$

Thus, by Prop. I.3.1.18 we have

$$I = \bigcup \left(\mathbf{P}\#\mathbf{P}'\right).$$

$\square$

A.Cor. I.11.1.2. Let I be a bounded interval, and let $\mathbf{P}, \mathbf{P}'$ be two partitions of I. Then every element $x \in I$ contains in exactly one of the element $\mathbf{P}\#\mathbf{P}'$. In other words, $\exists!\ S \in \mathbf{P}\#\mathbf{P}$ such that $x \in S$.

Proof. By A.Cor. I.11.1.1 we know that at least one element in $\mathbf{P}\#\mathbf{P}'$ contains x. Suppose for the sake of contradiction that $\exists S_1, S_2 \in \mathbf{P}\#\mathbf{P}'$ such that $x \in S_1$ and $x \in S_2$ but $S_1 \neq S_2$. By Def. I.11.1.16 we know that $S_1 = K \cap K'$ for some $K \in \mathbf{P}$ and $K' \in \mathbf{P}'$. Similarly, $S_2 = J \cap J'$ for some $J \in \mathbf{P}$ and $J' \in \mathbf{P}'$. We know that $x \in S_1$ implies $x \in K$. Similarly, $x \in S_2$ implies $x \in J$. But by Def. I.11.1.10 we know that $K = J$, and a similar argument holds for $K' = J'$. Thus, we must have $S_1 = S_2$, a contradiction. $\square$

A.Cor. I.11.1.3. Let I be a bounded interval, and let $\mathbf{P}, \mathbf{P}'$ be two partitions of I. Then $\mathbf{P}\#\mathbf{P}'$ is finite and every element in $\mathbf{P}\#\mathbf{P}'$ is a bounded interval.

Proof. Let $f : \mathbf{P} \times \mathbf{P}' \to \mathbf{P}\#\mathbf{P}'$ be a function where

$$f(K, K') = K \cap K' \text{ for every } (K, K') \in \mathbf{P} \times \mathbf{P}'.$$

By Def. I.11.1.16 we see that f is surjective. By Def. I.11.1.10 we know that both $\#(\mathbf{P}), \#(\mathbf{P}')$ are finite. Thus, by Prop. I.3.6.14(e) and Ex. I.8.4.3 we have

$$\#(\mathbf{P} \times \mathbf{P}') = \#(\mathbf{P}) \times \#(\mathbf{P}') \geq \#(\mathbf{P}\#\mathbf{P}').$$

This means $\mathbf{P}\#\mathbf{P}'$ is finite.

By Def. I.11.1.16 we know that for every $S \in \mathbf{P}\#\mathbf{P}'$, $S = K \cap K'$ for some $K \in \mathbf{P}$ and $K' \in \mathbf{P}'$. By Def. I.11.1.10 we know that both K, K' are bounded interval, thus by Cor. I.11.1.6 we know that S is also a bounded interval. Since S was arbitrary, we conclude that every element in $\mathbf{P}\#\mathbf{P}'$ is a bounded interval. □

Lem. I.11.1.18. Let I be a bounded interval, and let $\mathbf{P}$ and $\mathbf{P}'$ be two partitions of I. Then $\mathbf{P}\#\mathbf{P}'$ is also a partition of I, and is both finer than $\mathbf{P}$ and finer than $\mathbf{P}'$.

Proof. By A.Cor. I.11.1.1 we know that $I = \bigcup(\mathbf{P}\#\mathbf{P}')$. By A.Cor. I.11.1.2 we know that every element in I contains in exactly one of the element $\mathbf{P}\#\mathbf{P}'$. By A.Cor. I.11.1.3 we know that $\mathbf{P}\#\mathbf{P}'$ is finite and every element in $\mathbf{P}\#\mathbf{P}'$ is a bounded interval. Thus, by Def. I.11.1.10 $\mathbf{P}\#\mathbf{P}'$ is a partition of I.

By Def. I.11.1.16 we know that for every $S \in \mathbf{P}\#\mathbf{P}'$, $S = K \cap K'$ for some $K \in \mathbf{P}$ and $K' \in \mathbf{P}'$. This means $S \subseteq K$ and $S \subseteq K'$, thus by Def. I.11.1.14 $\mathbf{P}\#\mathbf{P}'$ is both finer than $\mathbf{P}$ and finer than $\mathbf{P}'$ □

A.Cor. I.11.1.4. Let I be a bounded interval, and let $\mathbf{P}, \mathbf{P}'$ be two partitions of I such that $\mathbf{P}'$ is finer than $\mathbf{P}$. For each $K \in \mathbf{P}$, we define $\mathbf{P}_K$ as follow:

$$\mathbf{P}_K = \left\{K' \in \mathbf{P}' : K' \subseteq K\right\}.$$

Then $\mathbf{P}_K$ is a partition of K for every $K \in \mathbf{P}$, and $\bigcup\limits_{K \in \mathbf{P}} \mathbf{P}_K = \mathbf{P}'$.

Proof. Since $\mathbf{P}_K \subseteq \mathbf{P}'$ and $\mathbf{P}'$ is a partition of I, by Def. I.11.1.10 we know the following facts:

- $\mathbf{P}_K$ is finite.
- All distinct elements in $\mathbf{P}_K$ are disjoint.
- All elements in $\mathbf{P}_K$ are bounded interval.

To show that $\mathbf{P}_K$ is a partition of K, by Def. I.11.1.10 it suffices to show that $K = \bigcup \mathbf{P}_K$.

Let $x \in K$. By Def. I.11.1.10 we know that $x \in I$, thus $\exists!\ K' \in \mathbf{P}'$ such that $x \in K'$. Since $\mathbf{P}'$ is finer than $\mathbf{P}$, we must have $K' \subseteq K$. If not, then we have some $J \in \mathbf{P}$ such that $K' \subseteq J$, but $x \in J$ implies $J = K$, a contradiction. Since $K' \in \mathbf{P}'$ and $K' \subseteq K$, we have $K' \in \mathbf{P}_K$. Since x was arbitrary, we have $K \subseteq \bigcup \mathbf{P}_K$. By the definition of $\mathbf{P}_K$ we know that $\bigcup \mathbf{P}_K \subseteq K$, thus by Prop. I.3.1.18 we have $K = \bigcup \mathbf{P}_K$.

Now we show that $\bigcup_{K \in \mathbf{P}} \mathbf{P}_K = \mathbf{P}'$. We know that $\bigcup_{K \in \mathbf{P}} \mathbf{P}_K \subseteq \mathbf{P}'$. Let $K' \in \mathbf{P}'$. By Lem. I.11.1.18 we know that $\mathbf{P}'$ is finer than $\mathbf{P}$. By Def. I.11.1.14 we know that $K' \subseteq K$ for some $K \in \mathbf{P}$. Thus, we have $K' \in \mathbf{P}_K$. Since K' was arbitrary, we have $\mathbf{P}' \subseteq \bigcup_{K \in \mathbf{P}} \mathbf{P}_K$. Thus, by Prop. I.3.1.18 we have $\bigcup_{K \in \mathbf{P}} \mathbf{P}_K = \mathbf{P}'$. □

A.Cor. I.11.1.5. Let I, J be bounded intervals such that $I \neq \emptyset$ and $I \subseteq J$, and let $\mathbf{P}$ be a partition of I. Let I_1, I_2 be the sets

$$I_1 = \big\{x \in J : \big(x \leq \inf(I)\big) \wedge (x \notin I)\big\}$$

and

$$I_2 = \big\{x \in J : \big(x \geq \sup(I)\big) \wedge (x \notin I)\big\}.$$

Then $\mathbf{P} \cup \{I_1, I_2\}$ is a partion of J.

Proof. First, we claim that I_1 is a bounded interval. If $I_1 = \emptyset$, then I_1 is a bounded interval. So suppose that $I_1 \neq \emptyset$. We know that $\inf(I) \in J$ since if $\inf(I) \notin J$, then by definition we would have $I_1 = \emptyset$, a contradiction. We must have $\inf(I_1) = \inf(J)$. If not, then we have $\inf(J) < \inf(I_1) \leq \inf(I)$. Since J is a bounded interval, we have $\inf(J) < x < \inf(I_1) \leq \inf(I)$ for some $x \in J$. But $x \in J$ and $x < \inf(I)$ implies $x \in I_1$, which contradict to $\inf(I_1) \leq x$. So we have $\inf(I_1) = \inf(J)$. Now we split into four cases:

- If $\inf(J) \in J$ and $\inf(I) \in I$, then $I_1 = \big[\inf(J), \inf(I)\big)$.
- If $\inf(J) \in J$ and $\inf(I) \notin I$, then $I_1 = [\inf(J), \inf(I)]$.
- If $\inf(J) \notin J$ and $\inf(I) \in I$, then $I_1 = \big(\inf(J), \inf(I)\big)$.
- If $\inf(J) \notin J$ and $\inf(I) \notin I$, then $I_1 = \big(\inf(J), \inf(I)]$.

From all cases above, we conclude that I_1 is a bounded interval.

Next we claim that I_2 is a bounded interval. If $I_2 = \emptyset$, then I_2 is a bounded interval. So suppose that $I_2 \neq \emptyset$. We know that $\sup(I) \in J$ since if $\sup(I) \notin J$, then by definition we would have $I_2 = \emptyset$, a contradiction. We must have $\sup(I_2) = \sup(J)$. If not, then we have $\sup(J) > \sup(I_2) \geq \sup(I)$. Since J is a bounded interval, we have $\sup(J) > x > \sup(I_2) \geq \sup(I)$ for some $x \in J$. But $x \in J$ and $x > \sup(I)$ implies $x \in I_2$, which contradict to $\sup(I_2) \geq x$. So we have $\sup(I_2) = \sup(J)$. Now we split into four cases:

- If $\sup(J) \in J$ and $\sup(I) \in I$, then $I_2 = \big(\sup(I), \sup(J)]$.
- If $\sup(J) \in J$ and $\sup(I) \notin I$, then $I_2 = [\sup(I), \sup(J)]$.
- If $\sup(J) \notin J$ and $\sup(I) \in I$, then $I_2 = \big(\sup(I), \sup(J)\big)$.
- If $\sup(J) \notin J$ and $\sup(I) \notin I$, then $I_2 = [\sup(I), \sup(J)\big)$.

From all cases above, we conclude that I_2 is a bounded interval.

Next we show that $I \cap I_1 = I \cap I_2 = I_1 \cap I_2 = \emptyset$. By definition we know that $I \cap I_1 = I \cap I_2 = \emptyset$. So we only need to show that $I_1 \cap I_2 = \emptyset$. If $(I_1 = \emptyset) \vee (I_2 = \emptyset)$, then we have $I_1 \cap I_2 = \emptyset$. So suppose that $(I_1 \neq \emptyset) \wedge (I_2 \neq \emptyset)$. Suppose for the sake of contradiction that $I_1 \cap I_2 \neq \emptyset$. Let $x \in I_1 \cap I_2$. Then we have $x \leq \inf(I) \leq \sup(I) \leq x$. Now we split into two cases:

- If $\inf(I) = \sup(I)$, then $I = \{a\}$ for some $a \in \mathbb{R}$. But $x \leq a \leq x$ implies $x = a$ and $x \in I$, which contradict to $x \notin I$.
- If $\inf(I) < \sup(I)$, then we have $x < x$, a contradiction.

From all cases above, we conclude that $I_1 \cap I_2 = \emptyset$.

Let $\mathbf{P}_J = \mathbf{P} \cup \{I_1, I_2\}$. By definition we know that $\bigcup \mathbf{P}_J \subseteq J$. Let $x \in J$. Now we split into two cases:

- If $x \in I$, then we have $x \in \bigcup \mathbf{P}$.
- If $x \notin I$, then we have $\big(x \leq \inf(I)\big) \vee \big(x \geq \sup(I)\big)$. Thus, $(x \in I_1) \vee (x \in I_2)$ and $x \in \bigcup \mathbf{P}$.

From all cases above, we conclude that $x \in \bigcup \mathbf{P}_J$. Since x was arbitrary, we have $J \subseteq \bigcup \mathbf{P}_J$. By Prop. I.3.1.18 we have $J = \bigcup \mathbf{P}_J$.

From the proofs above we have showed that $J = \bigcup \mathbf{P}_J$, all distinct element in $\mathbf{P}_J$ are disjoint, and all elements in $\mathbf{P}_J$ are bounded interval. Since $\mathbf{P}_J$ is finite ($\#(\mathbf{P}_J) = 3$), by Def. I.11.1.10 $\mathbf{P}_J$ is a partition of J. □

— Exercises —

Ex. I.11.1.1. Prove Lem. I.11.1.4.

Proof. See Lem. I.11.1.4. □

Ex. I.11.1.2. Prove Cor. I.11.1.6.

Proof. Prove Cor. I.11.1.6. □

Ex. I.11.1.3. Let I be a bounded interval of the form $I = (a, b)$ or $I = [a, b)$ for some real numbers $a < b$. Let $I_1, \dots, I_n$ be a partition of I. Prove that one of the intervals I_j in this partition is of the form $I_j = (c, b)$ or $I_j = [c, b)$ for some $a \leq c \leq b$.

Proof. Let $\mathbf{P} = \{I_1, \dots, I_n\}$. If $c = b$, then $(c, b) = \emptyset$, and thus by Def. I.11.1.10 $\mathbf{P} \cup \{\emptyset\}$ is a partition of I. So we only need to proof the cases where $a \leq c < b$. Suppose for the sake of contradiction that every interval I_j in the partition $\mathbf{P}$ is not of the form (c, b) or $[c, b)$. By Def. I.11.1.10 this means for every $j \in \{1, \dots, n\}$, $x \in I_j$ implies $x \geq b$ or $x < c$. Since $I = (a, b)$ or $I = [a, b)$, we cannot have $x \geq b$, thus we must have $x < c$. This means $\sup(I_j) \leq c < b$ for every $j \in \{1, \dots, n\}$. But then we have $\sup(I) = b > \max\{\sup(I_j) : j \in \{1, \dots, n\}\}$, a contradiction. Thus, we must have one interval $I_j \in \mathbf{P}$ such that $I_j = (c, b)$ for some $a \leq c < b$. □

Ex. I.11.1.4. Prove Lem. I.11.1.18.

Proof. Prove Lem. I.11.1.18. □

I.11.2 Piecewise constant functions

Def. I.11.2.1 (Constant functions)**.** Let X be a subset of $\mathbb{R}$, and let $f : X \to \mathbb{R}$ be a function. We say that f is *constant* iff there exists a real number c such that $f(x) = c$ for all $x \in X$. If E is a subset of X, we say that f is *constant on* E if the restriction $f|_E$ of f to E is constant, in other words there exists a real number c such that $f(x) = c$ for all $x \in E$. We refer to c as the *constant value* of f on E.

Rmk. I.11.2.2. If E is a non-empty set, then a function f which is constant on E can have only one constant value; However, if E is empty, every real number c is a constant value for f on E.

Def. I.11.2.3 (Piecewise constant functions I)**.** Let I be a bounded interval, let $f : I \to \mathbb{R}$ be a function, and let $\mathbf{P}$ be a partition of I. We say that f is *piecewise constant with respect to* $\mathbf{P}$ if for every $J \in \mathbf{P}$, f is constant on J.

Def. I.11.2.5 (Piecewise constant functions II)**.** Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be a function. We say that f is *piecewise constant on* I if there exists a partition $\mathbf{P}$ of I such that f is piecewise constant with respect to $\mathbf{P}$.

Lem. I.11.2.7. Let I be a bounded interval, let $\mathbf{P}$ be a partition of I, and let $f : I \to \mathbb{R}$ be a function which is piecewise constant with respect to $\mathbf{P}$. Let $\mathbf{P}'$ be a partition of I which is finer than $\mathbf{P}$. Then f is also piecewise constant with respect to $\mathbf{P}'$.

Proof. Let $K' \in \mathbf{P}'$. Since $\mathbf{P}'$ is finer than $\mathbf{P}$, by Def. I.11.1.14 $\exists K \in \mathbf{P}$ such that $K' \subseteq K$. Since f is piecewise constant with respect to $\mathbf{P}$, by Def. I.11.2.3 we know that $\forall x \in K$, $f(x)$ is constant. Thus, for every $x \in K'$, $x \in K$ and $f(x)$ is constant. Since K' was arbitrary, by Def. I.11.2.3 f is piecewise constant with respect to $\mathbf{P}'$. □

Lem. I.11.2.8. Let I be a bounded interval, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be piecewise constant functions on I. Then the functions $f+g$, $f-g$, $\max(f,g)$, $\min(f,g)$ and fg are also piecewise constant functions on I. Here of course $\max(f,g) : I \to \mathbb{R}$ is the function $\max(f,g)(x) := \max(f(x), g(x))$. If g does not vanish anywhere on I (i.e., $g(x) \neq 0$ for all $x \in I$) then f/g is also a piecewise constant function on I.

Proof. Since f is piecewise constant function on I, by Def. I.11.2.5 $\exists\mathbf{P}$ such that $\mathbf{P}$ is a partition of I and f is piecewise constant with respect to $\mathbf{P}$. Similarly, $\exists\mathbf{P}'$ such that $\mathbf{P}'$ is a partition of I and g is piecewise constant with respect to $\mathbf{P}'$. By Lem. I.11.1.18 we know that $\mathbf{P}\#\mathbf{P}'$ is also a partition of I and $\mathbf{P}\#\mathbf{P}'$ is both finer than $\mathbf{P}$ and finer than $\mathbf{P}'$. By Lem. I.11.2.7 we know that both f and g are piecewise constant with respect to $\mathbf{P}\#\mathbf{P}'$.

Now we show that f, g remain piecewise constant functions on I after algebraic operation. For every $J \in \mathbf{P}\#\mathbf{P}'$, we have $f(x)$ is constant and $g(x)$ is constant for every $x \in J$. Thus, we know that $f(x)+g(x)$, $f(x)-g(x)$, $\max\big(f(x), g(x)\big)$, $\min\big(f(x), g(x)\big)$ and $f(x)g(x)$ are constant. If $g(x) \neq 0$, then we also have $f(x)/g(x)$ is constant. Thus, by Def. I.11.2.3 $f+g$, $f-g$, $\max(f,g)$, $\min(f,g)$, fg is piecewise constant with respect to $\mathbf{P}\#\mathbf{P}'$, and when $g(x) \neq 0$ we have f/g is piecewise constant with respect to $\mathbf{P}\#\mathbf{P}'$. By Def. I.11.2.5 $f+g$, $f-g$, $\max(f,g)$, $\min(f,g)$, fg is piecewise constant on I, and when $g(x) \neq 0$ we have f/g is piecewise constant on I. $\square$

Def. I.11.2.9 (Piecewise constant integral I)**.** Let I be a bounded interval, let $\mathbf{P}$ be a partition of I. Let $f : I \to \mathbb{R}$ be a function which is piecewise constant with respect to $\mathbf{P}$. Then we define the *piecewise constant integral p.c.* $\displaystyle\int_{[\mathbf{P}]} f$ of f with respect to the partition $\mathbf{P}$ by the formula

$$p.c. \int_{[\mathbf{P}]} f := \sum_{J \in \mathbf{P}} c_J |J|,$$

where for each J in $\mathbf{P}$, we let c_J be the constant value of f on J.

Rmk. I.11.2.10. This definition seems like it could be ill-defined, because if J is empty then every number c_J can be the constant value of f on J, but fortunately in such cases $|J|$ is zero and so the choice of c_J is irrelevant. The notation $p.c. \displaystyle\int_{[\mathbf{P}]} f$ is rather artificial, but we shall only need it temporarily, en route to a more useful definition. Note that since $\mathbf{P}$ is finite, the sum $\displaystyle\sum_{J \in \mathbf{P}} c_J|J|$ is always well-defined (it is never divergent or infinite).

Rmk. I.11.2.11. The piecewise constant integral corresponds intuitively to one's notion of area, given that the area of a rectangle ought to be the product of the lengths of the sides. (Of course, if f is negative somewhere, then the "area" $c_J|J|$ would also be negative.)

Prop. I.11.2.13 (Piecewise constant integral is independent of partition)**.** Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be a function. Suppose that $\mathbf{P}$ and $\mathbf{P}'$ are partitions of

I such that f is piecewise constant both with respect to $\mathbf{P}$ and with respect to $\mathbf{P}'$. Then $p.c. \int_{[\mathbf{P}]} f = p.c. \int_{[\mathbf{P}']} f$.

Proof. By Lem. I.11.1.18 we know that $\mathbf{P}\#\mathbf{P}'$ is a partition of I and is both finer than $\mathbf{P}$ and finer than $\mathbf{P}'$, thus by Def. I.11.2.9 we have

$$p.c. \int_{[\mathbf{P}\#\mathbf{P}']} f = \sum_{J\in\mathbf{P}\#\mathbf{P}'} c_J|J|.$$

By Thm. I.11.1.13, we know that

$$|I| = \sum_{J\in\mathbf{P}}|J| = \sum_{J\in\mathbf{P}\#\mathbf{P}'}|J|.$$

For each $K \in \mathbf{P}$, let $\mathbf{P}_K$ be the set

$$\mathbf{P}_K = \{S \in \mathbf{P}\#\mathbf{P}' : S \subseteq K\}.$$

Since $\mathbf{P}\#\mathbf{P}'$ is finer than $\mathbf{P}$, by A.Cor. I.11.1.4 we know that $\mathbf{P}_K$ is a partition of K, and $\bigcup_{K\in\mathbf{P}} \mathbf{P}_K = \mathbf{P}\#\mathbf{P}'$. Since f is piecewise constant with respect to $\mathbf{P}$, by Lem. I.11.2.7 we know that f is piecewise constant with respect to $\mathbf{P}\#\mathbf{P}'$. So we have

$$\begin{aligned}
p.c. \int_{[\mathbf{P}\#\mathbf{P}']} f &= \sum_{J\in\mathbf{P}\#\mathbf{P}'} c_J|J| && \text{(by Def. I.11.2.9)}\\
&= \sum_{J\in\bigcup_{K\in\mathbf{P}}\mathbf{P}_K} c_J|J| \\
&= \sum_{K\in\mathbf{P}}\sum_{J\in\mathbf{P}_K} c_J|J| && \text{(by Prop. I.7.1.11(e))}\\
&= \sum_{K\in\mathbf{P}}\sum_{J\in\mathbf{P}_K} c_K|J| && (J \subseteq K)\\
&= \sum_{K\in\mathbf{P}} c_K\Bigg(\sum_{J\in\mathbf{P}_K}|J|\Bigg) \\
&= \sum_{K\in\mathbf{P}} c_K|K| && \text{(by Thm. I.11.1.13)}\\
&= p.c. \int_{[\mathbf{P}]} f. && \text{(by Def. I.11.2.9)}
\end{aligned}$$

Using similar arguments, we can show that $p.c. \int_{[\mathbf{P}']} f = p.c. \int_{[\mathbf{P}\#\mathbf{P}']} f$. Thus, we have $p.c. \int_{[\mathbf{P}]} f = p.c. \int_{[\mathbf{P}']} f$. □

Def. I.11.2.14 (Piecewise constant integral II)**.** Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be a piecewise constant function on I. We define the *piecewise constant integral* $p.c. \int_I f$ by the formula

$$p.c. \int_I f := p.c. \int_{[\mathbf{P}]} f,$$

where $\mathbf{P}$ is any partition of I with respect to which f is piecewise constant. (Note that Prop. I.11.2.13 tells us that the precise choice of this partition is irrelevant.)

Thm. I.11.2.16 (Laws of integration)**.** Let I be a bounded interval, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be piecewise constant functions on I.

(a) We have $p.c. \int_I (f+g) = p.c. \int_I f + p.c. \int_I g$.

(b) For any real number c, we have $p.c. \int_I (cf) = c(p.c. \int_I f)$.

(c) We have $p.c. \int_I (f-g) = p.c. \int_I f - p.c. \int_I g$.

(d) If $f(x) \geq 0$ for all $x \in I$, then $p.c. \int_I f \geq 0$.

(e) If $f(x) \geq g(x)$ for all $x \in I$, then $p.c. \int_I f \geq p.c. \int_I g$.

(f) If f is the constant function $f(x) = c$ for all $x \in I$, then $p.c. \int_I f = c|I|$.

(g) Let J be a bounded interval containing I (i.e., $I \subseteq J$), and let $F : J \to \mathbb{R}$ be the function

$$F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

Then F is piecewise constant on J, and $p.c. \int_J F = p.c. \int_I f$.

(h) Suppose that $\{J, K\}$ is a partition of I into two intervals J and K. Then the function $f|_J : J \to \mathbb{R}$ and $f|_K : K \to \mathbb{R}$ are piecewise constant on J and K respectively, and we have

$$p.c. \int_I f = p.c. \int_J f|_J + p.c. \int_K f|_K.$$

Proof. (a) Since f, g are both piecewise constant on I, by Def. I.11.2.3 f is piecewise constant with respect to $\mathbf{P}_f$ and g is piecewise constant with respect to $\mathbf{P}_g$ for some partitions $\mathbf{P}_f, \mathbf{P}_g$ of I. Let $\mathbf{P} = (\mathbf{P}_f \# \mathbf{P}_g) \setminus \{\emptyset\}$. Then by Lem. I.11.1.18 we know that $\mathbf{P}$ is a partition of

I and by Lem. I.11.2.7 f, g are piecewise constant with respect to $\mathbf{P}$. For each $J \in \mathbf{P}$, we define $c_{f|_J}, c_{g|_J} \in \mathbb{R}$ to be the constant value of $f|_J, g|_J$, respectively. Then by Def. I.11.2.1 $c_{f|_J} + c_{g|_J}$ is the constant value of $(f+g)|_J$ for each $J \in \mathbf{P}$. Thus, $f+g$ is piecewise constant with respect to $\mathbf{P}$ and

$$\begin{aligned}
p.c. \int_I f + p.c. \int_I g &= p.c. \int_{[\mathbf{P}]} f + p.c. \int_{[\mathbf{P}]} g && \text{(by Def. I.11.2.14)} \\
&= \sum_{J \in \mathbf{P}} c_{f|_J} |J| + \sum_{J \in \mathbf{P}} c_{g|_J} |J| && \text{(by Def. I.11.2.9)} \\
&= \sum_{J \in \mathbf{P}} (c_{f|_J} + c_{g|_J}) |J| && \text{(by Prop. I.7.1.11(f))} \\
&= p.c. \int_{[\mathbf{P}]} (f+g) && \text{(by Def. I.11.2.9)} \\
&= p.c. \int_I (f+g). && \text{(by Def. I.11.2.14)}
\end{aligned}$$

□

Proof. (b) By Def. I.11.2.3 f is piecewise constant with respect to $\mathbf{P}$ for some partition $\mathbf{P}$ of I. Without the loss of generality suppose that $\emptyset \notin \mathbf{P}$. For each $J \in \mathbf{P}$, we define $c_J \in \mathbb{R}$ to be the constant value of $f|_J$. Then by Def. I.11.2.1 $c \cdot c_J$ is the constant value of $(cf)|_J$. Thus, cf is piecewise constant with respect to $\mathbf{P}$ and

$$\begin{aligned}
c\left(p.c. \int_I f\right) &= c\left(p.c. \int_{[\mathbf{P}]} f\right) && \text{(by Def. I.11.2.14)} \\
&= c\left(\sum_{J \in \mathbf{P}} c_J |J|\right) && \text{(by Def. I.11.2.9)} \\
&= \sum_{J \in \mathbf{P}} c \cdot c_J |J| && \text{(by Prop. I.7.1.11(g))} \\
&= p.c. \int_{[\mathbf{P}]} (cf) && \text{(by Def. I.11.2.9)} \\
&= p.c. \int_I (cf). && \text{(by Def. I.11.2.14)}
\end{aligned}$$

□

Proof. (c) We have

$$\begin{aligned}
p.c. \int_I f - p.c. \int_I g &= p.c. \int_I f + (-1) p.c. \int_I g \\
&= p.c. \int_I f + p.c. \int_I (-g) && \text{(by Thm. I.11.2.16(b))}
\end{aligned}$$

$$\begin{aligned}
&= p.c. \int_I \big(f + (-g)\big) && \text{(by Thm. I.11.2.16(a))}\\
&= p.c. \int_I (f - g). && \text{(by Def. I.9.2.1)}
\end{aligned}$$

□

Proof. (d) By Def. I.11.2.3 f is piecewise constant with respect to $\mathbf{P}$ for some partition $\mathbf{P}$ of I. Without the loss of generality suppose that $\emptyset \notin \mathbf{P}$. For each $J \in \mathbf{P}$, we define $c_J \in \mathbb{R}$ to be the constant value of $f|_J$. Since $f(x) \geq 0$ for every $x \in I$, we have $c_J \geq 0$ and $c_J|J| \geq 0$ for every $J \in \mathbf{P}$. Thus

$$\begin{aligned}
p.c. \int_I f &= p.c. \int_{[\mathbf{P}]} f && \text{(by Def. I.11.2.14)}\\
&= \sum_{J \in \mathbf{P}} c_J |J| && \text{(by Def. I.11.2.9)}\\
&\geq \sum_{J \in \mathbf{P}} 0 && \text{(by Prop. I.7.1.11(h))}\\
&= 0.
\end{aligned}$$

□

Proof. (e) Since $f(x) \geq g(x)$ for all $x \in I$, we have $f(x) - g(x) \geq 0$ for all $x \in I$ and

$$\begin{aligned}
p.c. \int_I f - p.c. \int_I g &= p.c. \int_I (f - g) && \text{(by Thm. I.11.2.16(c))}\\
&\geq 0. && \text{(by Thm. I.11.2.16(d))}
\end{aligned}$$

Thus

$$p.c. \int_I f \geq p.c. \int_I g.$$

□

Proof. (f) Since $\{I\}$ is a partition of I, we have

$$\begin{aligned}
p.c. \int_I f &= p.c. \int_{[I]} f && \text{(by Def. I.11.2.14)}\\
&= \sum_{J \in I} c|J| && \text{(by Def. I.11.2.9)}\\
&= c \sum_{J \in I} |J| && \text{(by Prop. I.7.1.11(g))}\\
&= c|I|. && \text{(by Thm. I.11.1.13)}
\end{aligned}$$

□

Proof. (g) If $I = \emptyset$, then by Def. I.11.2.3 F is piecewise constant with respect to $\{J\}$, and by Thm. I.11.2.16(f) we have

$$p.c. \int_J F = 0|J| = 0 = p.c \int_I f.$$

So suppose that $I \neq \emptyset$. By Def. I.11.2.3, f is piecewise constant with respect to $\mathbf{P}$ for some partition $\mathbf{P}$ of I. Let I_1, I_2 be the sets

$$I_1 = \big\{x \in J, \big(x \leq \inf(I)\big) \wedge (x \notin I)\big\}$$

and

$$I_2 = \big\{x \in J, \big(x \geq \sup(I)\big) \wedge (x \notin I)\big\}.$$

By A.Cor. I.11.1.5 we know that $\mathbf{P} \cup \{I_1, I_2\}$ is a partition of J. By hypothesis we know that

$$\forall x \in J, F(x) = \begin{cases} f(x) & \text{if } x \in K \text{ for some } K \in \mathbf{P} \\ 0 & \text{if } x \in I_1 \text{ or } x \in I_2 \end{cases}$$

Thus, by Def. I.11.2.5 F is piecewise constant on J. For each $K \in \mathbf{P} \cup \{I_1, I_2\}$, we define $c_K \in \mathbb{R}$ to be the constant value of $F|_K$. Then we have

$$\begin{aligned}
p.c. \int_J F &= p.c. \int_{[\mathbf{P} \cup \{I_1, I_2\}]} F && \text{(by Def. I.11.2.14)} \\
&= \sum_{K \in \mathbf{P} \cup \{I_1, I_2\}} c_K |K| && \text{(by Def. I.11.2.9)} \\
&= c_{I_1}|I_1| + \sum_{K \in \mathbf{P}} c_K |K| + c_{I_2}|I_2| && \text{(by Prop. I.7.1.11(e))} \\
&= 0|I_1| + \sum_{K \in \mathbf{P}} c_K |K| + 0|I_2| && \text{(by hypothesis)} \\
&= \sum_{K \in \mathbf{P}} c_K |K| \\
&= p.c. \int_{[\mathbf{P}]} f && \text{(by Def. I.11.2.9)} \\
&= p.c. \int_I f. && \text{(by Def. I.11.2.14)}
\end{aligned}$$

□

Proof. (h) Let $\mathbf{P} = \{J, K\}$. By Def. I.11.2.3 f is piecewise constant with respect to $\mathbf{P}'$ for some partition $\mathbf{P}'$ of I. Now we define $\mathbf{P}_J$ as

$$\mathbf{P}_J = \big\{S \in \mathbf{P} \# \mathbf{P}' : S \subseteq J\big\}$$

and define $\mathbf{P}_K$ as

$$\mathbf{P}_K = \{S \in \mathbf{P}\#\mathbf{P}' : S \subseteq K\}.$$

By Def. I.11.1.8 we know that $\mathbf{P}\#\mathbf{P}'$ is a partition of I and is finer than $\mathbf{P}$. Since $\mathbf{P}\#\mathbf{P}'$ is finer than $\mathbf{P}$, by A.Cor. I.11.1.4 we know that $\mathbf{P}_J, \mathbf{P}_K$ are partitions of J, K, respectively. Again by A.Cor. I.11.1.4 we know that $\mathbf{P}_J \cup \mathbf{P}_K$ is a partition of I. Then by Lem. I.11.2.7 f is piecewise constant with respect to $\mathbf{P}_J \cup \mathbf{P}_K$. Without the loss of generality suppose that $\emptyset \notin \mathbf{P}_J \cup \mathbf{P}_K$. For each $S \in \mathbf{P}_J$, we define $c_S \in \mathbb{R}$ to be the constant value of $f|_J$. Similarly, for each $S \in \mathbf{P}_K$, we define $c_S \in \mathbb{R}$ to be the constant value of $f|_K$. Then we have

$$\begin{aligned}
p.c.\int_J f|_J + p.c.\int_K f|_K &= p.c.\int_{[\mathbf{P}_J]} f|_J + p.c.\int_{[\mathbf{P}_K]} f|_K && \text{(by Def. I.11.2.14)}\\
&= \sum_{S\in\mathbf{P}_J} c_S|S| + \sum_{S\in\mathbf{P}_K} c_S|S| && \text{(by Prop. I.7.1.11(e))}\\
&= \sum_{S\in\mathbf{P}_J\cup\mathbf{P}_K} c_S|S| && \text{(by Def. I.11.2.9)}\\
&= \sum_{S\in\mathbf{P}} c_S|S| \\
&= p.c.\int_{[\mathbf{P}]} f && \text{(by Def. I.11.2.9)}\\
&= p.c.\int_I f. && \text{(by Def. I.11.2.14)}
\end{aligned}$$

□

— Exercises —

Ex. I.11.2.1. Prove Lem. I.11.2.7.

Proof. See Lem. I.11.2.7. □

Ex. I.11.2.2. Prove Lem. I.11.2.8.

Proof. See Lem. I.11.2.8. □

Ex. I.11.2.3. Prove Prop. I.11.2.13.

Proof. See Prop. I.11.2.13. □

Ex. I.11.2.4. Prove Thm. I.11.2.16.

Proof. See Thm. I.11.2.16. □

I.11.3 Upper and lower Riemann integrals

Def. I.11.3.1 (Majorization of functions)**.** Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$. We say that g *majorizes* f on I if we have $g(x) \geq f(x)$ for all $x \in I$, and that g *minorizes* f on I if $g(x) \leq f(x)$ for all $x \in I$.

Def. I.11.3.2 (Upper and lower Riemann integrals)**.** Let $f : I \to \mathbb{R}$ be a bounded function defined on a bounded interval I. We define the *upper Riemann integral* $\overline{\int}_I f$ by the formula

$$\overline{\int}_I f := \inf\left\{ p.c. \int_I g : g \text{ is a piecewise constant function on } I \text{ which majorizes } f \right\}$$

and the *lower Riemann integral* $\underline{\int}_I f$ by the formula

$$\underline{\int}_I f := \sup\left\{ p.c. \int_I g : g \text{ is a piecewise constant function on } I \text{ which minorizes } f \right\}.$$

Lem. I.11.3.3. Let $f : I \to \mathbb{R}$ be a function on a bounded interval I which is bounded by some real number M, i.e., $-M \leq f(x) \leq M$ for all $x \in I$. Then we have

$$-M|I| \leq \underline{\int}_I f \leq \overline{\int}_I f \leq M|I|.$$

in particular, both the lower and upper Riemann integrals are real numbers (i.e., they are not infinite).

Proof. The function $g : I \to \mathbb{R}$ defined by $g(x) = M$ is constant, hence piecewise constant, and majorizes f; thus $\overline{\int}_I f \leq p.c. \int_I g = M|I|$ by definition of the upper Riemann integral. A similar argument gives $-M|I| \leq \underline{\int}_I f$. Finally, we have to show that $\underline{\int}_I f \leq \overline{\int}_I f$. Let g be any piecewise constant function majorizing f, and let h be any piecewise constant function minorizing f. Then g majorizes h, and hence $p.c. \int_I h \leq p.c. \int_I g$. Taking suprema in h, we obtain that $\underline{\int}_I f \leq p.c. \int_I g$. Taking infima in g, we thus obtain $\underline{\int}_I f \leq \overline{\int}_I f$, as desired. $\square$

Def. I.11.3.4 (Riemann integral)**.** Let $f : I \to \mathbb{R}$ be a bounded function on a bounded interval I. If $\underline{\int}_I f = \overline{\int}_I f$, then we say that f is *Riemann integrable on I* and define

$$\int_I f := \underline{\int}_I f = \overline{\int}_I f.$$

If the upper and lower Riemann integrals are unequal, we say that f is not Riemann integrable.

Rmk. I.11.3.5. Compare this definition to the relationship between the lim sup, *liminf*, and limit of a sequence a_n that was established in Prop. I.6.4.12(f); the lim sup is always greater than or equal to the lim inf, but they are only equal when the sequence converges, and in this case they are both equal to the limit of the sequence. The definition given above may differ from the definition you may have encountered in your calculus courses, based on Riemann sums. However, the two definitions turn out to be equivalent.

Rmk. I.11.3.6. Note that we do not consider unbounded functions to be Riemann integrable; an integral involving such functions is known as an *improper integral.* It is possible to still evaluate such integrals using more sophisticated integration methods (such as the Lebesgue integral).

Lem. I.11.3.7. Let $f : I \to \mathbb{R}$ be a piecewise constant function on a bounded interval I. Then f is Riemann integrable, and $\int_I f = p.c. \int_I f$.

Proof. Since $f(x) \leq f(x)$ for every $x \in I$, by Def. I.11.3.2 we have

$$\overline{\int}_I f \leq p.c. \int_I f$$

and

$$p.c. \int_I f \leq \underline{\int}_I f.$$

By Lem. I.11.3.3 we know that

$$p.c. \int_I f \leq \underline{\int}_I f \leq \overline{\int}_I f \leq p.c. \int_I f.$$

Thus, by Def. I.11.3.4 we have

$$\int_I f = \underline{\int}_I f = \overline{\int}_I f = p.c. \int_I f.$$

□

Rmk. I.11.3.8. Because of Lem. I.11.3.7, we will not refer to the piecewise constant integral $p.c. \int_I$ again, and just use the Riemann integral $\int_I$ throughout (until this integral is itself superceded by the Lebesgue integral). We observe one special case of Lem. I.11.3.7: if I is a point or the empty set, then $\int_I f = 0$ for all functions $f : I \to \mathbb{R}$. (Note that all such functions are automatically constant.)

Def. I.11.3.9 (Riemann sums)**.** Let $f : I \to \mathbb{R}$ be a bounded function on a bounded interval I, and let $\mathbf{P}$ be a partition of I. We define the *upper Riemann sum* $U(f, \mathbf{P})$ and the *lower Riemann sum* $L(f, \mathbf{P})$ by

$$U(f, \mathbf{P}) \coloneqq \sum_{J \in \mathbf{P} : J \neq \emptyset} \big(\sup_{x \in J} f(x)\big)|J|$$

and

$$L(f, \mathbf{P}) \coloneqq \sum_{J \in \mathbf{P} : J \neq \emptyset} \big(\inf_{x \in J} f(x)\big)|J|.$$

Rmk. I.11.3.10. The restriction $J \neq \emptyset$ is required because the quantities $\inf_{x \in J} f(x)$ and $\sup_{x \in J} f(x)$ are infinite (or negative infinite) if J is empty.

Lem. I.11.3.11. Let $f : I \to \mathbb{R}$ be a bounded function on a bounded interval I, and let g be a function which majorizes f and which is piecewise constant with respect to some partition $\mathbf{P}$ of I. Then

$$p.c. \int_I g \geq U(f, \mathbf{P}).$$

Similarly, if h is a function which minorizes f and is piecewise constant with respect to $\mathbf{P}$, then

$$p.c. \int_I h \leq L(f, \mathbf{P}).$$

Proof. Since g majorizes f and h minorizes f, by Def. I.11.3.1 we have $h(x) \leq f(x) \leq g(x)$ for every $x \in I$. Since $\mathbf{P}$ is a partition of I, by Def. I.11.1.10 for every $J \in \mathbf{P}$, we have $h(x) \leq f(x) \leq g(x)$ for all $x \in J$. In particular, when $J \neq \emptyset$ we have

$$h(x) \leq \inf_{x \in J} f(x) \leq f(x) \leq \sup_{x \in J} f(x) \leq g(x)$$

for every $x \in J$. Let $c_{g|_J}, c_{h|_J}$ be constant values of $g|_J, h|_J$, respectively. Then we have

$$\begin{aligned}
U(f, \mathbf{P}) &= \sum_{J \in \mathbf{P} : J \neq \emptyset} \big(\sup_{x \in J} f(x)\big)|J| && \text{(by Def. I.11.3.9)} \\
&\leq \sum_{J \in \mathbf{P} : J \neq \emptyset} c_{g|_J} |J| && \text{(by Prop. I.7.1.11(h))} \\
&= \sum_{J \in \mathbf{P}} c_{g|_J} |J| && \text{(by Prop. I.7.1.11(a)(e))} \\
&= p.c. \int_{[\mathbf{P}]} g && \text{(by Def. I.11.2.9)} \\
&= p.c. \int_I g && \text{(by Def. I.11.2.14)}
\end{aligned}$$

and

$$\begin{aligned}
L(f,\mathbf{P}) &= \sum_{J\in\mathbf{P}:J\neq\emptyset}\big(\inf_{x\in J} f(x)\big)|J| && \text{(by Def. I.11.3.9)}\\
&\geq \sum_{J\in\mathbf{P}:J\neq\emptyset} c_{h|_J}|J| && \text{(by Prop. I.7.1.11(h))}\\
&= \sum_{J\in\mathbf{P}} c_{h|_J}|J| && \text{(by Prop. I.7.1.11(a)(e))}\\
&= p.c.\int_{[\mathbf{P}]} h && \text{(by Def. I.11.2.9)}\\
&= p.c.\int_I h. && \text{(by Def. I.11.2.14)}
\end{aligned}$$

□

Prop. I.11.3.12. Let $f : I \to \mathbb{R}$ be a bounded function on a bounded interval I. Then

$$\overline{\int_I} f = \inf\{U(f,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}$$

and

$$\underline{\int_I} f = \sup\{L(f,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}.$$

Proof. Let g be a function which majorizes f and which is piecewise constant with respect to some partition $\mathbf{P}_g$ of I. Let h be a function which minorizes f and which is piecewise constant with respect to some partition $\mathbf{P}_h$ of I. Both functions are well defined since f is bounded function on a bounded interval I. By Lem. I.11.3.11 we have

$$\inf\{U(f,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \leq U(f,\mathbf{P}_g) \leq p.c.\int_I g$$

and

$$\sup\{L(f,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \geq L(f,\mathbf{P}_h) \geq p.c.\int_I h.$$

Since g, h were arbitrary, by Def. I.11.3.2 we have

$$\inf\{U(f,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \leq \overline{\int_I} f$$

and

$$\sup\{L(f,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \geq \underline{\int_I} f.$$

Let $\mathbf{P}$ be a partition of I. Let $G : I \to \mathbb{R}$ be a function where $G(x) = \sup_{x\in J} f(x)$ for all $J \in \mathbf{P}$. Let $H : I \to \mathbb{R}$ be a function where $H(x) = \inf_{x\in J} f(x)$ for all $J \in \mathbf{P}$. By Def. I.11.2.3 we know that G, H are piecewise constant functions with respect to $\mathbf{P}$. Thus, we have

$$\begin{aligned}
U(f,\mathbf{P}) &= \sum_{J\in\mathbf{P}:J\neq\emptyset} \big(\sup_{x\in J} f(x)\big)|J| && \text{(by Def. I.11.3.9)}\\
&= \sum_{J\in\mathbf{P}} \big(\sup_{x\in J} f(x)\big)|J| && \text{(by Prop. I.7.1.11(a)(e))}\\
&= p.c. \int_{[\mathbf{P}]} G && \text{(by Def. I.11.2.9)}\\
&= p.c. \int_{I} G && \text{(by Def. I.11.2.14)}
\end{aligned}$$

and

$$\begin{aligned}
L(f,\mathbf{P}) &= \sum_{J\in\mathbf{P}:J\neq\emptyset} \big(\inf_{x\in J} f(x)\big)|J| && \text{(by Def. I.11.3.9)}\\
&= \sum_{J\in\mathbf{P}} \big(\inf_{x\in J} f(x)\big)|J| && \text{(by Prop. I.7.1.11(a)(e))}\\
&= p.c. \int_{[\mathbf{P}]} H && \text{(by Def. I.11.2.9)}\\
&= p.c. \int_{I} H. && \text{(by Def. I.11.2.14)}
\end{aligned}$$

By Def. I.11.3.2 we have

$$\overline{\int}_I f \leq p.c. \int_I G = U(f,\mathbf{P})$$

and

$$\underline{\int}_I f \geq p.c. \int_I H = L(f,\mathbf{P}).$$

Since $\mathbf{P}$ was arbitrary, we have

$$\overline{\int}_I f \leq \inf\{U(f,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \leq U(f,\mathbf{P})$$

and

$$\underline{\int}_I f \geq \sup\{L(f,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \leq L(f,\mathbf{P}).$$

Combine all results above we have

$$\overline{\int}_I f = \inf\{U(f,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}$$

and
$$\underline{\int}_I f = \sup\{L(f,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}.$$
$\square$

— Exercises —

Ex. I.11.3.1. Let $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, and $h : I \to \mathbb{R}$ be functions. Show that if f majorizes g and g majorizes h, then f majorizes h. Show that if f and g majorize each other, then they must be equal.

Proof. We first show that if f majorizes g and g majorizes h, then f majorizes h. Since

$$\begin{aligned} &\forall x \in I, f(x) \geq g(x) \geq h(x) && \text{(by Def. I.11.3.1)} \\ \implies &f(x) \geq h(x), \end{aligned}$$

by Def. I.11.3.1 we know that f majorize h.

Now we show that if f and g majorize each other, then they must be equal. Since

$$\begin{aligned} &\forall x \in I, \big(f(x) \geq g(x)\big) \wedge \big(g(x) \geq f(x)\big) && \text{(by Def. I.11.3.1)} \\ \implies &f(x) = g(x), \end{aligned}$$

by Def. I.3.3.7 we know that $f = g$. $\square$

Ex. I.11.3.2. Let $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, and $h : I \to \mathbb{R}$ be functions. If f majorizes g, is it true that $f + h$ majorizes $g + h$? Is it true that $f \cdot h$ majorizes $g \cdot h$? If c is a real number, is it true that cf majorizes cg?

Proof. We first show that if f majorizes g, then $f + h$ majorizes $g + h$. Since

$$\begin{aligned} &\forall x \in I, f(x) \geq g(x) && \text{(by Def. I.11.3.1)} \\ \implies &f(x) + h(x) \geq g(x) + h(x) \\ \implies &(f + h)(x) \geq (g + h)(x), && \text{(by Def. I.9.2.1)} \end{aligned}$$

by Def. I.11.3.1 we know that $f + h$ majorizes $g + h$.

Now we show that $f \cdot h$ may not majorized $g \cdot h$ and cf may not majorize cg. Let $c = h(x) = -1$. Then we have

$$\begin{aligned} &\forall x \in I, f(x) \geq g(x) && \text{(by Def. I.11.3.1)} \\ \implies &cf(x) = f(x)h(x) \leq cg(x) = g(x)h(x) \\ \implies &(cf)(x) = (f \cdot h)(x) \leq (cg)(x) = (g \cdot h)(x). && \text{(by Def. I.9.2.1)} \end{aligned}$$

In this case $f \cdot h$ does not majorized $g \cdot h$ and cf does not majorized cg. $\square$

Ex. I.11.3.3. Prove Lem. I.11.3.7.

Proof. See Lem. I.11.3.7. □

Ex. I.11.3.4. Prove Lem. I.11.3.11.

Proof. See Lem. I.11.3.11. □

Ex. I.11.3.5. Prove Prop. I.11.3.12.

Prop. I.11.3.13. See Prop. I.11.3.12.

I.11.4 Basic properties of the Riemann integral

Thm. I.11.4.1 (Laws of Riemann integration)**.** Let I be a bounded interval, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be Riemann integrable functions on I.

(a) The function $f + g$ is Riemann integrable, and we have $\displaystyle\int_I (f+g) = \int_I f + \int_I g$.

(b) For any real number c, the function cf is Riemann integrable, and we have $\displaystyle\int_I (cf) = c(\int_I f)$.

(c) The function $f - g$ is Riemann integrable, and we have $\displaystyle\int_I (f-g) = \int_I f - \int_I g$.

(d) If $f(x) \geq 0$ for all $x \in I$, then $\displaystyle\int_I f \geq 0$.

(e) If $f(x) \geq g(x)$ for all $x \in I$, then $\displaystyle\int_I f \geq \int_I g$.

(f) If f is the constant function $f(x) = c$ for all $x \in I$, then $\displaystyle\int_I f = c|I|$.

(g) Let J be a bounded interval containing I (i.e., $I \subseteq J$), and let $F : J \to \mathbb{R}$ be the function

$$F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

Then F is Riemann integrable on J, and $\displaystyle\int_J F = \int_I f$.

(h) Suppose that $\{J,K\}$ is a partition of I into two intervals J and K. Then the functions $f|_J : J \to \mathbb{R}$ and $f|_K : K \to \mathbb{R}$ are Riemann integrable on J and K respectively, and we have
$$\int_I f = \int_J f|_J + \int_K f|_K.$$

Proof. (a) Since f, g are Riemann integrable on I, by Def. I.11.3.4 we have

$$\int_I f = \overline{\int}_I f = \underline{\int}_I f$$

and

$$\int_I g = \overline{\int}_I g = \underline{\int}_I g.$$

Let $f_U : I \to \mathbb{R}$ and $g_U : I \to \mathbb{R}$ be piecewise constant functions on I which majorizes f and g, respectively. Let $f_L : I \to \mathbb{R}$ and $g_L : I \to \mathbb{R}$ be piecewise constant functions on I which minorizes f and g, respectively. f_U, g_U, f_L, g_L are well-defined since by Def. I.11.3.4 f, g are bounded functions on a bounded interval I. By Def. I.11.3.2 we have

$$p.c. \int_I f_L \le \underline{\int}_I f = \int_I f = \overline{\int}_I f \le p.c. \int_I f_U$$

and

$$p.c. \int_I g_L \le \underline{\int}_I g = \int_I g = \overline{\int}_I g \le p.c. \int_I g_U.$$

By Def. I.11.3.4 both f, g are bounded functions, so $f+g$ is bounded function, and $\underline{\int}_I (f+g), \overline{\int}_I (f+g)$ are well-defined (by Def. I.11.3.2). By Ex. I.11.3.2 we know that $f_U + g_U$ majorizes $f + g_U$ and $f + g_U$ majorizes $f+g$, thus $f_U + g_U$ majorizes $f+g$. Similarly, $f_L + g_L$ minorizes $f+g$. Then we have

$$\begin{aligned}
&\overline{\int}_I (f+g) \le p.c. \int_I (f_U + g_U) && \text{(by Def. I.11.3.2)}\\
\Longrightarrow\ &\overline{\int}_I (f+g) \le p.c. \int_I f_U + p.c. \int_I g_U && \text{(by Thm. I.11.2.16(a))}\\
\Longrightarrow\ &\overline{\int}_I (f+g) - p.c. \int_I g_U \le p.c. \int_I f_U && \text{(note that } f_U \text{ was arbitrary)}\\
\Longrightarrow\ &\overline{\int}_I (f+g) - p.c. \int_I g_U \le \overline{\int}_I f && \text{(by Def. I.11.3.2)}\\
\Longrightarrow\ &\overline{\int}_I (f+g) - \overline{\int}_I f \le p.c. \int_I g_U && \text{(note that } g_U \text{ was arbitrary)}
\end{aligned}$$

$$
\begin{aligned}
&\implies \overline{\int_I}(f+g) - \overline{\int_I} f \le \overline{\int_I} g && \text{(by Def. I.11.3.2)}\\
&\implies \overline{\int_I}(f+g) \le \overline{\int_I} f + \overline{\int_I} g &&\\
&\implies \overline{\int_I}(f+g) \le \int_I f + \int_I g && \text{(by Def. I.11.3.4)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\underline{\int_I}(f+g) \ge p.c. \int_I (f_L + g_L) && \text{(by Def. I.11.3.2)}\\
&\implies \underline{\int_I}(f+g) \ge p.c. \int_I f_L + p.c. \int_I g_L && \text{(by Thm. I.11.2.16(a))}\\
&\implies \underline{\int_I}(f+g) - p.c. \int_I g_L \ge p.c. \int_I f_L && \text{(note that } f_L \text{ was arbitrary)}\\
&\implies \underline{\int_I}(f+g) - p.c. \int_I g_L \ge \underline{\int_I} f && \text{(by Def. I.11.3.2)}\\
&\implies \underline{\int_I}(f+g) - \underline{\int_I} f \ge p.c. \int_I g_L && \text{(note that } g_L \text{ was arbitrary)}\\
&\implies \underline{\int_I}(f+g) - \underline{\int_I} f \ge \underline{\int_I} g && \text{(by Def. I.11.3.2)}\\
&\implies \underline{\int_I}(f+g) \ge \underline{\int_I} f + \underline{\int_I} g &&\\
&\implies \underline{\int_I}(f+g) \ge \int_I f + \int_I g. && \text{(by Def. I.11.3.4)}
\end{aligned}
$$

By Lem. I.11.3.3 we have

$$\int_I f + \int_I g \le \underline{\int_I}(f+g) \le \overline{\int_I}(f+g) \le \int_I f + \int_I g$$

and thus by Def. I.11.3.4 we have

$$\int_I (f+g) = \underline{\int_I}(f+g) = \overline{\int_I}(f+g) = \int_I f + \int_I g.$$

□

Proof. (b) Since f is Riemann integrable on I, by Def. I.11.3.4 we have

$$\int_I f = \overline{\int_I} f = \underline{\int_I} f.$$

First, suppose that $c = 0$. Then we have $(cf)(x) = 0$ for all $x \in 0$, thus we have

$$\begin{aligned}\int_I (cf) &= p.c. \int_I (cf) && \text{(by Lem. I.11.3.7)}\\ &= 0\\ &= c \int_I f.\end{aligned}$$

Next suppose that $c > 0$. Let $f_U : I \to \mathbb{R}$ be a piecewise constant function on I which majorizes f. Let $f_L : I \to \mathbb{R}$ be a piecewise constant function on I which minorizes f. f_U, f_L are well-defined since by Def. I.11.3.4 f is a bounded function on a bounded interval I. Then by Def. I.11.3.2 we have

$$p.c. \int_I f_L \leq \underline{\int}_I f = \int_I f = \overline{\int}_I f \leq p.c. \int_I f_U.$$

Since f is a bounded function, cf is also a bounded function, by Def. I.11.3.2 both $\overline{\int}_I (cf), \underline{\int}_I (cf)$ are well-defined. Since $c > 0$, by Def. I.11.3.1 we know that cf_U majorizes cf and cf_L minorizes cf. Then we have

$$\begin{aligned}& \overline{\int}_I (cf) \leq p.c. \int_I (cf_U) && \text{(by Def. I.11.3.2)}\\ \Longrightarrow\ & \overline{\int}_I (cf) \leq c\left(p.c. \int_I f_U\right) && \text{(by Thm. I.11.2.16(b))}\\ \Longrightarrow\ & \frac{1}{c}\left(\overline{\int}_I (cf)\right) \leq p.c. \int_I f_U && \text{(note that } f_U \text{ was arbitrary)}\\ \Longrightarrow\ & \frac{1}{c}\left(\overline{\int}_I (cf)\right) \leq \overline{\int}_I f && \text{(by Def. I.11.3.2)}\\ \Longrightarrow\ & \overline{\int}_I (cf) \leq c\left(\overline{\int}_I f\right)\\ \Longrightarrow\ & \overline{\int}_I (cf) \leq c\left(\int_I f\right) && \text{(by Def. I.11.3.4)}\end{aligned}$$

and

$$\begin{aligned}& \underline{\int}_I (cf) \geq p.c. \int_I (cf_L) && \text{(by Def. I.11.3.2)}\\ \Longrightarrow\ & \underline{\int}_I (cf) \geq c\left(p.c. \int_I f_L\right) && \text{(by Thm. I.11.2.16(b))}\\ \Longrightarrow\ & \frac{1}{c}\left(\underline{\int}_I (cf)\right) \geq p.c. \int_I f_L && \text{(note that } f_L \text{ was arbitrary)}\end{aligned}$$

$$
\begin{aligned}
&\implies \frac{1}{c}\left(\underline{\int}_I (cf)\right) \geq \underline{\int}_I f && \text{(by Def. I.11.3.2)}\\
&\implies \underline{\int}_I (cf) \geq c\left(\underline{\int}_I f\right) && \\
&\implies \underline{\int}_I (cf) \geq c\left(\int_I f\right). && \text{(by Def. I.11.3.4)}
\end{aligned}
$$

By Lem. I.11.3.3 we have

$$c\left(\int_I f\right) \leq \underline{\int}_I (cf) \leq \overline{\int}_I (cf) \leq c\left(\int_I f\right)$$

and thus by Def. I.11.3.4 we have

$$\int_I (cf) = \underline{\int}_I (cf) = \overline{\int}_I (cf) = c\left(\int_I f\right).$$

Finally suppose that $c < 0$. Using the same definition of f_U, f_L we have

$$
\begin{aligned}
&\overline{\int}_I (cf) \leq p.c. \int_I (cf_U) && \text{(by Def. I.11.3.2)}\\
&\implies \overline{\int}_I (cf) \leq c\left(p.c. \int_I f_U\right) && \text{(by Thm. I.11.2.16(b))}\\
&\implies \frac{1}{c}\left(\overline{\int}_I (cf)\right) \geq p.c. \int_I f_U && \\
&\implies \frac{1}{c}\left(\overline{\int}_I (cf)\right) \geq p.c. \int_I f_U \geq \overline{\int}_I f && \text{(by Def. I.11.3.2)}\\
&\implies \overline{\int}_I (cf) \leq c\left(\overline{\int}_I f\right) && \\
&\implies \overline{\int}_I (cf) \leq c\left(\int_I f\right) && \text{(by Def. I.11.3.4)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\underline{\int}_I (cf) \geq p.c. \int_I (cf_L) && \text{(by Def. I.11.3.2)}\\
&\implies \underline{\int}_I (cf) \geq c\left(p.c. \int_I f_L\right) && \text{(by Thm. I.11.2.16(b))}\\
&\implies \frac{1}{c}\left(\underline{\int}_I (cf)\right) \leq p.c. \int_I f_L && \\
&\implies \frac{1}{c}\left(\underline{\int}_I (cf)\right) \leq p.c. \int_I f_L \leq \underline{\int}_I f && \text{(by Def. I.11.3.2)}
\end{aligned}
$$

$$\Longrightarrow \underline{\int}_I (cf) \geq c\Big(\underline{\int}_I f\Big)$$
$$\Longrightarrow \underline{\int}_I (cf) \geq c\Big(\int_I f\Big). \qquad \text{(by Def. I.11.3.4)}$$

By Lem. I.11.3.3 we have

$$c\Big(\int_I f\Big) \leq \underline{\int}_I (cf) \leq \overline{\int}_I (cf) \leq c\Big(\int_I f\Big)$$

and thus by Def. I.11.3.4 we have

$$\int_I (cf) = \underline{\int}_I (cf) = \overline{\int}_I (cf) = c\Big(\int_I f\Big).$$

We conclude that $\forall c \in \mathbb{R}$, $\displaystyle\int_I (cf) = c(\int_I f)$. □

Proof. (c) We have

$$\begin{aligned}
\int_I f - \int_I g &= \int_I f + \int_I (-g) && \text{(by Thm. I.11.4.1(b))}\\
&= \int_I \big(f + (-g)\big) && \text{(by Thm. I.11.4.1(a))}\\
&= \int_I (f-g). && \text{(by Def. I.9.2.1)}
\end{aligned}$$

□

Proof. (d) Let $f_U : I \to \mathbb{R}$ be a piecewise constant function on I which majorizes f. f_U is well-defined since by Def. I.11.3.4 f is a bounded function on a bounded interval I. Since $0 \leq f(x) \leq f_U(x)$ for every $x \in I$, we have

$$\begin{aligned}
& 0 \leq p.c. \int_I f_U && \text{(by Thm. I.11.2.16(d))}\\
\Longrightarrow\ & 0 \leq \overline{\int}_I f && \text{(by Def. I.11.3.2)}\\
\Longrightarrow\ & 0 \leq \int_I f. && \text{(by Def. I.11.3.4)}
\end{aligned}$$

□

Proof. (e) We have $f(x) - g(x) \geq 0$ for every $x \in I$ and by Thm. I.11.4.1(c) $f - g$ is Riemann integrable on I. Thus

$$\int_I (f-g) \geq 0 \qquad \text{(by Thm. I.11.4.1(d))}$$

$$\begin{aligned} &\implies \int_I f - \int_I g \geq 0 && \text{(by Thm. I.11.4.1(c))} \\ &\implies \int_I f \geq \int_I g. \end{aligned}$$

□

Proof. (f) We have

$$\begin{aligned} \int_I f &= p.c. \int_I f && \text{(by Lem. I.11.3.7)} \\ &= c|I|. && \text{(by Thm. I.11.2.16(f))} \end{aligned}$$

□

Proof. (g) Let $f_U : I \to \mathbb{R}$ be a piecewise constant function on I which majorizes f. Let $f_L : I \to \mathbb{R}$ be a piecewise constant function on I which minorizes f. f_U, f_L are well-defined since by Def. I.11.3.4 f is a bounded function on a bounded interval I. Then by Def. I.11.3.2 we have

$$p.c. \int_I f_L \leq \underline{\int}_I f = \int_I f = \overline{\int}_I f \leq p.c. \int_I f_U.$$

Let $F_U : J \to \mathbb{R}$ be the function

$$F_U(x) = \begin{cases} f_U(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

and let $F_L : J \to \mathbb{R}$ be the function

$$F_L(x) = \begin{cases} f_L(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I. \end{cases}$$

We know that F_U majorizes F and F_L minorizes F, and by Thm. I.11.2.16(g) we have $p.c. \int_J F_U = p.c. \int_I f_U$ and $p.c. \int_J F_L = p.c. \int_I f_L$. Thus, F is a bounded function on a bounded interval I, and we have

$$\begin{aligned} &\overline{\int}_J F \leq p.c. \int_J F_U && \text{(by Def. I.11.3.2)} \\ \implies &\overline{\int}_J F \leq p.c. \int_I f_U && \text{(by Thm. I.11.2.16(g))} \\ \implies &\overline{\int}_J F \leq \overline{\int}_I f && \text{(by Def. I.11.3.2 and } f_U \text{ was arbitrary)} \\ \implies &\overline{\int}_J F \leq \int_I f && \text{(by Def. I.11.3.4)} \end{aligned}$$

and

$$\begin{aligned}
&\underline{\int}_J F \geq p.c. \int_J F_L && \text{(by Def. I.11.3.2)}\\
\Longrightarrow\ &\underline{\int}_J F \geq p.c. \int_I f_L && \text{(by Thm. I.11.2.16(g))}\\
\Longrightarrow\ &\underline{\int}_J F \geq \underline{\int}_I f && \text{(by Def. I.11.3.2 and } f_L \text{ was arbitrary)}\\
\Longrightarrow\ &\underline{\int}_J F \geq \int_I f. && \text{(by Def. I.11.3.4)}
\end{aligned}$$

By Lem. I.11.3.3 we have

$$\int_I f \leq \underline{\int}_J F \leq \overline{\int}_J F \leq \int_I f$$

and thus by Def. I.11.3.4 we have

$$\int_J F = \underline{\int}_J F = \overline{\int}_J F = \int_I f.$$

□

Proof. (h) Let $f_U : I \to \mathbb{R}$ be a piecewise constant function on I which majorizes f. Let $f_L : I \to \mathbb{R}$ be a piecewise constant function on I which minorizes f. f_U, f_L are well-defined since by Def. I.11.3.4 f is a bounded function on a bounded interval I. Then by Def. I.11.3.2 we have

$$p.c. \int_I f_L \leq \underline{\int}_I f = \int_I f = \overline{\int}_I f \leq p.c. \int_I f_U.$$

By Thm. I.11.2.16(h) we know that $f_U|_J : J \to \mathbb{R}, f_L|_J : J \to \mathbb{R}$ are piecewise constant function on J and $f_U|_K : K \to \mathbb{R}$, $f_L|_K : K \to \mathbb{R}$ are piecewise constant functions on K. By Def. I.11.3.1 we know that $f_U|_J$ majorizes $f|_J$ and $f_L|_J$ minorizes $f|_J$, similarly $f_U|_K$ majorizes $f|_K$ and $f_L|_K$ minorizes $f|_K$. Thus, $f|_J, f|_K$ are bounded functions on bounded intervals J, K, respectively. So $\overline{\int}_J f|_J$, $\overline{\int}_K f|_K$, $\underline{\int}_J f|_J$, $\underline{\int}_K f|_K$ are well-defined. Then we have

$$\begin{aligned}
&\overline{\int}_J f|_J + \overline{\int}_K f|_K \leq p.c. \int_J f_U|_J + p.c. \int_K f_U|_K && \text{(by Def. I.11.3.2)}\\
\Longrightarrow\ &\overline{\int}_J f|_J + \overline{\int}_K f|_K \leq p.c. \int_I f_U && \text{(by Thm. I.11.2.16(h))}\\
\Longrightarrow\ &\overline{\int}_J f|_J + \overline{\int}_K f|_K \leq \overline{\int}_I f && \text{(by Def. I.11.3.2)}
\end{aligned}$$

$$\Longrightarrow \overline{\int}_J f|_J + \overline{\int}_K f|_K \leq \int_I f \qquad \text{(by Def. I.11.3.4)}$$

and

$$\begin{aligned}
&\underline{\int}_J f|_J + \underline{\int}_K f|_K \geq p.c. \int_J f_L|_J + p.c. \int_K f_L|_K && \text{(by Def. I.11.3.2)}\\
\Longrightarrow\ &\underline{\int}_J f|_J + \underline{\int}_K f|_K \geq p.c. \int_I f_L && \text{(by Thm. I.11.2.16(h))}\\
\Longrightarrow\ &\underline{\int}_J f|_J + \underline{\int}_K f|_K \geq \underline{\int}_I f && \text{(by Def. I.11.3.2)}\\
\Longrightarrow\ &\underline{\int}_J f|_J + \underline{\int}_K f|_K \geq \int_I f. && \text{(by Def. I.11.3.4)}
\end{aligned}$$

By Lem. I.11.3.3 we have

$$\int_I f \leq \underline{\int}_J f|_J + \underline{\int}_K f|_K \leq \overline{\int}_J f|_J + \overline{\int}_K f|_K \leq \int_I f$$

and thus we have

$$\underline{\int}_J f|_J + \underline{\int}_K f|_K = \overline{\int}_J f|_J + \overline{\int}_J f|_K = \int_I f.$$

Since

$$\begin{aligned}
&\underline{\int}_J f|_J + \underline{\int}_K f|_K = \overline{\int}_J f|_J + \overline{\int}_J f|_K\\
\Longrightarrow\ &0 \geq \underline{\int}_J f|_J - \overline{\int}_J f|_J = \overline{\int}_J f|_K - \underline{\int}_K f|_K \geq 0 && \text{(by Lem. I.11.3.3)}\\
\Longrightarrow\ &\underline{\int}_J f|_J - \overline{\int}_J f|_J = \overline{\int}_J f|_K - \underline{\int}_K f|_K = 0,
\end{aligned}$$

by Def. I.11.3.4 we have

$$\begin{aligned}
&\int_J f|_J = \underline{\int}_J f|_J = \overline{\int}_J f|_J,\\
&\int_K f|_K = \underline{\int}_K f|_K = \overline{\int}_K f|_K,\\
&\int_J f|_J + \int_K f|_K = \int_I f.
\end{aligned}$$

$\square$

Rmk. I.11.4.2. We often abbreviate $\displaystyle\int_J f|_J$ as $\displaystyle\int_J f$ even though f is really defined on a larger domain than just J. We also observe from Thm. I.11.4.1(h) and Rmk. I.11.3.8 that if $f : [a,b] \to \mathbb{R}$ is Riemann integrable on a closed interval $[a,b]$, then $\displaystyle\int_{[a,b]} f = \int_{(a,b]} f = \int_{[a,b)} f = \int_{(a,b)} f$.

Thm. I.11.4.3. and min preserve integrability] Let I be a bounded interval, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be a Riemann integrable function. Then the functions $\max(f,g) : I \to \mathbb{R}$ and $\min(f,g) : I \to \mathbb{R}$ defined by $\max(f,g)(x) := \max\big(f(x), g(x)\big)$ and $\min(f,g)(x) := \min\big(f(x), g(x)\big)$ are also Riemann integrable.

Proof. We shall just prove the claim for $\max(f,g)$, the case of $\min(f,g)$ being similar. First, note that since f and g are bounded, then $\max(f,g)$ is also bounded.

Let $\varepsilon > 0$. Since $\displaystyle\int_I f = \underline{\int}_I f$, there exists a piecewise constant function $\underline{f} : I \to \mathbb{R}$ which minorizes f on I such that

$$\int_I \underline{f} \geq \int_I f - \varepsilon.$$

Similarly, we can find a piecewise constant $g : I \to \mathbb{R}$ which minorizes g on I such that

$$\int_I \underline{g} \geq \int_I g - \varepsilon,$$

and we can find piecewise functions $\overline{f}, \overline{g}$ which majorize f, g respectively on I such that

$$\int_I \overline{f} \leq \int_I f + \varepsilon$$

and

$$\int_I \overline{g} \leq \int_I g + \varepsilon.$$

In particular, if $h : I \to \mathbb{R}$ denotes the function

$$h := (\overline{f} - \underline{f}) + (\overline{g} - \underline{g})$$

we have

$$\int_I h \leq 4\varepsilon.$$

On the other hand, $\max(\underline{f}, \underline{g})$ is a piecewise constant function on I which minorizes $\max(f,g)$, while $\max(\overline{f}, \overline{g})$ is similarly a piecewise constant function on I which majorizes $\max(f,g)$. Thus

$$\int_I \max(\underline{f}, \underline{g}) \leq \underline{\int}_I \max(f,g) \leq \overline{\int}_I \max(f,g) \leq \int_I \max(\overline{f}, \overline{g}),$$

and so

$$0 \leq \overline{\int}_I \max(f,g) - \underline{\int}_I \max(f,g) \leq \int_I \max(\overline{f},\overline{g}) - \max(\underline{f},\underline{g}).$$

But we have

$$\overline{f}(x) = \underline{f}(x) + (\overline{f} - \underline{f})(x) \leq \underline{f}(x) + h(x)$$

and similarly

$$\overline{g}(x) = \underline{g}(x) + (\overline{g} - \underline{g})(x) \leq \underline{g}(x) + h(x)$$

and thus

$$\max\left(\overline{f}(x), \overline{g}(x)\right) \leq \max\left(\underline{f}(x), \underline{g}(x)\right) + h(x).$$

Inserting this into the previous inequality, we obtain

$$0 \leq \overline{\int}_I \max(f,g) - \underline{\int}_I \max(f,g) \leq \int_I h \leq 4\varepsilon.$$

To summarize, we have shown that

$$0 \leq \overline{\int}_I \max(f,g) - \underline{\int}_I \max(f,g) \leq 4\varepsilon$$

for every ε. Since $\overline{\int}_I \max(f,g) - \underline{\int}_I \max(f,g)$ does not depend on ε, we thus see that

$$\overline{\int}_I \max(f,g) - \underline{\int}_I \max(f,g) = 0$$

and hence that $\max(f,g)$ is Riemann integrable. □

Cor. I.11.4.4 (Absolute values preserve Riemann integrability)**.** Let I be a bounded interval. If $f : I \to \mathbb{R}$ is a Riemann integrable function, then the positive part $f_+ := \max(f,0)$ and the negative part $f_- := \min(f,0)$ are also Riemann integrable on I. Also, the absolute value $|f|$, defined by $|f|(x) = |f(x)|$ is also Riemann integrable on I. (observe that $|f| = f_+ - f_-$)

Proof. By Thm. I.11.4.3 we know that f_+, f_- are Riemann integrable. Since $|f| = f_+ - f_-$, by Thm. I.11.4.1(a) we know that $|f|$ is Riemann integrable. □

Thm. I.11.4.5 (products preserve Riemann integrability)**.** Let I be a bounded interval. If $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are Riemann integrable, then $fg : I \to \mathbb{R}$ is also Riemann integrable.

Proof. We split $f = f_+ + f_-$ and $g = g_+ + g_-$ into positive and negative parts; by Cor. I.11.4.4, the functions f_+, f_-, g_+, g_- are Riemann integrable. Since

$$fg = f_+g_+ + f_+g_- + f_-g_+ + f_-g_-$$

then it suffices to show that the functions f_+g_+, f_+g_-, f_-g_+, f_-g_- are individually Riemann integrable. We will just show this for f_+g_+; the other three are similar.

Since f_+ and g_+ are bounded and positive, there are $M_1, M_2 > 0$ such that

$$0 \leq f_+(x) \leq M_1 \text{ and } 0 \leq g_+(x) \leq M_2$$

for all $x \in I$. Now let $\varepsilon > 0$ be arbitrary. Then, as in the proof of Thm. I.11.4.3, we can find a piecewise constant function $\underline{f_+}$ minorizing f_+ on I, and a piecewise constant function $\overline{f_+}$ majorizing f_+ on I, such that

$$\int_I \overline{f_+} \leq \int_I f_+ + \varepsilon$$

and

$$\int_I \underline{f_+} \geq \int_I f_+ - \varepsilon.$$

Note that $\underline{f_+}$ may be negative at places, but we can fix this by replacing $\underline{f_+}$ by $\max(\underline{f_+}, 0)$, since this still minorizes f_+ and still has integral greater than or equal to $\int_I f_+ - \varepsilon$. So without loss of generality we may assume that $\underline{f_+}(x) \geq 0$ for all $x \in I$. Similarly, we may assume that $\overline{f_+}(x) \leq M_1$ for all $x \in I$; thus

$$0 \leq \underline{f_+}(x) \leq f_+(x) \leq \overline{f_+}(x) \leq M_1$$

for all $x \in I$.

Similar reasoning allows us to find piecewise constant $\underline{g_+}$ minorizing g_+, and $\overline{g_+}$ majorizing g_+, such that

$$\int_I \overline{g_+} \leq \int_I g_+ + \varepsilon$$

and

$$\int_I \underline{g_+} \geq \int_I g_+ - \varepsilon,$$

and

$$0 \leq \underline{g_+}(x) \leq g_+(x) \leq \overline{g_+}(x) \leq M_2$$

for all $x \in I$.

Notice that $\underline{f_+ g_+}$ is piecewise constant and minorizes f_+g_+, while $\overline{f_+}\overline{g_+}$ is piecewise constant and majorizes f_+g_+. Thus

$$0 \leq \overline{\int}_I f_+g_+ - \underline{\int}_I f_+g_+ \leq \int_I \overline{f}_+\overline{g_+} - \underline{f_+g_+}.$$

However, we have

$$\begin{aligned}\overline{f_+}(x)\overline{g_+}(x) - \underline{f_+}(x)\underline{g_+}(x) &= \overline{f_+}(x)(\overline{g_+} - \underline{g_+})(x) + \underline{g_+}(x)(\overline{f_+} - \underline{f_+})(x) \\ &\leq M_1(\overline{g_+} - \underline{g_+})(x) + M_2(\overline{f_+} - \underline{f_+})(x)\end{aligned}$$

for all $x \in I$, and thus

$$\begin{aligned}0 \leq \overline{\int}_I f_+g_+ - \underline{\int}_I f_+g_+ &\leq M_1 \int_I (\overline{g_+} - \underline{g_+}) + M_2 \int_I (\overline{f_+} - \underline{f_+}) \\ &\leq M_1(2\varepsilon) + M_2(2\varepsilon).\end{aligned}$$

Again, since ε was arbitrary, we can conclude that f_+g_+ is Riemann integrable, as before. Similar arguments show that f_+g_-, f_-g_+, f_-g_- are Riemann integrable; combining them we obtain that fg is Riemann integrable. □

— Exercises —

Ex. I.11.4.1. Prove Thm. I.11.4.1.

Proof. See Thm. I.11.4.1. □

Ex. I.11.4.2. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous, non-negative function (so $f(x) \geq 0$ for all $x \in [a, b]$). Suppose that $\displaystyle\int_{[a,b]} f = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

Proof. Suppose for the sake of contradiction that $\exists x_0 \in [a, b]$ such that $f(x_0) > 0$. Since f is continuous, by Prop. I.9.4.7 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in [a, b], |x - x_0| < \delta \implies |f(x) - f(x_0)| \leq \varepsilon\big),$$

or equivalently

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in [a, b] \cap (x_0 - \delta, x_0 + \delta) \implies |f(x) - f(x_0)| \leq \varepsilon\big).$$

In particular, we have

$$\exists \delta \in \mathbb{R}^+ : \left(\forall x \in [a, b] \cap (x_0 - \delta, x_0 + \delta) \implies |f(x) - f(x_0)| \leq \frac{f(x_0)}{2}\right),$$

or equivalently

$$\exists \delta \in \mathbb{R}^+ : \left(\forall x \in [a, b] \cap (x_0 - \delta, x_0 + \delta) \implies \frac{f(x_0)}{2} \leq f(x) \leq \frac{3f(x_0)}{2}\right).$$

Since $\delta \neq 0$, we know that $[a,b] \cap (x_0-\delta, x_0+\delta) \neq \emptyset$. Since $a \neq b$, we know that

$$\sup\big([a,b] \cap (x_0-\delta, x_0+\delta)\big) \neq \inf\big([a,b] \cap (x_0-\delta, x_0+\delta)\big).$$

Thus, by Def. I.11.1.8 we have $|[a,b] \cap (x_0-\delta, x_0+\delta)| > 0$. By Cor. I.11.1.6 we know that $[a,b] \cap (x_0-\delta, x_0+\delta)$ is a bounded interval. Let $f_L : [a,b] \to \mathbb{R}$ be the function

$$f_L(x) = \begin{cases} \dfrac{f(x_0)}{2} & \text{if } x \in [a,b] \cap (x_0-\delta, x_0+\delta) \\ 0 & \text{if } x \notin [a,b] \cap (x_0-\delta, x_0+\delta) \end{cases}$$

Since $f(x) \geq 0$ for all $x \in [a,b]$, we know that f_L minorizes f. By Thm. I.11.2.16(g) we know that f_L is a piecewise constant function. By Lem. I.11.3.7 we have

$$\int_{[a,b]} f_L = p.c. \int_{[a,b]} f_L = \frac{f(x_0)}{2} |[a,b] \cap (x_0-\delta, x_0+\delta)| > 0.$$

But by Def. I.11.3.2 and Def. I.11.3.4 we have

$$0 < \int_{[a,b]} f_L \leq \underline{\int}_{[a,b]} f = \int_{[a,b]} f = 0,$$

a contradiction. Thus, we must have $f(x) = 0$ for all $x \in [a,b]$. $\square$

Ex. I.11.4.3. Let I be a bounded interval, let $f : I \to \mathbb{R}$ be a Riemann integrable function, and let $\mathbf{P}$ be a partition of I. Show that

$$\int_I f = \sum_{J \in \mathbf{P}} \int_J f|_J.$$

Proof. Let $P(n)$ be the statement "$\#(\mathbf{P}) = n$ and $\displaystyle\int_I f = \sum_{J \in \mathbf{P}} \int_J f|_J$." We induct on n to show that $P(n)$ is true $\forall n \in \mathbb{N}$. For $n = 0$, we have $\mathbf{P} = \emptyset$ and $I = \emptyset$. Thus

$$\begin{aligned}
p.c. \int_{[\emptyset]} f &= \sum_{J \in \emptyset} c_J |J| && \text{(by Def. I.11.2.9)} \\
&= 0 && \text{(by Prop. I.7.1.11(a))} \\
&= p.c. \int_{\emptyset} f && \text{(by Def. I.11.2.14)} \\
&= \int_{\emptyset} f && \text{(by Lem. I.11.3.7)} \\
&= \sum_{J \in \emptyset} \int_J f|_J && \text{(by Prop. I.7.1.11(a))}
\end{aligned}$$

and the base case holds. Suppose inductively that $P(n)$ is true for some $n \geq 0$. Then we need to show that $P(n+1)$ is true. Let $K \in \mathbf{P}$ such that $x < y$ for every $x \in K$ and $y \in I \setminus K$. Then $\left\{K, \bigcup(\mathbf{P} \setminus \{K\})\right\}$ is a partition of I, and

$$\begin{aligned}
\int_I f &= \int_K f|_K + \int_{\bigcup(\mathbf{P}\setminus\{K\})} f|_{\bigcup(\mathbf{P}\setminus\{K\})} && \text{(by Thm. I.11.4.1(h))} \\
&= \int_K f|_K + \sum_{J\in\mathbf{P}\setminus\{K\}} \int_J f|_J && \text{(by the induction hypothesis)} \\
&= \sum_{J\in\mathbf{P}} \int_J f|_J. && \text{(by Prop. I.7.1.11(e))}
\end{aligned}$$

This closes the induction. □

Ex. I.11.4.4. Without repeating all the computations in the above proofs, give a short explanation as to why the remaining cases of Thm. I.11.4.3 and Thm. I.11.4.5 follow automatically from the cases presented in the text.

Proof. We first show that the remaining case of Thm. I.11.4.3 is true. By Thm. I.11.4.1(b) $-f$ and $-g$ are Riemann integrable on I. Since $\max(-f,-g)$ is Riemann integrable and $\min(f,g) = -\max(-f,-g)$, by Thm. I.11.4.1(b) we know that $\min(f,g)$ is Riemann integrable.

Now we show that the remaining cases of Thm. I.11.4.5 are true. By Cor. I.11.4.4 $(-f)_+$ and $(-g)_+$ are Riemann integrable on I. Since for any Riemann integrable functions p and q, p_+q_+ are Riemann integrable (which is showed in the proof of Thm. I.11.4.5), we have

$$\begin{aligned}
f_+g_- &= f_+ \cdot \big(\min(g,0)\big) && \text{(by Cor. I.11.4.4)} \\
&= f_+ \cdot \big(-\max(-g,0)\big) \\
&= f_+ \cdot \big(-(-g)_+\big) && \text{(by Cor. I.11.4.4)} \\
&= -\big(f_+ \cdot (-g)_+\big) && \text{(by Def. I.9.2.1)} \\
f_-g_+ &= \big(\min(f,0)\big) \cdot g_+ && \text{(by Cor. I.11.4.4)} \\
&= \big(-\max(-f,0)\big) \cdot g_+ \\
&= \big(-(-f)_+\big) \cdot g_+ && \text{(by Cor. I.11.4.4)} \\
&= -\big((-f)_+ \cdot g_+\big) && \text{(by Def. I.9.2.1)} \\
f_-g_- &= \big(\min(f,0)\big) \cdot \big(\min(g,0)\big) && \text{(by Cor. I.11.4.4)} \\
&= \big(-\max(-f,0)\big) \cdot \big(-\max(-g,0)\big) \\
&= \big(-(-f)_+\big)\big(-(-g)_+\big) && \text{(by Cor. I.11.4.4)} \\
&= (-f)_+ \cdot (-g)_+ && \text{(by Def. I.9.2.1)}
\end{aligned}$$

and thus f_+g_-, f_-g_+, f_-g_- are Riemann integrable. □

I.11.5 Riemann integrability of continuous functions

Thm. I.11.5.1. Let I be a bounded interval, and let f be a function which is uniformly continuous on I. Then f is Riemann integrable.

Proof. From Prop. I.9.9.15 we see that f is bounded. Now we have to show that $\underline{\int}_I f = \overline{\int}_I f$.

If I is a point or the empty set then the theorem is trivial, so let us assume that I is one of the four intervals $[a, b]$, (a, b), $(a, b]$, or $[a, b)$ for some real numbers $a < b$.

Let $\varepsilon > 0$ be arbitrary. By uniform continuity, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in I$ are such that $|x - y| < \delta$. By the Archimedean principle, there exists an integer $N > 0$ such that $(b - a)/N < \delta$.

Note that we can partition I into N intervals $J_1, \ldots, J_N$, each of length $(b - a)/N$. (How? One has to treat each of the cases $[a, b], (a, b), (a, b], [a, b)$ slightly differently.) By Prop. I.11.3.12, we thus have

$$\overline{\int}_I f \leq \sum_{k=1}^{N} \big(\sup_{x \in J_k} f(x) \big) |J_k|$$

and

$$\underline{\int}_I f \geq \sum_{k=1}^{N} \big(\inf_{x \in J_k} f(x) \big) |J_k|$$

so in particular

$$\overline{\int}_I f - \underline{\int}_I f \leq \sum_{k=1}^{N} \big(\sup_{x \in J_k} f(x) - \inf_{x \in J_k} f(x) \big) |J_k|.$$

However, we have $|f(x) - f(y)| < \varepsilon$ for all $x, y \in J_k$, since $|J_k| = (b - a)/N < \delta$. In particular, we have

$$f(x) < f(y) + \varepsilon \text{ for all } x, y \in J_k.$$

Taking suprema in x, we obtain

$$\sup_{x \in J_k} f(x) \leq f(y) + \varepsilon \text{ for all } y \in J_k,$$

and then taking infima in y we obtain

$$\sup_{x \in J_k} f(x) \leq \inf_{y \in J_k} f(y) + \varepsilon.$$

Inserting this bound into our previous inequality, we obtain

$$\overline{\int}_I f - \underline{\int}_I f \leq \sum_{k=1}^{N} \varepsilon |J_k|,$$

but by Thm. I.11.1.13 we thus have

$$\overline{\int_I} f - \underline{\int_I} f \le \varepsilon(b-a).$$

But $\varepsilon > 0$ was arbitrary, while $(b-a)$ is fixed. Thus, $\overline{\int_I} f - \underline{\int_I} f$ cannot be positive. By Lem. I.11.3.3 and the definition of Riemann integrability we thus have that f is Riemann integrable. □

Cor. I.11.5.2. Let $[a, b]$ be a closed interval, and let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable.

Proof. Combining Thm. I.11.5.1 with Thm. I.9.9.16 we are done. □

Note. Note that Cor. I.11.5.2 is not true if $[a, b]$ is replaced by any other sort of interval, since it is not even guaranteed then that continuous functions are bounded. For instance, the function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) := 1/x$ is continuous but not Riemann integrable. However, if we assume that a function is both continuous *and* bounded, we can recover Riemann integrability (see Prop. I.11.5.3).

Prop. I.11.5.3. Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be both continuous and bounded. Then f is Riemann integrable on I.

Proof. If I is a point or an empty set then the claim is trivial; if I is a closed interval the claim follows from Cor. I.11.5.2. So let us assume that I is of the form $(a, b]$, (a, b), or $[a, b)$ for some $a < b$.

We have a bound M for f, so that $-M \le f(x) \le M$ for all $x \in I$. Now let $0 < \varepsilon < (b-a)/2$ be a small number. The function f when restricted to the interval $[a+\varepsilon, b-\varepsilon]$ is continuous, and hence Riemann integrable by Cor. I.11.5.2. In particular, we can find a piecewise constant function $h : [a+\varepsilon, b-\varepsilon] \to \mathbb{R}$ which majorizes f on $[a+\varepsilon, b-\varepsilon]$ such that

$$\int_{[a+\varepsilon,b-\varepsilon]} h \le \int_{[a+\varepsilon,b-\varepsilon]} f + \varepsilon.$$

Define $\tilde{h} : I \to \mathbb{R}$ by

$$\tilde{h}(x) := \begin{cases} h(x) & \text{if } x \in [a+\varepsilon, b-\varepsilon] \\ M & \text{if } x \in I \setminus [a+\varepsilon, b-\varepsilon] \end{cases}$$

Clearly, $\tilde{h}$ is piecewise constant on I and majorizes f; by Thm. I.11.2.16 we have

$$\int_I \tilde{h} = \varepsilon M + \int_{[a+\varepsilon,b-\varepsilon]} h + \varepsilon M \le \int_{[a+\varepsilon,b-\varepsilon]} f + (2M+1)\varepsilon.$$

In particular, we have

$$\overline{\int}_I f \leq \int_{[a+\varepsilon,b-\varepsilon]} f + (2M+1)\varepsilon.$$

This is true since $\tilde{h}$ majorize f. A similar argument gives

$$\underline{\int}_I f \geq \int_{[a+\varepsilon,b-\varepsilon]} f - (2M+1)\varepsilon.$$

and hence

$$\overline{\int}_I f - \underline{\int}_I f \leq (4M+2)\varepsilon.$$

But ε was arbitrary, and so we can argue as in the proof of Thm. I.11.5.1 to conclude Riemann integrability. □

Note. From Thm. I.11.5.1, Cor. I.11.5.2 and Prop. I.11.5.3 we see that if we can show a function f being *uniformly continuous* (not just continuous) on some bounded interval I, then f is Riemann integrable on I.

Def. I.11.5.4. Let I be a bounded interval, and let $f : I \to \mathbb{R}$. We say that f is *piecewise continuous on* I iff there exists a partition $\mathbf{P}$ of I such that $f|_J$ is continuous on J for all $J \in \mathbf{P}$.

Prop. I.11.5.6. Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be both piecewise continuous and bounded. Then f is Riemann integrable.

Proof. Since f is piecewise continuous on I, by Def. I.11.5.4 $\exists \mathbf{P}$ such that $\mathbf{P}$ is a partition of I and $f|_J$ is continuous on J for all $J \in \mathbf{P}$. Since f is bounded, we know that $f|_J$ is bounded for all $J \in \mathbf{P}$. Thus, by Prop. I.11.5.3 $f|_J$ is Riemann integrable on J for all $J \in \mathbf{P}$. For each $J \in \mathbf{P}$, we define $F_J : I \to \mathbb{R}$ to be the function

$$F_J(x) = \begin{cases} f|_J(x) & \text{if } x \in J \\ 0 & \text{if } x \notin J \end{cases}$$

Then by Thm. I.11.4.1(g) $F|_J$ is Riemann integrable for all $J \in \mathbf{P}$ and

$$\begin{aligned} \sum_{J \in \mathbf{P}} \int_I F_J &= \sum_{J \in \mathbf{P}} \int_J f|_J && \text{(by Thm. I.11.4.1(g))} \\ &= \int_I f. && \text{(by Ex. I.11.4.3)} \end{aligned}$$

Thus, f is Riemann integrable on I. □

— Exercises —

Ex. I.11.5.1. Prove Prop. I.11.5.6.

Proof. See Prop. I.11.5.6. □

I.11.6 Riemann integrability of monotone functions

Prop. I.11.6.1. Let $[a,b]$ be a closed and bounded interval and let $f : [a,b] \to \mathbb{R}$ be a monotone function. Then f is Riemann integrable on $[a,b]$.

Proof. Without loss of generality we may take f to be monotone increasing (instead of monotone decreasing). From Ex. I.9.8.1 we know that f is bounded. Now let $N > 0$ be an integer, and partition $[a,b]$ into N half-open intervals

$$\left\{ \left[a + \frac{b-a}{N}j, a + \frac{b-a}{N}(j+1)\right) : 0 \le j \le N-1 \right\}$$

of length $(b-a)/N$, together with the point $\{b\}$. Then by Prop. I.11.3.12 we have

$$\overline{\int}_I f \le \sum_{j=0}^{N-1} \left(\sup_{x \in \left[a + \frac{b-a}{N}j, a + \frac{b-a}{N}(j+1)\right)} f(x) \right) \frac{b-a}{N},$$

(the point $\{b\}$ clearly giving only a zero contribution). Since f is monotone increasing, we thus have

$$\overline{\int}_I f \le \sum_{j=0}^{N-1} f\left(a + \frac{b-a}{N}(j+1)\right) \frac{b-a}{N}.$$

Similarly, we have

$$\underline{\int}_I f \ge \sum_{j=0}^{N-1} f\left(a + \frac{b-a}{N}j\right) \frac{b-a}{N}.$$

Thus, we have

$$\overline{\int}_I f - \underline{\int}_I f \le \sum_{j=0}^{N-1} \left(f\left(a + \frac{b-a}{N}(j+1)\right) - f\left(a + \frac{b-a}{N}j\right)\right) \frac{b-a}{N}.$$

Using telescoping series (Lem. I.7.2.15) we thus have

$$\begin{aligned}\overline{\int}_I f - \underline{\int}_I f &\le \left(f\left(a + \frac{b-a}{N}N\right) - f\left(a + \frac{b-a}{N}0\right)\right) \frac{b-a}{N} \\ &= \big(f(b) - f(a)\big)\frac{b-a}{N}.\end{aligned}$$

But N was arbitrary, so we can conclude as in the proof of Thm. I.11.5.1 that f is Riemann integrable. □

Rmk. I.11.6.2. From Ex. I.9.8.5 we know that there exist monotone functions which are not piecewise continuous, so Prop. I.11.6.1 is not subsumed by Prop. I.11.5.6.

Cor. I.11.6.3. Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be both monotone and bounded. Then f is Riemann integrable on I.

Proof. Without loss of generality we may take f to be monotone increasing (instead of monotone decreasing). If I is a point or an empty set then the claim is trivial; if I is a closed interval the claim follows from Prop. I.11.6.1. So let us assume that I is of the form $(a, b]$, (a, b), or $[a, b)$ for some $a < b$.

We have a bound M for f, so that $-M \leq f(x) \leq M$ for all $x \in I$. Now let $0 < \varepsilon < (b-a)/2$ be a small number. The function f when restricted to the interval $[a+\varepsilon, b-\varepsilon]$ is monotone, and hence Riemann integrable by Prop. I.11.6.1. In particular, we can find a piecewise constant function $h : [a+\varepsilon, b-\varepsilon] \to \mathbb{R}$ which majorizes f on $[a+\varepsilon, b-\varepsilon]$ such that

$$\int_{[a+\varepsilon,b-\varepsilon]} h \leq \int_{[a+\varepsilon,b-\varepsilon]} f + \varepsilon.$$

Define $\tilde{h} : I \to \mathbb{R}$ by

$$\tilde{h}(x) := \begin{cases} h(x) & \text{if } x \in [a+\varepsilon, b-\varepsilon] \\ M & \text{if } x \in I \setminus [a+\varepsilon, b-\varepsilon] \end{cases}$$

Clearly, $\tilde{h}$ is piecewise constant on I and majorizes f; by Thm. I.11.2.16 we have

$$\int_I \tilde{h} = \varepsilon M + \int_{[a+\varepsilon,b-\varepsilon]} h + \varepsilon M \leq \int_{[a+\varepsilon,b-\varepsilon]} f + (2M+1)\varepsilon.$$

In particular, we have

$$\overline{\int}_I f \leq \int_{[a+\varepsilon,b-\varepsilon]} f + (2M+1)\varepsilon$$

This is true since $\tilde{h}$ majorize f. A similar argument gives

$$\underline{\int}_I f \geq \int_{[a+\varepsilon,b-\varepsilon]} f - (2M+1)\varepsilon.$$

and hence

$$\overline{\int}_I f - \underline{\int}_I f \leq (4M+2)\varepsilon.$$

But ε was arbitrary, and so we can argue as in the proof of Thm. I.11.5.1 to conclude Riemann integrability. $\square$

Prop. I.11.6.4 (Integral test)**.** Let $f : [0, \infty) \to \mathbb{R}$ be a monotone decreasing function which is non-negative (i.e., $f(x) \geq 0$ for all $x \geq 0$). Then the sum $\sum_{n=0}^{\infty} f(n)$ is convergent iff $\sup_{N>0} \int_{[0,N]} f$ is finite.

Proof. Let $N \in \mathbb{Z}^+$. Since f is monotone decreasing, by Prop. I.11.6.1 we know that f is Riemann integrable on both $[0, N]$ and every interval $[a, b] \subseteq [0, N]$. Then we have

$$\begin{aligned}
\int_{[0,N]} f &= \sum_{n=0}^{N-1} \int_{[n,n+1)} f|_{[n,n+1)} + \int_{[N,N]} f|_{[N,N]} && \text{(by Ex. I.11.4.3)} \\
&= \sum_{n=0}^{N-1} \int_{[n,n+1)} f|_{[n,n+1)} && \text{(by Def. I.11.1.8)} \\
&\leq \sum_{n=0}^{N-1} \int_{[n,n+1)} f(n) && \text{(by Thm. I.11.4.1(e))} \\
&= \sum_{n=0}^{N-1} f(n)|n+1-n| && \text{(by Def. I.11.2.9)} \\
&= \sum_{n=0}^{N-1} f(n) \\
&\leq \sum_{n=0}^{N} f(n) && (\forall x \in [0,\infty), f(x) \geq 0)
\end{aligned}$$

and

$$\begin{aligned}
\int_{[0,N]} f &= \sum_{n=0}^{N-1} \int_{[n,n+1)} f|_{[n,n+1)} + \int_{[N,N]} f|_{[N,N]} && \text{(by Ex. I.11.4.3)} \\
&= \sum_{n=0}^{N-1} \int_{[n,n+1)} f|_{[n,n+1)} && \text{(by Def. I.11.1.8)} \\
&\geq \sum_{n=0}^{N-1} \int_{[n,n+1)} f(n+1) && \text{(by Thm. I.11.4.1(e))} \\
&= \sum_{n=0}^{N-1} f(n+1)|n+1-n| && \text{(by Def. I.11.2.9)} \\
&= \sum_{n=0}^{N-1} f(n+1) \\
&= \sum_{n=1}^{N} f(n). && \text{(by Lem. I.7.1.4(b))}
\end{aligned}$$

Next we show that if $\sum_{n=0}^{\infty} f(n)$ is convergent, then $\sup_{N>0} \int_{[0,N]} f$ is finite. Suppose that

$\sum_{n=0}^{\infty} f(n)$ is convergent. Then by Def. I.7.2.2 we know that

$$\sum_{n=0}^{\infty} f(n) = \lim_{m\to\infty} \sum_{n=0}^{m} f(n)$$

and by Prop. I.6.1.12 $\big(\sum_{n=0}^{m} f(n)\big)_{m=0}^{\infty}$ is a Cauchy sequence. By Lem. I.5.1.15 we know that $\big(\sum_{n=0}^{m} f(n)\big)_{m=0}^{\infty}$ is bounded by some $M \in \mathbb{R}$. By comparison principle (Lem. I.6.4.13) we have

$$\int_{[0,N]} f \leq \sum_{n=0}^{N} f(n) \implies \sup\left(\int_{[0,N]} f\right)_{N=1}^{\infty} \leq \sup\left(\sum_{n=0}^{N} f(n)\right)_{N=1}^{\infty} \leq M$$

and thus $\sup_{N>0} \int_{[0,N]} f$ is finite.

Now we show that if $\sup_{N>0} \int_{[0,N]} f$ is finite, then $\sum_{n=0}^{\infty} f(n)$ is convergent. Suppose that $\sup_{N>0} \int_{[0,N]} f$ is finite. By comparison principle (Lem. I.6.4.13) we have

$$\sum_{n=1}^{N} f(n) \leq \int_{[0,N]} f \implies \sup\left(\sum_{n=1}^{N} f(n)\right)_{N=1}^{\infty} \leq \sup\left(\int_{[0,N]} f\right)_{N=1}^{\infty}$$

Thus, by Prop. I.7.3.1 $\sum_{n=0}^{\infty} f(n)$ is convergent. □

Cor. I.11.6.5. Let p be a real number. Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges absolutely when $p > 1$ and diverges when $p \leq 1$.

Proof. Let $f : [1, \infty) \to \mathbb{R}$ be the function $f(x) = \frac{1}{x^p}$. By Prop. I.6.7.3(a)(d) we know that f is positive and

$$\begin{cases} f \text{ is monotone decreasing if } p > 1; \\ f \text{ is monotone increasing if } p < 1; \\ f \text{ is both monotone increasing and decreasing if } p = 1. \end{cases}$$

By Prop. I.11.6.1 f is Riemann integrable on $[1, N]$ for every $N \in \mathbb{R}$ and $N \geq 1$. If $p \neq 1$, then we have

$$\int_{[1,N]} f = \frac{1}{1-p}(N^{1-p} - 1^{1-p}) \qquad \text{(by Thm. I.11.9.4)}$$

$$= \frac{1}{1-p}(N^{1-p} - 1).$$

If $p = 1$, then we have

$$\int_{[1,N]} f = \ln N - \ln 1 = \ln N.$$

Note that we use Thm. I.11.9.4 and logarithm without circularity.

First, suppose that $p > 1$. Since

$$\begin{aligned}
\int_{[1,N]} f &= \frac{1}{1-p}(N^{1-p} - 1) \\
&= \frac{1}{p-1}(1 - N^{1-p}) \\
&\leq \frac{1}{p-1} && (N^{1-p} \leq 1)
\end{aligned}$$

and N was arbitrary, we know that $\sup_{N>1} \int_{[1,N]} f \leq \frac{1}{p-1}$. Thus, $\sup_{N>1} \int_{[1,N]} f$ is finite and by Prop. I.11.6.4 $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

Next suppose that $p = 1$. Since $\int_{[1,N]} f = \ln N$ and $\ln N$ is unbounded, we know that $\sup_{N>1} \int_{[1,N]} f = +\infty$ and by Prop. I.11.6.4 $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent.

Next suppose that $0 < p < 1$. Since $\int_{[1,N]} f = \frac{1}{1-p}(N^{1-p} - 1)$ and $\left\{N^{1-p} : N \in \mathbb{R}^+\right\}$ is unbounded, we know that $\sup_{N>1} \int_{[1,N]} f = +\infty$ and by Prop. I.11.6.4 $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent.

Finally suppose that $p \leq 0$. By Prop. I.6.7.3(e) we know that $1 = x^0 \geq x^p$ for all $x \in [1, \infty)$, thus $1 = \frac{1}{x^0} \leq \frac{1}{x^p}$. By zero test (Cor. I.7.2.6) we know that $\lim_{n\to\infty} 1 \neq 0$ implies $\sum_{n=0}^{\infty} \frac{1}{x^0}$ diverges. Thus, by comparison test (Prop. I.7.3.1) $\sum_{n=1}^{N} \frac{1}{x^p}$ is divergent. □

— Exercises —

Ex. I.11.6.1. Use Prop. I.11.6.1 to prove Cor. I.11.6.3.

Proof. See Cor. I.11.6.3. □

Ex. I.11.6.2. Formulate a reasonable notion of a piecewise monotone function, and then show that all bounded piecewise monotone functions are Riemann integrable.

Proof. Let I be a bounded interval, and let $f : I \to \mathbb{R}$. We say that f is *piecewise monotone on* I iff there exists a partition $\mathbf{P}$ of I such that $f|_J$ is monotone on J for all $J \in \mathbf{P}$.

Now we show that all bounded piecewise monotone functions are Riemann integrable. Suppose that $f : I \to \mathbb{R}$ is a bounded piecewise monotone function. Then by definition $\exists \mathbf{P}$ such that $\mathbf{P}$ is a partition of I and $f|_J$ is monotone on J for all $J \in \mathbf{P}$. Since f is bounded, $f|_J$ is also bounded, by Cor. I.11.6.3 we know that $f|_J$ is Riemann integrable on J. Let $F_J : I \to \mathbb{R}$ be the function

$$F_J(x) = \begin{cases} f|_J(x) & \text{if } x \in J \\ 0 & \text{if } x \notin J \end{cases}$$

Then by Thm. I.11.4.1(g) we know that F_J is Riemann integrable and

$$\begin{aligned} \sum_{J\in\mathbf{P}} \int_I F_J &= \sum_{J\in\mathbf{P}} \int_J f|_J && \text{(by Thm. I.11.4.1(g))} \\ &= \int_I f. && \text{(by Ex. I.11.4.3)} \end{aligned}$$

Thus, f is Riemann integrable on I. □

Ex. I.11.6.3. Prove Prop. I.11.6.4.

Proof. See Prop. I.11.6.4. □

Ex. I.11.6.4. Give examples to show that both directions of the integral test break down if f is not assumed to be monotone decreasing.

Proof. Let $f_1 : [0, \infty) \to \mathbb{R}$ be the function

$$f_1(x) = \begin{cases} 1 & \text{if } x \in \mathbb{N} \\ 0 & \text{if } x \notin \mathbb{N} \end{cases}$$

Then we know that f_1 is not monotone decreasing and $\sum_{n=0}^{\infty} f_1(n)$ diverges. But $\int_{[0,N]} f_1 = 0$ for all $N \in \mathbb{Z}^+$, thus $\sup_{N>0} \int_{[0,N]} f_1$ is finite.

Let $f_2 : [0, \infty) \to \mathbb{R}$ be the function

$$f_2(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \in \mathbb{N} \\ \dfrac{1}{x} & \text{if } x \notin \mathbb{N} \end{cases}$$

Then we know that f_2 is not monotone decreasing. By Cor. I.11.6.5 we know that $\sup_{N>0}\int_{[0,N]}\frac{1}{x}$ is not finite, and since $\int_{[0,N]} f_2 = \int_{[0,N]}\frac{1}{x}$ we also have $\sup_{N>0}\int_{[0,N]} f_2$ is not finite. But by Cor. I.11.6.5 we know that $\sum_{n=0}^{\infty}\frac{1}{x^2}$ converges. ☐

Ex. I.11.6.5. Use Prop. I.11.6.4 to prove Cor. I.11.6.5.

Proof. See Cor. I.11.6.5. ☐

I.11.7 A non-Riemann integrable function

Prop. I.11.7.1. Let $f : [0, 1] \to \mathbb{R}$ be the discontinuous function

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then f is bounded but not Riemann integrable.

Proof. It is clear that f is bounded, so let us show that it is not Riemann integrable.

Let $\mathbf{P}$ be any partition of $[0, 1]$. For any $J \in \mathbf{P}$, observe that if J is not a point or the empty set, then

$$\sup_{x\in J} f(x) = 1$$

(by Prop. I.5.4.14). In particular, we have

$$\left(\sup_{x\in J} f(x)\right)|J| = |J|.$$

(Note this is also true when J is a point, since both sides are zero.) In particular, we see that

$$U(f, \mathbf{P}) = \sum_{J\in\mathbf{P}:J\neq\emptyset} |J| = |[0, 1]| = 1$$

by Thm. I.11.1.13; note that the empty set does not contribute anything to the total length. In particular, we have $\overline{\int}_{[0,1]} f = 1$, by Prop. I.11.3.12.

A similar argument gives that

$$\inf_{x\in J} f(x) = 0$$

for all J (other than points or the empty set), and so

$$L(f, \mathbf{P}) = \sum_{J\in\mathbf{P}:J\neq\emptyset} 0 = 0.$$

In particular, we have $\underline{\int}_{[0,1]} f = 0$, by Prop. I.11.3.12. Thus, the upper and lower Riemann integrals do not match, and so this function is not Riemann integrable. □

Rmk. I.11.7.2. It is only rather "artificial" bounded functions which are not Riemann integrable. Because of this, the Riemann integral is good enough for a large majority of cases. There are ways to generalize or improve this integral, though. One of these is the *Lebesgue integral.* Another is the *Riemann-Stieltjes integral* $\int_I f d\alpha$, where $\alpha : I \to \mathbb{R}$ is a monotone increasing function.

I.11.8 The Riemann-Stieltjes integral

A.Cor. I.11.8.1. Let I be a bounded interval, and let $f : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I. Then we have

$$f(x_0+) = \lim_{x \to x_0^+; x \in X} f(x) = \inf_{x \in X \cap (x_0, +\infty)} f(x)$$

and

$$f(x_0-) = \lim_{x \to x_0^-; x \in X} f(x) = \sup_{x \in X \cap (-\infty, x_0)} f(x)$$

for every $x_0 \in I$ and x_0 is not an endpoint of X.

Proof. If $I = \emptyset$, then the statement are vacuously true. So suppose that $I \neq \emptyset$. Define

$$U = \inf_{x \in X \cap (x_0, +\infty)} f(x);$$
$$L = \sup_{x \in X \cap (-\infty, x_0)} f(x).$$

Since $x_0 \in X$ and x_0 is not an endpoint of X, we know that $X \cap (x_0, +\infty) \neq \emptyset$ and $X \cap (-\infty, x_0) \neq \emptyset$. Since f is monotone increasing, we have

$$U = \inf_{x \in X \cap (x_0, +\infty)} f(x) \geq f(x_0)$$

and

$$L = \sup_{x \in X \cap (-\infty, x_0)} f(x) \leq f(x_0).$$

Thus, $U, L \in \mathbb{R}$.

First, we show that $f(x_0+) = U$. By the definition of U we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists x \in X \cap (x_0, +\infty) : 0 \leq f(x) - U \leq \varepsilon.$$

Now fix one pair of ε and x. Since f is monotone increasing, we know that

$$\forall y \in X \cap (x_0, +\infty), y < x \implies 0 \le f(y) - U \le f(x) - U \le \varepsilon.$$

Thus, by setting $\delta = x - x_0$ we have

$$\forall y \in X \cap (x_0, +\infty), |y - x_0| < \delta \implies |f(y) - U| \le \varepsilon.$$

Since ε was arbitrary, by Def. I.9.3.6 and Def. I.9.5.1 we have $f(x_0+) = U$.

Now we show that $f(x_0-) = L$. By the definition of L we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists x \in X \cap (-\infty, x_0) : 0 \le L - f(x) \le \varepsilon.$$

Now fix one pair of ε and x. Since f is monotone increasing, we know that

$$\forall y \in X \cap (-\infty, x_0), y > x \implies 0 \le L - f(y) \le L - f(x) \le \varepsilon.$$

Thus, by setting $\delta = x_0 - x$ we have

$$\forall y \in X \cap (-\infty, x_0), |y - x_0| < \delta \implies |f(y) - L| \le \varepsilon.$$

Since ε was arbitrary, by Def. I.9.3.6 and Def. I.9.5.1 we have $f(x_0-) = L$. $\square$

A.Cor. I.11.8.2. Let I be a bounded interval, and let $f : X \to \mathbb{R}$ be a monotone decreasing function defined on some interval X which contains I. Then we have

$$f(x_0+) = \lim_{x \to x_0^+; x \in X} f(x) = \sup_{x \in X \cap (x_0, \infty)} f(x)$$

and

$$f(x_0-) = \lim_{x \to x_0^-; x \in X} f(x) = \inf_{x \in X \cap (-\infty, x_0)} f(x)$$

for every $x_0 \in I$ and x_0 is not an endpoint of X.

Proof. If $I = \emptyset$, then the statement are vacuously true. So suppose that $I \neq \emptyset$. Define

$$U = \sup_{x \in X \cap (x_0, +\infty)} f(x);$$
$$L = \inf_{x \in X \cap (-\infty, x_0)} f(x).$$

Since $x_0 \in X$ and x_0 is not an endpoint of X, we know that $X \cap (x_0, +\infty) \neq \emptyset$ and $X \cap (-\infty, x_0) \neq \emptyset$. Since f is monotone decreasing, we have

$$U = \sup_{x \in X \cap (x_0, +\infty)} f(x) \le f(x_0)$$

and

$$L = \inf_{x \in X \cap (-\infty, x_0)} f(x) \geq f(x_0).$$

Thus, $U, L \in \mathbb{R}$.

First, we show that $f(x_0+) = U$. By the definition of U we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists x \in X \cap (x_0, +\infty) : 0 \leq U - f(x) \leq \varepsilon.$$

Now fix one pair of ε and x. Since f is monotone decreasing, we know that

$$\forall y \in X \cap (x_0, +\infty), y < x \implies 0 \leq U - f(y) \leq U - f(x) \leq \varepsilon.$$

Thus, by setting $\delta = x - x_0$ we have

$$\forall y \in X \cap (x_0, +\infty), |y - x_0| < \delta \implies |f(y) - U| \leq \varepsilon.$$

Since ε was arbitrary, by Def. I.9.3.6 and Def. I.9.5.1 we have $f(x_0+) = U$.

Now we show that $f(x_0-) = L$. By the definition of L we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists x \in X \cap (-\infty, x_0) : 0 \leq f(x) - L \leq \varepsilon.$$

Now fix one pair of ε and x. Since f is monotone decreasing, we know that

$$\forall y \in X \cap (-\infty, x_0), y > x \implies 0 \leq f(y) - L \leq f(x) - L \leq \varepsilon.$$

Thus, by setting $\delta = x_0 - x$ we have

$$\forall y \in X \cap (-\infty, x_0), |y - x_0| < \delta \implies |f(y) - L| \leq \varepsilon.$$

Since ε was arbitrary, by Def. I.9.3.6 and Def. I.9.5.1 we have $f(x_0-) = L$. □

Def. I.11.8.1 (α-length)**.** Let I be a bounded interval, and let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I. Then we define the *α-length* $\alpha[I]$ of I as follows.

- If I is the empty set, we set
$$\alpha[\emptyset] := 0.$$

- If I is a point of the form $\{a\}$ for some real number a, we set
$$\alpha\big[\{a\}\big] := \lim_{x \to a^+; x \in X} \alpha(x) - \lim_{x \to a^-; x \in X} \alpha(x),$$
with the convention that $\lim_{x \to a^+; x \in X} \alpha(x)$ (resp. $\lim_{x \to a^-; x \in X} \alpha(x)$) is $\alpha(a)$ when a is the right (resp. left) endpoint of X.

- If I is an interval of the form (a, b) for some real numbers $b > a$, set

$$\alpha\big[(a,b)\big] := \lim_{x\to b^-; x\in X} \alpha(x) - \lim_{x\to a^+; x\in X} \alpha(x).$$

- If I is an interval of the form $[a, b)$, $(a, b]$, or $[a, b]$ for some real numbers $b > a$, then we set

$$\alpha[I] = \begin{cases} \alpha\big[\{a\}\big] + \alpha\big[(a,b)\big] & \text{if } I = [a,b) \\ \alpha\big[(a,b)\big] + \alpha\big[\{b\}\big] & \text{if } I = (a,b] \\ \alpha\big[\{a\}\big] + \alpha\big[(a,b)\big] + \alpha\big[\{b\}\big] & \text{if } I = [a,b] \end{cases}$$

Note. In the special case when α is continuous, the definition of $\alpha[I]$ where I is of the form (a, b), $[a, b)$, $(a, b]$, or $[a, b]$ simplifies to $\alpha[I] = \alpha(b) - \alpha(a)$.

Note. We sometimes write $\alpha\big|_a^b$ or $\alpha(x)\big|_{x=a}^{x=b}$ instead of $\alpha\big[[a, b]\big]$.

Note. Def. I.11.8.1 is well-defined, thanks to A.Cor. I.11.8.1. Def. I.11.8.1 is can also be applied when α is monotone decreasing, thanks to A.Cor. I.11.8.2.

Lem. I.11.8.4. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I, and let $\mathbf{P}$ be a partition of I. Then we have

$$\alpha[I] = \sum_{J\in\mathbf{P}} \alpha[J].$$

Proof. We prove this by induction on n. More precisely, we let $P(n)$ be the property that whenever I is a bounded interval, and whenever $\mathbf{P}$ is a partition of I with cardinality n, that $\alpha[I] = \sum_{J\in\mathbf{P}} \alpha[J]$.

The base case $P(0)$ is trivial; the only way that I can be partitioned into an empty partition is if I is itself empty, so by Def. I.11.8.1 $\alpha[I] = 0$. The case $P(1)$ is also very easy; the only way that I can be partitioned into a singleton set $\{J\}$ is if $J = I$, at which point the claim is again very easy.

Now suppose inductively that $P(n)$ is true for some $n \geq 1$, and now we prove $P(n+1)$. Let I be a bounded interval, and let $\mathbf{P}$ be a partition of I of cardinality $n + 1$.

If I is the empty set or a point, then all the intervals in $\mathbf{P}$ must also be either the empty set or a point, and by Def. I.11.8.1 every interval either has α-length zero or

$$\alpha[\{a\}] = \lim_{x\to a^+; x\in X} \alpha(x) - \lim_{x\to a^-; x\in X} \alpha(x),$$

and the claim is trivial. Thus, we will assume that I is an interval of the form (a, b), $(a, b]$, $[a, b)$, or $[a, b]$.

Let us first suppose that $b \in I$, i.e., I is either $(a, b]$ or $[a, b]$. Since $b \in I$, we know that one of the intervals K in $\mathbf{P}$ contains b. Since K is contained in I, it must therefore be of the form $(c, b]$, $[c, b]$, or $\{b\}$ for some real number c, with $a \leq c \leq b$ (in the latter case of $K = \{b\}$, we set $c := b$). In particular, this means that the set $I \setminus K$ is also an interval of the form $[a, c], (a, c), (a, c], [a, c)$ when $c > a$, or a point or empty set when $a = c$. Either way, by Def. I.11.8.1 we see that

$$\begin{aligned}
\alpha\big[(a,b]\big] &= \alpha\big[(a,b)\big] + \alpha\big[\{b\}\big] \\
&= \lim_{x \to b^-; x \in X} \alpha(x) - \lim_{x \to a^+; x \in X} \alpha(x) + \alpha\big[\{b\}\big] \\
&= \lim_{x \to b^-; x \in X} \alpha(x) - \lim_{x \to c^+; x \in X} \alpha(x) \\
&\quad + \lim_{x \to c^+; x \in X} \alpha(x) - \lim_{x \to c^-; x \in X} \alpha(x) \\
&\quad + \lim_{x \to c^-; x \in X} \alpha(x) - \lim_{x \to a^+; x \in X} \alpha(x) + \alpha\big[\{b\}\big] \\
&= \alpha\big[(c,b)\big] + \alpha\big[\{c\}\big] + \alpha\big[(a,c)\big] + \alpha\big[\{b\}\big] \\
&= \begin{cases} \alpha\big[(a,c)\big] + \alpha\big[[c,b]\big] \\ \alpha\big[(a,c]\big] + \alpha\big[(c,b]\big] \end{cases} \\
&= \alpha[K] + \alpha[I \setminus K]
\end{aligned}$$

and

$$\begin{aligned}
\alpha\big[[a,b]\big] &= \alpha\big[\{a\}\big] + \alpha\big[(a,b]\big] \\
&= \begin{cases} \alpha\big[\{a\}\big] + \alpha\big[(a,c)\big] + \alpha\big[[c,b]\big] \\ \alpha\big[\{a\}\big] + \alpha\big[(a,c]\big] + \alpha\big[(c,b]\big] \end{cases} \\
&= \begin{cases} \alpha\big[[a,c)\big] + \alpha\big[[c,b]\big] \\ \alpha\big[[a,c]\big] + \alpha\big[(c,b]\big] \end{cases} \\
&= \alpha[K] + \alpha[I \setminus K].
\end{aligned}$$

On the other hand, since $\mathbf{P}$ forms a partition of I, we see that $\mathbf{P} \setminus \{K\}$ forms a partition of $I \setminus K$. By the induction hypothesis, we thus have

$$\alpha[I \setminus K] = \sum_{J \in \mathbf{P} \setminus \{K\}} \alpha[J].$$

Combining these two identities (and using the laws of addition for finite sets, see Prop. I.7.1.11(e)) we obtain

$$\alpha[I] = \sum_{J \in \mathbf{P}} \alpha[J]$$

as desired.

Now suppose that $b \notin I$, i.e., I is either (a, b) or $[a, b)$. Then one of the intervals K also is of the form (c, b) or $[c, b)$ (see Ex. I.11.1.3). In particular, this means that the set $I \setminus K$

is also an interval of the form $[a,c], (a,c), (a,c], [a,c)$ when $c > a$, or a point or empty set when $a = c$. By Def. I.11.8.1 we see that

$$\begin{aligned}
\alpha\big[(a,b)\big] &= \lim_{x\to b^-;x\in X}\alpha(x) - \lim_{x\to a^+;x\in X}\alpha(x) \\
&= \lim_{x\to b^-;x\in X}\alpha(x) - \lim_{x\to c^+;x\in X}\alpha(x) \\
&\quad + \lim_{x\to c^+;x\in X}\alpha(x) - \lim_{x\to c^-;x\in X}\alpha(x) \\
&\quad + \lim_{x\to c^-;x\in X}\alpha(x) - \lim_{x\to a^+;x\in X}\alpha(x) \\
&= \alpha\big[(c,b)\big] + \alpha\big[\{c\}\big] + \alpha\big[(a,c)\big] \\
&= \begin{cases} \alpha\big[(a,c)\big] + \alpha\big[[c,b)\big] \\ \alpha\big[(a,c]\big] + \alpha\big[(c,b)\big] \end{cases} \\
&= \alpha[K] + \alpha[I\setminus K]
\end{aligned}$$

and

$$\begin{aligned}
\alpha\big[[a,b)\big] &= \alpha\big[\{a\}\big] + \alpha\big[(a,b)\big] \\
&= \begin{cases} \alpha\big[\{a\}\big] + \alpha\big[(a,c)\big] + \alpha\big[[c,b)\big] \\ \alpha\big[\{a\}\big] + \alpha\big[(a,c]\big] + \alpha\big[(c,b)\big] \end{cases} \\
&= \begin{cases} \alpha\big[[a,c)\big] + \alpha\big[[c,b)\big] \\ \alpha\big[[a,c]\big] + \alpha\big[(c,b)\big] \end{cases} \\
&= \alpha[K] + \alpha[I\setminus K].
\end{aligned}$$

The rest of the argument then proceeds as above. □

A.Cor. I.11.8.3. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone decreasing function defined on some interval X which contains I, and let $\mathbf{P}$ be a partition of I. Then we have

$$\alpha[I] = \sum_{J\in\mathbf{P}} \alpha[J].$$

Proof. Since α is monotone decreasing, we know that $-\alpha$ is monotone increasing. Thus, by Lem. I.11.8.4 we have

$$\begin{aligned}
&(-\alpha)[I] = \sum_{J\in\mathbf{P}} (-\alpha)[J] \\
\implies &-\big(\alpha[I]\big) = \sum_{J\in\mathbf{P}} -\big(\alpha[J]\big) = -\sum_{J\in\mathbf{P}}\alpha[J] \qquad \text{(by limit laws)} \\
\implies &\alpha[I] = \sum_{J\in\mathbf{P}}\alpha[J].
\end{aligned}$$

□

Def. I.11.8.5 (piecewise constant Riemann-Stieltjes integral)**.** Let I be a bounded interval, and let $\mathbf{P}$ be a partition of I. Let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I, and let $f : I \to \mathbb{R}$ be a function which is piecewise constant with respect to $\mathbf{P}$. Then we define

$$p.c. \int_{[\mathbf{P}]} f \, d\alpha := \sum_{J \in \mathbf{P}} c_J \alpha[J]$$

where c_J is the constant value of f on J.

Note. When α is monotone decreasing, by Def. I.11.8.5 we have

$$p.c. \int_{[\mathbf{P}]} f \, d(-\alpha) = \sum_{J \in \mathbf{P}} c_J(-\alpha)[J] = -\sum_{J \in \mathbf{P}} c_J \alpha[J].$$

E.g. I.11.8.7. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be the identity function $\alpha(x) := x$. Then for any bounded interval I, any partition $\mathbf{P}$ of I, and any function f that is piecewise constant with respect to P, we have $p.c. \int_{[\mathbf{P}]} f \, d\alpha = p.c. \int_{[\mathbf{P}]} f$.

A.Cor. I.11.8.4. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I, and let $f : I \to \mathbb{R}$ be a function. Suppose that $\mathbf{P}$ and $\mathbf{P}'$ are partitions of I such that f is piecewise constant both with respect to $\mathbf{P}$ and with respect to $\mathbf{P}'$. Also suppose that both $p.c. \int_{[\mathbf{P}]} f \, d\alpha$ and $p.c. \int_{[\mathbf{P}']} f \, d\alpha$ are well-defined. Then $p.c. \int_{[\mathbf{P}]} f \, d\alpha = p.c. \int_{[\mathbf{P}']} f \, d\alpha$.

Proof. By Lem. I.11.1.18 we know that $\mathbf{P}\#\mathbf{P}'$ is a partition of I and is both finer than $\mathbf{P}$ and finer than $\mathbf{P}'$, thus by Def. I.11.8.5 we have

$$p.c. \int_{[\mathbf{P}\#\mathbf{P}']} f \, d\alpha = \sum_{J \in \mathbf{P}\#\mathbf{P}'} c_J \alpha[J].$$

By Lem. I.11.8.4 we know that

$$\alpha[I] = \sum_{J \in \mathbf{P}} \alpha[J] = \sum_{J \in \mathbf{P}\#\mathbf{P}'} \alpha[J].$$

For each $K \in \mathbf{P}$, let $\mathbf{P}_K$ be the set

$$\mathbf{P}_K = \{S \in \mathbf{P}\#\mathbf{P}' : S \subseteq K\}.$$

Since $\mathbf{P}\#\mathbf{P}'$ is finer than $\mathbf{P}$, by A.Cor. I.11.1.4 we know that $\mathbf{P}_K$ is a partition of K, and $\bigcup_{K \in \mathbf{P}} \mathbf{P}_K = \mathbf{P}\#\mathbf{P}'$. Since f is piecewise constant with respect to $\mathbf{P}$, by Lem. I.11.2.7 we know that f is piecewise constant with respect to $\mathbf{P}\#\mathbf{P}'$. So we have

$$\begin{aligned}
p.c. \int_{[\mathbf{P}\#\mathbf{P}']} f \, d\alpha &= \sum_{J \in \mathbf{P}\#\mathbf{P}'} c_J \alpha[J] && \text{(by Def. I.11.8.5)} \\
&= \sum_{J \in \bigcup_{K \in \mathbf{P}} \mathbf{P}_K} c_J \alpha[J] \\
&= \sum_{K \in \mathbf{P}} \sum_{J \in \mathbf{P}_K} c_J \alpha[J] && \text{(by Prop. I.7.1.11(e))} \\
&= \sum_{K \in \mathbf{P}} \sum_{J \in \mathbf{P}_K} c_K \alpha[J] \qquad (J \subseteq K) \\
&= \sum_{K \in \mathbf{P}} c_K \left(\sum_{J \in \mathbf{P}_K} \alpha[J] \right) \\
&= \sum_{K \in \mathbf{P}} c_K \alpha[K] && \text{(by Lem. I.11.8.4)} \\
&= p.c. \int_{[\mathbf{P}]} f \, d\alpha. && \text{(by Def. I.11.8.5)}
\end{aligned}$$

Using similar arguments, we can show that $p.c. \int_{[\mathbf{P}']} f \, d\alpha = p.c. \int_{[\mathbf{P}\#\mathbf{P}']} f \, d\alpha$. Thus, we have $p.c. \int_{[\mathbf{P}]} f \, d\alpha = p.c. \int_{[\mathbf{P}']} f \, d\alpha.$ □

A.Cor. I.11.8.5. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I, and let $f : I \to \mathbb{R}$ be a piecewise constant function on I. Then we define

$$p.c. \int_I f \, d\alpha := p.c. \int_{[\mathbf{P}]} f \, d\alpha,$$

where $\mathbf{P}$ is any partition of I with respect to which f is piecewise constant. (Note that A.Cor. I.11.8.4 tells us that the precise choice of this partition is irrelevant.)

A.Cor. I.11.8.6. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone function defined on some interval X which contains I.

- If α is monotone increasing, then $\alpha[I] \geq 0$.
- If α is monotone decreasing, then $\alpha[I] \leq 0$.

Proof. We split into four cases:

- $I = \emptyset$. Then by Def. I.11.8.1 we have $\alpha[\emptyset] = 0$.

- $I = \{x_0\}$ for some $x_0 \in \mathbb{R}$. If α is monotone increasing, then we have

$$\begin{aligned}
\alpha\big[\{x_0\}\big] &= \lim_{x \to x_0^+} \alpha(x) - \lim_{x \to x_0^-} \alpha(x) && \text{(by Def. I.11.8.1)} \\
&= \inf_{x \in X \cap (x_0, \infty)} f(x) - \sup_{x \in X \cap (-\infty, x_0)} f(x) && \text{(by A.Cor. I.11.8.1)} \\
&\geq f(x_0) - f(x_0) && \text{(since } \alpha \text{ is monotone increasing)} \\
&= 0.
\end{aligned}$$

If α is monotone decreasing, then we have

$$\begin{aligned}
\alpha\big[\{x_0\}\big] &= \lim_{x \to x_0^+} \alpha(x) - \lim_{x \to x_0^-} \alpha(x) && \text{(by Def. I.11.8.1)} \\
&= \sup_{x \in X \cap (x_0, \infty)} f(x) - \inf_{x \in X \cap (-\infty, x_0)} f(x) && \text{(by A.Cor. I.11.8.2)} \\
&\leq f(x_0) - f(x_0) && \text{(since } \alpha \text{ is monotone decreasing)} \\
&= 0.
\end{aligned}$$

- $I = (a, b)$ for some $a, b \in \mathbb{R}$ and $a < b$. If α is monotone increasing, then we have

$$\begin{aligned}
&\alpha\big[(a, b)\big] \\
&= \lim_{x \to b^-; x \in (a,b)} \alpha(x) - \lim_{x \to a^+; x \in (a,b)} \alpha(x) && \text{(by Def. I.11.8.1)} \\
&= \sup_{x \in (a,b) \cap (-\infty, b)} \alpha(x) - \inf_{x \in (a,b) \cap (a, \infty)} \alpha(x) && \text{(by A.Cor. I.11.8.1)} \\
&= \sup_{x \in (a,b)} \alpha(x) - \inf_{x \in (a,b)} \alpha(x) \\
&\geq 0.
\end{aligned}$$

If α is monotone decreasing, then we have

$$\begin{aligned}
&\alpha\big[(a, b)\big] \\
&= \lim_{x \to b^-; x \in (a,b)} \alpha(x) - \lim_{x \to a^+; x \in (a,b)} \alpha(x) && \text{(by Def. I.11.8.1)} \\
&= \inf_{x \in (a,b) \cap (-\infty, b)} \alpha(x) - \sup_{x \in (a,b) \cap (a, \infty)} \alpha(x) && \text{(by A.Cor. I.11.8.2)} \\
&= \inf_{x \in (a,b)} \alpha(x) - \sup_{x \in (a,b)} \alpha(x) \\
&\leq 0.
\end{aligned}$$

- I is one of $[a, b), (a, b], [a, b]$. If α is monotone increasing, then from the proof above we have

$$\alpha\big[[a, b)\big] = \alpha\big[\{a\}\big] + \alpha\big[(a, b)\big] \geq 0$$

$$\alpha\big[(a,b]\big] = \alpha\big[(a,b)\big] + \alpha\big[\{b\}\big] \geq 0$$
$$\alpha\big[[a,b]\big] = \alpha\big[\{a\}\big] + \alpha\big[(a,b]\big] \geq 0$$

If α is monotone decreasing, then from the proof above we have

$$\alpha\big[[a,b)\big] = \alpha\big[\{a\}\big] + \alpha\big[(a,b)\big] \leq 0$$
$$\alpha\big[(a,b]\big] = \alpha\big[(a,b)\big] + \alpha\big[\{b\}\big] \leq 0$$
$$\alpha\big[[a,b]\big] = \alpha\big[\{a\}\big] + \alpha\big[(a,b]\big] \leq 0$$

From all cases above, we conclude that $\alpha[I] \geq 0$ if α is monotone increasing and $\alpha[I] \leq 0$ if α is monotone decreasing. □

A.Cor. I.11.8.7. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be piecewise constant functions on I such that both $p.c. \int_I f \, d\alpha$ and $p.c. \int_I g \, d\alpha$ are well-defined.

(a) We have $p.c. \int_I (f+g) \, d\alpha = p.c. \int_I f \, d\alpha + p.c. \int_I g \, d\alpha$.

(b) For any real number c, we have $p.c. \int_I (cf) \, d\alpha = c(p.c. \int_I f \, d\alpha)$.

(c) We have $p.c. \int_I (f-g) \, d\alpha = p.c. \int_I f \, d\alpha - p.c. \int_I g \, d\alpha$.

(d) If $f(x) \geq 0$ for all $x \in I$, then $p.c. \int_I f \, d\alpha \geq 0$.

(e) If $f(x) \geq g(x)$ for all $x \in I$, then $p.c. \int_I f \, d\alpha \geq p.c. \int_I g \, d\alpha$.

(f) If f is the constant function $f(x) = c$ for all $x \in I$, then $p.c. \int_I f \, d\alpha = c\alpha[I]$.

(g) Let J be a bounded interval containing I (i.e., $I \subseteq J$), and let $F : J \to \mathbb{R}$ be the function

$$F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

Then F is piecewise constant on J, and $p.c. \int_J F \, d\alpha = p.c. \int_I f \, d\alpha$.

(h) Suppose that $\{J, K\}$ is a partition of I into two intervals J and K. Then the function $f|_J : J \to \mathbb{R}$ and $f|_K : K \to \mathbb{R}$ are piecewise constant on J and K respectively, and we have

$$p.c. \int_I f \, d\alpha = p.c. \int_J f|_J \, d\alpha + p.c. \int_K f|_K \, d\alpha.$$

Proof. (a) Let $\mathbf{P}$ be a partition of I. By Thm. I.11.2.16(a) we know that $f+g$ is piecewise constant with respect to $\mathbf{P}$. For each $J \in \mathbf{P}$, we define $c_{f|_J}, c_{g|_J} \in \mathbb{R}$ to be the constant value of $f|_J, g|_J$, respectively. Then by Def. I.11.2.1 $c_{f|_J} + c_{g|_J}$ is the constant value of $(f+g)|_J$ for each $J \in P$. Thus, we have

$$\begin{aligned}
&p.c. \int_I f \, d\alpha + p.c. \int_I g \, d\alpha \\
&= p.c. \int_{[\mathbf{P}]} f \, d\alpha + p.c. \int_{[\mathbf{P}]} g \, d\alpha && \text{(by A.Cor. I.11.8.5)} \\
&= \sum_{J \in \mathbf{P}} f_J \alpha[J] + \sum_{J \in \mathbf{P}} g_J \alpha[J] && \text{(by Def. I.11.8.5)} \\
&= \sum_{J \in \mathbf{P}} (f_J + g_J) \alpha[J] && \text{(by Prop. I.7.1.11(f))} \\
&= p.c. \int_{[\mathbf{P}]} (f_J + g_J) \, d\alpha && \text{(by Def. I.11.8.5)} \\
&= p.c. \int_I (f_J + g_J) \, d\alpha. && \text{(by A.Cor. I.11.8.5)}
\end{aligned}$$

□

Proof. (b) Let $\mathbf{P}$ be a partition of I. By Thm. I.11.2.16(b) we know that cf is piecewise constant with respect to $\mathbf{P}$. For each $J \in \mathbf{P}$, we define $c_J \in \mathbb{R}$ to be the constant value of $f|_J$. Then by Def. I.11.2.1 $c \cdot c_J$ is the constant value of $(cf)|_J$. Thus, we have

$$\begin{aligned}
c\left(p.c. \int_I f \, d\alpha\right) &= c\left(p.c. \int_{[\mathbf{P}]} f \, d\alpha\right) && \text{(by A.Cor. I.11.8.5)} \\
&= c\left(\sum_{J \in \mathbf{P}} c_J \alpha[J]\right) && \text{(by Def. I.11.8.5)} \\
&= \sum_{J \in \mathbf{P}} c \cdot c_J \alpha[J] && \text{(by Prop. I.7.1.11(g))} \\
&= p.c. \int_{[\mathbf{P}]} (cf) \, d\alpha && \text{(by Def. I.11.8.5)} \\
&= p.c. \int_I (cf) \, d\alpha. && \text{(by A.Cor. I.11.8.5)}
\end{aligned}$$

□

Proof. (c) We have

$$\begin{aligned}
&p.c. \int_I f \, d\alpha - p.c. \int_I g \, d\alpha \\
&= p.c. \int_I f \, d\alpha + (-1) p.c. \int_I g \, d\alpha
\end{aligned}$$

$$
\begin{aligned}
&= p.c. \int_I f \; d\alpha + p.c. \int_I (-g) \; d\alpha && \text{(by A.Cor. I.11.8.7(b))} \\
&= p.c. \int_I \big(f + (-g)\big) \; d\alpha && \text{(by A.Cor. I.11.8.7(a))} \\
&= p.c. \int_I (f - g) \; d\alpha. && \text{(by Def. I.9.2.1)}
\end{aligned}
$$

□

Proof. (d) By A.Cor. I.11.8.5 f is piecewise constant with respect to $\mathbf{P}$ for some partition $\mathbf{P}$ of I. Let $J \in \mathbf{P}$ and let $c_J \in \mathbb{R}$ be the constant value of $f|_J$. By A.Cor. I.11.8.6 we know that $\alpha[J] \geq 0$ for all $J \in \mathbf{P}$. Since $f(x) \geq 0$ for all $x \in I$, we have $c_J \geq 0$ and $c_J\alpha[J] \geq 0$ for all $J \in \mathbf{P}$. Thus

$$
\begin{aligned}
p.c. \int_I f \; d\alpha &= p.c. \int_{[\mathbf{P}]} f \; d\alpha && \text{(by A.Cor. I.11.8.5)} \\
&= \sum_{J \in \mathbf{P}} c_J \alpha[J] && \text{(by Def. I.11.8.5)} \\
&\geq \sum_{J \in \mathbf{P}} 0 && \text{(by Prop. I.7.1.11(h))} \\
&= 0.
\end{aligned}
$$

□

Proof. (e) Since $f(x) \geq g(x)$ for all $x \in I$, we have $f(x) - g(x) \geq 0$. By A.Cor. I.11.8.7(c) we have

$$p.c. \int_I f \; d\alpha - p.c. \int_I g \; d\alpha = p.c. \int_I (f - g) \; d\alpha.$$

Then by A.Cor. I.11.8.7(d) we have

$$p.c. \int_I (f - g) \; d\alpha \geq 0 \implies p.c. \int_I f \; d\alpha \geq p.c. \int_I g \; d\alpha.$$

□

Proof. (f) Since I is a partition of I, we have

$$
\begin{aligned}
p.c. \int_I f \; d\alpha &= p.c. \int_{[I]} f \; d\alpha && \text{(by A.Cor. I.11.8.5)} \\
&= \sum_{J \in I} c\alpha[J] && \text{(by Def. I.11.8.5)} \\
&= c \sum_{J \in I} \alpha[J] && \text{(by Prop. I.7.1.11(g))} \\
&= c\alpha[I]. && \text{(by Lem. I.11.8.4)}
\end{aligned}
$$

□

Proof. (g) If $I = \emptyset$, then by Def. I.11.2.3 F is piecewise constant with respect to $\{J\}$, and by A.Cor. I.11.8.7(f) we have

$$p.c. \int_J F \, d\alpha = 0\alpha[J] = 0 = p.c \int_I f \, d\alpha.$$

So suppose that $I \neq \emptyset$. By Def. I.11.2.3, f is piecewise constant with respect to $\mathbf{P}$ for some partition $\mathbf{P}$ of I. Let I_1, I_2 be the sets

$$I_1 = \big\{x \in J, \big(x \leq \inf(I)\big) \wedge (x \notin I)\big\}$$

and

$$I_2 = \big\{x \in J, \big(x \geq \sup(I)\big) \wedge (x \notin I)\big\}.$$

By A.Cor. I.11.1.5 we know that $\mathbf{P} \cup \{I_1, I_2\}$ is a partition of J. By hypothesis we know that

$$\forall x \in J, F(x) = \begin{cases} f(x) & \text{if } x \in K \text{ for some } K \in \mathbf{P} \\ 0 & \text{if } x \in I_1 \text{ or } x \in I_2 \end{cases}$$

Thus, by Def. I.11.2.5 F is piecewise constant on J. For each $K \in \mathbf{P} \cup \{I_1, I_2\}$, we define $c_K \in \mathbb{R}$ to be the constant value of $F|_K$. Then we have

$$\begin{aligned}
p.c. \int_J F \, d\alpha &= p.c. \int_{[\mathbf{P} \cup \{I_1, I_2\}]} F \, d\alpha && \text{(by A.Cor. I.11.8.5)} \\
&= \sum_{K \in \mathbf{P}} c_K \alpha[K] && \text{(by Def. I.11.8.5)} \\
&= c_{I_1}\alpha[I_1] + \sum_{K \in \mathbf{P}} c_K \alpha[K] + c_{I_2}\alpha[I_2] && \text{(by Prop. I.7.1.11(e))} \\
&= 0\alpha[I_1] + \sum_{K \in \mathbf{P}} c_K \alpha[K] + 0\alpha[I_2] && \text{(by hypothesis)} \\
&= \sum_{K \in \mathbf{P}} c_K \alpha[K] \\
&= p.c. \int_{[\mathbf{P}]} f \, d\alpha && \text{(by Def. I.11.8.5)} \\
&= p.c. \int_I f \, d\alpha. && \text{(by A.Cor. I.11.8.5)}
\end{aligned}$$

□

Proof. (h) Let $\mathbf{P} = \{J, K\}$. By Def. I.11.2.3 f is piecewise constant with respect to $\mathbf{P}'$ for some partition $\mathbf{P}'$ of I. Now we define $\mathbf{P}_J$ as

$$\mathbf{P}_J = \big\{S \in \mathbf{P} \# \mathbf{P}' : S \subseteq J\big\}$$

and define $\mathbf{P}_K$ as

$$\mathbf{P}_K = \{S \in \mathbf{P}\#\mathbf{P}' : S \subseteq K\}.$$

By Def. I.11.1.8 we know that $\mathbf{P}\#\mathbf{P}'$ is a partition of I and is finer than $\mathbf{P}$. Since $\mathbf{P}\#\mathbf{P}'$ is finer than $\mathbf{P}$, by A.Cor. I.11.1.4 we know that $\mathbf{P}_J, \mathbf{P}_K$ are partitions of J, K, respectively. Again by A.Cor. I.11.1.4 we know that $\mathbf{P}_J \cup \mathbf{P}_K$ is a partition of I. Then by Lem. I.11.2.7 f is piecewise constant with respect to $\mathbf{P}_J \cup \mathbf{P}_K$. Without the loss of generality suppose that $\emptyset \notin \mathbf{P}_J \cup \mathbf{P}_K$. For each $S \in \mathbf{P}_J$, we define $c_S \in \mathbb{R}$ to be the constant value of $f|_J$. Similarly, for each $S \in \mathbf{P}_K$, we define $c_S \in \mathbb{R}$ to be the constant value of $f|_K$. Then we have

$$\begin{aligned}
&p.c.\int_J f|_J \, d\alpha + p.c.\int_K f|_K \, d\alpha \\
=&p.c.\int_{[\mathbf{P}_J]} f|_J \, d\alpha + p.c.\int_{[\mathbf{P}_K]} f|_K \, d\alpha && \text{(by A.Cor. I.11.8.5)} \\
=&\sum_{S\in\mathbf{P}_J} c_S\alpha[S] + \sum_{S\in\mathbf{P}_K} c_S\alpha[S] && \text{(by Prop. I.7.1.11(e))} \\
=&\sum_{S\in\mathbf{P}_J\cup\mathbf{P}_K} c_S\alpha[S] && \text{(by Def. I.11.8.5)} \\
=&\sum_{S\in\mathbf{P}} c_S\alpha[S] \\
=&p.c.\int_{[\mathbf{P}]} f \, d\alpha && \text{(by Def. I.11.8.5)} \\
=&p.c.\int_I f \, d\alpha. && \text{(by A.Cor. I.11.8.5)}
\end{aligned}$$

□

A.Cor. I.11.8.8. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I, and let $f : I \to \mathbb{R}$ be a bounded function. We define the *upper Riemann-Stieltjes integral* $\overline{\int}_I f \, d\alpha$ by the formula

$$\overline{\int}_I f \, d\alpha := \inf\left\{p.c.\int_I g \, d\alpha : g \text{ is a p.c. function on } I \text{ which majorizes } f\right\}$$

and the *lower Riemann-Stieltjes integral* $\underline{\int}_I f \, d\alpha$ by the formula

$$\underline{\int}_I f \, d\alpha := \sup\left\{p.c.\int_I g \, d\alpha : g \text{ is a p.c. function on } I \text{ which minorizes } f\right\}.$$

If $\underline{\int}_I f\,d\alpha = \overline{\int}_I f\,d\alpha$, then we say that f is *Riemann-Stieltjes integrable on I with respect to α* and define

$$\int_I f\,d\alpha := \underline{\int}_I f\,d\alpha = \overline{\int}_I f\,d\alpha.$$

If the upper and lower Riemann-Stieltjes integrals are unequal, we say that f is not Riemann-Stieltjes integrable on I with respect to α.

A.Cor. I.11.8.9. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I. Let $f : I \to \mathbb{R}$ be a function which is bounded by some real number M, i.e., $-M \leq f(x) \leq M$ for all $x \in I$. Then we have

$$-M\alpha[I] \leq \underline{\int}_I f\,d\alpha \leq \overline{\int}_I f\,d\alpha \leq M\alpha[I].$$

in particular, both the lower and upper Riemann-Stieltjes integrals are real numbers (i.e., they are not infinite).

Proof. The function $g : I \to \mathbb{R}$ defined by $g(x) = M$ is constant, hence piecewise constant, and majorizes f; thus $\overline{\int}_I f\,d\alpha \leq p.c.\int_I g\,d\alpha = M\alpha[I]$ by definition of the upper Riemann-Stieltjes integral. A similar argument gives $-M\alpha[I] \leq \underline{\int}_I f\,d\alpha$. Finally, we have to show that $\underline{\int}_I f\,d\alpha \leq \overline{\int}_I f\,d\alpha$. Let g be any piecewise constant function majorizing f, and let h be any piecewise constant function minorizing f. Then g majorizes h, and hence $p.c.\int_I h\,d\alpha \leq p.c.\int_I g\,d\alpha$. Taking suprema in h, we obtain that $\underline{\int}_I f\,d\alpha \leq p.c.\int_I g\,d\alpha$. Taking infima in g, we thus obtain $\underline{\int}_I f\,d\alpha \leq \overline{\int}_I f\,d\alpha$, as desired. $\square$

Note. When α is the identity function $\alpha(x) := x$ then the Riemann-Stieltjes integral is identical to the Riemann integral; thus the Riemann-Stieltjes integral is a generalization of the Riemann integral. We sometimes write $\int_I f$ as $\int_I f\,dx$ or $\int_I f(x)\,dx$.

A.Cor. I.11.8.10. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I. Let $f : I \to \mathbb{R}$ be a piecewise constant function. Then f is Riemann-Stieltjes integrable on I with respect to α, and $\int_I f\,d\alpha = p.c.\int_I f\,d\alpha$.

Proof. Since $f(x) \leq f(x)$ for every $x \in I$, by A.Cor. I.11.8.8 and A.Cor. I.11.8.9 we have

$$p.c.\int_I f\,d\alpha \leq \underline{\int}_I f\,d\alpha \leq \overline{\int}_I f\,d\alpha \leq p.c.\int_I f\,d\alpha$$

Thus, by A.Cor. I.11.8.8 we have

$$\int_I f\ d\alpha = \underline{\int}_I f\ d\alpha = \overline{\int}_I f\ d\alpha = p.c. \int_I f\ d\alpha.$$

$\square$

A.Cor. I.11.8.11 (Laws of Riemann-Stieltjes integration)**.** Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be Riemann-Stieltjes integrable functions on I with respect to α.

(a) The function $f + g$ is Riemann-Stieltjes integrable, and we have $\int_I (f + g)\ d\alpha = \int_I f\ d\alpha + \int_I g\ d\alpha$.

(b) For any real number c, the function cf is Riemann-Stieltjes integrable, and we have $\int_I (cf)\ d\alpha = c(\int_I f\ d\alpha)$.

(c) The function $f - g$ is Riemann-Stieltjes integrable, and we have $\int_I (f - g)\ d\alpha = \int_I f\ d\alpha - \int_I g\ d\alpha$.

(d) If $f(x) \geq 0$ for all $x \in I$, then $\int_I f\ d\alpha \geq 0$.

(e) If $f(x) \geq g(x)$ for all $x \in I$, then $\int_I f\ d\alpha \geq \int_I g\ d\alpha$.

(f) If f is the constant function $f(x) = c$ for all $x \in I$, then $\int_I f\ d\alpha = c\alpha[I]$.

(g) Suppose that $\{J, K\}$ is a partition of I into two intervals J and K. Then the functions $f|_J : J \to \mathbb{R}$ and $f|_K : K \to \mathbb{R}$ are Riemann-Stieltjes integrable on J and K respectively, and we have
$$\int_I f\ d\alpha = \int_J f|_J\ d\alpha + \int_K f|_K\ d\alpha.$$

Proof. (a) Let $f_U : I \to \mathbb{R}$ and $g_U : I \to \mathbb{R}$ be piecewise constant functions on I which majorizes f and g, respectively. Let $f_L : I \to \mathbb{R}$ and $g_L : I \to \mathbb{R}$ be piecewise constant functions on I which minorizes f and g, respectively. f_U, g_U, f_L, g_L are well-defined since by A.Cor. I.11.8.8 f, g are bounded functions on a bounded interval I. Then we have

$$p.c. \int_I f_L\ d\alpha \leq \underline{\int}_I f\ d\alpha = \int_I f\ d\alpha = \overline{\int}_I f\ d\alpha \leq p.c. \int_I f_U\ d\alpha$$

and

$$p.c. \int_I g_L \, d\alpha \leq \underline{\int}_I g \, d\alpha = \int_I g \, d\alpha = \overline{\int}_I g \, d\alpha \leq p.c. \int_I g_U \, d\alpha.$$

By A.Cor. I.11.8.8 both f, g are bounded functions, so $f + g$ is bounded function, and $\underline{\int}_I (f+g) \, d\alpha, \overline{\int}_I (f+g) \, d\alpha$ are well-defined (by A.Cor. I.11.8.8). By Ex. I.11.3.2 we know that $f_U + g_U$ majorizes $f + g_U$ and $f + g_U$ majorizes $f + g$, thus $f_U + g_U$ majorizes $f + g$. Similarly, $f_L + g_L$ minorizes $f + g$. Then we have

$$\begin{aligned}
&\overline{\int}_I (f+g) \, d\alpha \leq p.c. \int_I (f_U + g_U) \, d\alpha && \text{(by A.Cor. I.11.8.8)} \\
\Longrightarrow &\overline{\int}_I (f+g) \, d\alpha \\
&\quad \leq p.c. \int_I f_U \, d\alpha + p.c. \int_I g_U \, d\alpha && \text{(by A.Cor. I.11.8.7(a))} \\
\Longrightarrow &\overline{\int}_I (f+g) \, d\alpha - p.c. \int_I g_U \, d\alpha \\
&\quad \leq p.c. \int_I f_U \, d\alpha && \text{(note that } f_U \text{ was arbitrary)} \\
\Longrightarrow &\overline{\int}_I (f+g) \, d\alpha - p.c. \int_I g_U \, d\alpha \leq \overline{\int}_I f \, d\alpha && \text{(by A.Cor. I.11.8.8)} \\
\Longrightarrow &\overline{\int}_I (f+g) \, d\alpha - \overline{\int}_I f \, d\alpha \leq p.c. \int_I g_U \, d\alpha && \text{(note that } g_U \text{ was arbitrary)} \\
\Longrightarrow &\overline{\int}_I (f+g) \, d\alpha - \overline{\int}_I f \, d\alpha \leq \overline{\int}_I g \, d\alpha && \text{(by A.Cor. I.11.8.8)} \\
\Longrightarrow &\overline{\int}_I (f+g) \, d\alpha \leq \overline{\int}_I f \, d\alpha + \overline{\int}_I g \, d\alpha \\
\Longrightarrow &\overline{\int}_I (f+g) \, d\alpha \leq \int_I f \, d\alpha + \int_I g \, d\alpha && \text{(by A.Cor. I.11.8.8)}
\end{aligned}$$

and

$$\begin{aligned}
&\underline{\int}_I (f+g) \, d\alpha \geq p.c. \int_I (f_L + g_L) \, d\alpha && \text{(by A.Cor. I.11.8.8)} \\
\Longrightarrow &\underline{\int}_I (f+g) \, d\alpha \\
&\quad \geq p.c. \int_I f_L \, d\alpha + p.c. \int_I g_L \, d\alpha && \text{(by A.Cor. I.11.8.7(a))} \\
\Longrightarrow &\underline{\int}_I (f+g) \, d\alpha - p.c. \int_I g_L \, d\alpha
\end{aligned}$$

$$
\begin{aligned}
&\geq p.c. \int_I f_L \, d\alpha && \text{(note that } f_L \text{ was arbitrary)} \\
\Longrightarrow \underline{\int}_I (f+g) \, d\alpha - p.c. \int_I g_L \, d\alpha \geq \underline{\int}_I f \, d\alpha && \text{(by A.Cor. I.11.8.8)} \\
\Longrightarrow \underline{\int}_I (f+g) \, d\alpha - \underline{\int}_I f \, d\alpha \geq p.c. \int_I g_L \, d\alpha && \text{(note that } g_L \text{ was arbitrary)} \\
\Longrightarrow \underline{\int}_I (f+g) \, d\alpha - \underline{\int}_I f \, d\alpha \geq \underline{\int}_I g \, d\alpha && \text{(by A.Cor. I.11.8.8)} \\
\Longrightarrow \underline{\int}_I (f+g) \, d\alpha \geq \underline{\int}_I f \, d\alpha + \underline{\int}_I g \, d\alpha && \\
\Longrightarrow \underline{\int}_I (f+g) \, d\alpha \geq \int_I f \, d\alpha + \int_I g \, d\alpha. && \text{(by A.Cor. I.11.8.8)}
\end{aligned}
$$

By A.Cor. I.11.8.9 we have

$$
\int_I f \, d\alpha + \int_I g \, d\alpha \leq \underline{\int}_I (f+g) \, d\alpha \leq \overline{\int}_I (f+g) \, d\alpha \leq \int_I f \, d\alpha + \int_I g \, d\alpha
$$

and thus by A.Cor. I.11.8.8 we have

$$
\int_I (f+g) \, d\alpha = \underline{\int}_I (f+g) \, d\alpha = \overline{\int}_I (f+g) \, d\alpha = \int_I f \, d\alpha + \int_I g \, d\alpha.
$$

□

Proof. (b) Since f is Riemann-Stieltjes integrable on I with respect to α, by A.Cor. I.11.8.8 we have

$$
\int_I f \, d\alpha = \overline{\int}_I f \, d\alpha = \underline{\int}_I f \, d\alpha.
$$

First, suppose that $c = 0$. Then we have $(cf)(x) = 0$ for all $x \in 0$, thus we have

$$
\begin{aligned}
\int_I (cf) \, d\alpha &= p.c. \int_I (cf) \, d\alpha && \text{(by A.Cor. I.11.8.10)} \\
&= 0 \\
&= c \int_I f \, d\alpha.
\end{aligned}
$$

Next suppose that $c > 0$. Let $f_U : I \to \mathbb{R}$ be a piecewise constant function on I which majorizes f. Let $f_L : I \to \mathbb{R}$ be a piecewise constant function on I which minorizes f. f_U, f_L are well-defined since by A.Cor. I.11.8.8 f is a bounded function on a bounded interval I. Then by A.Cor. I.11.8.8 we have

$$
p.c. \int_I f_L \, d\alpha \leq \underline{\int}_I f \, d\alpha = \int_I f \, d\alpha = \overline{\int}_I f \, d\alpha \leq p.c. \int_I f_U \, d\alpha.
$$

Since f is a bounded function, cf is also a bounded function, by A.Cor. I.11.8.8 both $\overline{\int_I} (cf)\ d\alpha, \underline{\int_I} (cf)\ d\alpha$ are well-defined. Since $c > 0$, by Def. I.11.3.1 we know that cf_U majorizes cf and cf_L minorizes cf. Then we have

$$\begin{aligned}
&\overline{\int_I}(cf)\ d\alpha \leq p.c. \int_I (cf_U)\ d\alpha && \text{(by A.Cor. I.11.8.8)}\\
\Longrightarrow &\overline{\int_I}(cf)\ d\alpha \leq c\Big(p.c. \int_I f_U\ d\alpha\Big) && \text{(by A.Cor. I.11.8.7(b))}\\
\Longrightarrow &\frac{1}{c}\Big(\overline{\int_I}(cf)\ d\alpha\Big) \leq p.c. \int_I f_U\ d\alpha && \text{(note that } f_U \text{ was arbitrary)}\\
\Longrightarrow &\frac{1}{c}\Big(\overline{\int_I}(cf)\ d\alpha\Big) \leq \overline{\int_I} f\ d\alpha && \text{(by A.Cor. I.11.8.8)}\\
\Longrightarrow &\overline{\int_I}(cf)\ d\alpha \leq c\Big(\overline{\int_I} f\ d\alpha\Big)\\
\Longrightarrow &\overline{\int_I}(cf)\ d\alpha \leq c\Big(\int_I f\ d\alpha\Big) && \text{(by A.Cor. I.11.8.8)}
\end{aligned}$$

and

$$\begin{aligned}
&\underline{\int_I}(cf)\ d\alpha \geq p.c. \int_I (cf_L)\ d\alpha && \text{(by A.Cor. I.11.8.8)}\\
\Longrightarrow &\underline{\int_I}(cf)\ d\alpha \geq c\Big(p.c. \int_I f_L\ d\alpha\Big) && \text{(by A.Cor. I.11.8.7(b))}\\
\Longrightarrow &\frac{1}{c}\Big(\underline{\int_I}(cf)\ d\alpha\Big) \geq p.c. \int_I f_L\ d\alpha && \text{(note that } f_L \text{ was arbitrary)}\\
\Longrightarrow &\frac{1}{c}\Big(\underline{\int_I}(cf)\ d\alpha\Big) \geq \underline{\int_I} f\ d\alpha && \text{(by A.Cor. I.11.8.8)}\\
\Longrightarrow &\underline{\int_I}(cf)\ d\alpha \geq c\Big(\underline{\int_I} f\ d\alpha\Big)\\
\Longrightarrow &\underline{\int_I}(cf)\ d\alpha \geq c\Big(\int_I f\ d\alpha\Big). && \text{(by A.Cor. I.11.8.8)}
\end{aligned}$$

By A.Cor. I.11.8.9 we have

$$c\Big(\int_I f\ d\alpha\Big) \leq \underline{\int_I}(cf)\ d\alpha \leq \overline{\int_I}(cf)\ d\alpha \leq c\Big(\int_I f\ d\alpha\Big)$$

and thus by A.Cor. I.11.8.8 we have

$$\int_I (cf)\ d\alpha = \underline{\int_I}(cf)\ d\alpha = \overline{\int_I}(cf)\ d\alpha = c\Big(\int_I f\ d\alpha\Big).$$

Finally suppose that $c < 0$. Using the same definition of f_U, f_L we have

$$\overline{\int}_I (cf\ d\alpha) \leq p.c. \int_I (cf_U\ d\alpha) \qquad \text{(by A.Cor. I.11.8.8)}$$
$$\implies \overline{\int}_I (cf\ d\alpha) \leq c\Big(p.c. \int_I f_U\ d\alpha\Big) \qquad \text{(by A.Cor. I.11.8.7(b))}$$
$$\implies \frac{1}{c}\Big(\overline{\int}_I (cf)\ d\alpha\Big) \geq p.c. \int_I f_U\ d\alpha$$
$$\implies \frac{1}{c}\Big(\overline{\int}_I (cf)\ d\alpha\Big) \geq p.c. \int_I f_U\ d\alpha \geq \overline{\int}_I f\ d\alpha \qquad \text{(by A.Cor. I.11.8.8)}$$
$$\implies \overline{\int}_I (cf)\ d\alpha \leq c\Big(\overline{\int}_I f\ d\alpha\Big)$$
$$\implies \overline{\int}_I (cf)\ d\alpha \leq c\Big(\int_I f\ d\alpha\Big) \qquad \text{(by A.Cor. I.11.8.8)}$$

and

$$\underline{\int}_I (cf)\ d\alpha \geq p.c. \int_I (cf_L)\ d\alpha \qquad \text{(by A.Cor. I.11.8.8)}$$
$$\implies \underline{\int}_I (cf)\ d\alpha \geq c\Big(p.c. \int_I f_L\ d\alpha\Big) \qquad \text{(by A.Cor. I.11.8.7(b))}$$
$$\implies \frac{1}{c}\Big(\underline{\int}_I (cf)\ d\alpha\Big) \leq p.c. \int_I f_L\ d\alpha$$
$$\implies \frac{1}{c}\Big(\underline{\int}_I (cf)\ d\alpha\Big) \leq p.c. \int_I f_L\ d\alpha \leq \underline{\int}_I f\ d\alpha \qquad \text{(by A.Cor. I.11.8.8)}$$
$$\implies \underline{\int}_I (cf)\ d\alpha \geq c\Big(\underline{\int}_I f\ d\alpha\Big)$$
$$\implies \underline{\int}_I (cf)\ d\alpha \geq c\Big(\int_I f\ d\alpha\Big). \qquad \text{(by A.Cor. I.11.8.8)}$$

By A.Cor. I.11.8.9 we have

$$c\Big(\int_I f\ d\alpha\Big) \leq \underline{\int}_I (cf)\ d\alpha \leq \overline{\int}_I (cf)\ d\alpha \leq c\Big(\int_I f\ d\alpha\Big)$$

and thus by A.Cor. I.11.8.8 we have

$$\int_I (cf)\ d\alpha = \underline{\int}_I (cf)\ d\alpha = \overline{\int}_I (cf)\ d\alpha = c\Big(\int_I f\ d\alpha\Big).$$

We conclude that $\forall c \in \mathbb{R}$, $\int_I (cf)\ d\alpha = c(\int_I f\ d\alpha)$. □

Proof. (c) We have

$$\begin{aligned}
\int_I f\, d\alpha - \int_I g\, d\alpha &= \int_I f\, d\alpha + \int_I (-g)\, d\alpha && \text{(by A.Cor. I.11.8.11(b))} \\
&= \int_I \big(f + (-g)\, d\alpha\big) && \text{(by A.Cor. I.11.8.11(a))} \\
&= \int_I (f-g)\, d\alpha. && \text{(by Def. I.9.2.1)}
\end{aligned}$$

□

Proof. (d) Let $f_U : I \to \mathbb{R}$ be a piecewise constant function on I which majorizes f. f_U is well-defined since by A.Cor. I.11.8.8 f is a bounded function on a bounded interval I. Since $0 \le f(x) \le f_U(x)$ for every $x \in I$, we have

$$\begin{aligned}
& 0 \le p.c. \int_I f_U\, d\alpha && \text{(by A.Cor. I.11.8.7(d))} \\
\Longrightarrow\ & 0 \le \overline{\int}_I f\, d\alpha && \text{(by A.Cor. I.11.8.8)} \\
\Longrightarrow\ & 0 \le \int_I f\, d\alpha. && \text{(by A.Cor. I.11.8.8)}
\end{aligned}$$

□

Proof. (e) We have $f(x) - g(x) \ge 0$ for every $x \in I$ and by A.Cor. I.11.8.11(c) $f - g$ is Riemann-Stieltjes integrable on I with respect to α. Thus

$$\begin{aligned}
& \int_I (f-g)\, d\alpha \ge 0 && \text{(by A.Cor. I.11.8.11(d))} \\
\Longrightarrow\ & \int_I f\, d\alpha - \int_I g\, d\alpha \ge 0 && \text{(by A.Cor. I.11.8.11(c))} \\
\Longrightarrow\ & \int_I f\, d\alpha \ge \int_I g\, d\alpha.
\end{aligned}$$

□

Proof. (f) We have

$$\begin{aligned}
\int_I f\, d\alpha &= p.c. \int_I f\, d\alpha && \text{(by A.Cor. I.11.8.10)} \\
&= c\alpha[I]. && \text{(by A.Cor. I.11.8.7(f))}
\end{aligned}$$

□

Proof. (g) Let $f_U : I \to \mathbb{R}$ be a piecewise constant function on I which majorizes f. Let $f_L : I \to \mathbb{R}$ be a piecewise constant function on I which minorizes f. f_U, f_L are well-defined since by A.Cor. I.11.8.8 f is a bounded function on a bounded interval I. Then we have

$$p.c. \int_I f_L \leq \underline{\int}_I f = \int_I f = \overline{\int}_I f \leq p.c. \int_I f_U.$$

By A.Cor. I.11.8.7(h) we know that $f_U|_J : J \to \mathbb{R}, f_L|_J : J \to \mathbb{R}$ are piecewise constant function on J and $f_U|_K : K \to \mathbb{R}$, $f_L|_K : K \to \mathbb{R}$ are piecewise constant functions on K. By Def. I.11.3.1 we know that $f_U|_J$ majorizes $f|_J$ and $f_L|_J$ minorizes $f|_J$, similarly $f_U|_K$ majorizes $f|_K$ and $f_L|_K$ minorizes $f|_K$. Thus, $f|_J, f|_K$ are bounded functions on bounded intervals J, K, respectively. So $\overline{\int}_J f|_J \, d\alpha$, $\overline{\int}_K f|_K \, d\alpha$, $\underline{\int}_J f|_J \, d\alpha$, $\underline{\int}_K f|_K \, d\alpha$ are well-defined. Then we have

$$\begin{aligned}
&\overline{\int}_J f|_J \, d\alpha + \overline{\int}_K f|_K \, d\alpha \\
&\qquad \leq p.c. \int_J f_U|_J \, d\alpha + p.c. \int_K f_U|_K \, d\alpha && \text{(by A.Cor. I.11.8.8)} \\
\Longrightarrow\ &\overline{\int}_J f|_J \, d\alpha + \overline{\int}_K f|_K \, d\alpha \leq p.c. \int_I f_U \, d\alpha && \text{(by A.Cor. I.11.8.7(h))} \\
\Longrightarrow\ &\overline{\int}_J f|_J \, d\alpha + \overline{\int}_K f|_K \, d\alpha \leq \overline{\int}_I f \, d\alpha && \text{(by A.Cor. I.11.8.8)} \\
\Longrightarrow\ &\overline{\int}_J f|_J \, d\alpha + \overline{\int}_K f|_K \, d\alpha \leq \int_I f \, d\alpha && \text{(by A.Cor. I.11.8.8)}
\end{aligned}$$

and

$$\begin{aligned}
&\underline{\int}_J f|_J \, d\alpha + \underline{\int}_K f|_K \, d\alpha \\
&\qquad \geq p.c. \int_J f_L|_J \, d\alpha + p.c. \int_K f_L|_K \, d\alpha && \text{(by A.Cor. I.11.8.8)} \\
\Longrightarrow\ &\underline{\int}_J f|_J \, d\alpha + \underline{\int}_K f|_K \, d\alpha \geq p.c. \int_I f_L \, d\alpha && \text{(by A.Cor. I.11.8.7(h))} \\
\Longrightarrow\ &\underline{\int}_J f|_J \, d\alpha + \underline{\int}_K f|_K \, d\alpha \geq \underline{\int}_I f \, d\alpha && \text{(by A.Cor. I.11.8.8)} \\
\Longrightarrow\ &\underline{\int}_J f|_J \, d\alpha + \underline{\int}_K f|_K \, d\alpha \geq \int_I f \, d\alpha. && \text{(by A.Cor. I.11.8.8)}
\end{aligned}$$

By A.Cor. I.11.8.9 we have

$$\int_I f \, d\alpha \leq \underline{\int}_J f|_J \, d\alpha + \underline{\int}_K f|_K \, d\alpha \leq \overline{\int}_J f|_J \, d\alpha + \overline{\int}_K f|_K \, d\alpha \leq \int_I f \, d\alpha$$

and thus we have

$$\underline{\int}_J f|_J \, d\alpha + \underline{\int}_K f|_K \, d\alpha = \overline{\int}_J f|_J \, d\alpha + \overline{\int}_J f|_K \, d\alpha = \int_I f \, d\alpha.$$

Since

$$\begin{aligned}
&\underline{\int}_J f|_J \, d\alpha + \underline{\int}_K f|_K \, d\alpha \\
&\quad = \overline{\int}_J f|_J \, d\alpha + \overline{\int}_J f|_K \, d\alpha \\
\implies 0 \geq &\underline{\int}_J f|_J \, d\alpha - \overline{\int}_J f|_J \, d\alpha \\
&\quad = \overline{\int}_J f|_K \, d\alpha - \underline{\int}_K f|_K \, d\alpha \geq 0 \qquad \text{(by A.Cor. I.11.8.9)} \\
\implies &\underline{\int}_J f|_J \, d\alpha - \overline{\int}_J f|_J \, d\alpha \\
&\quad = \overline{\int}_J f|_K \, d\alpha - \underline{\int}_K f|_K \, d\alpha = 0,
\end{aligned}$$

by A.Cor. I.11.8.8 we have

$$\begin{aligned}
\int_J f|_J \, d\alpha &= \underline{\int}_J f|_J \, d\alpha = \overline{\int}_J f|_J \, d\alpha, \\
\int_K f|_K \, d\alpha &= \underline{\int}_K f|_K \, d\alpha = \overline{\int}_K f|_K \, d\alpha, \\
\int_J f|_J \, d\alpha + \int_K f|_K \, d\alpha &= \int_I f \, d\alpha.
\end{aligned}$$

□

A.Cor. I.11.8.12 (Laws of Riemann-Stieltjes integration)**.** Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I. Let $f : I \to \mathbb{R}$ be a bounded function, and let $\mathbf{P}$ be a partition of I. We define the *upper Riemann-Stieltjes sum* $U(f, \alpha, \mathbf{P})$ and the *lower Riemann-Stieltjes sum* $L(f, \alpha, \mathbf{P})$ by

$$U(f, \alpha, \mathbf{P}) \coloneqq \sum_{J \in \mathbf{P} : J \neq \emptyset} \big(\sup_{x \in J} f(x)\big) \alpha[J]$$

and

$$L(f, \alpha, \mathbf{P}) \coloneqq \sum_{J \in \mathbf{P} : J \neq \emptyset} \big(\inf_{x \in J} f(x)\big) \alpha[J].$$

A.Cor. I.11.8.13. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I. Let $f : I \to \mathbb{R}$ be a bounded function, and let g be a function which majorizes f and which is piecewise constant with respect to some partition $\mathbf{P}$ of I. Then

$$p.c. \int_I g \, d\alpha \geq U(f, \alpha, \mathbf{P}).$$

Similarly, if h is a function which minorizes f and is piecewise constant with respect to $\mathbf{P}$, then

$$p.c. \int_I h \, d\alpha \leq L(f, \alpha, \mathbf{P}).$$

Proof. Since g majorizes f and h minorizes f, by Def. I.11.3.1 we have $h(x) \leq f(x) \leq g(x)$ for every $x \in I$. Since $\mathbf{P}$ is a partition of I, by Def. I.11.1.10 for every $J \in \mathbf{P}$, we have $h(x) \leq f(x) \leq g(x)$ for all $x \in J$. In particular, when $J \neq \emptyset$ we have

$$h(x) \leq \inf_{x \in J} f(x) \leq f(x) \leq \sup_{x \in J} f(x) \leq g(x)$$

for every $x \in J$. Let $c_{g|_J}, c_{h|_J}$ be constant values of $g|_J, h|_J$, respectively. Then we have

$$\begin{aligned}
U(f, \alpha, \mathbf{P}) &= \sum_{J \in \mathbf{P} : J \neq \emptyset} \big(\sup_{x \in J} f(x)\big) \alpha[J] && \text{(by A.Cor. I.11.8.12)} \\
&\leq \sum_{J \in \mathbf{P} : J \neq \emptyset} c_{g|_J} \alpha[J] && \text{(by Prop. I.7.1.11(h))} \\
&= \sum_{J \in \mathbf{P}} c_{g|_J} \alpha[J] && \text{(by Prop. I.7.1.11(a)(e))} \\
&= p.c. \int_{[\mathbf{P}]} g \, d\alpha && \text{(by Def. I.11.8.5)} \\
&= p.c. \int_I g \, d\alpha && \text{(by A.Cor. I.11.8.5)}
\end{aligned}$$

and

$$\begin{aligned}
L(f, \alpha, \mathbf{P}) &= \sum_{J \in \mathbf{P} : J \neq \emptyset} \big(\inf_{x \in J} f(x)\big) \alpha[J] && \text{(by A.Cor. I.11.8.12)} \\
&\geq \sum_{J \in \mathbf{P} : J \neq \emptyset} c_{h|_J} \alpha[J] && \text{(by Prop. I.7.1.11(h))} \\
&= \sum_{J \in \mathbf{P}} c_{h|_J} \alpha[J] && \text{(by Prop. I.7.1.11(a)(e))} \\
&= p.c. \int_{[\mathbf{P}]} h \, d\alpha && \text{(by Def. I.11.8.5)}
\end{aligned}$$

$$= p.c. \int_I h \; d\alpha. \qquad \text{(by A.Cor. I.11.8.5)}$$

□

A.Cor. I.11.8.14. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I. Let $f : I \to \mathbb{R}$ be a bounded function. Then

$$\overline{\int_I} f \; d\alpha = \inf\{U(f, \alpha, \mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}$$

and

$$\underline{\int_I} f \; d\alpha = \sup\{L(f, \alpha, \mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}.$$

Proof. Let g be a function which majorizes f and which is piecewise constant with respect to some partition $\mathbf{P}_g$ of I. Let h be a function which minorizes f and which is piecewise constant with respect to some partition $\mathbf{P}_h$ of I. Both functions are well defined since f is bounded function on a bounded interval I. By A.Cor. I.11.8.13 we have

$$\inf\{U(f, \alpha, \mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \le U(f, \alpha, \mathbf{P}_g) \le p.c. \int_I g \; d\alpha$$

and

$$\sup\{L(f, \alpha, \mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \ge L(f, \alpha, \mathbf{P}_h) \ge p.c. \int_I h \; d\alpha.$$

Since g, h were arbitrary, by A.Cor. I.11.8.8 we have

$$\inf\{U(f, \alpha, \mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \le \overline{\int_I} f \; d\alpha$$

and

$$\sup\{L(f, \alpha, \mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \ge \underline{\int_I} f \; d\alpha.$$

Let $\mathbf{P}$ be a partition of I. Let $G : I \to \mathbb{R}$ be a function where $G(x) = \sup_{x \in J} f(x)$ for all $J \in \mathbf{P}$. Let $H : I \to \mathbb{R}$ be a function where $H(x) = \inf_{x \in J} f(x)$ for all $J \in \mathbf{P}$. By Def. I.11.2.3 we know that G, H are piecewise constant functions with respect to $\mathbf{P}$. Thus, we have

$$\begin{aligned}
U(f, \alpha, \mathbf{P}) &= \sum_{J \in \mathbf{P} : J \ne \emptyset} \big(\sup_{x \in J} f(x)\big)\alpha[J] && \text{(by A.Cor. I.11.8.12)} \\
&= \sum_{J \in \mathbf{P}} \big(\sup_{x \in J} f(x)\big)\alpha[J] && \text{(by Prop. I.7.1.11(a)(e))} \\
&= p.c. \int_{[\mathbf{P}]} G \; d\alpha && \text{(by Def. I.11.8.5)}
\end{aligned}$$

$$= p.c. \int_I G\, d\alpha \qquad \text{(by A.Cor. I.11.8.5)}$$

and

$$\begin{aligned} L(f,\alpha,\mathbf{P}) &= \sum_{J\in\mathbf{P}: J\neq\emptyset} \big(\inf_{x\in J} f(x)\big)\alpha[J] && \text{(by A.Cor. I.11.8.12)} \\ &= \sum_{J\in\mathbf{P}} \big(\inf_{x\in J} f(x)\big)\alpha[J] && \text{(by Prop. I.7.1.11(a)(e))} \\ &= p.c. \int_{[\mathbf{P}]} H\, d\alpha && \text{(by Def. I.11.8.5)} \\ &= p.c. \int_I H\, d\alpha. && \text{(by A.Cor. I.11.8.5)} \end{aligned}$$

By A.Cor. I.11.8.8 we have

$$\overline{\int_I} f\, d\alpha \leq p.c. \int_I G\, d\alpha = U(f,\alpha,\mathbf{P})$$

and

$$\underline{\int_I} f\, d\alpha \geq p.c. \int_I H\, d\alpha = L(f,\alpha,\mathbf{P}).$$

Since $\mathbf{P}$ was arbitrary, we have

$$\overline{\int_I} f\, d\alpha \leq \inf\{U(f,\alpha,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \leq U(f,\alpha,\mathbf{P})$$

and

$$\underline{\int_I} f\, d\alpha \geq \sup\{L(f,\alpha,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\} \leq L(f,\alpha,\mathbf{P}).$$

Combine all results above we have

$$\overline{\int_I} f\, d\alpha = \inf\{U(f,\alpha,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}$$

and

$$\underline{\int_I} f\, d\alpha = \sup\{L(f,\alpha,\mathbf{P}) : \mathbf{P} \text{ is a partition of } I\}.$$

□

A.Cor. I.11.8.15. Let I be a bounded interval, let $\alpha : X \to \mathbb{R}$ be a monotone increasing function defined on some interval X which contains I. Let f be a function which is uniformly continuous on I. Then f is Riemann-Stieltjes integrable on I with respect to α.

Proof. From Prop. I.9.9.15 we see that f is bounded. By A.Cor. I.11.8.8 we have to show that $\underline{\int}_I f \, d\alpha = \overline{\int}_I f \, d\alpha$.

If I is a point or the empty set then the theorem is trivial, so let us assume that I is one of the four intervals $[a, b]$, (a, b), $(a, b]$, or $[a, b)$ for some real numbers $a < b$.

Let $\varepsilon > 0$ be arbitrary. By uniform continuity, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in I$ are such that $|x - y| < \delta$. By the Archimedean principle, there exists an integer $N > 0$ such that $(b - a)/N < \delta$ and

$$\frac{\lim_{x\to b^-;x\in X} \alpha(x) - \lim_{x\to a^+;x\in X} \alpha(x)}{N} < \delta.$$

Note that we can partition I into N intervals $J_1, \ldots, J_N$, each of length $(b - a)/N$. By A.Cor. I.11.8.14, we thus have

$$\overline{\int}_I f \, d\alpha \leq \sum_{k=1}^{N} \big(\sup_{x\in J_k} f(x) \big) \alpha[J_k]$$

and

$$\underline{\int}_I f \, d\alpha \geq \sum_{k=1}^{N} \big(\inf_{x\in J_k} f(x) \big) \alpha[J_k]$$

so in particular

$$\overline{\int}_I f \, d\alpha - \underline{\int}_I f \, d\alpha \leq \sum_{k=1}^{N} \big(\sup_{x\in J_k} f(x) - \inf_{x\in J_k} f(x) \big) \alpha[J_k].$$

However, we have $|f(x) - f(y)| < \varepsilon$ for all $x, y \in J_k$, since

$$\begin{aligned} \alpha[J_k] &= \frac{\lim_{x\to \sup(J_K)^-;x\in X} \alpha(x) - \lim_{x\to \inf(J_K)^+;x\in X} \alpha(x)}{N} \\ &\leq \frac{\lim_{x\to b^-;x\in X} \alpha(x) - \lim_{x\to a^+;x\in X} \alpha(x)}{N} && (\alpha \text{ is monotone increasing}) \\ &< \delta. \end{aligned}$$

In particular, we have

$$f(x) < f(y) + \varepsilon \text{ for all } x, y \in J_k.$$

Taking suprema in x, we obtain

$$\sup_{x\in J_k} f(x) \leq f(y) + \varepsilon \text{ for all } y \in J_k,$$

and then taking infima in y we obtain

$$\sup_{x\in J_k} f(x) \leq \inf_{y\in J_k} f(y) + \varepsilon.$$

Inserting this bound into our previous inequality, we obtain

$$\overline{\int_I} f\, d\alpha - \underline{\int_I} f\, d\alpha \le \sum_{k=1}^{N} \varepsilon \alpha[J_k],$$

but by Lem. I.11.8.4 we thus have

$$\overline{\int_I} f\, d\alpha - \underline{\int_I} f\, d\alpha \le \varepsilon \alpha[I].$$

But $\varepsilon > 0$ was arbitrary, while $\alpha[I]$ is fixed. Thus, $\overline{\int_I} f\, d\alpha - \underline{\int_I} f\, d\alpha$ cannot be positive. By A.Cor. I.11.8.8 we thus have that f is Riemann-Stieltjes integrable on I with respect to α. □

— Exercises —

Ex. I.11.8.1. Prove Lem. I.11.8.4.

Proof. See Lem. I.11.8.4. □

Ex. I.11.8.2. State and prove a version of Prop. I.11.2.13 for the Riemann-Stieltjes integral.

Proof. See A.Cor. I.11.8.4. □

Ex. I.11.8.3. State and prove a version of Thm. I.11.2.16 for the Riemann-Stieltjes integral.

Proof. See A.Cor. I.11.8.7. □

Ex. I.11.8.4. State and prove a version of Thm. I.11.5.1 for the Riemann-Stieltjes integral.

Proof. See A.Cor. I.11.8.15. □

Ex. I.11.8.5. Let $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ be the signum function

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when } x < 0. \end{cases}$$

Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function. Show that f is Riemann-Stieltjes integrable with respect to sgn, and that

$$\int_{[-1,1]} f\, d\operatorname{sgn} = 2f(0).$$

Proof. We first show that f is Riemann-Stieltjes integrable on $[-1, 1]$ with respect to sgn. By Thm. I.9.9.16 f is uniformly continuous, and thus by Prop. I.9.9.15 f is bounded. Since sgn is monotone increasing, by A.Cor. I.11.8.15 we know that f is Riemann-Stieltjes integrable on $[-1, 1]$ with respect to sgn.

Now we show that $\displaystyle\int_{[-1,1]} f \, d\,\mathrm{sgn} = 2f(0)$. Since f is continuous, by Prop. I.9.4.7 we have

$$\begin{aligned}
&\forall\varepsilon \in \mathbb{R}^+, \exists\delta \in \mathbb{R}^+ : \forall x \in [-1, 1], |x - 0| \le \delta \\
\implies &|f(x) - f(0)| \le \varepsilon \\
\implies &f(0) - \varepsilon \le f(x) \le f(0) + \varepsilon.
\end{aligned}$$

In particular, we can choose some $\delta \le 1$ such that

$$\begin{aligned}
&\forall\varepsilon \in \mathbb{R}^+, \exists\delta \in \mathbb{R}^+ : \\
&\forall x \in [-1, 1], |x - 0| \le \delta \le 1 \implies f(0) - \varepsilon \le f(x) \le f(0) + \varepsilon.
\end{aligned}$$

Since f is bounded, $\exists M \in \mathbb{R}^+$ such that $|f(x)| \le M$ for all $x \in [-1, 1]$. Let $f_U : [-1, 1] \to \mathbb{R}$ be the function

$$f_U(x) = \begin{cases} f(0) + \varepsilon & \text{if } x \in [-\delta, \delta] \\ M & \text{if } x \in [-1, 1] \setminus [-\delta, \delta] \end{cases}$$

and let $f_L : [-1, 1] \to \mathbb{R}$ be the function

$$f_L(x) = \begin{cases} f(0) - \varepsilon & \text{if } x \in [-\delta, \delta] \\ -M & \text{if } x \in [-1, 1] \setminus [-\delta, \delta] \end{cases}$$

Clearly, f_U, f_L are piecewise constant on $[-1, 1]$, f_U majorizes f and f_L minorizes f. Then we have

$$\begin{aligned}
\overline{\int}_{[-1,1]} f \, d\,\mathrm{sgn} &\le p.c. \int f_U \, d\,\mathrm{sgn} && \text{(by A.Cor. I.11.8.8)} \\
&= M\big(\mathrm{sgn}(-\delta) - \mathrm{sgn}(-1)\big) && \text{(by Def. I.11.8.5)} \\
&\quad + (f(0) + \varepsilon)(\mathrm{sgn}(\delta) - \mathrm{sgn}(-\delta)) \\
&\quad + M\big(\mathrm{sgn}(1) - \mathrm{sgn}(\delta)\big) \\
&= 2\big(f(0) + \varepsilon\big)
\end{aligned}$$

and

$$\begin{aligned}
\underline{\int}_{[-1,1]} f \, d\,\mathrm{sgn} &\ge p.c. \int f_L \, d\,\mathrm{sgn} && \text{(by A.Cor. I.11.8.8)} \\
&= M\big(\mathrm{sgn}(-\delta) - \mathrm{sgn}(-1)\big) && \text{(by Def. I.11.8.5)} \\
&\quad + (f(0) - \varepsilon)(\mathrm{sgn}(\delta) - \mathrm{sgn}(-\delta))
\end{aligned}$$

$$\begin{aligned}&+ M\big(\mathrm{sgn}(1) - \mathrm{sgn}(\delta)\big)\\ &= 2\big(f(0) - \varepsilon\big).\end{aligned}$$

Combining results above we have

$$2(f(0) - \varepsilon) \leq \underline{\int}_{[-1,1]} f\ d\mathrm{sgn} = \int_{[-1,1]} f\ d\mathrm{sgn} = \overline{\int}_{[-1,1]} f\ d\mathrm{sgn} \leq 2(f(0) + \varepsilon).$$

Since ε was arbitrary, we thus have $\displaystyle\int_{[-1,1]} f\ d\mathrm{sgn} = 2f(0)$. □

I.11.9 The two fundamental theorems of calculus

Thm. I.11.9.1 (First, Fundamental Theorem of Calculus)**.** Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Let $F : [a, b] \to \mathbb{R}$ be the function

$$F(x) := \int_{[a,x]} f.$$

Then F is continuous. Furthermore, if $x_0 \in [a, b]$ and f is continuous at x_0, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.

Proof. Since f is Riemann integrable, it is bounded (by Def. I.11.3.4). Thus, we have some real number M such that $-M \leq f(x) \leq M$ for all $x \in [a, b]$.

Now let $x < y$ be two elements of $[a, b]$. Then notice that

$$F(y) - F(x) = \int_{[a,y]} f - \int_{[a,x]} f = \int_{[x,y]} f$$

by Thm. I.11.4.1(h). By Thm. I.11.4.1(e) we thus have

$$\int_{[x,y]} f \leq \int_{[x,y]} M = p.c. \int_{[x,y]} M = M(y - x)$$

and

$$\int_{[x,y]} f \geq \int_{[x,y]} -M = p.c. \int_{[x,y]} -M = -M(y - x)$$

and thus

$$|F(y) - F(x)| \leq M(y - x).$$

This is for $y > x$. By interchanging x and y we thus see that

$$|F(y) - F(x)| \leq M(x - y)$$

when $x > y$. Also, we have $F(y) - F(x) = 0$ when $x = y$. Thus, in all three cases we have

$$|F(y) - F(x)| \leq M|x - y|.$$

Now let $z \in [a, b]$, and let $(z_n)_{n=0}^{\infty}$ be any sequence in $[a, b]$ converging to z. Then we have

$$-M|z_n - z| \leq F(z_n) - F(z) \leq M|z_n - z|$$

for each n. But $-M|z_n - z|$ and $M|z_n - z|$ both converge to 0 as $n \to \infty$, so by the squeeze test $F(z_n) - F(z)$ converges to 0 as $n \to \infty$, and thus $\lim_{n\to\infty} F(z_n) = F(z)$. Since this is true for all sequences $z_n \in [a, b]$ converging to z, we thus see that F is continuous at z (by Prop. I.9.4.7). Since z was an arbitrary element of $[a, b]$, we thus see that F is continuous (The above proof also show that when F is Lipschitz continuous, F is also continuous, see Ex. I.10.2.6).

Now suppose that $x_0 \in [a, b]$, and f is continuous at x_0. Choose any $\varepsilon > 0$. Then by continuity, we can find a $\delta > 0$ such that $|f(x) - f(x_0)| \leq \varepsilon$ for all x in the interval $I := [x_0 - \delta, x_0 + \delta] \cap [a, b]$, or in other words

$$f(x_0) - \varepsilon \leq f(x) \leq f(x_0) + \varepsilon \text{ for all } x \in I.$$

We now show that

$$|F(y) - F(x_0) - f(x_0)(y - x_0)| \leq \varepsilon|y - x_0|$$

for all $y \in I$, since Prop. I.10.1.7 will then imply that F is differentiable at x_0 with derivative $F'(x_0) = f(x_0)$ as desired.

Now fix $y \in I$. There are three cases. If $y = x_0$, then $F(y) - F(x_0) - f(x_0)(y - x_0) = 0$ and so the claim is obvious. If $y > x_0$, then

$$F(y) - F(x_0) = \int_{[x_0,y]} f.$$

Since x_0, $y \in I$, and I is a connected set (by Cor. I.11.1.6), then $[x_0, y]$ is a subset of I, and thus we have

$$f(x_0) - \varepsilon \leq f(x) \leq f(x_0) + \varepsilon \text{ for all } x \in [x_0, y],$$

and thus by Thm. I.11.4.1(e)

$$\big(f(x_0) - \varepsilon\big)(y - x_0) \leq \int_{[x_0,y]} f \leq \big(f(x_0) + \varepsilon\big)(y - x_0)$$

and so in particular

$$|F(y) - F(x_0) - f(x_0)(y - x_0)| \leq \varepsilon|y - x_0|$$

as desired. If $y < x_0$, then

$$F(y) - F(x_0) = -\big(F(x_0) - F(y)\big) = -\int_{[y,x_0]} f.$$

Since x_0, $y \in I$, and I is a connected set (by Cor. I.11.1.6), then $[y, x_0]$ is a subset of I, and thus we have

$$f(x_0) - \varepsilon \leq f(x) \leq f(x_0) + \varepsilon \text{ for all } x \in [y, x_0],$$

and thus by Thm. I.11.4.1(e)

$$\begin{aligned}
&\big(f(x_0) - \varepsilon\big)(x_0 - y) \leq \int_{[y,x_0]} f \leq \big(f(x_0) + \varepsilon\big)(x_0 - y) \\
\implies &\big(f(x_0) - \varepsilon\big)(y - x_0) \geq -\int_{[y,x_0]} f = F(y) - F(x_0) \geq \big(f(x_0) + \varepsilon\big)(y - x_0) \\
\implies &-\varepsilon(y - x_0) \geq F(y) - F(x_0) - f(x_0)(y - x_0) \geq \varepsilon(y - x_0) \\
\implies &|F(y) - F(x_0) - f(x_0)(y - x_0)| \leq \varepsilon|y - x_0|
\end{aligned}$$

as desired. □

Note. Informally, the first fundamental theorem of calculus asserts that

$$\left(\int_{[a,x]} f\right)'(x) = f(x)$$

given a certain number of assumptions on f. Roughly, this means that the derivative of an integral recovers the original function.

Def. I.11.9.3 (Antiderivatives)**.** Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be a function. We say that a function $F : I \to \mathbb{R}$ is an *antiderivative* of f if F is differentiable on I and $F'(x) = f(x)$ for all limit points x of I.

Thm. I.11.9.4. ond Fundamental Theorem of Calculus] Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. If $F : [a, b] \to \mathbb{R}$ is an antiderivative of f, then

$$\int_{[a,b]} f = F(b) - F(a).$$

Proof. The claim is trivial when $b = a$, so assume $b > a$. So, in particular, all points of $[a, b]$ are limit points. We will use Riemann sums. The idea is to show that

$$U(f, \mathbf{P}) \geq F(b) - F(a) \geq L(f, \mathbf{P})$$

for every partition $\mathbf{P}$ of $[a, b]$. The left inequality asserts that $F(b) - F(a)$ is a lower bound for $\{U(f, \mathbf{P}) : \mathbf{P}$ is a partition of $[a, b]\}$, while the right inequality asserts that $F(b) - F(a)$ is an upper bound for $\{L(f, \mathbf{P}) : \mathbf{P}$ is a partition of $[a, b]\}$. But by Prop. I.11.3.12, this means that

$$\overline{\int}_{[a,b]} f \geq F(b) - F(a) \geq \underline{\int}_{[a,b]} f,$$

but since f is assumed to be Riemann integrable, both the upper and lower Riemann integral equal $\int_{[a,b]} f$. The claim follows.

We have to show the bound $U(f, \mathbf{P}) \geq F(b) - F(a) \geq L(f, \mathbf{P})$. We shall just show the first inequality $U(f, \mathbf{P}) \geq F(b) - F(a)$; the other inequality is similar.

Let $\mathbf{P}$ be a partition of $[a, b]$. From Lem. I.11.8.4 we have

$$F(b) - F(a) = \sum_{J \in \mathbf{P}} F[J] = \sum_{J \in \mathbf{P}: J \neq \emptyset} F[J],$$

while from definition we have

$$U(f, \mathbf{P}) = \sum_{J \in \mathbf{P}: J \neq \emptyset} \sup_{x \in J} f(x)|J|.$$

Thus, it will suffice to show that

$$F[J] \leq \sup_{x \in J} f(x)|J|$$

for all $J \in \mathbf{P}$ (other than the empty set).

When J is a point then the claim is clear, since both sides are zero. Now suppose that $J = [c, d], (c, d], [c, d)$, or (c, d) for some $c < d$. Then the left-hand side is $F[J] = F(d) - F(c)$. Note that F, being differentiable, is continuous, so we may use the simplified formula for the F-length as opposed to the more complicated one in Def. I.11.8.1. By the mean-value theorem (Cor. I.10.2.9), this is equal to $(d - c)F'(e)$ for some $e \in J$. But since $F'(e) = f(e)$, we thus have

$$F[J] = (d - c)f(e) = f(e)|J| \leq \sup_{x \in J} f(x)|J|$$

as desired. □

Note. One can use the second fundamental theorem of calculus to compute integrals relatively easily provided that you can find an anti-derivative of the integrand f. The first fundamental theorem of calculus ensures that every *continuous* Riemann integrable function has an anti-derivative. For discontinuous functions, the situation is more complicated. Also, not every function with an anti-derivative is Riemann integrable.

Lem. I.11.9.5. Let I be a bounded interval, and let $f : I \to \mathbb{R}$ be a function. Let $F : I \to \mathbb{R}$ and $G : I \to \mathbb{R}$ be two antiderivatives of f. Then there exists a real number C such that $F(x) = G(x) + C$ for all $x \in I$.

Proof. If $I = \emptyset$, then the claim is trivially true. If $I = \{a\}$ for some $a \in \mathbb{R}$, then we can simply set $C = F(a) - G(a)$. So suppose that I is one of $(a, b), [a, b), (a, b], [a, b]$ for some $a, b \in \mathbb{R}$ and $a < b$. Since I is a bounded interval, for all $x \in I$ we know that x is a limit point. Since F, G are antiderivatives of f, by Def. I.11.9.3 we know that F, G are differentiable on I.

By Thm. I.10.1.13(f) we know that $F - G$ is differentiable on I, and thus by Cor. I.10.1.12 we know that $F - G$ are continuous on I. Let $x, y \in I$ and $x < y$. Since I is a bounded interval, by Def. I.11.1.10 we know that I is connected, and thus by Def. I.11.1.1 $[x, y] \subseteq I$. By the mean-value theorem (Cor. I.10.2.9) we know that

$$\exists c \in I : \frac{(F-G)(x) - (F-G)(y)}{x - y} = (F-G)'(c).$$

Thus, we have

$$\begin{aligned}
&\frac{(F-G)(x) - (F-G)(y)}{x - y} = (F-G)'(c) \\
\implies &\frac{(F-G)(x) - (F-G)(y)}{x - y} = F'(c) - G'(c) && \text{(by Thm. I.10.1.13(f))} \\
\implies &\frac{(F-G)(x) - (F-G)(y)}{x - y} = 0 && \text{(by hypothesis)} \\
\implies &(F-G)(x) = (F-G)(y) \\
\implies &F(x) - G(x) = F(y) - G(y) && \text{(by Def. I.9.2.1)} \\
\implies &F(x) = G(x) + F(y) - G(y).
\end{aligned}$$

By setting $C = F(y) - G(y)$ we are done. □

— Exercises —

Ex. I.11.9.1. Let $f : [0, 1] \to \mathbb{R}$ be the function in Ex. I.9.8.5. Show that for every rational number $q \in \mathbb{Q} \cap (0, 1)$, the function $F : [0, 1] \to \mathbb{R}$ defined by the formula $F(x) := \int_{[0,x]} f$ is not differentiable at q.

Proof. By Ex. I.9.8.5 we know that f is strictly monotone increasing, thus by Prop. I.11.6.1 we know that f is Riemann integrable and F is well-defined. By Ex. I.9.8.5 f is not continuous at q, thus by Ex. I.11.9.3 we know that F is not differentiable at q. □

Ex. I.11.9.2. Prove Lem. I.11.9.5.

Proof. See Lem. I.11.9.5. □

Ex. I.11.9.3. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a monotone increasing function. Let $F : [a, b] \to \mathbb{R}$ be the function $F(x) := \int_{[a,x]} f$. Let x_0 be an element of (a, b). Show that F is differentiable at x_0 iff f is continuous at x_0.

Proof. Since f is monotone increasing, by Prop. I.11.6.1 we know that f is Riemann integrable and F is well-defined. If f is continuous at x_0, then by Thm. I.11.9.1 we know that F is differentiable at x_0. So we only need to show that if F is differentiable at x_0, then f is continuous at x_0.

Suppose that F is differentiable at x_0. Suppose for the sake of contradiction that f is not continuous at x_0. Since f is monotone increasing, by A.Cor. I.11.8.1 we know that both $f(x_0+)$ and $f(x_0-)$ exist and $f(x_0-) \le f(x_0+)$. Since f is not continuous at x_0, by Prop. I.9.5.3 we have $f(x_0-) < f(x_0+)$.

If $x \in [a,b] \cap (-\infty, x_0)$, then we have $f(x) \le f(x_0-)$ and

$$\frac{F(x)-F(x_0)}{x-x_0} = \frac{-\int_{[x,x_0]} f}{x-x_0} = \frac{\int_{[x,x_0]} f}{x_0-x} \le \frac{p.c. \int_{[x,x_0]} f(x_0-)}{x_0-x} = f(x_0-).$$

If $x \in [a,b] \cap (x_0, \infty)$, then we have $f(x) \ge f(x_0+)$ and

$$\frac{F(x)-F(x_0)}{x-x_0} = \frac{\int_{[x_0,x]} f}{x-x_0} \ge \frac{p.c. \int_{[x_0,x]} f(x_0+)}{x-x_0} = f(x_0+).$$

Thus, by Prop. I.9.3.14 we have

$$F'(x_0-) \le f(x_0-) < f(x_0+) \le F'(x_0+).$$

But by Def. I.9.3.6 $F'(x_0-) \ne F'(x_0+)$ implies $F'(x_0)$ does not exist, a contradiction. So we conclude that f is continuous at x_0. □

I.11.10 Consequences of the fundamental theorems

Prop. I.11.10.1 (Integration by parts formula)**.** Let $I = [a,b]$, and let $F : [a,b] \to \mathbb{R}$ and $G : [a,b] \to \mathbb{R}$ be differentiable functions on $[a,b]$ such that F' and G' are Riemann integrable on I. Then we have

$$\int_{[a,b]} FG' = F(b)G(b) - F(a)G(a) - \int_{[a,b]} F'G.$$

Proof. Since F is an antiderivative of F' and F' is Riemann integrable on $[a,b]$, by Thm. I.11.9.1 we know that F is continuous on $[a,b]$. Similarly, G is continuous on $[a,b]$. By Cor. I.11.5.2 we know that F and G are Riemann integrable on $[a,b]$. By Thm. I.11.4.5 we know that FG' and $F'G$ are Riemann integrable on $[a,b]$. By Thm. I.10.1.13(d) we have $(FG)' = F'G + FG'$. Thus, by Thm. I.11.4.1(a) $(FG)'$ is Riemann integrable on $[a,b]$ and

$$\begin{aligned}
\int_{[a,b]} (FG') &= \int_{[a,b]} \big((FG)' - F'G\big) \\
&= \int_{[a,b]} \big((FG)'\big) - \int_{[a,b]} (F'G) && \text{(by Thm. I.11.4.1(c))} \\
&= F(b)G(b) - F(a)G(a) - \int_{[a,b]} (F'G). && \text{(by Thm. I.11.9.4)}
\end{aligned}$$

□

Thm. I.11.10.2. Let $\alpha : [a,b] \to \mathbb{R}$ be a monotone increasing function, and suppose that α is also differentiable on $[a,b]$, with α' being Riemann integrable. Let $f : [a,b] \to \mathbb{R}$ be a piecewise constant function on $[a,b]$. Then $f\alpha'$ is Riemann integrable on $[a,b]$, and

$$\int_{[a,b]} f \, d\alpha = \int_{[a,b]} f\alpha' .$$

Proof. Since f is piecewise constant, it is Riemann integrable, and since α' is also Riemann integrable, then $f\alpha'$ is Riemann integrable by Thm. I.11.4.5.

Suppose that f is piecewise constant with respect to some partition $\mathbf{P}$ of $[a,b]$; without loss of generality we may assume that $\mathbf{P}$ does not contain the empty set. Then we have

$$\int_{[a,b]} f \, d\alpha = p.c. \int_{[\mathbf{P}]} f \, d\alpha = \sum_{J \in \mathbf{P}} c_J \alpha[J]$$

where c_J is the constant value of f on J. On the other hand, from Thm. I.11.4.1(h) (and Ex. I.11.4.3) we have

$$\int_{[a,b]} f\alpha' = \sum_{J \in \mathbf{P}} \int_J f\alpha' = \sum_{J \in \mathbf{P}} \int_J c_J \alpha' = \sum_{J \in \mathbf{P}} c_J \int_J \alpha' .$$

But by the second fundamental theorem of calculus (Thm. I.11.9.4), $\int_J \alpha' = \alpha[J]$, and the claim follows. ☐

Cor. I.11.10.3. Let $\alpha : [a,b] \to \mathbb{R}$ be a monotone increasing function, and suppose that α is also differentiable on $[a,b]$, with α' being Riemann integrable. Let $f : [a,b] \to \mathbb{R}$ be a function which is Riemann-Stieltjes integrable with respect to α on $[a,b]$. Then $f\alpha'$ is Riemann integrable on $[a,b]$, and

$$\int_{[a,b]} f \, d\alpha = \int_{[a,b]} f\alpha' .$$

Proof. Note that since f and α' are bounded, then $f\alpha'$ must also be bounded. Also, since α is monotone increasing and differentable, α' is non-negative (by Prop. I.10.3.1).

Let $\varepsilon > 0$. Then, we can find a piecewise constant function $\overline{f}$ majorizing f on $[a,b]$, and a piecewise constant function $\underline{f}$ minorizing f on $[a,b]$, such that

$$\int_{[a,b]} f \, d\alpha - \varepsilon \le \int_{[a,b]} \underline{f} \, d\alpha \le \int_{[a,b]} \overline{f} \, d\alpha \le \int_{[a,b]} f \, d\alpha + \varepsilon .$$

Applying Thm. I.11.10.2, we obtain

$$\int_{[a,b]} f \, d\alpha - \varepsilon \le \int_{[a,b]} \underline{f}\alpha' \le \int_{[a,b]} \overline{f}\alpha' \le \int_{[a,b]} f \, d\alpha + \varepsilon .$$

Since α' is non-negative and $\underline{f}$ minorizes f, then $\underline{f}\alpha'$ minorizes $f\alpha'$. Thus, $\int_{[a,b]} \underline{f}\alpha' \leq \underline{\int}_{[a,b]} f\alpha'$. Thus

$$\int_{[a,b]} f\ d\alpha - \varepsilon \leq \underline{\int}_{[a,b]} f\alpha'.$$

Similarly, we have

$$\overline{\int}_{[a,b]} f\alpha' \leq \int_{[a,b]} f\ d\alpha + \varepsilon.$$

Since these statements are true for any $\varepsilon > 0$, we must have

$$\int_{[a,b]} f\ d\alpha \leq \underline{\int}_{[a,b]} f\alpha' \leq \overline{\int}_{[a,b]} f\alpha' \leq \int_{[a,b]} f\ d\alpha$$

and the claim follows. □

Rmk. I.11.10.4. Informally, Cor. I.11.10.3 asserts that $f\ d\alpha$ is essentially equivalent to $f\frac{d\alpha}{dx}dx$, when α is differentiable. However, the advantage of the Riemann-Stieltjes integral is that it still makes sense even when α is not differentiable.

Lem. I.11.10.5 (Change of variables formula I)**.** Let $[a, b]$ be a closed interval, and let $\phi : [a, b] \to [\phi(a), \phi(b)]$ be a continuous monotone increasing function. Let $f : [\phi(a), \phi(b)] \to \mathbb{R}$ be a piecewise constant function on $[\phi(a), \phi(b)]$. Then $f \circ \phi : [a, b] \to \mathbb{R}$ is also piecewise constant on $[a, b]$, and

$$\int_{[a,b]} f \circ \phi\ d\phi = \int_{[\phi(a),\phi(b)]} f.$$

Proof. Let $\mathbf{P}$ be a partition of $[\phi(a), \phi(b)]$ such that f is piecewise constant with respect to $\mathbf{P}$; we may assume that $\mathbf{P}$ does not contain the empty set. For each $J \in \mathbf{P}$, let c_J be the constant value of f on J, thus

$$\int_{[\phi(a),\phi(b)]} f = \sum_{J\in\mathbf{P}} c_J|J|.$$

For each interval J, let $\phi^{-1}(J)$ be the set $\phi^{-1}(J) \coloneqq \{x \in [a, b] : \phi(x) \in J\}$. Then $\phi^{-1}(J)$ is connected (by Prop. I.9.8.3 and Lem. I.11.1.4), and is thus an interval. Furthermore, c_J is the constant value of $f \circ \phi$ on $\phi^{-1}(J)$ (since $(f \circ \phi)\big(\phi^{-1}(J)\big) = f(J)$). Thus, if we define $\mathbf{S} \coloneqq \big\{\phi^{-1}(J) : J \in \mathbf{P}\big\}$, then $\mathbf{S}$ partitions $[a, b]$ ($\mathbf{S}$ is finite since $\mathbf{P}$ is finite; $\phi^{-1}(J)$ is an interval and ϕ is a bijection from $[a, b]$ to $[\phi(a), \phi(b)]$), and $f \circ \phi$ is piecewise constant with respect to $\mathbf{S}$ (for every $\phi^{-1}(J) \in \mathbf{S}$, f is constant on $\phi^{-1}(J)$). Thus

$$\int_{[a,b]} f \circ \phi\ d\phi = p.c. \int_{[\mathbf{S}]} f \circ \phi\ d\phi = \sum_{J\in\mathbf{P}} c_J\phi[\phi^{-1}(J)].$$

But $\phi[\phi^{-1}(J)] = |J|$ (since $\phi(\phi^{-1}(J)) = J$ and ϕ is continuous), and the claim follows. □

Prop. I.11.10.6 (Change of variables formula II)**.** Let $[a, b]$ be a closed interval, and let $\phi : [a, b] \to [\phi(a), \phi(b)]$ be a continuous monotone increasing function. Let $f : [\phi(a), \phi(b)] \to \mathbb{R}$ be a Riemann integrable function on $[\phi(a), \phi(b)]$. Then $f \circ \phi : [a, b] \to \mathbb{R}$ is Riemann-Stieltjes integrable with respect to ϕ on $[a, b]$, and

$$\int_{[a,b]} f \circ \phi \, d\phi = \int_{[\phi(a),\phi(b)]} f.$$

Proof. This will be obtained from Lem. I.11.10.5 in a similar manner to how Cor. I.11.10.3 was obtained from Thm. I.11.10.2. First, observe that since f is Riemann integrable, it is bounded, and then $f \circ \phi$ must also be bounded (by Prop. I.9.8.3 ϕ is a bijection).

Let $\varepsilon > 0$. Then, we can find a piecewise constant function $\overline{f}$ majorizing f on $[\phi(a), \phi(b)]$, and a piecewise constant function $\underline{f}$ minorizing f on $[\phi(a), \phi(b)]$, such that

$$\int_{[\phi(a),\phi(b)]} f - \varepsilon \leq \int_{[\phi(a),\phi(b)]} \underline{f} \leq \int_{[\phi(a),\phi(b)]} \overline{f} \leq \int_{[\phi(a),\phi(b)]} f + \varepsilon.$$

Applying Lem. I.11.10.5, we obtain

$$\int_{[\phi(a),\phi(b)]} f - \varepsilon \leq \int_{[a,b]} \underline{f} \circ \phi \, d\phi \leq \int_{[a,b]} \overline{f} \circ \ d\phi \leq \int_{[\phi(a),\phi(b)]} f + \varepsilon.$$

Since $\underline{f} \circ \phi$ is piecewise constant and minorizes $f \circ \phi$, we have

$$\int_{[a,b]} \underline{f} \circ \phi \, d\phi \leq \underline{\int}_{[a,b]} f \circ \phi \, d\phi$$

while similarly we have

$$\int_{[a,b]} \overline{f} \circ \phi \, d\phi \geq \overline{\int}_{[a,b]} f \circ \phi \, d\phi.$$

Thus

$$\int_{[\phi(a),\phi(b)]} f - \varepsilon \leq \underline{\int}_{[a,b]} f \circ \phi \, d\phi \leq \overline{\int}_{[a,b]} f \circ \ d\phi \leq \int_{[\phi(a),\phi(b)]} f + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this implies that

$$\int_{[\phi(a),\phi(b)]} f \leq \underline{\int}_{[a,b]} f \circ \phi \, d\phi \leq \overline{\int}_{[a,b]} f \circ \ d\phi \leq \int_{[\phi(a),\phi(b)]} f$$

and the claim follows. $\square$

Prop. I.11.10.7 (Change of variables formula III)**.** Let $[a, b]$ be a closed interval, and let $\phi : [a, b] \to [\phi(a), \phi(b)]$ be a differentiable monotone increasing function such that ϕ' is Riemann integrable. Let $f : [\phi(a), \phi(b)] \to \mathbb{R}$ be a Riemann integrable function on $[\phi(a), \phi(b)]$. Then $(f \circ \phi)\phi' : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, and

$$\int_{[a,b]} (f \circ \phi)\phi' = \int_{[\phi(a),\phi(b)]} f.$$

Proof. Since ϕ is differentable on $[a,b]$, by Cor. I.10.1.12 we know that ϕ is continuous on $[a,b]$. By Prop. I.11.10.6 we know that $f \circ \phi$ is Riemann-Stieltjes integrable with respect to ϕ on $[a,b]$. By Cor. I.11.10.3 we know that $(f \circ \phi)\phi'$ is Riemann integrable on $[a,b]$, and

$$\begin{aligned}\int_{[a,b]} (f \circ \phi)\phi' &= \int_{[a,b]} (f \circ \phi)\, d\phi && \text{(by Cor. I.11.10.3)}\\ &= \int_{[\phi(a),\phi(b)]} f. && \text{(by Prop. I.11.10.6)}\end{aligned}$$

□

A.Cor. I.11.10.1 (Change of variables formula IV)**.** Let $[a,b]$ be a closed interval, and let $\phi : [a,b] \to [\phi(a),\phi(b)]$ be a differentiable monotone increasing function such that ϕ' is Riemann integrable. Let $f : [\phi(a),\phi(b)] \to \mathbb{R}$ be a continuous function on $[\phi(a),\phi(b)]$. Then $(f \circ \phi)\phi' : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$, and

$$\int_{[a,b]} (f \circ \phi)\phi' = \int_{[\phi(a),\phi(b)]} f.$$

Proof. Since ϕ is differentiable on $[a,b]$, by Cor. I.10.1.12 we know that ϕ is continuous on $[a,b]$. Since f is continuous on $[\phi(a),\phi(b)]$ and ϕ is continuous on $[a,b]$, by Prop. I.9.4.13 we know that $f \circ \phi$ is continuous on $[a,b]$. By Cor. I.11.5.2 we know that $f \circ \phi$ is Riemann integrable on $[a,b]$. Since ϕ' is Riemann integrable on $[a,b]$, by Thm. I.11.4.5 we know that $(f \circ \phi)\phi'$ is Riemann integrable on $[a,b]$. Thus

$$\int_{[a,b]} (f \circ \phi)\phi'$$

is well-defined. Since f is continuous on $[\phi(a),\phi(b)]$, by Cor. I.11.5.2 we know that f is Riemann integrable on $[\phi(a),\phi(b)]$. Thus

$$\int_{[\phi(a),\phi(b)]} f$$

is well-defined. Let $F : [\phi(a),\phi(b)] \to \mathbb{R}$ be the function

$$F(x) = \int_{[\phi(a),x]} f$$

Since f is continuous on $[\phi(a),\phi(b)]$, by Thm. I.11.9.1 we know that $F'(x) = f(x)$ for each $x \in [\phi(a),\phi(b)]$. Then by Thm. I.10.1.15 we have

$$\forall x \in [a,b], (F \circ \phi)'(x) = F'\big(\phi(x)\big)\phi'(x) = f\big(\phi(x)\big)\phi'(x) = (f \circ \phi)(x) \cdot \phi(x)$$

and $(F \circ \phi)' = (f \circ \phi)\phi'$. Thus

$$\int_{[a,b]} (f \circ \phi)\phi' = \int_{[a,b]} (F \circ \phi)'$$

$$
\begin{aligned}
&= (F \circ \phi)(b) - (F \circ \phi)(a) && \text{(by Thm. I.11.9.4)} \\
&= F\big(\phi(b)\big) - F\big(\phi(a)\big) \\
&= \int_{[\phi(a),\phi(b)]} f - \int_{[\phi(a),\phi(a)]} f \\
&= \int_{[\phi(a),\phi(b)]} f.
\end{aligned}
$$

□

— Exercises —

Ex. I.11.10.1. Prove Prop. I.11.10.1.

Proof. See Prop. I.11.10.1. □

Ex. I.11.10.2. Fill in the gaps marked (why?) in the proof of Lem. I.11.10.5.

Proof. See Lem. I.11.10.5. □

Ex. I.11.10.3. Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Let $g : [-b, -a] \to \mathbb{R}$ be defined by $g(x) \coloneqq f(-x)$. Show that g is also Riemann integrable, and $\displaystyle\int_{[-b,-a]} g = \int_{[a,b]} f$.

Proof. Let $\varepsilon > 0$. Then, we can find a piecewise constant function $\overline{f}$ majorizing f on $[a, b]$, and a piecewise constant function $\underline{f}$ minorizing f on $[a, b]$, such that

$$\int_{[a,b]} f - \varepsilon \le \int_{[a,b]} \underline{f} = \int_{[a,b]} \overline{f} \le \int_{[a,b]} f + \varepsilon.$$

Let $\overline{g} : [-b, -a] \to \mathbb{R}$ be the function $\overline{g}(x) = \overline{f}(-x)$. Since $\overline{f}$ majorizes f on $[a, b]$, we know that $\overline{g}$ majorizes g on $[-b, -a]$ and

$$\int_{[-b,-a]} \overline{g} = \int_{[a,b]} \overline{f} \le \int_{[a,b]} f + \varepsilon.$$

Let $\underline{g} : [-b, -a] \to \mathbb{R}$ be the function $\underline{g}(x) = \underline{f}(-x)$. Since $\underline{f}$ minorizes f on $[a, b]$, we know that $\underline{g}$ minorizes g on $[-b, -a]$ and

$$\int_{[-b,-a]} \underline{g} = \int_{[a,b]} \underline{f} \ge \int_{[a,b]} f - \varepsilon.$$

By Def. I.11.3.2 and Lem. I.11.3.3 we have

$$\int_{[a,b]} f - \varepsilon \le \int_{[-b,-a]} \underline{g} \le \underline{\int}_{[-b,-a]} g \le \overline{\int}_{[-b,-a]} g \le \int_{[-b,-a]} \overline{g} \le \int_{[a,b]} f + \varepsilon.$$

Since these statements are true for any $\varepsilon > 0$, we must have

$$\int_{[a,b]} f \leq \underline{\int}_{[-b,-a]} g \leq \overline{\int}_{[-b,-a]} g \leq \int_{[a,b]} f$$

and the claim follows. □

Ex. I.11.10.4. What is the analogue of Prop. I.11.10.7 when ϕ is monotone decreasing instead of monotone increasing? (When ϕ is neither monotone increasing or monotone decreasing, the situation becomes significantly more complicated.)

Proof. Let $[a,b]$ be a closed interval, and let $\phi : [a,b] \to \big[\phi(b), \phi(a)\big]$ be a differentiable monotone decreasing function such that ϕ' is Riemann integrable. Let $f : \big[\phi(b), \phi(a)\big] \to \mathbb{R}$ be a Riemann integrable function on $\big[\phi(b), \phi(a)\big]$. Then $(f \circ \phi)\phi' : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$, and

$$\int_{[a,b]} (f \circ \phi)\phi' = -\int_{[\phi(b),\phi(a)]} f.$$

Now we proof the statement. Let $\eta : [-b,-a] \to [a,b]$ be the function $\eta = x \mapsto -x$. Let $\gamma : [-b,-a] \to [\phi(b), \phi(a)]$ be the function

$$\forall x \in [-b,-a], \gamma(x) = \phi(-x) = (\phi \circ \eta)(x).$$

Since η is differentiable on $[-b,-a]$, by chain rule (Thm. I.10.1.15) we know that

$$\forall x \in [-b,-a], \gamma'(x) = (\phi \circ \eta)'(x) = \phi'\big(\eta(x)\big)\eta'(x) = \phi'\big(\eta(x)\big)(-1) = -(\phi' \circ \eta)(x).$$

Observe that

$$\begin{aligned}
&\forall x, y \in [-b,-a], x \leq y \\
\implies& -x \geq -y \\
\implies& \phi(-x) \leq \phi(-y) && (\phi \text{ is monotone decreasing}) \\
\implies& \gamma(x) \leq \gamma(y).
\end{aligned}$$

Thus, γ is monotone increasing and by Prop. I.11.10.7 we have

$$\int_{[-b,-a]} (f \circ \gamma)\gamma' = \int_{[\gamma(-b),\gamma(-a)]} f.$$

Since

$$\begin{aligned}
&\int_{[-b,-a]} (f \circ \gamma)\gamma' \\
&= \int_{[-b,-a]} (f \circ \phi \circ \eta) \cdot (\phi \circ \eta)'
\end{aligned}$$

$$
\begin{aligned}
&= \int_{[-b,-a]} (f \circ \phi \circ \eta) \cdot \big(-(\phi' \circ \eta)\big) && \text{(from the proof above)} \\
&= -\int_{[-b,-a]} (f \circ \phi \circ \eta) \cdot (\phi' \circ \eta) && \text{(by Thm. I.11.4.1(b))}
\end{aligned}
$$

and

$$\int_{[\gamma(-b),\gamma(-a)]} f = \int_{\left[\phi(b),\phi(a)\right]} f,$$

we know that

$$\int_{[-b,-a]} (f \circ \phi \circ \eta) \cdot (\phi' \circ \eta) = -\int_{\left[\phi(b),\phi(a)\right]} f.$$

Since

$$
\begin{aligned}
\forall x \in [a,b], &\big((f \circ \phi) \cdot \phi'\big)(x) \\
&= (f \circ \phi)(x) \cdot \phi'(x) && \text{(by Def. I.9.2.1)} \\
&= (f \circ \phi \circ \eta)(-x) \cdot (\phi' \circ \eta)(-x) && \text{(by Def. I.3.3.10)} \\
&= \big((f \circ \phi \circ \eta) \cdot (\phi' \circ \eta)\big)(-x), && \text{(by Def. I.9.2.1)}
\end{aligned}
$$

By Ex. I.11.10.3 we know that

$$\int_{[a,b]} (f \circ \phi)\phi' = \int_{[-b,-a]} (f \circ \phi \circ \eta) \cdot (\phi' \circ \eta).$$

$\square$

Part II

Analysis-II

Chapter II.1

Metric spaces

II.1.1 Definitions and examples

Lem. II.1.1.1. Let $(x_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be another real number. Then $(x_n)_{n=m}^{\infty}$ converges to x iff $\lim_{n\to\infty} d(x_n, x) = 0$.

Proof.

$$
\begin{aligned}
& \lim_{n\to\infty} x_n = x \\
\iff & \lim_{n\to\infty} x_n - x = 0 \\
\iff & \lim_{n\to\infty} |x_n - x| = 0 && \text{(by Cor. I.6.4.17)} \\
\iff & \lim_{n\to\infty} d(x_n, x) = 0.
\end{aligned}
$$

□

Note. One would now like to generalize this notion of convergence, so that one can take limits not just of sequences of real numbers, but also sequences of complex numbers, or sequences of vectors, or sequences of matrices, or sequences of functions, even sequences of sequences. One way to do this is to redefine the notion of convergence each time we deal with a new type of object. A more efficient way is to work *abstractly*, defining a very general class of spaces - which includes such standard spaces as the real numbers, complex numbers, vectors, etc. - and define the notion of convergence on this entire class of spaces at once. (A *space* is just the set of all objects of a certain type. Mathematically, there is not much distinction between a space and a set, except that spaces tend to have much more structure than what a random set would have.)

Note. It turns out that there are two very useful classes of spaces which do the job. The first class is that of *metric spaces*. There is a more general class of spaces, called *topological spaces*.

Def. II.1.1.2 (Metric spaces)**.** A metric space (X, d) is a space X of objects (called *points*), together with a *distance function* or *metric* $d : X \times X \to [0, +\infty)$, which associates to each pair x, y of points in X a non-negative real number $d(x, y) \geq 0$. Furthermore, the metric must satisfy the following four axioms:

(a) For any $x \in X$, we have $d(x, x) = 0$.

(b) (Positivity) For any distinct $x, y \in X$, we have $d(x, y) > 0$.

(c) (Symmetry) For any $x, y \in X$, we have $d(x, y) = d(y, x)$.

(d) (Triangle inequality) For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Note. In many cases it will be clear what the metric d is, and we shall abbreviate (X, d) as just X.

Rmk. II.1.1.3. The conditions (a) and (b) of Lem. II.1.1.1 can be rephrased as follows: for any $x, y \in X$ we have $d(x, y) = 0$ iff $x = y$.

E.g. II.1.1.4 (The real line)**.** Let $\mathbb{R}$ be the real numbers, and let $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be the metric $d(x, y) := |x - y|$ mentioned earlier. Then $(\mathbb{R}, d)$ is a metric space. We refer to d as the *standard metric* on $\mathbb{R}$, and if we refer to $\mathbb{R}$ as a metric space, we assume that the metric is given by the standard metric d unless otherwise specified.

E.g. II.1.1.5 (Induced metric spaces)**.** Let (X, d) be any metric space, and let Y be a subset of X. Then we can restrict the metric function $d : X \times X \to [0, +\infty)$ to the subset $Y \times Y$ of $X \times X$ to create a restricted metric function $d|_{Y \times Y} : Y \times Y \to [0, +\infty)$ of Y; this is known as the metric on Y *induced* by the metric d on X. The pair $(Y, d|_{Y \times Y})$ is a metric space and is known the *subspace* of (X, d) induced by Y. Thus, for instance the metric on the real line in the E.g. II.1.1.4 induces a metric space structure on any subset of the reals, such as the integers Z, or an interval $[a, b]$, etc.

E.g. II.1.1.6 (Euclidean spaces)**.** Let $n \geq 1$ be a natural number, and let $\mathbb{R}^n$ be the space of n-tuples of real numbers:

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R}\}.$$

We define the *Euclidean metric* (also called the l^2 *metric*) $d_{l^2} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} d_{l^2}((x_1, \dots, x_n), (y_1, \dots, y_n)) &:= \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \\ &= \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}. \end{aligned}$$

Note. Euclidean metric corresponds to the geometric distance between the two points $(x_1, x_2, \dots, x_n)$, $(y_1, y_2, \dots, y_n)$ as given by Pythagoras' theorem. While geometry does give some very important examples of metric spaces, it is possible to have metric spaces which have no obvious geometry whatsoever. The verification that $(\mathbb{R}^n, d)$ is indeed a metric space can be seen geometrically (for instance, the triangle inequality now asserts that the length of one side of a triangle is always less than or equal to the sum of the lengths of the other two sides), but can also be proven algebraically. We refer to $(\mathbb{R}^n, d_{l^2})$ as the *Euclidean space* of *dimension* n. Extending the convention from E.g. II.1.1.4, if we refer to $\mathbb{R}^n$ as a metric space, we assume that the metric is given by the Euclidean metric unless otherwise specified.

E.g. II.1.1.7 (Taxi-cab metric)**.** Again let $n \geq 1$, and let $\mathbb{R}^n$ be as before. But now we use a different metric d_{l^1}, the so-called *taxicab metric* (or l^1 *metric*), defined by

$$\begin{aligned} d_{l^1}((x_1, \dots, x_n), (y_1, \dots, y_n)) &:= |x_1 - y_1| + \cdots + |x_n - y_n| \\ &= \sum_{i=1}^{n} |x_i - y_i|. \end{aligned}$$

Note. This metric is called the taxi-cab metric, because it models the distance a taxi-cab would have to traverse to get from one point to another if the cab was only allowed to move in cardinal directions (north, south, east, west) and not diagonally. As such it is always at least as large as the Euclidean metric, which measures distance "as the crow flies," as it were. We claim that the space $(\mathbb{R}^n, d_{l^1})$ is also a metric space. The metrics are not quite the same, but we do have the inequalities

$$d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y)$$

for all x, y.

Rmk. II.1.1.8. The taxi-cab metric is useful in several places, for instance in the theory of error correcting codes. A string of n binary digits can be thought of as an element of $\mathbb{R}^n$. The taxi-cab distance between two binary strings is then the number of bits in the two strings which do not match. The goal of error-correcting codes is to encode each piece of information (e.g., a letter of the alphabet) as a binary string in such a way that all the binary strings are as far away in the taxicab metric from each other as possible; this minimizes the chance that any distortion of the bits due to random noise can accidentally change one of the coded binary strings to another, and also maximizes the chance that any such distortion can be detected and correctly repaired.

E.g. II.1.1.9 (Sup norm metric)**.** Again let $n \geq 1$, and let $\mathbb{R}^n$ be as before. But now we use a different metric d_{l^∞}, the so-called *sup norm metric* (or l^∞ *metric*), defined by

$$d_{l^\infty}((x_1, \dots, x_n), (y_1, \dots, y_n)) := \sup\{|x_i - y_i| : 1 \leq i \leq n\}.$$

Note. The space $(\mathbb{R}^n, d_{l^\infty})$ is also a metric space, and is related to the l^2 metric by the inequalities

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y)$$

for all x, y.

Rmk. II.1.1.10. The l^1, l^2, and l^∞ metrics are special cases of the more general l^p *metrics*, where $p \in [1, +\infty)$.

E.g. II.1.1.11 (Discrete metric)**.** Let X be an arbitrary set (finite or infinite), and define the *discrete metric* d_{disc} by setting $d_{\mathrm{disc}}(x, y) := 0$ when $x = y$, and $d_{\mathrm{disc}}(x, y) := 1$ when $x \neq y$. Thus, in this metric, all points are equally far apart. The space (X, d_{disc}) is a metric space. Thus, every set X has at least one metric on it.

Def. II.1.1.14 (Convergence of sequences in metric spaces)**.** Let m be an integer, (X, d) be a metric space and let $(x^{(n)})_{n=m}^\infty$ be a sequence of points in X (i.e., for every natural number $n \geq m$, we assume that $x^{(n)}$ is an element of X). Let x be a point in X. We say that $(x^{(n)})_{n=m}^\infty$ *converges to x with respect to the metric d*, iff the limit $\lim_{n\to\infty} d(x^{(n)}, x)$ exists and is equal to 0. In other words, $(x^{(n)})_{n=m}^\infty$ converges to x with respect to d iff for every $\varepsilon > 0$, there exists an $N \geq m$ such that $d(x^{(n)}, x) \leq \varepsilon$ for all $n \geq N$.

Rmk. II.1.1.15. In view of Lem. II.1.1.1 we see that this definition generalizes our existing notion of convergence of sequences of real numbers. In many cases, it is obvious what the metric d is, and so we shall often just say "$(x^{(n)})_{n=m}^\infty$ converges to x" instead of "$(x^{(n)})_{n=m}^\infty$ converges to x with respect to the metric d" when there is no chance of confusion. We also sometimes write "$x^{(n)} \to x$ as $n \to \infty$ instead.

Rmk. II.1.1.16. There is nothing special about the superscript n in the above definition; it is a dummy variable. Saying that $(x^{(n)})_{n=m}^\infty$ converges to x is exactly the same statement as saying that $(x^{(k)})_{k=m}^\infty$ converges to x, for example; and sometimes it is convenient to change superscripts, for instance if the variable n is already being used for some other purpose. Similarly, it is not necessary for the sequence $x^{(n)}$ to be denoted using the superscript (n); the above definition is also valid for sequences x_n, or functions $f(n)$, or indeed of any expression which depends on n and takes values in X. We see that the starting point m of the sequence is unimportant for the purposes of taking limits; if $(x^{(n)})_{n=m}^\infty$ converges to x, then $(x^{(n)})_{n=m'}^\infty$ also converges to x for any $m' \geq m$.

Note. The convergence of a sequence can depend on what metric one uses.

Prop. II.1.1.18 (Equivalence of l^1, l^2, l^∞)**.** Let $\mathbb{R}^n$ be a Euclidean space, and let $(x^{(k)})_{k=m}^\infty$ be a sequence of points in $\mathbb{R}^n$. We write $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$, i.e., for $j = 1, 2, \ldots, n$, $x_j \in \mathbb{R}$ is the j^{th} coordinate of $x^{(k)} \in \mathbb{R}^n$. Let $x = (x_1, \ldots, x_n)$ be a point in $\mathbb{R}^n$. Then the following four statements are equivalent:

(a) $(x^{(k)})_{k=m}^\infty$ converges to x with respect to the Euclidean metric d_{l^2}.

(b) $(x^{(k)})_{k=m}^\infty$ converges to x with respect to the taxi-cab metric d_{l^1}.

(c) $(x^{(k)})_{k=m}^\infty$ converges to x with respect to the sup norm metric d_{l^∞}.

(d) For every $1 \leq j \leq n$, the sequence $(x_j^{(k)})_{k=m}^\infty$ converges to x_j. (Notice that this is a sequence of real numbers, not of points in $\mathbb{R}^n$.)

Proof. We have

$$
\begin{aligned}
&\lim_{k\to\infty} d_{l^2}(x^{(k)},x)=0 && \text{(by Def. II.1.1.14)}\\
\iff &\lim_{k\to\infty}\sqrt{\sum_{j=1}^{n}(x_j^{(k)}-x_j)^2}=0 && \text{(by E.g. II.1.1.6)}\\
\iff &\lim_{k\to\infty}\left(\sum_{j=1}^{n}(x_j^{(k)}-x_j)^2\right)=0\\
\iff &\sum_{j=1}^{n}\left(\lim_{k\to\infty}(x_j^{(k)}-x_j)^2\right)=0\\
\iff &\forall j\in\{i\in\mathbb{N}:1\le i\le n\},\ \lim_{k\to\infty}x_j^{(k)}-x_j=0\\
\iff &\forall j\in\{i\in\mathbb{N}:1\le i\le n\},\ \lim_{k\to\infty}x_j^{(k)}=x_j\\
\iff &\forall j\in\{i\in\mathbb{N}:1\le i\le n\},\ \lim_{k\to\infty}\left|x_j^{(k)}-x_j\right|=0 && \text{(by Lem. II.1.1.1)}\\
\iff &\sum_{j=1}^{n}\left(\lim_{k\to\infty}\left|x_j^{(k)}-x_j\right|\right)=0\\
\iff &\lim_{k\to\infty}\left(\sum_{j=1}^{n}\left|x_j^{(k)}-x_j\right|\right)=0\\
\iff &\lim_{k\to\infty} d_{l^1}(x^{(k)},x)=0 && \text{(by E.g. II.1.1.7)}\\
\iff &\lim_{k\to\infty}\sup\left\{\left|x_j^{(k)}-x_j\right| : j\in\{i\in\mathbb{N}:1\le i\le n\}\right\}=0\\
\iff &\lim_{k\to\infty} d_{l^\infty}(x^{(k)},x)=0. && \text{(by E.g. II.1.1.9)}
\end{aligned}
$$

□

Note. Because of the equivalence of Prop. II.1.1.18(a), (b) and (c), we say that the Euclidean, taxicab, and sup norm metrics on $\mathbb{R}^n$ are *equivalent*. (There are infinite-dimensional analogues of the Euclidean, taxicab, and sup norm metrics which are *not* equivalent.)

Prop. II.1.1.19 (Convergence in the discrete metric)**.** Let X be any set, and let d_{disc} be the discrete metric on X. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in X, and let x be a point in X. Then $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to the discrete metric d_{disc} iff there exists an $N \ge m$ such that $x^{(n)} = x$ for all $n \ge N$.

Proof. By Def. II.1.1.14 we know that $\lim_{n\to\infty} d_{\text{disc}}(x^{(n)},x)=0$ iff $\forall\varepsilon\in\mathbb{R}^+$, $\exists N\in\mathbb{N}$ and $N\ge m$ such that $d(x^{(n)},x)\le\varepsilon$ for all $n\ge N$. By E.g. II.1.1.11 we know that $d_{\text{disc}}(x^{(n)},x)\le\varepsilon$ iff $x^{(n)}=x$. □

Prop. II.1.1.20 (Uniqueness of limits)**.** Let (X, d) be a metric space, and let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in X. Suppose that there are two points $x, x' \in X$ such that $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to d, and $(x^{(n)})_{n=m}^{\infty}$ also converges to x' with respect to d. Then we have $x = x'$.

Proof. By Def. II.1.1.14 we have $\lim_{n\to\infty} d(x^{(n)}, x) = 0$ and $\lim_{n\to\infty} d(x^{(n)}, x') = 0$. So

$$\begin{aligned}
&\lim_{n\to\infty} d(x^{(n)}, x) + \lim_{n\to\infty} d(x^{(n)}, x') = 0 \\
\implies &\lim_{n\to\infty} \left(d(x^{(n)}, x) + d(x^{(n)}, x') \right) = 0 \\
\implies &d(x, x') = \lim_{n\to\infty} d(x, x') \leq 0 && \text{(by Def. II.1.1.2(d))} \\
\implies &d(x, x') = 0 && \text{(by Def. II.1.1.2(a)(b))} \\
\implies &x = x'. && \text{(by Def. II.1.1.2(a))}
\end{aligned}$$

□

Note. Because of Prop. II.1.1.20, it is safe to introduce the following notation: if $(x^{(n)})_{n=m}^{\infty}$ converges to x in the metric d, then we write $d - \lim_{n\to\infty} x^{(n)} = x$, or simply $\lim_{n\to\infty} x^{(n)} = x$ when there is no confusion as to what d is. The meaning of $d - \lim_{n\to\infty} x^{(n)}$ can depend on what d is; however Prop. II.1.1.20 assures us that once d is fixed, there can be at most one value of $d - \lim_{n\to\infty} x^{(n)}$. (Of course, it is still possible that this limit does not exist; some sequences are not convergent.)

Rmk. II.1.1.21. It is possible for a sequence to converge to one point using one metric, and another point using a different metric, although such examples are usually quite artificial. Thus, changing the metric on a space can greatly affect the nature of convergence (also called the *topology*) on that space.

A.Cor. II.1.1.1. Let $\vec{u} = (u_1, u_2), \vec{v} = (v_1, v_2)$ be two vectors in $\mathbb{R}^2$ such that $|\vec{v}| \neq 0$. Let $\alpha \in \mathbb{R}$ and let $\alpha\vec{v}$ be the vector $\vec{u}$ project onto $\vec{v}$. Then

$$\alpha = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}.$$

Proof. Let $\vec{z} = (v_2, -v_1)$. We know that $\vec{v} \perp \vec{z}$ since

$$\vec{v} \cdot \vec{z} = v_1 v_2 - v_1 v_2 = 0.$$

Since $\alpha\vec{v}$ is the vector $\vec{u}$ project onto $\vec{v}$, we know that $\exists \beta \in \mathbb{R}$ such that $\alpha\vec{v} + \beta\vec{z} = \vec{u}$. Then we have

$$\alpha\vec{v} + \beta\vec{z} = \vec{u}$$

$$\implies \begin{cases} \alpha v_1 + \beta v_2 = u_1 \\ \alpha v_2 - \beta v_1 = u_2 \end{cases}$$
$$\implies \begin{cases} \alpha v_1^2 + \beta v_1 v_2 = u_1 v_1 \\ \alpha v_2^2 - \beta v_1 v_2 = u_2 v_2 \end{cases}$$
$$\implies \alpha(v_1^2 + v_2^2) = u_1 v_1 + u_2 v_2$$
$$\implies \alpha = \frac{u_1 v_1 + u_2 v_2}{v_1^2 + v_2^2} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}.$$

□

A.Cor. II.1.1.2. Let $n \in \mathbb{Z}^+$ and let $\vec{u}, \vec{v}$ be two vectors in $\mathbb{R}^n$ such that $|\vec{v}| \neq 0$. Let $\alpha \in \mathbb{R}$ and let $\alpha\vec{v}$ be the vector $\vec{u}$ project onto $\vec{v}$. Then

$$\alpha = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}.$$

Proof. We can use $\vec{u}$ and $\vec{v}$ to form a linear combination $\vec{z} \in \mathbb{R}^n$ such that $\vec{z} \perp \vec{v}$. Since $\alpha\vec{v}$ is the vector $\vec{u}$ project onto $\vec{v}$, we know that $\exists \beta \in \mathbb{R}$ such that $\alpha\vec{v} + \beta\vec{z} = \vec{u}$. Then we have $\beta\vec{z} \cdot \vec{v} = 0$ and

$$\alpha\vec{v} + \beta\vec{z} = \vec{u}$$
$$\implies \alpha\vec{v} \cdot \vec{v} + \beta\vec{z} \cdot \vec{v} = \vec{u} \cdot \vec{v}$$
$$\implies \alpha = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}.$$

□

— Exercises —

Ex. II.1.1.1. Prove Lem. II.1.1.1.

Proof. See Lem. II.1.1.1. □

Ex. II.1.1.2. Show that the real line with the metric $d(x, y) := |x - y|$ is indeed a metric space.

Proof. Let $x, y, z \in \mathbb{R}$. For identity: We have $d(x, x) = |x - x| = 0$. For positivity: If $x \neq y$, then $d(x, y) = |x - y| > 0$. For symmetry: We have $d(x, y) = |x - y| = |y - x| = d(y, x)$. For triangle inequality: We have $d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z)$. Thus, by Def. II.1.1.2 $(\mathbb{R}, d)$ is a metric space. □

Ex. II.1.1.3. Let X be a set, and let $d : X \times X \to [0, \infty)$ be a function.

(a) Give an example of a pair (X, d) which obeys axioms (bcd) of Def. II.1.1.2, but not (a).

(b) Give an example of a pair (X, d) which obeys axioms (acd) of Def. II.1.1.2, but not (b).

(c) Give an example of a pair (X, d) which obeys axioms (abd) of Def. II.1.1.2, but not (c).

(d) Give an example of a pair (X, d) which obeys axioms (abc) of Def. II.1.1.2, but not (d).

Proof. (a) Let $X = \mathbb{R}$, and let $d(x, y) = 1$ for all $x, y \in \mathbb{R}$. Then (X, d) does not satisfy Def. II.1.1.2(a) but (bcd). $\square$

Proof. (b) Let $X = \mathbb{R}$ and let $d(x, y) = 0$ for all $x, y \in \mathbb{R}$. Then (X, d) does not satisfy Def. II.1.1.2(b) but (acd). $\square$

Proof. (c) Let $X = \{1, 2\}$, let $x, y \in X$ and let $d(x, y) = x^y$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Then (X, d) does not satisfy Def. II.1.1.2(c) but (abd). $\square$

Proof. (d) Let $X = \mathbb{R}^+$, let $x, y \in \mathbb{R}^+$ and let $d(x, y) = \max(x, y)$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Then (X, d) does not satisfy Def. II.1.1.2(d) but (abc). $\square$

Ex. II.1.1.4. Show that the pair $(Y, d|_{Y\times Y})$ defined in E.g. II.1.1.5 is indeed a metric space.

Proof. Let $x, y, z \in X$. Since $Y \subseteq X$, we know that $x, y, z \in X$. For identity: We have $d|_{Y\times Y}(x, x) = d(x, x) = 0$. For positivity: If $x \neq y$, then $d|_{Y\times Y}(x, y) = d(x, y) > 0$. For symmetry: We have $d|_{Y\times Y}(x, y) = d(x, y) = d(y, x) = d|_{Y\times Y}(y, x)$. For triangle inequality: We have $d|_{Y\times Y}(x, z) = d(x, z) \leq d(x, y) + d(y, z) = d|_{Y\times Y}(x, y) + d|_{Y\times Y}(y, z)$. Thus, by Def. II.1.1.2 $(Y, d|_{Y\times Y})$ is a metric space. $\square$

Ex. II.1.1.5. Let $n \geq 1$, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Verify the identity

$$\left(\sum_{i=1}^n a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right)$$

and conclude the *Cauchy-Schwarz inequality*

$$\left|\sum_{i=1}^n a_i b_i\right| \leq \left(\sum_{i=1}^n a_i^2\right)^{1/2} \left(\sum_{j=1}^n b_j^2\right)^{1/2}.$$

Then use the Cauchy-Schwarz inequality to prove the *triangle inequality*

$$\left(\sum_{i=1}^n (a_i + b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^n a_i^2\right)^{1/2} + \left(\sum_{j=1}^n b_j^2\right)^{1/2}.$$

Proof. We first show the identity is true by induction on n. For $n = 0$, we have

$$\left(\sum_{i=1}^{0} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{0}\sum_{j=1}^{0}(a_i b_j - a_j b_i)^2 = 0$$

and

$$\left(\sum_{i=1}^{0} a_i^2\right)\left(\sum_{j=1}^{0} b_j^2\right) = 0$$

so the base case holds. Suppose inductively that the identity is true for some $n \geq 0$. Then for $n+1$, we have

$$\begin{aligned}
&\left(\sum_{i=1}^{n+1} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 \\
=&\left(\sum_{i=1}^{n} a_i b_i + a_{n+1}b_{n+1}\right)^2 + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 \\
=&\left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right)(a_{n+1}b_{n+1}) + (a_{n+1}b_{n+1})^2 + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 \\
=&\left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right)(a_{n+1}b_{n+1}) + (a_{n+1}b_{n+1})^2 \\
&+ \frac{1}{2}\sum_{i=1}^{n+1}\left(\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + (a_i b_{n+1} - a_{n+1}b_i)^2\right) \\
=&\left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right)(a_{n+1}b_{n+1}) + (a_{n+1}b_{n+1})^2 \\
&+ \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \frac{1}{2}\sum_{i=1}^{n+1}(a_i b_{n+1} - a_{n+1}b_i)^2 \\
=&\left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right)(a_{n+1}b_{n+1}) + (a_{n+1}b_{n+1})^2 \\
&+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \frac{1}{2}\sum_{j=1}^{n}(a_{n+1} b_j - a_j b_{n+1})^2 + \frac{1}{2}\sum_{i=1}^{n+1}(a_i b_{n+1} - a_{n+1}b_i)^2 \\
=&\left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2\left(\sum_{i=1}^{n} a_i b_i\right)(a_{n+1}b_{n+1}) + (a_{n+1}b_{n+1})^2 \\
&+ \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \frac{1}{2}\sum_{j=1}^{n}(a_{n+1} b_j - a_j b_{n+1})^2 + \frac{1}{2}\sum_{i=1}^{n}(a_i b_{n+1} - a_{n+1}b_i)^2
\end{aligned}$$

$$
\begin{aligned}
=&\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 + 2\Big(\sum_{i=1}^{n} a_i b_i\Big)(a_{n+1}b_{n+1}) + (a_{n+1}b_{n+1})^2\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \sum_{i=1}^{n}(a_{n+1}b_i - a_i b_{n+1})^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big(\sum_{i=1}^{n+1} a_i^2\Big)\Big(\sum_{j=1}^{n+1} b_j^2\Big)\\
=&\Big(\sum_{i=1}^{n} a_i^2 + a_{n+1}^2\Big)\Big(\sum_{j=1}^{n} b_j^2 + b_{n+1}^2\Big)\\
=&\Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{j=1}^{n} b_j^2\Big) + a_{n+1}^2\Big(\sum_{j=1}^{n} b_j^2\Big) + b_{n+1}^2\Big(\sum_{i=1}^{n} a_i^2\Big) + (a_{n+1}b_{n+1})^2\\
=&\Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{j=1}^{n} b_j^2\Big) + \Big(\sum_{j=1}^{n} a_{n+1}^2 b_j^2\Big) + \Big(\sum_{i=1}^{n} a_i^2 b_{n+1}^2\Big) + (a_{n+1}b_{n+1})^2\\
=&\Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{j=1}^{n} b_j^2\Big) + \Big(\sum_{j=1}^{n} a_{n+1}^2 b_j^2 - 2a_{n+1}b_j a_j b_{n+1} + a_j^2 b_{n+1}^2\Big)\\
&+\sum_{j=1}^{n} 2a_{n+1}b_j a_j b_{n+1} - \sum_{j=1}^{n} a_j^2 b_{n+1}^2 + \Big(\sum_{i=1}^{n} a_i^2 b_{n+1}^2\Big) + (a_{n+1}b_{n+1})^2\\
=&\Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{j=1}^{n} b_j^2\Big) + \Big(\sum_{j=1}^{n}(a_{n+1}b_j - a_j b_{n+1})^2\Big)\\
&+2\sum_{j=1}^{n}\Big(b_j a_j\Big)(a_{n+1}b_{n+1}) + (a_{n+1}b_{n+1})^2.
\end{aligned}
$$

By the induction hypothesis we thus have

$$
\Big(\sum_{i=1}^{n+1} a_i b_i\Big)^2 + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 = \Big(\sum_{i=1}^{n+1} a_i^2\Big)\Big(\sum_{j=1}^{n+1} b_j^2\Big)
$$

and this closes the induction.

Next we show that Cauchy-Schwarz inequality is true. We have

$$
\begin{aligned}
&\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{j=1}^{n} b_j^2\Big)\\
\Longrightarrow&\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \le \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{j=1}^{n} b_j^2\Big)
\end{aligned}
$$

$$\Longrightarrow \left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}.$$

Finally we show that d_{l^2} satisfy triangle inequality. We have

$$\begin{aligned}
\sum_{i=1}^{n}(a_i+b_i)^2 &= \sum_{i=1}^{n}(a_i^2+2a_ib_i+b_i^2) \\
&= \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_ib_i + \sum_{i=1}^{n} b_i^2 \\
&\leq \sum_{i=1}^{n} a_i^2 + 2\left|\sum_{i=1}^{n} a_ib_i\right| + \sum_{i=1}^{n} b_i^2 \\
&\leq \sum_{i=1}^{n} a_i^2 + 2\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2} + \sum_{i=1}^{n} b_i^2 \\
&= \left(\left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}\right)^2
\end{aligned}$$

and thus

$$\left(\sum_{i=1}^{n}(a_i+b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}.$$

□

Ex. II.1.1.6. Show that $(\mathbb{R}^n, d_{l^2})$ in E.g. II.1.1.6 is indeed a metric space.

Proof. Let $n \in \mathbb{N}$ and let $x, y, z \in \mathbb{R}^n$. For each $n \in \mathbb{N}$, we define $I_n = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. For each $i \in I_n$, we define x_i, y_i, z_i to be the i^{th} coordinate of x, y, z, respectively. For identity: We have

$$d_{l^2}(x,x) = \left(\sum_{i=1}^{n}(x_i - x_i)^2\right)^{1/2} = 0.$$

For positivity: If $x \neq y$, then $\exists j \in I_n$ such that $x_j \neq y_j$. So

$$d_{l^2}(x,y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} \geq \left((x_j - y_j)^2\right)^{1/2} > 0.$$

For symmetry: We have

$$d_{l^2}(x,y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} = \left(\sum_{i=1}^{n}(y_i - x_i)^2\right)^{1/2} = d_{l^2}(y,x).$$

For triangle inequality: We define $a_i = x_i - y_i$ and $b_i = y_i - z_i$ for each $i \in I_n$. Then we have

$$\begin{aligned} d_{l^2}(x,z) &= \left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} \\ &= \left(\sum_{i=1}^{n}(x_i - y_i + y_i - z_i)^2\right)^{1/2} \\ &= \left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} \\ &\leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2} \\ &= \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2} \\ &= d_{l^2}(x,y) + d_{l^2}(y,z). \end{aligned}$$

Thus, by Def. II.1.1.2 $(\mathbb{R}^n, d_{l^2})$ is a metric space. □

Ex. II.1.1.7. Show that $(\mathbb{R}^n, d_{l^1})$ in E.g. II.1.1.7 is indeed a metric space.

Proof. Let $n \in \mathbb{N}$ and let $x, y, z \in \mathbb{R}^n$. For each $n \in \mathbb{N}$, we define $I_n = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. For each $i \in I_n$, we define x_i, y_i, z_i to be the i^{th} coordinate of x, y, z, respectively. For identity: We have

$$d_{l^1}(x,x) = \sum_{i=1}^{n}|x_i - x_i| = 0.$$

For positivity: If $x \neq y$, then $\exists j \in I_n$ such that $x_j \neq y_j$. So

$$d_{l^1}(x,y) = \sum_{i=1}^{n}|x_i - y_i| \geq |x_j - y_j| > 0.$$

For symmetry: We have

$$d_{l^1}(x,y) = \sum_{i=1}^{n}|x_i - y_i| = \sum_{i=1}^{n}|y_i - x_i| = d_{l^1}(y,x).$$

For triangle inequality: We have

$$\begin{aligned} d_{l^1}(x,z) &= \sum_{i=1}^{n}|x_i - z_i| \\ &= \sum_{i=1}^{n}|x_i - y_i + y_i - z_i| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n}(|x_i - y_i| + |y_i - z_i|) \\
&= d_{l^1}(x, y) + d_{l^1}(y, z).
\end{aligned}
$$

Thus, by Def. II.1.1.2 $(\mathbb{R}^n, d_{l^1})$ is a metric space. □

Ex. II.1.1.8. Prove the two inequalities

$$d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y)$$

for all $x, y \in \mathbb{R}^n$.

Proof. Let $n \in \mathbb{N}$ and let $x, y \in \mathbb{R}^n$. For each $n \in \mathbb{N}$, we define $I_n = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. For each $i \in I_n$, we define x_i, y_i to be the i^{th} coordinate of x, y, respectively. We have

$$
\begin{aligned}
\big(d_{l^2}(x, y)\big)^2 &= \sum_{i=1}^{n}(x_i - y_i)^2 \\
&= \sum_{i=1}^{n}|x_i - y_i|^2 \\
&\leq \sum_{i=1}^{n}|x_i - y_i|^2 + \sum_{i=1}^{n}\left(|x_i - y_i|\left(\sum_{j=1}^{i-1}|x_j - y_j| + \sum_{j=i+1}^{n}|x_j - y_j|\right)\right) \\
&= \sum_{i=1}^{n}\left(|x_i - y_i|\left(\sum_{j=1}^{i-1}|x_j - y_j| + |x_i - y_i| + \sum_{j=i+1}^{n}|x_j - y_j|\right)\right) \\
&= \left(\sum_{i=1}^{n}|x_i - y_i|\right)\left(\sum_{j=1}^{n}|x_j - y_j|\right) \\
&= \left(\sum_{i=1}^{n}|x_i - y_i|\right)^2 \\
&= \big(d_{l^1}(x, y)\big)^2
\end{aligned}
$$

and thus $d_{l^2}(x, y) \leq d_{l^1}(x, y)$. Now let $a_i = |x_i - y_i|$ and $b_i = 1$ for each $i \in I_n$. Then we have

$$
\begin{aligned}
d_{l^1}(x, y) &= \sum_{i=1}^{n}|x_i - y_i| \\
&= \left|\sum_{i=1}^{n}|x_i - y_i|\right| \\
&= \left|\sum_{i=1}^{n} a_i b_i\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \left(\sum_{i=1}^{n} b_i^2\right)^{1/2} && \text{(by Ex. II.1.1.5)} \\
&= \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} 1\right)^{1/2} \\
&= \sqrt{n} d_{l^2}(x, y).
\end{aligned}
$$

Combining the results we have

$$d_{l^2}(x, y) \le d_{l^1}(x, y) \le \sqrt{n} d_{l^2}(x, y).$$

□

Ex. II.1.1.9. Show that $(\mathbb{R}^n, d_{l^\infty})$ in E.g. II.1.1.9 is indeed a metric space.

Proof. Let $n \in \mathbb{N}$ and let $x, y, z \in \mathbb{R}^n$. For each $n \in \mathbb{N}$, we define $I_n = \{i \in \mathbb{N} : 1 \le i \le n\}$. For each $i \in I_n$, we define x_i, y_i, z_i to be the i^{th} coordinate of x, y, z, respectively. For identity: We have

$$d_{l^\infty}(x, x) = \sup\{|x_i - x_i| : i \in I_n\} = 0.$$

For positivity: If $x \ne y$, then $\exists j \in \mathbb{N}$ and $1 \le j \le n$ such that $x_j \ne y_j$. So

$$d_{l^\infty}(x, y) = \sup\{|x_i - y_i| : i \in I_n\} \ge |x_j - y_j| > 0.$$

For symmetry: We have

$$d_{l^\infty}(x, y) = \sup\{|x_i - y_i| : i \in I_n\} = \sup\{|y_i - x_i| : i \in I_n\} = d_{l^\infty}(y, x).$$

For triangle inequality: We have

$$
\begin{aligned}
d_{l^\infty}(x, z) &= \sup\{|x_i - z_i| : i \in I_n\} \\
&= \sup\{|x_i - y_i + y_i - z_i| : i \in I_n\} \\
&\le \sup\{|x_i - y_i| + |y_i - z_i| : i \in I_n\} \\
&\le \sup\{|x_i - y_i| : i \in I_n\} + \sup\{|y_i - z_i| : i \in I_n\} \\
&= d_{l^\infty}(x, y) + d_{l^\infty}(y, z).
\end{aligned}
$$

Thus, by Def. II.1.1.2 $(\mathbb{R}^n, d_{l^\infty})$ is a metric space. □

Ex. II.1.1.10. Prove the two inequalities

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \le d_{l^\infty}(x, y) \le d_{l^2}(x, y)$$

for all $x, y \in \mathbb{R}^n$.

Proof. Let $n \in \mathbb{N}$ and let $x, y \in \mathbb{R}^n$. For each $n \in \mathbb{N}$, we define $I_n = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. For each $i \in I_n$, we define x_i, y_i to be the i^{th} coordinate of x, y, respectively. Since

$$\begin{aligned}
\frac{1}{\sqrt{n}} d_{l^2}(x, y) &= \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} \\
&\leq \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} \big(\sup\{|x_i - y_i| : i \in I_n\} \big)^2 \right)^{1/2} \\
&= \frac{1}{\sqrt{n}} \Big(n \big(\sup\{|x_i - y_i| : i \in I_n\} \big)^2 \Big)^{1/2} \\
&= \sup\{|x_i - y_i| : i \in I_n\} \\
&= d_{l^\infty}(x, y) \\
&= \Big(\big(\sup\{|x_i - y_i| : i \in I_n\} \big)^2 \Big)^{1/2} \\
&\leq \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} \\
&= d_{l^2}(x, y),
\end{aligned}$$

we have

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y).$$

□

Ex. II.1.1.11. Show that (X, d_{disc}) in E.g. II.1.1.11 is indeed a metric space.

Proof. Let $x, y, z \in X$. For identity: We have

$$d_{\text{disc}}(x, x) = 0.$$

For positivity: If $x \neq y$, then we have

$$d_{\text{disc}}(x, y) = 1 > 0.$$

For symmetry: We have

$$x = y \iff d_{\text{disc}}(x, y) = 0 = d_{\text{disc}}(y, x)$$

and

$$x \neq y \iff d_{\text{disc}}(x, y) = 1 = d_{\text{disc}}(y, x).$$

For triangle inequality: We have

$$x = z \iff d_{\text{disc}}(x, z) = 0 \leq d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z)$$

and

$$x \neq z \iff d_{\text{disc}}(x, z) = 1 \leq d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z).$$

Thus, by Def. II.1.1.2 (X, d_{disc}) is a metric space. □

Ex. II.1.1.12. Prove Prop. II.1.1.18.

Proof. See Prop. II.1.1.18. □

Ex. II.1.1.13. Prove Prop. II.1.1.19.

Proof. See Prop. II.1.1.19. □

Ex. II.1.1.14. Prove Prop. II.1.1.20.

Proof. See Prop. II.1.1.20. □

Ex. II.1.1.15. Let

$$X := \left\{ (a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty \right\}$$

be the space of absolutely convergent sequences. Define the l^1 and l^∞ metrics on this space by

$$\begin{aligned} d_{l^1}\big((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\big) &:= \sum_{n=0}^{\infty} |a_n - b_n|; \\ d_{l^\infty}\big((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\big) &:= \sup_{n \in \mathbb{N}} |a_n - b_n|. \end{aligned}$$

Show that these are both metrics on X, but show that there exist sequences $x^{(1)}, x^{(2)}, \ldots$ of elements of X (i.e., sequences of sequences) which are convergent with respect to the d_{l^∞} metric but not with respect to the d_{l^1} metric. Conversely, show that any sequence which converges in the d_{l^1} metric automatically converges in the d_{l^∞} metric.

Proof. Let $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty} \in X$. Since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge absolutely, we know that

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n| + \sum_{n=0}^{\infty} |b_n| &= \lim_{N \to \infty} \sum_{n=0}^{N} |a_n| + \lim_{N \to \infty} \sum_{n=0}^{N} |b_n| \\ &= \lim_{N \to \infty} \sum_{n=0}^{N} \big(|a_n| + |b_n|\big) \\ &\geq \lim_{N \to \infty} \sum_{n=0}^{N} |a_n - b_n| \\ &= \sum_{n=0}^{\infty} |a_n - b_n| \\ &\geq \sup_{n \in N} |a_n - b_n|. \end{aligned}$$

Thus, both $\sum_{n=0}^{\infty}|a_n - b_n|$ and $\sup_{n\in N}|a_n - b_n|$ are well-defined and finite.

We first show that (X, d_{l^1}) and (X, d_{l^∞}) are metric spaces. For identity: We have

$$d_{l^1}\big((a_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\big) = \sum_{n=0}^{\infty}|a_n - a_n| = 0$$

and

$$d_{l^1}\big((a_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\big) = \sup_{n\in\mathbb{N}}|a_n - a_n| = 0.$$

For positivity: If $(a_n)_{n=0}^{\infty} \neq (b_n)_{n=0}^{\infty}$, then $\exists N \in \mathbb{N}$ such that $a_N \neq b_N$. So we have

$$d_{l^1}\big((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\big) = \sum_{n=0}^{\infty}|a_n - b_n| \geq |a_N - b_N| > 0$$

and

$$d_{l^\infty}\big((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\big) = \sup_{n\in\mathbb{N}}|a_n - b_n| \geq |a_N - b_N| > 0.$$

For symmetry: We have

$$d_{l^1}\big((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\big) = \sum_{n=0}^{\infty}|a_n - b_n| = \sum_{n=0}^{\infty}|b_n - a_n| = d_{l^1}\big((b_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\big)$$

and

$$d_{l^\infty}\big((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\big) = \sup_{n\in\mathbb{N}}|a_n - b_n| = \sup_{n\in\mathbb{N}}|b_n - a_n| = d_{l^\infty}\big((b_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\big).$$

For triangle inequality: We have

$$\begin{aligned}
d_{l^1}\big((a_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}\big) &= \sum_{n=0}^{\infty}|a_n - c_n| \\
&= \sum_{n=0}^{\infty}|a_n - b_n + b_n - c_n| \\
&\leq \sum_{n=0}^{\infty}|a_n - b_n| + \sum_{n=0}^{\infty}|b_n - c_n| \\
&= d_{l^1}\big((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}\big) + d_{l^1}\big((b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}\big)
\end{aligned}$$

and

$$d_{l^\infty}\big((a_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}\big) = \sup_{n\in\mathbb{N}}|a_n - c_n|$$

$$\begin{aligned}
&= \sup_{n\in\mathbb{N}} |a_n - b_n + b_n - c_n| \\
&\leq \sup_{n\in\mathbb{N}} (|a_n - b_n| + |b_n - c_n|) \\
&\leq \sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n| \\
&= d_{l^\infty}\big((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty\big) + d_{l^\infty}\big((b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty\big).
\end{aligned}$$

Thus, by Def. II.1.1.2 (X, d_{l^1}) and (X, d_{l^∞}) are metric spaces.

Next we show that there exist sequences of elements of X which are convergent with respect to the d_{l^∞} metric but not with respect to the d_{l^1} metric. Let $(x^{(k)})_{k=1}^\infty$ be the sequence of sequence $(x_n^{(k)})_{n=0}^\infty$ where

$$x_n^{(k)} = \begin{cases} 0 & \text{if } n = 0, \\ \dfrac{1}{n^2} + \dfrac{1}{k} & \text{if } n \leq k, \\ \dfrac{1}{n^2} & \text{if } n > k. \end{cases}$$

Then we know that $\sum_{n=0}^{\infty} \left| x_n^{(k)} \right|$ is absolutely convergent for all $k \in \mathbb{N}$ and $k \geq 1$. Let $(y_n)_{n=0}^\infty$ be a sequence where

$$y_n = \begin{cases} 0 & \text{if } n = 0, \\ \dfrac{1}{n^2} & \text{if } n > 0. \end{cases}$$

Then $\sum_{n=0}^{\infty} |y_n|$ is also absolutely convergent. Now we show that $(x^{(k)})_{k=1}^\infty$ converges to $(y_n)_{n=0}^\infty$ with respect to d_{l^∞}. By Def. II.1.1.14 we need to show that

$$\lim_{k\to\infty} d_{l^\infty}\big((x_n^{(k)})_{n=0}^\infty, (y_n)_{n=0}^\infty\big) = 0.$$

We have

$$\begin{aligned}
d_{l^\infty}\big((x_n^{(k)})_{n=0}^\infty, (y_n)_{n=0}^\infty\big) &= \sup_{n\in\mathbb{N}} \left| x_n^{(k)} - y_n \right| \\
&= \sup\left\{0, \frac{1}{k}\right\} \\
&= \frac{1}{k}
\end{aligned}$$

and thus

$$\lim_{k\to\infty} d_{l^\infty}\big((x_n^{(k)})_{n=0}^\infty, (y_n)_{n=0}^\infty\big) = \lim_{k\to\infty} \frac{1}{k} = 0.$$

But we also have

$$\begin{aligned} d_{l^1}\big((x_n^{(k)})_{n=0}^{\infty},(y_n)_{n=0}^{\infty}\big) &= \sum_{n=0}^{\infty}\Big|x_n^{(k)}-y_n\Big| \\ &= \sum_{n=1}^{k}\frac{1}{k} \\ &= 1 \end{aligned}$$

and thus

$$\lim_{k\to\infty} d_{l^1}\big((x_n^{(k)})_{n=0}^{\infty},(y_n)_{n=0}^{\infty}\big) = \lim_{k\to\infty} 1 = 1 \neq 0.$$

We conclude that $(x^{(k)})_{k=1}^{\infty}$ converges to $(y_n)_{n=0}^{\infty}$ with respect to d_{l^∞} but not d_{l^1}.

Finally we show that any sequence which converges in the d_{l^1} metric automatically converges in the d_{l^∞} metric. Suppose that $(x^{(k)})_{k=1}^{\infty}$ converges to $(y_n)_{n=0}^{\infty}$ with respect to d_{l^1}. Since

$$\begin{aligned} d_{l^1}\big((x_n^{(k)})_{n=0}^{\infty},(y_n)_{n=0}^{\infty}\big) &= \sum_{n=0}^{\infty}\Big|x_n^{(k)}-y_n\Big| \\ &\geq \sup_{n\in N}\Big|x_n^{(k)}-y_n\Big| \\ &= d_{l^\infty}\big((x_n^{(k)})_{n=0}^{\infty},(y_n)_{n=0}^{\infty}\big), \end{aligned}$$

by squeeze test we have

$$\begin{aligned} & 0 \leq d_{l^\infty}\big((x_n^{(k)})_{n=0}^{\infty},(y_n)_{n=0}^{\infty}\big) \leq d_{l^1}\big((x_n^{(k)})_{n=0}^{\infty},(y_n)_{n=0}^{\infty}\big) \\ \implies & 0 = \lim_{k\to\infty} 0 \leq \lim_{k\to\infty} d_{l^\infty}\big((x_n^{(k)})_{n=0}^{\infty},(y_n)_{n=0}^{\infty}\big) \leq \lim_{k\to\infty} d_{l^1}\big((x_n^{(k)})_{n=0}^{\infty},(y_n)_{n=0}^{\infty}\big) = 0 \\ \implies & \lim_{k\to\infty} d_{l^\infty}\big((x_n^{(k)})_{n=0}^{\infty},(y_n)_{n=0}^{\infty}\big) = 0. \end{aligned}$$

Thus, any sequence which converges in the d_{l^1} metric automatically converges in the d_{l^∞} metric. □

Ex. II.1.1.16. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two sequences in a metric space (X,d). Suppose that $(x_n)_{n=1}^{\infty}$ converges to a point $x\in X$, and $(y_n)_{n=1}^{\infty}$ converges to a point $y\in X$. Show that $\lim_{n\to\infty} d(x_n,y_n) = d(x,y)$.

Proof. We have

$$\begin{aligned} &\lim_{n\to\infty} d(x_n,x) = 0; && \text{(by Def. II.1.1.14)} \\ &\lim_{n\to\infty} d(y_n,y) = 0; && \text{(by Def. II.1.1.14)} \\ &\lim_{n\to\infty} d(x,y) = d(x,y). \end{aligned}$$

Since

$$\begin{aligned} d(x_n, y_n) &\le d(x_n, x) + d(x, y_n) && \text{(by Def. II.1.1.2(d))} \\ &\le d(x_n, x) + d(x, y) + d(y, y_n) && \text{(by Def. II.1.1.2(d))} \\ &= d(x_n, x) + d(x, y) + d(y_n, y) && \text{(by Def. II.1.1.2(c))} \end{aligned}$$

and

$$\begin{aligned} d(x, y) &\le d(x, x_n) + d(x_n, y) && \text{(by Def. II.1.1.2(d))} \\ &\le d(x, x_n) + d(x_n, y_n) + d(y_n, y) && \text{(by Def. II.1.1.2(d))} \\ &= d(x_n, x) + d(x_n, y_n) + d(y_n, y), && \text{(by Def. II.1.1.2(c))} \end{aligned}$$

we have

$$\begin{aligned} &\Big(d(x_n, y_n) - d(x, y) \le d(x_n, x) + d(y_n, y) \Big) \\ &\wedge \Big(d(x, y) - d(x_n, y_n) \le d(x_n, x) + d(y_n, y) \Big) \\ \implies & 0 \le |d(x_n, y_n) - d(x, y)| \le d(x_n, x) + d(y_n, y) \\ \implies & \lim_{n\to\infty} 0 \le \lim_{n\to\infty} |d(x_n, y_n) - d(x, y)| \le \lim_{n\to\infty} \big(d(x_n, x) + d(y_n, y) \big) \\ \implies & 0 \le \lim_{n\to\infty} |d(x_n, y_n) - d(x, y)| \le \lim_{n\to\infty} d(x_n, x) + \lim_{n\to\infty} d(y_n, y) = 0 \\ \implies & \lim_{n\to\infty} |d(x_n, y_n) - d(x, y)| = 0 \\ \implies & \lim_{n\to\infty} \big(d(x_n, y_n) - d(x, y) \big) = 0 \\ \implies & \lim_{n\to\infty} d(x_n, y_n) - \lim_{n\to\infty} d(x, y) = 0 \\ \implies & \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x, y) = d(x, y). \end{aligned}$$

□

II.1.2 Some point-set topology of metric spaces

Note. Having defined the operation of convergence on metric spaces, we now define a couple other related notions, including that of open set, closed set, interior, exterior, boundary, and adherent point. The study of such notions is known as *point-set topology*.

Def. II.1.2.1 (Balls)**.** Let (X, d) be a metric space, let x_0 be a point in X, and let $r > 0$. We define the *ball* $B_{(X,d)}(x_0, r)$ in X, centered at x_0, and with radius r, in the metric d, to be the set

$$B_{(X,d)}(x_0, r) := \{x \in X : d(x, x_0) < r\}.$$

When it is clear what the metric space (X, d) is, we shall abbreviate $B_{(X,d)}(x_0, r)$ as just $B(x_0, r)$.

Rmk. II.1.2.4. Note that the smaller the radius r, the smaller the ball $B(x_0, r)$. However, $B(x_0, r)$ always contains at least one point, namely the center x_0, as long as r stays positive, thanks to Def. II.1.1.2(a). (We don't consider balls of zero radius or negative radius since they are rather boring, being just the empty set.)

Def. II.1.2.5 (Interior, exterior, boundary)**.** Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an *interior point of* E if there exists a radius $r > 0$ such that $B(x_0, r) \subseteq E$. We say that x_0 is an *exterior point of* E if there exists a radius $r > 0$ such that $B(x_0, r) \cap E = \emptyset$. We say that x_0 is a *boundary point of* E if it is neither an interior point nor an exterior point of E.

Note. The set of all interior points of E is called the interior of E and is sometimes denoted $\mathrm{int}(E)$. The set of exterior points of E is called the exterior of E and is sometimes denoted $\mathrm{ext}(E)$. The set of boundary points of E is called the boundary of E and is sometimes denoted ∂E.

Note. We use the same notation of metric balls for the set of interior points $\mathrm{int}_{(X,d)}(E)$, the set of exterior points $\mathrm{ext}_{(X,d)}(E)$ and the set of boundary points $\partial_{(X,d)}(E)$.

Rmk. II.1.2.6. If x_0 is an interior point of E, then x_0 must actually be an element of E, since balls $B(x_0, r)$ always contain their center x_0. Conversely, if x_0 is an exterior point of E, then x_0 cannot be an element of E. In particular, it is not possible for x_0 to simultaneously be an interior and an exterior point of E. If x_0 is a boundary point of E, then it could be an element of E, but it could also not lie in E.

Note. By Rmk. II.1.2.6 we thus have

$$\begin{aligned}
&\mathrm{int}_{(X,d)}(E) \subseteq E \\
&\mathrm{ext}_{(X,d)}(E) \subseteq X \setminus E
\end{aligned}$$

for any metric space (X, d) and any subset E of X.

E.g. II.1.2.8. When we give a set X the discrete metric d_{disc}, and E is any subset of X, then every element of E is an interior point of E, every point not contained in E is an exterior point of E, and there are no boundary points.

Proof. We have

$$\begin{aligned}
&\forall x_0 \in E, d_{\mathrm{disc}}(x_0, x_0) = 0 && \text{(by E.g. II.1.1.11)} \\
\implies &\forall x_0 \in E, B_{(X,d_{\mathrm{disc}})}(x_0, 1) = \{x_0\} \subseteq E && \text{(by Def. II.1.2.1)} \\
\implies &\forall x_0 \in E, x_0 \in \mathrm{int}_{(X,d_{\mathrm{disc}})}(E) && \text{(by Def. II.1.2.5)} \\
\implies &E \subseteq \mathrm{int}_{(X,d_{\mathrm{disc}})}(E) \\
\implies &E = \mathrm{int}_{(X,d_{\mathrm{disc}})}(E) && \text{(by Rmk. II.1.2.6)}
\end{aligned}$$

and

$$\begin{aligned}
&E = \mathrm{int}_{(X,d_{\mathrm{disc}})}(E) \\
\implies &\forall x_0 \in X \setminus E, x_0 \notin \mathrm{int}_{(X,d_{\mathrm{disc}})}(E) \\
\implies &\forall x_0 \in X \setminus E, \forall r \in \mathbb{R}^+, B_{(X,d_{\mathrm{disc}})}(x_0, r) \not\subseteq E && \text{(by Def. II.1.2.5)} \\
\implies &\forall x_0 \in X \setminus E, \forall r \in \mathbb{R}^+, B_{(X,d_{\mathrm{disc}})}(x_0, r) \cap E = \emptyset \\
\implies &\forall x_0 \in X \setminus E, x_0 \in \mathrm{ext}_{(X,d_{\mathrm{disc}})}(E) \\
\implies &X \setminus E \subseteq \mathrm{ext}_{(X,d_{\mathrm{disc}})}(E) \\
\implies &X \setminus E = \mathrm{ext}_{(X,d_{\mathrm{disc}})}(E). && \text{(by Rmk. II.1.2.6)}
\end{aligned}$$

Since $X = (X \setminus E) \cup E = \mathrm{ext}(E) \cup \mathrm{int}(E)$, every point in (X, d_{disc}) is either a exterior point or interior point, thus by Def. II.1.2.5 there are no boundary points in (X, d_{disc}). ☐

Def. II.1.2.9 (Closure)**.** Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an *adherent point* of E if for every radius $r > 0$, the ball $B(x_0, r)$ has a non-empty intersection with E. The set of all adherent points of E is called the *closure* of E and is denoted $\overline{E}$.

Note. Notions in Def. II.1.2.9 are consistent with the corresponding notions on the real line.

Note. Since the closure of a set E depends on metric (X, d), we denote the closure of E with $\overline{E}_{(X,d)}$.

Prop. II.1.2.10. Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. Then the following statements are logically equivalent.

(a) x_0 is an adherent point of E.

(b) x_0 is either an interior point or a boundary point of E.

(c) There exists a sequence $(x_n)_{n=1}^{\infty}$ in E which converges to x_0 with respect to the metric d.

Proof. We first show that statement (a) implies statement (b).

$$\begin{aligned}
&\forall x_0 \in \overline{E}_{(X,d)}, \forall r \in \mathbb{R}^+, B_{(X,d)}(x_0, r) \cap E \neq \emptyset && \text{(by Def. II.1.2.9)} \\
\implies &\forall x_0 \in \overline{E}_{(X,d)}, x_0 \notin \mathrm{ext}_{(X,d)}(E) && \text{(by Def. II.1.2.5)} \\
\implies &\forall x_0 \in \overline{E}_{(X,d)}, x_0 \in \left(\mathrm{int}_{(X,d)}(E) \cup \partial_{(X,d)}(E)\right) && \text{(by Def. II.1.2.5)} \\
\implies &\overline{E}_{(X,d)} \subseteq \left(\mathrm{int}_{(X,d)}(E) \cup \partial_{(X,d)}(E)\right)
\end{aligned}$$

Next we show that statement (b) implies statement (c).

- Suppose that $x_0 \in \mathrm{int}_{(X,d)}(E)$. Then by setting $x_n = x_0$ for all $n \in \mathbb{Z}^+$ we have $\lim\limits_{n\to\infty} d(x_n, x_0) = \lim\limits_{n\to\infty} 0 = 0$.

- Suppose that $x_0 \in \partial_{(X,d)}(E)$. By Def. II.1.2.5 for all $r \in \mathbb{R}^+$, we have $B_{(X,d)}(x_0, r) \cap E \neq \emptyset$. In particular, we have $B_{(X,d)}(x_0, \frac{1}{n}) \cap E \neq \emptyset$ for all $n \in \mathbb{Z}^+$. Let $X_n = B_{(X,d)}(x_0, \frac{1}{n}) \cap E$. Since $X_n \neq \emptyset$, by axiom of choice we can choose $(x_n)_{n=1}^{\infty} \in \prod_{n \in \mathbb{Z}^+} X_n$. Thus, $d(x_n, x_0) < \frac{1}{n}$ for all $n \in \mathbb{Z}^+$ and by squeeze test we have $\lim_{n\to\infty} d(x_n, x_0) = 0$.

From all cases above, we conclude that there exists a sequence $(x_n)_{n=1}^{\infty}$ in E which converges to x_0 with respect to d.

Finally we show that statement (c) implies statement (a). Suppose that there exists a sequence $(x_n)_{n=1}^{\infty}$ in E which converges to x_0 with respect to d. By Def. II.1.1.14 we have

$$\forall r \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \in \mathbb{N}, n \geq N \implies d(x_n, x_0) \leq r.$$

Since $x_N \in E$ and $E \subseteq X$, we know that the set $B_{(X,d)}(x_0, r) \cap E \neq \emptyset$. Since r was arbitrary, by Def. II.1.2.9 we know that x_0 is an adherent point of E. $\square$

Cor. II.1.2.11. Let (X, d) be a metric space, and let E be a subset of X. Then $\overline{E} = \mathrm{int}(E) \cup \partial E = X \setminus \mathrm{ext}(E)$.

Proof. By Prop. II.1.2.10(a)(b) we are done. $\square$

Def. II.1.2.12 (Open and closed sets)**.** Let (X, d) be a metric space, and let E be a subset of X. We say that E is *closed* if it contains all of its boundary points, i.e., $\partial E \subseteq E$. We say that E is *open* if it contains none of its boundary points, i.e., $\partial E \cap E = \emptyset$. If E contains some of its boundary points but not others, then it is neither open nor closed.

Rmk. II.1.2.14. It is possible for a set to be simultaneously open and closed, if it has no boundary. For instance, in a metric space (X, d), the whole space X has no boundary (every point in X is an interior point), and so X is both open and closed. The empty set $\emptyset$ also has no boundary (every point in X is an exterior point), and so is both open and closed. In many cases these are the only sets that are simultaneously open and closed, but there are exceptions. For instance, using the discrete metric d_{disc}, *every* set is both open and closed! (See E.g. II.1.2.8)

Note. From Rmk. II.1.2.14 we see that the notions of being open and being closed are *not* negations of each other; there are sets that are both open and closed, and there are sets which are neither open nor closed. Thus, if one knew for instance that E was not an open set, it would be erroneous to conclude from this that E was a closed set, and similarly with the roles of open and closed reversed.

Prop. II.1.2.15 (Basic properties of open and closed sets)**.** Let (X, d) be a metric space.

(a) Let E be a subset of X. Then E is open iff $E = \mathrm{int}(E)$. In other words, E is open iff for every $x \in E$, there exists an $r > 0$ such that $B(x, r) \subseteq E$.

(b) Let E be a subset of X. Then E is closed iff E contains all its adherent points. In other words, E is closed iff for every convergent sequence $(x_n)_{n=m}^{\infty}$ in E, the limit $\lim_{n\to\infty} x_n$ of that sequence also lies in E.

(c) For any $x_0 \in X$ and $r > 0$, then the ball $B(x_0, r)$ is an open set. The set $\{x \in X : d(x, x_0) \leq$ is a closed set. (This set is sometimes called the *closed ball* of radius r centered at x_0.)

(d) Any singleton set $\{x_0\}$, where $x_0 \in X$, is automatically closed.

(e) If E is a subset of X, then E is open iff the complement $X \setminus E := \{x \in X : x \notin E\}$ is closed.

(f) If $E_1, \ldots, E_n$ are a finite collection of open sets in X, then $E_1 \cap E_2 \cap \cdots \cap E_n$ is also open. If $F_1, \ldots, F_n$ is a finite collection of closed sets in X, then $F_1 \cup F_2 \cup \cdots \cup F_n$ is also closed.

(g) If $\{E_\alpha\}_{\alpha \in I}$ is a collection of open sets in X (where the index set I could be finite, countable, or uncountable), then the union $\bigcup_{\alpha \in I} E_\alpha := \{x \in X : x \in E_\alpha \text{ for some } \alpha \in I\}$ is also open. If $\{F_\alpha\}_{\alpha \in I}$ is a collection of closed sets in X, then the intersection $\bigcap_{\alpha \in I} F_\alpha := \{x \in X : x \in F_\alpha \text{ for all } \alpha \in I\}$ is also closed.

(h) If E is any subset of X, then $\operatorname{int}(E)$ is the largest open set which is contained in E; in other words, $\operatorname{int}(E)$ is open, and given any other open set $V \subseteq E$, we have $V \subseteq \operatorname{int}(E)$. Similarly, $\overline{E}$ is the smallest closed set which contains E; in other words, $\overline{E}$ is closed, and given any other closed set $K \supseteq E$, $K \supseteq \overline{E}$.

Proof. (a)

$$
\begin{aligned}
& E \text{ is open in } (X, d) \\
\iff & \partial_{(X,d)}(E) \cap E = \emptyset && \text{(by Def. II.1.2.12)} \\
\iff & \forall x \in E, \big(x \notin \partial_{(X,d)}(E)\big) \wedge \big(x \notin \operatorname{ext}_{(X,d)}(E)\big) && \text{(by Rmk. II.1.2.6)} \\
\iff & \forall x \in E, x \in \operatorname{int}_{(X,d)}(E) && \text{(by Def. II.1.2.5)} \\
\iff & E \subseteq \operatorname{int}_{(X,d)}(E) \\
\iff & \operatorname{int}(E) = E. && \text{(by Rmk. II.1.2.6)}
\end{aligned}
$$

□

Proof. (b)

$$
\begin{aligned}
& E \text{ is closed in } (X, d) \\
\iff & \partial_{(X,d)}(E) \subseteq E && \text{(by Def. II.1.2.12)}
\end{aligned}
$$

$$
\begin{aligned}
&\iff E = \operatorname{int}_{(X,d)}(E) \cup \partial_{(X,d)}(E) && \text{(by Rmk. II.1.2.6)}\\
&\iff E = \overline{E}_{(X,d)}. && \text{(by Prop. II.1.2.10(a)(b))}
\end{aligned}
$$

□

Proof. (c) We first show that $B_{(X,d)}(x_0, r)$ is open in (X, d). Since $x_0 \in B_{(X,d)}(x_0, r)$, we know that $B_{(X,d)}(x_0, r) \neq \emptyset$. Let $x \in B_{(X,d)}(x_0, r)$ and let $r' = r - d(x, x_0)$. By Def. II.1.2.1 we have $d(x, x_0) < r$, so $r' > 0$. Then we have

$$
\begin{aligned}
&\forall y \in B_{(X,d)}(x, r')\\
\implies& d(y, x) < r' && \text{(by Def. II.1.2.1)}\\
\implies& d(y, x) < r - d(x, x_0)\\
\implies& d(y, x) + d(x, x_0) < r\\
\implies& d(y, x_0) \leq d(y, x) + d(x, x_0) < r && \text{(by Def. II.1.1.2(d))}\\
\implies& y \in B_{(X,d)}(x_0, r) && \text{(by Def. II.1.2.1)}
\end{aligned}
$$

and thus $B_{(X,d)}(x, r') \subseteq B_{(X,d)}(x_0, r)$. Since x was arbitrary, we have

$$
\begin{aligned}
&B_{(X,d)}(x_0, r) \subseteq \operatorname{int}_{(X,d)}\big(B_{(X,d)}(x_0, r)\big) && \text{(by Def. II.1.2.5)}\\
\implies& B_{(X,d)}(x_0, r) = \operatorname{int}_{(X,d)}\big(B_{(X,d)}(x_0, r)\big) && \text{(by Rmk. II.1.2.6)}\\
\implies& B_{(X,d)}(x_0, r) \text{ is open in } (X, d). && \text{(by Prop. II.1.2.15(a))}
\end{aligned}
$$

Let $E = \{x \in X : d(x, x_0) \leq r\}$. Now we show that E is closed in (X, d). By Prop. II.1.2.15(b) we know that E is closed in (X, d) iff $E = \overline{E}_{(X,d)}$. By Prop. II.1.2.10(c) we know that $E \subseteq \overline{E}_{(X,d)}$. So we only need to show that $\overline{E}_{(X,d)} \subseteq E$, or equivalently $\overline{E}_{(X,d)} \setminus E = \emptyset$. Suppose for the sake of contradiction that $\overline{E}_{(X,d)} \setminus E \neq \emptyset$. Let $y \in \overline{E}_{(X,d)} \setminus E$. By Prop. II.1.2.10(c), $\exists (y_n)_{n=1}^{\infty}$ such that $y_n \in E$ for all $n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} d(y_n, y) = 0$. Since $y \notin E$, we have $d(y, x_0) > r$. Then $d(y, x_0) - r > 0$ and we have

$$
\begin{aligned}
&\exists N \in \mathbb{Z}^+ : \forall n \geq N,\\
&d(y_n, y) \leq \frac{d(y, x_0) - r}{2} < d(y, x_0) - r && \text{(by Def. II.1.1.14)}\\
\implies& \exists N \in \mathbb{Z}^+ : \forall n \geq N,\\
&r < d(y, x_0) - d(y_n, y) \leq d(y, x_0) + d(y_n, y) \leq d(y_n, x_0) && \text{(by Def. II.1.1.2)}\\
\implies& \exists N \in \mathbb{Z}^+ : \forall n \geq N, r < d(y_n, x_0).
\end{aligned}
$$

But $d(y_n, x_0) > r$ means $y_n \notin E$, a contradiction. Thus, we must have $\overline{E}_{(X,d)} \setminus E = \emptyset$, as desired. □

Proof. (d) By Prop. II.1.2.15(b) we know that $\{x_0\}$ is closed in (X, d) iff $\{x_0\} = \overline{\{x_0\}}_{(X,d)}$. By Prop. II.1.2.10(c) we know that $\{x_0\} \subseteq \overline{\{x_0\}}_{(X,d)}$. So we only need to show that

$\overline{\{x_0\}}_{(X,d)} \subseteq \{x_0\}$. Let $y \in \overline{\{x_0\}}_{(X,d)}$. By Prop. II.1.2.10(c) we know that $\exists (y_n)_{n=1}^{\infty}$ such that $y_n \in \{x_0\}$ for all $n \in \mathbb{Z}^+$ and $\lim\limits_{n\to\infty} d(y_n, y) = 0$. But $y_n \in \{x_0\}$ implies $y_n = x_0$ for all $n \in \mathbb{Z}^+$, thus

$$\lim_{n\to\infty} d(y_n, y) = \lim_{n\to\infty} d(x_0, y) = d(x_0, y) = 0.$$

By Def. II.1.1.2(a) we have $x_0 = y$. This means $\overline{\{x_0\}}_{(X,d)} \subseteq \{x_0\}$, as desired. □

Proof. (e) Since

$$\begin{aligned}
&x_0 \in \operatorname{int}_{(X,d)}(E) \\
\iff &\exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \subseteq E && \text{(by Def. II.1.2.5)} \\
\iff &\exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \cap (X \setminus E) = \emptyset \\
\iff &x_0 \in \operatorname{ext}_{(X,d)}(X \setminus E), && \text{(by Def. II.1.2.5)}
\end{aligned}$$

we know that $\operatorname{int}_{(X,d)}(E) = \operatorname{ext}_{(X,d)}(X \setminus E)$ for any subset E of X. Then we have

$$\begin{aligned}
\partial_{(X,d)}(E) &= X \setminus \big(\operatorname{int}_{(X,d)}(E) \cup \operatorname{ext}_{(X,d)}(E)\big) && \text{(by Def. II.1.2.5)} \\
&= X \setminus \big(\operatorname{ext}_{(X,d)}(X \setminus E) \cup \operatorname{int}_{(X,d)}(X \setminus E)\big) \\
&= \partial_{(X,d)}(X \setminus E) && \text{(by Def. II.1.2.5)}
\end{aligned}$$

and

$$\begin{aligned}
&E \text{ is open in } (X, d) \\
\iff &\partial_{(X,d)}(E) \cap E = \emptyset && \text{(by Def. II.1.2.12)} \\
\iff &\partial_{(X,d)}(E) \subseteq (X \setminus E) && \text{(by Def. II.1.2.5)} \\
\iff &\partial_{(X,d)}(X \setminus E) \subseteq (X \setminus E) \\
\iff &X \setminus E \text{ is closed in } (X, d). && \text{(by Def. II.1.2.12)}
\end{aligned}$$

□

Proof. (f) Let $I_n = \{i \in \mathbb{N} : 1 \le i \le n\}$. First, suppose that E_i is open in (X, d) for every $i \in I_n$. Let $x_0 \in \bigcap\limits_{i \in I_n} E_i$. Then we have

$$\begin{aligned}
&x_0 \in \bigcap_{i \in I_n} E_i \\
\implies &\forall i \in I_n, x_0 \in E_i \\
\implies &\forall i \in I_n, x_0 \in \operatorname{int}_{(X,d)}(E_i) && \text{(by Prop. II.1.2.15(a))} \\
\implies &\forall i \in I_n, \exists r_i \in \mathbb{R}^+ : B_{(X,d)}(x_0, r_i) \subseteq E_i && \text{(by Def. II.1.2.5)} \\
\implies &\forall i \in I_n, B_{(X,d)}\big(x_0, \min_{j \in I_n}(r_j)\big) \subseteq E_i && (I_n \text{ is finite})
\end{aligned}$$

$$
\begin{aligned}
&\implies B_{(X,d)}\big(x_0, \min_{j\in I_n}(r_j)\big) \subseteq \bigcap_{i\in I_n} E_i \\
&\implies x_0 \in \operatorname{int}_{(X,d)}\bigg(\bigcap_{i\in I_n} E_i\bigg). && \text{(by Def. II.1.2.5)}
\end{aligned}
$$

Since x_0 was arbitrary, we have

$$
\begin{aligned}
&\bigcap_{i\in I_n} E_i \subseteq \operatorname{int}_{(X,d)}\bigg(\bigcap_{i\in I_n} E_i\bigg) \\
\implies &\bigcap_{i\in I_n} E_i = \operatorname{int}_{(X,d)}\bigg(\bigcap_{i\in I_n} E_i\bigg) && \text{(by Rmk. II.1.2.6)} \\
\implies &\bigcap_{i\in I_n} E_i \text{ is open in } (X,d). && \text{(by Prop. II.1.2.15(a))}
\end{aligned}
$$

Now suppose that F_i is closed in (X,d) for every $i \in I_n$. Then we have

$$
\begin{aligned}
&\forall i \in I_n, F_i \text{ is closed in } (X,d) \\
\implies &\forall i \in I_n, X \setminus F_i \text{ is open in } (X,d) && \text{(by Prop. II.1.2.15(e))} \\
\implies &\bigcap_{i\in I_n} (X \setminus F_i) \text{ is open in } (X,d) && \text{(from the proof above)} \\
\implies &X \setminus \bigg(\bigcap_{i\in I_n} (X \setminus F_i)\bigg) \text{ is closed in } (X,d) && \text{(by Prop. II.1.2.15(e))} \\
\implies &\bigcup_{i\in I_n} F_i \text{ is closed in } (X,d).
\end{aligned}
$$

□

Proof. (g) First, that E_α is open in (X,d) for every $\alpha \in I$. Let $x_0 \in \bigcup_{\alpha\in I} E_\alpha$. Then we have

$$
\begin{aligned}
&x_0 \in \bigcup_{\alpha\in I} E_\alpha \\
\implies &\exists \beta \in I : x_0 \in E_\beta \\
\implies &\exists \beta \in I : x_0 \in \operatorname{int}_{(X,d)}(E_\beta) && \text{(by Prop. II.1.2.15(a))} \\
\implies &\exists \beta \in I : \exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \subseteq E_\beta && \text{(by Def. II.1.2.5)} \\
\implies &\exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \subseteq \bigcup_{\alpha\in I} E_\alpha \\
\implies &x_0 \in \operatorname{int}_{(X,d)}\bigg(\bigcup_{\alpha\in I} E_\alpha\bigg). && \text{(by Def. II.1.2.5)}
\end{aligned}
$$

Since x_0 was arbitrary, we have

$$\begin{aligned}
&\bigcup_{\alpha\in I} E_\alpha \subseteq \operatorname{int}_{(X,d)}\Big(\bigcup_{\alpha\in I} E_\alpha\Big) \\
\implies &\bigcup_{\alpha\in I} E_\alpha = \operatorname{int}_{(X,d)}\Big(\bigcup_{\alpha\in I} E_\alpha\Big) && \text{(by Rmk. II.1.2.6)} \\
\implies &\bigcup_{\alpha\in I} E_\alpha \text{ is open in } (X,d). && \text{(by Prop. II.1.2.15(a))}
\end{aligned}$$

Now suppose that F_α is closed in (X,d) for every $\alpha \in I$. Then we have

$$\begin{aligned}
&\forall \alpha \in I, F_\alpha \text{ is closed in } (X,d) \\
\implies &\forall \alpha \in I, X \setminus F_\alpha \text{ is open in } (X,d) && \text{(by Prop. II.1.2.15(e))} \\
\implies &\bigcup_{\alpha\in I}(X \setminus F_\alpha) \text{ is open in } (X,d) \\
\implies &X \setminus \Big(\bigcup_{\alpha\in I}(X \setminus F_\alpha)\Big) \text{ is closed in } (X,d) && \text{(by Prop. II.1.2.15(e))} \\
\implies &\bigcap_{\alpha\in I} F_\alpha \text{ is closed in } (X,d).
\end{aligned}$$

□

Proof. (h) We first show that $\operatorname{int}_{(X,d)}(E)$ is open in (X,d). Let $x_0 \in \operatorname{int}_{(X,d)}(E)$. Then we have

$$\begin{aligned}
&x_0 \in \operatorname{int}_{(X,d)}(E) \\
\implies &\exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \subseteq E && \text{(by Def. II.1.2.5)} \\
\implies &\exists r \in \mathbb{R}^+ : \forall y \in B_{(X,d)}(x_0, r), \exists r' \in \mathbb{R}^+ : \\
&B_{(X,d)}(y, r') \subseteq B_{(X,d)}(x_0, r) \subseteq E && \text{(by Prop. II.1.2.15(c))} \\
\implies &\exists r \in \mathbb{R}^+ : \forall y \in B_{(X,d)}(x_0, r), y \in \operatorname{int}_{(X,d)}(E) && \text{(by Def. II.1.2.5)} \\
\implies &\exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \subseteq \operatorname{int}_{(X,d)}(E) \\
\implies &x_0 \in \operatorname{int}_{(X,d)}\big(\operatorname{int}_{(X,d)}(E)\big). && \text{(by Def. II.1.2.5)}
\end{aligned}$$

Since x_0 was arbitrary, we have

$$\begin{aligned}
&\operatorname{int}_{(X,d)}(E) \subseteq \operatorname{int}_{(X,d)}\big(\operatorname{int}_{(X,d)}(E)\big) \\
\implies &\operatorname{int}_{(X,d)}(E) = \operatorname{int}_{(X,d)}\big(\operatorname{int}_{(X,d)}(E)\big) && \text{(by Rmk. II.1.2.6)} \\
\implies &\operatorname{int}_{(X,d)}(E) \text{ is open in } (X,d). && \text{(by Prop. II.1.2.15(a))}
\end{aligned}$$

Next we show that if $V \subseteq E$ and (V, d) is open in X, then $V \subseteq \mathrm{int}_{(X,d)}(E)$.

$$\begin{aligned}
&(V \subseteq E) \wedge \big(V \text{ is open in } (X,d)\big) \\
\implies& V = \mathrm{int}_{(X,d)}(V) \subseteq E && \text{(by Prop. II.1.2.15(a))} \\
\implies& \forall x_0 \in V, \exists r \in \mathbb{R}^+ : \\
& B_{(X,d)}(x_0, r) \subseteq V \subseteq E && \text{(by Def. II.1.2.5)} \\
\implies& \forall x_0 \in V, x_0 \in \mathrm{int}_{(X,d)}(E) && \text{(by Def. II.1.2.5)} \\
\implies& V \subseteq \mathrm{int}_{(X,d)}(E).
\end{aligned}$$

Next we show that $\overline{E}_{(X,d)}$ is closed in (X, d). Let $x_0 \in X \setminus \overline{E}_{(X,d)}$. Then we have

$$\begin{aligned}
& x_0 \in X \setminus \overline{E}_{(X,d)} \\
\implies& \exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \cap E = \emptyset && \text{(by Def. II.1.2.9)} \\
\implies& \exists r \in \mathbb{R}^+ : \forall y \in B_{(X,d)}(x_0, r), \exists r' \in \mathbb{R}^+ : \\
& \begin{cases} B_{(X,d)}(y, r') \subseteq B_{(X,d)}(x_0, r) \\ B_{(X,d)}(y, r') \cap E = \emptyset \end{cases} && \text{(by Prop. II.1.2.15(c))} \\
\implies& \exists r \in \mathbb{R}^+ : \forall y \in B_{(X,d)}(x_0, r), y \in X \setminus \overline{E}_{(X,d)} && \text{(by Def. II.1.2.9)} \\
\implies& \exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \subseteq X \setminus \overline{E}_{(X,d)} \\
\implies& x_0 \in \mathrm{int}_{(X,d)}(X \setminus \overline{E}_{(X,d)}). && \text{(by Def. II.1.2.5)}
\end{aligned}$$

Since x_0 was arbitrary, we have

$$\begin{aligned}
& X \setminus \overline{E}_{(X,d)} \subseteq \mathrm{int}_{(X,d)}(X \setminus \overline{E}_{(X,d)}) \\
\implies& X \setminus \overline{E}_{(X,d)} = \mathrm{int}_{(X,d)}(X \setminus \overline{E}_{(X,d)}) && \text{(by Rmk. II.1.2.6)} \\
\implies& X \setminus \overline{E}_{(X,d)} \text{ is open in } (X,d) && \text{(by Prop. II.1.2.15(a))} \\
\implies& \overline{E}_{(X,d)} \text{ is closed in } (X,d). && \text{(by Prop. II.1.2.15(e))}
\end{aligned}$$

Finally we show that if $E \subseteq K \subseteq X$ and K is closed in (X, d), then $\overline{E}_{(X,d)} \subseteq K$.

$$\begin{aligned}
& (E \subseteq K) \wedge \big(K \text{ is closed in } (X,d)\big) \\
\implies& \forall x_0 \in \overline{E}_{(X,d)}, \forall r \in \mathbb{R}^+, B_{(X,d)}(x_0, r) \cap E \neq \emptyset && \text{(by Def. II.1.2.9)} \\
\implies& \forall x_0 \in \overline{E}_{(X,d)}, \forall r \in \mathbb{R}^+, B_{(X,d)}(x_0, r) \cap K \neq \emptyset && (E \subseteq K) \\
\implies& \forall x_0 \in \overline{E}_{(X,d)}, x_0 \in \overline{K}_{(X,d)} && \text{(by Def. II.1.2.9)} \\
\implies& \overline{E}_{(X,d)} \subseteq \overline{K}_{(X,d)} \\
\implies& \overline{E}_{(X,d)} \subseteq K. && \text{(by Prop. II.1.2.15(b))}
\end{aligned}$$

□

— Exercises —

Ex. II.1.2.1. Verify the claims in E.g. II.1.2.8.

Proof. See E.g. II.1.2.8. □

Ex. II.1.2.2. Prove Prop. II.1.2.10.

Proof. See Prop. II.1.2.10. □

Ex. II.1.2.3. Prove Prop. II.1.2.15.

Proof. See Prop. II.1.2.15. □

Ex. II.1.2.4. Let (X, d) be a metric space, x_0 be a point in X, and $r > 0$. Let B be the open ball $B := B(x_0, r) = \{x \in X : d(x, x_0) < r\}$, and let C be the closed ball $C := \{x \in X : d(x, x_0) \leq r\}$.

(a) Show that $\overline{B} \subseteq C$.

(b) Give an example of a metric space (X, d), a point x_0, and a radius $r > 0$ such that $\overline{B}$ is *not* equal to C.

Proof. (a) Since $B \subseteq C$ and C is closed in (X, d), by Prop. II.1.2.15(h) we have $\overline{B}_{(X,d)} \subseteq C$. □

Proof. (b) Let $X = \mathbb{R}$ and let $d = d_{\text{disc}}$ be the metric function. By Ex. II.1.1.11 we know that $(\mathbb{R}, d_{\text{disc}})$ is a metric space. Let $B = B_{(\mathbb{R},d_{\text{disc}})}(0, 1)$ and let $C = \{x \in \mathbb{R} : d_{\text{disc}}(x, 0) \leq 1\}$. Then we know that $B = \{0\}$ and $C = \mathbb{R}$. But by E.g. II.1.2.8 we have $\overline{B} = B \neq C$. □

II.1.3 Relative topology

Note. Consider the plane $\mathbb{R}^2$ with the Euclidean metric d_{l^2}. Inside the plane, we can find the x-axis $X := \{(x, 0) : x \in \mathbb{R}\}$. The metric d_{l^2} can be restricted to X, creating a subspace $(X, d_{l^2}|_{X\times X})$ of $(\mathbb{R}^2, d_{l^2})$. This subspace is essentially the same as the real line $(\mathbb{R}, d)$ with the usual metric; the precise way of stating this is that $(X, d_{l^2}|_{X\times X})$ is *isometric* to $(\mathbb{R}, d)$.

Def. II.1.3.3 (Relative topology)**.** Let (X, d) be a metric space, let Y be a subset of X, and let E be a subset of Y. We say that E is *relatively open with respect to* Y if it is open in the metric subspace $(Y, d|_{Y\times Y})$. Similarly, we say that E is *relatively closed with respect to* Y if it is closed in the metric space $(Y, d|_{Y\times Y})$.

Prop. II.1.3.4. Let (X, d) be a metric space, let Y be a subset of X, and let E be a subset of Y.

(a) E is relatively open with respect to Y iff $E = V \cap Y$ for some set $V \subseteq X$ which is open in X.

(b) E is relatively closed with respect to Y iff $E = K \cap Y$ for some set $K \subseteq X$ which is closed in X.

Proof. (a) First, suppose that E is relatively open with respect to Y. Then, E is open in the metric space $(Y, d|_{Y\times Y})$. Thus, for every $x \in E$, there exists a radius $r > 0$ such that the ball $B_{(Y,d|_{Y\times Y})}(x, r)$ is contained in E. This radius r depends on x; to emphasize this we write r_x instead of r, thus for every $x \in E$ the ball $B_{(Y,d|_{Y\times Y})}(x, r_x)$ is contained in E. (Note that we have used the axiom of choice to do this.)

Now consider the set

$$V := \bigcup_{x\in E} B_{(X,d)}(x, r_x).$$

This is a subset of X. By Prop. II.1.2.15(c) and (g), V is open in (X, d). Now we prove that $E = V \cap Y$. Certainly any point x in E lies in $V \cap Y$, since it lies in Y and it also lies in $B_{(X,d)}(x, r_x)$, and hence in V. Now suppose that y is a point in $V \cap Y$. Then $y \in V$, which implies that there exists an $x \in E$ such that $y \in B_{(X,d)}(x, r_x)$. But since y is also in Y , this implies that $y \in B_{(Y,d|_{Y\times Y})}(x, r_x)$. But by definition of r_x, this means that $y \in E$, as desired. Thus, we have found an open set V in (X, d) for which $E = V \cap Y$ as desired.

Now we do the converse. Suppose that $E = V \cap Y$ for some open set V in (X, d); we have to show that E is relatively open with respect to Y. Let x be any point in E; we have to show that x is an interior point of E in the metric space $(Y, d|_{Y\times Y})$. Since $x \in E$, we know $x \in V$. Since V is open in (X, d), we know that there is a radius $r > 0$ such that $B_{(X,d)}(x, r)$ is contained in V. Strictly speaking, r depends on x, and so we could write r_x instead of r, but for this argument we will only use a single choice of x (as opposed to the argument in the previous paragraph) and so we will not bother to subscript r here. Since $E = V \cap Y$, this means that $B_{(X,d)}(x, r) \cap Y$ is contained in E. But $B_{(X,d)}(x, r) \cap Y$ is exactly the same as $B_{(Y,d|_{Y\times Y})}(x, r)$, and so $B_{(Y,d|_{Y\times Y})}(x, r)$ is contained in E. Thus, x is an interior point of E in the metric space $(Y, d|_{Y\times Y})$, as desired. □

Proof. (b) First, suppose that E is relatively closed with respect to Y. Then by Def. II.1.3.3 E is closed in $(Y, d|_{Y\times Y})$. By Prop. II.1.2.15(e) $Y \setminus E$ is open in $(Y, d|_{Y\times Y})$. By Prop. II.1.3.4(a) we know that $Y \setminus E = V \cap Y$ for some set $V \subseteq X$ which is open in (X, d). Let $K = X \setminus V$. By Prop. II.1.2.15(e) we know that K is closed in (X, d). Then we have

$$\begin{aligned} K \cap Y &= (X \setminus V) \cap Y \\ &= (X \cap Y) \setminus (V \cap Y) \\ &= Y \setminus (V \cap Y) \\ &= Y \setminus (Y \setminus E) \\ &= E. \end{aligned}$$

Now suppose that $E = K \cap Y$ for some set $K \subseteq X$ which is closed in (X, d). Then by Prop. II.1.2.15(e) $X \setminus K$ is open in (X, d). By Prop. II.1.3.4(a) we know that $F = (X \setminus K) \cap Y$

is relatively open with respect to Y. Then by Prop. II.1.2.15(e) $Y \setminus F$ is relatively closed with respect to Y and

$$\begin{aligned}
Y \setminus F &= Y \setminus \big((X \setminus K) \cap Y\big) \\
&= Y \setminus \big((X \cap Y) \setminus (K \cap Y)\big) \\
&= Y \setminus \big(Y \setminus (K \cap Y)\big) \\
&= Y \setminus (Y \setminus E) \\
&= E.
\end{aligned}$$

□

— Exercises —

Ex. II.1.3.1. Prove Prop. II.1.3.4(b).

Proof. See Prop. II.1.3.4. □

II.1.4 Cauchy sequences and complete metric spaces

Def. II.1.4.1 (Subsequences)**.** Suppose that $(x^{(n)})_{n=m}^{\infty}$ is a sequence of points in a metric space (X, d). Suppose that $n_1, n_2, n_3, \ldots$ is an increasing sequence of integers which are at least as large as m, thus

$$m \leq n_1 < n_2 < n_3 < \ldots .$$

Then we call the sequence $(x^{(n_j)})_{j=1}^{\infty}$ a *subsequence* of the original sequence $(x^{(n)})_{n=m}^{\infty}$.

Lem. II.1.4.3. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in (X, d) which converges to some limit x_0. Then every subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of that sequence also converges to x_0.

Proof.

$$\begin{aligned}
&\lim_{n\to\infty} d(x^{(n)}, x_0) = 0 \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \geq N, d(x^{(n)}, x_0) \leq \varepsilon && \text{(by Def. II.1.1.14)} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists j \in \mathbb{N} : \forall j \geq N, (n_j \geq j) \wedge \big(d(x^{(n_j)}, x_0) \leq \varepsilon\big) \\
\implies &\lim_{j\to\infty} d(x^{(n_j)}, x_0) = 0. && \text{(by Def. II.1.1.14)}
\end{aligned}$$

□

Def. II.1.4.4 (Limit points)**.** Suppose that $(x^{(n)})_{n=m}^{\infty}$ is a sequence of points in a metric space (X, d), and let $L \in X$. We say that L is a *limit point* of $(x^{(n)})_{n=m}^{\infty}$ iff for every $N \geq m$ and $\varepsilon > 0$ there exists an $n \geq N$ such that $d(x^{(n)}, L) \leq \varepsilon$.

Prop. II.1.4.5. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d), and let $L \in X$. Then the following are equivalent:

- L is a limit point of $(x^{(n)})_{n=m}^{\infty}$.
- There exists a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of the original sequence $(x^{(n)})_{n=m}^{\infty}$ which converges to L.

Proof. We first show that if L is a limit point of $(x^{(n)})_{n=m}^{\infty}$ in (X, d), then there exists a subsequence of $(x^{(n)})_{n=m}^{\infty}$ which converges to L with respect to d. Let $N \in \mathbb{N}$. Since L is a limit point of $(x^{(n)})_{n=m}^{\infty}$ in (X, d), by Def. II.1.4.4 we know that

$$\forall \varepsilon \in \mathbb{R}^+, \forall N \geq m, \exists n \geq N : d(x^{(n)}, L) \leq \varepsilon.$$

In particular, for every $j \in \mathbb{Z}^+$, we have

$$\forall N \geq m, \exists n \geq N : d(x^{(n)}, L) \leq \frac{1}{j}.$$

Now we define n_j as follow:

$$n_j = \begin{cases} \min\Big\{n \in \mathbb{Z}^+ : d(x^{(n)}, L) \leq 1\Big\} & \text{if } j = 1 \\ \min\Big\{n \in \mathbb{Z}^+ : (n > n_{j-1}) \wedge \big(d(x^{(n)}, L) \leq \frac{1}{j}\big)\Big\} & \text{if } j \neq 1 \end{cases}$$

By well-ordering theorem n_j is well-defined for every $j \in \mathbb{Z}^+$. By applying squeeze test to the subsequence $(x^{(n_j)})_{j=1}^{\infty}$ we have

$$\begin{aligned} & 0 \leq d(x^{(n_j)}, L) \leq \frac{1}{j} \\ \implies & 0 = \lim_{j\to\infty} 0 \leq \lim_{j\to\infty} d(x^{(n_j)}, L) \leq \lim_{j\to\infty} \frac{1}{j} = 0 \\ \implies & \lim_{j\to\infty} d(x^{(n_j)}, L) = 0 \end{aligned}$$

and thus by Def. II.1.1.14 the sequence $(x^{(n_j)})_{j=1}^{\infty}$ converges to L with respect to d.

Now we show that if a subsequence of $(x^{(n)})_{n=m}^{\infty}$ converges to L with respect to d, then L is a limit point of $(x^{(n)})_{n=m}^{\infty}$ in (X, d). Let $N \in \mathbb{N}$ and let $(x^{(n_j)})_{j=1}^{\infty}$ be a subsequence of $(x^{(n)})_{n=0}^{\infty}$ where $\lim_{j\to\infty} d(x^{(n_j)}, L) = 0$. Then we have

$$\begin{aligned} & \lim_{n\to\infty} d(x^{(n_j)}, L) = 0 \\ \implies & \forall \varepsilon \in \mathbb{R}^+, \exists j \geq 1 : d(x^{(n_j)}, L) \leq \varepsilon && \text{(by Def. II.1.1.14)} \\ \implies & \forall \varepsilon \in \mathbb{R}^+, \forall N \geq m, \exists n \geq n_j \geq N : d(x^{(n)}, L) \leq \varepsilon \\ \implies & L \text{ is a limit point of } (x^{(n)})_{n=m}^{\infty} \text{ in } (X, d). && \text{(by Def. II.1.4.4)} \end{aligned}$$

$\square$

Def. II.1.4.6 (Cauchy sequences)**.** Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d). We say that this sequence is a *Cauchy sequence* iff for every $\varepsilon > 0$, there exists an $N \geq m$ such that $d(x^{(j)}, x^{(k)}) \leq \varepsilon$ for all $j, k \geq N$.

Note. Here the book use $<$ instead of $\leq$, but the two inequalities are more or less the same. I use $\leq$ to ensure consistency with Definition 5.1.8 in Analysis I.

Lem. II.1.4.7 (Convergent sequences are Cauchy sequences)**.** Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in (X, d) which converges to some limit x_0. Then $(x^{(n)})_{n=m}^{\infty}$ is also a Cauchy sequence.

Proof. Let $N \in \mathbb{N}$. Since $\lim_{n\to\infty} d(x^{(n)}, x_0) = 0$, by Def. II.1.1.14 we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \geq m : \forall n \geq N, d(x^{(n)}, x_0) \leq \frac{\varepsilon}{2}.$$

Let $k \in \mathbb{N}$ and $k \geq N$. Then we have

$$\begin{aligned} d(x^{(n)}, x^{(k)}) &\leq d(x^{(n)}, x_0) + d(x_0, x^{(k)}) && \text{(by Def. II.1.1.2(d))} \\ &= d(x^{(n)}, x_0) + d(x^{(k)}, x_0) && \text{(by Def. II.1.1.2(c))} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

and by Def. II.1.4.6 $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence in (X, d). $\square$

Lem. II.1.4.9. Let $(x^{(n)})_{n=m}^{\infty}$ be a Cauchy sequence in (X, d). Suppose that there is some subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of this sequence which converges to a limit x_0 in X. Then the original sequence $(x^{(n)})_{n=m}^{\infty}$ also converges to x_0.

Proof. Let $N_1, N_2, i, k \in \mathbb{N}$. Since $\lim_{j\to\infty} d(x^{(n_j)}, x_0) = 0$, by Def. II.1.1.14 we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists N_1 \geq 1 : \forall j \geq N_1, d(x^{(n_j)}, x_0) \leq \frac{\varepsilon}{2}.$$

Now fix such ε. Since $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence in (X, d), by Def. II.1.4.6 we know that

$$\exists N_2 \geq m : \forall i, k \geq N_2, d(x^{(i)}, x^{(k)}) \leq \frac{\varepsilon}{2}.$$

Let $N = \max(N_1, N_2)$. Then $\forall i, j \geq N$, we have

$$\begin{aligned} d(x^{(i)}, x_0) &\leq d(x^{(i)}, x^{(n_j)}) + d(x^{(n_j)}, x_0) && \text{(by Def. II.1.1.2(d))} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (n_j \geq j \geq N) \\ &= \varepsilon. \end{aligned}$$

Since ε was arbitrary, by Def. II.1.1.14 $\lim_{n\to\infty} d(x^{(n)}, x_0) = 0$. $\square$

Def. II.1.4.10 (Complete metric spaces)**.** A metric space (X, d) is said to be *complete* iff every Cauchy sequence in (X, d) is in fact convergent in (X, d).

Prop. II.1.4.12.

(a) Let (X, d) be a metric space, and let $(Y, d|_{Y\times Y})$ be a subspace of (X, d). If $(Y, d|_{Y\times Y})$ is complete, then Y must be closed in X.

(b) Conversely, suppose that (X, d) is a complete metric space, and Y is a closed subset of X. Then the subspace $(Y, d|_{Y\times Y})$ is also complete.

Proof. We first show that the statement (a) is true. Suppose that (X, d) is a metric space and $(Y, d|_{Y\times Y})$ is a complete subspace of (X, d). Let $x_0 \in \partial_{(X,d)}(Y)$. Then we have

$$\begin{aligned}
&\begin{cases} (Y, d|_{Y\times Y}) \text{ is complete} \\ x_0 \in \partial_{(X,d)}(Y) \end{cases} \\
\implies &\begin{cases} (Y, d|_{Y\times Y}) \text{ is complete} \\ \exists (x^{(n)})_{n=m}^{\infty} \text{ in } Y : \lim_{n\to\infty} d(x^{(n)}, x_0) = 0 \end{cases} && \text{(by Prop. II.1.2.10)} \\
\implies &x_0 \in Y. && \text{(by Def. II.1.4.10)}
\end{aligned}$$

Since x_0 was arbitrary, we have $\partial_{(X,d)}(Y) \subseteq Y$ and by Def. II.1.2.12 Y is closed in (X, d).

Now we show that the statement (b) is true. Suppose that (X, d) is a complete metric space, $Y \subseteq X$ and Y is closed in (X, d). Let $(x^{(n)})_{n=m}^{\infty}$ be a Cauchy sequence in $(Y, d|_{Y\times Y})$. Since $Y \subseteq X$, we know that $(x^{(n)})_{n=m}^{\infty}$ is also a Cauchy sequence in (X, d). Since (X, d) is complete, by Def. II.1.4.10 we know that $\lim_{n\to\infty} d(x^{(n)}, x_0) = 0$ for some $x_0 \in X$. Since $(x^{(n)})_{n=m}^{\infty}$ is in Y and $\lim_{n\to\infty} d(x^{(n)}, x_0) = 0$, by Prop. II.1.2.10(c) we know that $x_0 \in \overline{Y}_{(X,d)}$. But Y is closed in (X, d), thus by Prop. II.1.2.15(b) we know that $x_0 \in Y$. Since $(x^{(n)})_{n=m}^{\infty}$ was arbitrary Cauchy sequence in $(Y, d|_{Y\times Y})$, by Def. II.1.4.10 $(Y, d|_{Y\times Y})$ is complete. □

— Exercises —

Ex. II.1.4.1. Prove Lem. II.1.4.3.

Proof. See Lem. II.1.4.3. □

Ex. II.1.4.2. Prove Proposition 1.4.5.

Proof. See Prop. II.1.4.5. □

Ex. II.1.4.3. Prove Lem. II.1.4.7.

Proof. See Lem. II.1.4.7. □

Ex. II.1.4.4. Prove Lem. II.1.4.9.

Proof. See Lem. II.1.4.9. □

Ex. II.1.4.5. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d), and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then L is an adherent point of the set $\left\{x^{(n)} : n \geq m\right\}$. Is the converse true?

Proof. Let $E = \left\{x^{(n)} : n \geq m\right\}$. We first show that if L is a limit point of $(x^{(n)})_{n=m}^{\infty}$ in (X, d), then $L \in \overline{E}_{(X,d)}$. Suppose that L is a limit point of $(x^{(n)})_{n=m}^{\infty}$ in (X, d). By Prop. II.1.4.5 we know that $\exists (x^{(n_j)})_{j=1}^{\infty}$ in E such that $\lim_{j\to\infty} d(x^{(n_j)}, L) = 0$. Thus, by Prop. II.1.2.10(c) $L \in \overline{E}_{(X,d)}$.

Now we show that if $L \in \overline{E}_{(X,d)}$, then L may not be a limit point of $(x^{(n)})_{n=m}^{\infty}$ in (X, d). Let $(X, d) = (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ and let $x^{(n)} = 1/n$. Then by Def. II.1.2.9 we know that $1 \in \overline{E}_{(X,d)}$. But by Def. II.1.4.4 we know that 1 is not an limit point of $(x^{(n)})_{n=m}^{\infty}$ in $(\mathbb{R}, d_{l^1})$ since every subsequence of $(x^{(n)})_{n=1}^{\infty}$ converges to 0 with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Thus, if L is an adherent point of E in (X, d), then L may not be a limit point of $(x^{(n)})_{n=m}^{\infty}$ in (X, d). □

Ex. II.1.4.6. Show that every Cauchy sequence can have at most one limit point.

Proof. Suppose for the sake of contradiction that there exists a Cauchy sequence $(x^{(n)})_{n=m}^{\infty}$ in some metric space (X, d) which has two limit points L and L'. Then by Prop. II.1.4.5 $\exists (x^{(n_i)})_{i=1}^{\infty}, (x^{(n_j)})_{j=1}^{\infty}$, which converges to L and L' respectively. Since $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence in (X, d), by Lem. II.1.4.9 we know that $(x^{(n)})_{n=m}^{\infty}$ converges to L and L' with respect to d, which contradict to Prop. II.1.1.20. Thus, every Cauchy sequence can have at most one limit point. □

Ex. II.1.4.7. Prove Prop. II.1.4.12.

Proof. See Prop. II.1.4.12. □

Ex. II.1.4.8. The following construction generalizes the construction of the reals from the rationals in Chapter 5, allowing one to view any metric space as a subspace of a complete metric space. In what follows we let (X, d) be a metric space.

(a) Given any Cauchy sequence $(x^{(n)})_{n=m}^{\infty}$ in X, we introduce the *formal limit* $\operatorname{LIM}_{n\to\infty} x_n$. We say that two formal limits $\operatorname{LIM}_{n\to\infty} x_n$ and $\operatorname{LIM}_{n\to\infty} y_n$ are equal if $\operatorname{LIM}_{n\to\infty} d(x_n, y_n)$ is equal to zero. Show that this equality relation obeys the reflexive, symmetry, and transitive axioms.

(b) Let $\overline{X}$ be the space of all formal limits of Cauchy sequences in X, with the above equality relation. Define a metric $d_{\overline{X}} : \overline{X} \times \overline{X} \to [0, \infty)$ by setting

$$d_{\overline{X}}(\operatorname{LIM}_{n\to\infty} x_n, \operatorname{LIM}_{n\to\infty} y_n) \coloneqq \lim_{n\to\infty} d(x_n, y_n).$$

Show that this function is well-defined (this means not only that the limit $\lim_{n\to\infty} d(x_n, y_n)$ exists, but also that the axiom of substitution is obeyed; cf. Lemma 5.3.7), and gives $\overline{X}$ the structure of a metric space.

(c) Show that the metric space $(\overline{X}, d_{\overline{X}})$ is complete.

(d) We identify an element $x \in X$ with the corresponding formal limit $\operatorname*{LIM}_{n\to\infty} x$ in X; show that this is legitimate by verifying that $x = y \iff \operatorname*{LIM}_{n\to\infty} x = \operatorname*{LIM}_{n\to\infty} y$. With this identification, show that $d(x, y) = d_{\overline{X}}(x, y)$, and thus (X, d) can now be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$.

(e) Show that the closure of X in $\overline{X}$ is $\overline{X}$ (which explains the choice of notation $\overline{X}$).

(f) Show that the formal limit agrees with the actual limit, thus if $(x_n)_{n=1}^{\infty}$ is any Cauchy sequence in X, then we have $\lim_{n\to\infty} x_n = \operatorname*{LIM}_{n\to\infty} x_n$ in $\overline{X}$.

Proof. (a) Let $(x^{(n)})_{n=m}^{\infty}$, $(y^{(n)})_{n=m}^{\infty}$, $(z^{(n)})_{n=m}^{\infty}$ be Cauchy sequences in (X, d).

First, suppose that $\operatorname*{LIM}_{n\to\infty} x^{(n)}$ is well-defined. Then we have

$$\begin{aligned} & \lim_{n\to\infty} d(x^{(n)}, x^{(n)}) = \lim_{n\to\infty} 0 = 0 && \text{(by Def. II.1.1.2(a))} \\ \Longrightarrow\ & \operatorname*{LIM}_{n\to\infty} x^{(n)} = \operatorname*{LIM}_{n\to\infty} x^{(n)} && \text{(by definition)} \end{aligned}$$

and thus the equality relation of Ex. II.1.4.8(a) is reflexive.

Next suppose that $\operatorname*{LIM}_{n\to\infty} x^{(n)}, \operatorname*{LIM}_{n\to\infty} y^{(n)}$ are well-defined and $\operatorname*{LIM}_{n\to\infty} x^{(n)} = \operatorname*{LIM}_{n\to\infty} y^{(n)}$. Then we have

$$\begin{aligned} & \operatorname*{LIM}_{n\to\infty} x^{(n)} = \operatorname*{LIM}_{n\to\infty} y^{(n)} && \\ \iff\ & \lim_{n\to\infty} d(x^{(n)}, y^{(n)}) = 0 && \text{(by definition)} \\ \iff\ & \lim_{n\to\infty} d(y^{(n)}, x^{(n)}) = 0 && \text{(by Def. II.1.1.2(c))} \\ \iff\ & \operatorname*{LIM}_{n\to\infty} y^{(n)} = \operatorname*{LIM}_{n\to\infty} x^{(n)} && \text{(by definition)} \end{aligned}$$

and thus the equality relation of Ex. II.1.4.8(a) is symmetry.

Finally suppose that $\operatorname*{LIM}_{n\to\infty} x^{(n)}, \operatorname*{LIM}_{n\to\infty} y^{(n)}, \operatorname*{LIM}_{n\to\infty} z^{(n)}$ are well-defined. Suppose also that $\operatorname*{LIM}_{n\to\infty} x^{(n)} = \operatorname*{LIM}_{n\to\infty} y^{(n)}$ and $\operatorname*{LIM}_{n\to\infty} y^{(n)} = \operatorname*{LIM}_{n\to\infty} z^{(n)}$. Then we have

$$\begin{cases} \operatorname*{LIM}_{n\to\infty} x^{(n)} = \operatorname*{LIM}_{n\to\infty} y^{(n)} \\ \operatorname*{LIM}_{n\to\infty} y^{(n)} = \operatorname*{LIM}_{n\to\infty} z^{(n)} \end{cases}$$

$$\begin{aligned}
&\implies \begin{cases} \lim_{n\to\infty} d(x^{(n)}, y^{(n)}) = 0 \\ \lim_{n\to\infty} d(y^{(n)}, z^{(n)}) = 0 \end{cases} && \text{(by definition)} \\
&\implies \lim_{n\to\infty} \big(d(x^{(n)}, y^{(n)}) + d(y^{(n)}, z^{(n)})\big) = 0 \\
&\implies 0 \le \lim_{n\to\infty} d(x^{(n)}, z^{(n)}) \\
&\qquad \le \lim_{n\to\infty} \big(d(x^{(n)}, y^{(n)}) + d(y^{(n)}, z^{(n)})\big) = 0 && \text{(by Def. II.1.1.2)} \\
&\implies \lim_{n\to\infty} d(x^{(n)}, z^{(n)}) = 0 && \text{(by squeeze test)} \\
&\implies \operatorname*{LIM}_{n\to\infty} x^{(n)} = \operatorname*{LIM}_{n\to\infty} z^{(n)} && \text{(by definition)}
\end{aligned}$$

and thus the equality relation of Ex. II.1.4.8(a) is transitive. □

Proof. (b) Let $(x^{(n)})_{n=m}^\infty$, $(y^{(n)})_{n=m}^\infty$, $(z^{(n)})_{n=m}^\infty$ be Cauchy sequences in (X, d) with formal limits in $\overline{X}$. We first show that the limit

$$d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} x^{(n)}, \operatorname*{LIM}_{n\to\infty} y^{(n)}) := \lim_{n\to\infty} d(x^{(n)}, y^{(n)})$$

exists. Let $(a^{(n)})_{n=m}^\infty$ be the sequence $a^{(n)} = d(x^{(n)}, y^{(n)})$. To show that the above limit exists, it will suffice to show that $(a^{(n)})_{n=m}^\infty$ converges in $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Let $N_1, N_2, j, k \in \mathbb{N}$. Since $(x^{(n)})_{n=m}^\infty$ is a Cauchy sequence in (X, d), by Def. II.1.4.6 we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists N_1 \ge m : \forall j, k \ge N_1, d(x^{(j)}, x^{(k)}) \le \frac{\varepsilon}{2}.$$

Similarly,

$$\forall \varepsilon \in \mathbb{R}^+, \exists N_2 \ge m : \forall j, k \ge N_2, d(y^{(j)}, y^{(k)}) \le \frac{\varepsilon}{2}.$$

Let $N = \max(N_1, N_2)$. Then by Def. II.1.1.2 we have

$$\begin{aligned}
&\forall j, k \ge N, |a_j - a_k| \\
&= \left| d(x^{(j)}, y^{(j)}) - d(x^{(k)}, y^{(k)}) \right| \\
&= \left| d(x^{(j)}, y^{(j)}) + d(y^{(j)}, x^{(k)}) - d(y^{(j)}, x^{(k)}) - d(x^{(k)}, y^{(k)}) \right| \\
&\le \left| d(x^{(j)}, y^{(j)}) + d(y^{(j)}, x^{(k)}) + d(y^{(j)}, x^{(k)}) + d(x^{(k)}, y^{(k)}) \right| \\
&= d(x^{(j)}, y^{(j)}) + d(y^{(j)}, x^{(k)}) + d(y^{(j)}, x^{(k)}) + d(x^{(k)}, y^{(k)}) \\
&\le d(x^{(j)}, x^{(k)}) + d(y^{(j)}, y^{(k)}) \\
&\le \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon
\end{aligned}$$

and thus $(a^{(n)})_{n=m}^\infty$ is a Cauchy sequence in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Since $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ is complete (see Theorem 6.4.18, Analysis I), we know that $(a^{(n)})_{n=m}^\infty$ converges in $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$.

Next we show that $d_{\overline{X}}$ obeys the axiom of substitution. Suppose that $\operatorname*{LIM}_{n\to\infty} x^{(n)} = \operatorname*{LIM}_{n\to\infty} z^{(n)}$ and $\lim_{n\to\infty} d(x^{(n)}, y^{(n)})$ exists. Then we have

$$\begin{aligned}
&\begin{cases} d(x^{(n)}, y^{(n)}) \le d(x^{(n)}, z^{(n)}) + d(z^{(n)}, y^{(n)}) \\ d(z^{(n)}, y^{(n)}) \le d(x^{(n)}, y^{(n)}) + d(x^{(n)}, z^{(n)}) \end{cases} && \text{(by Def. II.1.1.2(c)(d))}\\
\Longrightarrow &\begin{cases} d(x^{(n)}, y^{(n)}) - d(z^{(n)}, y^{(n)}) \le d(x^{(n)}, z^{(n)}) \\ d(z^{(n)}, y^{(n)}) - d(x^{(n)}, y^{(n)}) \le d(x^{(n)}, z^{(n)}) \end{cases}\\
\Longrightarrow &\ 0 \le \left| d(x^{(n)}, y^{(n)}) - d(z^{(n)}, y^{(n)}) \right| \le d(x^{(n)}, z^{(n)})\\
\Longrightarrow &\ 0 = \lim_{n\to\infty} 0 \le \lim_{n\to\infty} \left| d(x^{(n)}, y^{(n)}) - d(z^{(n)}, y^{(n)}) \right|\\
&\qquad \le \lim_{n\to\infty} d(x^{(n)}, z^{(n)}) = 0 && \text{(by Ex. II.1.4.8(a))}\\
\Longrightarrow &\ \lim_{n\to\infty} \left| d(x^{(n)}, y^{(n)}) - d(z^{(n)}, y^{(n)}) \right| = 0 && \text{(by squeeze test)}\\
\Longrightarrow &\ \lim_{n\to\infty} \big(d(x^{(n)}, y^{(n)}) - d(z^{(n)}, y^{(n)}) \big) = 0\\
\Longrightarrow &\ \lim_{n\to\infty} d(x^{(n)}, y^{(n)}) = \lim_{n\to\infty} d(z^{(n)}, y^{(n)}) && (\lim_{n\to\infty} d(x^{(n)}, y^{(n)}) \text{ exists})
\end{aligned}$$

and thus $d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} x^{(n)}, \operatorname*{LIM}_{n\to\infty} y^{(n)}) = d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} z^{(n)}, \operatorname*{LIM}_{n\to\infty} y^{(n)})$.

Now we show that $(\overline{X}, d_{\overline{X}})$ is a metric space. For identify: By Def. II.1.1.2(a) we have

$$d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} x^{(n)}, \operatorname*{LIM}_{n\to\infty} x^{(n)}) = \lim_{n\to\infty} d(x^{(n)}, x^{(n)}) = 0.$$

For positivity: If $\operatorname*{LIM}_{n\to\infty} x^{(n)} \ne \operatorname*{LIM}_{n\to\infty} y^{(n)}$, then by Ex. II.1.4.8(a) we have

$$d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} x^{(n)}, \operatorname*{LIM}_{n\to\infty} y^{(n)}) = \lim_{n\to\infty} d(x^{(n)}, y^{(n)}) \ne 0.$$

For symmetry: We have

$$\begin{aligned}
&d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} x^{(n)}, \operatorname*{LIM}_{n\to\infty} y^{(n)})\\
&= \lim_{n\to\infty} d(x^{(n)}, y^{(n)}) && \text{(by definition)}\\
&= \lim_{n\to\infty} d(y^{(n)}, x^{(n)}) && \text{(by Def. II.1.1.2(c))}\\
&= d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} y^{(n)}, \operatorname*{LIM}_{n\to\infty} x^{(n)}). && \text{(by definition)}
\end{aligned}$$

For transitive: We have

$$\begin{aligned}
&d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} x^{(n)}, \operatorname*{LIM}_{n\to\infty} z^{(n)})\\
&= \lim_{n\to\infty} d(x^{(n)}, z^{(n)})
\end{aligned}$$

$$\begin{aligned}
&\le \lim_{n\to\infty} \left(d(x^{(n)}, y^{(n)}) + d(y^{(n)}, z^{(n)})\right) \\
&= \lim_{n\to\infty} d(x^{(n)}, y^{(n)}) + \lim_{n\to\infty} d(y^{(n)}, z^{(n)}) \\
&= d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} x^{(n)}, \operatorname*{LIM}_{n\to\infty} y^{(n)}) + d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} y^{(n)}, \operatorname*{LIM}_{n\to\infty} z^{(n)}).
\end{aligned}$$

Thus, by Def. II.1.1.2 $(\overline{X}, d_{\overline{X}})$ is a metric space. $\square$

Proof. (c) Let $I, J, K, N, k_1, k_2, n_1, n_2 \in \mathbb{Z}^+$. Let $(a^{(n)})_{n=m}^{\infty}$ be arbitrary Cauchy sequence in $(\overline{X}, d_{\overline{X}})$. Since $(a^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence in $(\overline{X}, d_{\overline{X}})$, by Def. II.1.4.6 we know that

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists N \ge m : \forall n_1, n_2 \ge N, \\
&d_{\overline{X}}(a^{(n_1)}, a^{(n_2)}) \le \frac{\varepsilon}{4} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \ge m : \forall n_1, n_2 \ge N, \\
&d_{\overline{X}}(\operatorname*{LIM}_{k\to\infty} a_k^{(n_1)}, \operatorname*{LIM}_{k\to\infty} a_k^{(n_2)}) \le \frac{\varepsilon}{4} && \text{(by Ex. II.1.4.8(a))} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \ge m : \forall n_1, n_2 \ge N, \\
&\lim_{k\to\infty} d(a_k^{(n_1)}, a_k^{(n_2)}) \le \frac{\varepsilon}{4}. && \text{(by Ex. II.1.4.8(b))}
\end{aligned}$$

Since the choice of N depends on ε, we denote such N as N_ε. We can use axiom of choice to fix N_ε for each $\varepsilon \in \mathbb{R}^+$, and we rewrite the above statement as

$$\forall \varepsilon \in \mathbb{R}^+, \forall n_1, n_2 \ge N_\varepsilon, \lim_{k\to\infty} d(a_k^{(n_1)}, a_k^{(n_2)}) \le \frac{\varepsilon}{4}.$$

Let $L = \lim_{k\to\infty} d(a_k^{(n_1)}, a_k^{(n_2)})$. Since $\left(d(a_k^{(n_1)}, a_k^{(n_2)})\right)_{k=1}^{\infty}$ is a sequence in $\mathbb{R}$ and converges to L with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$, we have

$$\begin{aligned}
&\exists I \ge 1 : \forall k \ge I, \left|d(a_k^{(n_1)}, a_k^{(n_2)}) - L\right| \le \frac{\varepsilon}{4} \\
\implies &\exists I \ge 1 : \forall k \ge I, -\frac{\varepsilon}{4} \le d(a_k^{(n_1)}, a_k^{(n_2)}) - L \le \frac{\varepsilon}{4} \\
\implies &\exists I \ge 1 : \forall k \ge I, 0 \le d(a_k^{(n_1)}, a_k^{(n_2)}) - L + \frac{\varepsilon}{4} \le \frac{\varepsilon}{2} \\
\implies &\exists I \ge 1 : \forall k \ge I, 0 \le d(a_k^{(n_1)}, a_k^{(n_2)}) \le d(a_k^{(n_1)}, a_k^{(n_2)}) - L + \frac{\varepsilon}{4} \le \frac{\varepsilon}{2} && (L \le \frac{\varepsilon}{4})
\end{aligned}$$

Since such I depends on the choice of ε, we denote such I as I_ε. Again we can use axiom of choice to fix I_ε for each $\varepsilon \in \mathbb{R}^+$, and we rewrite the above statement as

$$\forall \varepsilon \in \mathbb{R}^+, \forall n_1, n_2 \ge N_\varepsilon, \forall k \ge I_\varepsilon, d(a_k^{(n_1)}, a_k^{(n_2)}) \le \frac{\varepsilon}{2}.$$

If we let $M_\varepsilon = \max(N_\varepsilon, I_\varepsilon)$, then we can further reduce the statement as

$$\forall \varepsilon \in \mathbb{R}^+, \forall n_1, n_2, k \ge M_\varepsilon, d(a_k^{(n_1)}, a_k^{(n_2)}) \le \frac{\varepsilon}{2}.$$

Since $a^{(n)} \in \overline{X}$ for each $n \geq m$, by Ex. II.1.4.8(b) we know that there exists a Cauchy sequence $(a_k^{(n)})_{k=1}^{\infty}$ in (X, d) such that $\operatorname*{LIM}_{k\to\infty} a_k^{(n)} = a^{(n)}$ for each $n \geq m$. By Def. II.1.4.6 we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists J \geq 1 : \forall k_1, k_2 \geq J, d(a_{k_1}^{(n)}, a_{k_2}^{(n)}) \leq \frac{\varepsilon}{2}.$$

Since such J depends on the choice of n and ε, we denote such J as $J_\varepsilon^{(n)}$. We can use axiom of choice to fix $J_\varepsilon^{(n)}$ for each $\varepsilon \in \mathbb{R}^+$ and for each $n \geq m$, and we rewrite the above statement as

$$\forall \varepsilon \in \mathbb{R}^+, \forall k_1, k_2 \geq J_\varepsilon^{(n)}, d(a_{k_1}^{(n)}, a_{k_2}^{(n)}) \leq \frac{\varepsilon}{2}.$$

We now define a sequence $(b_k)_{k=1}^{\infty}$ in (X, d) by setting $b_k = a_k^{(m+k-1)}$ for all $k \geq 1$. Informally, $(b_k)_{k=1}^{\infty}$ is consist of diagonal elements in $\big((a_k^{(n)})_{k=1}^{\infty}\big)_{n=m}^{\infty}$. We claim that $(b_k)_{k=1}^{\infty}$ is a Cauchy sequence in (X, d). By Def. II.1.4.6 it suffices to show that

$$\forall \varepsilon \in \mathbb{R}^+, \exists K \geq 1 : \forall k_1, k_2 \geq K, d(b_{k_1}, b_{k_2}) \leq \varepsilon.$$

For each $\varepsilon \in \mathbb{R}^+$, we have

$$\begin{aligned}
&\forall k_1, k_2 \geq M_\varepsilon, d(b_{k_1}, b_{k_2}) \\
&= d\Big(a_{k_1}^{(m+k_1-1)}, a_{k_2}^{(m+k_2-1)}\Big) \\
&\leq d\Big(a_{k_1}^{(m+k_1-1)}, a_{k_2}^{(m+k_1-1)}\Big) + d\Big(a_{k_2}^{(m+k_1-1)}, a_{k_2}^{(m+k_2-1)}\Big) && \text{(by Def. II.1.1.2(d))} \\
&\leq d\Big(a_{k_1}^{(m+k_1-1)}, a_{k_2}^{(m+k_1-1)}\Big) + \frac{\varepsilon}{2}. && \text{(by the definition of } M_\varepsilon)
\end{aligned}$$

By choosing $K = \max(M_\varepsilon, J_\varepsilon^{(M_\varepsilon)})$ we have

$$\begin{aligned}
&\forall k_1, k_2 \geq K, d(b_{k_1}, b_{k_2}) \\
&\leq d\Big(a_{k_1}^{(m+k_1-1)}, a_{k_2}^{(m+k_1-1)}\Big) + \frac{\varepsilon}{2} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{(by the definition of } J_\varepsilon^{(M_\varepsilon)}) \\
&= \varepsilon.
\end{aligned}$$

Since ε was arbitrary, we have showed that

$$\forall \varepsilon \in \mathbb{R}^+, \exists K \geq 1 : \forall k_1, k_2 \geq K, d(b_{k_1}, b_{k_2}) \leq \varepsilon.$$

Thus, by Def. II.1.4.6 $(b_k)_{k=1}^{\infty}$ is a Cauchy sequence in (X, d).

Now we show that $(a^{(n)})_{n=m}^{\infty}$ converges in $(\overline{X}, d_{\overline{X}})$. From the proof above we know that $(b_k)_{k=1}^{\infty}$ is a Cauchy sequence in (X, d), so $\operatorname*{LIM}_{k\to\infty} b_k \in \overline{X}$. We claim that $(a^{(n)})_{n=m}^{\infty}$ converges to $\operatorname*{LIM}_{k\to\infty} b_k$ with respect to $d_{\overline{X}}$. It suffices to show that

$$\lim_{n\to\infty} d_{\overline{X}}\Big(a^{(n)}, \operatorname*{LIM}_{k\to\infty} b_k\Big) = 0 \qquad \text{(by Def. II.1.1.14)}$$

$$\begin{aligned}
&\iff \lim_{n\to\infty} d_{\overline{X}}\Big(\operatorname*{LIM}_{k\to\infty} a_k^{(n)}, \operatorname*{LIM}_{k\to\infty} b_k\Big) = 0 && \text{(by Ex. II.1.4.8(a))}\\
&\iff \lim_{n\to\infty} \big(\lim_{k\to\infty} d(a_k^{(n)}, b_k)\big) = 0 && \text{(by Ex. II.1.4.8(b))}\\
&\iff \forall \varepsilon \in \mathbb{R}^+, \exists N \geq m : \forall n \geq m,\\
&\qquad \left|\lim_{k\to\infty} d(a_k^{(n)}, b_k) - 0\right| \leq \varepsilon.
\end{aligned}$$

Since

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists M_\varepsilon \geq 1 : \forall k \geq M_\varepsilon, d(a_k^{(M_\varepsilon)}, b_k)\\
&= d\Big(a_k^{(M_\varepsilon)}, a_k^{(m+k-1)}\Big)\\
&\leq \frac{\varepsilon}{2} && \text{(by the definition of } M_\varepsilon)\\
&< \varepsilon,
\end{aligned}$$

we know that $\lim_{k\to\infty} d(a_k^{(M_\varepsilon)}, b_k)$ exists. Since

$$\begin{aligned}
&\forall n, k \geq M_\varepsilon, 0 \leq d(a_k^{(n)}, b_k) \leq \frac{\varepsilon}{2} < \varepsilon && \text{(by the definition of } M_\varepsilon)\\
\implies &\forall n \geq M_\varepsilon, 0 \leq \lim_{k\to\infty} d(a_k^{(n)}, b_k) \leq \varepsilon && \text{(by comparison test)}\\
\implies &\forall n \geq M_\varepsilon, \left|\lim_{k\to\infty} d(a_k^{(n)}, b_k) - 0\right| \leq \varepsilon.
\end{aligned}$$

by setting $N = M_\varepsilon$ we are done.

Since for arbitrary Cauchy sequence $(a^{(n)})_{n=m}^\infty$ in $(\overline{X}, d_{\overline{X}})$, $(a^{(n)})_{n=m}^\infty$ converges in $\overline{X}$ with respect to $d_{\overline{X}}$, by Def. II.1.4.10 we know that $(\overline{X}, d_{\overline{X}})$ is complete. □

Proof. (d) Since for any $x, y \in X$, we have

$$\begin{aligned}
&x = y\\
\iff &d(x, y) = 0 && \text{(by Def. II.1.1.2(a))}\\
\iff &\lim_{n\to\infty} d(x, y) = 0\\
\iff &d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} x, \operatorname*{LIM}_{n\to\infty} y) = 0 && \text{(by Ex. II.1.4.8(b))}\\
\iff &\operatorname*{LIM}_{n\to\infty} x = \operatorname*{LIM}_{n\to\infty} y. && \text{(by Def. II.1.1.2(a))}
\end{aligned}$$

Thus

$$\begin{aligned}
d_{\overline{X}}(x, y) &= d_{\overline{X}}(\operatorname*{LIM}_{n\to\infty} x, \operatorname*{LIM}_{n\to\infty} y)\\
&= \lim_{n\to\infty} d(x, y) && \text{(by Ex. II.1.4.8(b))}\\
&= d(x, y).
\end{aligned}$$

□

Proof. (e) From Ex. II.1.4.8(d) have $d = d_{\overline{X}}|_{X\times X}$. Let Y be the closure of X in $(\overline{X}, d_{\overline{X}})$. We want to show that $Y = \overline{X}$. By Def. II.1.2.9 we know that $Y \subseteq \overline{X}$. Thus, we only need to show that $\overline{X} \subseteq Y$.

Let $x_0 \in \overline{X}$. By Ex. II.1.4.8(b) there exists a Cauchy sequence $(a_n)_{n=1}^{\infty}$ in $(X, d_{\overline{X}})$ such that $\operatorname*{LIM}_{n\to\infty} a_n = x_0$. Since $(\overline{X}, d_{\overline{X}})$ is complete, by Def. II.1.4.10 we know that $(a_n)_{n=1}^{\infty}$ converges in $\overline{X}$ with respect to $d_{\overline{X}}$. But we know that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(X, d_{\overline{X}})$, thus by Prop. II.1.2.10(c) x_0 is an adherent point of X in $(\overline{X}, d_{\overline{X}})$ and $x_0 \in Y$. Since x_0 was arbitrary, we thus have $\overline{X} \subseteq Y$. □

Proof. (f) By Ex. II.1.4.8(d) we know that if $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in (X, d), then $(x_n)_{n=1}^{\infty}$ is also a Cauchy sequence in $(\overline{X}, d_{\overline{X}})$. Thus, by Ex. II.1.4.8(c)(e) we have $\lim_{n\to\infty} x_n = \operatorname*{LIM}_{n\to\infty} x_n$. □

II.1.5 Compact metric spaces

Def. II.1.5.1 (Compactness)**.** A metric space (X, d) is said to be *compact* iff every sequence in (X, d) has at least one convergent subsequence. A subset Y of a metric space X is said to be *compact* if the subspace $(Y, d|_{Y\times Y})$ is compact.

Rmk. II.1.5.2. The notion of a set Y being compact is *intrinsic*, in the sense that it only depends on the metric function $d|_{Y\times Y}$ restricted to Y, and not on the choice of the ambient space X. The notions of completeness in Def. II.1.4.10, and of boundedness in Def. II.1.5.3, are also intrinsic, but the notions of open and closed are not (see the discussion in Sec. II.1.3).

Note. The notion of a set Y being compact only depends on the metric function $d|_{Y\times Y}$ but not ambient space X since the elements of a sequence in Y stays the same no matter which spaces Y is subset to. But the notion of a set being open or closed depends on the definition of a metric ball, which may be different when given different ambient spaces.

Note. Heine-Borel theorem shows that in the real line $\mathbb{R}$ with the usual metric, every closed and bounded set is compact, and conversely every compact set is closed and bounded.

Def. II.1.5.3 (Bounded sets)**.** Let (X, d) be a metric space, and let Y be a subset of X. We say that Y is *bounded* iff for every $x \in X$ there exists a ball $B(x, r)$ in X which contains Y. We call (X, d) bounded if X is bounded.

Rmk. II.1.5.4. Def. II.1.5.3 is compatible with the definition of a bounded set in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$.

Prop. II.1.5.5. Let (X, d) be a compact metric space. Then (X, d) is both complete and bounded.

Proof. We first show that (X, d) is complete. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence in (X, d). Since (X, d) is compact, by Def. II.1.5.1 we know that there exists a subsequence of $(a_n)_{n=1}^{\infty}$

which converges to some $x_0 \in X$ with respect to d. Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence in (X,d), by Lem. II.1.4.9 we know that $(a_n)_{n=1}^{\infty}$ must converge to x_0 with respect to d. Since $(a_n)_{n=1}^{\infty}$ was arbitrary, by Def. II.1.4.10 we know that (X,d) is complete.

Now we show that (X,d) is bounded by contradiction. Suppose for the sake of contradiction that (X,d) is not bounded. Then by Def. II.1.5.3 we have

$$\begin{aligned}
&\neg\big(\forall x_0 \in X, \exists r \in \mathbb{R}^+ : X \subseteq B_{(X,d)}(x_0,r)\big)\\
\implies &\exists x_0 \in X : \forall r \in \mathbb{R}^+, X \not\subseteq B_{(X,d)}(x_0,r)\\
\implies &\exists x_0 \in X : \forall r \in \mathbb{R}^+, \big(X \setminus B_{(X,d)}(x_0,r)\big) \neq \emptyset\\
\implies &\exists x_0 \in X : \forall n \in \mathbb{Z}^+, \big(X \setminus B_{(X,d)}(x_0,n)\big) \neq \emptyset.
\end{aligned}$$

If $X = \emptyset$, then $(\emptyset, d)$ is bound since $x_0 \notin \emptyset$. So we only considered the cases $X \neq \emptyset$. Let $(a_n)_{n=1}^{\infty}$ be the sequence where $a_n \in X \setminus B_{(X,d)}(x_0,n)$ for all $n \in \mathbb{Z}^+$. Note that such sequence is well-defined by axiom of choice. Since $a_n \in X \setminus B_{(X,d)}(x_0,n)$, by Def. II.1.2.1 we know that $d(a_n,x_0) \geq n$ for all $n \in \mathbb{Z}^+$. Since (X,d) is compact, by Def. II.1.5.1 there exists a subsequence $(a_{n_j})_{j=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ such that $(a_{n_j})_{j=1}^{\infty}$ converges in X. Let $\lim\limits_{j\to\infty} d(a_{n_j},L) = 0$ for some $L \in X$. By Def. II.1.1.14 we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists J \in \mathbb{Z}^+ : \forall j \geq J, d(a_{n_j},L) \leq \varepsilon.$$

In particular,

$$\exists J \in \mathbb{Z}^+ : \forall j \geq J, d(a_{n_j},L) \leq 1.$$

Now we fix such J and let $i = \max\big(J+1, \lceil d(L,x_0)\rceil + 2\big)$. Then we have

$$\begin{aligned}
d(a_{n_i},x_0) &\leq d(a_{n_i},L) + d(L,x_0) && \text{(by Def. II.1.1.2(d))}\\
&\leq 1 + d(L,x_0) && (i > J)
\end{aligned}$$

and

$$\begin{aligned}
d(a_{n_i},x_0) &\geq n_i && \text{(by the definition of } a_{n_i})\\
&\geq i\\
&\geq d(L,x_0) + 2. && \text{(by the definition of } i)
\end{aligned}$$

But this means $d(L,x_0) + 2 \leq d(L,x_0) + 1$, a contradiction. Thus, (X,d) is bounded. □

Cor. II.1.5.6 (Compact sets are closed and bounded)**.** Let (X,d) be a metric space, and let Y be a compact subset of X. Then Y is closed and bounded.

Proof. Since $(Y, d|_{Y\times Y})$ is compact, by Prop. II.1.5.5 we know that (Y,d) is complete and bounded. Thus, by Prop. II.1.4.12(a) we know that Y is closed in (X,d). □

Thm. II.1.5.7 (Heine-Borel theorem)**.** Let $(\mathbb{R}^n, d)$ be a Euclidean space with either the Euclidean metric, the taxicab metric, or the supnorm metric. Let E be a subset of $\mathbb{R}^n$. Then E is compact iff it is closed and bounded.

Proof. We first show that for any $E \subseteq \mathbb{R}^n$, E is closed and bounded in $(\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})$ iff $(E, d_{l^1}|_{E \times E})$ is compact. By Ex. II.1.1.7 we know that $(\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})$ is a metric space, and by Cor. II.1.5.6 we know that if $(E, d_{l^1}|_{E \times E})$ is compact, then E is closed and bounded in $(\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})$. So we only need to show that if E is closed and bounded in $(\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})$, then $(E, d_{l^1}|_{E \times E})$ is compact. Suppose that $E \subseteq \mathbb{R}^n$, E is closed and bounded in $(\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})$. Since $E \subseteq \mathbb{R}^n$, we know that for every $x \in E$, x is in the form $x = (x_1, \ldots, x_n) = (x_i)_{i=1}^n \in \mathbb{R}^n$. Let $I_n = \{i \in \mathbb{N} : 1 \leq i \leq n\}$. For each $i \in I_n$, let E_i be the set

$$E_i = \{y \in \mathbb{R} | \exists x \in E : x_i = y\},$$

i.e., E_i is the collection of i^{th} coordinate of all element $x \in E$. We claim that for every $i \in I_n$, E_i is a subset of some closed interval and thus E_i is bounded in $(\mathbb{R}, d_{l^1}|_{\mathbb{R} \times \mathbb{R}})$. This is true since

$$\begin{aligned}
& E \text{ is bounded in } (\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n}) \\
\implies & \forall y \in \mathbb{R}^n, \exists r \in \mathbb{R}^+ : E \subseteq B_{(\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})}(y, r) && \text{(by Def. II.1.5.3)} \\
\implies & \forall y \in \mathbb{R}^n, \exists r \in \mathbb{R}^+ : \forall x \in E, x \in B_{(\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})}(y, r) \\
\implies & \forall y \in \mathbb{R}^n, \exists r \in \mathbb{R}^+ : \forall x \in E, \\
& d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n}(x, y) = \sum_{i=1}^n |x_i - y_i| < r && \text{(by Def. II.1.2.1)} \\
\implies & \forall y \in \mathbb{R}^n, \exists r \in \mathbb{R}^+ : \forall x \in E, \forall i \in I_n, \\
& d_{l^1}|_{\mathbb{R} \times \mathbb{R}}(x_i, y_i) = |x_i - y_i| < r \\
\implies & \forall y \in \mathbb{R}^n, \exists r \in \mathbb{R}^+ : \forall x \in E, \forall i \in I_n, \\
& x_i \in (y_i - r, y_i + r) \\
\implies & \forall y \in \mathbb{R}^n, \exists r \in \mathbb{R}^+ : \forall i \in I_n, \\
& E_i \subseteq (y_i - r, y_i + r) \subseteq [y_i - r, y_i + r] \\
\implies & \forall y \in \mathbb{R}^n, \exists r \in \mathbb{R}^+ : \forall i \in I_n, \\
& E_i \subseteq B_{(\mathbb{R}, d_{l^1}|_{\mathbb{R} \times \mathbb{R}})}(y_i, r) && \text{(by Def. II.1.2.1)} \\
\implies & \forall i \in I_n, E_i \text{ is bounded in } (\mathbb{R}, d_{l^1}|_{\mathbb{R} \times \mathbb{R}}). && \text{(by Def. II.1.5.3)}
\end{aligned}$$

Let $P(n)$ be the statement "If $F \subseteq \mathbb{R}^n$ such that for every $i \in I_n$, F_i is bounded in $(\mathbb{R}, d_{l^1}|_{\mathbb{R} \times \mathbb{R}})$ and $F_i \subseteq C_i$ for some closed interval $C_i \subseteq \mathbb{R}$, then for any sequence in F there exists a subsequence which converges in $\mathbb{R}^n$ with respect to $d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n}$." We induct on n to show that $P(n)$ is true for all $n \in \mathbb{Z}^+$.

For $n = 1$, by hypothesis we have $F = F_1$ is bounded in $(\mathbb{R}, d_{l^1}|_{\mathbb{R} \times \mathbb{R}})$ and $F_1 \subseteq C_1$ for some closed interval $C_1 \subseteq \mathbb{R}$. By Heine-Borel theorem on real line (Theorem 9.1.24 in Analysis I) we know that for every sequence $(a^{(k)})_{k=1}^\infty$ in F, there exists a subsequence $(a^{(k_j)})_{j=1}^\infty$ which converges in $C_1 \subseteq \mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R} \times \mathbb{R}}$. Thus, the base case holds.

Suppose inductively that $P(n)$ is true for some $n \geq 1$. Then we need to show that $P(n+1)$ is true. Let $F \subseteq \mathbb{R}^{n+1}$ such that for every $i \in I_{n+1}$, F_i is bounded in $(\mathbb{R}, d_{l^1}|_{\mathbb{R} \times \mathbb{R}})$

and $F_i \subseteq C_i$ for some closed interval $C_i \in \mathbb{R}$. Let $(a^{(k)})_{k=1}^{\infty}$ be arbitrary sequence in F. We define $(b^{(k)})_{k=1}^{\infty}$ by setting $b^{(k)} = (a_1^{(k)}, \ldots, a_n^{(k)})$ for each $k \geq 1$, i.e., $b^{(k)}$ is the first n coordinates of $a^{(k)}$. Since for all $k \geq 1$, $b^{(k)} \in \mathbb{R}^n$ and $b_i^{(k)} \in F_i$ for all $i \in I_n$, by the induction hypothesis there exists a subsequence $(b^{(k_j)})_{j=1}^{\infty}$ which converges in $\mathbb{R}^n$ with respect to $d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n}$. Since $(a_{n+1}^{(k_j)})_{j=1}^{\infty}$ is in F_{n+1} and $F_{n+1} \subseteq C_{n+1}$ for some closed interval $C_{n+1} \subseteq \mathbb{R}$, by Heine-Borel theorem on real line (Theorem 9.1.24 in Analysis I) we know that there exists a subsequence $(a^{(k_{j_p})})_{p=1}^{\infty}$ which converges in C_{n+1} with respect to $d_{l^1}|_{\mathbb{R} \times \mathbb{R}}$. But by Lem. II.1.4.3 we know that every subsequence of $(b^{(k_j)})_{j=1}^{\infty}$ also converges in $\mathbb{R}^n$ with respect to $d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n}$. In particular, $(b^{k_{j_p}})_{p=1}^{\infty}$ converges in $\mathbb{R}^n$ with respect to $d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n}$. Thus, by Prop. II.1.1.18(b)(d) we know that $(a^{(k_{j_p})})_{p=1}^{\infty}$ converges in $\mathbb{R}^{n+1}$ with respect to $d_{l^1}|_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}}$, and this closes the induction.

From the proof above we know that if $E \subseteq \mathbb{R}^n$ such that E is closed and bounded in $(\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})$, then for every $i \in I_n$, E_i is bounded in $(\mathbb{R}, d_{l^1}|_{\mathbb{R} \times \mathbb{R}})$ and $E_i \subseteq C_i$ for some closed interval $C_i \subseteq \mathbb{R}$. Thus, we know that for every sequence in E there exists a subsequence which converges in $\mathbb{R}^n$ with respect to $d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n}$. Since E is closed in $(\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})$, by Prop. II.1.2.15(b) we know that such subsequence must converges in E with respect to $d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n}$. Thus, by Def. II.1.5.1 $(E, d_{l^1}|_{E \times E})$ is compact.

Since every sequence in E has a subsequence which converges in E with respect to $d_{l^1}|_{E \times E}$, by Prop. II.1.1.18 we know that such subsequence also converges with respect to $d_{l^2}|_{E \times E}$ and $d_{l^\infty}|_{E \times E}$. Thus, $(E, d_{l^2}|_{E \times E})$ and $(E, d_{l^\infty}|_{E \times E})$ are compact iff $(E, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})$ is compact. □

Note. The Heine-Borel theorem is not true for more general metric spaces. However, a version of the Heine-Borel theorem is available if one is willing to replace closedness with the stronger notion of completeness, and boundedness with the stronger notion of *total boundedness*.

Note. One can characterize compactness topologically via Thm. II.1.5.8: every open cover of a compact set has a finite subcover.

Thm. II.1.5.8. Let (X, d) be a metric space, and let Y be a compact subset of X. Let $(V_\alpha)_{\alpha \in A}$ be a collection of open sets in X, and suppose that

$$Y \subseteq \bigcup_{\alpha \in A} V_\alpha.$$

(i.e., the collection $(V_\alpha)_{\alpha \in A}$ *covers* Y). Then there exists a *finite* subset F of A such that

$$Y \subseteq \bigcup_{\alpha \in F} V_\alpha.$$

Proof. We assume for the sake of contradiction that there does not exist any finite subset F of A for which $Y \subseteq \bigcup_{\alpha \in F} V_\alpha$.

Let y be any element of Y. Then y must lie in at least one of the sets V_α. Since each V_α is open in (X, d), by Prop. II.1.2.15(a) there must therefore be an $r > 0$ such that $B_{(X,d)}(y, r) \subseteq V_\alpha$. Now let $r(y)$ denote the quantity

$$r(y) := \sup\{r \in (0, \infty) : B_{(X,d)}(y, r) \subseteq V_\alpha \text{ for some } \alpha \in A\}.$$

By the above discussion, we know that $r(y) > 0$ for all $y \in Y$. Now, let r_0 denote the quantity

$$r_0 := \inf\{r(y) : y \in Y\}.$$

Since $r(y) > 0$ for all $y \in Y$, we have $r_0 \geq 0$. There are three cases: $r_0 = 0$, $0 < r_0 < \infty$ and $r_0 = \infty$.

- Case 1: $r_0 = 0$. Then for every integer $n \geq 1$, there is at least one point y in Y such that $r(y) < 1/n$ (otherwise the infimum cannot be 0). We thus choose, for each $n \geq 1$, a point $y^{(n)}$ in Y such that $r(y^{(n)}) < 1/n$ (we can do this because of the axiom of choice, see Proposition 8.4.7 in Analysis I). In particular, we have $\lim_{n\to\infty} r(y^{(n)}) = 0$, by the squeeze test. The sequence $(y^{(n)})_{n=1}^\infty$ is a sequence in Y; since $(Y, d|_{Y\times Y})$ is compact, we can thus find a subsequence $(y^{(n_j)})_{j=1}^\infty$ which converges to a point $y_0 \in Y$ with respect to $d|_{Y\times Y}$.

  As before, we know that there exists some $\alpha \in A$ such that $y_0 \in V_\alpha$, and hence (since V_α is open in (X, d)) there exists some $\varepsilon > 0$ such that $B_{(X,d)}(y_0, \varepsilon) \subseteq V_\alpha$. Since $(y^{(n_j)})_{j=1}^\infty$ converges to y_0, there must exist an $N \geq 1$ such that $y^{(n_j)} \in B_{(X,d)}(y_0, \varepsilon/2)$ for all $n_j \geq N$. In particular, by the triangle inequality we have $B_{(X,d)}(y^{(n_j)}, \varepsilon/2) \subseteq B_{(X,d)}(y_0, \varepsilon)$, and thus $B_{(X,d)}(y^{(n_j)}, \varepsilon/2) \subseteq V_\alpha$. By definition of $r(y^{(n_j)})$, this implies that $r(y^{(n_j)}) \geq \varepsilon/2$ for all $n_j \geq N$. But this contradicts the fact that $\lim_{n\to\infty} r(y^{(n)}) = 0$.

- Case 2: $0 < r_0 < \infty$. In this case we now have $r(y) > r_0/2$ for all $y \in Y$. This implies that for every $y \in Y$ there exists an $\alpha \in A$ such that $B_{(X,d)}(y, r_0/2) \subseteq V_\alpha$ (by the definition of $r(y)$).

  We now construct a sequence $y^{(1)}, y^{(2)}, \ldots$ by the following recursive procedure. We let $y^{(1)}$ be any point in Y. The ball $B_{(X,d)}(y^{(1)}, r_0/2)$ is contained in one of the V_α and thus cannot cover all of Y, since we would then obtain a finite cover, a contradiction. Thus, there exists a point $y^{(2)}$ which does not lie in $B_{(X,d)}(y^{(1)}, r_0/2)$, so, in particular, $d(y^{(2)}, y^{(1)}) \geq r_0/2$. Choose such a point $y^{(2)}$. The set $B_{(X,d)}(y^{(1)}, r_0/2) \cup B_{(X,d)}(y^{(2)}, r_0/2)$ cannot cover all of Y, since we would then obtain two sets V_{α_1} and V_{α_2} which covered Y, a contradiction again. So we can choose a point $y^{(3)}$ which does not lie in $B_{(X,d)}(y^{(1)}, r_0/2) \cup B_{(X,d)}(y^{(2)}, r_0/2)$, so, in particular, $d(y^{(3)}, y^{(1)}) \geq r_0/2$ and $d(y^{(3)}, y^{(2)}) \geq r_0/2$. Continuing in this fashion we obtain a sequence $(y^{(n)})_{n=1}^\infty$ in Y with the property that $d(y^{(k)}, y^{(j)}) \geq r_0/2$ for all $k > j$. In particular, the sequence $(y^{(n)})_{n=1}^\infty$ is not a Cauchy sequence, and in fact no subsequence of $(y^{(n)})_{n=1}^\infty$ can be a Cauchy sequence either. But this contradicts the assumption that $(Y, d|_{Y\times Y})$ is compact (by Lem. II.1.4.7).

- Case 3: $r_0 = \infty$. For this case we argue as in Case 2, but replacing the role of $r_0/2$ by (say) 1.

$\square$

Note. It turns out that Thm. II.1.5.8 has a converse: if Y has the property that every open cover has a finite sub-cover, then it is compact. In fact, this property is often considered the more fundamental notion of compactness than the sequence-based one. (For metric spaces, the two notions, that of compactness and sequential compactness, are equivalent, but for more general *topological spaces*, the two notions are slightly different.)

Cor. II.1.5.9. Let (X, d) be a metric space, and let $K_1, K_2, K_3, \dots$ be a sequence of non-empty compact subsets of X such that

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots.$$

Then the intersection $\bigcap_{n=1}^{\infty} K_n$ is non-empty.

Proof. Since $(K_1, d|_{K_1\times K_1})$ is compact and $K_1 \subseteq X$, by Cor. II.1.5.6 we know that K_1 is closed in (X, d). Since K_1 is closed in (X, d) and $K_1 \cap K_n = K_n$ for every $n \geq 1$, by Prop. II.1.3.4(b) we know that K_n is relatively closed in $(K_1, d|_{K_1\times K_1})$. Let $V_n = K_1 \setminus K_n$ for every $n \geq 1$. Then for every $n \geq 1$, we have $V_n \subseteq K_1$ and by Prop. II.1.2.15(e) V_n is open in $(K_1, d|_{K_1\times K_1})$.

Suppose for the sake of contradiction that $\bigcap_{n=1}^{\infty} K_n = \emptyset$. Since

$$\bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} (K_1 \setminus K_n) = K_1 \setminus \left(\bigcap_{n=1}^{\infty} K_n\right) = K_1$$

and (K_1, d) is compact, by Thm. II.1.5.8 we know that there exists a finite set $F \subseteq \mathbb{Z}^+$ such that

$$K_1 \subseteq \bigcup_{i\in F} V_i.$$

Since F is finite subset of $\mathbb{Z}^+$, we know that $\min(F)$ is well-defined. Then we have

$$\begin{aligned} & K_1 \subseteq \bigcup_{i\in F} V_i \subseteq \bigcup_{n=1}^{\infty} V_i = K_1 \\ \implies & K_1 = \bigcup_{i\in F} V_i \end{aligned}$$

$$\begin{aligned}
&\implies K_1 = \bigcup_{i\in F}(K_1 \setminus K_i) \\
&\implies K_1 = K_1 \setminus \left(\bigcap_{i\in F} K_i\right) \\
&\implies \bigcap_{i\in F} K_i = \emptyset && \text{(since } \bigcap_{i\in F} K_i \subseteq K_1\text{)} \\
&\implies K_{\min(F)} = \emptyset. && \text{(since } K_{\min(F)} = \bigcap_{i\in F} K_i\text{)}
\end{aligned}$$

But by hypothesis we know that $K_{\min(F)} \neq \emptyset$, a contradiction. Thus, $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. □

Thm. II.1.5.10. Let (X, d) be a metric space.

(a) If Y is a compact subset of X, and $Z \subseteq Y$, then Z is compact iff Z is closed.

(b) If $Y_1, \ldots, Y_n$ are a finite collection of compact subsets of X, then their union $Y_1 \cup \cdots \cup Y_n$ is also compact.

(c) Every finite subset of X (including the empty set) is compact.

Proof. (a) By Cor. II.1.5.6 we know that if $(Z, d|_{Z\times Z})$ is compact then Z is closed in $(Y, d|_{Y\times Y})$. Now we show that if Z is closed in $(Y, d|_{Y\times Y})$ then $(Z, d|_{Z\times Z})$ is compact. Since $(Y, d|_{Y\times Y})$ is compact and $Z \subseteq Y$, by Def. II.1.5.1 we know that every sequence $(z^{(n)})_{n=1}^{\infty}$ in Z has a convergent subsequence $(z^{(n_j)})_{j=1}^{\infty}$ which converges in Y with respect to $d|_{Y\times Y}$. Since Z is closed in $(Y, d|_{Y\times Y})$, by Prop. II.1.2.15(b) we know that $(z^{(n_j)})_{j=1}^{\infty}$ converges in Z with respect to $d|_{Y\times Y}$. Since $(z^{(n)})_{n=1}^{\infty}$ was arbitrary, by Def. II.1.5.1 we know that $(Z, d|_{Z\times Z})$ is compact. □

Proof. (b) We induct on n to show that $(\bigcup_{i=1}^{n} Y_i, d)$ is compact for every $n \in \mathbb{Z}^+$. For $n = 1$, we know that $\bigcup_{i=1}^{1} Y_i = Y_1$ and by hypothesis (Y_1, d) is compact, Thus, the base case holds. Suppose inductively that $(\bigcup_{i=1}^{n} Y_i, d)$ is compact for some $n \geq 1$. Then for $n+1$, we need to show that $(\bigcup_{i=1}^{n+1} Y_i, d)$ is compact. Let $(y^{(k)})_{k=1}^{\infty}$ be a sequence in $\bigcup_{i=1}^{n+1} Y_i$. We now split into two cases:

- If there exists a subsequence $(y^{(k_j)})_{j=1}^{\infty}$ whose elements are in $\bigcup_{i=1}^{n} Y_i$, then by the induction hypothesis we know that $(y^{(k_j)})_{j=1}^{\infty}$ converges in $\bigcup_{i=1}^{n} Y_i$ with respect to d. This means $(y^{(k_j)})_{j=1}^{\infty}$ also converges in $\bigcup_{i=1}^{n+1} Y_i$ with respect to d since $\bigcup_{i=1}^{n} Y_i \subseteq \bigcup_{i=1}^{n+1} Y_i$.

- If there does not exist a subsequence $(y^{(k_j)})_{j=1}^{\infty}$ whose elements are in $\bigcup_{i=1}^{n} Y_i$, then there is only finitely many elements in $(y^{(k)})_{k=1}^{\infty}$ which are in $\bigcup_{i=1}^{n} Y_i$. This means there exists a subsequence $(y^{(k_j)})_{j=1}^{\infty}$ whose elements are in Y_{n+1}. By hypothesis we know that (Y_{n+1}, d) is compact, thus by Def. II.1.5.1 there exists a subsequence $(y^{(k_{j_p})})_{p=1}^{\infty}$ of $(y^{(k_j)})_{j=1}^{\infty}$ converges in Y_{n+1} with respect to d. Since $Y_{n+1} \subseteq \bigcup_{i=1}^{n+1} Y_i$, we know that $(y^{(k_{j_p})})_{p=1}^{\infty}$ also converges in $\bigcup_{i=1}^{n+1} Y_i$ with respect to d.

From all cases above, we conclude that there exists a subsequence of $(y^{(k)})_{k=1}^{\infty}$ which converges in $\bigcup_{i=1}^{n+1} Y_i$ with respect to d. Since $(y^{(k)})_{k=1}^{\infty}$ was arbitrary, by Def. II.1.5.1 $(\bigcup_{i=1}^{n+1} Y_i, d)$ is compact. This closes the induction. □

Proof. (c) Let $Y \subseteq X$ and $\#(Y) = n$. Let $P(n)$ be the statement "$\#(Y) = n$ and for every sequence $(y^{(k)})_{k=1}^{\infty}$ in Y, there exists a subsequence of $(y^{(k)})_{k=1}^{\infty}$ which converges in Y with respect to d." We induct on n to show that $P(n)$ is true for all $n \in \mathbb{N}$. For $n = 0$, we have $Y = \emptyset$ and the statement $P(0)$ is trivially true. Thus, by Def. II.1.5.1 $(\emptyset, d)$ is compact and the base case holds. Suppose inductively that $P(n)$ is true for some $n \geq 0$. Then we need to show that $P(n+1)$ is true. Let $Y \subseteq X$ such that $\#(Y) = n+1$ and let $x_0 \in Y$. Let $(y^{(k)})_{k=1}^{\infty}$ be arbitrary sequence in Y. Now we split into two cases:

- If the set $\left\{k \in \mathbb{N} : y^{(k)} = x_0\right\}$ is finite, then we can have a subsequence $(y^{(k_j)})_{j=1}^{\infty}$ whose elements are in $Y \setminus \{x_0\}$. Since $(y^{(k_j)})_{j=1}^{\infty}$ is in $Y \setminus \{x_0\}$ and $\#(Y \setminus \{x_0\}) = n$, by the induction hypothesis we know that there exists a subsequence of $(y^{(k_j)})_{j=1}^{\infty}$ which converges in $Y \setminus \{x_0\}$ with respect to d. But $(y^{(k_j)})_{j=1}^{\infty}$ is also in Y, thus we know that there exists a subsequence of $(y^{(k_j)})_{j=1}^{\infty}$ which converges in Y with respect to d.

- If the set $\left\{k \in \mathbb{N} : y^{(k)} = x_0\right\}$ is infinite, then we can have a subsequence $(y^{(k_j)})_{j=1}^{\infty}$

whose elements are all x_0 and obviously $(y^{(k_j)})_{j=1}^{\infty}$ converges to x_0 with respect to d. Since $x_0 \in Y$, we know that $(y^{(k_j)})_{j=1}^{\infty}$ converges in Y with respect to d.

From all cases above, we conclude that there exists a subsequence of $(y^{(k)})_{k=1}^{\infty}$ which converges in Y with respect to d. Since $(y^{(k)})_{k=1}^{\infty}$ was arbitrary, we conclude that $P(n+1)$ is true and this closes the induction. Since $P(n)$ is true for every $n \in \mathbb{N}$, by Def. II.1.5.1 we know that if Y is a finite subset of X, then $(Y, d|_{Y\times Y})$ is compact. $\square$

— Exercises —

Ex. II.1.5.1. Show that Definitions 9.1.22 in Analysis I and Def. II.1.5.3 match when talking about subsets of the real line with the standard metric.

Proof. Let $X \subseteq \mathbb{R}$ and let $d = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. By Ex. II.1.1.2 we know that $(\mathbb{R}, d)$ is a metric space. Then we have

$$\begin{aligned}
&X \text{ is bounded in the sense of Definition 9.1.22 in Analysis I}\\
\iff &\exists M \in \mathbb{R}^+ : X \subseteq [-M, M] \subseteq (-M-1, M+1)\\
\iff &\exists M \in \mathbb{R}^+ : \forall x \in X, |x-0| < M+1\\
\iff &\forall y \in \mathbb{R}, \exists M \in \mathbb{R}^+ : \forall x \in X,\\
&|x-y| \le |x-0| + |y-0| < M+1+|y|\\
\iff &\forall y \in \mathbb{R}, \exists M \in \mathbb{R}^+ : X \subseteq B_{(\mathbb{R},d)}(y, M+1+|y|)\\
\iff &X \text{ is bounded in the sense of Def. II.1.5.3.}
\end{aligned}$$

$\square$

Ex. II.1.5.2. Prove Prop. II.1.5.5.

Proof. See Prop. II.1.5.5. $\square$

Ex. II.1.5.3. Prove Thm. II.1.5.7.

Proof. See Thm. II.1.5.7. $\square$

Ex. II.1.5.4. Let $(\mathbb{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$, and an open set $V \subseteq \mathbb{R}$, such that the image $f(V) \coloneqq \{f(x) : x \in V\}$ of V is *not* open.

Proof. Let $f(x) = 1$ for all $x \in \mathbb{R}$. Since f is a constant function, we know that f is continuous. By Def. II.1.2.1 and Prop. II.1.2.15(c) we know that

$$B_{(\mathbb{R},d_{l^1}|_{\mathbb{R}\times\mathbb{R}})}(1,1) = (0,2)$$

is open in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ and by Prop. II.1.2.15(d)

$$f\big((0,2)\big) = \{1\}$$

is closed in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. By Prop. II.1.2.15(a) we know that $\{1\}$ is not open in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ since we cannot find an $r \in \mathbb{R}^+$ such that $B_{(\mathbb{R},d_{l^1})}(1, r) \subseteq \{1\}$. Thus, f satisfys the requirements. □

Ex. II.1.5.5. Let $(\mathbb{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$, and a closed set $F \subseteq \mathbb{R}$, such that $f(F)$ is *not* closed.

Proof. Let $f(x) = 2^x$ for all $x \in \mathbb{R}$. We know that 2^x is continuous on $(-\infty, \infty)$. Let $F = (-\infty, 0]$. If $x_0 \in \partial_{(\mathbb{R},d)}(F)$, then by Def. II.1.2.5 we must have $B(x_0, r) \nsubseteq F$ and $B(x_0, r) \cap F \neq \emptyset$ for every $r \in \mathbb{R}^+$. Thus, we must have $\partial_{(\mathbb{R},d)}(F) = \{0\}$. Since $0 \in F$, by Def. II.1.2.12 we know that F is closed in $(\mathbb{R}, d)$. Since

$$f(F) = f\big((-\infty, 0]\big) = (0, 1]$$

and $0 \notin (0, 1]$, by Def. II.1.2.12 we know that $f(F)$ is not closed in $(\mathbb{R}, d)$. Thus, the function f and the closed set F satisfy the requirements. □

Ex. II.1.5.6. Prove Cor. II.1.5.9.

Proof. See Cor. II.1.5.9. □

Ex. II.1.5.7. Prove Thm. II.1.5.10.

Proof. See Thm. II.1.5.10. □

Ex. II.1.5.8. Let (X, d_{l^1}) be the metric space from Ex. II.1.1.15. For each natural number n, let $e^{(n)} = (e_j^{(n)})_{j=0}^{\infty}$ be the sequence in X such that $e_j^{(n)} := 1$ when $n = j$ and $e_j^{(n)} := 0$ when $n \neq j$. Show that the set $\left\{e^{(n)} : n \in \mathbb{N}\right\}$ is a closed and bounded subset of X, but is not compact. (This is despite the fact that (X, d_{l^1}) is even a complete metric space - a fact which we will not prove here. The problem is that not that X is incomplete, but rather that it is "infinite-dimensional," in a sense that we will not discuss here.)

Proof. Let $E = \left\{e^{(n)} : n \in \mathbb{N}\right\}$. We first show that E is bounded in (X, d_{l^1}). Since

$$\sum_{j=0}^{\infty}\left|e_j^{(n)}\right| = 1$$

for every $n \in \mathbb{N}$, we know that $e^{(n)}$ is absolutely convergent and by Ex. II.1.1.15 $e^{(n)} \in X$. By Ex. II.1.1.15 we know that

$$\forall (a_j)_{j=0}^{\infty}, (b_j)_{j=0}^{\infty} \in X, d_{l^1}\big((a_j)_{j=0}^{\infty}, (b_j)_{j=0}^{\infty}\big)$$

is well-defined, thus $d_{l^1}\big((a_j)_{j=0}^{\infty}, (e_j^{(n)})_{j=0}^{\infty}\big)$ is well-defined for every $n \in \mathbb{N}$. Since

$$\forall (a_j)_{j=0}^{\infty} \in X, d_{l^1}\big((a_j)_{j=0}^{\infty}, (e_j^{(n)})_{j=0}^{\infty}\big)$$

$$
\begin{aligned}
&= \sum_{j=0}^{\infty}\left|a_j - e_j^{(n)}\right| \\
&= \sum_{j=0:j\neq n}^{\infty} |a_j| + |a_n - 1| \\
&\leq \sum_{j=0}^{\infty}|a_j| + |a_n - 1| && \text{(well-defined since } (a_j)_{j=0}^{\infty} \in X\text{)} \\
&\leq \sum_{j=0}^{\infty}|a_j| + \sup_{n\in\mathbb{N}}|a_n - 1| && \text{(well-defined since } (a_j)_{j=0}^{\infty} \in X\text{)} \\
&< \sum_{j=0}^{\infty}|a_j| + \sup_{n\in\mathbb{N}}|a_n - 1| + 1,
\end{aligned}
$$

by Def. II.1.2.1 we know that the ball

$$B_{(X,d_{l^1})}\left((a_j)_{j=0}^{\infty}, \sum_{j=0}^{\infty}|a_j| + \sup_{n\in\mathbb{N}}|a_n - 1| + 1\right)$$

contains the set E for every $(a_j)_{j=0}^{\infty} \in X$. Thus, by Def. II.1.5.3 we know that E is bounded in (X, d_{l^1}).

Next we show that E is closed in (X, d_{l^1}). Let $\overline{E}_{(X,d_{l^1})}$ be the closure of E and let $x \in \overline{E}_{(X,d_{l^1})}$. By Prop. II.1.2.10(c) we know that there exists a sequence $(a^{(k)})_{k=0}^{\infty}$ in E such that $\lim_{k\to\infty} d_{l^1}\big((a_j^{(k)})_{j=0}^{\infty}, x\big) = 0$. By Lem. II.1.4.7 we know that $(a^{(k)})_{k=0}^{\infty}$ is a Cauchy sequence in (X, d_{l^1}). Let $k, k' \in \mathbb{N}$. By Def. II.1.4.6 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall k, k' \geq N, d_{l^1}(a^{(k)}, a^{(k')}) \leq \varepsilon.$$

In particular,

$$\exists N \in \mathbb{N} : \forall k, k' \geq N, d_{l^1}(a^{(k)}, a^{(k')}) \leq \frac{1}{2} < 1.$$

Since $a^{(k)}, a^{(k')} \in E$, we know that $a^{(k)} = e^{(n)}$ and $a^{(k')} = e^{(n')}$ for some $n, n' \in \mathbb{N}$ and

$$d_{l^1}(a^{(k)}, a^{(k')}) = d_{l^1}(e^{(n)}, e^{(n')}) = \begin{cases} 0 & \text{if } n = n'; \\ 2 & \text{if } n \neq n'. \end{cases}$$

This means

$$
\begin{aligned}
&\exists N \in \mathbb{N} : \forall k, k' \geq N, d_{l^1}(a^{(k)}, a^{(k')}) \leq \frac{1}{2} < 1 \\
\Longrightarrow\ &\exists N \in \mathbb{N} : \forall k, k' \geq N, d_{l^1}(a^{(k)}, a^{(k')}) = 0 \\
\Longrightarrow\ &\exists N \in \mathbb{N} : \forall k \geq N, d_{l^1}(a^{(k)}, a^{(N)}) = 0
\end{aligned}
$$

$$\begin{aligned}
&\implies \exists N \in \mathbb{N} : \lim_{k\to\infty} d_{l^1}(a^{(k)}, a^{(N)}) = 0 && \text{(by Def. II.1.1.14)}\\
&\implies \exists N \in \mathbb{N} : a^{(N)} = x && \text{(by Prop. II.1.1.20)}\\
&\implies x \in E.
\end{aligned}$$

Since x was arbitrary adherent point of E in (X, d_{l^1}), we have

$$\begin{aligned}
&\overline{E}_{(X,d_{l^1})} \subseteq E\\
\implies &\overline{E}_{(X,d_{l^1})} = E && \text{(by Prop. II.1.2.10(c))}\\
\implies &E \text{ is closed in } (X, d_{l^1}). && \text{(by Prop. II.1.2.15(b))}
\end{aligned}$$

Finally we show that $(E, d_{l^1}|_{E\times E})$ is not compact. Let $(e^{(n)})_{n=0}^{\infty}$ be a sequence and let $n, n' \in \mathbb{N}$. Since $(e^{(n)})_{n=0}^{\infty}$ is a sequence in E, we know that

$$\forall N \in \mathbb{N}, \forall n, n' \geq N, d_{l^1}(e^{(n)}, e^{(n')}) = 2.$$

Thus, by Def. II.1.4.6 $(e^{(n)})_{n=0}^{\infty}$ is not a Cauchy sequence in $(E, d_{l^1}|_{E\times E})$. Similarly, any subsequence of $(e^{(n)})_{n=0}^{\infty}$ is not a Cauchy sequence in $(E, d_{l^1}|_{E\times E})$. By Lem. II.1.4.7 this means no subsequence is convergent in $(E, d_{l^1}|_{E\times E})$, and by Def. II.1.5.1 $(E, d_{l^1}|_{E\times E})$ is not compact. □

Ex. II.1.5.9. Show that a metric space (X, d) is compact iff every sequence in X has at least one limit point.

Proof.

$$\begin{aligned}
&(X, d) \text{ is compact}\\
\iff &\text{every sequence in } X \text{ has a convergent subsequence}\\
&\text{which converges in } X && \text{(by Def. II.1.5.1)}\\
\iff &\text{every sequence in } X \text{ has at least one limit point.} && \text{(by Prop. II.1.4.5)}
\end{aligned}$$

□

Ex. II.1.5.10. A metric space (X, d) is called *totally bounded* if for every $\varepsilon > 0$, there exists a natural number n and a finite number of balls $B(x^{(1)}, \varepsilon), \ldots, B(x^{(n)}, \varepsilon)$ which cover X (i.e., $X = \bigcup_{i=1}^{n} B(x^{(i)}, \varepsilon)$). (Note that $x^{(1)}, \ldots, x^{(n)} \in X$)

(a) Show that every totally bounded space is bounded.

(b) Show the following stronger version of Prop. II.1.5.5: if (X, d) is compact, then complete and totally bounded.

(c) Conversely, show that if X is complete and totally bounded, then X is compact.

Proof. (a) Suppose that X is totally bounded in (X,d). Let $i,j,n \in \mathbb{N}$. Then by definition we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists n \in \mathbb{N} : X = \bigcup_{i=1}^{n} B_{(X,d)}(x^{(i)}, \varepsilon).$$

In particular, we have

$$\exists n \in \mathbb{N} : X = \bigcup_{i=1}^{n} B_{(X,d)}(x^{(i)}, 1).$$

If $n = 0$, then we have $X = \emptyset$ and by Def. II.1.5.3 $\emptyset$ is bounded in $(\emptyset, d)$. So suppose that $n > 0$. Now we use axiom of choice to choose one finite collection of $x^{(1)}, \ldots, x^{(n)} \in X$. Let $I_n = \{i \in \mathbb{N} : 1 \le i \le n\}$ and let $r = \max\Big\{d(x^{(i)}, x^{(1)}) : i \in I_n\Big\}$. We know that r is well-defined since I_n is finite. Then have

$$\begin{aligned}
& \forall i \in I_n, \forall y \in B_{(X,d)}(x^{(i)}, 1), d(y, x^{(i)}) < 1 && \text{(by Def. II.1.2.1)}\\
\implies & \forall i \in I_n, \forall y \in B_{(X,d)}(x^{(i)}, 1),\\
& d(y, x^{(1)}) \le d(y, x^{(i)}) + d(x^{(i)}, x^{(1)}) < 1 + r && \text{(by Def. II.1.1.2(d))}\\
\implies & \forall y \in \bigcup_{i=1}^{n} B_{(X,d)}(x^{(i)}, 1), d(y, x^{(1)}) < 1 + r\\
\implies & \forall y \in X, d(y, x^{(1)}) < 1 + r && (X = \bigcup_{i=1}^{n} B_{(X,d)}(x^{(i)}, 1))\\
\implies & \forall y \in X, y \in B_{(X,d)}(x^{(1)}, 1 + r) && \text{(by Def. II.1.2.1)}\\
\implies & X \subseteq B_{(X,d)}(x^{(1)}, 1 + r)\\
\implies & \forall y \in X, X \subseteq B_{(X,d)}\big(y, d(y, x^{(1)}) + 1 + r\big) && \text{(by Def. II.1.1.2(d))}\\
\implies & X \text{ is bounded in } (X,d). && \text{(by Def. II.1.5.3)}
\end{aligned}$$

□

Proof. (b) By Prop. II.1.5.5 we know that if (X,d) is compact then (X,d) is complete. Thus, we only need to show that X is totally bound in (X,d). Let $\varepsilon \in \mathbb{R}^+$. Then we have

$$\begin{aligned}
& \forall y \in X, y \in B_{(X,d)}(y, \varepsilon) && \text{(by Def. II.1.2.1)}\\
\implies & X \subseteq \bigcup_{y \in X} B_{(X,d)}(y, \varepsilon)\\
\implies & \bigcup_{y \in X} B_{(X,d)}(y, \varepsilon) \text{ is an open cover of } X \text{ in } (X,d) && \text{(by Prop. II.1.2.15(c)(g))}\\
\implies & \exists F \subseteq X : (F \text{ is finite}) \wedge \left(X \subseteq \bigcup_{y \in F} B_{(X,d)}(y, \varepsilon)\right). && \text{(by Thm. II.1.5.8)}
\end{aligned}$$

Since ε was arbitrary, by definition we know that X is totally bounded in (X,d). □

Proof. (c) Suppose that (X, d) is complete and X is totally bounded in (X, d). If $X = \emptyset$, then we know that $(\emptyset, d)$ is compact is trivially true. So suppose that $X \neq \emptyset$. To show that (X, d) is compact, by Def. II.1.5.1 we need to show that every sequence in X has a convergent subsequence which converges in X with respect to d. So let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in X. If we have

$$\exists N \in \mathbb{Z}^+ : \left\{n \in \mathbb{Z}^+ : x^{(n)} = x^{(N)}\right\} \text{ is infinite,}$$

then there must exist a subsequence of $(x^{(n)})_{n=1}^{\infty}$ which converges to $x^{(N)}$ with respect to d for some $N \in \mathbb{Z}^+$. So suppose that

$$\forall N \in \mathbb{Z}^+, \left\{n \in \mathbb{Z}^+ : x^{(n)} = x^{(N)}\right\} \text{ is finite.}$$

Let $E = \left\{x^{(n)} : n \in \mathbb{Z}^+\right\}$. For each $\varepsilon \in \mathbb{R}^+$, we define F_ε to be the set

$$F_\varepsilon = \left\{F \subseteq X : (F \text{ is finite}) \wedge \left(X \subseteq \bigcup_{y \in F} B_{(X,d)}(y, \varepsilon)\right)\right\}.$$

Since X is totally bounded in (X, d), by definition we know that $F_\varepsilon \neq \emptyset$ for every $\varepsilon \in \mathbb{R}^+$. For arbitrary $\varepsilon \in \mathbb{R}^+$ and arbitrary $F \in F_\varepsilon$, we claim that

$$\exists y \in F : E \cap B_{(X,d)}(y, \varepsilon) \text{ is infinite.}$$

Suppose for the sake of contradiction that the claim is false. Then we have

$$\begin{aligned}
&\forall y \in F, E \cap B_{(X,d)}(y, \varepsilon) \text{ is finite} \\
\Longrightarrow\ &\bigcup_{y \in F} \left(E \cap B_{(X,d)}(y, \varepsilon)\right) \text{ is finite} && \text{(since } F \text{ is finite)} \\
\Longrightarrow\ &E \cap \left(\bigcup_{y \in F} B_{(X,d)}(y, \varepsilon)\right) \text{ is finite} \\
\Longrightarrow\ &E = E \cap X \subseteq E \cap \left(\bigcup_{y \in F} B_{(X,d)}(y, \varepsilon)\right)
\end{aligned}$$

and thus E is finite. But by the definition of E we know that E is infinite, a contradiction. Thus, the claim is true.

Using the claim above we can now define the set A_ε for each $\varepsilon \in \mathbb{R}^+$.

$$A_\varepsilon = \left\{y \in F : (F \in F_\varepsilon) \wedge (E \cap B_{(X,d)}(y, \varepsilon) \text{ is infinite})\right\}$$

From the claim above we know that $A_\varepsilon \neq \emptyset$ for every $\varepsilon \in \mathbb{R}^+$. Now we claim that

$$\forall \delta, \varepsilon \in \mathbb{R}^+, \delta < \varepsilon \implies A_\delta \subseteq A_\varepsilon.$$

Let $\delta, \varepsilon \in \mathbb{R}^+$ and $\delta < \varepsilon$. Then we have

$$\begin{aligned}
&\forall y \in A_\delta, E \cap B_{(X,d)}(y, \delta) \text{ is infinite} \\
\implies &\forall y \in A_\delta, B_{(X,d)}(y, \delta) \text{ is infinite} && \text{(since } E \text{ is infinite)} \\
\implies &\forall y \in A_\delta, B_{(X,d)}(y, \delta) \subseteq B_{(X,d)}(y, \varepsilon) && \text{(by Def. II.1.2.1)} \\
\implies &\forall y \in A_\delta, E \cap B_{(X,d)}(y, \delta) \subseteq E \cap B_{(X,d)}(y, \varepsilon) \\
\implies &\forall y \in A_\delta, E \cap B_{(X,d)}(y, \varepsilon) \text{ is infinite} \\
\implies &\forall y \in A_\delta, y \in A_\varepsilon \\
\implies &A_\delta \subseteq A_\varepsilon
\end{aligned}$$

and thus the claim is true.

Now we construct a subsequence of $x^{(n)}$. For each $j \in \mathbb{Z}^+$ we define N_j as follow:

$$N_j = \left\{ n \in \mathbb{Z}^+ : x^{(n)} \in E \cap B_{(X,d)}(y, \frac{1}{j}) \text{ for some } y \in A_{\frac{1}{j}} \right\}.$$

From the claim above we know that $A_{\frac{1}{j}}$ is infinite for every $j \in \mathbb{Z}^+$, thus N_j is infinite and we have

$$\forall i, j \in \mathbb{Z}^+, i < j \implies \frac{1}{j} < \frac{1}{i} \implies A_{\frac{1}{j}} \subseteq A_{\frac{1}{i}} \implies N_j \subseteq N_i.$$

Now we recursively define n_j for each $j \in \mathbb{Z}^+$ as follow:

$$n_j = \begin{cases} \min N_1 & \text{if } j = 1 \\ \min\{n \in N_{j-1} : n > n_{j-1}\} & \text{if } j > 1 \end{cases}$$

Since N_j is infinite for each $j \in \mathbb{Z}^+$, we know that the set $\{n \in N_{j-1} : n > n_{j-1}\}$ is also infinite for every $j \geq 2$. If not, then the maximum element of N_{j-1} would be n_{j-1} and N_{j-1} is finite, a contradiction. Since $\{n \in N_j : n > n_{j-1}\} \subseteq \mathbb{Z}^+$ for each $j \in \mathbb{Z}^+$ and $j \geq 2$, by well-ordering principle we know that $\min\{n \in N_j : n > n_{j-1}\}$ is well-defined. Thus, n_j is well-defined for each $j \in \mathbb{Z}^+$ and $(x^{(n_j)})_{j=1}^\infty$ is a subsequence of $(x^{(n)})_{n=1}^\infty$.

Now we claim that the subsequence $(x^{(n_j)})_{j=1}^\infty$ converges in X with respect to d. We have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists J \in \mathbb{Z}^+ : \frac{1}{J} < \varepsilon && \text{(by Archimedean property)} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists J \in \mathbb{Z}^+ : \\
&(\frac{1}{J} < \varepsilon) \wedge (\forall j \geq J, N_j \subseteq N_J) && \text{(from the claim above)} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists J \in \mathbb{Z}^+ :
\end{aligned}$$

$$
\begin{aligned}
&(\frac{1}{J} < \varepsilon) \\
&\wedge \big(\exists y \in A_{\frac{1}{j}} : \forall j \geq J, d(x^{(n_j)}, y) < \frac{1}{j} \leq \frac{1}{J} < \varepsilon\big). \qquad \text{(by the definition of } A_{\frac{1}{j}}\text{)}
\end{aligned}
$$

If we fix J for each ε, then we have

$$
\begin{aligned}
&\forall i, j \geq 2J, \exists y \in A_{\frac{1}{2J}} : \\
&d(x^{(n_i)}, y) + d(x^{(n_j)}, y) < \frac{1}{2J} + \frac{1}{2J} = \frac{1}{J} < \varepsilon \\
\implies &\forall i, j \geq 2J, \exists y \in A_{\frac{1}{2J}} : \\
&d(x^{(n_i)}, x^{(n_j)}) \leq d(x^{(n_i)}, y) + d(x^{(n_j)}, y) < \varepsilon \qquad \text{(by Def. II.1.1.2(c)(d))} \\
\implies &\forall i, j \geq 2J, d(x^{(n_i)}, x^{(n_j)}) < \varepsilon.
\end{aligned}
$$

Thus, we conclude that

$$
\forall \varepsilon \in \mathbb{R}^+, \exists J \in \mathbb{Z}^+ : \forall i, j \geq J, d(x^{(n_i)}, d^{(n_j)}) < \varepsilon
$$

and by Def. II.1.4.6 $(x^{(n_j)})_{j=1}^{\infty}$ is a Cauchy sequence in (X, d). Since (X, d) is compatible, by Def. II.1.4.10 we know that $(x^{(n_j)})_{j=1}^{\infty}$ converges in X with respect to d. Since $(x^{(n)})_{n=1}^{\infty}$ was arbitrary, we know that any sequence in X has a subsequence converges in X with respect to d, and by Def. II.1.5.1 (X, d) is compact. □

Ex. II.1.5.11. Let (X, d) have the property that every open cover of X has a finite subcover. Show that X is compact.

Proof. Suppose that (X, d) is a metric space such that every open cover of X has a finite subcover. We want to show that (X, d) is compact. Suppose for the sake of contradiction that (X, d) is not compact. Then by Def. II.1.5.1 we know that there exists a sequence $(x^{(n)})_{n=1}^{\infty}$ in X which has no convergent subsequence which converges in X with respect to d. We know that

$$
\forall N \in \mathbb{Z}^+, \left\{n \in \mathbb{Z}^+ : x^{(n)} = x^{(N)}\right\} \text{ is finite.}
$$

If not, then we would have a subsequence which converges to $x^{(N)}$ with respect to d for some $N \in \mathbb{Z}^+$, a contradiction. Thus, we know that $\left\{x^{(n)} : n \in \mathbb{Z}^+\right\}$ is infinite. By Ex. II.1.5.9 we know that $(x^{(n)})_{n=1}^{\infty}$ cannot have a limit point in X with respect to d. By Def. II.1.4.4 this means

$$
\begin{aligned}
&\neg\big(\exists y \in X : \forall \varepsilon \in \mathbb{R}^+, \forall N \in \mathbb{Z}^+, \exists n \geq N : d(x^{(n)}, y) \leq \varepsilon\big) \\
\implies &\forall y \in X, \exists \varepsilon \in \mathbb{R}^+ : \exists N \in \mathbb{Z}^+ : \forall n \geq N, d(x^{(n)}, y) > \varepsilon
\end{aligned}
$$

$$\implies \forall y \in X, \exists \varepsilon \in \mathbb{R}^+ : \left\{x^{(n)} : n \in \mathbb{Z}^+\right\} \cap B_{(X,d)}(y, \varepsilon) \text{ is finite.}$$

Now we fix such ε. Since

$$\begin{aligned} &\forall y \in X, B_{(X,d)}(y, \varepsilon) \text{ is open in } (X, d) && \text{(by Prop. II.1.2.15(c))} \\ \implies &X = \bigcup_{y \in X} B_{(X,d)}(y, \varepsilon) && \text{(by Def. II.1.2.1)} \end{aligned}$$

and $\bigcup_{y \in X} B_{(X,d)}(y, \varepsilon)$ is an open cover of X in (X, d), we know that

$$\exists F \subseteq X : (F \text{ is finite}) \wedge \left(X = \bigcup_{y \in F} B_{(X,d)}(y, \varepsilon)\right).$$

Now we fix such F. Then we have

$$\begin{aligned} \left\{x^{(n)} : n \in \mathbb{Z}^+\right\} &= \left\{x^{(n)} : n \in \mathbb{Z}^+\right\} \cap X \\ &= \left\{x^{(n)} : n \in \mathbb{Z}^+\right\} \cap \left(\bigcup_{y \in F} B_{(X,d)}(y, \varepsilon)\right) \\ &= \bigcup_{y \in F} \left(\left\{x^{(n)} : n \in \mathbb{Z}^+\right\} \cap B_{(X,d)}(y, \varepsilon)\right). \end{aligned}$$

But we know that $\bigcup_{y \in F} \left(\left\{x^{(n)} : n \in \mathbb{Z}^+\right\} \cap B_{(X,d)}(y, \varepsilon)\right)$ is finite since F is finite and $\left\{x^{(n)} : n \in \mathbb{Z}^+\right\} \cap B_{(X,d)}(y, \varepsilon)$ is finite for every $y \in F$. This means $\left\{x^{(n)} : n \in \mathbb{Z}^+\right\}$ is finite, a contradiction. Thus, (X, d) is compact. $\square$

Ex. II.1.5.12. Let (X, d_{disc}) be a metric space with the discrete metric d_{disc}.

(a) Show that X is always complete.

(b) When is X compact, and when is X not compact? Prove your claim.

Proof. (a) Let $(x^{(n)})_{n=1}^{\infty}$ be a Cauchy sequence in (X, d_{disc}). Let $i, j \in \mathbb{Z}^+$. By Def. II.1.4.6 we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall i, j \geq N, d_{\text{disc}}(x^{(i)}, x^{(j)}) \leq \varepsilon.$$

In particular, we have

$$\exists N \in \mathbb{Z}^+ : \forall i, j \geq N, d_{\text{disc}}(x^{(i)}, x^{(j)}) \leq \frac{1}{2}.$$

But by E.g. II.1.1.11 we know that

$$d_{\text{disc}}(x^{(i)}, x^{(j)}) \leq \frac{1}{2} \iff x^{(i)} = x^{(j)}.$$

Thus, we have
$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall i \geq N, x^{(i)} = x^{(N)}.$$
and by Def. II.1.1.14 we have $\lim_{n\to\infty} d_{\text{disc}}(x^{(n)}, x^{(N)}) = 0$. This means $(x^{(n)})_{n=1}^{\infty}$ converges to some $x^{(N)} \in X$ with respect to d_{disc}. Since $(x^{(n)})_{n=1}^{\infty}$ was arbitrary, by Def. II.1.4.10 we know that (X, d_{disc}) is complete. □

Proof. (b) We claim that (X, d_{disc}) is compact iff X is finite. By Thm. II.1.5.10(c) we know that if X is finite then (X, d_{disc}) is compact. So we only need to show that if (X, d_{disc}) is compact then X is finite. Suppose that (X, d_{disc}) is compact. By Thm. II.1.5.8 we know that every open cover of X in (X, d_{disc}) has a finite subcover. In particular, we know that
$$\exists F \subseteq X : (F \text{ is finite}) \wedge \left(X = \bigcup_{y\in F} B_{(X,d_{\text{disc}})}(y, \frac{1}{2})\right).$$
Now we fix such F. We know that
$$\begin{aligned}
&\forall y \in F, B_{(X,d)}(y, \frac{1}{2}) = \left\{z \in X : d(z, y) < \frac{1}{2}\right\} && \text{(by Def. II.1.2.1)}\\
\implies &\forall y \in F, B_{(X,d)}(y, \frac{1}{2}) = \{y\} && \text{(by E.g. II.1.1.11)}\\
\implies &X = \bigcup_{y\in F} B_{(X,d)}(y, \frac{1}{2}) = F
\end{aligned}$$
Thus, X is finite. □

Ex. II.1.5.13. Let E and F be two compact subsets of $\mathbb{R}$ (with the standard metric $d(x, y) = |x - y|$). Show that the Cartesian product $E \times F := \{(x, y) : x \in E, y \in F\}$ is a compact subset of $\mathbb{R}^2$ (with the Euclidean metric d_{l^2}).

Proof. Since $E \times F \subseteq \mathbb{R}^2$, we know that every element $x \in E \times F$ is in the form $x = (x_1, x_2)$ where $x_1 \in E$ and $x_2 \in F$. Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in $E \times F$. Then we know that $(x_1^{(n)})_{n=1}^{\infty}$ is a sequence in E and $(x_2^{(n)})_{n=1}^{\infty}$ is a sequence in F. Since $(E, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ is compact, by Def. II.1.5.1 we know that there exists a subsequence $(x_1^{(n_j)})_{j=1}^{\infty}$ which converges to some $L_E \in E$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Since $(x_2^{(n_j)})_{j=1}^{\infty}$ is a subsequence of $(x_2^{(n)})_{n=1}^{\infty}$ and $(F, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ is compact, by Def. II.1.5.1 we know that there exists a subsequence $(x_2^{(n_{j_p})})_{p=1}^{\infty}$ which converges to some $L_F \in F$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Since $\lim_{j\to\infty} d_{l^1}|_{\mathbb{R}\times\mathbb{R}}(x_1^{(n_j)}, L_E) = 0$, by Lem. II.1.4.9 we know that $\lim_{p\to\infty} d_{l^1}|_{\mathbb{R}\times\mathbb{R}}(x_1^{(n_{j_p})}, L_E) = 0$. Thus, by Prop. II.1.1.18(a)(b)(d) we have
$$\lim_{p\to\infty} d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2}\big(x^{(n_{j_p})}, (L_E, L_F)\big) = \lim_{p\to\infty} d_{l^2}|_{\mathbb{R}^2\times\mathbb{R}^2}\big(x^{(n_{j_p})}, (L_E, L_F)\big) = 0.$$
Since $(x^{(n)})_{n=1}^{\infty}$ was arbitrary, by Def. II.1.5.1 we know that $(E\times F, d_{l^2}|_{\mathbb{R}^2\times\mathbb{R}^2})$ is compact. □

Ex. II.1.5.14. Let (X, d) be a metric space, let E be a non-empty compact subset of X, and let x_0 be a point in X. Show that there exists a point $x \in E$ such that

$$d(x_0, x) = \inf\{d(x_0, y) : y \in E\},$$

i.e., x is the closest point in E to x_0.

Proof. Let $R = \inf\{d(x_0, y) : y \in E\}$. Since $d(x_0, y) \geq 0$ for every $y \in E$, we know that the set $\{d(x_0, y) : y \in E\}$ is bounded below and thus R is well-defined. Since R is well-defined, we can construct a sequence $(x^{(n)})_{n=1}^{\infty}$ in E such that $d(x_0, x^{(n)}) \leq R + \frac{1}{n}$ for every $n \in \mathbb{Z}^+$. If not, then we would have

$$\exists n \in \mathbb{Z}^+ : \forall y \in E, d(x_0, y) > R + \frac{1}{n}$$

and thus $\inf\{d(x_0, y) : y \in E\} \geq R + \frac{1}{n} > R$, a contradiction. Since $\big(d(x_0, x^{(n)})\big)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$, we have

$$\begin{aligned}
&\forall n \in \mathbb{Z}^+, R \leq d(x_0, x^{(n)}) \leq R + \frac{1}{n} \\
\Longrightarrow\ &\lim_{n\to\infty} d(x_0, x^{(n)}) = R. && \text{(by squeeze test)}
\end{aligned}$$

Since (E, d) is compact, by Def. II.1.5.1 we know that there exists a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} d(x^{(n_j)}, x) = 0$$

for some $x \in E$. Then we have

$$\begin{aligned}
&\lim_{n\to\infty} d(x_0, x^{(n)}) = R \\
\Longrightarrow\ &\lim_{j\to\infty} d(x_0, x^{(n_j)}) = R && \text{(by Lem. II.1.4.3)} \\
\Longrightarrow\ &\lim_{j\to\infty} d(x_0, x^{(n_j)}) + \lim_{j\to\infty} d(x^{(n_j)}, x) = R + 0 = R \\
\Longrightarrow\ &\lim_{j\to\infty} \big(d(x_0, x^{(n_j)}) + d(x^{(n_j)}, x)\big) = R \\
\Longrightarrow\ &\lim_{j\to\infty} d(x_0, x) \leq \lim_{j\to\infty} \big(d(x_0, x^{(n_j)}) + d(x^{(n_j)}, x)\big) = R && \text{(by Def. II.1.1.2(d))} \\
\Longrightarrow\ &R \leq d(x_0, x) \leq R \\
\Longrightarrow\ &d(x_0, x) = R
\end{aligned}$$

and thus x is the closest point in E to x_0. □

Ex. II.1.5.15. Let (X, d) be a compact metric space. Suppose that $(K_\alpha)_{\alpha \in I}$ is a collection of closed sets in X with the property that any finite subcollection of these sets necessarily has non-empty intersection, thus $\bigcap_{\alpha \in F} K_\alpha \neq \emptyset$ for all finite $F \subseteq I$. (This property is known as the *finite intersection property*.) Show that the *entire* collection has non-empty intersection, thus $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$. Show by counterexample that this statement fails if X is not compact.

Proof. Suppose for the sake of contradiction that $\bigcap_{\alpha \in I} K_\alpha = \emptyset$. Since

$$X = X \setminus \emptyset = X \setminus \bigcap_{\alpha \in I} K_\alpha = \bigcup_{\alpha \in I} (X \setminus K_\alpha),$$

we know that

$$\begin{aligned}
&\forall \alpha \in I, K_\alpha \text{ is closed in } (X,d) \\
\implies &\forall \alpha \in I, X \setminus K_\alpha \text{ is open in } (X,d) && \text{(by Prop. II.1.2.15(e))} \\
\implies &X = \bigcup_{\alpha \in I} (X \setminus K_\alpha) \text{ is an open cover of } X \text{ in } (X,d) \\
\implies &\exists F \subseteq I : (F \text{ is finite}) \wedge \left(X = \bigcup_{\alpha \in F} (X \setminus K_\alpha) \right) && \text{(by Thm. II.1.5.8)} \\
\implies &\exists F \subseteq I : (F \text{ is finite}) \wedge \left(X = X \setminus \bigcap_{\alpha \in F} K_\alpha \right) \\
\implies &\exists F \subseteq I : (F \text{ is finite}) \wedge \left(\bigcap_{\alpha \in F} K_\alpha = \emptyset \right).
\end{aligned}$$

But by hypothesis we know such F does not exist, a contradiction. Thus, we must have $\bigcap_{\alpha \in I} K_\alpha \neq \emptyset$.

Now we show an counterexample when X, d is not compact. Let $X = I = \mathbb{R}^+$ and let $d = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. We know that the sequence $(n)_{n=1}^\infty$ in $\mathbb{R}^+$ has no convergent subsequence with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$, thus by Def. II.1.5.1 $(\mathbb{R}^+, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ is not compact. For each $\varepsilon \in \mathbb{R}^+$, we define $K_\varepsilon = [\varepsilon, \infty)$ (which is an interval in $\mathbb{R}$). Then we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \overline{K_\varepsilon}_{(\mathbb{R}^+, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})} = \{\varepsilon\} && \text{(by Def. II.1.2.5)} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, K_\varepsilon \text{ is closed in } (\mathbb{R}^+, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}) && \text{(by Prop. II.1.2.15(b))}
\end{aligned}$$

Let $F \subseteq \mathbb{R}^+$ such that $F \neq \emptyset$ and F is finite. Since F is finite, we know that $\max(F)$ is well-defined. Then we have

$$\begin{aligned}
&\forall i \in F, i \leq \max(F) \\
\implies &\forall i \in F, [\max(F), \infty) \subseteq [i, \infty) \\
\implies &\forall i \in F, K_i \subseteq K_{\max(F)}
\end{aligned}$$

$$\implies \bigcap_{i \in F} K_i = K_{\max(F)}. \qquad \text{(since } F \text{ is finite)}$$

Since F was arbitrary, we know that every finite subcollection of $(K_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$ has non-empty intersection. Suppose for the sake of contradiction that $\bigcap\limits_{\varepsilon \in \mathbb{R}^+} K_\varepsilon \neq \emptyset$. Let $r \in \bigcap\limits_{\varepsilon \in \mathbb{R}^+} K_\varepsilon$. Then we have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, r \in K_\varepsilon \\ \implies& r \in K_{2r} \\ \implies& r \in [2r, \infty) \\ \implies& r = 0. \end{aligned}$$

But we know that $0 \notin K_\varepsilon$ for every $\varepsilon \in \mathbb{R}^+$. Thus, $0 \notin \bigcap\limits_{\varepsilon \in \mathbb{R}^+} K_\varepsilon$, a contradiction. So we must have $\bigcap\limits_{\varepsilon \in \mathbb{R}^+} K_\varepsilon = \emptyset$. $\square$

Chapter II.2

Continuous functions on metric spaces

II.2.1 Continuous functions

Def. II.2.1.1 (Continuous functions)**.** Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space, and let $f : X \to Y$ be a function. If $x_0 \in X$, we say that f is *continuous at* x_0 iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. We say that f is *continuous* iff it is continuous at every point $x \in X$.

Rmk. II.2.1.2. Continuous functions are also sometimes called *continuous maps*. Mathematically, there is no distinction between the two terminologies.

Rmk. II.2.1.3. If $f : X \to Y$ is continuous, and K is any subset of X, then the restriction $f|_K : K \to Y$ of f to K is also continuous.

Proof. Let $x_0 \in K$. Suppose that $f : X \to Y$ is continuous at x_0 from (X, d_X) to (Y, d_Y). Then we have

$$\begin{aligned}
&f : X \to Y \text{ is continuous at } x_0 \\
&\text{from } (X, d_X) \text{ to } (Y, d_Y) \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0) < \varepsilon\big)\Big) && \text{(by Def. II.2.1.1)} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x \in K, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0) < \varepsilon\big)\Big) \quad (K \subseteq X) \\
\implies &f|_K : K \to Y \text{ is continuous at } x_0 \\
&\text{from } (K, d_X|_{K \times K}) \text{ to } (Y, d_Y). && \text{(by Def. II.2.1.1)}
\end{aligned}$$

Now suppose that $f : X \to Y$ is continuous from (X, d_X) to (Y, d_Y). Then we have

$$f : X \to Y \text{ is continuous from } (X, d_X) \text{ to } (Y, d_Y)$$

$$
\begin{aligned}
&\implies \forall x_0 \in X, f \text{ is continuous at } x_0 \text{ from } (X, d_X) \text{ to } (Y, d_Y) && \text{(by Def. II.2.1.1)}\\
&\implies \forall x_0 \in K, f \text{ is continuous at } x_0\\
&\qquad \text{from } (K, d_X|_{K\times K}) \text{ to } (Y, d_Y) && (K \subseteq X)\\
&\implies f|_K : K \to Y \text{ is continuous from } (K, d_X|_{K\times K}) \text{ to } (Y, d_Y). && \text{(by Def. II.2.1.1)}
\end{aligned}
$$

□

Thm. II.2.1.4 (Continuity preserves convergence)**.** Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Let $f : X \to Y$ be a function, and let $x_0 \in X$ be a point in X. Then the following three statements are logically equivalent:

(a) f is continuous at x_0.

(b) Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X which converges to x_0 with respect to the metric d_X, the sequence $\big(f(x^{(n)})\big)_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.

(c) For every open set $V \subseteq Y$ that contains $f(x_0)$, there exists an open set $U \subseteq X$ containing x_0 such that $f(U) \subseteq V$.

Proof. We first show that statement (a) implies statement (b). Suppose that $f : X \to Y$ is continuous at x_0 from (X, d_X) to (Y, d_Y). Then by Def. II.2.1.1 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists d \in \mathbb{R}^+ : \Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon\Big).$$

Now we choose δ for each $\varepsilon \in \mathbb{R}^+$ and denoted it as δ_ε. Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in X such that $\lim_{n\to\infty} d_X(x^{(n)}, x_0) = 0$. Then we have

$$
\begin{aligned}
&\lim_{n\to\infty} d_X(x^{(n)}, x_0) = 0\\
&\implies \forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, d_X(x^{(n)}, x_0) \leq \delta && \text{(by Def. II.1.1.14)}\\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists \delta_\varepsilon \in \mathbb{R}^+ :\\
&\quad \left(\exists N \in \mathbb{Z}^+ : \forall n \geq N, d_X(x^{(n)}, x_0) \leq \frac{\delta_\varepsilon}{2} < \delta_\varepsilon\right)\\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists \delta_\varepsilon \in \mathbb{R}^+ :\\
&\quad \Big(\exists N \in \mathbb{Z}^+ : \forall n \geq N, d_Y\big(f(x^{(n)}), f(x_0)\big) < \varepsilon\Big) && \text{(by hypothesis)}\\
&\implies \lim_{n\to\infty} d_Y\big(f(x^{(n)}), f(x_0)\big) = 0. && \text{(by Def. II.1.1.14)}
\end{aligned}
$$

Since $(x^{(n)})_{n=1}^{\infty}$ was arbitrary, we know that statement (a) implies statement (b).

Next we show that statement (b) implies statement (c). Suppose that

$$\forall (x^{(n)})_{n=1}^{\infty} \text{ in } X, \lim_{n\to\infty} d_X(x^{(n)}, x_0) = 0 \implies \lim_{n\to\infty} d_Y\big(f(x^{(n)}), f(x_0)\big) = 0.$$

Let V be an open set in (Y, d_Y) such that $f(x_0) \in V$. Then we have

$$\begin{aligned}
&V \text{ is open in } (Y, d_Y) \\
\implies &V = \operatorname{int}_{(Y,d_Y)}(V) && \text{(by Prop. II.1.2.15(a))} \\
\implies &\exists \varepsilon \in \mathbb{R}^+ : B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big) \subseteq V. && \text{(by Def. II.1.2.5)}
\end{aligned}$$

Now we choose one ε and define $V_\varepsilon = B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big)$. By Rmk. II.1.2.4 we know that $f(x_0) \in V_\varepsilon$, thus we have $x_0 \in f^{-1}(V_\varepsilon)$ and $f^{-1}(V_\varepsilon) \neq \emptyset$. Now we claim that

$$\exists \delta \in \mathbb{R}^+ : B_{(X,d_X)}(x_0, \delta) \subseteq f^{-1}(V_\varepsilon).$$

Suppose the claim is false. Then we have

$$\begin{aligned}
&\forall \delta \in \mathbb{R}^+, B_{(X,d_X)}(x_0, \delta) \not\subseteq f^{-1}(V_\varepsilon) \\
\implies &\forall \delta \in \mathbb{R}^+, B_{(X,d_X)}(x_0, \delta) \setminus f^{-1}(V_\varepsilon) \neq \emptyset \\
\implies &\forall \delta \in \mathbb{R}^+, \exists x \in X : \\
&\big(d_X(x, x_0) < \delta\big) \wedge \Big(d_Y\big(f(x), f(x_0)\big) \geq \varepsilon\Big) && \text{(by Def. II.1.2.1)} \\
\implies &\forall n \in \mathbb{Z}^+, \exists x \in X : \\
&\big(d_X(x, x_0) < \frac{1}{n}\big) \wedge \Big(d_Y\big(f(x), f(x_0)\big) \geq \varepsilon\Big).
\end{aligned}$$

For each $n \in \mathbb{Z}^+$, we define $X_n = B_{(X,d_X)}(x_0, \frac{1}{n}) \setminus f^{-1}(V_\varepsilon)$. We choose one sequence $(x^{(n)})_{n=1}^\infty \in \prod_{n \in \mathbb{Z}^+} X_n$. Then we have

$$\begin{aligned}
&\forall n \in \mathbb{Z}^+, d_X(x^{(n)}, x_0) < \frac{1}{n} \\
\implies &\lim_{n \to \infty} d_X(x^{(n)}, x_0) = 0 \\
\implies &\lim_{n \to \infty} d_Y\big(f(x^{(n)}), f(x_0)\big) = 0 && \text{(by hypothesis)} \\
\implies &\exists N \in \mathbb{Z}^+ : \forall n \geq N, d_Y\big(f(x^{(n)}), f(x_0)\big) \leq \frac{\varepsilon}{2}. && \text{(by Def. II.1.1.14)}
\end{aligned}$$

But by the definition of $(x^{(n)})_{n=1}^\infty$ we know that

$$\forall n \in \mathbb{Z}^+, d_Y\big(f(x^{(n)}), f(x_0)\big) \geq \varepsilon,$$

a contradiction. Thus, the claim is true. Using the claim we choose one δ and define $U = B_{(X,d_X)}(x_0, \delta)$. By Prop. II.1.2.15(c) we know that U is open in (X, d_X). By Rmk. II.1.2.4 we know that $x_0 \in U$. Since $U \subseteq f^{-1}(V_\varepsilon) \subseteq X$, we know that $f(U) \subseteq V$.

Finally we show that statement (c) implies statement (a). Suppose that

$$\forall V \subseteq Y, \big(V \text{ is open in } (Y, d_Y)\big) \wedge \big(f(x_0) \in V\big)$$

$$\implies \exists U \subseteq X : \big(U \text{ is open in } (X, d_X)\big) \wedge (x_0 \in U) \wedge \big(f(U) \subseteq V\big).$$

Let $\varepsilon \in \mathbb{R}^+$. By Prop. II.1.2.15(c) we know that $B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big)$ is open in (Y, d_Y). By hypothesis we know that

$$\exists U \subseteq X : \big(U \text{ is open in } (X, d_X)\big) \wedge (x_0 \in U) \wedge \Big(f(U) \subseteq B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big)\Big).$$

Now we choose one such U. Since U is open in (X, d_X) and $x_0 \in U$, we have

$$\begin{aligned}
& x_0 \in \operatorname{int}_{(X,d_X)}(U) && \text{(by Prop. II.1.2.15(a))}\\
\implies & \exists \delta \in \mathbb{R}^+ : B_{(X,d_X)}(x_0, \delta) \subseteq U && \text{(by Def. II.1.2.5)}\\
\implies & \exists \delta \in \mathbb{R}^+ : f\big(B_{(X,d_X)}(x_0, \delta)\big) \subseteq f(U)\\
\implies & \exists \delta \in \mathbb{R}^+ : f\big(B_{(X,d_X)}(x_0, \delta)\big) \subseteq B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big)\\
\implies & \exists \delta \in \mathbb{R}^+ :\\
& \Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon\Big). && \text{(by Def. II.1.2.1)}
\end{aligned}$$

Since ε was arbitrary, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon\Big).$$

Thus, by Def. II.2.1.1 f is continuous at x_0 from (X, d_X) to (Y, d_Y). We conclude that statements (a)(b)(c) are equivalent. □

Thm. II.2.1.5. Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space. Let $f : X \to Y$ be a function. Then the following four statements are equivalent:

(a) f is continuous.

(b) Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X which converges to some point $x_0 \in X$ with respect to the metric d_X, the sequence $\big(f(x^{(n)})\big)_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.

(c) Whenever V is an open set in Y, the set $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is an open set in X.

(d) Whenever F is a closed set in Y, the set $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is a closed set in X.

Proof. We first show that statements (a)(b) are equivalent.

$$\begin{aligned}
& f \text{ is continuous from } (X, d_X) \text{ to } (Y, d_Y)\\
\iff & \forall x_0 \in X, f \text{ is continuous at } x_0\\
& \text{from } (X, d_X) \text{ to } (Y, d_Y) && \text{(by Def. II.2.1.1)}
\end{aligned}$$

$$\begin{aligned}
&\iff \forall x_0 \in X, \text{ every sequence } (x^{(n)})_{n=1}^{\infty} \text{ in } X \text{ satisfies} \\
&\qquad \lim_{n\to\infty} d_X(x^{(n)}, x_0) = 0 \text{ implies} \\
&\qquad \lim_{n\to\infty} d_Y\big(f(x^{(n)}), f(x_0)\big) = 0. && \text{(by Thm. II.2.1.4(a)(b))}
\end{aligned}$$

Next we show that statements (a) implies statement (c). Suppose that f is continuous from (X, d_X) to (Y, d_Y). Let V be an open set in (Y, d_Y) and let $E = f^{-1}(V)$. Then we have

$$\begin{aligned}
&f \text{ is continuous from } (X, d_X) \text{ to } (Y, d_Y) \\
\Longrightarrow\ & f|_E \text{ is continuous from } \big(E, d_X|_{E\times E}\big) \text{ to } (Y, d_Y) && \text{(by Rmk. II.2.1.3)} \\
\Longrightarrow\ & \forall x_0 \in E, f \text{ is continuous at } x_0 \\
& \text{from } \big(E, d_X|_{E\times E}\big) \text{ to } (Y, d_Y) && \text{(by Def. II.2.1.1)} \\
\Longrightarrow\ & \forall x_0 \in E, \exists U \subseteq X : \\
& \big(U \text{ is open in } (X, d_X)\big) \wedge (x_0 \in U) \wedge \big(f(U) \subseteq V\big) && \text{(by Thm. II.2.1.4(a)(c))} \\
\Longrightarrow\ & \forall x_0 \in E, \exists U \subseteq X : \\
& \big(U \text{ is open in } (X, d_X)\big) \wedge (x_0 \in U) \wedge \big(U \subseteq E\big) && \text{(by the definition of } E) \\
\Longrightarrow\ & \forall x_0 \in E, \exists U \subseteq X : \\
& \big(\exists r \in \mathbb{R}^+ : B_{(X,d_X)}(x_0, r) \subseteq U \subseteq E\big) && \text{(by Prop. II.1.2.15(a))} \\
\Longrightarrow\ & E \text{ is open in } (X, d_X). && \text{(by Prop. II.1.2.15(a))}
\end{aligned}$$

Since V was arbitrary, we know that statement (a) implies statement (c).

Next we show that statements (c) implies statement (a). Suppose that

$$\forall V \subseteq Y, V \text{ is open in } (Y, d_Y) \implies f^{-1}(V) \text{ is open in } (X, d_X).$$

Let $x_0 \in X$. Then we have

$$\begin{aligned}
&\forall V \subseteq Y, \big(V \text{ is open in } (Y, d_Y)\big) \wedge \big(f(x_0) \in V\big) \\
\Longrightarrow\ & \big(f^{-1}(V) \text{ is open in } (X, d_X)\big) \wedge \big(x_0 \in f^{-1}(V)\big) && \text{(by hypothesis)}
\end{aligned}$$

and by Thm. II.2.1.4(a)(c) we know that f is continuous at x_0 from (X, d_X) to (Y, d_Y). Since x_0 was arbitrary, we know that f is continuous from (X, d_X) to (Y, d_Y). Thus, statements (c) implies statement (a) and from the proof above we conclude that statements (a)(c) are equivalent.

Next we show that statements (c) implies statement (d). Suppose that

$$\forall V \subseteq Y, V \text{ is open in } (Y, d_Y) \implies f^{-1}(V) \text{ is open in } (X, d_X).$$

Let F be an closed set in (Y, d_Y). Then we have

$$F \text{ is closed in } (Y, d_Y)$$

$$
\begin{aligned}
&\implies Y \setminus F \text{ is open in } (Y, d_Y) && \text{(by Prop. II.1.2.15(e))}\\
&\implies f^{-1}(Y \setminus F) \text{ is open in } (X, d_X) && \text{(by hypothesis)}\\
&\implies X \setminus f^{-1}(Y \setminus F) \text{ is closed in } (X, d_X) && \text{(by Prop. II.1.2.15(e))}\\
&\implies X \setminus \{x \in X : f(x) \in Y \setminus F\} \text{ is closed in } (X, d_X)\\
&\implies \{x \in X : f(x) \in F\} \text{ is closed in } (X, d_X)\\
&\implies f^{-1}(F) \text{ is closed in } (X, d_X).
\end{aligned}
$$

Since F was arbitrary, we know that statement (c) implies statement (d).

Finally we show that statements (d) implies statement (c). Suppose that

$$\forall F \subseteq Y, F \text{ is closed in } (Y, d_Y) \implies f^{-1}(F) \text{ is closed in } (X, d_X).$$

Let V be an open set in (Y, d_Y). Then we have

$$
\begin{aligned}
&V \text{ is open in } (Y, d_Y)\\
&\implies Y \setminus V \text{ is closed in } (Y, d_Y) && \text{(by Prop. II.1.2.15(e))}\\
&\implies f^{-1}(Y \setminus V) \text{ is closed in } (X, d_X) && \text{(by hypothesis)}\\
&\implies X \setminus f^{-1}(Y \setminus V) \text{ is open in } (X, d_X) && \text{(by Prop. II.1.2.15(e))}\\
&\implies X \setminus \{x \in X : f(x) \in Y \setminus V\} \text{ is open in } (X, d_X)\\
&\implies \{x \in X : f(x) \in V\} \text{ is open in } (X, d_X)\\
&\implies f^{-1}(V) \text{ is open in } (X, d_X).
\end{aligned}
$$

Since V was arbitrary, we know that statement (d) implies statement (c). We conclude that statements (a)(b)(c)(d) are all equivalent. □

Rmk. II.2.1.6. It may seem strange that continuity ensures that the *inverse* image of an open set is open. One may guess instead that the reverse should be true, that the *forward* image of an open set is open; but this is not true; see Ex. II.1.5.4 and II.1.5.5.

Cor. II.2.1.7 (Continuity preserved by composition)**.** Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces.

(a) If $f : X \to Y$ is continuous at a point $x_0 \in X$, and $g : Y \to Z$ is continuous at $f(x_0)$, then the composition $g \circ f : X \to Z$, defined by $g \circ f(x) := g(f(x))$, is continuous at x_0.

(b) If $f : X \to Y$ is continuous, and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is also continuous.

Proof. (a) Since f is continuous at x_0 from (X, d_X) to (Y, d_Y), by Thm. II.2.1.4(a)(c) we know that

$$\forall V \subseteq Y, \big(V \text{ is open in } (Y, d_Y)\big) \wedge \big(f(x_0) \in V\big)$$

$$\implies \exists U \subseteq X : \big(U \text{ is open in } (X, d_X)\big) \wedge (x_0 \in U) \wedge \big(f(U) \subseteq V\big)$$

Now we choose such U for each open set V in (Y, d_Y) and denote it as U_V. Since g is continuous at $f(x_0)$ from (Y, d_Y) to (Z, d_Z), by Thm. II.2.1.4(a)(c) we know that

$$\begin{aligned}
&\forall W \subseteq Z, \big(W \text{ is open in } (Z, d_Z)\big) \wedge \Big(g\big(f(x_0)\big) \in W\Big) \\
\implies &\exists V \subseteq Y : \big(V \text{ is open in } (Y, d_Y)\big) \wedge (y_0 \in V) \wedge \big(g(V) \subseteq W\big) \\
\implies &\exists U_V \subseteq X : \big(U_V \text{ is open in } (X, d_X)\big) \wedge (x_0 \in U_V) \wedge \big(f(U_V) \subseteq V\big) \\
\implies &\exists U_V \subseteq X : \big(U_V \text{ is open in } (X, d_X)\big) \wedge (x_0 \in U_V) \wedge \Big(g\big(f(U_V)\big) \subseteq g(V) \subseteq W\Big).
\end{aligned}$$

Thus, by Thm. II.2.1.4(a)(c) we know that $g \circ f$ is continuous at x_0 from (X, d_X) to (Z, d_Z). □

Proof. (b) Let $x_0 \in X$. Then we have

$$\begin{aligned}
&f \text{ is continuous from } (X, d_X) \text{ to } (Y, d_Y) \\
\implies &f \text{ is continuous at } x_0 \text{ from } (X, d_X) \text{ to } (Y, d_Y). && \text{(by Def. II.2.1.1)}
\end{aligned}$$

Since $f(x_0) \in Y$, we have

$$\begin{aligned}
&g \text{ is continuous from } (Y, d_Y) \text{ to } (Z, d_Z) \\
\implies &g \text{ is continuous at } f(x_0) \text{ from } (Y, d_Y) \text{ to } (Z, d_Z) && \text{(by Def. II.2.1.1)} \\
\implies &g \circ f \text{ is continuous at } x_0 \text{ from } (X, d_X) \text{ to } (Z, d_Z). && \text{(by Cor. II.2.1.7(a))}
\end{aligned}$$

Since x_0 was arbitrary, by Def. II.2.1.1 we know that $g \circ f$ is continuous from (X, d_X) to (Z, d_Z). □

— Exercises —

Ex. II.2.1.1. Prove Thm. II.2.1.4.

Proof. See Thm. II.2.1.4. □

Ex. II.2.1.2. Prove Thm. II.2.1.5.

Proof. See Thm. II.2.1.5. □

Ex. II.2.1.3. Use Thm. II.2.1.4 and Thm. II.2.1.5 to prove Cor. II.2.1.7.

Proof. See Cor. II.2.1.7. □

Ex. II.2.1.4. Give an example of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

(a) f is not continuous, but g and $g \circ f$ are continuous;

(b) g is not continuous, but f and $g \circ f$ are continuous;

(c) f and g are not continuous, but $g \circ f$ is continuous.

Explain briefly why these examples do not contradict Cor. II.2.1.7.

Proof. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$\forall x \in \mathbb{R}, f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

and let $g : \mathbb{R} \to \mathbb{R}$ be the function $g(x) = 0$ for all $x \in \mathbb{R}^+$. Then we know that f is not continuous at 0 from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ and thus f is not continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Since g is constant function, we know that g is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Since $g \circ f$ is also a constant function, we know that $g \circ f$ is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. This does not contradict to Cor. II.2.1.7 since f is not continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. □

Proof. (b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = 0$ for all $x \in \mathbb{R}^+$. Let $g : \mathbb{R} \to \mathbb{R}$ be the function

$$\forall x \in \mathbb{R}, g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Since f is constant function, we know that f is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Since g is not continuous at 0 from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$, we know that g is not continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Since $g \circ f$ is a constant function, we know that $g \circ f$ is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. This does not contradict to Cor. II.2.1.7 since g is not continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. □

Proof. (c) Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$\forall x \in \mathbb{R}, f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

and let $g : \mathbb{R} \to \mathbb{R}$ be the function

$$\forall x \in \mathbb{R}, g(x) = \begin{cases} 1 & \text{if } x = 2 \\ 0 & \text{if } x \neq 2 \end{cases}$$

Since f is not continuous at 0 from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$, we know that f is not continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Similarly, g is not continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Since $g \circ f$ is a constant function, we know that $g \circ f$ is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. This does not contradict to Cor. II.2.1.7 since f, g are not continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. □

Ex. II.2.1.5. Let (X, d) be a metric space, and let $(E, d|_{E\times E})$ be a subspace of (X, d). Let $\iota_{E\to X} : E \to X$ be the inclusion map, defined by setting $\iota_{E\to X}(x) := x$ for all $x \in E$. Show that $\iota_{E\to X}$ is continuous.

Proof. Let $x_0 \in E$. Since

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \forall x \in E, d|_{E\times E}(x, x_0) < \varepsilon \\
\implies &d|_{E\times E}\big(\iota_{E\to X}(x), \iota_{E\to X}(x_0)\big) = d|_{E\times E}(x, x_0) < \varepsilon, \qquad \text{(by the definition of } \iota_{E\to X}\text{)}
\end{aligned}$$

by setting $\delta = \varepsilon$ we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x \in E, d|_{E\times E}(x, x_0) < \delta \implies d|_{E\times E}\big(\iota_{E\to X}(x), \iota_{E\to X}(x_0)\big) < \varepsilon\Big)
\end{aligned}$$

and thus by Def. II.2.1.1 $\iota_{E\to X}$ is continuous at x_0 from $(E, d|_{E\times E})$ to (X, d). Since x_0 was arbitrary, by Def. II.2.1.1 $\iota_{E\to X}$ is continuous from $(E, d|_{E\times E})$ to (X, d). □

Ex. II.2.1.6. Let $f : X \to Y$ be a function from one metric space (X, d_X) to another (Y, d_Y). Let E be a subset of X (which we give the induced metric $d_X|_{E\times E}$), and let $f|_E : E \to Y$ be the restriction of f to E, thus $f|_E(x) := f(x)$ when $x \in E$. If $x_0 \in E$ and f is continuous at x_0, show that $f|_E$ is also continuous at x_0. (Is the converse of this statement true? Explain.) Conclude that if f is continuous, then $f|_E$ is continuous. Thus, restriction of the domain of a function does not destroy continuity.

Proof. See Rmk. II.2.1.3. The converse is not true since the statement

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \Big(\forall x \in E, d_X|_{E\times E}(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon\Big)$$

does not imply the statement

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon\Big).$$

□

Ex. II.2.1.7. Let $f : X \to Y$ be a function from one metric space (X, d_X) to another (Y, d_Y). Suppose that the image $f(X)$ of X is contained in some subset $E \subseteq Y$ of Y. Let $g : X \to E$ be the function which is the same as f but with the codomain restricted from Y to E, thus $g(x) = f(x)$ for all $x \in X$. We give E the metric $d_Y|_{E\times E}$ induced from Y. Show that for any $x_0 \in X$, that f is continuous at x_0 iff g is continuous at x_0. Conclude that f is continuous iff g is continuous. (Thus, the notion of continuity is not affected if one restricts the codomain of the function.)

Proof. Since $f(X) \subseteq E \subseteq Y$, we have

$$\forall x \in X, d_Y(x, x_0) = d_Y|_{E\times E}(x, x_0).$$

Thus, by Def. II.2.1.1 we have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon\Big) \\ \iff &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y|_{E\times E}\big(f(x), f(x_0)\big) < \varepsilon\Big). \end{aligned}$$

This means f is continuous at x_0 from (X, d_X) to (Y, d_Y) iff g is continuous at x_0 from (X, d_X) to $(E, d_Y|_{E\times E})$. By Def. II.2.1.1 we conclude that f is continuous from (X, d_X) to (Y, d_Y) iff g is continuous from (X, d_X) to $(E, d_Y|_{E\times E})$. □

II.2.2 Continuity and product spaces

Note. Given two functions $f : X \to Y$ and $g : X \to Z$, one can define their *direct sum* $f \oplus g : X \to Y \times Z$ defined by $f \oplus g(x) := \big(f(x), g(x)\big)$, i.e., this is the function taking values in the Cartesian product $Y \times Z$ whose first co-ordinate is $f(x)$ and whose second co-ordinate is $g(x)$ (cf. Exercise 3.5.7 in Analysis I).

Lem. II.2.2.1. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions, and let $f \oplus g : X \to \mathbb{R}^2$ be their direct sum. We give $\mathbb{R}^2$ the Euclidean metric.

(a) If $x_0 \in X$, then f and g are both continuous at x_0 iff $f \oplus g$ is continuous at x_0.

(b) f and g are both continuous iff $f \oplus g$ is continuous.

Proof. (a) Let (X, d) be a metric space. Then we have

$$\begin{aligned} &f, g \text{ are continuous at } x_0 \\ &\text{from } (X, d) \text{ to } (\mathbb{R}, d_{l^2}|_{\mathbb{R}\times\mathbb{R}}) \\ \iff &\text{every sequence } (x^{(n)})_{n=1}^{\infty} \text{ in } X \text{ satisfies the following:} \\ &\lim_{n\to\infty} d\big(x^{(n)}, x_0\big) = 0 \text{ implies} \\ &\begin{cases} \lim\limits_{n\to\infty} d_{l^2}|_{\mathbb{R}\times\mathbb{R}}\big(f(x^{(n)}), f(x_0)\big) = 0 \\ \lim\limits_{n\to\infty} d_{l^2}|_{\mathbb{R}\times\mathbb{R}}\big(g(x^{(n)}), g(x_0)\big) = 0 \end{cases} \qquad \text{(by Thm. II.2.1.4(a)(b))} \\ \iff &\text{every sequence } (x^{(n)})_{n=1}^{\infty} \text{ in } X \text{ satisfies the following:} \\ &\lim_{n\to\infty} d\big(x^{(n)}, x_0\big) = 0 \text{ implies} \\ &\lim_{n\to\infty} d_{l^2}|_{\mathbb{R}^2\times\mathbb{R}^2}\Big(\big(f(x^{(n)}), g(x^{(n)})\big), \big(f(x_0), g(x_0)\big)\Big) = 0 \quad \text{(by Prop. II.1.1.18)} \\ \iff &f \oplus g \text{ is continuous at } x_0 \end{aligned}$$

from (X, d) to $(\mathbb{R}^2, d_{l^2}|_{\mathbb{R}^2\times\mathbb{R}^2})$. (by Thm. II.2.1.4(a)(b))

□

Proof. (b) Let (X, d) be a metric space. Then we have

$$\begin{aligned}
& f, g \text{ are continuous from } (X, d) \text{ to } (\mathbb{R}, d_{l^2}|_{\mathbb{R}\times\mathbb{R}}) \\
\iff & \forall x_0 \in X, f, g \text{ are continuous at } x_0 \\
& \text{from } (X, d) \text{ to } (\mathbb{R}, d_{l^2}|_{\mathbb{R}\times\mathbb{R}}) && \text{(by Def. II.2.1.1)} \\
\iff & \forall x_0 \in X, f \oplus g \text{ is continuous at } x_0 && \text{(by Lem. II.2.2.1(a))} \\
& \text{from } (X, d) \text{ to } (\mathbb{R}^2, d_{l^2}|_{\mathbb{R}^2\times\mathbb{R}^2}) \\
\iff & f \oplus g \text{ is continuous from } (X, d) \text{ to } (\mathbb{R}^2, d_{l^2}|_{\mathbb{R}^2\times\mathbb{R}^2}). && \text{(by Def. II.2.1.1)}
\end{aligned}$$

□

A.Cor. II.2.2.1. Let (X, d) be a metric space. Let $(\mathbb{R}, d_\mathbb{R})$ be a metric space where $d_\mathbb{R}$ can be $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$, $d_{l^2}|_{\mathbb{R}\times\mathbb{R}}$ or $d_{l^\infty}|_{\mathbb{R}\times\mathbb{R}}$. Let $(\mathbb{R}^2, d_{\mathbb{R}^2})$ be a metric space where $d_{\mathbb{R}^2}$ can be $d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2}$, $d_{l^2}|_{\mathbb{R}^2\times\mathbb{R}^2}$ or $d_{l^\infty}|_{\mathbb{R}^2\times\mathbb{R}^2}$. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions, and let $f \oplus g : X \to \mathbb{R}^2$ be their direct sum.

(a) If $x_0 \in X$, then f and g are both continuous at x_0 from (X, d) to $(\mathbb{R}, d_\mathbb{R})$ iff $f \oplus g$ is continuous at x_0 from (X, d) to $(\mathbb{R}^2, d_{\mathbb{R}^2})$.

(b) f and g are both continuous from (X, d) to $(\mathbb{R}, d_\mathbb{R})$ iff $f \oplus g$ is continuous from (X, d) to $(\mathbb{R}^2, d_{\mathbb{R}^2})$.

Proof. By Prop. II.1.1.18 and Lem. II.2.2.1 we are done. □

Lem. II.2.2.2. The addition function $(x, y) \mapsto x+y$, the subtraction function $(x, y) \mapsto x-y$, the multiplication function $(x, y) \mapsto xy$, the maximum function $(x, y) \mapsto \max(x, y)$, and the minimum function $(x, y) \mapsto \min(x, y)$, are all continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. The division function $(x, y) \mapsto x/y$ is a continuous function from $\mathbb{R} \times (\mathbb{R} \setminus \{0\}) = \big\{(x, y) \in \mathbb{R}^2 : y \neq 0\big\}$ to $\mathbb{R}$. For any real number c, the function $x \mapsto cx$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}$.

Proof. First, we show that the addition, subtraction, multiplication, maximum and minimum functions from $\mathbb{R}^2$ to $\mathbb{R}$ are continuous from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Let $(x, y) \in \mathbb{R}^2$ and let $(x^{(n)}, y^{(n)})_{n=1}^\infty$ be a sequence in $\mathbb{R}^2$ such that

$$\lim_{n\to\infty} d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2}\big((x^{(n)}, y^{(n)}), (x, y)\big) = 0$$

By limit laws we know that

$$\lim_{n\to\infty} d_{l^1}|_{\mathbb{R}\times\mathbb{R}}(x^{(n)} + y^{(n)}, x + y) = 0$$
$$\lim_{n\to\infty} d_{l^1}|_{\mathbb{R}\times\mathbb{R}}(x^{(n)} - y^{(n)}, x - y) = 0$$

$$\lim_{n\to\infty} d_{l^1}|_{\mathbb{R}\times\mathbb{R}}(x^{(n)}y^{(n)}, xy) = 0$$
$$\lim_{n\to\infty} d_{l^1}|_{\mathbb{R}\times\mathbb{R}}\big(\max(x^{(n)}, y^{(n)}), \max(x,y)\big) = 0$$
$$\lim_{n\to\infty} d_{l^1}|_{\mathbb{R}\times\mathbb{R}}\big(\min(x^{(n)}, y^{(n)}), \min(x,y)\big) = 0$$

Since $(x^{(n)}, y^{(n)})_{n=1}^{\infty}$ was arbitrary, by Thm. II.2.1.4(a)(b) we know that the addition, subtraction, multiplication, maximum and minimum functions from $\mathbb{R}^2$ to $\mathbb{R}$ are continuous at (x,y) from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Since (x,y) was arbitrary, by Thm. II.2.1.5(a)(b) we know that the addition, subtraction, multiplication, maximum and minimum functions from $\mathbb{R}^2$ to $\mathbb{R}$ are continuous from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$.

Next we show that the division function from $E = \mathbb{R}\times(\mathbb{R}\setminus\{0\})$ to $\mathbb{R}$ is continuous from $(E, d_{l^1}|_{E\times E})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Let $(x,y)\in E$ and let $(x^{(n)}, y^{(n)})_{n=1}^{\infty}$ be a sequence in E such that

$$\lim_{n\to\infty} d_{l^1}|_{E\times E}\big((x^{(n)}, y^{(n)}), (x,y)\big) = 0$$

By limit laws we know that

$$\lim_{n\to\infty} d_{l^1}|_{\mathbb{R}\times\mathbb{R}}(x^{(n)}/y^{(n)}, x/y) = 0.$$

Thus, using similar arguments as above, we know that the division function from E to $\mathbb{R}$ is continuous from $(E, d_{l^1}|_{E\times E})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$.

Finally we show that the constant multiplication function from $\mathbb{R}$ to $\mathbb{R}$ is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Let $c, x\in\mathbb{R}$ and let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that

$$\lim_{n\to\infty} d_{l^1}|_{\mathbb{R}\times\mathbb{R}}(x^{(n)}, x) = 0.$$

By limit laws we know that

$$\lim_{n\to\infty} d_{l^1}|_{\mathbb{R}\times\mathbb{R}}(cx^{(n)}, cx) = 0.$$

Thus, using similar arguments as above, we know that the constant function from $\mathbb{R}$ to $\mathbb{R}$ is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. $\square$

Cor. II.2.2.3. Let (X,d) be a metric space, let $f: X\to\mathbb{R}$ and $g: X\to\mathbb{R}$ be functions. Let c be a real number.

(a) If $x_0\in X$ and f and g are continuous at x_0, then the functions $f+g: X\to\mathbb{R}$, $f-g: X\to\mathbb{R}$, $fg: X\to\mathbb{R}$, $\max(f,g): X\to\mathbb{R}$, $\min(f,g): X\to\mathbb{R}$, and $cf: X\to\mathbb{R}$ (see Definition 9.2.1 in Analysis I for definitions) are also continuous at x_0. If $g(x)\neq 0$ for all $x\in X$, then $f/g: X\to\mathbb{R}$ is also continuous at x_0.

(b) If f and g are continuous, then the functions $f+g: X\to\mathbb{R}$, $f-g: X\to\mathbb{R}$, $fg: X\to\mathbb{R}$, $\max(f,g): X\to\mathbb{R}$, $\min(f,g): X\to\mathbb{R}$, and $cf: X\to\mathbb{R}$ are also continuous. If $g(x)\neq 0$ for all $x\in X$, then $f/g: X\to\mathbb{R}$ is also continuous.

Proof. We first prove (a). Since f, g are continuous at x_0 from (X, d) to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$, then by A.Cor. II.2.2.1(a) $f \oplus g : X \to \mathbb{R}^2$ is also continuous at x_0 from (X, d) to $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$. On the other hand, from Lem. II.2.2.2 the function $(x, y) \mapsto x + y$ is continuous from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$, and in particular, is continuous at $f \oplus g(x_0)$ from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. If we then compose these two functions using Cor. II.2.1.7 we conclude that $f + g : X \to \mathbb{R}$ is continuous from (X, d) to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. A similar argument gives the continuity of $f - g$, fg, $\max(f, g)$, $\min(f, g)$ and cf. To prove the claim for f/g, we first use Ex. II.2.1.7 to restrict the codomain of g from $\mathbb{R}$ to $\mathbb{R} \setminus \{0\}$, and then one can argue as before. The claim (b) follows immediately from (a). □

— Exercises —

Ex. II.2.2.1. Prove Lem. II.2.2.1.

Proof. See Lem. II.2.2.1. □

Ex. II.2.2.2. Prove Lem. II.2.2.2.

Proof. See Lem. II.2.2.2. □

Ex. II.2.2.3. Show that if $f : X \to \mathbb{R}$ is a continuous function, so is the function $|f| : X \to \mathbb{R}$ defined by $|f|(x) := |f(x)|$.

Proof. Let (X, d_X) be a metric space and let $f : X \to \mathbb{R}$ be a function which is continuous from (X, d_X) to . Since

$$\begin{aligned} &\forall x_0 \in X, |f|(x) = |f(x)| = \max\big(-f(x), f(x)\big) \\ \implies &|f| = \max(-f, f), \end{aligned}$$

we have

$$\begin{aligned} &f \text{ is continuous from } (X, d_X) \text{ to } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}) \\ \implies &-f \text{ is continuous from } (X, d_X) \text{ to } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}) && \text{(by Cor. II.2.2.3(b))} \\ \implies &\max(f, -f) \text{ is continuous from } (X, d_X) \text{ to } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}) && \text{(by Cor. II.2.2.3(b))} \\ \implies &|f| \text{ is continuous from } (X, d_X) \text{ to } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}). \end{aligned}$$

□

Ex. II.2.2.4. Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the functions $\pi_1(x, y) := x$ and $\pi_2(x, y) := y$ (these two functions are sometimes called the *co-ordinate functions* on $\mathbb{R}^2$). Show that π_1 and π_2 are continuous. Conclude that if $f : \mathbb{R} \to X$ is any continuous function into a metric space (X, d), then the functions $g_1 : \mathbb{R}^2 \to X$ and $g_2 : \mathbb{R}^2 \to X$ defined by $g_1(x, y) := f(x)$ and $g_2(x, y) := f(y)$ are also continuous.

Proof. Let $(x, y) \in \mathbb{R}^2$. We know that

$$
\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \forall (x', y') \in \mathbb{R}^2, \\
&d_{l^1}|_{\mathbb{R}^2 \times \mathbb{R}^2}\big((x, y), (x', y')\big) < \varepsilon \\
\implies &\left|x - x'\right| + \left|y - y'\right| < \varepsilon && \text{(by E.g. II.1.1.7)} \\
\implies &\left|x - x'\right| < \varepsilon && \text{(by Def. II.1.1.2(a)(b))} \\
\implies &d_{l^1}|_{\mathbb{R} \times \mathbb{R}}(x, x') < \varepsilon && \text{(by E.g. II.1.1.7)} \\
\implies &d_{l^1}|_{\mathbb{R} \times \mathbb{R}}\big(\pi_1(x, y), \pi_1(x', y')\big) < \varepsilon. && \text{(by the definition of } \pi_1\text{)}
\end{aligned}
$$

Thus, by setting $\delta = \varepsilon$ we have

$$
\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall (x', y') \in \mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2 \times \mathbb{R}^2}\big((x, y), (x', y')\big) < \delta \implies d_{l^1}|_{\mathbb{R} \times \mathbb{R}}\big(\pi_1(x, y), \pi_1(x', y')\big) < \varepsilon\Big).
\end{aligned}
$$

Since (x, y) was arbitrary, by Def. II.2.1.1 π_1 is continuous from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2 \times \mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R} \times \mathbb{R}})$. Using similar arguments, we can show that π_2 is continuous from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2 \times \mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R} \times \mathbb{R}})$.

Let $f : \mathbb{R} \to X$ be a function which is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R} \times \mathbb{R}})$ to (X, d). Let $g_1 : \mathbb{R}^2 \to X$ and $g_2 : \mathbb{R}^2 \to X$ be functions where

$$
\forall (x, y) \in \mathbb{R}^2, \begin{cases} g_1(x, y) = f(x) \\ g_2(x, y) = f(y) \end{cases}
$$

Since

$$
\begin{aligned}
&\forall (x, y) \in \mathbb{R}^2, \\
&f \circ \pi_1(x, y) = f\big(\pi_1(x, y)\big) = f(x) = g_1(x, y); \\
&f \circ \pi_2(x, y) = f\big(\pi_2(x, y)\big) = f(y) = g_2(x, y),
\end{aligned}
$$

we know that $g_1 = f \circ \pi_1$ and $g_2 = f \circ \pi_2$. Thus, by Cor. II.2.1.7(b) g_1, g_2 are continuous from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2 \times \mathbb{R}^2})$ to (X, d). □

Ex. II.2.2.5. Let $n, m \geq 0$ be integers. Suppose that for every $0 \leq i \leq n$ and $0 \leq j \leq m$ we have a real number c_{ij}. Form the function $P : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$
P(x, y) := \sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij} x^i y^j.
$$

(Such a function is known as a *polynomial of two variables*) Show that P is continuous. Conclude that if $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous functions, then the function $P(f, g) : X \to \mathbb{R}$ defined by $P(f, g)(x) := P\big(f(x), g(x)\big)$ is also continuous.

Proof. First, we show that P is continuous from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Let $(x,y) \in \mathbb{R}^2$. Let π_1, π_2 be the functions defined in Ex. II.2.2.4. Since π_1 is continuous from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$, by Cor. II.2.2.3(b) we know that

$$x^i = \prod_{k=1}^{i} x = \prod_{k=1}^{i} \pi_1(x,y)$$

is continuous at (x,y) from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ for every $0 \le i \le n$. Similarly

$$y^j = \prod_{k=1}^{j} y = \prod_{k=1}^{j} \pi_2(x,y)$$

is continuous at (x,y) from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ for every $0 \le j \le m$. Thus, by Cor. II.2.2.3(b) we know that $c_{ij}x^i y^j$ is continuous at (x,y) from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ for every $0 \le i \le n$ and $0 \le j \le m$, and

$$\sum_{i=0}^{n}\sum_{j=0}^{m} c_{ij}x^i y^j = P(x,y)$$

is continuous at (x,y) from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Since (x,y) was arbitrary, by Def. II.2.1.1 we know that P is continuous from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$.

Now suppose that $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are to continuous functions from (X,d) to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Then we have

$$\begin{aligned}
&f \oplus g \text{ is continuous} \\
&\text{from } (X,d) \text{ to } (\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2}) && \text{(by A.Cor. II.2.2.1(b))} \\
\Longrightarrow\ &P \circ (f \oplus g) \text{ is continuous} \\
&\text{from } (X,d) \text{ to } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}) && \text{(by Cor. II.2.1.7(b))} \\
\Longrightarrow\ &P(f,g) \text{ is continuous} \\
&\text{from } (X,d) \text{ to } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}). && \text{(by the definition of } P\text{)}
\end{aligned}$$

□

Ex. II.2.2.6. Let $\mathbb{R}^m$ and $\mathbb{R}^n$ be Euclidean spaces. If $f : X \to \mathbb{R}^m$ and $g : X \to \mathbb{R}^n$ are continuous functions, show that $f \oplus g : X \to \mathbb{R}^{m+n}$ is also continuous, where we have identified $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$ in the obvious manner. Is the converse statement true?

Proof. Let (X,d) be a metric space. For each $k \in \mathbb{Z}^+$, let d_k be one of the metric functions $d_{l^1}|_{\mathbb{R}^k\times\mathbb{R}^k}$, $d_{l^2}|_{\mathbb{R}^k\times\mathbb{R}^k}$ or $d_{l^\infty}|_{\mathbb{R}^k\times\mathbb{R}^k}$. Let $f : X \to \mathbb{R}^m$ be a continuous function from (X,d) to $(\mathbb{R}^m, d_m)$, and let $g : X \to \mathbb{R}^n$ be a continuous function from (X,d) to $(\mathbb{R}^n, d_n)$. Let $x_0 \in X$. Then we have

$$\begin{cases} f \text{ is continuous from } (X,d) \text{ to } (\mathbb{R}^m, d_m) \\ g \text{ is continuous from } (X,d) \text{ to } (\mathbb{R}^n, d_n) \end{cases}$$

$$\begin{aligned}
&\iff \text{every sequence } (x^{(k)})_{k=1}^{\infty} \text{ in } X \text{ satisfies the following:}\\
&\quad \lim_{k\to\infty} d\big(x^{(k)}, x_0\big) = 0 \text{ implies}\\
&\quad \begin{cases} \lim_{k\to\infty} d_m\big(f(x^{(k)}), f(x_0)\big) = 0 \\ \lim_{k\to\infty} d_n\big(g(x^{(k)}), g(x_0)\big) = 0 \end{cases} && \text{(by Thm. II.2.1.4(a)(b))}\\
&\iff \text{every sequence } (x^{(k)})_{k=1}^{\infty} \text{ in } X \text{ satisfies the following:}\\
&\quad \lim_{k\to\infty} d\big(x^{(k)}, x_0\big) = 0 \text{ implies}\\
&\quad \lim_{k\to\infty} d_{m+n}\Big(\big(f(x^{(k)}) \oplus g(x^{(k)})\big), \big(f(x_0) \oplus g(x_0)\big)\Big) = 0 && \text{(by Prop. II.1.1.18)}\\
&\iff f \oplus g \text{ is continuous at } x_0\\
&\quad \text{from } (X, d) \text{ to } (\mathbb{R}^{m+n}, d_{m+n}). && \text{(by Thm. II.2.1.4(a)(b))}
\end{aligned}$$

Thus, the statment is true and the converse is also true. □

Ex. II.2.2.7. Let $k \geq 1$, let I be a finite subset of $\mathbb{N}^k$, and let $c : I \to \mathbb{R}$ be a function. Form the function $P : \mathbb{R}^k \to \mathbb{R}$ defined by

$$P(x_1, \dots, x_k) \coloneqq \sum_{(i_1,\dots,i_k)\in I} c(i_1, \dots, i_k) x_1^{i_1} \dots x_k^{i_k}.$$

(Such a function is known as a *polynomial of k variables*; Show that P is continuous.

Proof. For each $k \in \mathbb{Z}^+$, let I_k be the finite subset of $\mathbb{N}^k$, let $d_k = d_{l^1}|_{\mathbb{R}^k \times \mathbb{R}^k}$, and let P_k be a polynomial of k variables, i.e.,

$$P_k(x_1, \dots, x_k) = \sum_{(i_1,\dots,i_k)\in I_k} \big(c(i_1, \dots, i_k) x_1^{i_1} \cdots x_k^{i_k}\big).$$

We induct on k to show that P_k is continuous from $(\mathbb{R}^k, d_k)$ to $(\mathbb{R}, d_1)$ for every $k \in \mathbb{Z}^+$. We start with $k = 1$. For $k = 1$, we have

$$\begin{aligned}
&\forall i_1 \in I_1, c(i_1) x_1^{i_1} \text{ is continuous from } (\mathbb{R}, d_1) \text{ to } (\mathbb{R}, d_1) && \text{(by Lem. II.2.2.2(b))}\\
\implies & P_1(x_1) = \sum_{i_1 \in I_1} \big(c(i_1) x_1^{i_1}\big) \text{ is continuous} && \text{(note that } I_1 \text{ is finite)}\\
&\text{from } (\mathbb{R}, d_1) \text{ to } (\mathbb{R}, d_1) && \text{(by Lem. II.2.2.2(b))}
\end{aligned}$$

and Thus, the base case holds. Suppose inductively that for some $k \geq 1$, P_k is continuous from $(\mathbb{R}^k, d_k)$ to $(\mathbb{R}, d_1)$. Then for $k+1$, we need to show that P_{k+1} is continuous from $(\mathbb{R}^{k+1}, d_{k+1})$ to $(\mathbb{R}, d_1)$. Let $F : \mathbb{R}^k \to 2^{\mathbb{R}}$ be the function

$$\forall (x_1, \dots, x_k, x_{k+1}) \in \mathbb{R}^{k+1}, F(x_1, \dots, x_k) = \{x_{k+1} \in \mathbb{R} : (x_1, \dots, x_k, x_{k+1}) \in I_{k+1}\},$$

and let $A = \Big\{(i_1, \ldots, i_k) \in \mathbb{R}^k : (i_1, \ldots, i_k, i_{k+1}) \in I_{k+1}\Big\}$. Then we have

$$\begin{aligned}
&P_{k+1}(x_1, \ldots, x_k, x_{k+1}) \\
&= \sum_{(i_1 \ldots, i_k, i_{k+1}) \in I_{k+1}} c(i_1, \ldots, i_k, i_{k+1}) x_1^{i_1} \cdots x_k^{i_k} x_{k+1}^{i_{k+1}} \\
&= \sum_{(i_1 \ldots, i_k) \in A} \left(\sum_{i_{k+1} \in \mathbb{R} : (i_1, \ldots, i_k, i_{k+1}) \in I_{k+1}} c(i_1, \ldots, i_k, i_{k+1}) x_1^{i_1} \cdots x_k^{i_k} x_{k+1}^{i_{k+1}} \right) \\
&= \sum_{(i_1 \ldots, i_k) \in A} \left(x_1^{i_1} \cdots x_k^{i_k} \left(\sum_{i_{k+1} \in \mathbb{R} : (i_1, \ldots, i_k, i_{k+1}) \in I_{k+1}} c(i_1, \ldots, i_k, i_{k+1}) x_{k+1}^{i_{k+1}} \right) \right).
\end{aligned}$$

By the induction hypothesis we know that

$$\sum_{i_{k+1} \in \mathbb{R} : (i_1, \ldots, i_k, i_{k+1}) \in I_{k+1}} c(i_1, \ldots, i_k, i_{k+1}) x_{k+1}^{i_{k+1}}$$

is continuous from $(\mathbb{R}, d_1)$ to $(\mathbb{R}, d_1)$ and

$$x_1^{i_1} \cdots x_k^{i_k}$$

is continuous from $(\mathbb{R}^k, d_k)$ to $(\mathbb{R}, d_1)$. Thus, by Lem. II.2.2.2(b) we know that

$$\begin{aligned}
&\forall (i_1, \ldots, i_k, i_{k+1}) \in I_{k+1}, \\
&(x_1^{i_1} \cdots x_k^{i_k}) \left(\sum_{i_{k+1} \in \mathbb{R} : (i_1, \ldots, i_k, i_{k+1}) \in I_{k+1}} c(i_1, \ldots, i_k, i_{k+1}) x_{k+1}^{i_{k+1}} \right) \\
&\text{is continuous from } (\mathbb{R}^{k+1}, d_{k+1}) \text{ to } (\mathbb{R}, d_1) \\
\implies &\sum_{(i_1 \ldots, i_k) \in A} \left(x_1^{i_1} \cdots x_k^{i_k} \left(\sum_{i_{k+1} \in \mathbb{R} : (i_1, \ldots, i_k, i_{k+1}) \in I_{k+1}} c(i_1, \ldots, i_k, i_{k+1}) x_{k+1}^{i_{k+1}} \right) \right) \\
&\text{is continuous from } (\mathbb{R}^{k+1}, d_{k+1}) \text{ to } (\mathbb{R}, d_1) \\
\implies &P_{k+1} \text{ is continuous from } (\mathbb{R}^{k+1}, d_{k+1}) \text{ to } (\mathbb{R}, d_1)
\end{aligned}$$

and this closes the induction. □

Ex. II.2.2.8. Let (X, d_X) and (Y, d_Y) be metric spaces. Define the metric $d_{X \times Y} : (X \times Y) \times (X \times Y) \to [0, \infty)$ by the formula

$$d_{X \times Y}\big((x, y), (x', y')\big) \coloneqq d_X(x, x') + d_Y(y, y').$$

Show that $(X \times Y, d_{X \times Y})$ is a metric space, and deduce an analogue of Prop. II.1.1.18 and Lem. II.2.2.1.

Proof. We first show that $(X \times Y, d_{X\times Y})$ is a metric space. For any $(x, y) \in X \times Y$, we have

$$\begin{aligned} d_{X\times Y}\big((x,y),(x,y)\big) &= d_X(x,x) + d_Y(y,y) \\ &= 0+0=0 \qquad\qquad \text{(by Def. II.1.1.2(a))} \end{aligned}$$

and thus $(X \times Y, d_{X\times Y})$ satisfies Def. II.1.1.2(a). For any $(x_1, y_1), (x_2, y_2) \in X \times Y$, we have

$$\begin{aligned} & (x_1,y_1) \neq (x_2,y_2) \\ \Longrightarrow\ & (x_1 \neq x_2) \vee (y_1 \neq y_2) \\ \Longrightarrow\ & d_{X\times Y}\big((x_1,y_1),(x_2,y_2)\big) = d_X(x_1,x_2) + d_Y(y_1,y_2) \neq 0 \qquad \text{(by Def. II.1.1.2(b))} \end{aligned}$$

and thus $(X \times Y, d_{X\times Y})$ satisfies Def. II.1.1.2(b). For any $(x_1, y_1), (x_2, y_2) \in X \times Y$, we have

$$\begin{aligned} & d_{X\times Y}\big((x_1,y_1),(x_2,y_2)\big) \\ &= d_X(x_1,x_2) + d_Y(y_1,y_2) \\ &= d_X(x_2,x_1) + d_Y(y_2,y_1) \qquad\qquad \text{(by Def. II.1.1.2(c))} \\ &= d_{X\times Y}\big((x_2,y_2),(x_1,y_1)\big) \end{aligned}$$

and thus $(X \times Y, d_{X\times Y})$ satisfies Def. II.1.1.2(c). For any $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$, we have

$$\begin{aligned} & d_{X\times Y}\big((x_1,y_1),(x_2,y_2)\big) + d_{X\times Y}\big((x_2,y_2),(x_3,y_3)\big) \\ &= d_X(x_1,x_2) + d_Y(y_1,y_2) + d_X(x_2,x_3) + d_Y(y_2,y_3) \\ &\geq d_X(x_1,x_3) + d_Y(y_1,y_3) \qquad\qquad \text{(by Def. II.1.1.2(d))} \\ &= d_{X\times Y}\big((x_1,y_1),(x_3,y_3)\big) \end{aligned}$$

and thus $(X \times Y, d_{X\times Y})$ satisfies Def. II.1.1.2(d). By Def. II.1.1.2 we conclude that $(X \times Y, d_{X\times Y})$ is a metric space.

Next we propose an analogue of Prop. II.1.1.18 and proof it. Let (X, d_X), (Y, d_Y) be two metric spaces, let $(x, y) \in X \times Y$ and let $(x^{(n)}, y^{(n)})_{n=1}^{\infty}$ be a sequence in $X \times Y$. We claim that the follow two statements are equivalent:

- $\lim\limits_{n\to\infty} d_{X\times Y}\big((x^{(n)}, y^{(n)}), (x, y)\big) = 0$.
- $\lim\limits_{n\to\infty} d_X(x^{(n)}, x) = 0$ and $\lim\limits_{n\to\infty} d_Y(y^{(n)}, y) = 0$.

The claim is true since

$$\begin{aligned} & \lim_{n\to\infty} d_{X\times Y}\big((x^{(n)},y^{(n)}),(x,y)\big) = 0 \\ \Longleftrightarrow\ & \lim_{n\to\infty} \big(d_X(x^{(n)},x) + d_Y(y^{(n)},y)\big) = 0 \\ \Longleftrightarrow\ & \begin{cases} 0 \leq \lim\limits_{n\to\infty} d_X(x^{(n)},x) \leq \lim\limits_{n\to\infty} \big(d_X(x^{(n)},x) + d_Y(y^{(n)},y)\big) \leq 0 \\ 0 \leq \lim\limits_{n\to\infty} d_Y(y^{(n)},y) \leq \lim\limits_{n\to\infty} \big(d_X(x^{(n)},x) + d_Y(y^{(n)},y)\big) \leq 0 \end{cases} \end{aligned}$$

$$\iff \begin{cases} \lim\limits_{n\to\infty} d_X(x^{(n)}, x) = 0 \\ \lim\limits_{n\to\infty} d_Y(y^{(n)}, y) = 0 \end{cases}$$

Finally we propose an analogue of Lem. II.2.2.1 and proof it. Let (X, d_X), (Y_1, d_{Y_1}), (Y_2, d_{Y_2}) be metric spaces, let $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ be two functions, let $f_1 \oplus f_2 : X \to (Y_1 \times Y_2)$ and let $x_0 \in X$. We claim that the follow two statements are equivalent:

- f_1 is continuous at x_0 from (X, d_X) to (Y_1, d_{Y_1}) and f_2 is continuous at x_0 from (X, d_X) to (Y_2, d_{Y_2}).
- $f_1 \oplus f_2$ is continuous at x_0 from (X, d_X) to $(Y_1 \times Y_2, d_{Y_1\times Y_2})$.

The claim is true since by Def. II.2.1.1 we have

$$\begin{aligned}
&\begin{cases} f_1 \text{ is continuous at } x_0 \text{ from } (X, d_X) \text{ to } (Y_1, d_{Y_1}) \\ f_2 \text{ is continuous at } x_0 \text{ from } (X, d_X) \text{ to } (Y_2, d_{Y_2}) \end{cases} \\
\iff& \forall \varepsilon \in \mathbb{R}^+, \exists \delta_1, \delta_2 \in \mathbb{R}^+ : \\
&\begin{cases} \big(\forall x \in X, d_X(x, x_0) < \delta_1 \implies d_{Y_1}\big(f_1(x), f_1(x_0)\big) < \dfrac{\varepsilon}{2}\big) \\ \big(\forall x \in X, d_X(x, x_0) < \delta_2 \implies d_{Y_2}\big(f_2(x), f_2(x_0)\big) < \dfrac{\varepsilon}{2}\big) \end{cases} \\
\iff& \forall \varepsilon \in \mathbb{R}^+, \exists \delta = \min(\delta_1, \delta_2) \in \mathbb{R}^+ : \\
&\big(\forall x \in X, d_X(x, x_0) < \delta \implies d_{Y_1\times Y_2}\Big(\big(f_1(x), f_2(x)\big), \big(f_1(x_0), f_2(x_0)\big)\Big) < \varepsilon\big) \\
\iff& f_1 \oplus f_2 \text{ is continuous at } x_0 \text{ from } (X, d_X) \text{ to } (Y_1 \times Y_2, d_{Y_1\times Y_2}).
\end{aligned}$$

□

Ex. II.2.2.9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function from $\mathbb{R}^2$ to $\mathbb{R}$. Let (x_0, y_0) be a point in $\mathbb{R}^2$. If f is continuous at (x_0, y_0), show that

$$\lim_{x\to x_0} \limsup_{y\to y_0} f(x, y) = \lim_{y\to y_0} \limsup_{x\to x_0} f(x, y) = f(x_0, y_0)$$

and

$$\lim_{x\to x_0} \liminf_{y\to y_0} f(x, y) = \lim_{y\to y_0} \liminf_{x\to x_0} f(x, y) = f(x_0, y_0).$$

Recall that

$$\limsup_{x\to x_0} f(x) \coloneqq \inf_{r>0} \sup_{|x-x_0|<r} f(x)$$

$$\liminf_{x\to x_0} f(x) \coloneqq \sup_{r>0} \inf_{|x-x_0|<r} f(x)$$

In particular, we have

$$\lim_{x \to x_0} \lim_{y \to y_0} f(x,y) = \lim_{y \to y_0} \lim_{x \to x_0} f(x,y)$$

whenever the limits on both sides exist. (Note that the limits do not necessarily exist in general.) Discuss the comparison between this result and Example 1.2.7.

Proof. Let $d_1 = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$ and let $d_2 = d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2}$. Since f is continuous at (x_0, y_0) from $(\mathbb{R}^2, d_2)$ to $(\mathbb{R}, d_1)$, by Def. II.2.1.1 we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall (x,y) \in \mathbb{R}^2,\\
&d_2\big((x,y),(x_0,y_0)\big) < \delta \text{ implies } d_1\big(f(x,y), f(x_0,y_0)\big) < \varepsilon\\
\Longrightarrow\ &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall (x,y) \in \mathbb{R}^2,\\
&|x - x_0| + |y - y_0| < \delta \text{ implies } |f(x,y) - f(x_0,y_0)| < \varepsilon\\
\Longrightarrow\ &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall (x,y) \in \mathbb{R}^2,\\
&|x - x_0| + |y - y_0| < \delta \text{ implies}\\
&f(x_0,y_0) - \varepsilon < f(x,y) < f(x_0,y_0) + \varepsilon\\
\Longrightarrow\ &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in \mathbb{R}, |x - x_0| < \frac{\delta}{2} \text{ implies}\\
&f(x_0,y_0) - \varepsilon \leq \inf_{|y-y_0|<\frac{\delta}{2}} f(x,y) \leq \sup_{|y-y_0|<\frac{\delta}{2}} f(x,y) \leq f(x_0,y_0) + \varepsilon\\
\Longrightarrow\ &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in \mathbb{R}, |x - x_0| < \frac{\delta}{2} \text{ implies}\\
&\begin{cases}
\inf\left\{\sup\limits_{|y-y_0|<r} f(x,y) : r \in \mathbb{R}^+\right\} \leq \sup\limits_{|y-y_0|<\frac{\delta}{2}} f(x,y) \leq f(x_0,y_0) + \varepsilon\\[2ex]
f(x_0,y_0) - \varepsilon \leq \inf\limits_{|y-y_0|<\frac{\delta}{2}} f(x,y) \leq \sup\left\{\inf\limits_{|y-y_0|<r} f(x,y) : r \in \mathbb{R}^+\right\}
\end{cases}\\
\Longrightarrow\ &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in \mathbb{R}, |x - x_0| < \frac{\delta}{2} \text{ implies}\\
&\begin{cases}
\limsup\limits_{y \to y_0} f(x,y) \leq f(x_0,y_0) + \varepsilon\\
f(x_0,y_0) - \varepsilon \leq \liminf\limits_{y \to y_0} f(x,y)
\end{cases}\\
\Longrightarrow\ &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in \mathbb{R}, |x - x_0| < \frac{\delta}{2} \text{ implies}\\
&\begin{cases}
\left|\limsup\limits_{y \to y_0} f(x,y) - f(x_0,y_0)\right| = \limsup\limits_{y \to y_0} f(x,y) - f(x_0,y_0) \leq \varepsilon\\
\left|\liminf\limits_{y \to y_0} f(x,y) - f(x_0,y_0)\right| = f(x_0,y_0) - \liminf\limits_{y \to y_0} f(x,y) \leq \varepsilon
\end{cases}
\end{aligned}$$

$$\implies \begin{cases} \lim\limits_{x\to x_0}\big(\limsup\limits_{y\to y_0} f(x,y)\big) = f(x_0,y_0) \\ \lim\limits_{x\to x_0}\big(\liminf\limits_{y\to y_0} f(x,y)\big) = f(x_0,y_0) \end{cases}$$

Then we have

$$\begin{aligned} &\forall x\in X, \liminf_{y\to y_0} f(x,y) \le \lim_{y\to y_0} f(x,y) \le \limsup_{y\to y_0} f(x,y) \\ \implies & f(x_0,y_0) = \lim_{x\to x_0}\big(\liminf_{y\to y_0} f(x,y)\big) \\ &\qquad \le \lim_{x\to x_0}\big(\lim_{y\to y_0} f(x,y)\big) \le \lim_{x\to x_0}\big(\limsup_{y\to y_0} f(x,y)\big) = f(x_0,y_0) \\ \implies & \lim_{x\to x_0}\big(\lim_{y\to y_0} f(x,y)\big) = f(x_0,y_0). \end{aligned}$$

Using similar arguments, we can show that

$$\begin{aligned} &\lim_{y\to y_0}\big(\limsup_{x\to x_0} f(x,y)\big) = f(x_0,y_0) \\ &\lim_{y\to y_0}\big(\liminf_{x\to x_0} f(x,y)\big) = f(x_0,y_0) \\ &\lim_{y\to y_0}\big(\lim_{x\to x_0} f(x,y)\big) = f(x_0,y_0) \end{aligned}$$

and we conclude that

$$\lim_{x\to x_0}\big(\lim_{y\to y_0} f(x,y)\big) = \lim_{y\to y_0}\big(\lim_{x\to x_0} f(x,y)\big) = f(x_0,y_0).$$

□

Ex. II.2.2.10. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a continuous function. Show that for each $x\in\mathbb{R}$, the function $y\mapsto f(x,y)$ is continuous on $\mathbb{R}$, and for each $y\in\mathbb{R}$, the function $x\mapsto f(x,y)$ is continuous on $\mathbb{R}$. Thus, a function $f(x,y)$ which is jointly continuous in (x,y) is also continuous in each variable x,y separately.

Proof. Let $d_1 = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$ and let $d_2 = d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2}$. Let $x_0,y_0\in\mathbb{R}$. Since f is continuous from $(\mathbb{R}^2,d_2)$ to $(\mathbb{R},d_1)$, by Def. II.2.1.1 we know that f is continuous at (x_0,y_0) from $(\mathbb{R}^2,d_2)$ to $(\mathbb{R},d_1)$ and we have

$$\begin{aligned} &\forall\varepsilon\in\mathbb{R}^+,\exists\delta\in\mathbb{R}^+:\forall(x,y)\in\mathbb{R}^2, \\ &d_2\big((x,y),(x_0,y_0)\big)<\delta \text{ implies } d_1\big(f(x,y),f(x_0,y_0)\big)<\varepsilon \\ \implies &\forall\varepsilon\in\mathbb{R}^+,\exists\delta\in\mathbb{R}^+:\forall(x,y)\in\mathbb{R}^2, \\ &|x-x_0|+|y-y_0|<\delta \text{ implies } |f(x,y)-f(x_0,y_0)|<\varepsilon \\ \implies &\forall\varepsilon\in\mathbb{R}^+,\exists\delta\in\mathbb{R}^+: \\ &\begin{cases} \forall x\in\mathbb{R}, |x-x_0|<\delta \implies |f(x,y_0)-f(x_0,y_0)|<\varepsilon \\ \forall y\in\mathbb{R}, |y-y_0|<\delta \implies |f(x_0,y)-f(x_0,y_0)|<\varepsilon \end{cases} \end{aligned}$$

$$\implies \begin{cases} x \mapsto f(x, y_0) \text{ is continuous at } x_0 \text{ from } (\mathbb{R}, d_1) \text{ to } (\mathbb{R}, d_1) \\ y \mapsto f(x_0, y) \text{ is continuous at } y_0 \text{ from } (\mathbb{R}, d_1) \text{ to } (\mathbb{R}, d_1) \end{cases}$$

Since x_0, y_0 were arbitrary, we conclude thatt $x \mapsto f(x, y)$ is continuous from $(\mathbb{R}, d_1)$ to $(\mathbb{R}, d_1)$ and $y \mapsto f(x, y)$ is continuous from $(\mathbb{R}, d_1)$ to $(\mathbb{R}, d_1)$. $\square$

Ex. II.2.2.11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = \dfrac{xy}{x^2 + y^2}$ when $(x, y) \neq (0, 0)$, and $f(x, y) = 0$ otherwise. Show that for each fixed $x \in \mathbb{R}$, the function $y \mapsto f(x, y)$ is continuous on $\mathbb{R}$, and that for each fixed $y \in \mathbb{R}$, the function $x \mapsto f(x, y)$ is continuous on $\mathbb{R}$, but that the function $f : \mathbb{R}^2 \to \mathbb{R}$ is not continuous on $\mathbb{R}^2$. This shows that the converse to Ex. II.2.2.10 fails; it is possible to be continuous in each variable separately without being jointly continuous.

Proof. We have

$$\forall y \in \mathbb{R}, f(0, y) = 0$$

and thus by Lem. II.2.2.2 $y \mapsto f(0, y)$ is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. By Lem. II.2.2.2 and Ex. II.2.2.10 we also have

$$\begin{cases} (x, y) \mapsto xy \text{ is continuous from } (\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2}) \text{ to } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}); \\ (x, y) \mapsto x^2 + y^2 \text{ is continuous} \\ \text{from } \big((\mathbb{R} \setminus \{0\} \times \mathbb{R}), d_{l^1}|_{\left((\mathbb{R}\setminus\{0\})\times\mathbb{R}\right)\times\left((\mathbb{R}\setminus\{0\})\times\mathbb{R}\right)}\big) \text{ to } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}) \end{cases}$$
$$\implies (x, y) \mapsto \frac{xy}{x^2 + y^2} \text{ is continuous}$$
$$\text{from } \big((\mathbb{R} \setminus \{0\} \times \mathbb{R}), d_{l^1}|_{\left((\mathbb{R}\setminus\{0\})\times\mathbb{R}\right)\times\left((\mathbb{R}\setminus\{0\})\times\mathbb{R}\right)}\big) \text{ to } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$$
$$\implies \forall x \in \mathbb{R} \setminus \{0\}, y \mapsto \frac{xy}{x^2 + y^2} \text{ is continuous}$$
$$\text{from } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}) \text{ to } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}).$$

Thus, we conclude that $\forall x \in \mathbb{R}$, $y \mapsto f(x, y)$ is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Using similar arguments as above, we also have $\forall y \in \mathbb{R}$, $x \mapsto f(x, y)$ is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$.

Now we show that f is not continuous from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. In particular, we do this by showing f is not continuous at $(0, 0)$ from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Consider the sequence $(\frac{1}{n}, \frac{1}{n})_{n=1}^{\infty}$ in $\mathbb{R}^2$. We have

$$\lim_{n\to\infty} d_{l^1}|_{\mathbb{R},\mathbb{R}}(\frac{1}{n}, 0) = 0$$
$$\implies \lim_{n\to\infty} d_{l^1}|_{\mathbb{R}^2,\mathbb{R}^2}\big((\frac{1}{n}, \frac{1}{n}), (0, 0)\big) = 0 \qquad \text{(by Prop. II.1.1.18(b)(d))}$$

and

$$\forall n \in \mathbb{Z}^+, f(\frac{1}{n}, \frac{1}{n}) = \frac{\frac{1}{n^2}}{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{1}{2}$$
$$\implies \lim_{n\to\infty} f(\frac{1}{n}, \frac{1}{n}) = \frac{1}{2} \neq 0 = f(0,0).$$

Thus, by Thm. II.2.1.4(a)(b) f is not continuous at $(0,0)$ from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. □

Ex. II.2.2.12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x,y) := x^2/y$ when $y \neq 0$, and $f(x,y) := 0$ when $y = 0$. Show that $\lim_{t\to 0} f(tx,ty) = f(0,0)$ for every $(x,y) \in \mathbb{R}^2$, but that f is not continuous at the origin. Thus, being continuous on every line through the origin is not enough to guarantee continuity at the origin!

Proof. Let $(x_0, y_0) \in \mathbb{R}^2$. We split into two cases:

- If $y_0 = 0$, then we have

$$\forall t \in \mathbb{R}, f(tx_0, t0) = f(tx_0, 0) = 0 = f(0,0)$$

  and thus $t \mapsto f(tx_0, t0)$ is constant function and is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$.

- If $x_0 = 0$, then we have

$$\forall t \in \mathbb{R}, f(t0, ty_0) = f(0, ty_0) = \frac{0}{ty_0} = 0 = f(0,0)$$

  and thus $t \mapsto f(t0, ty_0)$ is constant function and is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$.

- If $x_0 \neq 0$ and $y_0 \neq 0$, then we have

$$\forall t \in \mathbb{R}, f(tx_0, ty_0) = \begin{cases} \frac{t^2x_0^2}{ty_0} = \frac{tx_0^2}{y_0} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \left(\forall t \in \mathbb{R}, |t-0| < \varepsilon\left|\frac{y_0}{x_0^2}\right| \implies \left|\frac{tx_0^2}{y_0} - 0\right| < \varepsilon\right)$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall t \in \mathbb{R}, |t-0| < \delta \implies |f(tx_0, ty_0) - f(0,0)| < \varepsilon\big)$$
$$\implies \lim_{t\to\infty} f(tx_0, ty_0) = 0 = f(0,0).$$

  Thus, $t \mapsto f(tx_0, ty_0)$ is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$.

From all cases above, we conclude that $t \mapsto f(tx_0, ty_0)$ is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Since (x_0, y_0) was arbitrary, we conclude that for any $(x, y) \in \mathbb{R}^2$, $t \mapsto f(tx, ty)$ is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$.

Now we show that f is not continuous at $(0, 0)$ from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Consider the sequence $(\frac{1}{n}, \frac{1}{n^2})_{n=1}^{\infty}$ in $\mathbb{R}^2$. Since

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^2} = 0 \\
\implies &\lim_{n\to\infty} d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2}\big((\frac{1}{n}, \frac{1}{n^2}), (0,0)\big) = 0 && \text{(by Prop. II.1.1.18(b)(d))}
\end{aligned}$$

and

$$\begin{aligned}
&\forall n \in \mathbb{Z}^+, f(\frac{1}{n}, \frac{1}{n^2}) = \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1 \\
\implies &\lim_{n\to\infty} f(\frac{1}{n}, \frac{1}{n^2}) = 1 \neq 0 = f(0,0),
\end{aligned}$$

by Thm. II.2.1.4(a)(b) we know that f is not continuous at $(0, 0)$ from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. $\square$

II.2.3 Continuity and compactness

Thm. II.2.3.1 (Continuous maps preserve compactness)**.** Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let $K \subseteq X$ be any compact subset of X. Then the image $f(K) \coloneqq \{f(x) : x \in K\}$ of K is also compact.

Proof. Let $\bigcup_{\alpha\in I} V_\alpha$ be an open cover of $f(K)$ in (Y, d_Y), i.e., $I \subseteq Y$ and for each $\alpha \in I$, V_α is an open set in (Y, d_Y) such that $f(K) \subseteq \bigcup_{\alpha\in I} V_\alpha$. Since

$$\begin{aligned}
&f \text{ is continuous from } (X, d_X) \text{ to } (Y, d_Y) \\
\implies &f|_K \text{ is continuous from } (K, d_X|_{K\times K}) \text{ to } (Y, d_Y) && \text{(by Rmk. II.2.1.3)} \\
\implies &\forall \alpha \in I, f|_K^{-1}(V_\alpha) \text{ is open in } (K, d_X|_{K\times K}) && \text{(by Thm. II.2.1.5(a)(d))} \\
\implies &K = \bigcup_{\alpha\in I} f|_K^{-1}(V_\alpha) \text{ is open in } (K, d_X|_{K\times K}) && \text{(by Prop. II.1.2.15(g))} \\
\implies &\exists F \subseteq I : (F \text{ is finite}) \wedge \left(K = \bigcup_{\alpha\in F} f|_K^{-1}(V_\alpha)\right) && \text{(by Thm. II.1.5.8)} \\
\implies &\exists F \subseteq I : (F \text{ is finite}) \wedge \left(f(K) = f\big(\bigcup_{\alpha\in F} f|_K^{-1}(V_\alpha)\big)\right)
\end{aligned}$$

$$\implies \exists F \subseteq I : (F \text{ is finite}) \wedge \left(f(K) \subseteq \bigcup_{\alpha \in F} V_\alpha) \right),$$

we know that there exists an finite subcover of $f(K)$ with respect to $\bigcup_{\alpha \in I} V_\alpha$ in (Y, d_Y). Since I was arbitrary open cover of $f(K)$ in (Y, d_Y), by Ex. II.1.5.11 we know that $\big(f(K), d_Y|_{Y\times Y}\big)$ is compact. $\square$

Prop. II.2.3.2 (Maximum principle)**.** Let (X, d) be a compact metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Then f is bounded. Furthermore, if X is non-empty, f attains its maximum at some point $x_{\max} \in X$, and also attains its minimum at some point $x_{\min} \in X$.

Proof. We have

$$\begin{aligned}
&\begin{cases} (X, d) \text{ is compact} \\ f \text{ is continuous from } (X, d) \text{ to } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}) \end{cases} \\
\implies & \big(f(X), d_{l^1}|_{f(X)\times f(X)}\big) \text{ is compact} && \text{(by Thm. II.2.3.1)} \\
\implies & \begin{cases} f(X) \text{ is closed in } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}) \\ \big(f(X), d_{l^1}|_{f(X)\times f(X)}\big) \text{ is bounded} \end{cases} && \text{(by Cor. II.1.5.6)} \\
\implies & \begin{cases} f(X) \text{ is closed in } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}}) \\ f(X) \text{ is bounded subset of } \mathbb{R} \end{cases} && \text{(by Ex. II.1.5.1)}
\end{aligned}$$

Now we show that if $X \neq \emptyset$, then

$$\exists x_{\min}, x_{\max} \in X : \forall x \in X, f(x_{\min}) \leq f(x) \leq f(x_{\max}).$$

Let $U = \sup\big(f(X)\big)$ and let $L = \inf\big(f(X)\big)$. Since f is bounded subset of $\mathbb{R}$, we know that $U, L \in \mathbb{R}$. By the definition of U and L we know that

$$\forall n \in \mathbb{Z}^+, \exists u_n, l_n \in f(X) : \begin{cases} U - u_n < \dfrac{1}{n} \\ l_n - L < \dfrac{1}{n} \end{cases}$$

Thus, we have

$$\begin{aligned}
&\begin{cases} 0 = \lim\limits_{n\to\infty} U - u_n = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0 \\ 0 = \lim\limits_{n\to\infty} l_n - L = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0 \end{cases} \\
\implies &\begin{cases} \lim\limits_{n\to\infty} u_n = U \\ \lim\limits_{n\to\infty} l_n = L \end{cases}
\end{aligned}$$

Since $f(X)$ is closed in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ and $(u_n)_{n=1}^\infty$, $(l_n)_{n=1}^\infty$ are convergent sequences in $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$, by Prop. II.1.2.15(b) we know that $U, L \in f(X)$. Since $U, L \in f(X)$, we know that

$$\exists x_{\min}, x_{\max} \in X : \big(f(x_{\min}) = L\big) \wedge \big(f(x_{\max}) = U\big).$$

□

Rmk. II.2.3.3. As was already noted in Exercise 9.6.1 in Analysis I, this principle can fail if X is not compact. Prop. II.2.3.2 should be compared with Lemma 9.6.3 in Analysis I and Proposition 9.6.7 in Analysis I.

Def. II.2.3.4 (Uniform continuity)**.** Let $f : X \to Y$ be a map from one metric space (X, d_X) to another (Y, d_Y). We say that f is *uniformly continuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y\big(f(x), f(x')\big) < \varepsilon$ whenever $x, x' \in X$ are such that $d_X(x, x') < \delta$.

Note. Every uniformly continuous function is continuous, but not conversely. But if the domain X is compact, then the two notions are equivalent.

Thm. II.2.3.5. Let (X, d_X) and (Y, d_Y) be metric spaces, and suppose that (X, d_X) is compact. If $f : X \to Y$ is function, then f is continuous iff it is uniformly continuous.

Proof. If f is uniformly continuous then it is also continuous by Ex. II.2.3.3. Now suppose that f is continuous. Fix $\varepsilon > 0$. For every $x_0 \in X$, the function f is continuous at x_0 from (X, d_X) to (Y, d_Y). Thus, there exists a $\delta(x_0) > 0$, depending on x_0, such that $d_Y\big(f(x), f(x_0)\big) < \varepsilon/2$ whenever $d_X(x, x_0) < \delta(x_0)$. In particular, by the triangle inequality this implies that $d_Y(f(x), f(x')) < \varepsilon$ whenever $x \in B_{(X,d_X)}\big(x_0, \delta(x_0)/2\big)$ and $d_X(x', x) < \delta(x_0)/2$.

Now consider the (possibly infinite) collection of balls

$$\big\{B_{(X,d_X)}\big(x_0, \delta(x_0)/2\big) : x_0 \in X\big\}.$$

Each ball in this collection is of course open in (X, d_X), and the union of all these balls covers X, since each point x_0 in X is contained in its own ball $B_{(X,d_X)}\big(x_0, \delta(x_0)/2\big)$. Hence, by Thm. II.1.5.8, there exist a finite number of points $x_1, \ldots, x_n$ such that the balls $B_{(X,d_X)}\big(x_j, \delta(x_j)/2\big)$ for $j = 1, \ldots, n$ cover X:

$$X \subseteq \bigcup_{j=1}^{n} B_{(X,d_X)}\big(x_j, \delta(x_j)/2\big).$$

Now let $\delta := \min_{j=1}^{n} \delta(x_j)/2$. Since each of the $\delta(x_j)$ are positive, and there are only a finite number of j, we see that $\delta > 0$. Now let x, x' be any two points in X such that $d_X(x, x') < \delta$. Since the balls $B_{(X,d_X)}\big(x_j, \delta(x_j)/2\big)$ cover X, we see that there must exist $1 \le j \le n$ such that $x \in B_{(X,d_X)}\big(x_j, \delta(x_j)/2\big)$. Since $d_X(x, x') < \delta$, we have $d_X(x, x') < \delta(x_j)/2$, and so by the previous discussion we have $d_Y\big(f(x), f(x')\big) < \varepsilon$. We have thus found a δ such that $d_Y\big(f(x), f(x')\big) < \varepsilon$ whenever $d(x, x') < \delta$, and this proves uniform continuity as desired. □

— Exercises —

Ex. II.2.3.1. Prove Thm. II.2.3.1.

Proof. See Thm. II.2.3.1. □

Ex. II.2.3.2. Prove Prop. II.2.3.2.

Proof. See Prop. II.2.3.2. □

Ex. II.2.3.3. Show that every uniformly continuous function is continuous, but give an example that shows that not every continuous function is uniformly continuous.

Proof. Let (X, d_X), (Y, d_Y) be two metric spaces and let $f : X \to Y$ be a function which is uniformly continuous from (X, d_X) to (Y, d_Y). Then we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x, x_0 \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon\Big) && \text{(by Def. II.2.3.4)} \\
\implies &\forall x_0 \in X, f \text{ is continuous at } x_0 \text{ from } (X, d_X) \text{ to } (Y, d_Y) && \text{(by Def. II.2.1.1)} \\
\implies &f \text{ is continuous from } (X, d_X) \text{ to } (Y, d_Y). && \text{(by Def. II.2.1.1)}
\end{aligned}$$

For the converse example, see Example 9.9.11 in Analysis I. □

Ex. II.2.3.4. Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces, and let $f : X \to Y$ and $g : Y \to Z$ be two uniformly continuous functions. Show that $g \circ f : X \to Z$ is also uniformly continuous.

Proof. Since f is uniformly continuous from (X, d_X) to (Y, d_Y) and g is uniformly continuous from (Y, d_Y) to (Z, d_Z), by Def. II.2.3.4 we have

$$\begin{aligned}
&\begin{cases}
\forall \delta' \in \mathbb{R}^+, \exists \delta, \in \mathbb{R}^+ : \\
\Big(\forall x_1, x_2 \in X, d_X(x_1, x_2) < \delta \implies d_Y\big(f(x_1), f(x_2)\big) < \delta'\Big); \\
\forall \varepsilon \in \mathbb{R}^+, \exists \delta' \in \mathbb{R}^+ : \\
\Big(\forall y_1, y_2 \in Y, d_Y(y_1, y_2) < \delta' \implies d_Z\big(g(y_1), g(y_2)\big) < \varepsilon\Big);
\end{cases} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
&\bigg(\forall x_1, x_2 \in X, d_X(x_1, x_2) < \delta \implies d_Z\Big(g\big(f(x_1)\big), g\big(f(x_2)\big)\Big) < \varepsilon\bigg).
\end{aligned}$$

Thus, by Def. II.2.3.4 $g \circ f$ is uniformly continuous from (X, d_X) to (Z, d_Z). □

Ex. II.2.3.5. Let (X, d_X) be a metric space, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be uniformly continuous functions. Show that the direct sum $f \oplus g : X \to \mathbb{R}^2$ defined by $f \oplus g(x) := \big(f(x), g(x)\big)$ is uniformly continuous.

Proof. Let $d_1 = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$ and let $d_2 = d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2}$. Since f, g are uniformly continuous from (X, d_X) to $(\mathbb{R}, d_1)$, by Def. II.2.3.4 we have

$$\begin{aligned}
&\forall\varepsilon \in \mathbb{R}^+, \exists\delta \in \mathbb{R}^+ :\\
&\begin{cases}\forall x_1, x_2 \in X, d_X(x_1, x_2) < \delta \implies d_1\big(f(x_1), f(x_2)\big) < \dfrac{\varepsilon}{2}\\ \forall x_1, x_2 \in X, d_X(x_1, x_2) < \delta \implies d_1\big(g(x_1), g(x_2)\big) < \dfrac{\varepsilon}{2}\end{cases}\\
\implies &\forall\varepsilon \in \mathbb{R}^+, \exists\delta \in \mathbb{R}^+ :\\
&\Big(\forall x_1, x_2 \in X, d_X(x_1, x_2) < \delta \implies d_1\big(f(x_1), f(x_2)\big) + d_1\big(g(x_1), g(x_2)\big) < \varepsilon\Big)\\
\implies &\forall\varepsilon \in \mathbb{R}^+, \exists\delta \in \mathbb{R}^+ :\\
&\bigg(\forall x_1, x_2 \in X, d_X(x_1, x_2) < \delta \implies d_2\Big(\big(f(x_1), g(x_1)\big), \big(f(x_2), g(x_2)\big)\Big) < \varepsilon\bigg).
\end{aligned}$$

Thus, by Def. II.2.3.4 $f \oplus g$ is uniformly continuous from (X, d_X) to $(\mathbb{R}^2, d_2)$. □

Ex. II.2.3.6. Show that the addition function $(x, y) \mapsto x + y$ and the subtraction function $(x, y) \mapsto x - y$ are uniformly continuous from $\mathbb{R}^2$ to $\mathbb{R}$, but the multiplication function $(x, y) \mapsto xy$ is not. Conclude that if $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are uniformly continuous functions on a metric space (X, d), then $f + g : X \to \mathbb{R}$ and $f - g : X \to \mathbb{R}$ are also uniformly continuous. Give an example to show that $fg : X \to \mathbb{R}$ need not be uniformly continuous. What is the situation for $\max(f, g)$, $\min(f, g)$, f/g, and cf for a real number c?

Proof. Let $d_1 = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$ and let $d_2 = d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2}$. We first show that $(x, y) \mapsto x + y$ and $(x, y) \mapsto x - y$ are uniformly continuous from $(\mathbb{R}^2, d_2)$ to $(\mathbb{R}, d_1)$. Since

$$\begin{aligned}
&\forall\varepsilon \in \mathbb{R}^+, \forall(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2,\\
&\big(|x_1 - x_2| < \frac{\varepsilon}{2}\big) \wedge \big(|y_1 - y_2| < \frac{\varepsilon}{2}\big)\\
\implies &\begin{cases}|x_1 - x_2 + y_1 - y_2| \le |x_1 - x_2| + |y_1 - y_2| < \varepsilon\\ |x_1 - x_2 - y_1 + y_2| \le |x_1 - x_2| + |y_2 - y_1| < \varepsilon\end{cases}\\
\implies &\begin{cases}d_1(x_1 + y_1, x_2 + y_2) \le d_2\big((x_1, y_1), (x_2, y_2)\big) < \varepsilon\\ d_1(x_1 - y_1, x_2 - y_2) \le d_2\big((x_1, y_1), (x_2, y_2)\big) < \varepsilon\end{cases}
\end{aligned}$$

by choosing $\delta = \varepsilon$ we have

$$\begin{aligned}
&\forall\varepsilon \in \mathbb{R}^+, \exists\delta \in \mathbb{R}^+ : \forall(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2,\\
&d_2\big((x_1, y_1), (x_2, y_2)\big) < \delta \implies \begin{cases}d_1(x_1 + y_1, x_2 + y_2) < \varepsilon\\ d_1(x_1 - y_1, x_2 - y_2) < \varepsilon\end{cases}
\end{aligned}$$

and thus by Def. II.2.3.4 $(x, y) \mapsto x + y$ and $(x, y) \mapsto x - y$ are uniformly continuous from $(\mathbb{R}^2, d_2)$ to $(\mathbb{R}, d_1)$.

Next we show that $(x,y) \mapsto xy$ is not uniformly continuous from $(\mathbb{R}^2, d_2)$ to $(\mathbb{R}, d_1)$. Since

$$\forall n \in \mathbb{Z}^+, \begin{cases} d_2\big((n,n),(n+\dfrac{1}{n},n)\big) = \dfrac{1}{n} \\ d_1\big(n \times n, (n+\dfrac{1}{n}) \times n\big) = 1 \end{cases}$$

no matter which $\delta \in \mathbb{R}^+$ we choose, we cannot make $d_1(n^2, n^2+1) < \dfrac{1}{2}$ for every $n \in \mathbb{Z}^+$. Thus, by Def. II.2.3.4 $(x,y) \mapsto xy$ is not uniformly continuous from $(\mathbb{R}^2, d_2)$ to $(\mathbb{R}, d_1)$.

Next we show that $(x,y) \mapsto \max(x,y)$ and $(x,y) \mapsto \min(x,y)$ are uniformly continuous from (X,d) to $(\mathbb{R}, d_1)$. Since

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \forall (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2, \\ &\big(|x_1 - x_2| < \varepsilon\big) \wedge \big(|y_2 - y_1| < \varepsilon\big) \\ \implies &\begin{cases} x_2 - \varepsilon < x_1 < x_2 + \varepsilon \\ y_2 - \varepsilon < y_1 < y_2 + \varepsilon \end{cases} \\ \implies &\begin{cases} \max(x_2,y_2) - \varepsilon < \max(x_1,y_1) < \max(x_2,y_2) + \varepsilon \\ \min(x_2,y_2) - \varepsilon < \min(x_1,y_1) < \min(x_2,y_2) + \varepsilon \end{cases} \\ \implies &\begin{cases} |\max(x_1,y_1) - \max(x_2,y_2)| < \varepsilon \\ |\min(x_1,y_1) - \min(x_2,y_2)| < \varepsilon \end{cases} \end{aligned}$$

by choosing $\delta = \varepsilon$ we have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\ &\forall (x_1,y_1),(x_2,y_2) \in \mathbb{R}^2, d_2\big((x_1,y_1),(x_2,y_2)\big) < \delta \\ &\quad \implies \begin{cases} d_1\big(\max(x_1,y_1), \max(x_2,y_2)\big) < \varepsilon \\ d_1\big(\min(x_1,y_1), \min(x_2,y_2)\big) < \varepsilon \end{cases} \end{aligned}$$

and thus by Def. II.2.3.4 $(x,y) \mapsto \max(x,y)$ and $(x,y) \mapsto \min(x,y)$ are uniformly continuous from $(\mathbb{R}^2, d_2)$ to $(\mathbb{R}, d_1)$.

Next we show that $f+g$, $f-g$, $\max(f,g)$, $\min(f,g)$ are uniformly continuous from (X,d) to $(\mathbb{R}, d_1)$ given f,g are uniformly continuous from (X,d) to $(\mathbb{R}, d_1)$.

$$\begin{aligned} &f,g \text{ are uniformly continuous from } (X,d) \text{ to } (\mathbb{R}, d_1) \\ \implies &f \oplus g \text{ is uniformly continuous from } (X,d) \text{ to } (\mathbb{R}^2, d_2) && \text{(by Ex. II.2.3.5)} \end{aligned}$$

$$\implies \begin{cases} f \oplus g(x) \mapsto f(x) + g(x) \text{ is uniformly continuous} \\ \text{from } (X,d) \text{ to } (\mathbb{R}, d_1); \\ f \oplus g(x) \mapsto f(x) - g(x) \text{ is uniformly continuous} \\ \text{from } (X,d) \text{ to } (\mathbb{R}, d_1); \\ f \oplus g(x) \mapsto \max\big(f(x), g(x)\big) \text{ is uniformly continuous} \\ \text{from } (X,d) \text{ to } (\mathbb{R}, d_1); \\ f \oplus g(x) \mapsto \min\big(f(x), g(x)\big) \text{ is uniformly continuous} \\ \text{from } (X,d) \text{ to } (\mathbb{R}, d_1). \end{cases} \qquad \text{(by Ex. II.2.3.4)}$$
$$\implies f+g, f-g, \max(f,g), \min(f,g) \text{ are uniformly continuous from } (X,d) \text{ to } (\mathbb{R}, d_1).$$

Next we give a example where fg is not continuous from (X, d) to $(\mathbb{R}, d_1)$ given f, g are uniformly continuous from (X, d) to $(\mathbb{R}, d_1)$. Let $f(x) = g(x) = x$. Then $fg(x) = x^2$ and by Example 9.9.11 in Analysis I we know that x^2 is not uniformly continuous from $(\mathbb{R}, d_1)$ to $(\mathbb{R}, d_1)$.

Next we give a example where f/g is not continuous from (X, d) to $(\mathbb{R}, d_1)$ given f, g are uniformly continuous from (X, d) to $(\mathbb{R}, d_1)$. Let $f(x) = 1$ and let $g(x) = x$. Then $f/g(x) = 1/x$ and by Example 9.9.10 in Analysis I we know that $1/x$ is not uniformly continuous from $(\mathbb{R}, d_1)$ to $(\mathbb{R}, d_1)$.

Finally we show that for each $c \in \mathbb{R}$, cf is uniformly continuous from (X, d) to $(\mathbb{R}, d_1)$ given f is uniformly continuous from (X, d) to $(\mathbb{R}, d_1)$. If $c = 0$, then f is a constant function. Thus

$$\forall \varepsilon \in \mathbb{R}^+, \forall x \in X, d(x_1, x_2) < \varepsilon \implies d_1\big(f(x_1), f(x_2)\big) = 0 < \varepsilon$$

and by Def. II.2.3.4 f is uniformly continuous from (X, d) to $(\mathbb{R}, d_1)$. Suppose that $c \neq 0$. Since f is uniformly continuous from (X, d) to $(\mathbb{R}, d_1)$, by Def. II.2.3.4 we have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\ &\left(\forall x_1, x_2 \in X, d(x_1, x_2) < \delta \implies d_1\big(f(x_1), f(x_2)\big) < \frac{\varepsilon}{|c|}\right) \\ \implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\ &\Big(\forall x_1, x_2 \in X, d(x_1, x_2) < \delta \implies d_1\big(cf(x_1), cf(x_2)\big) < \varepsilon\Big) \end{aligned}$$

and thus by Def. II.2.3.4 cf is uniformly continuous from (X, d) to $(\mathbb{R}, d_1)$. □

II.2.4 Continuity and connectedness

Def. II.2.4.1 (Connected spaces)**.** Let (X, d) be a metric space. We say that X is *disconnected* iff there exist disjoint non-empty open sets V and W in X such that $V \cup W = X$. (Equivalently, X is disconnected iff X contains a non-empty proper subset which is simultaneously closed and open, see Prop. II.1.2.15(e).) We say that X is *connected* iff it is non-empty and not disconnected.

Note. We declare the empty set $\emptyset$ as being special - it is neither connected nor disconnected; one could think of the empty set as "unconnected."

Def. II.2.4.3 (Connected sets)**.** Let (X, d) be a metric space, and let Y be a subset of X. We say that Y is *connected* iff the metric space $(Y, d|_{Y\times Y})$ is connected, and we say that Y is *disconnected* iff the metric space $(Y, d|_{Y\times Y})$ is disconnected.

Rmk. II.2.4.4. This definition is intrinsic; whether a set Y is connected or not depends only on what the metric is doing on Y, but not on what ambient space X one placing Y in.

Thm. II.2.4.5. Let X be a non-empty subset of the real line $\mathbb{R}$. Then the following statements are equivalent.

(a) X is connected.

(b) Whenever $x, y \in X$ and $x < y$, the interval $[x, y]$ is also contained in X.

(c) X is an interval (in the sense of Definition 9.1.1 in Analysis I).

Proof. First, we show that (a) implies (b). Let $d = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Suppose that (X, d) is connected, and suppose for the sake of contradiction that we could find points $x < y$ in X such that $[x, y]$ is not contained in X. Then there exists a real number $x < z < y$ such that $z \notin X$. Thus, the sets $(-\infty, z) \cap X$ and $(z, \infty) \cap X$ will cover X. But these sets are non-empty (because they contain x and y respectively) and are open relative to (X, d), and so X is disconnected, a contradiction.

Now we show that (b) implies (a). Let X be a set obeying the property (b). Suppose for the sake of contradiction that X is disconnected. Then there exist disjoint non-empty sets V, W which are open relative to X, such that $V \cup W = X$. Since V and W are non-empty, we may choose an $x \in V$ and $y \in W$. Since V and W are disjoint, we have $x \neq y$; without loss of generality we may assume $x < y$. By property (b), we know that the entire interval $[x, y]$ is contained in X.

Now consider the set $[x, y] \cap V$. This set is both bounded and non-empty (because it contains x). Thus, it has a supremum

$$z := \sup([x, y] \cap V).$$

Clearly, $z \in [x, y]$, and hence $z \in X$. Thus, either $z \in V$ or $z \in W$. Suppose first that $z \in V$. Then $z \neq y$ (since $y \in W$ and V is disjoint from W). But V is open relative to X, which contains $[x, y]$, so there is some ball $B_{\left([x,y],d\right)}(z, r) = (z - r, z + r)$ which is contained in V. But this contradicts the fact that z is the supremum of $[x, y] \cap V$. Now suppose that $z \in W$. Then $z \neq x$ (since $x \in V$ and V is disjoint from W). But W is open relative to X, which contains $[x, y]$, so there is some ball $B_{\left([x,y],d\right)}(z, r) = (z - r, z + r)$ which is contained in W.

But this again contradicts the fact that z is the supremum of $[x, y] \cap V$. Thus, in either case we obtain a contradiction, which means that X cannot be disconnected, and must therefore be connected.

Next we show that (b) implies (c). Suppose that

$$\forall x, y \in X, x < y \implies [x, y] \subseteq X.$$

Suppose for the sake of contradiction that X is not an interval. Then we would have

$$\exists x, y \in X : x < y \implies \exists z \in \mathbb{R} \setminus X : x < z < y.$$

Clearly, this contradicts to hypothesis, thus X is an interval.

Finally we show that (c) implies (b). Suppose that X is an interval. Then we have

$$\begin{aligned}
&\forall x, y \in X, x < y \\
\implies &\inf(X) \leq x < y \leq \sup(X) \\
\implies &\big(\forall z \in \mathbb{R}, x \leq z \leq y \implies \inf(X) \leq z \leq \sup(X)\big) \\
\implies &\big(\forall z \in \mathbb{R}, x \leq z \leq y \implies z \in X\big) \\
\implies &[x, y] \subseteq X.
\end{aligned}$$

□

Thm. II.2.4.6. tinuity preserves connectedness] Let f : $X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let E be any connected subset of X. Then $f(E)$ is also connected.

Proof. Suppose for the sake of contradiction that $\big(f(E), d_Y|_{f(E)\times f(E)}\big)$ is disconnected. Then there exists two open set V_1, V_2 in $\big(f(E), d_Y|_{f(E)\times f(E)}\big)$ such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = f(E)$. But then we have

$$\begin{aligned}
&f \text{ is continuous from } (X, d_X) \text{ to } (Y, d_Y) \\
\implies &f^{-1}(V_1), f^{-1}(V_2) \text{ are open in } (E, d_X|_{E\times E}) && \text{(by Thm. II.2.1.5(a)(c))} \\
\implies &\big(f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset\big) \wedge \big(f^{-1}(V_1) \cup f^{-1}(V_2) = E\big) \\
\implies &(E, d_X|_{E\times E}) \text{ is disconnected,} && \text{(by Def. II.2.4.1)}
\end{aligned}$$

a contradiction. Thus, $\big(f(E), d_Y|_{f(E)\times f(E)}\big)$ is connected. □

Cor. II.2.4.7 (Intermediate value theorem)**.** Let $f : X \to \mathbb{R}$ be a continuous map from one metric space (X, d_X) to the real line. Let E be any connected subset of X, and let a, b be any two elements of E. Let y be a real number between $f(a)$ and $f(b)$, i.e., either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists $c \in E$ such that $f(c) = y$.

Proof. Since f is continuous from (X, d_X) to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ and $(E, d_X|_{E\times E})$ is connected, by Thm. II.2.4.6 we know that $\big(f(E), d_{l^1}|_{f(E)\times f(E)}\big)$ is connected. By Thm. II.2.4.5(a)(c) we know that $f(E)$ is an interval. Thus, we have

$$\begin{aligned}
&\forall a, b \in E, \forall y \in \Big[\min\big(f(a), f(b)\big), \max\big(f(a), f(b)\big)\Big] \\
\Longrightarrow\ & y \in f(E) \\
\Longrightarrow\ & \exists c \in X : f(c) = y.
\end{aligned}$$

□

— Exercises —

Ex. II.2.4.1. Let (X, d_{disc}) be a metric space with the discrete metric. Let E be a subset of X which contains at least two elements. Show that E is disconnected.

Proof. Let $x, y \in E$ such that $x \neq y$, let $V = \{x\}$ and let $W = E \setminus V$. Since $x \in V$ and $y \in W$, we know that $V \neq \emptyset \neq W$. By E.g. II.1.2.8 we know that V, W are open in $(E, d_{\text{disc}}|_{E\times E})$. Since $V \neq \emptyset \neq W$, $V \cup W = E$ and $V \cap W = \emptyset$, by Def. II.2.4.3 we know that $(E, d_{\text{disc}}|_{E\times E})$ is disconnected. □

Ex. II.2.4.2. Let $f : X \to Y$ be a function from a connected metric space (X, d) to a metric space (Y, d_{disc}) with the discrete metric. Show that f is continuous iff it is constant.

Proof. We first show that if f is continuous from (X, d) to (Y, d_{disc}), then f is a constant function. Suppose for the sake of contradiction that f is not a constant function. Then we have

$$\exists x_1, x_2 \in X : (x_1 \neq x_2) \wedge \big(f(x_1) \neq f(x_2)\big)$$

and by Ex. II.2.4.1 we know that $\big(f(E), d_{\text{disc}}|_{E\times E}\big)$ is disconnected. But (X, d) is connected, thus by Thm. II.2.4.6 we know that $\big(f(E), d_{\text{disc}}|_{E\times E}\big)$ is connected, a contradiction. Thus, f is a constant function.

Now we show that if f is a constant function, then f is continuous from (X, d) to (Y, d_{disc}). We have

$$\begin{aligned}
&\forall x_1, x_2 \in X, f(x_1) = f(x_2) \\
\Longrightarrow\ & \forall \varepsilon \in \mathbb{R}^+, \forall x_1, x_2 \in X, d_{\text{disc}}\big(f(x_1), f(x_2)\big) = 0 < \varepsilon \\
\Longrightarrow\ & f \text{ is uniformly continuous from } (X, d) \text{ to } (Y, d_{\text{disc}}) && \text{(by Def. II.2.3.4)} \\
\Longrightarrow\ & f \text{ is continuous from } (X, d) \text{ to } (Y, d_{\text{disc}}). && \text{(by Ex. II.2.3.3)}
\end{aligned}$$

□

Ex. II.2.4.3. Prove the equivalence of statements (b) and (c) in Thm. II.2.4.5.

Proof. See Thm. II.2.4.5. □

Ex. II.2.4.4. Prove Thm. II.2.4.6.

Proof. See Thm. II.2.4.6. □

Ex. II.2.4.5. Use Thm. II.2.4.6 to prove Cor. II.2.4.7.

Proof. See Cor. II.2.4.7. □

Ex. II.2.4.6. Let (X, d) be a metric space, and let $(E_\alpha)_{\alpha \in I}$ be a collection of connected sets in X. Suppose also that $\bigcap_{\alpha \in I} E_\alpha$ is non-empty. Show that $\bigcup_{\alpha \in I} E_\alpha$ is connected.

Proof. Let

$$d_E = d|_{(\bigcup_{\alpha \in I} V_\alpha) \times (\bigcup_{\alpha \in I} V_\alpha)}.$$

Suppose for the sake of contradiction that $(\bigcup_{\alpha \in I} E_\alpha, d_E)$ is disconnected. By Def. II.2.4.1 we know that

$$\exists V, W \subseteq \bigcup_{\alpha \in I} E_\alpha : \begin{cases} V, W \text{ are open in } \left(\bigcup_{\alpha \in I} E_\alpha, d_E \right) \\ V \neq \emptyset \neq W \\ V \cap W = \emptyset \\ V \cup W = \bigcup_{\alpha \in I} E_\alpha \end{cases}$$

Since $\bigcap_{\alpha \in I} E_\alpha \neq \emptyset$, we know that $V_\alpha \neq \emptyset$ for each $\alpha \in I$. We claim that there exists some $\beta \in I$ such that $V \cap E_\beta \neq \emptyset$ and $W \cap E_\beta \neq \emptyset$. If not, then we would have

$$\begin{aligned} &(\forall \alpha \in I, V \cap E_\alpha = \emptyset) \vee (\forall \alpha \in I, W \cap E_\alpha = \emptyset) \\ \implies &(V = \emptyset) \vee (W = \emptyset), \end{aligned}$$

which contradicts to $V \neq \emptyset \neq W$. Now let $\beta \in I$ such that $V \cap E_\beta \neq \emptyset$ and $W \cap E_\beta \neq \emptyset$. But then we have

$$\begin{aligned} &E_\beta \subseteq \bigcup_{\alpha \in I} E_\alpha = V \cup W \\ \implies &\begin{cases} V \cap E_\beta \text{ is open in } (E_\beta, d_E|_{E_\beta \times E_\beta}) \\ W \cap E_\beta \text{ is open in } (E_\beta, d_E|_{E_\beta \times E_\beta}) \\ V \cap E_\beta \neq \emptyset \neq W \cap E_\beta \\ (V \cap E_\beta) \cap (W \cap E_\beta) = \emptyset \\ (V \cap E_\beta) \cup (W \cap E_\beta) = E_\beta \end{cases} && \text{(by Prop. II.1.3.4(a))} \\ \implies &(E_\beta, d_E|_{E_\beta \times E_\beta}) \text{ is disconnected,} && \text{(by Def. II.2.4.1)} \end{aligned}$$

which contradict to the hypothesis that $(E_\beta, d_E|_{E_\beta \times E_\beta})$ is connected. Thus, we know that $(\bigcup_{\alpha \in I} E_\alpha, d_E)$ is connected. $\square$

Ex. II.2.4.7. Let (X, d) be a metric space, and let E be a subset of X. We say that E is *path-connected* iff, for every $x, y \in E$, there exists a continuous function $\gamma : [0,1] \to E$ from the unit interval $[0,1]$ to E such that $\gamma(0) = x$ and $\gamma(1) = y$. Show that every non-empty path-connected set is connected. (The converse is false, but is a bit tricky to show and will not be detailed here.)

Proof. Suppose for the sake of contradiction that $(E, d|_{E\times E})$ is disconnected. Then by Def. II.2.4.1 we have

$$\exists V, W \subseteq E : \begin{cases} V, W \text{ are open in } (E, d|_{E\times E}) \\ V \neq \emptyset \neq W \\ V \cap W = \emptyset \\ V \cup W = E \end{cases}$$

Let $x \in V$ and let $y \in W$. Since $(E, d|_{E\times E})$ is path-connected and $x, y \in E$, by definition we know that there exists a function $\gamma : [0,1] \to E$ which is continuous from $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ to $(E, d|_{E\times E})$ and

$$\begin{cases} \gamma(0) = x; \\ \gamma(1) = y; \\ x \in \gamma([0,1]) \cap V; \\ y \in \gamma([0,1]) \cap W. \end{cases}$$

But then we have

$$\begin{aligned} & ([0,1], d_{l^1}|_{\mathbb{R}\times\mathbb{R}}) \text{ is connected} && \text{(by Thm. II.2.4.5(a)(c))} \\ \implies & \Big(\gamma([0,1]), d|_{\gamma([0,1])\times\gamma([0,1])}\Big) \text{ is connected} && \text{(by Thm. II.2.4.6)} \end{aligned}$$

and

$$\begin{aligned} & \begin{cases} \gamma([0,1]) \cap V \text{ is open in } \big(r([0,1]), d|_{\gamma([0,1])\times\gamma([0,1])}\big) \\ \gamma([0,1]) \cap W \text{ is open in } \big(r([0,1]), d|_{\gamma([0,1])\times\gamma([0,1])}\big) \\ \gamma([0,1]) \cap V \neq \emptyset \neq \gamma([0,1]) \cap W \\ \big(\gamma([0,1]) \cap V\big) \cap \big(\gamma([0,1]) \cap W\big) = \emptyset \\ \big(\gamma([0,1]) \cap V\big) \cup \big(\gamma([0,1]) \cap W\big) = \gamma([0,1]) \end{cases} && \text{(by Prop. II.1.3.4(a))} \\ \implies & \big(r([0,1]), d|_{\gamma([0,1])\times\gamma([0,1])}\big) \text{ is disconnected,} && \text{(by Def. II.2.4.1)} \end{aligned}$$

a contradiction. Thus, we know that $(E, d|_{E\times E})$ is connected. $\square$

Ex. II.2.4.8. Let (X, d) be a metric space, and let E be a subset of X. Show that if E is connected, then the closure $\overline{E}$ of E is also connected. Is the converse true?

Proof. Let $\overline{E}$ be the closure of E in (X, d). Suppose for the sake of contradiction that $(\overline{E}, d|_{\overline{E}\times\overline{E}})$ is disconnected. Then by Def. II.2.4.1 we have

$$\exists V, W \subseteq \overline{E} : \begin{cases} V, W \text{ are open in } (\overline{E}, d|_{\overline{E}\times\overline{E}}) \\ V \neq \emptyset \neq W \\ V \cap W = \emptyset \\ V \cup W = \overline{E} \end{cases}$$

By Prop. II.1.2.10(a)(b) we know that $E \subseteq \overline{E}$, thus we know that at least one of $E \cap V$ and $E \cap W$ is not empty. Now we split into two cases:

- $E \cap V = \emptyset$. But then we have $E \subseteq W$ and

$$\begin{aligned} &\forall x \in V, x \in \overline{E} \setminus E \\ \implies &\forall x \in V, \exists r \in \mathbb{R}^+ : B_{(\overline{E}, d|_{\overline{E}\times\overline{E}})}(x, r) \subseteq V && \text{(by Prop. II.1.2.15(a))} \\ \implies &\forall y \in E, \forall x \in V, \exists r \in \mathbb{R}^+ : d|_{\overline{E}\times\overline{E}}(x, y) \geq r \end{aligned}$$

  which contradict to Prop. II.1.2.10(c). Thus, we must have $E \cap V \neq \emptyset$.

- $E \cap W = \emptyset$. But using similar arguments as above, we know that we must have $E \cap W \neq \emptyset$.

From all cases above, we conclude that $E \cap V \neq \emptyset \neq E \cap W$. But then we have

$$\begin{aligned} & E \subseteq \overline{E} && \text{(by Prop. II.1.2.10(a)(b))} \\ \implies & \begin{cases} E \cap V \text{ is open in } \big(E, d|_{E\times E}\big) \\ E \cap W \text{ is open in } \big(E, d|_{E\times E}\big) \\ E \cap V \neq \emptyset \neq E \cap W \\ \big(E \cap V\big) \cap \big(E \cap W\big) = \emptyset \\ \big(E \cap V\big) \cup \big(E \cap W\big) = E \end{cases} && \text{(by Prop. II.1.3.4(a))} \\ \implies & (E, d|_{E\times E}) \text{ is disconnected,} && \text{(by Def. II.2.4.1)} \end{aligned}$$

a contradiction. Thus, we know that $(\overline{E}, d|_{\overline{E}\times\overline{E}})$ is connected.

Now we give an example to show that the converse is not true. Let $E = (0, 1) \cup (1, 2)$. Then we have $\overline{E}_{(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})} = [0, 2]$. But by Thm. II.2.4.5(a)(c) we know that $(\overline{E}_{(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ is connected but $(E, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ is not. $\square$

Ex. II.2.4.9. Let (X, d) be a metric space. Let us define a relation $x \sim y$ on X by declaring $x \sim y$ iff there exists a connected subset of X which contains both x and y. Show that this is an equivalence relation (i.e., it obeys the reflexive, symmetric, and transitive axioms). Also, show that the equivalence classes of this relation (i.e., the sets of the form $\{y \in X : y \sim x\}$ for some $x \in X$) are all closed and connected. These sets are known as the *connected components* of X.

Proof. Note that the relation $\sim$ depends on the metric function d and we denoted it as $\sim_d$. Since

$$\begin{aligned} & \forall x \in X, \{x\} \subseteq X \\ \implies & \left(\{x\}, d|_{\{x\}\times\{x\}}\right) \text{ is connected} && \text{(by Def. II.2.4.1)} \\ \implies & x \sim_d x, && \text{(by definition)} \end{aligned}$$

we know that $(X, \sim_d)$ is reflexive. Since

$$\begin{aligned} & \forall x, y \in X, x \sim_d y \\ \implies & \exists S \subseteq X : \begin{cases} x, y \in S \\ (S, d|_{S\times S}) \text{ is connected} \end{cases} && \text{(by definition)} \\ \implies & y \sim_d x, && \text{(by definition)} \end{aligned}$$

we know that $(X, \sim_d)$ is symmetric. Since

$$\begin{aligned} & \forall x, y, z \in X, \begin{cases} x \sim_d y \\ y \sim_d z \end{cases} \\ \implies & \exists S_1, S_2 \subseteq X : \begin{cases} x, y \in S_1 \\ (S_1, d|_{S_1\times S_1}) \text{ is connected} \\ y, z \in S_2 \\ (S_2, d|_{S_2\times S_2}) \text{ is connected} \end{cases} && \text{(by definition)} \\ \implies & S_1 \cap S_2 \neq \emptyset && (y \in S_1 \cap S_2) \\ \implies & \left((S_1 \cup S_2), d|_{(S_1\cup S_2)\times(S_1\cup S_2)}\right) \text{ is connected} && \text{(by Ex. II.2.4.6)} \\ \implies & x \sim_d z, && \text{(by definition)} \end{aligned}$$

we know that $(X, \sim_d)$ is transitive. Since $(X, \sim_d)$ is reflexive, symmetric and transitive, we know that $(X, \sim_d)$ is an equivalence relation.

Next we show that every equivalence class of $(X, \sim_d)$ is connected. Let $x \in X$ and let $E_x = \{y \in X : x \sim_d y\}$. By definition we know that

$$\forall y \in E_x, \exists S_y \subseteq X : \begin{cases} (S_y, d|_{S_y\times S_y}) \text{ is connected} \\ x, y \in S \end{cases}$$

Since

$$\forall y \in E_x, x \in S_y$$

$$\implies x \in \bigcap_{y \in E_x} S_y$$
$$\implies \bigcap_{y \in E_x} S_y \neq \emptyset$$
$$\implies \Big(\bigcup_{y \in E_x} S_y, d|_{(\bigcup_{y\in E_x} S_y)\times(\bigcup_{y\in E_x} S_y)}\Big) \text{ is connected,} \qquad \text{(by Ex. II.2.4.6)}$$

by definition we know that $E_x = \bigcup_{y \in E_x} S_y$. Thus, $(E_x, d|_{E_x \times E_x})$ is connected. Since x was arbitrary, we know that every equivalent class of $(X, \sim_d)$ is connected.

Finally we show that every equivalence class of $(X, \sim_d)$ is closed. Let $x \in X$, let $E_x = \{y \in X : x \sim_d y\}$ and let $\overline{E}_x$ be the closure of E_x in (X, d). To show that E_x is closed in (X, d), by Prop. II.1.2.15(b) we need to show that $E_x = \overline{E}_x$. Since $(E_x, d|_{E_x \times E_x})$ is connected, by Ex. II.2.4.8 we know that $(\overline{E}_x, d|_{\overline{E}_x \times \overline{E}_x})$ is connected. Thus, we have

$$\begin{aligned} & E_x \subseteq \overline{E}_x && \text{(by Prop. II.1.2.10(a)(b))} \\ \implies & (x \in \overline{E}_x) \wedge (\forall y \in \overline{E}_x, y \sim_d x) && \text{(by definition)} \\ \implies & \forall y \in \overline{E}_x, y \in E_x \\ \implies & E_x = \overline{E}_x. \end{aligned}$$

□

Ex. II.2.4.10. Combine Prop. II.2.3.2 and Cor. II.2.4.7 to deduce a theorem for continuous functions on a compact connected domain which generalizes Corollary 9.7.4 in Analysis I.

Proof. First, we deduce a theorem was asked. Let (X, d) be a metric space. Let $f : X \to \mathbb{R}$ be continuous map from (X, d_X) to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Let $E \subseteq X$ such that $E \neq \emptyset$ and $(E, d|_{E\times E})$ is compact and connected. Then we have

$$\exists x_{\min}, x_{\max} \in E : f(E) = \big[f(x_{\min}), f(x_{\max})\big].$$

Now we prove the theorem. Since $(E, d|_{E\times E})$ is compact, by Prop. II.2.3.2 we know that

$$\exists x_{\min}, x_{\max} \in E : \forall x \in E, f(x_{\min}) \leq f(x) \leq f(x_{\max}).$$

Then by Cor. II.2.4.7 we have

$$\forall y \in \big[f(x_{\min}), f(x_{\max})\big], \exists x \in E : f(x) = y.$$

□

II.2.5 Topological spaces

Note. The concept of a metric space can be generalized to that of a *topological space*. The idea here is not to view the metric d as the fundamental object; indeed, in a general topological space there is no metric at all. Instead, it is the collection of *open sets* which is the fundamental concept. Thus, whereas in a metric space one introduces the metric d first, and then uses the metric to define first the concept of an open ball and then the concept of an open set, in a topological space one starts just with the notion of an open set. As it turns out, starting from the open sets, one cannot necessarily reconstruct a usable notion of a ball or metric (thus not all topological spaces will be metric spaces), but remarkably one can still define many of the concepts in the preceding sections.

Def. II.2.5.1 (Topological spaces)**.** A *topological space* is a pair $(X, \mathcal{F})$, where X is a set, and $\mathcal{F} \subseteq 2^X$ is a collection of subsets of X, whose elements are referred to as *open sets*. Furthermore, the collection $\mathcal{F}$ must obey the following properties:

- The empty set $\emptyset$ and the whole set X are open; in other words, $\emptyset \in \mathcal{F}$ and $X \in \mathcal{F}$.
- Any finite intersection of open sets is open. In other words, if $V_1, \ldots, V_n$ are elements of $\mathcal{F}$, then $V_1 \cap \cdots \cap V_n$ is also in $\mathcal{F}$.
- Any arbitrary union of open sets is open (including infinite unions). In other words, if $(V_\alpha)_{\alpha \in I}$ is a family of sets in $\mathcal{F}$, then $\displaystyle\bigcup_{\alpha \in I} V_\alpha$ is also in $\mathcal{F}$.

Note. In many cases, the collection $\mathcal{F}$ of open sets can be deduced from context, and we shall refer to the topological space $(X, \mathcal{F})$ simply as X.

Note. From Prop. II.1.2.15 we see that every metric space (X, d) is automatically also a topological space (if we set $\mathcal{F}$ equal to the collection of sets which are open in (X, d)). However, there do exist topological spaces which do not arise from metric spaces.

Def. II.2.5.2 (Neighbourhoods)**.** Let $(X, \mathcal{F})$ be a topological space, and let $x \in X$. A *neighbourhood* of x is defined to be any open set in $\mathcal{F}$ which contains x.

E.g. II.2.5.3. If (X, d) is a metric space, $x \in X$, and $r > 0$, then $B_{(X,d)}(x, r)$ is a neighbourhood of x (see Prop. II.1.2.15(c)).

Def. II.2.5.4 (Topological convergence)**.** Let m be an integer, $(X, \mathcal{F})$ be a topological space and let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in X. Let x be a point in X. We say that $(x^{(n)})_{n=m}^{\infty}$ *converges to* x iff, for every neighbourhood V of x, there exists an $N \geq m$ such that $x^{(n)} \in V$ for all $n \geq N$.

Note. Def. II.2.5.4 is consistent with that of convergence in metric spaces (Def. II.1.1.14). One can then ask whether one has the basic property of uniqueness of limits (Prop. II.1.1.20). The answer turns out to usually be yes - if the topological space has an additional property known as the Hausdorff property - but the answer can be no for other topologies.

Def. II.2.5.5 (Interior, exterior, boundary)**.** Let $(X, \mathcal{F})$ be a topological space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an *interior point of* E if there exists a neighbourhood V of x_0 such that $V \subseteq E$. We say that x_0 is an *exterior point of* E if there exists a neighbourhood V of x_0 such that $V \cap E = \emptyset$. We say that x_0 is a *boundary point of* E if it is neither an interior point nor an exterior point of E.

Note. Def. II.2.5.5 is consistent with the corresponding notion for metric spaces (Def. II.1.2.5).

Def. II.2.5.6 (Closure)**.** Let $(X, \mathcal{F})$ be a topological space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an adherent point of E if every neighbourhood V of x_0 has a non-empty intersection with E. The set of all adherent points of E is called the closure of E and is denoted $\overline{E}$.

Note. We define a set K in a topological space $(X, \mathcal{F})$ to be closed iff its complement $X \setminus K$ is open; this is consistent with the metric space definition, thanks to Prop. II.1.2.15(e).

Def. II.2.5.7 (Relative topology)**.** Let $(X, \mathcal{F})$ be a topological space, and Y be a subset of X. Then we define $\mathcal{F}_Y := \{V \cap Y : V \in F\}$, and refer this as the topology on Y *induced* by $(X, \mathcal{F})$. We call $(Y, \mathcal{F}_Y)$ a *topological subspace* of $(X, \mathcal{F})$.

Note. From Prop. II.1.3.4 we see that Def. II.2.5.7 is compatible with the one for metric spaces.

Def. II.2.5.8 (Continuous functions)**.** Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. If $x_0 \in X$, we say that f is *continuous at* x_0 iff for every neighbourhood V of $f(x_0)$, there exists a neighbourhood U of x_0 such that $f(U) \subseteq V$. We say that f is *continuous* iff it is continuous at every point $x \in X$.

Note. Def. II.2.5.8 is consistent with that in Def. II.2.1.1. In particular, a function is continuous iff the pre-images of every open set is open.

Note. There is unfortunately no notion of a Cauchy sequence, a complete space, or a bounded space, for topological spaces. However, there is certainly a notion of a compact space.

Def. II.2.5.9 (Compact topological spaces)**.** Let $(X, \mathcal{F})$ be a topological space. We say that this space is *compact* if every open cover of X has a finite subcover. If Y is a subset of X, we say that Y is compact if the topological space on Y induced by $(X, \mathcal{F})$ is compact.

Note. Many basic facts about compact metric spaces continue to hold true for compact topological spaces, notably Thm. II.2.3.1 and Prop. II.2.3.2. However, there is no notion of uniform continuity, and so there is no analogue of Thm. II.2.3.5.

Note. We can also define the notion of connectedness by repeating Def. II.2.4.1 verbatim, and also repeating Def. II.2.4.3 (but with Def. II.2.5.7 instead of Def. II.1.3.3). Many of the results and exercises in Sec. II.2.4 continue to hold for topological spaces (with almost no changes to any of the proofs!).

— Exercises —

Ex. II.2.5.1. Let X be an arbitrary set, and let $\mathcal{F} := \{\emptyset, X\}$. Show that $(X, \mathcal{F})$ is a topology (called the *trivial topology* on X). If X contains more than one element, show that the trivial topology cannot be obtained from by placing a metric d on X. Show that this topological space is both compact and connected.

Proof. Let X be a set and let $\mathcal{F} = \{\emptyset, X\}$. First, we show that $(X, \mathcal{F})$ is a topology. Let $n \in \mathbb{N}$, let $S_1, \dots, S_n \in \mathcal{F}$ and let $i, j \in \mathbb{Z}^+$. If there exists some $1 \le j \le n$ such that $S_j = \emptyset$, then we know that $\bigcap_{i=1}^{n} S_i = \emptyset \in \mathcal{F}$. If such j does not exist, then we have $S_i = X$ for every $1 \le i \le n$ and $\bigcap_{i=1}^{n} S_i = X \in \mathcal{F}$. Since n was arbitrary, we conclude that for arbitrary finite collection of element in $\mathcal{F}$ there intersection is still in $\mathcal{F}$.

Let $S \subseteq 2^{\mathcal{F}}$. Then we have

$$\forall s \in S, (s = \emptyset) \vee (s = X) \implies \bigcup S \in F$$

and we conclude that any union of open sets is open.

Since $\emptyset, X \in \mathcal{F}$ and the claims above, by Def. II.2.5.1 we know that $(X, \mathcal{F})$ is a topology.

Next we show that if X contains more than one element, then $(X, \mathcal{F})$ cannot be obtained from by placing a metric d on X. Let $(X, \mathcal{F})$ be a trivial topology and let $x, y \in X$ such that $x \neq y$. Given arbitrary metric d, by Def. II.1.1.2(b) we know that $d(x, y) > \mathbb{R}^+$. But by Prop. II.1.2.15(c) we know that $B_{(X,d)}\big(x, \dfrac{d(x,y)}{2}\big)$ is open in (X, d), thus by Def. II.2.5.1 we must have $B_{(X,d)}\big(x, \dfrac{d(x,y)}{2}\big) \in \mathcal{F}$, which means $(X, \mathcal{F})$ is a not trivial topology.

Finally we show that if X contains more than one element, then $(X, \mathcal{F})$ is compact and connected. Since $\emptyset, X$ are the only two open sets in $\mathcal{F}$, we know that an open cover of X is either $\{X\}$ or $\{\emptyset, X\}$, and both are finite. Thus, by Def. II.2.5.9 $(X, \mathcal{F})$ is compact. Since X is the only non-empty open set in $\mathcal{F}$, by Def. II.2.4.3 we know that $(X, \mathcal{F})$ is connected. $\square$

Ex. II.2.5.2. Let (X, d) be a metric space (and hence a topological space). Show that the two notions of convergence of sequences in Def. II.1.1.14 and Def. II.2.5.4 coincide.

Proof. Let $\mathcal{F}$ be the set of all open sets in (X, d) and let $N \in \mathbb{N}$. First, suppose that $(x^{(n)})_{n=m}^{\infty}$ converges to x in the sense of Def. II.1.1.14. Let $V \in \mathcal{F}$ be a neighbourhood of x. By Def. II.2.5.2 we know that V is open in (X, d). Since $x \in V$, by Prop. II.1.2.15(a) we know that

$$\exists \varepsilon \in \mathbb{R}^+ : B_{(X,d)}(x, \varepsilon) \subseteq V.$$

Now we fix such ε. Then we have

$$\lim_{n\to\infty} d(x^{(n)}, x) = 0$$

$$\Longrightarrow \exists N \geq m : \forall n \geq N, d(x^{(n)}, x) < \varepsilon \qquad \text{(by Def. II.1.1.14)}$$
$$\Longrightarrow \exists N \geq m : \forall n \geq N, x^{(n)} \in B_{(X,d)}(x, \varepsilon) \qquad \text{(by Def. II.1.2.1)}$$
$$\Longrightarrow \exists N \geq m : \forall n \geq N, x^{(n)} \in V. \qquad (B_{(X,d)}(x, \varepsilon) \subseteq V)$$

Since V was arbitrary, we know that $(x^{(n)})_{n=m}^{\infty}$ converges to x in the sense of Def. II.2.5.4.

Now suppose that $(x^{(n)})_{n=m}^{\infty}$ converges to x in the sense of Def. II.2.5.4. Then we have

$$\forall \varepsilon \in \mathbb{R}^+, B_{(X,d)}(x, \varepsilon) \in \mathcal{F} \qquad \text{(by Prop. II.1.2.15(c))}$$
$$\Longrightarrow \forall \varepsilon \in \mathbb{R}^+, \exists N \geq m : \forall n \geq N, x^{(n)} \in B_{(X,d)}(x, \varepsilon) \qquad \text{(by Def. II.2.5.4)}$$
$$\Longrightarrow \forall \varepsilon \in \mathbb{R}^+, \exists N \geq m : \forall n \geq N, d(x^{(n)}, x) < \varepsilon \qquad \text{(by Def. II.1.2.1)}$$
$$\Longrightarrow \lim_{n \to \infty} d(x^{(n)}, x) = 0. \qquad \text{(by Def. II.1.1.14)}$$

Thus, Def. II.1.1.14 and Def. II.2.5.4 coincide. □

Ex. II.2.5.3. Let (X, d) be a metric space (and hence a topological space). Show that the two notions of interior, exterior, and boundary in Def. II.1.2.5 and II.2.5.5 coincide.

Proof. Let $\mathcal{F}$ be the set of all open sets in (X, d), let $E \subseteq X$ and let $x_0 \in X$. First, suppose that x_0 is an interior point of E in the sense of Def. II.1.2.5. Then we have

$$x_0 \in \operatorname{int}_{(X,d)}(E)$$
$$\Longrightarrow \exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \subseteq E \qquad \text{(by Def. II.1.2.5)}$$
$$\Longrightarrow \exists r \in \mathbb{R}^+ : \begin{cases} B_{(X,d)}(x_0, r) \in \mathcal{F} \\ B_{(X,d)}(x_0, r) \subseteq E \end{cases} \qquad \text{(by Prop. II.1.2.15(c))}$$
$$\Longrightarrow x_0 \in \operatorname{int}_{(X,\mathcal{F})}(E). \qquad \text{(by Def. II.2.5.5)}$$

Next suppose that x_0 is an interior point of E in the sense of Def. II.2.5.5. Then we have

$$x_0 \in \operatorname{int}_{(X,\mathcal{F})}(E)$$
$$\Longrightarrow \exists V \in \mathcal{F} : (x_0 \in V) \wedge (V \subseteq E) \qquad \text{(by Def. II.2.5.5)}$$
$$\Longrightarrow \exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \subseteq V \subseteq E \qquad \text{(by Prop. II.1.2.15(a))}$$
$$\Longrightarrow x_0 \in \operatorname{int}_{(X,d)}(E). \qquad \text{(by Def. II.1.2.5)}$$

Next suppose that x_0 is an exterior point of E in the sense of Def. II.1.2.5. Then we have

$$x_0 \in \operatorname{ext}_{(X,d)}(E)$$
$$\Longrightarrow \exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \cap E = \emptyset \qquad \text{(by Def. II.1.2.5)}$$
$$\Longrightarrow \exists r \in \mathbb{R}^+ : \begin{cases} B_{(X,d)}(x_0, r) \in \mathcal{F} \\ B_{(X,d)}(x_0, r) \cap E = \emptyset \end{cases} \qquad \text{(by Prop. II.1.2.15(c))}$$

$$\implies x_0 \in \operatorname{ext}_{(X,\mathcal{F})}(E). \qquad \text{(by Def. II.2.5.5)}$$

Next suppose that x_0 is an exterior point of E in the sense of Def. II.2.5.5. Then we have

$$\begin{aligned}
& x_0 \in \operatorname{ext}_{(X,\mathcal{F})}(E) \\
\implies & \exists V \in \mathcal{F} : (x_0 \in V) \wedge (V \cap E = \emptyset) && \text{(by Def. II.2.5.5)} \\
\implies & \exists r \in \mathbb{R}^+ : \begin{cases} B_{(X,d)}(x_0, r) \subseteq V \\ B_{(X,d)}(x_0, r) \cap E = \emptyset \end{cases} && \text{(by Prop. II.1.2.15(a))} \\
\implies & x_0 \in \operatorname{ext}_{(X,d)}(E). && \text{(by Def. II.1.2.5)}
\end{aligned}$$

Next suppose that x_0 is an boundary point of E in the sense of Def. II.1.2.5. Then we have

$$\begin{aligned}
& x_0 \in \partial_{(X,d)}(E) \\
\implies & \forall r \in \mathbb{R}^+, \begin{cases} B_{(X,d)}(x_0, r) \not\subseteq E \\ B_{(X,d)}(x_0, r) \cap E \neq \emptyset \end{cases} && \text{(by Def. II.1.2.5)} \\
\implies & \forall V \in \mathcal{F}, x_0 \in V \text{ implies} \\
& \begin{cases} \exists r \in \mathbb{R}^+ : B_{(X,d)}(x_0, r) \subseteq V \\ V \not\subseteq E \\ V \cap E \neq \emptyset \end{cases} && \text{(by Prop. II.1.2.15(a))} \\
\implies & x_0 \in \partial_{(X,\mathcal{F})}(E). && \text{(by Def. II.2.5.5)}
\end{aligned}$$

Finally suppose that x_0 is an boundary point of E in the sense of Def. II.2.5.5. Then we have

$$\begin{aligned}
& x_0 \in \partial_{(X,\mathcal{F})}(E) \\
\implies & \forall V \in \mathcal{F}, x_0 \in V \text{ implies} \begin{cases} V \not\subseteq E \\ V \cap E \neq \emptyset \end{cases} && \text{(by Def. II.2.5.5)} \\
\implies & \forall r \in \mathbb{R}^+, \begin{cases} B_{(X,d)}(x_0, r) \in \mathcal{F} \\ B_{(X,d)}(x_0, r) \not\subseteq E \\ B_{(X,d)}(x_0, r) \cap E \neq \emptyset \end{cases} && \text{(by Prop. II.1.2.15(c))} \\
\implies & x_0 \in \partial_{(X,d)}(E). && \text{(by Def. II.1.2.5)}
\end{aligned}$$

Thus, Def. II.1.2.5 and Def. II.2.5.5 coincide. □

Ex. II.2.5.4. A topological space $(X, \mathcal{F})$ is said to be *Hausdorff* if given any two distinct points $x, y \in X$, there exists a neighbourhood V of x and a neighbourhood W of y such that $V \cap W = \emptyset$. Show that any topological space coming from a metric space is Hausdorff, and show that the trivial topology is not Hausdorff if the space contains at least two elements. Show that the analogue of Prop. II.1.1.20 holds for Hausdorff topological spaces, but give an example of a non-Hausdorff topological space in which Prop. II.1.1.20 fails. (In practice, most topological spaces one works with are Hausdorff; non-Hausdorff topological spaces tend to be so pathological that it is not very profitable to work with them.)

Proof. We first show that every topological space coming from a metric space is Hausdorff. Let (X, d) be a metric space and let $\mathcal{F}$ be the set of all open sets in (X, d). Let $x, y \in X$. Then we have

$$
\begin{aligned}
& x \neq y \\
\implies & d(x,y) \in \mathbb{R}^+ && \text{(by Def. II.1.1.2(b))} \\
\implies & \begin{cases} B_{(X,d)}\big(x, \dfrac{d(x,y)}{2}\big) \in \mathcal{F} \\ B_{(X,d)}\big(y, \dfrac{d(x,y)}{2}\big) \in \mathcal{F} \end{cases} && \text{(by Prop. II.1.2.15(c))} \\
\implies & \left(B_{(X,d)}\big(x, \frac{d(x,y)}{2}\big)\right) \cap \left(B_{(X,d)}\big(y, \frac{d(x,y)}{2}\big)\right) = \emptyset && \text{(by Def. II.1.1.2(d))} \\
\implies & (X, \mathcal{F}) \text{ is a Hausdorff space.} && \text{(by definition)}
\end{aligned}
$$

Next we show that if a trivial topological space contains at least two elements, then it is not Hausdorff. Let X be a set such that $x, y \in X$ and $x \neq y$. Let $\mathcal{F} = \{\emptyset, X\}$. By Def. II.2.5.2 the only neighbourhood of x in $\mathcal{F}$ is X, similarly the only neighbourhood of y in $\mathcal{F}$ is X. But $X \cap X \neq \emptyset$ implies $(X, \mathcal{F})$ is not Hausdorff.

Next we show that every convergent sequence in a Hausdorff space has only one limit. Let $(X, \mathcal{F})$ be a Hausdorff space, let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in X and let $x, x' \in X$ such that $(x^{(n)})_{n=1}^{\infty}$ converges to x, x', respectively. Suppose for the sake of contradiction that $x \neq x'$. Since $x \neq x'$ and $(X, \mathcal{F})$ is Hausdorff, by definition we know that

$$
\exists V, V' \in \mathcal{F} : \begin{cases} x \in V \\ x' \in V' \\ V \cap V' = \emptyset \end{cases}
$$

But then we have

$$
\begin{aligned}
& \begin{cases} (x^{(n)})_{n=1}^{\infty} \text{ converges to } x \\ (x^{(n)})_{n=1}^{\infty} \text{ converges to } x' \end{cases} \\
\implies & \begin{cases} \exists N \in \mathbb{Z}^+ : \forall n \geq N, x^{(n)} \in V \\ \exists N' \in \mathbb{Z}^+ : \forall n \geq N', x^{(n)} \in V' \end{cases} && \text{(by Def. II.2.5.4)} \\
\implies & \exists N, N' \in \mathbb{Z}^+ : \forall n \geq \max(N, N'), x^{(n)} \in V \cap V' \\
\implies & V \cap V' \neq \emptyset,
\end{aligned}
$$

a contradiction. Thus, we must have $x = x'$.

Finally we give an non-Hausdorff topology space in which Prop. II.1.1.20 fails. Let $X = \{0, 1\}$ and let $\mathcal{F} = \{\emptyset, X\}$. From the proof above we know that $(X, \mathcal{F})$ is not Hausdorff. Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in X. By Def. II.2.5.2 we know that the only neighbourhood of 0 in $\mathcal{F}$ is X. Similarly, the only neighbourhood of 1 in $\mathcal{F}$ is X. Thus, we have

$$\begin{aligned} & \forall n \in \mathbb{Z}^+, x^{(n)} \in X \\ \implies & \begin{cases} (x^{(n)})_{n=1}^{\infty} \text{ converges to } 0 \\ (x^{(n)})_{n=1}^{\infty} \text{ converges to } 1 \end{cases} & \text{(by Def. II.2.5.4)} \end{aligned}$$

but $0 \neq 1$. $\square$

Ex. II.2.5.5. Given any totally ordered set X with order relation $\leq$, declare a set $V \subseteq X$ to be *open* if for every $x \in V$ there exists a set I which is an interval $\{y \in X : a < y < b\}$ for some $a, b \in X$, a ray $\{y \in X : a < y\}$ for some $a \in X$, the ray $\{y \in X : y < b\}$ for some $b \in X$, or the whole space X, which contains x and is contained in V. Let $\mathcal{F}$ be the set of all open subsets of X. Show that $(X, \mathcal{F})$ is a topology (this is the *order topology* on the totally ordered set $(X, \leq)$) which is Hausdorff in the sense of Ex. II.2.5.4. Show that on the real line $\mathbb{R}$ (with the standard ordering $\leq$), the order topology matches the standard topology (i.e., the topology arising from the standard metric). If instead one applies this to the extended real line $\mathbb{R}^*$, show that $\mathbb{R}$ is an open set with boundary $\{-\infty, +\infty\}$. If $(x_n)_{n=1}^{\infty}$ is a sequence of numbers in $\mathbb{R}$ (and hence in $\mathbb{R}^*$), show that x_n converges to $+\infty$ iff $\liminf_{n\to\infty} x_n = +\infty$, and x_n converges to $-\infty$ iff $\limsup_{n\to\infty} x_n = -\infty$.

Proof. We first show that $(X, \mathcal{F})$ is a topology space. By definition we know that X is open and $\emptyset$ is open trivially, thus $X, \emptyset \in \mathcal{F}$. Let $n \in \mathbb{N}$ and let $S_n \subseteq \mathcal{F}$ such that $\#(S_n) = n$. We induct on n to show that $\bigcap S_n \in \mathcal{F}$ for every $n \in \mathbb{N}$. For $n = 0$, we have $S_0 = \emptyset$ and $\bigcap S_0 = \emptyset$. From the proof above we know that $\emptyset \in \mathcal{F}$, Thus, the base case holds. Suppose inductively that $\bigcap S_n \in \mathcal{F}$ for some $n \geq 0$. Let $S_{n+1} \subseteq \mathcal{F}$ such that $\#(S_{n+1}) = n+1$. Then we have $S_{n+1} = \{V_1, \ldots, V_{n+1} : \forall i \in \mathbb{Z}^+, V_i \in \mathcal{F}\}$ and $\bigcap S_{n+1} = \bigcap_{i=1}^{n+1} V_i$. If $\bigcap S_{n+1} = \emptyset$, then from the proof above we know that $\emptyset \in \mathcal{F}$. So suppose that $\bigcap S_{n+1} \neq \emptyset$. Let $x \in \bigcap S_{n+1}$. Since $x \in \bigcap_{i=1}^{n} V_i$ and $\#(\{V_1, \ldots, V_n\}) = n$, by the induction hypothesis we know that there exists a set I in one of the following forms

$$I = \begin{cases} \{y \in X : a < y < b\} \text{ for some } a, b \in X \\ \{y \in X : a < y\} \text{ for some } a \in X \\ \{y \in X : y < b\} \text{ for some } b \in X \\ X \end{cases}$$

such that $x \in I$ and $I \subseteq \bigcap_{i=1}^{n} V_i$. Since $x \in V_{n+1}$ and $V_{n+1} \in \mathcal{F}$, we know that there exists a set I' in one of the following forms

$$I' = \begin{cases} \left\{y \in X : a' < y < b'\right\} \text{ for some } a', b' \in X \\ \left\{y \in X : a' < y\right\} \text{ for some } a' \in X \\ \left\{y \in X : y < b'\right\} \text{ for some } b' \in X \\ X \end{cases}$$

such that $x \in I'$ and $I' \subseteq V_{n+1}$. Then we have $x \in I \cap I'$ and $I \cap I' \subseteq \bigcap_{i=1}^{n+1} V_i$. Since $(X, \leq)$ is totally ordered, we know that $I \cap I'$ is in one of the following forms

$$I \cap I' = \begin{cases} \left\{y \in X : \max\limits_{(X,\leq)}(a, a') < y < \min\limits_{(X,\leq)}(b, b')\right\} \\ \left\{y \in X : \max\limits_{(X,\leq)}(a, a') < y < b\right\} \\ \left\{y \in X : a < y < \min\limits_{(X,\leq)}(b, b')\right\} \\ \{y \in X : a < y < b\} \\ \left\{y \in X : \max\limits_{(X,\leq)}(a, a') < y < b'\right\} \\ \left\{y \in X : \max\limits_{(X,\leq)}(a, a') < y\right\} \\ \left\{y \in X : a < y < b'\right\} \\ \{y \in X : a < y\} \\ \left\{y \in X : a' < y < \min\limits_{(X,\leq)}(b, b')\right\} \\ \left\{y \in X : a' < y < b\right\} \\ \left\{y \in X : y < \min\limits_{(X,\leq)}(b, b')\right\} \\ \{y \in X : y < b\} \\ \left\{y \in X : a' < y < b'\right\} \\ \left\{y \in X : a' < y\right\} \\ \left\{y \in X : y < b'\right\} \\ X \end{cases}$$

Thus, by definition $I \cap I'$ is an interval. Since x was arbitrary, we know that $\bigcap S_{n+1}$ is open in $(X, \mathcal{F})$, and this closes the induction. We conclude that for any finite collection of open sets, their intersection is again open in $(X, \mathcal{F})$.

Let $S \subseteq \mathcal{F}$. If $\bigcup S = \emptyset$, then from the proof above we know that $\emptyset \in \mathcal{F}$. So suppose that $\bigcup S \neq \emptyset$. Let $x \in \bigcup S$. We know that there exists an $V \in S$ such that $x \in V$. Since $V \in S$, we know that V is open in $(X, \mathcal{F})$ and by definition there exists an interval I such that $x \in I$ and $I \subseteq V$. Then we have $I \subseteq V \subseteq \bigcup S$. Since x was arbitrary, we know that $\bigcup S$ is open in $(X, \mathcal{F})$. Combine all the results above we know that $(X, \mathcal{F})$ is a topological space by Def. II.2.5.1.

Next we show that $(X, \mathcal{F})$ is Hausdorff. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since $(X, \leq)$ is totally ordered, we have either $x_1 < x_2$ or $x_2 < x_1$. Without the loss of generality suppose that $x_1 < x_2$. Let $I_1 = \{y \in X : y < x_2\}$ and let $I_2 = \{y \in X : x_1 < y\}$. Then we have $x_1 \in I_1$ and $x_2 \in I_2$. By definition we know that $I_1, I_2 \in \mathcal{F}$. If $I_1 \cap I_2 = \emptyset$, then we are done. So suppose that $I_1 \cap I_2 \neq \emptyset$. Let $x \in I_1 \cap I_2$, let $J_1 = \{y \in X : y < x\}$ and let $J_2 = \{y \in X : x < y\}$. Since $I_1 \cap I_2 = \{y \in X : x_1 < y < x_2\}$, we know that $x \neq x_1$ and $x \neq x_2$. Since $x_1 < x$, we have $x_1 \in J_1$. Similarly, we have $x_2 \in J_2$. By definition we know that $J_1, J_2 \in \mathcal{F}$. Since $(X, \leq)$ is totally ordered, we know that $J_1 \cap J_2 = \emptyset$. Since x_1, x_2 were arbitrary, by Ex. II.2.5.4 we know that $(X, \mathcal{F})$ is Hausdorff.

Next we show that the order topology in $\mathbb{R}$ with order relation $\leq$ matches standard topology. Let $\mathcal{F}_o$ be the order topology in $\mathbb{R}$ and let $\mathcal{F}_s$ be the standard topology in $\mathbb{R}$. We want to show that $\mathcal{F}_o = \mathcal{F}_s$.

Let $V \in \mathcal{F}_o$ and let $x \in V$. Then we have

$$\exists I \subseteq \mathbb{R} : \begin{cases} I \text{ is an open interval in } \mathbb{R} \\ x \in I \\ I \subseteq V \end{cases}$$

Now we split into four cases:

- If $I = (a, b)$ for some $a, b \in \mathbb{R}$, then we have

$$\begin{aligned} & x \in (a, b) \\ \Longrightarrow\ & r = \min(|x-a|, |x-b|) = \min(x-a, b-x) > 0 \\ \Longrightarrow\ & (x-r, x+r) \subseteq (a, b) \subseteq V \\ \Longrightarrow\ & B_{(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})}(x, r) \subseteq (a, b) \subseteq V && \text{(by Def. II.1.2.1)} \\ \Longrightarrow\ & x \in \operatorname{int}_{(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})}(V). && \text{(by Def. II.1.2.5)} \end{aligned}$$

- If $I = (a, \infty)$ for some $a \in \mathbb{R}$, then we have

$$\begin{aligned} & x \in (a, \infty) \\ \Longrightarrow\ & r = |x-a| = x-a > 0 \end{aligned}$$

$$
\begin{aligned}
&\implies (x-r, x+r) \subseteq (a,\infty) \subseteq V \\
&\implies B_{(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})}(x,r) \subseteq (a,\infty) \subseteq V && \text{(by Def. II.1.2.1)} \\
&\implies x \in \operatorname{int}_{(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})}(V). && \text{(by Def. II.1.2.5)}
\end{aligned}
$$

- If $I = (-\infty, b)$ for some $b \in \mathbb{R}$, then we have

$$
\begin{aligned}
&x \in (-\infty, b) \\
\implies &r = |x-b| = b-x > 0 \\
\implies &(x-r, x+r) \subseteq (-\infty, b) \subseteq V \\
\implies &B_{(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})}(x,r) \subseteq (-\infty, b) \subseteq V && \text{(by Def. II.1.2.1)} \\
\implies &x \in \operatorname{int}_{(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})}(V). && \text{(by Def. II.1.2.5)}
\end{aligned}
$$

- If $I = \mathbb{R}$, then we have $V = \mathbb{R}$ and $x \in \operatorname{int}_{(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})}(V) = \mathbb{R}$.

From all cases above, we conclude that $x \in \operatorname{int}_{(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})}(V)$. Since x was arbitrary, by Prop. II.1.2.15(a) we know that V is open in $(X, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ and thus $V \in \mathcal{F}_s$. Since V was arbitrary, we have $\mathcal{F}_o \subseteq \mathcal{F}_s$.

Let $W \in \mathcal{F}_s$. Since W is open in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$, we have

$$
\begin{aligned}
&\forall x \in W, \exists r \in \mathbb{R}^+ : B_{(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})}(x,r) \subseteq W && \text{(by Prop. II.1.2.15(a))} \\
\implies &\forall x \in W, \exists r \in \mathbb{R}^+ : (x-r, x+r) \subseteq W && \text{(by Def. II.1.2.1)} \\
\implies &W \in \mathcal{F}_o. && \text{(by definition)}
\end{aligned}
$$

Since W was arbitrary, we have $\mathcal{F}_s \subseteq \mathcal{F}_o$. From the proof above we thus have $\mathcal{F}_o = \mathcal{F}_s$.

Next we show that if $(\mathbb{R}^*, \mathcal{F})$ is an order topology with order relation $\leq$, then $\mathbb{R}$ is open in $(\mathbb{R}^*, \mathcal{F})$. Since $(\mathbb{R}^*, \leq)$ is totally ordered, we know that $(\mathbb{R}^*, \mathcal{F})$ is an order topology. Since

$$\forall x \in \mathbb{R}, (x-1, x+1) \subseteq \mathbb{R},$$

by definition we know that $\mathbb{R}$ is open in $(\mathbb{R}^*, \mathcal{F})$.

Next we show that if $(\mathbb{R}^*, \mathcal{F})$ is an order topology with order relation $\leq$, then the boundary points of $\mathbb{R}$ in $(\mathbb{R}^*, \mathcal{F})$ are $-\infty$ and ∞. Since $\infty \notin \mathbb{R}$ and $\mathbb{R}$ is open in $(\mathbb{R}^*, \mathcal{F})$, by Def. II.2.5.5 we know that ∞ is either an exterior point or a boundary point of $\mathbb{R}$ in $(\mathbb{R}^*, \mathcal{F})$. Suppose for the sake of contradiction that ∞ is an exterior point of $\mathbb{R}$ in $(\mathbb{R}^*, \mathcal{F})$. Then by Def. II.2.5.5 we have

$$\exists V \in \mathcal{F} : (\infty \in V) \wedge (\mathbb{R} \cap V = \emptyset).$$

Since $V \cap \mathbb{R} = \emptyset$, we know that the only possible choices of V are $\{\infty\}$ or $\{-\infty, \infty\}$. But in either case, we cannot find an open interval $I \subseteq \mathbb{R}^*$ such that $I \subseteq V$, a contradiction. Thus, ∞ is a boundary point of $\mathbb{R}$ in $(\mathbb{R}^*, \mathcal{F})$. Using similar arguments as above, we can show that $-\infty$ is a boundary point of $\mathbb{R}$ in $(\mathbb{R}^*, \mathcal{F})$.

Finally we show that if $(\mathbb{R}^*, \mathcal{F})$ is an order topology with order relation $\leq$ and $(x^{(n)})_{n=1}^{\infty}$, $(y^{(n)})_{n=1}^{\infty}$ are sequence in $\mathbb{R}$, then we have

$$\begin{cases} x_n \text{ converges to } \infty \text{ in } (\mathbb{R}^*, \mathcal{F}) \\ y_n \text{ converges to } -\infty \text{ in } (\mathbb{R}^*, \mathcal{F}) \end{cases} \iff \begin{cases} \liminf\limits_{n\to\infty} x_n = \infty \\ \limsup\limits_{n\to\infty} y_n = -\infty \end{cases}$$

This is true since

$$\begin{aligned} & \begin{cases} \liminf\limits_{n\to\infty} x^{(n)} = \infty \\ \limsup\limits_{n\to\infty} y^{(n)} = -\infty \end{cases} \\ \iff & \begin{cases} \sup\Big\{\inf\Big\{x^{(n)} : n \geq N\Big\} : N \geq 1\Big\} = \infty \\ \inf\Big\{\sup\Big\{y^{(n)} : n \geq N\Big\} : N \geq 1\Big\} = -\infty \end{cases} \\ \iff & \begin{cases} \sup\Big\{x^{(n)} : n \geq 1\Big\} = \infty \\ \inf\Big\{y^{(n)} : n \geq 1\Big\} = -\infty \end{cases} \\ \iff & \begin{cases} \forall \varepsilon \in \mathbb{R}^+, \exists N \geq 1 : \forall n \geq N, x^{(n)} > \varepsilon \\ \forall \varepsilon \in \mathbb{R}^+, \exists N \geq 1 : \forall n \geq N, y^{(n)} < -\varepsilon \end{cases} \\ \iff & \begin{cases} \forall \varepsilon \in \mathbb{R}^+, \exists N \geq 1 : \forall n \geq N, x^{(n)} \in (\varepsilon, \infty] \\ \forall \varepsilon \in \mathbb{R}^+, \exists N \geq 1 : \forall n \geq N, y^{(n)} \in [-\infty, -\varepsilon) \end{cases} \\ \iff & \begin{cases} \forall V \in \mathcal{F}, \infty \in V \implies \exists \varepsilon \in \mathbb{R}^+ : \begin{cases} (\varepsilon, \infty] \subseteq V \\ \exists N \geq 1 : \forall n \geq N, x^{(n)} \in V \end{cases} \\ \forall V \in \mathcal{F}, -\infty \in V \implies \exists \varepsilon \in \mathbb{R}^+ : \begin{cases} [-\infty, -\varepsilon) \subseteq V \\ \exists N \geq 1 : \forall n \geq N, y^{(n)} \in V \end{cases} \end{cases} \\ \iff & \begin{cases} x_n \text{ converges to } \infty \text{ in } (\mathbb{R}^*, \mathcal{F}) \\ y_n \text{ converges to } -\infty \text{ in } (\mathbb{R}^*, \mathcal{F}) \end{cases} \end{aligned}$$

□

Ex. II.2.5.6. Let X be an uncountable set, and let $\mathcal{F}$ be the collection of all subsets E in X which are either empty or co-finite (which means that $X \setminus E$ is finite). Show that $(X, \mathcal{F})$ is a topology (this is called the *co-finite topology* on X) which is not Hausdorff in the sense of Ex. II.2.5.4, and is compact and connected. Also, show that if $x \in X$ and $(V_n)_{n=1}^{\infty}$ is any countable collection of open sets containing x, then $\bigcap\limits_{n=1}^{\infty} V_n \neq \{x\}$. Use this to show that the co-finite topology cannot be obtained by placing a metric d on X.

Proof. We first show that $(X, \mathcal{F})$ is a topological space. By definition we have $\emptyset \in \mathcal{F}$. Since $X \setminus X = \emptyset$ is finite, we know that X is co-finite and $X \in \mathcal{F}$. Let $n \in \mathbb{N}$, let $I_n = \{i \in \mathbb{N} : 1 \leq i \leq n\}$ and let $A = \{V_i \in \mathcal{F} : i \in I_n\}$ be a finite collection of open sets in $(X, \mathcal{F})$. If $\bigcap A = \emptyset$, then from the proof above we know that $\emptyset \in \mathcal{F}$. So suppose that $A \neq \emptyset$. Then we have

$$\begin{aligned}
&\forall i \in I_n, V_i \text{ is co-finite} \\
\implies &\forall i \in I_n, X \setminus V_i \text{ is finite} \\
\implies &\bigcup_{i=1}^{n} (X \setminus V_i) \text{ is finite} \\
\implies &X \setminus \left(\bigcap_{i=1}^{n} V_i \right) \text{ is finite} \\
\implies &X \setminus \left(\bigcap A \right) \text{ is finite} \\
\implies &\bigcap A \text{ is co-finite}
\end{aligned}$$

and thus $\bigcap A \in \mathcal{F}$. Since n was arbitrary, we conclude that the intersection of any finite collection of open sets in $(X, \mathcal{F})$ is open in $(X, \mathcal{F})$.

Let $S \subseteq \mathcal{F}$. Then we have

$$\begin{aligned}
&\forall V \in S, V \text{ is co-finite} \\
\implies &\forall V \in S, X \setminus V \text{ is finite} \\
\implies &\bigcap_{V \in S} (X \setminus V) \text{ is finite} \\
\implies &X \setminus \left(\bigcup S \right) \text{ is finite} \\
\implies &\bigcup S \text{ is co-finite}
\end{aligned}$$

and thus $\bigcup S \in \mathcal{F}$. Since S was arbitrary, we conclude that the union of arbitrary open sets in $(X, \mathcal{F})$ is open in $(X, \mathcal{F})$. Combine all the proofs above we conclude that $(X, \mathcal{F})$ is a topological space by Def. II.2.5.1.

Next we show that $(X, \mathcal{F})$ is not Hausdorff. Suppose for the sake of contradiction that $(X, \mathcal{F})$ is Hausdorff. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. By Ex. II.2.5.4 we know that

$$\exists V_1, V_2 \in \mathcal{F} : \begin{cases} x_1 \in V_1 \\ x_2 \in V_2 \\ V_1 \cap V_2 = \emptyset \end{cases}$$

But then we have

$$V_1, V_2 \text{ are co-finite}$$

$$\begin{aligned}
&\implies X \setminus V_1, X \setminus V_2 \text{ are finite} \\
&\implies (X \setminus V_1) \cup (X \setminus V_2) \text{ is finite} \\
&\implies X \setminus (V_1 \cap V_2) \text{ is finite} \\
&\implies X \text{ is finite,} && (V_1 \cap V_2 = \emptyset)
\end{aligned}$$

a contradiction. Thus, $(X, \mathcal{F})$ is not Hausdorff.

Next we show that $(X, \mathcal{F})$ is compact. Let S be an open cover of X in $(X, \mathcal{F})$. Let $V_0 \in S$. Since V_0 is co-finite, we know that $X \setminus V_0$ is finite. Let $n = \#(X \setminus V_0)$, let $I_n = \{i \in \mathbb{N} : 1 \leq i \leq n\}$ and let $X \setminus V_0 = \{x_i \in X : i \in I_n\}$. Then we have

$$\begin{aligned}
&\forall i \in I_n, x_i \in X \\
\implies &\forall i \in I_n, \exists V_i \in S : x_i \in V_i \\
\implies &X = V_0 \cup \left(\bigcup_{i \in I_n} V_i \right).
\end{aligned}$$

Since S was arbitrary, by Def. II.2.5.9 we know that $(X, \mathcal{F})$ is compact.

Next we show that $(X, \mathcal{F})$ is connected. Suppose for the sake of contradiction that $(X, \mathcal{F})$ is disconnected. Then by Def. II.2.4.1 we have

$$\exists V, W \in \mathcal{F} : \begin{cases} V \neq \emptyset \neq W \\ V \cup W = X \\ V \cap W = \emptyset \end{cases}$$

But then we have

$$\begin{aligned}
&V, W \text{ are co-finite} \\
\implies &X \setminus V, X \setminus W \text{ are finite} \\
\implies &(X \setminus V) \cup (X \setminus W) \text{ is finite} \\
\implies &X \setminus (V \cap W) \text{ is finite} \\
\implies &X \text{ is finite,} && (V \cap W = \emptyset)
\end{aligned}$$

a contradiction. Thus, $(X, \mathcal{F})$ is connected.

Next we show that if $x \in X$ and $(V_n)_{n=1}^{\infty}$ is any countable collection of open sets in $(X, \mathcal{F})$ such that $x \in V_n$ for all $n \in \mathbb{Z}^+$, then $\bigcap_{n=1}^{\infty} V_n \neq \{x\}$. This is true since

$$\begin{aligned}
&\forall n \in \mathbb{Z}^+, V_n \text{ is co-finite} \\
\implies &\forall n \in \mathbb{Z}^+, X \setminus V_n \text{ is finite} \\
\implies &\bigcup_{n=1}^{\infty} (X \setminus V_n) \text{ is at most countable} && \text{(by Exercise 8.1.9 in Analysis I)}
\end{aligned}$$

$$\begin{aligned}
&\implies X \setminus \left(\bigcap_{n=1}^{\infty} V_n\right) \text{ is at most countable} \\
&\implies X \setminus \left(X \setminus \left(\bigcap_{n=1}^{\infty} V_n\right)\right) \text{ is uncountable} \qquad (X \text{ is uncountable}) \\
&\implies \bigcap_{n=1}^{\infty} V_n \text{ is uncountable} \\
&\implies \bigcap_{n=1}^{\infty} V_n \neq \{x\}.
\end{aligned}$$

Finally we show that $(X, \mathcal{F})$ cannot be obtained by placing a metric d on X. Suppose for the sake of contradiction that there exists some metric d such that $\mathcal{F} = \{V \subseteq X : V \text{ is open in } (X$ Let $x \in X$ and for each $n \in \mathbb{Z}^+$ let $V_n = B_{(X,d)}(x, \frac{1}{n})$. By Prop. II.1.2.15(c) we know that V_n is open in (X, d) for each $n \in \mathbb{Z}^+$. From the proof above we must have $\bigcap_{n=1}^{\infty} V_n \neq \{x\}$. So let $y \in \bigcap_{n=1}^{\infty} V_n$ such that $y \neq x$. By Def. II.1.1.2(b) we know that $d(y, x) \in \mathbb{R}^+$. But then we have

$$\begin{aligned}
&\exists n \in \mathbb{Z}^+ : d(y,x) > \frac{1}{n} && \text{(by Archimedean property)} \\
\implies &\exists n \in \mathbb{Z}^+ : y \notin B_{(X,d)}(x, \frac{1}{n}) && \text{(by Def. II.1.2.1)} \\
\implies &\exists n \in \mathbb{Z}^+ : y \notin V_n \\
\implies &y \notin \bigcap_{n=1}^{\infty} V_n,
\end{aligned}$$

a contradiction. Thus, $(X, \mathcal{F})$ cannot be obtained by placing a metric d on X. $\square$

Ex. II.2.5.7. Let X be an uncountable set, and let $\mathcal{F}$ be the collection of all subsets E in X which are either empty or co-countable (which means that $X \setminus E$ is at most countable). Show that $(X, \mathcal{F})$ is a topology (this is called the *co-countable topology* on X) which is not Hausdorff in the sense of Ex. II.2.5.4, and connected, but cannot arise from a metric space and is not compact.

Proof. We first show that $(X, \mathcal{F})$ is a topological space. By definition we have $\emptyset \in \mathcal{F}$. Since $X \setminus X = \emptyset$ is finite, we know that X is co-countable and $X \in \mathcal{F}$. Let $n \in \mathbb{N}$, let $I_n = \{i \in \mathbb{N} : 1 \leq i \leq n\}$ and let $A = \{V_i \in \mathcal{F} : i \in I_n\}$ be a finite collection of open sets in $(X, \mathcal{F})$. If $\bigcap A = \emptyset$, then from the proof above we know that $\emptyset \in \mathcal{F}$. So suppose that $A \neq \emptyset$. Then we have

$$\forall i \in I_n, V_i \text{ is co-countable}$$

$$\begin{aligned}
&\implies \forall i \in I_n, X \setminus V_i \text{ is at most countable} \\
&\implies \bigcup_{i=1}^{n} (X \setminus V_i) \text{ is at most countable} && \text{(by Exercise 8.1.9 in Analysis I)} \\
&\implies X \setminus \left(\bigcap_{i=1}^{n} V_i \right) \text{ is at most countable} \\
&\implies X \setminus \left(\bigcap A \right) \text{ is at most countable} \\
&\implies \bigcap A \text{ is co-countable}
\end{aligned}$$

and thus $\bigcap A \in \mathcal{F}$. Since n was arbitrary, we conclude that the intersection of any finite collection of open sets in $(X, \mathcal{F})$ is open in $(X, \mathcal{F})$.

Let $S \subseteq \mathcal{F}$. Then we have

$$\begin{aligned}
&\forall V \in S, V \text{ is co-countable} \\
\implies &\forall V \in S, X \setminus V \text{ is at most countable} \\
\implies &\bigcap_{V \in S} (X \setminus V) \text{ is at most countable} \\
\implies &X \setminus \left(\bigcup S \right) \text{ is at most countable} \\
\implies &\bigcup S \text{ is co-countable}
\end{aligned}$$

and thus $\bigcup S \in \mathcal{F}$. Since S was arbitrary, we conclude that the union of arbitrary open sets in $(X, \mathcal{F})$ is open in $(X, \mathcal{F})$. Combine all the proofs above we conclude that $(X, \mathcal{F})$ is a topological space by Def. II.2.5.1.

Next we show that $(X, \mathcal{F})$ is not Hausdorff. Suppose for the sake of contradiction that $(X, \mathcal{F})$ is Hausdorff. Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. By Ex. II.2.5.4 we know that

$$\exists V_1, V_2 \in \mathcal{F} : \begin{cases} x_1 \in V_1 \\ x_2 \in V_2 \\ V_1 \cap V_2 = \emptyset \end{cases}$$

But then we have

$$\begin{aligned}
&V_1, V_2 \text{ are co-countable} \\
\implies &X \setminus V_1, X \setminus V_2 \text{ are at most countable} \\
\implies &(X \setminus V_1) \cup (X \setminus V_2) \text{ is at most countable} \\
\implies &X \setminus (V_1 \cap V_2) \text{ is at most countable} \\
\implies &X \text{ is at most countable,} && (V_1 \cap V_2 = \emptyset)
\end{aligned}$$

a contradiction. Thus, $(X, \mathcal{F})$ is not Hausdorff.

Next we show that $(X, \mathcal{F})$ is connected. Suppose for the sake of contradiction that $(X, \mathcal{F})$ is disconnected. Then by Def. II.2.4.1 we have

$$\exists V, W \in \mathcal{F} : \begin{cases} V \neq \emptyset \neq W \\ V \cup W = X \\ V \cap W = \emptyset \end{cases}$$

But then we have

$$\begin{aligned} & V, W \text{ are co-countable} \\ \implies & X \setminus V, X \setminus W \text{ are at most countable} \\ \implies & (X \setminus V) \cup (X \setminus W) \text{ is at most countable} \\ \implies & X \setminus (V \cap W) \text{ is at most countable} \\ \implies & X \text{ is at most countable,} && (V \cap W = \emptyset) \end{aligned}$$

a contradiction. Thus, $(X, \mathcal{F})$ is connected.

Next we show that if $x \in X$ and $(V_n)_{n=1}^{\infty}$ is any countable collection of open sets in $(X, \mathcal{F})$ such that $x \in V_n$ for all $n \in \mathbb{Z}^+$, then $\bigcap\limits_{n=1}^{\infty} V_n \neq \{x\}$. This is true since

$$\begin{aligned} & \forall n \in \mathbb{Z}^+, V_n \text{ is co-countable} \\ \implies & \forall n \in \mathbb{Z}^+, X \setminus V_n \text{ is at most countable} \\ \implies & \bigcup_{n=1}^{\infty} (X \setminus V_n) \text{ is at most countable} && \text{(by Exercise 8.1.9 in Analysis I)} \\ \implies & X \setminus \left(\bigcap_{n=1}^{\infty} V_n \right) \text{ is at most countable} \\ \implies & X \setminus \left(X \setminus \left(\bigcap_{n=1}^{\infty} V_n \right) \right) \text{ is uncountable} && (X \text{ is uncountable}) \\ \implies & \bigcap_{n=1}^{\infty} V_n \text{ is uncountable} \\ \implies & \bigcap_{n=1}^{\infty} V_n \neq \{x\}. \end{aligned}$$

Next we show that $(X, \mathcal{F})$ cannot be obtained by placing a metric d on X. Suppose for the sake of contradiction that there exists some metric d such that $\mathcal{F} = \{V \subseteq X : V \text{ is open in } (X, d)$ Let $x \in X$ and for each $n \in \mathbb{Z}^+$ let $V_n = B_{(X,d)}(x, \frac{1}{n})$. By Prop. II.1.2.15(c) we know that

V_n is open in (X, d) for each $n \in \mathbb{Z}^+$. From the proof above we must have $\bigcap_{n=1}^{\infty} V_n \neq \{x\}$. So let $y \in \bigcap_{n=1}^{\infty} V_n$ such that $y \neq x$. By Def. II.1.1.2(b) we know that $d(y, x) \in \mathbb{R}^+$. But then we have

$$\begin{aligned}
&\exists n \in \mathbb{Z}^+ : d(y, x) > \frac{1}{n} && \text{(by Archimedean property)} \\
\implies &\exists n \in \mathbb{Z}^+ : y \notin B_{(X,d)}(x, \frac{1}{n}) && \text{(by Def. II.1.2.1)} \\
\implies &\exists n \in \mathbb{Z}^+ : y \notin V_n \\
\implies &y \notin \bigcap_{n=1}^{\infty} V_n,
\end{aligned}$$

a contradiction. Thus, $(X, \mathcal{F})$ cannot be obtained by placing a metric d on X.

Finally we show that $(X, \mathcal{F})$ is not compact. Let $(x^{(n)})_{n=1}^{\infty}$ be a countable collection of elements in X. For each $n \in \mathbb{Z}^+$, we define $E_n = (X \setminus \left\{x^{(i)} : i \in \mathbb{Z}^+\right\}) \cup \left\{x^{(n)}\right\}$. Since

$$\begin{aligned}
&\forall n \in \mathbb{Z}^+, X \setminus E_n = \left\{x^{(i)} : i \in \mathbb{N}\right\} \setminus \left\{x^{(n)}\right\} \text{ is countable} \\
\implies &\forall n \in \mathbb{Z}^+, E_n \in \mathcal{F}
\end{aligned}$$

and

$$\begin{aligned}
\bigcup_{n=1}^{\infty} E_n &= \bigcup_{n=1}^{\infty} \left((X \setminus \left\{x^{(i)} : i \in \mathbb{Z}^+\right\}) \cup \left\{x^{(n)}\right\}\right) \\
&= \left(\bigcup_{n=1}^{\infty} (X \setminus \left\{x^{(i)} : i \in \mathbb{Z}^+\right\})\right) \cup \left(\bigcup_{n=1}^{\infty} \left\{x^{(n)}\right\}\right) \\
&= \left(X \setminus \left(\bigcap_{n=1}^{\infty} \left\{x^{(i)} : i \in \mathbb{Z}^+\right\}\right)\right) \cup \left(\bigcup_{n=1}^{\infty} \left\{x^{(n)}\right\}\right) \\
&= (X \setminus \left\{x^{(i)} : i \in \mathbb{Z}^+\right\}) \cup \left(\bigcup_{n=1}^{\infty} \left\{x^{(n)}\right\}\right) \\
&= X,
\end{aligned}$$

we know that $\bigcup_{n=1}^{\infty} E_n$ is an open cover of X in $(X, \mathcal{F})$. Let $(E_{n_i})_{i=1}^{k}$ be a finite subset of $(E_n)_{n=1}^{\infty}$. Then we have

$$\forall 1 \leq i \leq k, x_{n_i} \in \bigcup_{j=1}^{k} E_{n_j}$$

$$\implies \forall 1 \le i \le k, x_{n_i} \notin X \setminus \left(\bigcup_{i=1}^{k} E_{n_j} \right).$$

Since $(E_{n_i})_{i=1}^{k}$ was arbitrary, we know that every finite subset of $(E_n)_{n=1}^{\infty}$ cannot cover X in $(X, \mathcal{F})$. Thus, by Def. II.2.5.9 $(X, \mathcal{F})$ is not compact. $\square$

Ex. II.2.5.9. Let $(X, \mathcal{F})$ be a compact topological space. Assume that this space is *first countable*, which means that for every $x \in X$ there exists a countable collection $V_1, V_2, \ldots$ of neighbourhoods of x, such that every neighbourhood of x contains one of the V_n. Show that every sequence in X has a convergent subsequence, by modifying Ex. II.1.5.11.

Proof. If $X = \emptyset$, then the statement is trivial. So suppose that $X \neq \emptyset$. Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in X. If the set $E = \left\{ x^{(n)} : n \in \mathbb{Z}^+ \right\}$ is finite, then there exists a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ such that $x^{(n_j)} = x^{(n_1)}$ for every $j \in \mathbb{Z}^+$. By Def. II.2.5.4 we know that $(x^{(n_j)})_{j=1}^{\infty}$ converges to x.

Now suppose that E is infinite. We claim that

$$\exists y \in X : \forall W \in \mathcal{F}, y \in W \implies W \cap E \text{ is infinite.}$$

Suppose for the sake of contradiction that the claim is false. Then we have

$$\forall y \in X, \exists W \in \mathcal{F} : \begin{cases} y \in W; \\ W \cap E \text{ is finite.} \end{cases}$$

We choose one such W for each $y \in X$ and denote it as W_y. But then we have

$$\begin{aligned} & X = \bigcup_{y \in X} W_y \\ \implies & \exists Y \subseteq X : \begin{cases} Y \text{ is finite} \\ X = \bigcup_{y \in Y} W_y \end{cases} && \text{(by Def. II.2.5.9)} \\ \implies & \exists Y \subseteq X : \begin{cases} Y \text{ is finite} \\ E = \bigcup_{y \in Y} (W_y \cap E) \text{ is finite} \end{cases} \end{aligned}$$

which contradict to the fact that E is infinite. Thus, the claim is true and we can choose one $y \in X$ such that every neighbourhood of y contains infinitely many elements of $(x^{(n)})_{n=1}^{\infty}$. Since $(X, \mathcal{F})$ is first countable, we know that there exists a countable collection $(V_j)_{j=1}^{\infty}$ of neighbourhoods of y such that

$$\forall W \in \mathcal{F}, y \in W \implies \exists j \in \mathbb{Z}^+ : V_j \subseteq W.$$

We fix such $(V_j)_{j=1}^{\infty}$ and define $U_j = \bigcap_{i=1}^{j} V_j$ for each $j \in \mathbb{Z}^+$. Since $y \in V_j$ for every $j \in \mathbb{Z}^+$, we know that $y \in \bigcap_{j=1}^{\infty} V_j$. Thus, we have $y \in U_j$ and $U_j \neq \emptyset$ for every $j \in \mathbb{Z}^+$. Observe that

$$\forall p_1, p_2 \in \mathbb{Z}^+, p_1 < p_2 \implies \bigcap_{i=1}^{p_2} V_i \subseteq \bigcap_{i=1}^{p_1} V_i \implies U_{p_2} \subseteq U_{p_1}.$$

Now we construct a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ which converges to y. Let

$$A_1 = \left\{ n \in \mathbb{Z}^+ : x^{(n)} \in U_1 \right\}.$$

Since $U_1 = V_1$ is a neighbourhood of y, by the definition of y we know that A_1 is infinite. Since $A_1 \subseteq \mathbb{Z}^+$, by well-ordering principle we know that $\min(A_1)$ is well-defined. Let $n_1 = \min(A_1)$. Suppose that n_j is already defined for some $j \geq 1$. Then we define n_{j+1} as follow:

$$\begin{aligned} A_{j+1} &= \left\{ n \in \mathbb{Z}^+ : (n > n_j) \wedge (x^{(n)} \in U_{j+1}) \right\} \\ n_{j+1} &= \min(A_{j+1}) \end{aligned}$$

Since $U_{j+1} = \bigcap_{i=1}^{j+1} V_i$, by Def. II.2.5.1 we know that U_{j+1} is a neighbourhood of y. Thus, by the definition of y we know that A_{j+1} is infinite. Since $A_{j+1} \subseteq \mathbb{Z}^+$, by well-ordering principle we know that n_{j+1} is well-defined. Thus, we have construct a subsequence $(x^{(n_j)})_{j=1}^{\infty}$. Let W be a neighbourhood of y. Since $(X, \mathcal{F})$ is first countable, we know that

$$\begin{aligned} & \exists N \in \mathbb{Z}^+ : V_N \subseteq W \\ \implies & \exists N \in \mathbb{Z}^+ : U_N \subseteq V_N \subseteq W && \text{(by the definition of } U_N) \\ \implies & \exists N \in \mathbb{Z}^+ : \forall j \geq N, U_j \subseteq U_N \subseteq W && \text{(by the definition of } U_N) \\ \implies & \exists N \in \mathbb{Z}^+ : \forall j \geq N, x^{(n_j)} \in W. && \text{(by the definition of } x^{(n_j)}) \end{aligned}$$

Since W was arbitrary, by Def. II.2.5.4 we know that $(x^{(n_j)})_{j=1}^{\infty}$ converges to y in $(X, \mathcal{F})$. Thus, we have found a subsequence of $(x^{(n)})_{n=1}^{\infty}$ which converges in $(X, \mathcal{F})$. Since $(x^{(n)})_{n=1}^{\infty}$ was arbitrary, we conclude that $(X, \mathcal{F})$ is sequentially compact, i.e., every sequence in X has a convergent subsequence. □

Ex. II.2.5.10. Prove the following partial analogue of Prop. II.1.2.10 for topological spaces: (c) implies both (a) and (b), which are equivalent to each other. Show that in the co-countable topology in Ex. II.2.5.7, it is possible for (a) and (b) to hold without (c) holding.

Proof. Let $(X, \mathcal{F})$ be a topological space, let $E \subseteq X$, let $x_0 \in X$. We first show that if there exists a sequence $(x^{(n)})_{n=1}^{\infty}$ in E which converges to x_0 in $(X, \mathcal{F})$, then x_0 is an adherent point of E in $(X, \mathcal{F})$. This is true since

$$\begin{aligned}
&(x^{(n)})_{n=1}^{\infty} \text{ converges to } x_0 \text{ in } (X, \mathcal{F}) \\
\implies &\big(\forall V \in \mathcal{F}, x_0 \in V \implies \exists N \in \mathbb{Z}^+ : \forall n \geq N, x^{(n)} \in V\big) && \text{(by Def. II.2.5.4)} \\
\implies &\big(\forall V \in \mathcal{F}, x_0 \in V \implies \exists N \in \mathbb{Z}^+ : x^{(N)} \in V \cap E\big) \\
\implies &\big(\forall V \in \mathcal{F}, x_0 \in V \implies V \cap E \neq \emptyset\big) \\
\implies &x_0 \in \overline{E}_{(X,\mathcal{F})}. && \text{(by Def. II.2.5.6)}
\end{aligned}$$

Next we show that x_0 is an adherent point of E in $(X, \mathcal{F})$ iff x_0 is an interior point or a boundary point of E in $(X, \mathcal{F})$. This is true since

$$\begin{aligned}
&x_0 \in \overline{E}_{(X,\mathcal{F})} \\
\iff &\forall V \in \mathcal{F}, x_0 \in V \implies V \cap E \neq \emptyset && \text{(by Def. II.2.5.6)} \\
\iff &x_0 \notin \operatorname{ext}_{(X,\mathcal{F})}(E) && \text{(by Def. II.2.5.5)} \\
\iff &\big(x_0 \in \operatorname{int}_{(X,\mathcal{F})}(E)\big) \vee \big(x_0 \in \partial_{(X,\mathcal{F})}(E)\big). && \text{(by Def. II.2.5.5)}
\end{aligned}$$

Finally we show that if X is uncountable, $(X, \mathcal{F})$ is a co-countable topology and $x_0 \in E$ for some $E \in \mathcal{F}$, then there may not exist a sequence in E which coverges to x_0 in $(X, \mathcal{F})$. Let $C \subseteq X$ such that C is countable and let $E = X \setminus C$. Then by Ex. II.2.5.7 we know that $E \in \mathcal{F}$ and $E \neq \emptyset$.

Let $x_0 \in C$. Since $x_0 \in C$, we know that $x_0 \notin E$ and by Def. II.2.5.5 we know that $x_0 \notin \operatorname{int}_{(X,\mathcal{F})}(E)$. This means $x_0 \in \operatorname{ext}_{(X,\mathcal{F})}(E)$ or $x_0 \in \partial_{(X,\mathcal{F})}(E)$. Now we claim that $x_0 \in \partial_{(X,\mathcal{F})}(E)$. Suppose for the sake of contradiction that the claim is false. Then we have $x_0 \in \operatorname{ext}_{(X,\mathcal{F})}(E)$, i.e.,

$$\exists V \in \mathcal{F} : x_0 \in V \implies V \cap E = \emptyset.$$

Fix this V. Since $(X, \mathcal{F})$ is a co-countable, by Ex. II.2.5.7 we know that $X \setminus V$ is at most countable. But then we have

$$\begin{aligned}
&(X \setminus V) \cup C \text{ is countable} \\
\implies &X \setminus \big((X \setminus V) \cup C\big) \text{ is co-countable} \\
\implies &V \cap (X \setminus C) \text{ is co-countable} \\
\implies &V \cap E \text{ is co-countable} \\
\implies &X \setminus (V \cap E) \text{ is at most countable} \\
\implies &X \text{ is at most countable,} && (V \cap E = \emptyset)
\end{aligned}$$

which contradict to the hypothesis that X is uncountable. Thus, the claim is true. From the proof above we know that $x_0 \in \partial_{(X,\mathcal{F})}(E)$ implies $x_0 \in \overline{E}_{(X,\mathcal{F})}$.

Suppose for the sake of contradiction that there exists a sequence $(x^{(n)})_{n=1}^{\infty}$ in E which converges to x_0 in $(X, \mathcal{F})$. Then we have

$$\begin{aligned} &\left\{x^{(n)} : n \in \mathbb{Z}^+\right\} \text{ is countable} \\ \implies &X \setminus \left\{x^{(n)} : n \in \mathbb{Z}^+\right\} \text{ is co-countable} \\ \implies &X \setminus \left\{x^{(n)} : n \in \mathbb{Z}^+\right\} \in \mathcal{F}. \end{aligned}$$

Since $x_0 \notin E$, we have

$$\begin{aligned} &x_0 \notin \left\{x^{(n)} : n \in \mathbb{Z}^+\right\} \\ \implies &x_0 \in X \setminus \left\{x^{(n)} : n \in \mathbb{Z}^+\right\}. \end{aligned}$$

But $X \setminus \left\{x^{(n)} : n \in \mathbb{Z}^+\right\} \in \mathcal{F}$ means we have found one neighbourhood of x_0 which does not contain any elements of $(x^{(n)})_{n=1}^{\infty}$. By Def. II.2.5.4 this means $(x^{(n)})_{n=1}^{\infty}$ does not coverges to x_0 in $(X, \mathcal{F})$, a contradiction. Thus, we conclude that there does not exist a sequence in E which converges to x_0 when x_0 is an adherent point of E in $(X, \mathcal{F})$. $\square$

Ex. II.2.5.11. Let E be a subset of a topological space $(X, \mathcal{F})$. Show that E is open iff every element of E is an interior point, and show that E is closed iff E contains all of its adherent points. Prove analogues of Prop. II.1.2.15(e)-(h) (some of these are automatic by definition). If we assume in addition that X is Hausdorff, prove an analogue of Prop. II.1.2.15(d) also, but give an example to show that (d) can fail when X is not Hausdorff.

Proof. We first show that E is open in $(X, \mathcal{F})$ iff $E = \operatorname{int}_{(X,\mathcal{F})}(E)$.

$$\begin{aligned} &E \text{ is open in } (X, \mathcal{F}) \\ \iff &E \in \mathcal{F} && \text{(by Def. II.2.5.1)} \\ \iff &\forall x \in E, \exists V_x \in \mathcal{F} : \begin{cases} x \in V_x \\ V_x \subseteq E \end{cases} && (\bigcup_{x \in E} V_x = E) \\ \iff &\forall x \in E, x \in \operatorname{int}_{(X,\mathcal{F})} && \text{(by Def. II.2.5.5)} \\ \iff &E = \operatorname{int}_{(X,\mathcal{F})}. && \text{(by Def. II.2.5.5)} \end{aligned}$$

Next we show that E is closed in $(X, \mathcal{F})$ iff $E = \overline{E}_{(X,\mathcal{F})}$.

$$\begin{aligned} &E \text{ is closed in } (X, \mathcal{F}) \\ \iff &X \setminus E \text{ is open in } (X, \mathcal{F}) \\ \iff &X \setminus E = \operatorname{int}_{(X,\mathcal{F})}(X \setminus E) && \text{(from the proof above)} \end{aligned}$$

$$
\begin{aligned}
&\iff \forall x \in X \setminus E, \exists V \in \mathcal{F} : \begin{cases} x \in V \\ V \subseteq X \setminus E \end{cases} && \text{(by Def. II.2.5.5)} \\
&\iff \forall x \in X \setminus E, \exists V \in \mathcal{F} : \begin{cases} x \in V \\ V \cap E = \emptyset \end{cases} \\
&\iff \forall x \in X \setminus E, x \in \operatorname{ext}_{(X,\mathcal{F})}(E) && \text{(by Def. II.2.5.5)} \\
&\iff X \setminus E = \operatorname{ext}_{(X,\mathcal{F})}(E) && \text{(by Def. II.2.5.5)} \\
&\iff E = \big(\operatorname{int}_{(X,\mathcal{F})}(E)\big) \cup \big(\partial_{(X,\mathcal{F})}(E)\big) && \text{(by Def. II.2.5.5)} \\
&\iff E = \overline{E}_{(X,\mathcal{F})}. && \text{(by Ex. II.2.5.10)}
\end{aligned}
$$

Next we show that E is open in $(X, \mathcal{F})$ iff $X \setminus E$ is closed in $(X, \mathcal{F})$. This is true by definition.

Next we show that if $\{E_1, \ldots, E_n\}$ is a finite collection of open sets in $(X, \mathcal{F})$, then $\bigcap\limits_{i=1}^{n} E_i$ is open in $(X, \mathcal{F})$. This is true by Def. II.2.5.1.

Next we show that if $\{E_1, \ldots, E_n\}$ is a finite collection of closed sets in $(X, \mathcal{F})$, then $\bigcup\limits_{i=1}^{n} E_i$ is closed in $(X, \mathcal{F})$.

$$
\begin{aligned}
&\forall 1 \leq i \leq n, E_i \text{ is closed in } (X, \mathcal{F}) \\
\implies &\forall 1 \leq i \leq n, X \setminus E_i \text{ is open in } (X, \mathcal{F}) \\
\implies &\bigcap_{i=1}^{n} (X \setminus E_i) \text{ is open in } (X, \mathcal{F}) && \text{(by Def. II.2.5.1)} \\
\implies &X \setminus \left(\bigcup_{i=1}^{n} E_i \right) \text{ is open in } (X, \mathcal{F}) \\
\implies &\bigcup_{i=1}^{n} E_i \text{ is closed in } (X, \mathcal{F}).
\end{aligned}
$$

Next we show that if $(E_\alpha)_{\alpha \in I}$ is a collection of open sets in $(X, \mathcal{F})$ where I is some index set, then $\bigcup\limits_{\alpha \in I} E_\alpha$ is open in $(X, \mathcal{F})$. This is true by Def. II.2.5.1.

Next we show that if $(E_\alpha)_{\alpha \in I}$ is a collection of closed sets in $(X, \mathcal{F})$ where I is some index set, then $\bigcap\limits_{\alpha \in I} E_\alpha$ is closed in $(X, \mathcal{F})$.

$$
\begin{aligned}
&\forall \alpha \in I, E_\alpha \text{ is closed in } (X, \mathcal{F}) \\
\implies &\forall \alpha \in I, X \setminus E_\alpha \text{ is open in } (X, \mathcal{F}) \\
\implies &\bigcup_{\alpha \in I} (X \setminus E_\alpha) \text{ is open in } (X, \mathcal{F}) && \text{(by Def. II.2.5.1)}
\end{aligned}
$$

$$\begin{aligned}&\implies X \setminus \Big(\bigcap_{\alpha\in I} E_\alpha\Big) \text{ is open in } (X,\mathcal{F})\\&\implies \bigcap_{\alpha\in I} E_\alpha \text{ is closed in } (X,\mathcal{F}).\end{aligned}$$

Next we show that if $E \subseteq X$, then $\operatorname{int}_{(X,\mathcal{F})}(E)$ is open in $(X,\mathcal{F})$. Suppose for the sake of contradiction that $\operatorname{int}_{(X,\mathcal{F})}(E) \notin \mathcal{F}$. Then by Def. II.2.5.5 we know that

$$\exists x \in \operatorname{int}_{(X,\mathcal{F})}(E) : \forall V \in \mathcal{F}, x \in V \implies V \not\subseteq \operatorname{int}_{(X,\mathcal{F})}(E).$$

Fix this x. By Def. II.2.5.5 we know that $\operatorname{int}_{(X,\mathcal{F})}(E) \subseteq E$. But then we have

$$\forall V \in \mathcal{F}, x \in V \implies V \not\subseteq \operatorname{int}_{(X,\mathcal{F})}(E) \subseteq E,$$

which means $x \notin \operatorname{int}_{(X,\mathcal{F})}(E)$ by Def. II.2.5.5, a contradiction. Thus, we must have

$$\forall x \in \operatorname{int}_{(X,\mathcal{F})}(E), \exists V \in \mathcal{F} : \begin{cases} x \in V \\ V \subseteq \operatorname{int}_{(X,\mathcal{F})}(E)\end{cases}$$

and by Def. II.2.5.5 we have $\operatorname{int}_{(X,\mathcal{F})}\big(\operatorname{int}_{(X,\mathcal{F})}(E)\big) = \operatorname{int}_{(X,\mathcal{F})}(E)$. From the proof above we conclude that $\operatorname{int}_{(X,\mathcal{F})}(E)$ is open in $(X,\mathcal{F})$.

Next we show that if $E \subseteq X$, then $\operatorname{int}_{(X,\mathcal{F})}(E)$ is the largest open set in $(X,\mathcal{F})$ which is contained in E. Let $V \in \mathcal{F}$ such that $V \subseteq E$. Then we have

$$\begin{aligned}&\forall x \in V, x \in E\\ \implies &\forall x \in V, x \in \operatorname{int}_{(X,\mathcal{F})}(E) && \text{(by Def. II.2.5.5)}\\ \implies &V \subseteq \operatorname{int}_{(X,\mathcal{F})}(E).\end{aligned}$$

Since V was arbitrary, we conclude that $\operatorname{int}_{(X,\mathcal{F})}(E)$ is the largest open set in $(X,\mathcal{F})$ which is contained in E.

Next we show that if $E \subseteq X$, then $\partial_{(X,\mathcal{F})}(E) = \partial_{(X,\mathcal{F})}(X \setminus E)$. We have

$$\begin{aligned}&x \in \partial_{(X,\mathcal{F})}(E)\\ \iff &\forall V \in \mathcal{F}, x \in V \implies \begin{cases} V \cap E \neq \emptyset \\ V \not\subseteq E\end{cases} && \text{(by Def. II.2.5.5)}\\ \iff &\forall V \in \mathcal{F}, x \in V \implies \begin{cases} V \cap (X\setminus E) \neq \emptyset \\ V \not\subseteq (X \setminus E)\end{cases}\\ \iff &x \in \partial_{(X,\mathcal{F})}(X \setminus E). && \text{(by Def. II.2.5.5)}\end{aligned}$$

Thus, $\partial_{(X,\mathcal{F})}(E) = \partial_{(X,\mathcal{F})}(X \setminus E)$.

Next we show that if $E \subseteq X$, then $\mathrm{int}_{(X,\mathcal{F})}(E) = \mathrm{ext}_{(X,\mathcal{F})}(X \setminus E)$. We have

$$\begin{aligned}
& x \in \mathrm{int}_{(X,\mathcal{F})}(E) \\
\iff & \exists V \in \mathcal{F} : \begin{cases} x \in V \\ V \subseteq E \end{cases} && \text{(by Def. II.2.5.5)} \\
\iff & \exists V \in \mathcal{F} : \begin{cases} x \in V \\ V \cap (X \setminus E) = \emptyset \end{cases} \\
\iff & x \in \mathrm{ext}_{(X,\mathcal{F})}(X \setminus E). && \text{(by Def. II.2.5.5)}
\end{aligned}$$

Thus, $\mathrm{int}_{(X,\mathcal{F})}(E) = \mathrm{ext}_{(X,\mathcal{F})}(X \setminus E)$.

Next we show that if $E \subseteq X$, then $\overline{E}_{(X,\mathcal{F})}$ is closed in $(X, \mathcal{F})$.

$$\begin{aligned}
& \mathrm{int}_{(X,\mathcal{F})}(X \setminus E) \text{ is open in } (X, \mathcal{F}) && \text{(from the proof above)} \\
\iff & X \setminus \mathrm{int}_{(X,\mathcal{F})}(X \setminus E) \text{ is closed in } (X, \mathcal{F}) \\
\iff & \big(\mathrm{ext}_{(X,\mathcal{F})}(X \setminus E)\big) \cup \big(\partial_{(X,\mathcal{F})}(X \setminus E)\big) \text{ is closed in } (X, \mathcal{F}) && \text{(by Def. II.2.5.5)} \\
\iff & \big(\mathrm{int}_{(X,\mathcal{F})}(E)\big) \cup \big(\partial_{(X,\mathcal{F})}(E)\big) \text{ is closed in } (X, \mathcal{F}) && \text{(from the proof above)} \\
\iff & \overline{E}_{(X,\mathcal{F})}(E) \text{ is closed in } (X, \mathcal{F}). && \text{(by Ex. II.2.5.10)}
\end{aligned}$$

Next we show that if $E \subseteq X$, then $\overline{E}_{(X,\mathcal{F})}$ is the smallest closed set in $(X, \mathcal{F})$ which contains E. Let $V \in \mathcal{F}$ such that $E \subseteq X \setminus V$. Then we have

$$\begin{aligned}
& E \subseteq (X \setminus V) \\
\implies & E \cap V = \emptyset \\
\implies & \overline{E}_{(X,\mathcal{F})} \cap V = \emptyset && \text{(by Def. II.2.5.6)} \\
\implies & \overline{E}_{(X,\mathcal{F})} \subseteq (X \setminus V).
\end{aligned}$$

Since $X \setminus V$ was arbitrary, we know that $\overline{E}_{(X,\mathcal{F})}$ is the smallest closed set in $(X, \mathcal{F})$ which contains E.

Next we show that if $(X, \mathcal{F})$ is a Hausdorff space, then $\{x_0\}$ is closed in $(X, \mathcal{F})$ for any $x_0 \in X$. Let $x_0 \in X$. We have

$$\begin{aligned}
& \forall y \in X \setminus \{x_0\}, y \neq x_0 \\
\implies & \forall y \in X \setminus \{x_0\}, \exists V, W \in \mathcal{F} : \begin{cases} V \neq \emptyset \neq W \\ x_0 \in V \\ y \in W \\ V \cap W = \emptyset \end{cases} && \text{(by Ex. II.2.5.4)} \\
\implies & \forall y \in X \setminus \{x_0\}, \exists W \in \mathcal{F} : \begin{cases} y \in W \\ W \subseteq (X \setminus \{x_0\}) \end{cases}
\end{aligned}$$

$$\begin{aligned}
&\implies \forall y \in X \setminus \{x_0\}, y \in \operatorname{int}_{(X,\mathcal{F})}(X \setminus \{x_0\}) && \text{(by Def. II.2.5.5)}\\
&\implies X \setminus \{x_0\} = \operatorname{int}_{(X,\mathcal{F})}(X \setminus \{x_0\}) && \text{(by Def. II.2.5.5)}\\
&\implies X \setminus \{x_0\} \text{ is open in } (X,\mathcal{F}) && \text{(from the proof above)}\\
&\implies \{x_0\} \text{ is closed in } (X,\mathcal{F}).
\end{aligned}$$

Since x_0 was arbitrary, we conclude that $\{x_0\}$ is closed in $(X, \mathcal{F})$ for any $x_0 \in X$.

Finally we give an counterexample of Prop. II.1.2.15(d) when $(X, \mathcal{F})$ is not Hausdorff. Let $X \neq \emptyset$ and let $(X, \mathcal{F})$ be a trivial topology. Then by Ex. II.2.5.1 we know that $\mathcal{F} = \{\emptyset, X\}$ and by Ex. II.2.5.4 $(X, \mathcal{F})$ is not Hausdorff. For any $x_0 \in X$, we have $X \setminus \{x_0\} \notin \mathcal{F}$, thus $X \setminus \{x_0\}$ is not open in $(X, \mathcal{F})$ and $\{x_0\}$ is not closed in $(X, \mathcal{F})$. □

Ex. II.2.5.12. Show that the pair $(Y, \mathcal{F}_Y)$ defined in Def. II.2.5.7 is indeed a topological space.

Proof. We have

$$\begin{aligned}
&\begin{cases} X \in \mathcal{F} \\ \emptyset \in \mathcal{F} \end{cases} && \text{(by Def. II.2.5.1)}\\
\implies &\begin{cases} Y \cap X = Y \in \mathcal{F}_Y \\ Y \cap \emptyset = \emptyset \in \mathcal{F}_Y \end{cases}
\end{aligned}$$

Let $n \in \mathbb{N}$ and let $(V_Y^{(i)})_{i=1}^n$ be a finite collection of open sets in $\mathcal{F}_Y$. Then we have

$$\begin{aligned}
&\forall 1 \leq i \leq n, \exists V_X^{(i)} \in \mathcal{F} : V_X^{(i)} \cap Y = V_Y^{(i)} && \text{(by Def. II.2.5.7)}\\
\implies &\bigcap_{i=1}^n V_X^{(i)} \in \mathcal{F} && \text{(by Def. II.2.5.1)}\\
\implies &Y \cap \left(\bigcap_{i=1}^n V_X^{(i)} \right) \in \mathcal{F}_Y && \text{(by Def. II.2.5.7)}\\
\implies &\bigcap_{i=1}^n (V_X^{(i)} \cap Y) = \bigcap_{i=1}^n V_Y^{(i)} \in \mathcal{F}_Y.
\end{aligned}$$

Since n was arbitrary, we conclude that the intersection of any finite collection of open sets in $(Y, \mathcal{F}_Y)$ is open in $(Y, \mathcal{F}_Y)$. Let $S \subseteq \mathcal{F}_Y$. Then we have

$$\begin{aligned}
&\forall V_Y \in S, \exists V_X \in \mathcal{F} : V_X \cap Y = V_Y && \text{(by Def. II.2.5.7)}\\
\implies &\bigcup_{V_X \in \mathcal{F} : V_X \cap Y \in S} V_X \in \mathcal{F} && \text{(by Def. II.2.5.1)}\\
\implies &Y \cap \left(\bigcup_{V_X \in \mathcal{F} : V_X \cap Y \in S} V_X \right) \in \mathcal{F}_Y && \text{(by Def. II.2.5.7)}
\end{aligned}$$

$$\implies \bigcup_{V_X \in \mathcal{F}: V_X \cap Y \in S} (V_X \cap Y) = \bigcup_{V_Y \in S} V_Y \in \mathcal{F}_Y.$$

Since S was arbitrary, we conclude that the union of arbitrary many open sets in $(Y, \mathcal{F}_Y)$ is open in $(Y, \mathcal{F}_Y)$. Combine all the proofs above we conclude by Def. II.2.5.1 that $(Y, \mathcal{F}_Y)$ is a topological space. ☐

Ex. II.2.5.13. Generalize Cor. II.1.5.9 to compact sets in a Hausdorff topological space.

Proof. Let $(X, \mathcal{F})$ be a Hausdorff space, and let $(K_n)_{n=1}^{\infty}$ be a countable collection of non-empty compact topological subspaces of X such that

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots.$$

We want to show that the intersection $\bigcap\limits_{n=1}^{\infty} K_n$ is non-empty.

Since $K_n \subseteq K_1$ for each $n \in \mathbb{Z}^+$, by Def. II.2.5.7 we know that

$$\begin{aligned}
\forall n \in \mathbb{Z}^+, \mathcal{F}_{K_n} &= \{K_n \cap V : V \in \mathcal{F}\} \\
&= \{(K_1 \cap K_n) \cap V : V \in \mathcal{F}\} && (K_n \subseteq K_1) \\
&= \{K_n \cap (K_1 \cap V) : V \in \mathcal{F}\} \\
&= \{K_n \cap V : V \in \mathcal{F}_{K_1}\}.
\end{aligned}$$

Thus, by Def. II.2.5.7 we know that for every $n \in \mathbb{Z}^+$, $(K_n, \mathcal{F}_{K_n})$ induced by $(X, \mathcal{F})$ can also induced by $(K_1, \mathcal{F}_{K_1})$. By Def. II.2.5.9 we have

$$\begin{aligned}
&\forall n \in \mathbb{Z}^+, (K_n, \mathcal{F}_{K_n}) \text{ is compact topological subspace of } (X, \mathcal{F}) \\
\implies &\forall n \in \mathbb{Z}^+, \exists S_n \subseteq \mathcal{F}_{K_n} : \begin{cases} S_n \text{ is finite} \\ K_n = \bigcup \mathcal{F}_{K_n} = \bigcup S_n \end{cases} \\
\implies &\forall n \in \mathbb{Z}^+, \exists S_n \subseteq \mathcal{F}_{K_n} : \begin{cases} S_n' = \{V \cap W : (V, W) \in S_1 \times S_n\} \text{ is finite} \\ S_n' \subseteq \mathcal{F}_{K_n} \text{ by Def. II.2.5.7} \\ \forall S \subseteq \mathcal{F}, K_n \subseteq K_1 \subseteq \bigcup S \implies K_n \subseteq \bigcup S_n' \end{cases} \\
\implies &\forall n \in \mathbb{Z}^+, (K_n, \mathcal{F}_{K_n}) \text{ is compact topological subspace of } (K_1, \mathcal{F}_{K_1}).
\end{aligned}$$

Now we fix one $n \in \mathbb{Z}^+$. Since $(X, \mathcal{F})$ is Hausdorff, we know that

$$\begin{aligned}
&\forall x, y \in K_n, x \neq y \\
\implies &\exists V, W \in \mathcal{F} : \begin{cases} V \neq \emptyset \neq W \\ x \in V \\ y \in W \\ V \cap W = \emptyset \end{cases} && (K_n \subseteq X)
\end{aligned}$$

$$\Longrightarrow \exists V_{K_n}, W_{K_n} \in \mathcal{F}_{K_n} : \begin{cases} V_{K_n} = V \cap K_n \\ W_{K_n} = W \cap K_n \\ x \in V_{K_n} \\ y \in W_{K_n} \\ V_{K_n} \neq \emptyset \neq W_{K_n} \\ V_{K_n} \cap W_{K_n} = \emptyset \end{cases} \qquad \text{(by Def. II.2.5.7)}$$

Thus, $(K_n, \mathcal{F}_{K_n})$ is Hausdorff. Since n was arbitrary, we know that for each $n \in \mathbb{Z}^+$, $(K_n, \mathcal{F}_{K_n})$ is Hausdorff.

We claim that for each $n \in \mathbb{Z}^+$, $(K_n, \mathcal{F}_{K_n})$ is closed in $(K_1, \mathcal{F}_{K_1})$. Suppose for the sake of contradiction that there exists some $n \in \mathbb{Z}^+$ such that $(K_n, \mathcal{F}_{K_n})$ is not closed in $(K_1, \mathcal{F}_{K_1})$. Then by Ex. II.2.5.11 we know that $\overline{K_n}_{(K_1,\mathcal{F}_{K_1})} \setminus K_n = \emptyset$. Let $y \in \overline{K_n}_{(K_1,\mathcal{F}_{K_1})} \setminus K_n$. Since $(K_1, \mathcal{F}_{K_1})$ is Hausdorff, we know that

$$\forall x \in K_n, \exists V_x, W_x \in \mathcal{F}_{K_1} : \begin{cases} V_x \neq \emptyset \neq W_x \\ x \in V_x \\ y \in W_x \\ V_x \cap W_x = \emptyset \end{cases}$$

Since $(K_n, \mathcal{F}_{K_n})$ is a compact topological subspace of $(K_1, \mathcal{F}_{K_1})$, by Def. II.2.5.9 we have

$$K_n \subseteq \bigcup_{x \in K_n} V_x \implies \exists S \subseteq K_n : \begin{cases} S \text{ is finite} \\ K_n \subseteq \bigcup_{x \in S} V_x \end{cases}$$

By Def. II.2.5.1 we have

$$\begin{aligned} &\begin{cases} \forall x \in S, V_x \in \mathcal{F}_{K_1} \\ y \in \bigcap_{x \in S} W_x \in \mathcal{F}_{K_1} \end{cases} \\ \Longrightarrow\ & \forall x \in S, V_x \cap \left(\bigcap_{x' \in S} W_{x'} \right) = \emptyset \\ \Longrightarrow\ & \bigcup_{x \in S} \left(V_x \cap \left(\bigcap_{x' \in S} W_{x'} \right) \right) = \emptyset \\ \Longrightarrow\ & \left(\bigcup_{x \in S} V_x \right) \cap \left(\bigcap_{x' \in S} W_{x'} \right) = \emptyset \\ \Longrightarrow\ & K_n \cap \left(\bigcap_{x' \in S} W_{x'} \right) = \emptyset \end{aligned}$$

But this contradicts the fact that $y \in \overline{K_n}_{(K_1,\mathcal{F}_{K_1})}$. Thus, $(K_n, \mathcal{F}_{K_n})$ is closed in $(K_1, \mathcal{F}_{K_1})$ for each $n \in \mathbb{Z}^+$.

Let $V_n = K_1 \setminus K_n$ for every $n \geq 1$. Then for every $n \geq 1$, we have $V_n \subseteq K_1$ and V_n is open in $(K_1, \mathcal{F}_{K_1})$. Suppose for the sake of contradiction that $\bigcap_{n=1}^{\infty} K_n = \emptyset$. Since

$$\bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} (K_1 \setminus K_n) = K_1 \setminus \left(\bigcap_{n=1}^{\infty} K_n \right) = K_1$$

and $(K_1, \mathcal{F}_{K_1})$ is compact, by Thm. II.1.5.8 we know that there exists a finite set $F \subseteq \mathbb{Z}^+$ such that

$$K_1 \subseteq \bigcup_{i \in F} V_i.$$

Since F is finite subset of $\mathbb{Z}^+$, we know that $\min(F)$ is well-defined. Then we have

$$\begin{aligned}
& K_1 \subseteq \bigcup_{i \in F} V_i \subseteq \bigcup_{n=1}^{\infty} V_i = K_1 \\
\implies & K_1 = \bigcup_{i \in F} V_i \\
\implies & K_1 = \bigcup_{i \in F} (K_1 \setminus K_i) \\
\implies & K_1 = K_1 \setminus \left(\bigcap_{i \in F} K_i \right) \\
\implies & \bigcap_{i \in F} K_i = \emptyset && \left(\text{since } \bigcap_{i \in F} K_i \subseteq K_1\right) \\
\implies & K_{\min(F)} = \emptyset. && \left(\text{since } K_{\min(F)} = \bigcap_{i \in F} K_i\right)
\end{aligned}$$

But by hypothesis we know that $K_{\min(F)} \neq \emptyset$, a contradiction. Thus, $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. □

Ex. II.2.5.14. Generalize Thm. II.1.5.10 to compact sets in a Hausdorff topological space.

Proof. Let $(X, \mathcal{F})$ be a Hausdorff topological space. We first show that if $Z \subseteq Y \subseteq X$ such that $(Y, \mathcal{F}_Y)$ is compact, then $(Z, \mathcal{F}_Z)$ is compact iff Z is closed in $(Y, \mathcal{F}_Y)$. By Ex. II.2.5.13 we know that if $(Z, \mathcal{F}_Z)$ is compact then Z is closed in $(Y, \mathcal{F}_Y)$. So we only need to show that if Z is closed in $(Y, \mathcal{F}_Y)$ then $(Z, \mathcal{F}_Z)$ is compact. Let $S_Z \subseteq \mathcal{F}_Z$ be an open cover of Z. Then we have

$$Z \text{ is closed in } (Y, \mathcal{F}_Y)$$

$$\implies Y \setminus Z \text{ is open in } (Y, \mathcal{F}_Y)$$

and

$$\begin{aligned}
&Z = \bigcup S_Z \\
\implies &Z \subseteq \bigcup \{V \in \mathcal{F}_Y : V \cap Z \in S_Z\} && \text{(by Def. II.2.5.7)} \\
\implies &Y = \left(\bigcup \{V \in \mathcal{F}_Y : V \cap Z \in S_Z\}\right) \cup (Y \setminus Z) \\
\implies &\exists S_Y \subseteq \mathcal{F}_Y : \begin{cases} S_Y \text{ is finite} \\ Y = \left(\bigcup \{V \in S_Y : V \cap Z \in S_Z\}\right) \\ \quad \cup (Y \setminus Z) \end{cases} && \text{(by Def. II.2.5.9)} \\
\implies &\exists S_Y \subseteq \mathcal{F}_Y : \begin{cases} S_Y \text{ is finite} \\ Z = \bigcup \{V \cap Z : V \in S_Y\} \end{cases} \\
\implies &\{V \cap Z : V \in S_Y\} \text{ is an finite subcover of } Z \text{ in } (Z, \mathcal{F}_Z).
\end{aligned}$$

Since S_Z was arbitrary, by Def. II.2.5.9 we know that $(Z, \mathcal{F}_Z)$ is compact.

Next we show that if $(Y_i)_{i=1}^n$ is a finite collection of compact topological subspaces of $(X, \mathcal{F})$, then $\left(\bigcup_{i=1}^n Y_i, \mathcal{F}_{\bigcup_{i=1}^n Y_i}\right)$ is also a compact topological subspace of $(X, \mathcal{F})$. By Def. II.2.5.7 we have

$$\mathcal{F}_{\bigcup_{i=1}^n Y_i} = \left\{V \cap \left(\bigcup_{i=1}^n Y_i\right) : V \in \mathcal{F}\right\}.$$

Observe that

$$\begin{cases} X \in \mathcal{F} \\ \emptyset \in \mathcal{F} \end{cases}$$

$$\implies \begin{cases} X \cap \left(\bigcup_{i=1}^n Y_i\right) = \bigcup_{i=1}^n Y_i \in \mathcal{F}_{\bigcup_{i=1}^n Y_i} \\ \emptyset \cap \left(\bigcup_{i=1}^n Y_i\right) = \emptyset \in \mathcal{F}_{\bigcup_{i=1}^n Y_i} \end{cases}$$

Let $(V^{(j)})_{j=1}^m$ be a finite collection of elements in $\mathcal{F}_{\bigcup_{i=1}^n Y_i}$. Then we have

$$\begin{aligned}
&\forall 1 \leq j \leq m, \exists V_X^{(j)} \in \mathcal{F} : V_X^{(j)} \cap \left(\bigcup_{i=1}^n Y_i\right) = V^{(j)} && \text{(by Def. II.2.5.7)} \\
\implies &\bigcap_{j=1}^m V_X^{(j)} \in \mathcal{F} && \text{(by Def. II.2.5.1)}
\end{aligned}$$

$$\Longrightarrow \left(\bigcap_{j=1}^{m} V_X^{(j)}\right) \cap \left(\bigcup_{i=1}^{n} Y_i\right) \in \mathcal{F}_{\bigcup_{i=1}^{n} Y_i} \qquad \text{(by Def. II.2.5.7)}$$

$$\Longrightarrow \bigcap_{j=1}^{m} \left(V_X^{(j)} \cap \left(\bigcup_{i=1}^{n} Y_i\right)\right) = \bigcap_{j=1}^{m} V^{(j)} \in \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}.$$

Since $(V^{(j)})_{j=1}^{m}$ was arbitrary, we conclude that the intersection of any finite collection of open sets in $\left(\bigcup_{i=1}^{n} Y_i, \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}\right)$ is open in $\left(\bigcup_{i=1}^{n} Y_i, \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}\right)$. Let $S \subseteq \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}$. Then we have

$$\forall V \in S, \exists V_X \in \mathcal{F} : V_X \cap \left(\bigcup_{i=1}^{n} Y_i\right) = V \qquad \text{(by Def. II.2.5.7)}$$

$$\Longrightarrow \bigcup_{V_x \in \mathcal{F} : V_X \cap (\bigcup_{i=1}^{n} Y_i) \in \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}} V_x \in \mathcal{F} \qquad \text{(by Def. II.2.5.1)}$$

$$\Longrightarrow \left(\bigcup_{V_x \in \mathcal{F} : V_X \cap (\bigcup_{i=1}^{n} Y_i) \in \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}} V_x\right) \cap \left(\bigcup_{i=1}^{n} Y_i\right) \in \mathcal{F}_{\bigcup_{i=1}^{n} Y_i} \qquad \text{(by Def. II.2.5.7)}$$

$$\Longrightarrow \bigcup_{V \in S} V \in \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}.$$

Since S was arbitrary, we conclude that the union of arbitrary many open sets in $\left(\bigcup_{i=1}^{n} Y_i, \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}\right)$ is open in $\left(\bigcup_{i=1}^{n} Y_i, \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}\right)$. By Def. II.2.5.1 and all the proofs above we conclude that $\left(\bigcup_{i=1}^{n} Y_i, \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}\right)$ is a topological subspace of $(X, \mathcal{F})$.

Let $S \subseteq \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}$ be an open cover of $\bigcup_{i=1}^{n} Y_i$ in $\left(\bigcup_{i=1}^{n} Y_i, \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}\right)$. Then we have

$$\forall 1 \leq i \leq n, Y_i \subseteq \bigcup_{j=1}^{n} Y_i = \bigcup S$$

$$\Longrightarrow \forall 1 \leq i \leq n, \exists S_i \in \mathcal{F}_{\bigcup_{j=1}^{n} Y_j} :$$

$$\begin{cases} S_i \text{ is finite} \\ Y_i \subseteq \bigcup S_i \subseteq \bigcup_{i=1}^{n} Y_i = \bigcup S \end{cases} \qquad \text{(by Def. II.2.5.9)}$$

$$\Longrightarrow \bigcup_{i=1}^{n} Y_i = \bigcup \left(\bigcup_{i=1}^{n} S_i\right).$$

Since S_i is finite for each $1 \leq i \leq n$, we know that $\bigcup_{i=1}^{n} S_i$ is a finite subcover of $\bigcup_{i=1}^{n} Y_i$ in $\big(\bigcup_{i=1}^{n} Y_i, \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}\big)$. Since S was arbitrary, by Def. II.2.5.9 we know that $\big(\bigcup_{i=1}^{n} Y_i, \mathcal{F}_{\bigcup_{i=1}^{n} Y_i}\big)$ is a compact topological subspace of $(X, \mathcal{F})$.

Finally we show that every finite subset of X is compact. Let $x_0 \in X$ and let $\mathcal{F}_{\{x_0\}} = \{V \cap \{x_0\} : V \in \mathcal{F}\}$. We have

$$\begin{aligned}
&\begin{cases} X \in \mathcal{F} \\ \emptyset \in \mathcal{F} \end{cases} && \text{(by Def. II.2.5.1)} \\
\implies &\begin{cases} X \cap \{x_0\} = \{x_0\} \in \mathcal{F}_{\{x_0\}} \\ \emptyset \cap \{x_0\} = \emptyset \in \mathcal{F}_{\{x_0\}} \end{cases}
\end{aligned}$$

Since $\mathcal{F}_{\{x_0\}} = \{\emptyset, \{x_0\}\}$, by Ex. II.2.5.1 we know that $(\{x_0\}, \mathcal{F}_{\{x_0\}})$ is a topological space. Let $S \subseteq \mathcal{F}_{\{x_0\}}$ such that $\{x_0\} \subseteq \bigcup S$. Then we have

$$\begin{aligned}
&\{x_0\} \subseteq \bigcup S \\
\implies &\exists V \in S : x_0 \in V \\
\implies &\{x_0\} \subseteq V.
\end{aligned}$$

Since S was arbitrary, we conclude that every open cover of $\{x_0\}$ has a finite subcover in $(\{x_0\}, \mathcal{F}_{\{x_0\}})$, and by Def. II.2.5.9 we know that $(\{x_0\}, \mathcal{F}_{\{x_0\}})$ is compact. Since x_0 was arbitrary, we conclude that every singleton subset of X is compact. And from the proof above we conclude that every finite subset of X is compact. □

Ex. II.2.5.15. Let (X, d_X) and (Y, d_Y) be metric spaces (and hence a topological space). Show that the two notions continuity (both at a point, and on the whole domain) of a function $f : X \to Y$ in Def. II.2.1.1 and Def. II.2.5.8 coincide.

Proof. Let

$$\begin{aligned}
\mathcal{F}_X &= \{V \subseteq X : V \text{ is open in } (X, d_X)\}; \\
\mathcal{F}_Y &= \{V \subseteq Y : V \text{ is open in } (Y, d_Y)\}.
\end{aligned}$$

Since (X, d_X) and (Y, d_Y) are metric spaces, we know that $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ are topological spaces.

Let $x_0 \in X$. First, suppose that f is continuous at x_0 in the sense of Def. II.2.1.1. Then we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon\Big)
\end{aligned}$$

$$\begin{aligned}
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
&\qquad \Big(\forall x \in X, x \in B_{(X,d_X)}(x_0, \delta) \implies f(x) \in B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big) \Big) \\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
&\qquad f\big(B_{(X,d_X)}(x_0, \delta)\big) \subseteq B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big)
\end{aligned}$$

Let $V \in \mathcal{F}_Y$ such that $f(x_0) \in V$. Then we have

$$\begin{aligned}
&f(x_0) \in V \\
\implies &\exists \varepsilon \in \mathbb{R}^+ : B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big) \subseteq V && \text{(by Prop. II.1.2.15(a))} \\
\implies &\exists \delta \in \mathbb{R}^+ : f\big(B_{(X,d_X)}(x_0, \delta)\big) \subseteq B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big) \subseteq V.
\end{aligned}$$

By Prop. II.1.2.15(c) we know that $B_{(X,d_X)}(x_0, \delta) \in \mathcal{F}_X$. Since V was arbitrary, by Def. II.2.5.8 we know that f is continuous at x_0 from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$.

Now suppose that f is continuous at x_0 in the sense of Def. II.2.5.8. Then we have

$$\forall V \in \mathcal{F}_Y, f(x_0) \in V \implies \exists U \in \mathcal{F}_X : \begin{cases} x_0 \in U \\ f(U) \subseteq V \end{cases}$$

Let $\varepsilon \in \mathbb{R}^+$. Then we have

$$\begin{aligned}
&B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big) \in \mathcal{F}_Y && \text{(by Prop. II.1.2.15(c))} \\
\implies &\exists U \in \mathcal{F}_X : \begin{cases} x_0 \in U \\ f(U) \subseteq B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big) \end{cases} \\
\implies &\exists \delta \in \mathbb{R}^+ : \begin{cases} B_{(X,d_X)}(x_0, \delta) \subseteq U \\ f(U) \subseteq B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big) \end{cases} && \text{(by Prop. II.1.2.15(a))} \\
\implies &\exists \delta \in \mathbb{R}^+ : \\
&f\big(B_{(X,d_X)}(x_0, \delta)\big) \subseteq B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big) \\
\implies &\exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon \Big).
\end{aligned}$$

Since ε was arbitrary, by Def. II.2.1.1 we know that f is continuous at x_0 from (X, d_X) to (Y, d_Y).

Since x_0 was arbitrary, we conclude that f is continuous on X from (X, d_X) to (Y, d_Y) iff f is continuous on X from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$. □

Ex. II.2.5.16. Show that when Thm. II.2.1.4 is extended to topological spaces, that (a) implies (b). (The converse is false, but constructing an example is diffcult.) Show that when Thm. II.2.1.5 is extended to topological spaces, that (a), (c), (d) are all equivalent to each other, and imply (b). (Again, the converse implications are false, but diffcult to prove.)

Proof. Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be two topological spaces. Let $f : X \to Y$ be a function. Let $x_0 \in X$. We first show that if f is continuous at x_0 from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$ and $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X which converges to x_0 in $(X, \mathcal{F}_X)$, then $\big(f(x)\big)_{n=1}^{\infty}$ converges to $f(x_0)$ in $(Y, \mathcal{F}_Y)$. Since f is continuous at x_0 from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$, by Def. II.2.5.8 we know that

$$\forall V \in \mathcal{F}_Y, f(x_0) \in V \implies \exists U \in \mathcal{F}_X : \begin{cases} x_0 \in U \\ f(U) \subseteq V \end{cases}$$

Since $(x^{(n)})_{n=1}^{\infty}$ converges to x_0 in $(X, \mathcal{F}_X)$, by Def. II.2.5.4 we have

$$\forall U \in \mathcal{F}_X, x_0 \in U \implies \exists N \in \mathbb{Z}^+ : \forall n \geq N, x^{(n)} \in U$$

This means

$$\forall V \in \mathcal{F}_Y, f(x_0) \in V \implies \exists U \in \mathcal{F}_X : \begin{cases} x_0 \in U \\ f(U) \subseteq V \\ \exists N \in \mathbb{Z}^+ : \forall n \geq N, f(x^{(n)}) \in f(U) \subseteq V \end{cases}$$

and we have

$$\forall V \in \mathcal{F}_Y, f(x_0) \in V \implies \exists N \in \mathbb{Z}^+ : \forall n \geq N, f(x^{(n)}) \in V.$$

By Def. II.2.5.4 we know that $\big(f(x^{(n)})\big)_{n=1}^{\infty}$ converges to $f(x_0)$ in $(Y, \mathcal{F}_Y)$. Since x_0 was arbitrary, we conclude that if f is continuous on X from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$, then whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X which converges to some $x_0 \in X$ in $(X, \mathcal{F})$, the sequence $\big(f(x^{(n)})\big)_{n=1}^{\infty}$ converges to $f(x_0)$ in $(Y, \mathcal{F}_Y)$.

Next we show that if f is continuous on X from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$, then whenever $V \in \mathcal{F}_Y$, the set $f^{-1}(V) \in \mathcal{F}_X$. Let $V \in \mathcal{F}_Y$ and let $x_0 \in f^{-1}(V)$. Since f is continuous at x_0 from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$, we know that

$$\exists U \in \mathcal{F}_X : \begin{cases} x_0 \in U \\ f(U) \subseteq V \end{cases}$$

Since $f(U) \subseteq V$, we know that $U \subseteq f^{-1}(V)$. Since $U \in \mathcal{F}_X$, by Def. II.2.5.5 we know that $x_0 \in \operatorname{int}_{(X,\mathcal{F}_X)}\big(f^{-1}(V)\big)$. Since x_0 was arbitrary, by Ex. II.2.5.11 we know that $f^{-1}(V) \in \mathcal{F}_X$.

Next we show that if $V \in \mathcal{F}_Y$ implies $f^{-1}(V) \in \mathcal{F}_X$, then U is closed in $(Y, \mathcal{F}_Y)$ implies $f^{-1}(U)$ is closed in $(X, \mathcal{F}_X)$. Let $U \subseteq Y$ such that U is closed in $(Y, \mathcal{F}_Y)$. Then we have

$$\begin{aligned} & U \text{ is closed in } (Y, \mathcal{F}_Y) \\ \implies & Y \setminus U \text{ is open in } (Y, \mathcal{F}_Y) \\ \implies & f^{-1}(Y \setminus U) \text{ is open in } (X, \mathcal{F}_X) && \text{(from the proof above)} \\ \implies & X \setminus f^{-1}(Y \setminus U) \text{ is closed in } (X, \mathcal{F}_X) \end{aligned}$$

$$\implies f^{-1}(U) \text{ is closed in } (X, \mathcal{F}_X).$$

Next we show that if U is closed in $(Y, \mathcal{F}_Y)$ implies $f^{-1}(U)$ is closed in $(X, \mathcal{F}_X)$, then f is continuous on X from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$. Let $x_0 \in X$ and let $V \in \mathcal{F}_Y$ such that $f(x_0) \in V$. We have

$$\begin{aligned}
& V \in \mathcal{F}_Y \\
\implies & V \text{ is open in } (Y, \mathcal{F}_Y) \\
\implies & Y \setminus V \text{ is closed in } (Y, \mathcal{F}_Y) \\
\implies & f^{-1}(Y \setminus V) \text{ is closed in } (Y, \mathcal{F}_Y) && \text{(by hypothesis)} \\
\implies & X \setminus f^{-1}(Y \setminus V) \text{ is open in } (Y, \mathcal{F}_Y) \\
\implies & f^{-1}(V) \text{ is open in } (Y, \mathcal{F}_Y) \\
\implies & f^{-1}(V) \in \mathcal{F}_X.
\end{aligned}$$

Since V was arbitrary, we know that

$$\forall V \in \mathcal{F}_Y, f(x_0) \in V \implies \exists U \in \mathcal{F}_X : \begin{cases} x_0 \in U \\ f(U) \subseteq V \end{cases}$$

and by Def. II.2.5.8 f is continuous at x_0 from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$. Since x_0 was arbitrary, we conclude that f is continuous on X from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$.

From all proofs above, we conclude that Thm. II.2.1.5(a)(c)(d) are equivalent in the topological version. □

Ex. II.2.5.17. Generalize both Thm. II.2.3.1 and Prop. II.2.3.2 to compact sets in a topological space.

Proof. Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be two topological spaces. Let $f : X \to Y$ be continuous function from $(X, \mathcal{F}_X)$ to $(Y, \mathcal{F}_Y)$, and let $K \subseteq X$ such that $(K, \mathcal{F}_K)$ is compact. We want to show that $\big(f(K), \mathcal{F}_{f(K)}\big)$ is also compact. Let $S \subseteq \mathcal{F}_{f(K)}$ be an open cover of $f(K)$, i.e., $f(K) = \bigcup_{V_K \in S} V_K$. By Def. II.2.5.7 we know that the set $S_Y = \{V \in \mathcal{F}_Y : V \cap f(K) \in S\}$ is non-empty. Then we have

$$\begin{aligned}
& \forall V \in S_Y, f^{-1}(V) \in \mathcal{F}_X && \text{(by Ex. II.2.5.16)} \\
\implies & \forall V \in S_Y, f^{-1}(V) \cap K \in \mathcal{F}_K && \text{(by Def. II.2.5.7)} \\
\implies & \forall V \in S_Y, f\big(f^{-1}(V) \cap K\big) = V \cap f(K) \\
\implies & f(K) = \bigcup_{V \in S_Y} f\big(f^{-1}(V) \cap K\big) = \bigcup_{V \in S_Y} \big(V \cap f(K)\big) \\
\implies & K \subseteq \bigcup_{V \in S_Y} \big(f^{-1}(V) \cap K\big)
\end{aligned}$$

$$\implies \exists S'_Y \subseteq S_Y : \begin{cases} S'_Y \text{ is finite} \\ K \subseteq \bigcup\limits_{V \in S'_Y} \left(f^{-1}(V) \cap K\right) \\ f(K) = \bigcup\limits_{V \in S'_Y} \left(V \cap f(K)\right) \end{cases} \qquad \text{(by Def. II.2.5.9)}$$

Since S was arbitrary, by Def. II.2.5.9 we know that $\left(f(K), \mathcal{F}_{f(K)}\right)$ is compact.

Now let $(X, \mathcal{F})$ be a compact topological space. Define

$$\mathcal{F}_{\mathbb{R}} = \{V \subseteq \mathbb{R} : V \text{ is open in } (\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})\}.$$

Let $f : X \to \mathbb{R}$ be a continuous function from $(X, \mathcal{F})$ to $(\mathbb{R}, \mathcal{F}_{\mathbb{R}})$. We want to show that f is bounded. Furthermore, if $X \neq \emptyset$, then there exists some $x_{\min}, x_{\max} \in X$ such that $f(x_{\min}) \leq f(x) \leq f(x_{\max})$ for each $x \in X$. We have

$$\begin{aligned} &\begin{cases} (X, \mathcal{F}) \text{ is compact} \\ f \text{ is continuous from } (X, \mathcal{F}) \text{ to } (\mathbb{R}, \mathcal{F}_{\mathbb{R}}) \end{cases} \\ \implies &\left(f(X), \mathcal{F}_{f(X)}\right) \text{ is compact} && \text{(from the proof above)} \\ \implies &\begin{cases} f(X) \text{ is closed in } (\mathbb{R}, \mathcal{F}_{\mathbb{R}}) \\ \left(f(X), d_{l^1}|_{f(X)\times f(X)}\right) \text{ is bounded} \end{cases} && \text{(by Thm. II.1.5.7)} \\ \implies &\begin{cases} f(X) \text{ is closed in } (\mathbb{R}, \mathcal{F}_{\mathbb{R}}) \\ f(X) \text{ is bounded subset of } \mathbb{R} \end{cases} && \text{(by Ex. II.1.5.1)} \end{aligned}$$

The rest follows as in Prop. II.2.3.2. $\square$

Chapter II.3

Uniform convergence

Note. It turns out that there are several different concepts of convergence of functions; here we describe the two most important ones, *pointwise convergence* and *uniform convergence*. (There are other types of convergence for functions, such as L^1 convergence, L^2 convergence, convergence in measure, almost everywhere convergence, and so forth, but these are beyond the scope of this text.) The two notions are related, but not identical; the relationship between the two is somewhat analogous to the relationship between continuity and uniform continuity.

II.3.1 Limiting values of functions

Def. II.3.1.1 (Limiting value of a function)**.** Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, and let $f : E \to Y$ be a function. If $x_0 \in X$ is an adherent point of E, and $L \in Y$, we say that $f(x)$ *converges to* L *in* Y *as* x *converges to* x_0 *in* E, or write $\lim\limits_{x \to x_0; x \in E} f(x) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y\big(f(x), L\big) < \varepsilon$ for all $x \in E$ such that $d_X(x, x_0) < \delta$.

Rmk. II.3.1.2. Some authors exclude the case $x = x_0$ from the above definition, thus requiring $0 < d_X(x, x_0) < \delta$. In our current notation, this would correspond to removing x_0 from E, thus one would consider

$$\lim_{x \to x_0; x \in E \setminus \{x_0\}} f(x)$$

instead of

$$\lim_{x \to x_0; x \in E} f(x).$$

Note. Comparing this with Def. II.2.1.1, we see that f is continuous at x_0 iff

$$\lim_{x \to x_0; x \in X} f(x) = f(x_0).$$

Thus, f is continuous on X iff we have

$$\lim_{x \to x_0; x \in X} f(x) = f(x_0) \text{ for all } x_0 \in X.$$

Rmk. II.3.1.4. Often we shall omit the condition $x \in X$, and abbreviate

$$\lim_{x \to x_0; x \in X} f(x)$$

as simply

$$\lim_{x \to x_0} f(x)$$

when it is clear what space x will range in.

Prop. II.3.1.5. Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, and let $f : E \to Y$ be a function. Let $x_0 \in X$ be an adherent point of E and $L \in Y$. Then the following four statements are logically equivalent:

(a) $\lim_{x \to x_0; x \in E} f(x) = L$.

(b) For every sequence $(x^{(n)})_{n=1}^{\infty}$ in E which converges to x_0 with respect to the metric d_X, the sequence $\big(f(x^{(n)})\big)_{n=1}^{\infty}$ converges to L with respect to the metric d_Y.

(c) For every open set $V \subseteq Y$ which contains L, there exists an open set $U \subseteq X$ containing x_0 such that $f(U \cap E) \subseteq V$.

(d) If one defines the function $g : E \cup \{x_0\} \to Y$ by defining $g(x_0) := L$, and $g(x) := f(x)$ for $x \in E \setminus \{x_0\}$, then g is continuous at x_0. Furthermore, if $x_0 \in E$, then $f(x_0) = L$.

Proof. We first show that statement (a) implies statement (b). Suppose that

$$d_Y - \lim_{x \to x_0; x \in E} f(x) = L.$$

By Def. II.3.1.1 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \Big(\forall x \in E, d_X(x, x_0) < \delta \implies d_Y\big(f(x), L\big) < \varepsilon\Big).$$

Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in E such that $\lim_{n\to\infty} d_X(x^{(n)}, x_0) = 0$. By Def. II.1.1.14 we have

$$\forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, d_X(x^{(n)}, x_0) < \delta.$$

Since $(x^{(n)})_{n=1}^{\infty}$ is in E, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \begin{cases} \exists N \in \mathbb{Z}^+ : \forall n \geq N, d_X(x^{(n)}, x_0) < \delta \\ d_X(x^{(n)}, x_0) < \delta \implies d_Y\big(f(x^{(n)}), L\big) < \varepsilon \end{cases}$$

and

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, d_Y\big(f(x^{(n)}, L)\big) < \varepsilon.$$

By Def. II.1.1.14 we have $\lim\limits_{n\to\infty} d_Y\big(f(x^{(n)}), L\big) = 0$. Since $(x^{(n)})_{n=1}^{\infty}$ was arbitrary, we conclude that (a) implies (b).

Next we show that statement (b) implies statement (a). Suppose that if $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X such that $\lim\limits_{n\to\infty} d_X(x^{(n)}, x_0) = 0$, then $\lim\limits_{n\to\infty} d_Y\big(f(x), L\big) = 0$. Suppose for the sake of contradiction that

$$d_Y - \lim_{x\to x_0; x\in X} f(x) \neq L.$$

Then by Def. II.3.1.1 we have

$$\exists \varepsilon \in \mathbb{R}^+ : \forall \delta \in \mathbb{R}^+, \exists x \in X : \begin{cases} d_X(x, x_0) < \delta \\ d_Y\big(f(x), L\big) \geq \varepsilon \end{cases}$$

Thus, we can choose one sequence $(x^{(n)})_{n=1}^{\infty}$ which satsifies

$$\forall n \in \mathbb{Z}^+, \begin{cases} d_X(x^{(n)}, x_0) < \dfrac{1}{n} \\ d_Y\big(f(x^{(n)}), L\big) \geq \varepsilon \end{cases}$$

By squeeze test we have $\lim\limits_{n\to\infty} d_X(x^{(n)}, x_0) = 0$. But by hypothesis we know that $\lim\limits_{n\to\infty} d_Y\big(f(x^{(n)}),$ 0, which means

$$\exists N \in \mathbb{Z}^+ : \forall n \geq N, d_Y\big(f(x^{(n)}), L\big) < \varepsilon,$$

a contradiction. Thus, we have

$$d_Y - \lim_{x\to x_0; x\in X} f(x) = L$$

and we conclude that statements (a)(b) are equivalent.

Next we show that statement (a) implies statement (c). Suppose that

$$d_Y - \lim_{x\to x_0; x\in E} f(x) = L.$$

By Def. II.3.1.1 we have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \Big(\forall x \in E, d_X(x, x_0) < \delta \implies d_Y\big(f(x), L\big) < \varepsilon\Big) \\ \implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \Big(x \in B_{(X, d_X)}(x_0, \delta) \cap E \implies d_Y\big(f(x), L\big) < \varepsilon\Big). \end{aligned}$$

Let V be an open set in (Y, d_Y) such that $L \in V$. Then we have

$$V = \operatorname{int}_{(Y, d_Y)}(V) \qquad \text{(by Prop. II.1.2.15(a))}$$

$$
\begin{aligned}
&\implies \exists \varepsilon \in \mathbb{R}^+ : B_{(Y,d_Y)}(L,\varepsilon) \subseteq V && \text{(by Def. II.1.2.5)}\\
&\implies \exists \delta \in \mathbb{R}^+ :\\
&\qquad \begin{cases} x \in B_{(X,d_X)}(x_0,\delta) \cap E \implies d_Y\big(f(x),L\big) < \varepsilon \\ f\big(B_{(X,d_X)}(x_0,\delta) \cap E\big) \subseteq B_{(Y,d_Y)}\big(L,\varepsilon\big) \subseteq V \end{cases}
\end{aligned}
$$

and by Prop. II.1.2.15(c) we know that $B_{(X,d_X)}(x_0,\delta)$ is open in (X,d_X). Since V was arbitrary, we conclude that statement (a) implies statement (c).

Next we show that statement (c) implies statement (a). Suppose that

$$
\forall V \subseteq Y, \begin{cases} L \in V \\ V \text{ is open in } (Y,d_Y) \end{cases} \implies \exists U \subseteq X : \begin{cases} x_0 \in U \\ U \text{ is open in } (X,d_X) \\ f(U \cap E) \subseteq V \end{cases}
$$

Let $\varepsilon \in \mathbb{R}^+$. By Prop. II.1.2.15(c) we know that $B_{(Y,d_Y)}(L,\varepsilon)$ is open in (Y,d_Y). By hypothesis we know that there exists some $U \subseteq X$ such that

$$
\begin{cases} x_0 \in U \\ U \text{ is open in } (X,d_X) \\ f(U \cap E) \subseteq B_{(Y,d_Y)}(L,\varepsilon) \end{cases}
$$

Then we have

$$
\begin{aligned}
&\begin{cases} x_0 \in U \\ U = \operatorname{int}_{(X,d_X)}(U) \end{cases} && \text{(by Prop. II.1.2.15(a))}\\
&\implies \exists \delta \in \mathbb{R}^+ : B_{(X,d_X)}(x_0,\delta) \subseteq U && \text{(by Def. II.1.2.5)}\\
&\implies \exists \delta \in \mathbb{R}^+ : B_{(X,d_X)}(x_0,\delta) \cap E \subseteq U \cap E\\
&\implies \exists \delta \in \mathbb{R}^+ :\\
&\qquad f\big(B_{(X,d_X)}(x_0,\delta) \cap E\big) \subseteq f(U \cap E) \subseteq B_{(Y,d_Y)}(L,\varepsilon)\\
&\implies \exists \delta \in \mathbb{R}^+ :\\
&\qquad \Big(\forall x \in E, d_X(x,x_0) < \delta \implies d_Y\big(f(x),L\big) < \varepsilon\Big).
\end{aligned}
$$

Since ε was arbitrary, by Def. II.3.1.1 we have

$$
d_Y - \lim_{x \to x_0; x \in E} f(x) = L
$$

and we conclude that statements (a)(c) are equivalent.

Next we show that statement (a) implies statement (d). Suppose that

$$
d_Y - \lim_{x \to x_0; x \in E} f(x) = L.
$$

Then by Def. II.3.1.1 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \Big(\forall x \in E, d_X(x, x_0) < \delta \implies d_Y\big(f(x), L\big) < \varepsilon\Big).$$

Let $g : E \cup \{x_0\} \to Y$ be a function where

$$\forall x \in E \cup \{x_0\}, g(x) = \begin{cases} L & \text{if } x = x_0 \\ f(x) & \text{if } x \neq x_0 \end{cases}$$

Then we have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\ &\Big(\forall x \in E, d_X(x, x_0) < \delta \implies d_Y\big(f(x), L\big) < \varepsilon\Big) \\ \implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\ &\Big(\forall x \in E \cup \{x_0\}, d_X(x, x_0) < \delta \implies d_Y\big(g(x), g(x_0)\big) < \varepsilon\Big). \end{aligned}$$

Thus, by Def. II.2.1.1 g is continuous at x_0 from $(E \cup \{x_0\}, d_X|_{(E\cup\{x_0\})\times(E\cup\{x_0\})})$ to (Y, d_Y).

Now suppose that $x_0 \in E$. We claim that $f(x_0) = L$. Suppose for the sake of contradiction that $f(x_0) \neq L$. Then by Def. II.1.1.2(b) we have $d_Y\big(f(x_0), L\big) > 0$. Let $r = d_Y\big(f(x_0), L\big)$. By Def. II.3.1.1 we have

$$\exists \delta \in \mathbb{R}^+ : \forall x \in E, d_X(x, x_0) < \delta \implies d_Y\big(f(x), L\big) < r.$$

Since $x_0 \in E$, we have $d_X(x_0, x_0) = 0 < \delta$. But then we have $d_Y\big(f(x_0), L\big) < r = d_Y\big(f(x_0), L\big)$, a contradiction. Thus, we have $f(x_0) = L$.

Finally we show that statement (d) implies statement (a). Suppose that $g : E\cup\{x_0\} \to Y$ is a function where

$$\forall x \in E \cup \{x_0\}, g(x) = \begin{cases} L & \text{if } x = x_0 \\ f(x) & \text{if } x \neq x_0 \end{cases}$$

and g is continuous from $(E \cup \{x_0\}, d_X|_{(E\cup\{x_0\})\times(E\cup\{x_0\})})$ to (Y, d_Y). Suppose also that if $x_0 \in E$, then $f(x_0) = L$. Then by Def. II.2.1.1 we have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\ &\Big(\forall x \in E \cup \{x_0\}, d_X(x, x_0) < \delta \implies d_Y\big(g(x), g(x_0)\big) < \varepsilon\Big) \\ \implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\ &\begin{cases} \forall x \in E \setminus \{x_0\}, d_X(x, x_0) < \delta \implies d_Y\big(f(x), L\big) < \varepsilon \\ x_0 \in E \implies d_X(x_0, x_0) = 0 < \delta \implies f(x_0) = L \implies d_Y\big(f(x_0), L\big) < \varepsilon \end{cases} \\ \implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\ &\Big(\forall x \in E, d_X(x, x_0) < \delta \implies d_Y\big(f(x), L\big) < \varepsilon\Big). \end{aligned}$$

By Def. II.3.1.1 this means
$$d_Y - \lim_{x \to x_0; x \in E} f(x) = L.$$
We conclude that statements (a)(b)(c)(d) are all equivalent. $\square$

Rmk. II.3.1.6. Observe from Prop. II.3.1.5(b) and Prop. II.1.1.20 that a function $f(x)$ can converge to at most one limit L as x converges to x_0. In other words, if the limit

$$\lim_{x \to x_0; x \in E} f(x)$$

exists at all, then it can only take at most one value.

Rmk. II.3.1.7. The requirement that x_0 be an adherent point of E is necessary for the concept of limiting value to be useful, otherwise x_0 will lie in the exterior of E, the notion that $f(x)$ converges to L as x converges to x_0 in E is vacuous (for δ sufficiently small, there are no points $x \in E$ so that $d(x, x_0) < \delta$).

Rmk. II.3.1.8. Strictly speaking, we should write

$$d_Y - \lim_{x \to x_0; x \in E} f(x) \text{ instead of } \lim_{x \to x_0; x \in E} f(x),$$

since the convergence depends on the metric d_Y. However in practice it will be obvious what the metric d_Y is and so we will omit the d_Y- prefix from the notation.

— Exercises —

Ex. II.3.1.1. Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, let $f : E \to Y$ be a function, and let x_0 be an element of E. Assume that x_0 is an adherent point of $E \setminus \{x_0\}$ (or equivalently, that x_0 is not an *isolated point* of E). Show that the limit $\lim_{x \to x_0; x \in E} f(x)$ exists iff the limit $\lim_{x \to x_0; x \in E \setminus \{x_0\}} f(x)$ exists and is equal to $f(x_0)$. Also, show that if the limit $\lim_{x \to x_0; x \in E} f(x)$ exists at all, then it must equal $f(x_0)$.

Proof. Let $L \in Y$. By Def. II.1.1.2(a) we know that

$$\forall \varepsilon \in \mathbb{R}^+, d_Y\big(f(x_0), L\big) < \varepsilon \iff L = f(x_0).$$

Thus, we have

$$\begin{aligned}
&d_Y - \lim_{x \to x_0; x \in E \setminus \{x_0\}} f(x) = f(x_0) \\
\iff &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x \in E \setminus \{x_0\}, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon\Big) \qquad \text{(by Def. II.3.} \\
\iff &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ :
\end{aligned}$$

$$\Big(\forall x \in E, d_X(x,x_0) < \delta \implies d_Y\big(f(x),f(x_0)\big) < \varepsilon\Big) \qquad (E \setminus \{x_0\} \subseteq E)$$
$$\iff d_Y - \lim_{x \to x_0; x \in E} f(x) = f(x_0). \qquad \text{(by Def. II.3.}$$

□

Ex. II.3.1.2. Prove Prop. II.3.1.5.

Proof. See Prop. II.3.1.5. □

Ex. II.3.1.3. Use Prop. II.3.1.5(c) to define a notion of a limiting value of a function $f : E \to Y$ from one topological space $(X, \mathcal{F}_X)$ to another $(Y, \mathcal{F}_Y)$ where $E \subseteq X$. If X is a topological space and Y is a Hausdorff topological space (see Ex. II.2.5.4), prove the equivalence of Prop. II.3.1.5(c)(d) in this setting, as well as an analogue of Rmk. II.3.1.6. What happens to these statements of Y is not Hausdorff?

Proof. Let $(X, \mathcal{F}_X)$, $(Y, \mathcal{F}_Y)$ be topological spaces, let $E \subseteq X$, let $f : E \to Y$ be a function, let $x_0 \in \overline{E}_{(X,\mathcal{F}_X)}$, and let $L \in Y$. We say that $f(x)$ converges to L in Y as x converges to x_0 in E iff

$$\forall V \in \mathcal{F}_Y, L \in V \implies \exists U \in \mathcal{F}_X : \begin{cases} x_0 \in U \\ f(U \cap E) \subseteq V \end{cases}$$

We want to show that if $(Y, \mathcal{F}_Y)$ is Hausdorff, then the definition above is equivalent to the follow: If $g : E \cup \{x_0\} \to Y$ is a function such that

$$\forall x \in E \cup \{x_0\}, g(x) = \begin{cases} L & \text{if } x = x_0 \\ f(x) & \text{if } x \neq x_0 \end{cases}$$

and $(E \cup \{x_0\}, \mathcal{F}_{E\cup\{x_0\}})$ is a topological subspace induced by $(X, \mathcal{F}_X)$, then g is continuous at x_0 from $(E \cup \{x_0\}, \mathcal{F}_{E\cup\{x_0\}})$ to $(Y, \mathcal{F}_Y)$.

First, suppose that $f(x)$ converges to L in Y as x converges to x_0 in E. Let g be the function in the definition and let $V \in \mathcal{F}_Y$ such that $L \in V$. By hypothesis we know that

$$\exists U \in \mathcal{F}_X : \begin{cases} x_0 \in U \\ f(U \cap E) \subseteq V \end{cases}$$

Then by Def. II.2.5.7 we have $U \cap (E \cup \{x_0\}) \in \mathcal{F}_{E\cup\{x_0\}}$ and

$$\begin{aligned} g\big(U \cap (E \cup \{x_0\})\big) &= g\big((U \cap E) \cup \{x_0\}\big) \\ &= g\big((U \cap E) \setminus \{x_0\}\big) \cup g(\{x_0\}) \\ &= f\big((U \cap E) \setminus \{x_0\}\big) \cup \{L\} \\ &\subseteq f(U \cap E) \cup \{L\} \\ &\subseteq V. \end{aligned}$$

Since V was arbitrary, by Def. II.2.5.8 we know that g is continuous at x_0 from $(E \cup \{x_0\}, \mathcal{F}_{E\cup\{x_0\}})$ to $(Y, \mathcal{F}_Y)$.

Next suppose that $x_0 \in E$ and $f(x)$ converges to L in Y as x converges to x_0 in E. We want to show that $f(x_0) = L$. Suppose for the sake of contradiction that $f(x_0) \neq L$. Since $(Y, \mathcal{F}_Y)$ is Hausdorff, by Ex. II.2.5.4 we know that

$$\exists V, W \in \mathcal{F}_Y : \begin{cases} L \in V \\ f(x_0) \in W \\ V \cap W = \emptyset \end{cases}$$

By definition we have

$$\exists U_V, U_W \in \mathcal{F}_X : \begin{cases} x_0 \in U_V \\ x_0 \in U_W \\ f(U_V \cap E) \subseteq V \\ f(U_W \cap E) \subseteq W \end{cases}$$

By Def. II.2.5.1 we know that $U_V \cap U_W \in \mathcal{F}_X$. But then we have

$$\begin{cases} x_0 \in U_V \cap U_W \\ f(U_V \cap U_W \cap E) \subseteq f(U_V \cap E) \subseteq V \\ f(U_V \cap U_W \cap E) \subseteq f(U_W \cap E) \subseteq W \end{cases}$$

which means $V \cap W \neq \emptyset$, a contradiction. Thus, we have $f(x_0) = L$.

Now suppose that g is the function in the definition such that g is continuous at x_0 from $(E \cup \{x_0\}, \mathcal{F}_{E\cup\{x_0\}})$ to $(Y, \mathcal{F}_Y)$. Also suppose that if $x_0 \in E$, then $f(x_0) = L$. Let $V \in \mathcal{F}_Y$ such that $g(x_0) = L \in V$. By Def. II.2.5.8 we know that

$$\exists U \in \mathcal{F}_{E\cup\{x_0\}} : \begin{cases} x_0 \in U \\ g(U) \subseteq V \end{cases}$$

By Def. II.2.5.7 we know that

$$\exists U_X \in \mathcal{F}_X : U_X \cap (E \cup \{x_0\}) = U.$$

Since $x_0 \in U$, we know that $x_0 \in U_X$. Thus, we have

$$\begin{aligned} f(U_X \cap E) &= f\big((U_X \cap E) \setminus \{x_0\}\big) \cup f(E \cap \{x_0\}) && (f(E \cap \{x_0\}) = \emptyset \iff x_0 \notin E) \\ &\subseteq g\big((U_X \cap E) \setminus \{x_0\}\big) \cup g(\{x_0\}) && (x_0 \in E \iff f(x_0) = L = g(x_0)) \\ &= g\big(U_X \cap (E \cup \{x_0\})\big) \\ &= g(U) \\ &\subseteq V. \end{aligned}$$

Since V was arbitrary, we conclude that $f(x)$ converges to L in Y as x converges to x_0 in E.

If $(Y, \mathcal{F}_Y)$ is not Hausdorff, then $x_0 \in E$ may not implies $f(x_0) = L$. $\square$

Ex. II.3.1.4. Recall from Ex. II.2.5.5 that the extended real line $\mathbb{R}^*$ comes with a standard topology (the order topology). We view the natural numbers $\mathbb{N}$ as a subspace of this topological space, and $+\infty$ as an adherent point of $\mathbb{N}$ in $\mathbb{R}^*$. Let $(a_n)_{n=0}^{\infty}$ be a sequence taking values in a topological space $(Y, \mathcal{F}_Y)$, and let $L \in Y$. Show that $\lim_{n\to+\infty; n\in\mathbb{N}} a_n = L$ (in the sense of Ex. II.3.1.3) iff $\lim_{n\to\infty} a_n = L$ (in the sense of Def. II.2.5.4). This shows that the notions of limiting values of a sequence, and limiting values of a function, are compatible.

Proof. Let $(\mathbb{R}^*, \mathcal{F}_{\mathbb{R}^*})$ be the order topology in Ex. II.2.5.5. Let $f : \mathbb{N} \to Y$ be the function where $f(n) = a_n$ for each $n \in \mathbb{N}$. First, suppose that

$$\lim_{n\to+\infty; n\in\mathbb{N}} a_n = \lim_{n\to+\infty; n\in\mathbb{N}} f(n) = L.$$

By Ex. II.3.1.3 we have

$$\forall V \in \mathcal{F}_Y, L \in V \implies \exists U \in \mathcal{F}_{\mathbb{R}^*} : \begin{cases} +\infty \in U \\ f(U \cap \mathbb{N}) \subseteq V \end{cases}$$

Since $U \in \mathcal{F}_{\mathbb{R}^*}$, by Ex. II.2.5.5 we know that there exists an interval $I \subseteq \mathbb{R}^*$ such that $I \subseteq U$ and $+\infty \in I$. We know that I must in the form $(a, +\infty]$ for some $a \in \mathbb{R}$. By Archimedean property we know that there exists some $N \in \mathbb{N}$ such that $N > a$. Then we have

$$\begin{aligned} &\begin{cases} I \subseteq U \subseteq \mathbb{R}^* \\ I = (a, +\infty] \in \mathcal{F}_{\mathbb{R}^*} \\ N > a \end{cases} \\ &\implies \forall n \geq N+1, n \in I \\ &\implies \forall n \geq N+1, n \in U \cap \mathbb{N} \\ &\implies \forall n \geq N+1, f(n) \in f(U \cap \mathbb{N}) \\ &\implies \forall n \geq N+1, f(n) \in V. \end{aligned}$$

Since this is true for arbitrary V, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = L$$

in the sense of Def. II.2.5.4.

Now suppose that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} f(n) = L$$

in the sense of Def. II.2.5.4. Then we have

$$\forall V \in \mathcal{F}_Y, L \in V \implies \exists N \in \mathbb{N} : \forall n \geq N, f(n) \in V.$$

Let $I = (N, +\infty]$. Then we know that I is an interval of $\mathbb{R}^*$ and by Ex. II.2.5.5 we have $I \in \mathcal{F}_{\mathbb{R}^*}$. Observe that

$$I \cap \mathbb{N} = \{m \in \mathbb{N} : m \geq N+1\}$$

$$\begin{aligned}&\Longrightarrow \forall n \in I \cap \mathbb{N}, f(n) \in V\\&\Longrightarrow f(I \cap \mathbb{N}) \subseteq V.\end{aligned}$$

Since this is true for arbitrary $V \in \mathcal{F}_Y$, by Ex. II.3.1.3 we have

$$\lim_{n \to +\infty; n \in \mathbb{N}} a_n = \lim_{n \to +\infty; n \in \mathbb{N}} f(n) = L.$$

□

Ex. II.3.1.5. Let (X, d_X), (Y, d_Y), (Z, dZ) be metric spaces, and let $x_0 \in X$, $y_0 \in Y$, $z_0 \in Z$. let $E \subseteq X$ and let $f : E \to Y$ and $g : Y \to Z$ be functions. If we have $\lim\limits_{x \to x_0; x \in E} f(x) = y_0$ and $\lim\limits_{y \to y_0; y \in f(E)} g(y) = z_0$, conclude that $\lim\limits_{x \to x_0; x \in E} g \circ f(x) = z_0$.

Proof. By Def. II.3.1.1 we have

$$\begin{aligned}&d_Z - \lim_{y \to y_0; y \in f(E)} g(y) = z_0\\&\Longrightarrow \forall \varepsilon \in \mathbb{R}^+, \exists \delta' \in \mathbb{R}^+ :\\&\quad \Big(\forall y \in f(E), d_Y(y, y_0) < \delta' \implies d_Z\big(g(y), z_0\big) < \varepsilon\Big)\end{aligned}$$

and

$$\begin{aligned}&d_Y - \lim_{x \to x_0; x \in E} f(x) = y_0\\&\Longrightarrow \forall \delta' \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ :\\&\quad \Big(\forall x \in E, d_X(x, x_0) < \delta \implies d_Y\big(f(x), y_0\big) < \delta'\Big).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ :\\&\Big(\forall x \in E, d_X(x, x_0) < \delta \implies d_Y\big(f(x), y_0\big) < \delta' \implies d_Z\Big(g\big(f(x)\big), z_0\Big) < \varepsilon\Big)\end{aligned}$$

and by Def. II.3.1.1 $d_Z - \lim\limits_{x \to x_0; x \in E} g \circ f(x) = z_0$. □

Ex. II.3.1.6. State and prove an analogue of the limit laws in Proposition 9.3.14 in Analysis I when X is now a metric space rather than a subset of $\mathbb{R}$.

Proof. Let (X, d) be a metric space, let $d_1 = d_{l^1}|_{\mathbb{R} \times \mathbb{R}}$, let $E \subseteq X$, let $x_0 \in \overline{E}_{(X,d)}$, let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be functions, and let $c \in \mathbb{R}$. Suppose that

$$d_1 - \lim_{x \to x_0; x \in E} f(x) = L$$

$$d_1 - \lim_{x \to x_0; x \in E} g(x) = M$$

We want to show that

$$d_1 - \lim_{x \to x_0; x \in E} (f+g)(x) = L + M$$
$$d_1 - \lim_{x \to x_0; x \in E} (f-g)(x) = L - M$$
$$d_1 - \lim_{x \to x_0; x \in E} (fg)(x) = LM$$
$$d_1 - \lim_{x \to x_0; x \in E} \min(f,g)(x) = \min(L,M)$$
$$d_1 - \lim_{x \to x_0; x \in E} \max(f,g)(x) = \max(L,M)$$
$$d_1 - \lim_{x \to x_0; x \in E} (cf)(x) = cL$$
$$d_1 - \lim_{x \to x_0; x \in E} (f/g)(x) = L/M \text{ if } \begin{cases} \forall x \in E, g(x) \neq 0 \\ M \neq 0 \end{cases}$$

Let $f^* : E \cup \{x_0\} \to \mathbb{R}$ be the function

$$\forall x \in E, f^*(x) = \begin{cases} L & \text{if } x = x_0 \\ f(x) & \text{if } x \neq x_0 \end{cases}$$

and let $g^* : E \cup \{x_0\} \to \mathbb{R}$ be the function

$$\forall x \in E, g^*(x) = \begin{cases} M & \text{if } x = x_0 \\ g(x) & \text{if } x \neq x_0 \end{cases}$$

By Prop. II.3.1.5(c) we know that f^* and g^* are continuous at x_0 from (X,d) to $(\mathbb{R}, d_1)$. Thus, by Cor. II.2.2.3 we have

$f^* + g^*$ is continuous at x_0 from (X,d) to $(\mathbb{R}, d_1)$
$f^* - g^*$ is continuous at x_0 from (X,d) to $(\mathbb{R}, d_1)$
$f^* g^*$ is continuous at x_0 from (X,d) to $(\mathbb{R}, d_1)$
$\min(f^*, g^*)$ is continuous at x_0 from (X,d) to $(\mathbb{R}, d_1)$
$\max(f^*, g^*)$ is continuous at x_0 from (X,d) to $(\mathbb{R}, d_1)$
cf^* is continuous at x_0 from (X,d) to $(\mathbb{R}, d_1)$
f^*/g^* is continuous at x_0 from (X,d) to $(\mathbb{R}, d_1)$ if $\begin{cases} \forall x \in E, g(x) \neq 0 \\ M \neq 0 \end{cases}$

Since

$$\forall x \in E \cup \{x_0\}, (f^* + g^*)(x) = \begin{cases} L + M & \text{if } x = x_0 \\ f(x) + g(x) & \text{if } x \neq x_0 \end{cases}$$

$$(f^* - g^*)(x) = \begin{cases} L - M & \text{if } x = x_0 \\ f(x) - g(x) & \text{if } x \neq x_0 \end{cases}$$

$$(f^* g^*)(x) = \begin{cases} LM & \text{if } x = x_0 \\ f(x)g(x) & \text{if } x \neq x_0 \end{cases}$$

$$\min(f^*, g^*)(x) = \begin{cases} \min(L, M) & \text{if } x = x_0 \\ \min\big(f(x), g(x)\big) & \text{if } x \neq x_0 \end{cases}$$

$$\max(f^*, g^*)(x) = \begin{cases} \max(L, M) & \text{if } x = x_0 \\ \max\big(f(x), g(x)\big) & \text{if } x \neq x_0 \end{cases}$$

$$(cf^*)(x) = \begin{cases} cL & \text{if } x = x_0 \\ cf(x) & \text{if } x \neq x_0 \end{cases}$$

$$(f^*/g^*)(x) = \begin{cases} L/M & \text{if } x = x_0 \\ f(x)/g(x) & \text{if } x \neq x_0 \end{cases} \text{ when } \begin{cases} \forall x \in E, g(x) \neq 0 \\ M \neq 0 \end{cases}$$

by Prop. II.3.1.5(a)(d) we know that

$$d_1 - \lim_{x \to x_0; x \in E} (f+g)(x) = L + M$$

$$d_1 - \lim_{x \to x_0; x \in E} (f-g)(x) = L - M$$

$$d_1 - \lim_{x \to x_0; x \in E} (fg)(x) = LM$$

$$d_1 - \lim_{x \to x_0; x \in E} \min(f,g)(x) = \min(L, M)$$

$$d_1 - \lim_{x \to x_0; x \in E} \max(f,g)(x) = \max(L, M)$$

$$d_1 - \lim_{x \to x_0; x \in E} (cf)(x) = cL$$

$$d_1 - \lim_{x \to x_0; x \in E} (f/g)(x) = L/M \text{ if } \begin{cases} \forall x \in E, g(x) \neq 0 \\ M \neq 0 \end{cases}$$

□

II.3.2 Pointwise and uniform convergence

Def. II.3.2.1 (Pointwise convergence)**.** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^{\infty}$ *converges pointwise to* f *on* X if we have

$$\lim_{n \to \infty} f^{(n)}(x) = f(x)$$

for all $x \in X$, i.e.,

$$\lim_{n\to\infty} d_Y\big(f^{(n)}(x), f(x)\big) = 0.$$

Or in other words, for every x and every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y\big(f^{(n)}(x), f(x)\big) < \varepsilon$ for every $n > N$. We call the function f the *pointwise limit* of the functions $f^{(n)}$.

Rmk. II.3.2.2. Note that $f^{(n)}(x)$ and $f(x)$ are points in Y, rather than functions, so we are using our prior notion of convergence in metric spaces to determine convergence of functions. Also note that we are not really using the fact that (X, d_X) is a metric space (i.e., we are not using the metric d_X); for this definition it would suffice for X to just be a plain old set with no metric structure. However, later on we shall want to restrict our attention to *continuous* functions from X to Y, and in order to do so we need a metric on X (and on Y), or at least a topological structure. Also when we introduce the concept of *uniform convergence*, then we will definitely need a metric structure on X and Y; there is no comparable notion for topological spaces.

Note. From Prop. II.1.1.20 we see that a sequence $(f^{(n)})_{n=1}^{\infty}$ of functions from one metric space (X, d_X) to another (Y, d_Y) can have at most one pointwise limit f (this explains why we can refer to f as *the* pointwise limit). However, it is of course possible for a sequence of functions to have no pointwise limit, just as a sequence of points in a metric space do not necessarily have a limit.

Note. Pointwise convergence is a very natural concept, but it has a number of disadvantages: it does not preserve continuity, derivatives, limits, or integrals. The problem is that while $f^{(n)}(x)$ converges to $f(x)$ for each x, the *rate* of that convergence varies substantially with x. To put things another way, the convergence of $f^{(n)}$ to f is not *uniform* in x - the N that one needs to get $f^{(n)}(x)$ within ε of f depends on x as well as on ε. This motivates a stronger notion of convergence.

Def. II.3.2.7 (Uniform convergence)**.** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^{\infty}$ *converges uniformly to f on X* if for every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y\big(f^{(n)}(x), f(x)\big) < \varepsilon$ for every $n > N$ and $x \in X$. We call the function f the *uniform limit* of the functions $f^{(n)}$.

Rmk. II.3.2.8. Note that Def. II.3.2.7 is subtly different from the definition for pointwise convergence in Def. II.3.2.1. In the definition of pointwise convergence, N was allowed to depend on x; now it is not. The reader should compare this distinction to the distinction between continuity and uniform continuity (i.e., between Def. II.2.1.1 and Def. II.2.3.4).

Note. If $f^{(n)}$ converges uniformly to f on X, then it also converges pointwise to the same function f. Thus, when the uniform limit and pointwise limit both exist, then they have to be equal. However, the converse is not true.

Note. If a sequence $f^{(n)} : X \to Y$ of functions converges pointwise (or uniformly) to a function $f : X \to Y$, then the restrictions $f^{(n)}|_E : E \to Y$ of $f^{(n)}$ to some subset E of X will also converge pointwise (or uniformly) to $f|_E$.

— Exercises —

Ex. II.3.2.1. The purpose of this exercise is to demonstrate a concrete relationship between continuity and pointwise convergence, and between uniform continuity and uniform convergence. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For any $a \in \mathbb{R}$, let $f_a : \mathbb{R} \to \mathbb{R}$ be the shifted function $f_a(x) := f(x - a)$.

(a) Show that f is continuous iff, whenever $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge pointwise to f.

(b) Show that f is uniformly continuous iff, whenever $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge uniformly to f.

Proof. (a) Suppose that f is continuous on $\mathbb{R}$. Let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$ such that $\lim\limits_{n\to\infty} a_n = 0$. Let $x_0 \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\lim_{n\to\infty} a_n = 0 \\
\Longrightarrow\ &\lim_{n\to\infty} (x_0 - a_n) = x_0 \\
\Longrightarrow\ &\lim_{n\to\infty} f_{a_n}(x_0) = \lim_{n\to\infty} f(x_0 - a_n) = f(x_0). \qquad (f \text{ is continuous at } x_0 \text{ on } \mathbb{R})
\end{aligned}$$

Since x_0 was arbitrary, by Def. II.3.2.1 we know that $(f_{a_n})_{n=0}^{\infty}$ converges pointwise to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Since $(a_n)_{n=0}^{\infty}$ was arbitrary, we conclude that if $(a_n)_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$ such that $\lim\limits_{n\to\infty} a_n = 0$, then $(f_{a_n})_{n=0}^{\infty}$ converges pointwise to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$.

Now suppose that if $(a_n)_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$ such that $\lim\limits_{n\to\infty} a_n = 0$, then $(f_{a_n})_{n=0}^{\infty}$ converges pointwise to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Suppose for the sake of contradiction that there exists some $x_0 \in \mathbb{R}$ such that $\lim\limits_{x\to x_0; x\in\mathbb{R}} f(x) \neq f(x_0)$. Then we have

$$\exists \varepsilon \in \mathbb{R}^+ : \forall \delta \in \mathbb{R}^+, \exists x \in \mathbb{R} : \begin{cases} |x - x_0| < \delta \\ |f(x) - f(x_0)| > \varepsilon \end{cases}$$

Fix such ε. We choose a sequence $(a_n)_{n=0}^{\infty}$ in $\mathbb{R}$ such that $|a_n - x_0| < \dfrac{1}{n+1}$ for all $n \in \mathbb{N}$. Then we have $\lim\limits_{n\to\infty} |a_n - x_0| = \lim\limits_{n\to\infty} (a_n - x_0) = 0$. By hypothesis we have

$$\forall x \in \mathbb{R}, f(x) = \lim_{n\to\infty} f_{a_n - x_0}(x) = \lim_{n\to\infty} f(x - a_n + x_0).$$

But this means

$$\begin{cases} |x - a_n + x_0 - x| = |-a_n + x_0| < \dfrac{1}{n} \\ |f(x - a_n + x_0) - f(x)| < \varepsilon \end{cases}$$

a contradiction. Thus, we have $\lim_{x \to x_0; x \in \mathbb{R}} f(x) = f(x_0)$ for every $x_0 \in \mathbb{R}$ and f is continuous on $\mathbb{R}$. □

Proof. (b) Suppose that f is uniformly continuous on $\mathbb{R}$. Let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$ such that $\lim_{n \to \infty} a_n = 0$. Since f is uniformly continuous on $\mathbb{R}$, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon.$$

Fix one pair of ε and δ. Then we have

$$\begin{aligned} &\exists N \in \mathbb{N} : \forall n \geq N, |a_n| < \delta \\ \implies &\exists N \in \mathbb{N} : \forall n \geq N, |-a_n| < \delta \\ \implies &\exists N \in \mathbb{N} : \forall n \geq N, \forall x \in \mathbb{R}, |x - a_n - x| < \delta \\ \implies &\exists N \in \mathbb{N} : \forall n \geq N, \forall x \in \mathbb{R}, |f(x - a_n) - f(x)| < \varepsilon \\ \implies &\exists N \in \mathbb{N} : \forall n \geq N, \forall x \in \mathbb{R}, |f_{a_n}(x) - f(x)| < \varepsilon. \end{aligned}$$

Since this is true for arbitrary ε, by Def. II.3.2.7 we know that $(f_{a_n})_{n=0}^{\infty}$ converges uniformly to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R} \times \mathbb{R}}$. Since $(a_n)_{n=0}^{\infty}$ was arbitrary, we conclude that if $(a_n)_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$ such that $\lim_{n \to \infty} a_n = 0$, then $(f_{a_n})_{n=0}^{\infty}$ uniformly converges to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R} \times \mathbb{R}}$.

Now suppose that if $(a_n)_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$ such that $\lim_{n \to \infty} a_n = 0$, then $(f_{a_n})_{n=0}^{\infty}$ uniformly converges to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R} \times \mathbb{R}}$. Suppose for the sake of contradiction that f is not uniformly continuous on $\mathbb{R}$. Then we have

$$\exists \varepsilon \in \mathbb{R}^+ : \forall \delta \in \mathbb{R}^+, \exists x_1, x_2 \in \mathbb{R} : \begin{cases} |x_1 - x_2| < \delta \\ |f(x_1) - f(x_2)| > \varepsilon \end{cases}$$

Let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$ such that $\lim_{n \to \infty} |a_n| = 0$. Then we have

$$\begin{aligned} &\lim_{n \to \infty} |a_n| = 0 = \lim_{n \to \infty} a_n = \lim_{n \to \infty} -a_n \\ \implies &\forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \geq N, |-a_n| < \delta \\ \implies &\forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in \mathbb{R}, |x - a_n - x| < \delta \\ \implies &\forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in \mathbb{R}, |f(x - a_n) - f(x)| > \varepsilon \\ \implies &\forall \delta \in \mathbb{R}^+, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in \mathbb{R}, |f_{a_n}(x) - f(x)| > \varepsilon. \end{aligned}$$

But by hypothesis we know that $(f_{a_n})_{n=0}^{\infty}$ uniformly converges to f on $\mathbb{R}$, which by Def. II.3.2.7 means

$$\exists N' \in \mathbb{N} : \forall n \geq N', \forall x \in \mathbb{R}, |f_{a_n}(x) - f(x)| < \varepsilon,$$

a contradiction. Thus, f is uniformly continuous on $\mathbb{R}$. $\square$

Ex. II.3.2.2.

(a) Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function from X to Y. Show that if $f^{(n)}$ converges uniformly to f, then $f^{(n)}$ also converges pointwise to f.

(b) For each integer $n \geq 1$, let $f^{(n)} : (-1, 1) \to \mathbb{R}$ be the function $f^{(n)}(x) \coloneqq x^n$. Prove that $f^{(n)}$ converges pointwise to the zero function 0, but does not converge uniformly to any function $f : (-1, 1) \to \mathbb{R}$.

(c) Let $g : (-1, 1) \to \mathbb{R}$ be the function $g(x) \coloneqq x/(1-x)$. With the notation as in (b), show that the partial sums $\displaystyle\sum_{n=1}^{N} f^{(n)}$ converges pointwise as $N \to \infty$ to g, but does not converge uniformly to g, on the open interval $(-1, 1)$. What would happen if we replaced the open interval $(-1, 1)$ with the closed interval $[-1, 1]$?

Proof. (a) Let $x_0 \in X$. Since

$$\begin{aligned}
&(f^{(n)})_{n=1}^{\infty} \text{ converges uniformly to } f \text{ on } X \\
&\text{ with respect to } d_Y \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
&\forall n \geq N, \forall x \in X, d_Y\big(f^{(n)}(x), f(x)\big) < \varepsilon && \text{(by Def. II.3.2.7)} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
&\forall n \geq N, d_Y\big(f^{(n)}(x_0), f(x_0)\big) < \varepsilon \\
\implies &\lim_{n\to\infty} d_Y\big(f^{(n)}(x_0), f(x_0)\big) = 0 && \text{(by Def. II.1.1.14)}
\end{aligned}$$

and x_0 was arbitrary, by Def. II.3.2.1 we know that $(f^{(n)})_{n=1}^{\infty}$ converges pointwise to f on X with respect to d_Y. $\square$

Proof. (b) Let $z : (-1, 1) \to \mathbb{R}$ be the zero function, i.e., $z(x) = 0$ for each $x \in (-1, 1)$. Then we have

$$\begin{aligned}
&\forall x \in (-1, 1), \lim_{n\to\infty} x^n = 0 \\
\implies &\forall x \in (-1, 1), \lim_{n\to\infty} f^{(n)}(x) = 0 = z(x) \\
\implies &(f^{(n)})_{n=1}^{\infty} \text{ converges pointwise to } z \text{ on } (-1, 1)
\end{aligned}$$

with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. (by Def. II.3.2.1)

Suppose for the sake of contradiction that there exists a $f : (-1,1) \to \mathbb{R}$ such that $(f^{(n)})_{n=1}^{\infty}$ converges uniformly to f on $(-1,1)$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. By Ex. II.3.2.2(a) and Prop. II.1.1.20 we know that $f = z$. Then we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
&\forall n \geq N, \forall x \in (-1,1), \left| f^{(n)}(x) - z(x) \right| < \varepsilon \qquad \text{(by Def. II.3.2.7)} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
&\forall n \geq N, \forall x \in (-1,1), |x^n| < \varepsilon.
\end{aligned}$$

Now consider $\varepsilon = \dfrac{1}{2}$. Then we have

$$\begin{aligned}
&\exists N \in \mathbb{Z}^+ : \forall n \geq N, \forall x \in (-1,1), |x^n| < \frac{1}{2} \\
\implies &\exists N \in \mathbb{Z}^+ : \begin{cases} (\frac{1}{2})^{\frac{1}{n}} \in (-1,1) \\ \left| ((\frac{1}{2})^{\frac{1}{n}})^n \right| = \frac{1}{2} < \frac{1}{2} \end{cases}
\end{aligned}$$

a contradiction. Thus, $(f^{(n)})_{n=1}^{\infty}$ does not converge uniformly to any f on $(-1,1)$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. $\square$

Proof. (c) By Lemma 7.3.3 in Analysis I we have

$$\begin{aligned}
\forall x \in (-1,1), \lim_{N\to\infty} \sum_{n=1}^{N} f^{(n)}(x) &= \lim_{N\to\infty} \sum_{n=1}^{N} x^n \\
&= \lim_{N\to\infty} \left(\sum_{n=0}^{N} x^n - 1 \right) \\
&= \left(\lim_{N\to\infty} \sum_{n=0}^{N} x^n \right) - 1 \\
&= \frac{1}{1-x} - 1 \\
&= \frac{x}{1-x} \\
&= g(x).
\end{aligned}$$

Thus, by Def. II.3.2.1 we know that $(\sum_{n=1}^{N} f^{(n)})_{N=1}^{\infty}$ converges pointwise to g on $(-1,1)$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Suppose for the sake of contradiction that $(\sum_{n=1}^{N} f^{(n)})_{N=1}^{\infty}$ converges uniformly to g on $(-1,1)$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Then by Def. II.3.2.7 we know that

$$\begin{aligned}
&\forall\varepsilon\in\mathbb{R}^+,\exists M\in\mathbb{Z}^+:\forall N\geq M,\forall x\in(-1,1),\left|\sum_{n=1}^{N} f^{(n)}(x)-g(x)\right|<\varepsilon\\
\implies&\forall\varepsilon\in\mathbb{R}^+,\exists M\in\mathbb{Z}^+:\forall N\geq M,\forall x\in(-1,1),\left|\frac{x(1-x^N)}{1-x}-\frac{x}{1-x}\right|<\varepsilon\\
\implies&\forall\varepsilon\in\mathbb{R}^+,\exists M\in\mathbb{Z}^+:\forall N\geq M,\forall x\in(-1,1),\left|\frac{-x^{N+1}}{1-x}\right|=\left|\frac{x^{N+1}}{1-x}\right|<\varepsilon.
\end{aligned}$$

Now consider $\varepsilon=\frac{1}{2}$. Then we have

$$\begin{aligned}
&\exists N\geq M:\forall n\geq N,\forall x\in(-1,1),\left|\frac{x^{N+1}}{1-x}\right|<\frac{1}{2}\\
\implies&\exists N\geq M:\begin{cases}(\frac{1}{2})^{\frac{1}{N+1}}\in(-1,1)\\ \left|\dfrac{((\frac{1}{2})^{\frac{1}{N+1}})^{N+1}}{1-(\frac{1}{2})^{\frac{1}{N+1}}}\right|=\dfrac{\frac{1}{2}}{1-(\frac{1}{2})^{\frac{1}{N+1}}}<\frac{1}{2}\end{cases}
\end{aligned}$$

But we know that

$$\begin{aligned}
&(\frac{1}{2})^{\frac{1}{N+1}}>\frac{1}{2}\\
\implies&1-(\frac{1}{2})^{\frac{1}{N+1}}<1-\frac{1}{2}=\frac{1}{2}\\
\implies&\frac{\frac{1}{2}}{1-(\frac{1}{2})^{\frac{1}{N+1}}}>1,
\end{aligned}$$

a contradiction. Thus, $(\sum_{n=1}^{N} f^{(n)})_{N=1}^{\infty}$ does not converge uniformly to g on $(-1,1)$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$.

If we replace $(-1,1)$ with $[-1,1]$, then by Lemma 7.3.3 in Analysis I $\sum_{n=1}^{\infty} 1$ and $\sum_{n=1}^{\infty} -1$ does not converge, thus by Def. II.3.2.1 $(\sum_{n=1}^{N} f^{(n)})_{N=1}^{\infty}$ does not converge pointwise to g on $[-1,1]$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. □

Ex. II.3.2.3. Let (X, d_X) a metric space, and for every integer $n \geq 1$, let $f_n : X \to \mathbb{R}$ be a real-valued function. Suppose that f_n converges pointwise to another function $f : X \to \mathbb{R}$ on X (in this question we give $\mathbb{R}$ the standard metric $d(x,y) = |x-y|$). Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the functions $h \circ f_n$ converge pointwise to $h \circ f$ on X, where $h \circ f_n : X \to \mathbb{R}$ is the function $h \circ f_n(x) := h\big(f_n(x)\big)$, and similarly for $h \circ f$.

Proof. Let $x_0 \in X$. We have

$$\begin{aligned}
& h \text{ is continuous on } \mathbb{R} \text{ with respect to } d \\
\implies & h \text{ is continuous at } x_0 \text{ with respect to } d \\
\implies & \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
& \big(\forall x \in \mathbb{R}, |x - x_0| < \delta \implies |h(x) - h(x_0)| < \varepsilon\big).
\end{aligned}$$

Fix one pair of ε and δ. Then we have

$$\begin{aligned}
& (f_n)_{n=1}^{\infty} \text{ converges pointwise to } f \text{ on } X \text{ with respect to } d \\
\implies & \lim_{n\to\infty} |f_n(x_0) - f(x_0)| = 0 && \text{(by Def. II.3.2.1)} \\
\implies & \exists N \in \mathbb{Z}^+ : \forall n \geq N, |f_n(x_0) - f(x_0)| < \delta \\
\implies & \exists N \in \mathbb{Z}^+ : \forall n \geq N, \big|h\big(f_n(x_0)\big) - h\big(f(x_0)\big)\big| < \varepsilon.
\end{aligned}$$

Since ε was arbitrary, by Def. II.1.1.14 we know that

$$\lim_{n\to\infty} \big|h\big(f_n(x_0)\big) - h\big(f(x_0)\big)\big| = \lim_{n\to\infty} |(h \circ f_n)(x_0) - (h \circ f)(x_0)| = 0.$$

Since x_0 was arbitrary, by Def. II.3.2.1 we know $(h \circ f_n)_{n=1}^{\infty}$ converges pointwise to $h \circ f$ on X with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. □

Ex. II.3.2.4. Let $f_n : X \to Y$ be a sequence of bounded functions from one metric space (X, d_X) to another metric space (Y, d_Y). Suppose that f_n converges uniformly to another function $f : X \to Y$. Suppose that f is a bounded function; i.e., there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$. Show that the sequence f_n is *uniformly bounded*; i.e., there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f_n(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$ and all positive integers n.

Proof. Since f is bounded, by Def. II.1.5.3 we have

$$\forall y \in Y, \exists \varepsilon \in \mathbb{R}^+ : f(X) \subseteq B_{(Y,d_Y)}(y, \varepsilon).$$

We choose one pair of y and ε. Since $(f_n)_{n=1}^{\infty}$ converges uniformly to f on X with respect to d_Y, we have

$$\begin{aligned}
&\exists N \in \mathbb{Z}^+ : \forall n \geq N, \forall x \in X, d_Y\big(f_n(x), f(x)\big) < \varepsilon && \text{(by Def. II.3.2.7)}\\
\implies &\exists N \in \mathbb{Z}^+ : \forall n \geq N, \forall x \in X, \\
&d_Y\big(f_n(x), y\big) \leq d_Y\big(f_n(x), f(x)\big) + d_Y\big(f(x), y\big) < \varepsilon + \varepsilon && \text{(by Def. II.1.1.2(d))}\\
\implies &\exists N \in \mathbb{Z}^+ : \forall n \geq N, \forall x \in X, f_n(x) \in B_{(Y,d_Y)}\big(y, 2\varepsilon\big)\\
\implies &\exists N \in \mathbb{Z}^+ : \forall n \geq N, f_n(X) \subseteq B_{(Y,d_Y)}\big(y, 2\varepsilon\big)
\end{aligned}$$

Now fix N. Let $S = \big\{n \in \mathbb{Z}^+ : n < N\big\}$. Then S is finite. If $S = \emptyset$, then we have $N = 1$ and thus by definition $(f_n)_{n=1}^{\infty}$ is uniformly bounded. So suppose that $S \neq \emptyset$. By hypothesis we know that f_n is bounded for each $n \in S$, thus by Def. II.1.5.3 we have

$$\forall n \in S, \exists \delta_n \in \mathbb{R}^+ : f_n(X) \subseteq B_{(Y,d_Y)}(y, \delta_n).$$

We choose one δ_n for each $n \in S$. Since S is finite, we know that $\delta_{\max} = \max\{\delta_n : n \in S\}$ is well-defined. Then we have

$$\begin{aligned}
&\begin{cases} \forall n \in S, f_n(X) \subseteq B_{(Y,d_Y)}(y, \delta_{\max}) \\ \forall n \geq N, f_n(X) \subseteq B_{(Y,d_Y)}(y, 2\varepsilon) \end{cases}\\
\implies &\forall n \in \mathbb{Z}^+, f_n(X) \subseteq B_{(Y,d_Y)}(y, 2\varepsilon + \delta_{\max}).
\end{aligned}$$

Since y was arbitrary, by Def. II.1.5.3 we know that f_n is bounded for each $n \in \mathbb{Z}^+$, i.e.,

$$\forall y \in Y, \exists r \in \mathbb{R}^+ : \forall n \in \mathbb{Z}^+, f_n(X) \subseteq B_{(Y,d_Y)}(y, r).$$

And by definition $(f_n)_{n=1}^{\infty}$ is uniformly bounded. $\square$

II.3.3 Uniform convergence and continuity

Thm. II.3.3.1 (Uniform limits preserve continuity I). Suppose $(f^{(n)})_{n=1}^{\infty}$ is a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. Let x_0 be a point in X. If the functions $f^{(n)}$ are continuous at x_0 for each n, then the limiting function f is also continuous at x_0.

Proof. We have

$$\begin{aligned}
&(f^{(n)})_{n=1}^{\infty} \text{ converges uniformly to } f \text{ on } X\\
&\text{with respect to } d_Y\\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ :\\
&\forall n \geq N, \forall x \in X, d_Y\big(f^{(n)}(x), f(x)\big) < \frac{\varepsilon}{3}. && \text{(by Def. II.3.2.7)}
\end{aligned}$$

We choose one pair of ε and N. For each $n \in \mathbb{Z}^+$, since $f^{(n)}$ is continuous at x_0 from (X, d_X) to (Y, d_Y), by Def. II.2.1.1 we have

$$\begin{aligned}
&\forall n \geq N, f^{(n)} \text{ is continuous at } x_0 \text{ from } (X, d_X) \text{ to } (Y, d_Y) \\
\implies &\forall n \geq N, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f^{(n)}(x), f^{(n)}(x_0)\big) < \frac{\varepsilon}{3}\Big) \\
\implies &\forall n \geq N, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) \\
&\leq d_Y\big(f(x), f^{(n)}(x)\big) + d_Y\big(f^{(n)}(x), f^{(n)}(x_0)\big) + d_Y\big(f^{(n)}(x_0), f(x_0)\big) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\Big) \\
\implies &\forall n \geq N, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), f(x_0)\big) < \varepsilon.
\end{aligned}$$

Since ε was arbitrary, by Def. II.2.1.1 we know that f is continuous at x_0 from (X, d_X) to (Y, d_Y). □

Cor. II.3.3.2 (Uniform limits preserve continuity II)**.** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. If the functions $f^{(n)}$ are continuous on X for each n, then the limiting function f is also continuous on X.

Proof. By applying Thm. II.3.3.1 to each $x \in X$ we conclude that f is continuous on X from (X, d_X) to (Y, d_Y). □

Prop. II.3.3.3 (Interchange of limits and uniform limits)**.** Let (X, d_X) and (Y, d_Y) be metric spaces, with Y complete, and let E be a subset of X. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from E to Y, and suppose that this sequence converges uniformly in E to some function $f : E \to Y$. Let $x_0 \in X$ be an adherent point of E, and suppose that for each n the limit $\lim_{x \to x_0; x \in E} f^{(n)}(x)$ exists. Then the limit $\lim_{x \to x_0; x \in E} f(x)$ also exists, and is equal to the limit of the sequence $\big(\lim_{x \to x_0; x \in E} f^{(n)}(x)\big)_{n=1}^{\infty}$; in other words we have the interchange of limits

$$\lim_{n \to \infty} \lim_{x \to x_0; x \in E} f^{(n)}(x) = \lim_{x \to x_0; x \in E} \lim_{n \to \infty} f^{(n)}(x).$$

Proof. For each $n \in \mathbb{Z}^+$, we define $d_Y - \lim_{x \to x_0; x \in E} f^{(n)}(x) = L^{(n)}$. We claim that the sequence $(L^{(n)})_{n=1}^{\infty}$ converges in Y with respect to d_Y. Since (Y, d_Y) is complete, by Def. II.1.4.10 it suffices to show that $(L^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in (Y, d_Y). Let $n_1, n_2 \in \mathbb{Z}^+$. Then by Def. II.3.2.7 we have

$$(f^{(n)})_{n=1}^{\infty} \text{ converges uniformly to } f \text{ on } X \text{ with respect to } d_Y$$

$$\Longrightarrow \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \forall x \in X, d_Y\big(f^{(n)}(x), f(x)\big) < \frac{\varepsilon}{4}$$

Now fix one pair of ε and N. Since $L^{(n)}$ exists for each $n \in \mathbb{N}$, by Def. II.3.1.1 we have

$$\begin{aligned}
&\forall n \geq N, d_Y - \lim_{x \to x_0; x \in E} f^{(n)}(x) = L^{(n)} \\
\Longrightarrow\ &\forall n \geq N, \exists \delta \in \mathbb{R}^+ : \Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f^{(n)}(x), L^{(n)}\big) < \frac{\varepsilon}{4}\Big) \\
\Longrightarrow\ &\forall n_1, n_2 \geq N, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(L^{(n_1)}, L^{(n_2)}\big) \\
&\leq d_Y\big(L^{(n_1)}, f^{(n_1)}(x)\big) + d_Y\big(f^{(n_1)}(x), f(x)\big) \\
&\quad + d_Y\big(f(x), f^{(n_2)}(x)\big) + d_Y\big(f^{(n_2)}(x), L^{(n_2)}\big) \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}\Big) \\
\Longrightarrow\ &\forall n_1, n_2 \geq N, d_Y\big(L^{(n_1)}, L^{(n_2)}\big) < \varepsilon
\end{aligned}$$

Since ε was arbitrary, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n_1, n_2 \geq N, d_Y\big(L^{(n_1)}, L^{(n_2)}\big) < \varepsilon$$

and by Def. II.1.4.6 $(L^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in (Y, d_Y).

Let $L \in Y$ such that $d_Y - \lim_{n \to \infty} L^{(n)} = L$. Again by Def. II.3.2.7 we have

$$\begin{aligned}
&(f^{(n)})_{n=1}^{\infty} \text{ converges uniformly to } f \text{ on } X \text{with respect to } d_Y \\
\Longrightarrow\ &\forall \varepsilon \in \mathbb{R}^+, \exists N_1 \in \mathbb{Z}^+ : \forall n \geq N_1, \forall x \in X, d_Y\big(f^{(n)}(x), f(x)\big) < \frac{\varepsilon}{3}.
\end{aligned}$$

Again we choose one pair of ε and N_1. Since L exists, by Def. II.3.1.1 we have

$$\begin{aligned}
&\lim_{n \to \infty} d_Y\big(L^{(n)}, L\big) = 0 \\
\Longrightarrow\ &\exists N_2 \in \mathbb{Z}^+ : \forall n \geq N_2, d_Y(L^{(n)}, L) < \frac{\varepsilon}{3} \\
\Longrightarrow\ &\exists N = \max(N_1, N_2) : \forall n \geq N, \\
&\begin{cases}
\exists \delta \in \mathbb{R}^+ : \forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f^{(n)}(x), L^{(n)}\big) < \dfrac{\varepsilon}{3} \\
d_Y(L^{(n)}, L) < \dfrac{\varepsilon}{3} \\
\forall x \in X, d_Y\big(f^{(n)}(x), f(x)\big) < \dfrac{\varepsilon}{3}
\end{cases} \\
\Longrightarrow\ &\exists N = \max(N_1, N_2) : \forall n \geq N, \exists \delta \in \mathbb{R}^+ : \\
&\Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), L\big)
\end{aligned}$$

$$\leq d_Y\big(f(x), f^{(n)}(x)\big) + d_Y\big(f^{(n)}(x), L^{(n)}\big) + d_Y\big(L^{(n)}, L\big) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\Big)$$
$$\implies \exists \delta \in \mathbb{R}^+ : \Big(\forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f(x), L\big) < \varepsilon\Big).$$

Since ε was arbitrary, by Def. II.3.1.1 we know that $d_Y - \lim\limits_{x \to x_0; x \in E} f(x) = L$. ☐

Prop. II.3.3.4. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of continuous functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. Let $x^{(n)}$ be a sequence of points in X which converge to some limit x. Then $f^{(n)}(x^{(n)})$ converges (in Y) to $f(x)$.

Proof. Let $x_0 \in X$. Suppose that $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X such that

$$\lim_{n \to \infty} d_X(x^{(n)}, x_0) = 0.$$

By Thm. II.3.3.1 we know that f is continuous at x_0 from (X, d_X) to (Y, d_Y). Thus, by Thm. II.2.1.4(a)(b) we have

$$\lim_{n \to \infty} d_Y\big(f(x^{(n)}), f(x_0)\big) = 0$$

and by Def. II.1.1.14 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N_1 \in \mathbb{Z}^+ : \forall n \geq N_1, d_Y\big(f(x^{(n)}), f(x_0)\big) < \frac{\varepsilon}{2}.$$

Now we choose one pair of ε and N_1. Since $(f^{(n)})_{n=1}^{\infty}$ converges uniformly to f on X with respect to d_Y, by Def. II.3.2.7 we have

$$\exists N_2 \in \mathbb{Z}^+ : \forall n \geq N_2, \forall x \in X, d_Y\big(f^{(n)}(x), f(x)\big) < \frac{\varepsilon}{2}$$
$$\implies \exists N_2 \in \mathbb{Z}^+ : \forall n \geq N_2, d_Y\big(f^{(n)}(x^{(n)}), f(x^{(n)})\big) < \frac{\varepsilon}{2}$$
$$\implies \exists N = \max(N_1, N_2) : \forall n \geq N,$$
$$d_Y\big(f^{(n)}(x^{(n)}), f(x_0)\big) \leq d_Y\big(f^{(n)}(x^{(n)}), f(x^{(n)})\big) + d_Y\big(f(x^{(n)}), f(x_0)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$\implies \exists N = \max(N_1, N_2) : \forall n \geq N, d_Y\big(f^{(n)}(x^{(n)}), f(x_0)\big) < \varepsilon.$$

Since ε was arbitrary, by Def. II.1.1.14 we know that

$$\lim_{n \to \infty} d_Y\big(f^{(n)}(x^{(n)}), f(x_0)\big) = 0.$$

Since x_0 was arbitrary, we conclude that for any $x_0 \in X$, if $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X such that

$$\lim_{n \to \infty} d_X(x^{(n)}, x_0) = 0,$$

then we have

$$\lim_{n \to \infty} d_Y\big(f^{(n)}(x^{(n)}), f(x_0)\big) = 0.$$

☐

Def. II.3.3.5 (Bounded functions)**.** A function $f : X \to Y$ from one metric space (X, d_X) to another (Y, d_Y) is *bounded* if $f(X)$ is a bounded set, i.e., there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$.

Prop. II.3.3.6 (Uniform limits preserve boundedness)**.** Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. If the functions $f^{(n)}$ are bounded on X for each n, then the limiting function f is also bounded on X.

Proof. Since $f^{(n)}$ is bounded in (Y, d_Y) for each $n \in \mathbb{Z}^+$, by Def. II.3.3.5 and Def. II.1.5.3 we have

$$\begin{aligned}
&\forall n \in \mathbb{Z}^+, \forall y \in Y, \exists \varepsilon \in \mathbb{R}^+ : f^{(n)}(X) \subseteq B_{(Y,d_Y)}(y, \varepsilon) \\
\implies &\forall n \in \mathbb{Z}^+, \forall y \in Y, \exists \varepsilon \in \mathbb{R}^+ : \forall x \in X, d_Y\big(f^{(n)}(x), y\big) < \varepsilon.
\end{aligned}$$

Now we choose y and ε for each $n \in \mathbb{Z}^+$ and we denote them as $y^{(n)}$ and $\varepsilon^{(n)}$. Since $(f^{(n)})_{n=1}^{\infty}$ converges uniformly to f on X with respect to d_Y, by Def. II.3.2.7 we have

$$\begin{aligned}
&\exists N \in \mathbb{Z}^+ : \forall n \geq N, \forall x \in X, d_Y\big(f^{(n)}(x), f(x)\big) < 1 \\
&\exists N \in \mathbb{Z}^+ : \forall x \in X, d_Y\big(f^{(N)}(x), f(x)\big) < 1 \\
\implies &\exists N \in \mathbb{Z}^+ : \forall x \in X, \\
&d_Y\big(f(x), y^{(N)}\big) \leq d_Y\big(f(x), f^{(N)}(x)\big) + d_Y\big(f^{(N)}(x), y^{(N)}\big) < \varepsilon^{(N)} + 1 \\
\implies &\exists N \in \mathbb{Z}^+ : \forall x \in X, d_Y\big(f(x), y^{(N)}\big) < \varepsilon^{(N)} + 1 \\
\implies &\exists N \in \mathbb{Z}^+ : f(X) \subseteq B_{(Y,d_Y)}(y^{(N)}, \varepsilon^{(N)} + 1).
\end{aligned}$$

Since $y^{(N)}$ was arbitrary, we have

$$\forall y \in Y, \exists \varepsilon \in \mathbb{R}^+ : f(X) \subseteq B_{(Y,d_Y)}(y, \varepsilon)$$

and by Def. II.1.5.3 and Def. II.3.3.5 f is bounded in (Y, d_Y). □

Rmk. II.3.3.7. The above propositions sound very reasonable, but one should caution that it only works if one assumes uniform convergence; pointwise convergence is not enough.

— Exercises —

Ex. II.3.3.1. Prove Thm. II.3.3.1. Explain briefly why your proof requires uniform convergence, and why pointwise convergence would not suffice.

Proof. See Thm. II.3.3.1. Without uniform convergence we cannot make $f^{(n)}(x)$ and $f(x)$ arbitrary close. □

Ex. II.3.3.2. Prove Prop. II.3.3.3.

Proof. See Prop. II.3.3.3. □

Ex. II.3.3.3. Compare Prop. II.3.3.3 with Example 1.2.8 in Analysis I. Can you now explain why the interchange of limits in Example 1.2.8 in Analysis I led to a false statement, whereas the interchange of limits in Prop. II.3.3.3 is justified?

Proof. By Ex. II.3.2.2(b) we know that $f^{(n)}(x) = x^{(n)}$ does not converge uniformly to any function $f : (-1, 1) \to \mathbb{R}$, thus the interchange of limits in Example 1.2.8 in Analysis I failed. □

Ex. II.3.3.4. Prove Prop. II.3.3.4.

Proof. See Prop. II.3.3.4. □

Ex. II.3.3.5. Give an example to show that Prop. II.3.3.4 fails if the phrase "converges uniformly" is replaced by "converges pointwise."

Proof. For each $n \in \mathbb{Z}^+$, let $f^{(n)} : [0, 1] \to \mathbb{R}$ be the function where $f^{(n)}(x) = x^n$ for each $x \in [0, 1]$. Let $f : [0, 1] \to \mathbb{R}$ be the function where

$$\forall x \in [0,1], f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1) \end{cases}$$

By Example 3.2.4 in Analysis II we know that $(f^{(n)})_{n=1}^{\infty}$ converges pointwise to f on X with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. By Ex. II.3.2.2(b) we know that $f^{(n)}(x) = x^{(n)}$ does not converge uniformly to f on X with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Let $(x^{(n)})_{n=1}^{\infty}$ be the sequence where $x^{(n)} = (\frac{1}{2})^{\frac{1}{n}}$ for each $n \in \mathbb{Z}^+$. Then we have

$$\lim_{n\to\infty} x^{(n)} = 1 = f(1).$$

But

$$\lim_{n\to\infty} f^{(n)}(x^{(n)}) = \lim_{n\to\infty} \left((\frac{1}{2})^{\frac{1}{n}}\right)^n = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 1.$$

Thus, Prop. II.3.3.4 fails when the phrase "converges uniformly" is replaced by "converges pointwise." □

Ex. II.3.3.6. Prove Prop. II.3.3.6.

Proof. See Prop. II.3.3.6. □

Ex. II.3.3.7. Give an example to show that Prop. II.3.3.6 fails if the phrase "converges uniformly" is replaced by "converges pointwise."

Proof. By Ex. II.3.2.2(c) we know that g is unbounded since

$$\lim_{x\to -1; x\in(-1,1)} g(x) = \lim_{x\to -1; x\in(-1,1)} \frac{x}{1-x} = \lim_{x\to -1; x\in(-1,1)} \frac{1}{1-x} - 1 = +\infty.$$

□

Ex. II.3.3.8. Let (X,d) be a metric space, and for every positive integer n, let $f_n : X \to \mathbb{R}$ and $g_n : X \to \mathbb{R}$ be functions. Suppose that $(f_n)_{n=1}^{\infty}$ converges uniformly to another function $f : X \to \mathbb{R}$, and that $(g_n)_{n=1}^{\infty}$ converges uniformly to another function $g : X \to \mathbb{R}$. Suppose also that the functions $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are uniformly bounded, i.e., there exists an $M > 0$ such that $|f_n(x)| \le M$ and $|g_n(x)| \le M$ for all $n \ge 1$ and $x \in X$. Prove that the functions $f_n g_n : X \to \mathbb{R}$ converge uniformly to $fg : X \to \mathbb{R}$.

Proof. Let $d_1 = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Since $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are uniformly bounded, by Def. II.3.3.5 we know that f_n and g_n are bounded in $(\mathbb{R}, d_1)$ for each $n \in \mathbb{Z}^+$. By Prop. II.3.3.6 we know that f and g are bounded in $(\mathbb{R}, d_1)$, i.e.,

$$\exists U \in \mathbb{R}^+ : \forall x \in X, \begin{cases} |f(x)| < U \\ |g(x)| < U \end{cases}$$

Since $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ converge uniformly to f on X with respect to d_1, by Def. II.3.2.7 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N_1 \in \mathbb{Z}^+ : \forall n \ge N_1, \forall x \in X, |f_n(x) - f(x)| < \frac{\varepsilon}{2M};$$
$$\forall \varepsilon \in \mathbb{R}^+, \exists N_2 \in \mathbb{Z}^+ : \forall n \ge N_2, \forall x \in X, |g_n(x) - g(x)| < \frac{\varepsilon}{2U}.$$

Now we fix one ε and its corresponding N_1, N_2. Let $N = \max(N_1, N_2)$. Then we have

$$\begin{aligned}
\forall n \ge N, \forall x \in X, &|f_n(x)g_n(x) - f(x)g(x)| \\
&= |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)| \\
&\le |f_n(x)g_n(x) - f(x)g_n(x)| + |f(x)g_n(x) - f(x)g(x)| \\
&= |f_n(x) - f(x)||g_n(x)| + |f(x)||g_n(x) - g(x)| \\
&< \frac{\varepsilon}{2M} M + U\frac{\varepsilon}{2U} \\
&= \varepsilon.
\end{aligned}$$

Since ε was arbitrary, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \ge N, \forall x \in X, |f_n(x)g_n(x) - f(x)g(x)| < \varepsilon$$

and by Def. II.3.2.7 $(f_n g_n)_{n=1}^{\infty}$ converges uniformly to fg on X with respect to d_1. □

II.3.4 The metric of uniform convergence

Note. We have now developed at least four, apparently separate, notions of limit in this text:

(a) limits $\lim_{n\to\infty} x^{(n)}$ of sequences of points in a metric space (Def. II.1.1.14; see also Def. II.2.5.4);

(b) limiting values $\lim_{x\to x_0; x\in E} f(x)$ of functions at a point (Def. II.3.1.1);

(c) pointwise limits f of functions $f^{(n)}$ (Def. II.3.2.1); and

(d) uniform limits f of functions $f^{(n)}$ (Def. II.3.2.7).

This proliferation of limits may seem rather complicated. However, we can reduce the complexity slightly by observing that (d) can be viewed as a special case of (a), though in doing so it should be cautioned that because we are now dealing with functions instead of points, the convergence is not in X or in Y, but rather in a new space, the space of functions from X to Y.

Rmk. II.3.4.1. If one is willing to work in topological spaces instead of metric spaces, we can also view (a) as a special case of (b), see Ex. II.3.1.4, and (c) is also a special case of (a), see Ex. II.3.4.4. Thus, the notion of convergence in a topological space can be used to unify all the notions of limits we have encountered so far.

Def. II.3.4.2 (Metric space of bounded functions)**.** Suppose (X, d_X) and (Y, d_Y) are metric spaces. We let $B(X \to Y)$ denote the space of bounded functions from X to Y :

$$B(X \to Y) \coloneqq \{f | f : X \to Y \text{ is a bounded function}\}.$$

We define a metric $d_\infty : B(X \to Y) \times B(X \to Y) \to [0, +\infty)$ by defining

$$d_\infty(f, g) \coloneqq \sup_{x\in X} d_Y\big(f(x), g(x)\big) = \sup\big\{d_Y\big(f(x), g(x)\big) : x \in X\big\}$$

for all $f, g \in B(X \to Y)$. This metric is sometimes known as the *uniform metric*, or *sup norm metric*, or the L^∞ *metric*. We will also use $d_{B(X\to Y)}$ as a synonym for d_∞. We restrict the definition of d_∞ to the case when $X \neq \emptyset$. If $X = \emptyset$, then we instead define $d_\infty(f, g) = 0$.

Note. $B(X \to Y)$ is a set, thanks to the power set axiom (Axiom 3.10 in Analysis I) and the axiom of specification (Axiom 3.5 in Analysis I).

Note. The distance $d_\infty(f, g)$ is always finite because f and g are assumed to be bounded on X.

Prop. II.3.4.4. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f^{(n)})_{n=1}^\infty$ be a sequence of functions in $B(X \to Y)$, and let f be another function in $B(X \to Y)$. Then $(f^{(n)})_{n=1}^\infty$ converges to f in the metric $d_{B(X\to Y)}$ iff $(f^{(n)})_{n=1}^\infty$ converges uniformly to f.

Proof. We have

$$\begin{aligned}
& d_{B(X\to Y)} - \lim_{n\to\infty} f^{(n)} = f \\
\iff & \forall\varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
& \forall n \geq N, d_{B(X\to Y)}(f^{(n)}, f) \leq \frac{\varepsilon}{2} < \varepsilon && \text{(by Def. II.1.1.14)} \\
\iff & \forall\varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
& \forall n \geq N, \sup_{x\in X} d_Y\big(f^{(n)}(x), f(x)\big) \leq \frac{\varepsilon}{2} < \varepsilon && \text{(by Def. II.3.4.2)} \\
\iff & \forall\varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \\
& \forall n \geq N, \forall x \in X, d_Y\big(f^{(n)}(x), f(x)\big) \leq \frac{\varepsilon}{2} < \varepsilon \\
\iff & (f^{(n)})_{n=1}^{\infty} \text{ converges uniformly to } f \text{ on } X \\
& \text{with respect to } d_Y. && \text{(by Def. II.3.2.7)}
\end{aligned}$$

□

Note. Now let $C(X \to Y)$ be the space of bounded continuous functions from X to Y :

$$C(X \to Y) \coloneqq \{f \in B(X \to Y) | f \text{ is continuous}\}.$$

This set $C(X \to Y)$ is clearly a subset of $B(X \to Y)$. Cor. II.3.3.2 asserts that this space $C(X \to Y)$ is closed in $\big(B(X \to Y), d_{B(X\to Y)}\big)$.

Thm. II.3.4.5 (The space of continuous functions is complete)**.** Let (X, d_X) be a metric space, and let (Y, d_Y) be a complete metric space. The space $\big(C(X \to Y), d_{B(X\to Y)}|_{C(X\to Y)\times C(X\to Y)}\big)$ is a complete subspace of $\big(B(X \to Y), d_{B(X\to Y)}\big)$. In other words, every Cauchy sequence of functions in $C(X \to Y)$ converges to a function in $C(X \to Y)$.

Proof. Let $d_{C(X\to Y)} = d_{B(X\to Y)}|_{C(X\to Y)\times C(X\to Y)}$ and let $n_1, n_2 \in \mathbb{Z}^+$. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\big(C(X \to Y), d_{C(X\to Y)}\big)$. Observe that

$$\begin{aligned}
& \forall\varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n_1, n_2 \geq N, \\
& d_{C(X\to Y)}\big(f^{(n_1)}, f^{(n_2)}\big) < \varepsilon && \text{(by Def. II.1.4.6)} \\
\implies & \forall\varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n_1, n_2 \geq N, \\
& \sup_{x\in X} d_Y\big(f^{(n_1)}(x), f^{(n_2)}(x)\big) < \varepsilon && \text{(by Def. II.3.4.2)} \\
\implies & \forall x \in X, \forall\varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n_1, n_2 \geq N, \\
& d_Y\big(f^{(n_1)}(x), f^{(n_2)}(x)\big) \leq \sup_{x\in X} d_Y\big(f^{(n_1)}(x), f^{(n_2)}(x)\big) < \varepsilon \\
\implies & \forall x \in X, \big(f_n(x)\big)_{n=1}^{\infty} \text{ is a Cauchy sequence in } (Y, d_Y). && \text{(by Def. II.1.4.6)}
\end{aligned}$$

By hypothesis we know that (Y, d_Y) is complete, thus by Def. II.1.4.10 we have

$$\forall x \in X, d_Y - \lim_{n\to\infty} f_n(x) \in Y$$

and we can define a function $f : X \to Y$ such that

$$\forall x \in X, f(x) = d_Y - \lim_{n\to\infty} f_n(x).$$

By Def. II.1.1.14 we have

$$\forall x \in X, \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, d_Y\big(f_n(x), f(x)\big) < \frac{\varepsilon}{3}.$$

We choose one N for each pairs of x and ε and denote it as $N_{x,\varepsilon}$. Since $f_n \in C(X \to Y)$ for all $n \in \mathbb{Z}^+$, by Def. II.2.1.1 we know that

$$\forall x_0 \in X, \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in X, d_X(x, x_0) < \delta \implies d_Y\big(f_n(x), f_n(x_0)\big) < \frac{\varepsilon}{3}.$$

If we denote $M_{x,x_0,\varepsilon} = \max(N_{x,\varepsilon}, N_{x_0,\varepsilon})$, then by Def. II.1.1.2(d) we have

$$\begin{aligned}
&\forall x_0 \in X, \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in X, d_X(x, x_0) < \delta \\
\implies &\begin{cases} \forall n \geq M_{x,x_0,\varepsilon}, d_Y\big(f_n(x), f(x)\big) < \frac{\varepsilon}{3} \\ \forall n \geq M_{x,x_0,\varepsilon}, d_Y\big(f_n(x_0), f(x_0)\big) < \frac{\varepsilon}{3} \\ d_Y\big(f_n(x), f_n(x_0)\big) < \frac{\varepsilon}{3} \end{cases} \\
\implies &\forall n \geq M_{x,x_0,\varepsilon}, \\
&d_Y\big(f(x), f(x_0)\big) \leq d_Y\big(f_n(x), f(x)\big) + d_Y\big(f_n(x), f_n(x_0)\big) + d_Y\big(f_n(x_0), f(x_0)\big) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \\
\implies &d_Y\big(f(x), f(x_0)\big) < \varepsilon.
\end{aligned}$$

By Def. II.2.1.1 this means $f \in C(X \to Y)$. Since $(f_n)_{n=1}^{\infty}$ was arbitrary, by Def. II.1.4.10 $\big(C(X \to Y), d_{C(X\to Y)}\big)$ is complete. □

— Exercises —

Ex. II.3.4.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Show that the space $B(X \to Y)$ defined in Def. II.3.4.2, with the metric $d_{B(X\to Y)}$, is indeed a metric space.

Proof. If $X = \emptyset$, then by Def. II.3.4.2 we have

- If $f \in B(\emptyset \to Y)$, then $d_{B(X\to Y)}(f, f) = 0$.
- If $f, g \in B(\emptyset \to Y)$, then $d_{B(X\to Y)}(f, g) = 0 = d_{B(X\to Y)}(g, f)$.

- If $f,g,h \in B(\emptyset \to Y)$, then $d_{B(X\to Y)}(f,h) = 0 = d_{B(X\to Y)}(f,g) + d_{B(X\to Y)}(g,h)$.

Thus, by Def. II.1.1.2 $\big(B(\emptyset \to Y), d_{B(X\to Y)}\big)$ is a metric space. Now suppose that $X \neq \emptyset$. Since

$$\begin{aligned}
\forall f \in B(X\to Y), d_{B(X\to Y)}(f,f) &= \sup_{x\in X} d_Y\big(f(x),f(x)\big) && \text{(by Def. II.3.4.2)}\\
&= \sup\{0\} && \text{(by Def. II.1.1.2(a))}\\
&= 0,
\end{aligned}$$

by Def. II.1.1.2(a) we know that $\big(B(X\to Y), d_{B(X\to Y)}\big)$ is reflexive. Since

$$\begin{aligned}
&\forall f,g \in B(X\to Y), f \neq g\\
\implies& \exists x\in X : f(x) \neq g(x)\\
\implies& \exists x\in X : d_Y\big(f(x),g(x)\big) > 0 && \text{(by Def. II.1.1.2(b))}\\
\implies& d_{B(X\to Y)}(f,g) = \sup_{x\in X} d_Y\big(f(x),g(x)\big) > 0, && \text{(by Def. II.3.4.2)}
\end{aligned}$$

by Def. II.1.1.2(b) we know that $\big(B(X\to Y), d_{B(X\to Y)}\big)$ is positive. Since

$$\begin{aligned}
\forall f,g \in B(X\to Y), d_{B(X\to Y)}(f,g) &= \sup_{x\in X} d_Y\big(f(x),g(x)\big) && \text{(by Def. II.3.4.2)}\\
&= \sup_{x\in X} d_Y\big(g(x),f(x)\big) && \text{(by Def. II.1.1.2(c))}\\
&= d_{B(X\to Y)}(g,f), && \text{(by Def. II.3.4.2)}
\end{aligned}$$

by Def. II.1.1.2(c) we know that $\big(B(X\to Y), d_{B(X\to Y)}\big)$ is symmetry. Since

$$\begin{aligned}
&\forall f,g,h \in B(X\to Y),\\
&d_{B(X\to Y)}(f,g) + d_{B(X\to Y)}(g,h)\\
&= \sup_{x\in X} d_Y\big(f(x),g(x)\big) + \sup_{x\in X} d_Y\big(g(x),h(x)\big) && \text{(by Def. II.3.4.2)}\\
&\geq \sup_{x\in X}\Big(d_Y\big(f(x),g(x)\big) + d_Y\big(g(x),h(x)\big)\Big)\\
&\geq \sup_{x\in X} d_Y\big(f(x),h(x)\big) && \text{(by Def. II.1.1.2(d))}\\
&= d_{B(X\to Y)}(f,h), && \text{(by Def. II.3.4.2)}
\end{aligned}$$

by Def. II.1.1.2(d) we know that $\big(B(X\to Y), d_{B(X\to Y)}\big)$ is transitive. Combine all the proofs above we conclude by Def. II.1.1.2 that $\big(B(X\to Y), d_{B(X\to Y)}\big)$ is a metric space. □

Ex. II.3.4.2. Prove Prop. II.3.4.4.

Proof. See Prop. II.3.4.4. □

Ex. II.3.4.3. Prove Thm. II.3.4.5.

Proof. See Thm. II.3.4.5. $\square$

Ex. II.3.4.4. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $Y^X := \{f | f : X \to Y\}$ be the space of all functions from X to Y (cf. Axiom 3.10 in Analysis I). If $x_0 \in X$ and V is an open set in Y, let $V^{(x_0)} \subseteq Y^X$ be the set

$$V^{(x_0)} := \big\{f \in Y^X : f(x_0) \in V\big\}.$$

If E is a subset of Y^X, we say that E is *open* if for every $f \in E$, there exists a finite number of points $x_1, \ldots, x_n \in X$ and open sets $V_1, \ldots, V_n \subseteq Y$ such that

$$f \in V_1^{(x_1)} \cap \cdots \cap V_n^{(x_n)} \subseteq E.$$

- Show that if $\mathcal{F}$ is the collection of open sets in Y^X, then $(Y^X, \mathcal{F})$ is a topological space.
- For each natural number n, let $f^{(n)} : X \to Y$ be a function from X to Y, and let $f : X \to Y$ be another function from X to Y. Show that $f^{(n)}$ converges to f in the topology $\mathcal{F}$ (in the sense of Def. II.2.5.4) iff $f^{(n)}$ converges to f pointwise (in the sense of Def. II.3.2.1).

The topology $\mathcal{F}$ is known as the *topology of pointwise convergence*, for obvious reasons; it is also known as the *product topology*. It shows that the concept of pointwise convergence can be viewed as a special case of the more general concept of convergence in a topological space.

Proof. We know that $\emptyset \in \mathcal{F}$ trivially. First, we show that $Y^X \in \mathcal{F}$. Let $f \in Y^X$ and let $x_0 \in X$. By Prop. II.1.2.15(c) we know that $B_{(Y,d_Y)}\big(f(x_0), 1\big)$ is open in (Y, d_Y). Then we have

$$f \in \Big(B_{(Y,d_Y)}\big(f(x_0), 1\big)\Big)^{(x_0)} \subseteq Y^X.$$

Since f was arbitrary, by definition we know that $Y^X \in \mathcal{F}$.

Next we show that the intersection of any finite collection of open sets in $(Y^X, \mathcal{F})$ is open in $(Y^X, \mathcal{F})$. Let $n \in \mathbb{N}$ and let $(U_i)_{i=1}^n$ be a finite collection of open sets in $(Y^X, \mathcal{F})$. If $\displaystyle\bigcap_{i=1}^n U_i = \emptyset$, then from the proof above we know that $\emptyset \in \mathcal{F}$. So suppose that $\displaystyle\bigcap_{i=1}^n U_i \neq \emptyset$. Let $f \in \displaystyle\bigcap_{i=1}^n U_i$. Since

$$\begin{aligned}
&\forall 1 \le i \le n, f \in U_i \\
\implies &\forall 1 \le i \le n, \exists m_i \in \mathbb{Z}^+ : \begin{cases} x_{(i,1)}, \ldots, x_{(i,m_i)} \in X \\ V_{(i,1)}, \ldots, V_{(i,m_i)} \text{ are open sets in } (Y, d_Y) \\ f \in \displaystyle\bigcap_{j=1}^{m_i} V_{(i,j)}^{(x_{(i,j)})} \subseteq U_i \end{cases}
\end{aligned}$$

$$\implies f \in \bigcap_{i=1}^{n}\left(\bigcap_{j=1}^{m_i} V_{(i,j)}^{(x_{(i,j)})}\right) \subseteq \bigcap_{i=1}^{n} U_i$$

and f was arbitrary, we know that $\bigcap_{i=1}^{n} U_i \in \mathcal{F}$. Since n was arbitrary, we know that the intersection of any finite collection of open sets in $(Y^X, \mathcal{F})$ is open in $(Y^X, \mathcal{F})$.

Next we show that the union of arbitrary open sets in $(Y^X, \mathcal{F})$ is open in $(Y^X, \mathcal{F})$. Let $S \subseteq \mathcal{F}$ and let $f \in \bigcup S$. Since

$$\begin{aligned}
&f \in \bigcup S \\
\implies &\exists U \in S : f \in U \\
\implies &\exists U \in S : \begin{cases} x_1, \ldots, x_n \in X \\ V_1, \ldots, V_n \text{ are open sets in } (Y, d_Y) \\ f \in \bigcap_{i=1}^{n} V_i^{(x_i)} \subseteq U \subseteq \bigcup S \end{cases}
\end{aligned}$$

and f was arbitrary, we know that $\bigcup S \in \mathcal{F}$. Since S was arbitrary, we know that the union of arbitrary open sets in $(Y^X, \mathcal{F})$ is open in $(Y^X, \mathcal{F})$. Combine all the proofs above we conclude by Def. II.2.5.1 that $(Y^X, \mathcal{F})$ is a topological space.

Next suppose that $(f^{(n)})_{n=1}^{\infty}$ is a sequence in Y^X and $f \in Y^X$. Suppose also that $(f^{(n)})_{n=1}^{\infty}$ converges to f in $(Y^X, \mathcal{F})$. Then by Def. II.2.5.4 we have

$$\forall E \in \mathcal{F}, f \in E \implies \exists N \in \mathbb{Z}^+ : \forall n \geq N, f^{(n)} \in E.$$

Let $x_0 \in X$. Then we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big) \text{ is open in } (Y, d_Y) && \text{(by Prop. II.1.2.15(c))} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, f \in \Big(B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big)\Big)^{(x_0)} \in \mathcal{F} && \text{(by definition)} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \\
&f^{(n)} \in \Big(B_{(Y,d_Y)}\big(f(x_0), \varepsilon\big)\Big)^{(x_0)} && \text{(by Def. II.2.5.4)} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \\
&d_Y\big(f^{(n)}(x_0), f(x_0)\big) < \varepsilon && \text{(by definition)} \\
\implies &\lim_{n\to\infty} d_Y\big(f^{(n)}(x_0), f(x_0)\big). && \text{(by Def. II.1.1.14)}
\end{aligned}$$

Since x_0 was arbitrary, by Def. II.3.2.1 $(f^{(n)})_{n=1}^{\infty}$ converges pointwise to f on X with respect to d_Y.

Finally suppose that $(f^{(n)})_{n=1}^{\infty}$ is a sequence in Y^X and $f \in Y^X$. Suppose also that $(f^{(n)})_{n=1}^{\infty}$ converges pointwise to f on X with respect to d_Y. Then we have

$$\begin{aligned}
&\forall x \in X, \lim_{n\to\infty} d_Y\big(f^{(n)}(x), f(x)\big) && \text{(by Def. II.3.2.1)}\\
\implies &\forall x \in X, \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N,\\
&d_Y\big(f^{(n)}(x), f(x)\big) < \varepsilon. && \text{(by Def. II.1.1.14)}
\end{aligned}$$

We choose one N for each pair of (x, ε) and denote it as $N_{(x,\varepsilon)}$. Let $E \in \mathcal{F}$ such that $f \in E$. By definition we know that

$$\exists m \in \mathbb{Z}^+ : \begin{cases} x_1, \ldots, x_m \in X \\ V_1, \ldots, V_m \text{ are open sets in } (Y, d_Y) \\ f \in \displaystyle\bigcap_{i=1}^{m} V_i^{(x_i)} \subseteq E \end{cases}$$

Then we have

$$\begin{aligned}
&\forall 1 \leq i \leq m, f \in V_i^{(x_i)}\\
\implies &\forall 1 \leq i \leq m, \begin{cases} f(x_i) \in V_i \\ V_i \text{ is open in } (Y, d_Y) \end{cases}\\
\implies &\forall 1 \leq i \leq m, \exists \varepsilon_i \in \mathbb{R}^+ : B_{(Y,d_Y)}\big(f(x_i), \varepsilon_i\big) \subseteq V_i && \text{(by Prop. II.1.2.15(a))}\\
\implies &\forall 1 \leq i \leq m, \exists \varepsilon_i \in \mathbb{R}^+ : \exists N_{(x_i,\varepsilon_i)} \in \mathbb{Z}^+ :\\
&\forall n \geq N_{(x_i,\varepsilon_i)}, f^{(n)}(x_i) \in B_{(Y,d_Y)}\big(f(x_i), \varepsilon_i\big) \subseteq V_i\\
\implies &\exists N = \max_{1\leq i\leq m} N_{(x_i,\varepsilon_i)} : \forall n \geq N, f^{(n)} \in \bigcap_{i=1}^{m} V_i^{(x_i)} \subseteq E.
\end{aligned}$$

Since E was arbitrary, by Def. II.2.5.4 we know that $(f^{(n)})_{n=1}^{\infty}$ converges to f in $(Y^X, \mathcal{F})$. $\square$

II.3.5 Series of functions; the Weierstrass M-test

Note. Functions whose codomain is $\mathbb{R}$ are sometimes called *real-valued* functions.

Note. given any finite collection $f^{(1)}, \ldots, f^{(N)}$ of functions from X to $\mathbb{R}$, we can define the finite sum $\displaystyle\sum_{i=1}^{N} f^{(i)} : X \to \mathbb{R}$ by

$$\left(\sum_{i=1}^{N} f^{(i)}\right)(x) := \sum_{i=1}^{N} f^{(i)}(x).$$

Def. II.3.5.2 (Infinite series)**.** Let (X, d_X) be a metric space. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from X to $\mathbb{R}$, and let f be another function from X to $\mathbb{R}$. If the partial sums $\sum_{n=1}^{N} f^{(n)}$ converges pointwise to f on X as $N \to \infty$, we say that the infinite series $\sum_{n=1}^{\infty} f^{(n)}$ *converges pointwise* to f, and write $f = \sum_{n=1}^{\infty} f^{(n)}$. If the partial sums $\sum_{n=1}^{N} f^{(n)}$ converge uniformly to f on X as $N \to \infty$, we say that the infinite series $\sum_{n=1}^{\infty} f^{(n)}$ *converges uniformly* to f, and again write $f = \sum_{n=1}^{\infty} f^{(n)}$. (Thus, when one sees an expression such as $f = \sum_{n=1}^{\infty} f^{(n)}$, one should look at the context to see in what sense this infinite series converges.)

Rmk. II.3.5.3. A series $\sum_{n=1}^{\infty} f^{(n)}$ converges pointwise to f on X iff $\sum_{n=1}^{\infty} f^{(n)}(x)$ converges to $f(x)$ for *every* $x \in X$. (Thus, if $\sum_{n=1}^{\infty} f^{(n)}$ does not converge pointwise to f, this does not mean that it diverges pointwise; it may just be that it converges for some points x but diverges at other points x.)

Note. If a series $\sum_{n=1}^{\infty} f^{(n)}$ converges uniformly to f, then it also converges pointwise to f; but not vice versa.

Def. II.3.5.5 (Sup norm)**.** If $f : X \to \mathbb{R}$ is a bounded real-valued function, we define the *sup norm* $\|f\|_{\infty}$ of f to be the number

$$\|f\|_{\infty} := \sup\{|f(x)| : x \in X\}.$$

In other words, $\|f\|_{\infty} = d_{\infty}(f, 0)$, where $0 : X \to \mathbb{R}$ is the zero function $0(x) := 0$, and d_{∞} was defined in Def. II.3.4.2. We restrict the definition of $\|f\|_{\infty}$ to the case when $X \neq \emptyset$. If $X = \emptyset$, then we instead define $\|f\|_{\infty} = 0$.

Note. When f is bounded then $\|f\|_{\infty}$ will always be a non-negative real number.

Thm. II.3.5.7 (Weierstrass M-test)**.** Let (X, d) be a metric space, and let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of bounded real-valued continuous functions on X such that the series $\sum_{n=1}^{\infty} \left\| f^{(n)} \right\|_{\infty}$ is convergent. (Note that this is a series of plain old real numbers, not of functions.) Then the series $\sum_{n=1}^{\infty} f^{(n)}$ converges uniformly to some function f on X, and that function f is also continuous.

Proof. Let $N_1, N_2 \in \mathbb{Z}^+$. Let $d_{C(X\to\mathbb{R})} = d_{B(X\to\mathbb{R})}|_{C(X\to\mathbb{R})\times C(X\to\mathbb{R})}$. We have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\|f^{(n)}\right\|_\infty = \lim_{N\to\infty}\sum_{n=1}^{N}\left\|f^{(n)}\right\|_\infty \\
\implies &\forall\varepsilon\in\mathbb{R}^+, \exists M\in\mathbb{Z}^+ : \forall N\geq M, \\
&\left|\sum_{n=1}^{\infty}\left\|f^{(n)}\right\|_\infty - \sum_{n=1}^{N}\left\|f^{(n)}\right\|_\infty\right| < \varepsilon && \text{(by Def. II.1.1.14)} \\
\implies &\forall\varepsilon\in\mathbb{R}^+, \exists M\in\mathbb{Z}^+ : \forall N\geq M, \\
&\left|\sum_{n=N+1}^{\infty}\left\|f^{(n)}\right\|_\infty\right| < \varepsilon && \text{(by Proposition 7.2.14(c) in Analysis I)} \\
\implies &\forall\varepsilon\in\mathbb{R}^+, \exists M\in\mathbb{Z}^+ : \forall N\geq M, \\
&\sum_{n=N+1}^{\infty}\left\|f^{(n)}\right\|_\infty < \varepsilon && \left(\left\|f^{(n)}\right\|_\infty \geq 0\right) \\
\implies &\forall\varepsilon\in\mathbb{R}^+, \exists M\in\mathbb{Z}^+ : \forall N\geq M, \\
&\sum_{n=N+1}^{\infty}\sup_{x\in X}\left|f^{(n)}(x)\right| < \varepsilon. && \text{(by Def. II.3.5.5)}
\end{aligned}$$

Fix one ε and M. Since $f^{(n)} \in C(X\to\mathbb{R})$, by Ex. II.3.5.1 we know that $\sum_{n=1}^{N} f^{(n)} \in C(X\to\mathbb{R})$ for each $N\in\mathbb{Z}^+$. Thus, $d_{C(X\to\mathbb{R})}\left(\sum_{n=1}^{N_1} f^{(n)}, \sum_{n=1}^{N_2} f^{(n)}\right)$ is well defined for each $N_1, N_2 \geq M$ and

$$\begin{aligned}
\forall N_1, N_2 \geq M, &d_{C(X\to\mathbb{R})}\left(\sum_{n=1}^{N_1} f^{(n)}, \sum_{n=1}^{N_2} f^{(n)}\right) \\
&= \sup_{x\in X}\left|\sum_{n=1}^{N_1} f^{(n)}(x) - \sum_{n=1}^{N_2} f^{(n)}(x)\right| && \text{(by Def. II.3.4.2)} \\
&= \sup_{x\in X}\left|\sum_{n=\min(N_1,N_2)+1}^{\max(N_1,N_2)} f^{(n)}(x)\right| \\
&\leq \sup_{x\in X}\left(\sum_{n=\min(N_1,N_2)+1}^{\max(N_1,N_2)} \left|f^{(n)}(x)\right|\right) \\
&\leq \sup_{x\in X}\left(\sum_{n=M+1}^{\infty} \left|f^{(n)}(x)\right|\right)
\end{aligned}$$

$$\leq \sum_{n=M+1}^{\infty} \sup_{x\in X}\left|f^{(n)}(x)\right| < \varepsilon.$$

Since ε was arbitrary, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists M \in \mathbb{Z}^+ : \forall N_1, N_2 \geq M, d_{C(X\to\mathbb{R})}\left(\sum_{n=1}^{N_1} f^{(n)}, \sum_{n=1}^{N_2} f^{(n)}\right) < \varepsilon.$$

By Def. II.1.4.6 $\left(\sum_{n=1}^{N} f^{(n)}\right)_{N=1}^{\infty}$ is a Cauchy sequence in $\left(C(X \to \mathbb{R}), d_{C(X\to\mathbb{R})}\right)$. Since $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ is complete, by Thm. II.3.4.5 we know that $\left(\sum_{n=1}^{N} f^{(n)}\right)_{N=1}^{\infty}$ converges uniformly to some $f \in C(X \to \mathbb{R})$ on X with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. □

Note. To put the Weierstrass M-test succinctly: absolute convergence of sup norms implies uniform convergence of functions.

E.g. II.3.5.8. Let $0 < r < 1$ be a real number, and let $f^{(n)} : [-r, r] \to \mathbb{R}$ be the series of functions $f^{(n)}(x) := x^n$. Then each $f^{(n)}$ is continuous and bounded, and $\left\|f^{(n)}\right\|_{\infty} = r^n$. Since the series $\sum_{n=1}^{\infty} r^n$ is absolutely convergent (e.g., by the root test, Theorem 7.5.1 in Analysis I), we thus see that $\sum_{n=1}^{\infty} f^{(n)}$ converges uniformly in $[-r, r]$ to some continuous function; in Ex. II.3.2.2(c) we see that this function must in fact be the function $f : [-r, r] \to \mathbb{R}$ defined by $f(x) := x/(1-x)$. In other words, the series $\sum_{n=1}^{\infty} x^n$ is pointwise convergent, but not uniformly convergent, on $(-1, 1)$, but is uniformly convergent on the smaller interval $[-r, r]$ for any $0 < r < 1$.

Note. The Weierstrass M-test is especially useful in relation to power series.

— Exercises —

Ex. II.3.5.1. Let $f^{(1)}, \ldots, f^{(N)}$ be a finite sequence of bounded functions from a metric space (X, d_X) to $\mathbb{R}$. Show that $\sum_{i=1}^{N} f^{(i)}$ is also bounded. Prove a similar claim when "bounded" is replaced by "continuous." What if "continuous" was replaced by "uniformly continuous"?

Proof. Let $d_1 = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. We first show that $\sum_{n=1}^{N} f^{(n)}$ is bounded on X with respect to d_1 for each $N \in \mathbb{Z}^+$. Suppose that $f^{(n)}$ is bounded on X with respect to d_1 for each $n \in \mathbb{Z}^+$.

We induct on N. For $N=1$, by hypothesis we know that $\sum_{n=1}^{1} f^{(n)} = f^{(1)}$ is bounded on X. Thus, the base case holds. Suppose inductively that $\sum_{n=1}^{N} f^{(n)}$ is bounded on X with respect to d_1 for some $N \geq 1$. By the induction hypothesis we have

$$\exists M \in \mathbb{R}^+ : \left(\sum_{n=1}^{N} f^{(n)}\right)(X) \subseteq [-M, M].$$

By hypothesis we know that $f^{(N+1)}$ is bounded on X with respect to d_1, thus we have

$$\exists M' \in \mathbb{R}^+ : f^{(N+1)}(X) \subseteq [-M', M'].$$

Then we have

$$\begin{aligned}\left(\sum_{n=1}^{N+1} f^{(n)}\right)(X) &= \left\{\sum_{n=1}^{N+1} f^{(n)}(x) : x \in X\right\} \\ &= \left\{\sum_{n=1}^{N} f^{(n)}(x) + f^{(N+1)}(x) : x \in X\right\} \\ &\subseteq [-(M+M'), M+M'].\end{aligned}$$

This closes the induction.

Next we show that $\sum_{n=1}^{N} f^{(n)}$ is continuous from (X, d_X) to $(\mathbb{R}, d_1)$ for each $N \in \mathbb{Z}^+$. Suppose that $f^{(n)}$ is continuous from (X, d_X) to $(\mathbb{R}, d_1)$ for each $n \in \mathbb{Z}^+$. We induct on N. For $N=1$, by hypothesis we know that $\sum_{n=1}^{1} f^{(n)} = f^{(1)}$ is continuous from (X, d_X) to $(\mathbb{R}, d_1)$. Thus, the base case holds. Suppose inductively that $\sum_{n=1}^{N} f^{(n)}$ is continuous from (X, d_X) to $(\mathbb{R}, d_1)$ for some $N \geq 1$. Then by A.Cor. II.2.2.1

$$\sum_{n=1}^{N+1} f^{(n)} = \left(\sum_{n=1}^{N} f^{(n)}\right) \oplus f^{(N+1)}$$

is continuous from (X, d_X) to $(\mathbb{R}, d_1)$. This closes the induction.

Finally we show that $\sum_{n=1}^{N} f^{(n)}$ is uniformly continuous from (X, d_X) to $(\mathbb{R}, d_1)$ for each $N \in \mathbb{Z}^+$. Suppose that $f^{(n)}$ is uniformly continuous from (X, d_X) to $(\mathbb{R}, d_1)$ for each $n \in \mathbb{Z}^+$.

We induct on N. For $N = 1$, by hypothesis we know that $\sum_{n=1}^{1} f^{(n)} = f^{(1)}$ is uniformly continuous from (X, d_X) to $(\mathbb{R}, d_1)$. Thus, the base case holds. Suppose inductively that $\sum_{n=1}^{N} f^{(n)}$ is uniformly continuous from (X, d_X) to $(\mathbb{R}, d_1)$ for some $N \geq 1$. Then by Ex. II.2.3.5

$$\sum_{n=1}^{N+1} f^{(n)} = \left(\sum_{n=1}^{N} f^{(n)}\right) \oplus f^{(N+1)}$$

is uniformly continuous from (X, d_X) to $(\mathbb{R}, d_1)$. This closes the induction. □

Ex. II.3.5.2. Prove Thm. II.3.5.7.

Proof. See Thm. II.3.5.7. □

II.3.6 Uniform convergence and integration

Thm. II.3.6.1. Let $[a, b]$ be an interval, and for each integer $n \geq 1$, let $f^{(n)} : [a, b] \to \mathbb{R}$ be a Riemann-integrable function. Suppose $f^{(n)}$ converges uniformly on $[a, b]$ to a function $f : [a, b] \to \mathbb{R}$. Then f is also Riemann integrable, and

$$\lim_{n\to\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} f.$$

Proof. We first show that f is Riemann integrable on $[a, b]$. This is the same as showing that the upper and lower Riemann integrals of f match: $\underline{\int}_{[a,b]} f = \overline{\int}_{[a,b]} f$.

Let $\varepsilon > 0$. Since $f^{(n)}$ converges uniformly to f, we see that there exists an $N > 0$ such that $\left|f^{(n)}(x) - f(x)\right| < \varepsilon$ for all $n > N$ and $x \in [a, b]$. In particular, we have

$$f^{(n)}(x) - \varepsilon < f(x) < f^{(n)}(x) + \varepsilon$$

for all $x \in [a, b]$. Integrating this on $[a, b]$ we obtain

$$\underline{\int}_{[a,b]} (f^{(n)} - \varepsilon) \leq \underline{\int}_{[a,b]} f \leq \overline{\int}_{[a,b]} f \leq \overline{\int}_{[a,b]} (f^{(n)} + \varepsilon).$$

Since $f^{(n)}$ is assumed to be Riemann integrable, we thus see

$$\left(\int_{[a,b]} f^{(n)}\right) - \varepsilon(b - a) \leq \underline{\int}_{[a,b]} f \leq \overline{\int}_{[a,b]} f \leq \left(\int_{[a,b]} f^{(n)}\right) + \varepsilon(b - a).$$

In particular, we see that

$$0 \leq \overline{\int}_{[a,b]} f - \underline{\int}_{[a,b]} f \leq 2\varepsilon(b-a).$$

Since this is true for every $\varepsilon > 0$, we obtain $\underline{\int}_{[a,b]} f = \overline{\int}_{[a,b]} f$ as desired.

The above argument also shows that for every $\varepsilon > 0$ there exists an $N > 0$ such that

$$\left| \int_{[a,b]} f^{(n)} - \int_{[a,b]} f \right| \leq \varepsilon(b-a)$$

for all $n \geq N$. Since ε was arbitrary, we see that $\int_{[a,b]} f^{(n)}$ converges to $\int_{[a,b]} f$ as desired. □

Note. To rephrase Thm. II.3.6.1: we can rearrange limits and integrals (on compact intervals $[a,b]$),

$$\lim_{n\to\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} \lim_{n\to\infty} f^{(n)},$$

provided that the convergence is uniform.

Cor. II.3.6.2. Let $[a,b]$ be an interval, and let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of Riemann integrable functions on $[a,b]$ such that the series $\sum_{n=1}^{\infty} f^{(n)}$ is uniformly convergent. Then we have

$$\sum_{n=1}^{\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} \sum_{n=1}^{\infty} f^{(n)}.$$

Proof. By Theorem 11.4.1(a) in Analysis I we know that

$$\forall N \in \mathbb{Z}^+, \int_{[a,b]} \sum_{n=1}^{N} f^{(n)} = \sum_{n=1}^{N} \int_{[a,b]} f^{(n)}.$$

Let $f : [a,b] \to \mathbb{R}$ be the function such that $\sum_{n=1}^{\infty} f^{(n)}$ converges uniformly to f on $[a,b]$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. By Thm. II.3.6.1 we have

$$\sum_{n=1}^{\infty} \int_{[a,b]} f^{(n)} = \lim_{N\to\infty} \sum_{n=1}^{N} \int_{[a,b]} f^{(n)} = \lim_{N\to\infty} \int_{[a,b]} \sum_{n=1}^{N} f^{(n)} = \int_{[a,b]} f = \int_{[a,b]} \sum_{n=1}^{\infty} f^{(n)}.$$

□

Note. Cor. II.3.6.2 works particularly well in conjunction with the Weierstrass M-test (Thm. II.3.5.7).

— Exercises —

Ex. II.3.6.1. Use Thm. II.3.6.1 to prove Cor. II.3.6.2.

Proof. See Cor. II.3.6.2. □

II.3.7 Uniform convergence and derivatives

Note. In particular, we have

$$\frac{d}{dx} \lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty} \frac{d}{dx} f_n(x)$$

So, in summary, uniform convergence of the functions f_n says nothing about the convergence of the derivatives f_n'.

Thm. II.3.7.1. Let $[a,b]$ be an interval, and for every integer $n \geq 1$, let $f_n : [a,b] \to \mathbb{R}$ be a differentiable function whose derivative $f_n' : [a,b] \to \mathbb{R}$ is continuous. Suppose that the derivatives f_n' converge uniformly to a function $g : [a,b] \to \mathbb{R}$. Suppose also that there exists a point x_0 such that the limit $\lim\limits_{n\to\infty} f_n(x_0)$ exists. Then the functions f_n converge uniformly to a differentiable function f, and the derivative of f equals g.

Proof. Since f_n' is continuous, by Corollary 11.5.2 in Analysis I we know that f_n' is Riemann integrable. We see from the fundamental theorem of calculus (Theorem 11.9.4 in Analysis I) that

$$f_n(x) - f_n(x_0) = \int_{[x_0,x]} f_n'$$

when $x \in [x_0, b]$, and

$$f_n(x) - f_n(x_0) = -\int_{[x,x_0]} f_n'$$

when $x \in [a, x_0]$. Let L be the limit of $f_n(x_0)$ as $n \to \infty$:

$$L \coloneqq \lim_{n\to\infty} f_n(x_0).$$

By hypothesis, L exists. Now, since each f_n' is continuous on $[a,b]$, and f_n' converges uniformly to g, we see by Cor. II.3.3.2 that g is also continuous. By Thm. II.3.6.1 we have

$$\forall x \in [a,b], \lim_{n\to\infty} \big(f_n(x) - f_n(a)\big) = \lim_{n\to\infty} \int_{[a,x]} f_n' = \int_{[a,x]} \big(\lim_{n\to\infty} f_n'\big) = \int_{[a,x]} g.$$

Now define the function $f : [a, b] \to \mathbb{R}$ by setting

$$f(x) := L - \int_{[a,x_0]} g + \int_{[a,x]} g$$

for all $x \in [a, b]$. To finish the proof, we have to show that f_n converges uniformly to f, and that f is differentiable with derivative g.

We know that $a \neq b$ since if $a = b$, then we have $x_0 = a = b$ and

$$\forall n \in \mathbb{Z}^+, \lim_{x \to x_0; x \in \{x_0\} \setminus \{x_0\}} \frac{f_n(x) - f_n(x_0)}{x - x_0} \text{ is undefined}$$

which contradict to the hypothesis that f_n is differentiable on $[a, b]$. Observe that

$$\begin{aligned} & L = \lim_{n \to \infty} f_n(x_0) \\ \implies & \forall \varepsilon \in \mathbb{R}^+, \exists N_1 \in \mathbb{Z}^+ : \forall n \geq N_1, |f_n(x_0) - L| < \frac{\varepsilon}{3(b-a)}. \end{aligned}$$

Now we fix one pair of ε and N_1. Since $(f_n')_{n=1}^{\infty}$ converges uniformly to g on $[a, b]$ with respect to $d_{l^1}|_{\mathbb{R} \times \mathbb{R}}$, by Def. II.3.2.7 we have

$$\begin{aligned} & \exists N_2 \in \mathbb{Z}^+ : \forall n \geq N_2, \forall x \in [a, b], \\ & \left|f_n'(x) - g(x)\right| < \frac{\varepsilon}{3(b-a)} \\ \implies & \exists N_2 \in \mathbb{Z}^+ : \forall n \geq N_2, \forall x \in [a, b], \\ & \frac{-\varepsilon}{3(b-a)} < f_n'(x) - g(x) < \frac{\varepsilon}{3(b-a)} \\ \implies & \exists N_2 \in \mathbb{Z}^+ : \forall n \geq N_2, \forall x \in [a, b], \\ & \frac{-\varepsilon(x-a)}{3(b-a)} \leq \int_{[a,x]} f_n'(x) - \int_{[a,x]} g(x) \leq \frac{\varepsilon(x-a)}{3(b-a)} \\ \implies & \exists N_2 \in \mathbb{Z}^+ : \forall n \geq N_2, \forall x \in [a, b], \\ & \frac{-\varepsilon(x-a)}{3(b-a)} \leq f_n(x) - f_n(a) - \int_{[a,x]} g(x) \leq \frac{\varepsilon(x-a)}{3(b-a)} \\ \implies & \exists N_2 \in \mathbb{Z}^+ : \forall n \geq N_2, \forall x \in [a, b], \\ & \left| f_n(x) - f_n(a) - \int_{[a,x]} g(x) \right| \leq \frac{\varepsilon|x-a|}{3(b-a)}. \end{aligned}$$

Let $N = \max(N_1, N_2)$. Then we have

$$\begin{aligned} & \forall n \geq N, \forall x \in [a, b], |f_n(x) - f(x)| \\ = & \left| f_n(x) - f_n(x_0) + f_n(x_0) - f_n(a) + f_n(a) - L + \int_{[a,x_0]} g - \int_{[a,x]} g \right| \end{aligned}$$

$$\begin{aligned}
&\le \left| f_n(x) - f_n(a) - \int_{[a,x]} g \right| + |f_n(x_0) - L| + \left| f_n(x_0) - f_n(a) - \int_{[a,x_0]} g \right| \\
&< \frac{\varepsilon|x-a|}{3(b-a)} + \frac{\varepsilon|x_0-a|}{3(b-a)} + \frac{\varepsilon}{3(b-a)} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since ε was arbitrary, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \ge N, \forall x \in [a,b], |f_n(x) - f(x)| < \varepsilon$$

and by Def. II.3.2.7 $(f_n)_{n=1}^{\infty}$ converges uniformly to f with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$.

Since f_n is continuous on $[a,b]$ for each $n \in \mathbb{Z}^+$, by Cor. II.3.3.2 we know that f is also continuous on $[a,b]$. Since g is continuous on $[a,b]$, by fundamental theorem of calculus (Theorem 11.9.1 in Analysis) we know that

$$\forall x \in X, G(x) = \int_{[a,x]} g \text{ is differentiable at } x.$$

Since $L + \int_{[a,x_0]} g$ is constant, we know that

$$\forall x \in X, f(x) = L + \int_{[a,x_0]} g + G(x) = L + \int_{[a,x_0]} g + \int_{[a,x]} g \text{ is differentiable at } x$$

and by fundamental theorem of calculus (Theorem 11.9.1 in Analysis) we have

$$\forall x \in X, f'(x) = \left(\int_{[a,x]} g \right)' = g(x).$$

□

Note. Informally, Thm. II.3.7.1 says that if f_n' converges uniformly, and $f_n(x_0)$ converges for some x_0, then f_n also converges uniformly, and

$$\frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x)$$

Rmk. II.3.7.2. It turns out that Thm. II.3.7.1 is still true when the functions f_n' are not assumed to be continuous, but the proof is more difficult; see Ex. II.3.7.2.

Cor. II.3.7.3. Let $[a,b]$ be an interval, and for every integer $n \ge 1$, let $f_n : [a,b] \to \mathbb{R}$ be a differentiable function whose derivative $f_n' : [a,b] \to \mathbb{R}$ is continuous. Suppose that the series $\sum_{n=1}^{\infty} \|f_n'\|_\infty$ is absolutely convergent, where

$$\|f_n'\|_\infty \coloneqq \sup_{x\in[a,b]} |f_n'(x)|$$

is the sup norm of f_n', as defined in Def. II.3.5.5. Suppose also that the series $\sum_{n=1}^{\infty} f_n(x_0)$ is convergent for some $x_0 \in [a,b]$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a,b]$ to a differentiable function, and in fact

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x)$$

for all $x \in [a,b]$.

Proof. Let $F_N = \sum_{n=1}^{N} f_n$ for each $N \in \mathbb{Z}^+$. Then by Theorem 10.1.13(c) in Analysis I we have

$$\forall N \in \mathbb{Z}^+, F_N' = \left(\sum_{n=1}^{N} f_n\right)' = \sum_{n=1}^{N} f_n'.$$

Since f_n' is continuous on $[a,b]$ for each $n \in \mathbb{Z}^+$, by Proposition 9.6.7 in Analysis I we know that $f_n' \in B\big([a,b] \to \mathbb{R}\big)$ and thus $f_n' \in C\big([a,b] \to \mathbb{R}\big)$. By Ex. II.3.5.1 we know that

$$\forall N \in \mathbb{Z}^+, F_N' = \sum_{n=1}^{N} f_n' \in C([a,b] \to \mathbb{R}).$$

Since $\sum_{n=1}^{\infty} \|f_n'\|_\infty$ converges and $f_n' \in C\big([a,b] \to \mathbb{R}\big)$ for each $n \in \mathbb{Z}^+$, by Thm. II.3.5.7 we know that there exists some $G : [a,b] \to \mathbb{R}$ such that $\big(\sum_{n=1}^{N} f_n'\big)_{N=1}^{\infty}$ converges uniformly to G on $[a,b]$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Equivalently, $(F_N')_{N=1}^{\infty}$ converges uniformly to G on $[a,b]$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. Since

$$\sum_{n=1}^{\infty} f_n(x_0) = \lim_{N\to\infty} \sum_{n=1}^{N} f_n(x_0) = \lim_{N\to\infty} F_N(x_0),$$

by Thm. II.3.7.1 we know that there exists some $F : [a,b] \to \mathbb{R}$ such that $(F_N)_{N=1}^{\infty}$ converges uniformly to F on $[a,b]$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$ and $F' = G$. Then we have

$$\forall x \in [a,b], \begin{cases} F(x) = \lim\limits_{N\to\infty} F_N(x) = \lim\limits_{N\to\infty} \sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{\infty} f_n(x) \\ G(x) = \lim\limits_{N\to\infty} F_N'(x) = \lim\limits_{N\to\infty} \sum_{n=1}^{N} f_n'(x) = \sum_{n=1}^{\infty} f_n'(x) \\ F'(x) = G(x) \end{cases}$$

$$\implies \forall x \in [a,b], \left(\sum_{n=1}^{\infty} f_n\right)'(x) = \sum_{n=1}^{\infty} f_n'(x)$$
$$\implies \left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'.$$

□

Note. E.g. II.3.7.4 was discovered by Weierstrass.

E.g. II.3.7.4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) := \sum_{n=1}^{\infty} 4^{-n} \cos(32^n \pi x).$$

Note that this series is uniformly convergent, thanks to the Weierstrass M-test, and since each individual function $4^{-n}\cos(32^n\pi x)$ is continuous, the function f is also continuous. However, it is not differentiable; in fact it is a *nowhere differentiable function*, one which is not differentiable at any point, despite being continuous everywhere!

— Exercises —

Ex. II.3.7.1. Complete the proof of Thm. II.3.7.1. Compare this theorem with Example 1.2.10 in Analysis I, and explain why this example does not contradict the theorem.

Proof. See Thm. II.3.7.1. Since $\lim_{n\to\infty} \dfrac{x^3}{\frac{1}{n}+x^2}$ is not continuous at $x=0$, is does not contradict to Thm. II.3.7.1. □

Ex. II.3.7.2. Prove Thm. II.3.7.1 without assuming that f_n' is continuous. (This means that you cannot use the fundamental theorem of calculus. However, the mean value theorem (Corollary 10.2.9 in Analysis I) is still available. Use this to show that if $d_\infty(f_n', f_m') \le \varepsilon$, then $\left|\big(f_n(x) - f_m(x)\big) - \big(f_n(x_0) - f_m(x_0)\big)\right| \le \varepsilon|x - x_0|$ for all $x \in [a,b]$, and then use this to complete the proof of Thm. II.3.7.1.)

Proof. Let $m \in \mathbb{Z}^+$. Let $d_{C([a,b]\to\mathbb{R})} = d_{B([a,b]\to\mathbb{R})}|_{C([a,b]\to\mathbb{R})\times C([a,b]\to\mathbb{R})}$. Since f_n is differentiable on $[a,b]$, by Corollary 10.1.12 and Proposition 9.6.7 in Analysis I we know that $f_n \in C([a,b]\to\mathbb{R})$ for each $n\in\mathbb{Z}^+$. Since $(f_n')_{n=1}^{\infty}$ converges uniformly to g on $[a,b]$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$, by Def. II.3.2.7 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N_1 \in \mathbb{Z}^+ : \forall n \ge N_1, \forall x \in [a,b], \left|f_n'(x) - g(x)\right| < \frac{\varepsilon}{6(b-a)}$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists N_1 \in \mathbb{Z}^+ : \forall n,m \ge N_1, \forall x \in [a,b],$$
$$\left|f_n'(x) - f_m'(x)\right| \le \left|f_n'(x) - g(x)\right| + \left|f_m'(x) - g(x)\right| < \frac{\varepsilon}{6(b-a)} + \frac{\varepsilon}{6(b-a)}$$

$$\begin{aligned}
&\Longrightarrow \forall \varepsilon \in \mathbb{R}^+, \exists N_1 \in \mathbb{Z}^+ : \forall n, m \geq N_1, \forall x \in [a,b], \\
&\quad \left|(f_n' - f_m')(x)\right| < \frac{\varepsilon}{3(b-a)}.
\end{aligned}$$

Now we fix one such ε and N_1. Let $x \in [a,b]$. We split into two cases:

- $x = x_0$. Then we have

$$\forall n, m \geq N_1, \left|\big(f_n(x_0) - f_m(x_0)\big) - \big(f_n(x_0) - f_m(x_0)\big)\right| = 0 = \frac{\varepsilon|x_0 - x_0|}{3(b-a)}.$$

- $x \neq x_0$. Suppose that $x < x_0$. Since $[x, x_0] \subseteq [a,b]$, by mean value theorem we know that

$$\forall n, m \geq N_1, \exists y \in (x, x_0) : (f_n - f_m)'(y) = \frac{(f_n - f_m)(x_0) - (f_n - f_m)(x)}{x_0 - x}.$$

  Now suppose that $x > x_0$. Since $[x_0, x] \subseteq [a,b]$, by mean value theorem we know that

$$\forall n, m \geq N_1, \exists y \in (x_0, x) : (f_n - f_m)'(y) = \frac{(f_n - f_m)(x) - (f_n - f_m)(x_0)}{x - x_0}.$$

  In either case, we have

$$\begin{aligned}
&\forall n, m \geq N_1, \exists y \in (a,b) : \\
&\left|\frac{(f_n - f_m)(x) - (f_n - f_m)(x_0)}{x - x_0}\right| = \left|(f_n - f_m)'(y)\right| \\
\Longrightarrow\ &\forall n, m \geq N_1, \exists y \in (a,b) : \\
&\left|\big(f_n(x) - f_m(x)\big) - \big(f_n(x_0) - f_m(x_0)\big)\right| \\
&= \left|f_n'(y) - f_m'(y)\right||x - x_0| \\
&\leq \frac{\varepsilon|x - x_0|}{3(b-a)} \\
\Longrightarrow\ &\forall n, m \geq N_1, \left|\big(f_n(x) - f_m(x)\big) - \big(f_n(x_0) - f_m(x_0)\big)\right| \leq \frac{\varepsilon|x - x_0|}{3(b-a)}.
\end{aligned}$$

From all cases above, we conclude that

$$\forall n, m \geq N_1, \forall x \in [a,b], \left|\big(f_n(x) - f_m(x)\big) - \big(f_n(x_0) - f_m(x_0)\big)\right| \leq \frac{\varepsilon|x - x_0|}{3(b-a)}.$$

Let $L = \lim\limits_{n\to\infty} f_n(x_0)$. Then we have

$$\lim_{n\to\infty} f_n(x_0) = L$$

$$\implies \exists N_2 \in \mathbb{Z}^+ : \forall n \geq N_2, |f_n(x_0) - L| < \frac{\varepsilon}{3}$$

Let $N = \max(N_1, N_2)$. Then we have

$$\begin{aligned}
&\forall n, m \geq N, \forall x \in [a,b], \\
&|f_n(x) - f_m(x)| \\
&= \Big|\big(f_n(x) - f_m(x)\big) - \big(f_n(x_0) - f_m(x_0)\big) + \big(f_n(x_0) - f_m(x_0)\big) + L - L\Big| \\
&\leq \Big|\big(f_n(x) - f_m(x)\big) - \big(f_n(x_0) - f_m(x_0)\big)\Big| + |f_n(x_0) - L| + |f_m(x_0) - L| \\
&< \frac{\varepsilon |x - x_0|}{3(b-a)} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since ε was arbitrary, by Def. II.3.4.2 we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n, m \geq N, \forall x \in [a,b], |f_n(x) - f_m(x)| < \varepsilon \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n, m \geq N, \sup_{x \in [a,b]} |f_n(x) - f_m(x)| < \varepsilon \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n, m \geq N, d_{C([a,b] \to \mathbb{R})}(f_n, f_m) < \varepsilon.
\end{aligned}$$

Thus, by Def. II.1.4.6 $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $\big(C([a,b] \to \mathbb{R}), d_{C([a,b]\to\mathbb{R})}\big)$. Since $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$ is complete, by Thm. II.3.4.5 we know that there exists some $f : [a,b] \to \mathbb{R}$ such that

$$\begin{cases} f \in C([a,b] \to \mathbb{R}) \\ d_{C([a,b]\to\mathbb{R})} - \lim\limits_{n\to\infty} f_n = f \end{cases}$$

By Prop. II.3.4.4 we know that $(f_n)_{n=1}^\infty$ convergent uniformly to f on $[a,b]$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$.

Now we show that f is differentiable on $[a,b]$ and $f' = g$. Let $y \in [a,b]$. Since $(f_n')_{n=1}^\infty$ converges uniformly to g on $[a,b]$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$, we have

$$\begin{aligned}
g(y) &= \lim_{n\to\infty} f_n'(y) && \text{(by Ex. II.3.2.2(a))} \\
&= \lim_{n\to\infty} \left(\lim_{x \to y; x \in [a,b]\setminus\{y\}} \frac{f_n(x) - f_n(y)}{x - y} \right) \\
&= \lim_{x \to y; x \in [a,b]\setminus\{y\}} \left(\lim_{n\to\infty} \frac{f_n(x) - f_n(y)}{x - y} \right). && \text{(by Prop. II.3.3.3)}
\end{aligned}$$

Since $(f_n)_{n=1}^\infty$ converges uniformly to f on $[a,b]$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$, we have

$$\begin{aligned}
&\forall x \in [a,b], f(x) = \lim_{n\to\infty} f_n(x) && \text{(by Ex. II.3.2.2(a))} \\
\implies &\forall x \in [a,b], f(x) - f(y) = \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(y)
\end{aligned}$$

$$\Longrightarrow \forall x \in [a,b], f(x) - f(y) = \lim_{n\to\infty} \big(f_n(x) - f_n(y)\big)$$
$$\Longrightarrow \forall x \in [a,b] \setminus \{y\}, \frac{f(x)-f(y)}{x-y} = \frac{\lim_{n\to\infty} \big(f_n(x) - f_n(y)\big)}{x-y}$$
$$\Longrightarrow \forall x \in [a,b] \setminus \{y\}, \frac{f(x)-f(y)}{x-y} = \lim_{n\to\infty} \frac{f_n(x)-f_n(y)}{x-y}.$$

Thus, we have

$$g(y) = \lim_{x\to y; x\in[a,b]\setminus\{y\}} \left(\lim_{n\to\infty} \frac{f_n(x)-f_n(y)}{x-y} \right) = \lim_{x\to y; x\in[a,b]\setminus\{y\}} \frac{f(x)-f(y)}{x-y} = f'(y).$$

Since y was arbitrary, we conclude that f is differentiable on $[a,b]$ and $f' = g$. □

Ex. II.3.7.3. Prove Cor. II.3.7.3.

Proof. See Cor. II.3.7.3. □

II.3.8 Uniform approximation by polynomials

Note. As we have just seen, continuous functions can be very badly behaved, for instance they can be nowhere differentiable (E.g. II.3.7.4). On the other hand, functions such as polynomials are always very well behaved, in particular, being always differentiable. Fortunately, while most continuous functions are not as well behaved as polynomials, they can always be *uniformly approximated* by polynomials; this important (but difficult) result is known as the *Weierstrass approximation theorem*,

Def. II.3.8.1. Let $[a,b]$ be an interval. A *polynomial on* $[a,b]$ is a function $f : [a,b] \to \mathbb{R}$ of the form $f(x) := \sum_{j=0}^{n} c_j x^j$, where $n \geq 0$ is an integer and $c_0, \dots, c_n$ are real numbers. If $c_n \neq 0$, then n is called the *degree* of f.

Thm. II.3.8.3 (Weierstrass approximation theorem)**.** If $[a,b]$ is an interval, $f : [a,b] \to \mathbb{R}$ is a continuous function, and $\varepsilon > 0$, then there exists a polynomial P on $[a,b]$ such that $d_\infty(P,f) \leq \varepsilon$ (i.e., $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [a,b]$).

Proof. Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Let $g : [0,1] \to \mathbb{R}$ denote the function

$$g(x) := f\big(a + (b-a)x\big) \text{ for all } x \in [0,1]$$

Observe then that

$$f(y) = g(\frac{y-a}{b-a}) \text{ for all } y \in [a,b].$$

The function g is continuous on $[0,1]$ since $y \mapsto \dfrac{y-a}{b-a}$ is bijective on $[a,b]$, and so by Cor. II.3.8.19 we may find a polynomial $Q : [0,1] \to \mathbb{R}$ such that $|Q(x) - g(x)| \leq \varepsilon$ for all $x \in [0,1]$. In particular, for any $y \in [a,b]$, we have

$$\left| Q(\frac{y-a}{b-a}) - g(\frac{y-a}{b-a}) \right| \leq \varepsilon.$$

If we thus set $P(y) := Q(\dfrac{y-a}{b-a})$, then we observe that P is also a polynomial since $y \mapsto \dfrac{y-a}{b-a}$ is bijective on $[a,b]$, and so we have $|P(y) - f(y)| \leq \varepsilon$ for all $y \in [a,b]$, as desired. $\square$

Note. Another way of stating Thm. II.3.8.3 is as follows. Recall that $C([a,b] \to \mathbb{R})$ was the space of continuous functions from $[a,b]$ to $\mathbb{R}$, with the uniform metric d_∞. Let $P([a,b] \to \mathbb{R})$ be the space of all polynomials on $[a,b]$; this is a subspace of $C([a,b] \to \mathbb{R})$, since all polynomials are continuous (Exercise 9.4.7 in Analysis I). The Weierstrass approximation theorem then asserts that every continuous function is an adherent point of $P([a,b] \to \mathbb{R})$; or in other words, that the closure of the space of polynomials is the space of continuous functions (see Cor. II.3.3.2):

$$\overline{P([a,b] \to \mathbb{R})}_{\left(C([a,b] \to \mathbb{R}), d_\infty\right)} = C([a,b] \to \mathbb{R}).$$

In particular, every continuous function on $[a,b]$ is the uniform limit of polynomials (see Prop. II.3.4.4). Another way of saying this is that the space of polynomials is *dense* in the space of continuous functions, in the *uniform topology*.

Def. II.3.8.4 (Compactly supported functions)**.** Let $[a,b]$ be an interval. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *supported* on $[a,b]$ iff $f(x) = 0$ for all $x \notin [a,b]$. We say that f is *compactly supported* iff it is supported on some interval $[a,b]$. If f is continuous and supported on $[a,b]$, we define the improper integral $\displaystyle\int_{-\infty}^{\infty} f$ to be $\displaystyle\int_{-\infty}^{\infty} f := \int_{[a,b]} f$.

Note. A function can be supported on more than one interval, for instance a function which is supported on $[3,4]$ is also automatically supported on $[2,5]$.

Lem. II.3.8.5. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and supported on an interval $[a,b]$, and is also supported on another interval $[c,d]$, then $\displaystyle\int_{[a,b]} f = \int_{[c,d]} f$.

Proof. Since

$$\begin{aligned} &\begin{cases} f \text{ is supported on } [a,b] \\ f \text{ is supported on } [c,d] \end{cases} \\ \implies &\begin{cases} \forall x \notin [a,b], f(x) = 0 \\ \forall x \notin [c,d], f(x) = 0 \end{cases} && \text{(by Def. II.3.8.4)} \end{aligned}$$

$$\implies \begin{cases} \forall x \in \mathbb{R}, (x < a) \vee (x > b) \implies f(x) = 0 \\ \forall x \in \mathbb{R}, (x < c) \vee (x > d) \implies f(x) = 0 \end{cases}$$

we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f &= \int_{[a,b]} f && \text{(by Def. II.3.8.4)} \\
&= \begin{cases} \displaystyle\int_{[a,c]} f + \int_{[c,b]} f & \text{if } a \leq c \\ \displaystyle 0 + \int_{[a,b]} f & \text{if } a > c \end{cases} \\
&= \begin{cases} \displaystyle 0 + \int_{[c,b]} f & \text{if } a \leq c \\ \displaystyle\int_{[c,a]} f + \int_{[a,b]} f & \text{if } a > c \end{cases} \\
&= \int_{[c,b]} f \\
&= \begin{cases} \displaystyle\int_{[c,b]} f + 0 & \text{if } b \leq d \\ \displaystyle\int_{[c,d]} f + \int_{[d,b]} f & \text{if } b > d \end{cases} \\
&= \begin{cases} \displaystyle\int_{[c,b]} f + \int_{[b,d]} f & \text{if } b \leq d \\ \displaystyle\int_{[c,d]} f + 0 & \text{if } b > d \end{cases} \\
&= \int_{[c,d]} f.
\end{aligned}$$

□

Def. II.3.8.6 (Approximation to the identity)**.** Let $\varepsilon > 0$ and $0 < \delta < 1$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be an (ε, δ)-*approximation to the identity* if it obeys the following three properties:

(a) f is supported on $[-1, 1]$, and $f(x) \geq 0$ for all $-1 \leq x \leq 1$.

(b) f is continuous, and $\displaystyle\int_{-\infty}^{\infty} f = 1$.

(c) $|f(x)| \leq \varepsilon$ for all $\delta \leq |x| \leq 1$.

Rmk. II.3.8.7. For those of you who are familiar with the Dirac delta function, approximations to the identity are ways to approximate this (very discontinuous) delta function by a continuous function (which is easier to analyze).

Lem. II.3.8.8 (Polynomials can approximate the identity)**.** For every $\varepsilon > 0$ and $0 < \delta < 1$ there exists an (ε, δ)-approximation to the identity which is a polynomial P on $[-1, 1]$.

Proof. Let $\varepsilon \in \mathbb{R}^+$ and let $\delta \in (0, 1)$. We have

$$\begin{aligned} & \forall x \in [-1,1], \delta \leq |x| \leq 1 \\ \implies & \delta^2 \leq x^2 \leq 1 \\ \implies & 0 \leq 1 - x^2 \leq 1 - \delta^2 < 1 \\ \implies & \lim_{n\to\infty} \sqrt{n}(1-\delta^2)^n = 0 \qquad \text{(by Exercise 7.5.2 in Analysis I)} \\ \implies & \exists N \in \mathbb{Z}^+ : \forall n \geq N, \sqrt{n}(1-\delta^2)^n < \varepsilon. \end{aligned}$$

Now we fix such N. Define $g : \mathbb{R} \to \mathbb{R}$ to be the function

$$\forall x \in \mathbb{R}, g(x) = \begin{cases} (1-x^2)^N & \text{if } x \in [-1,1] \\ 0 & \text{if } x \notin [-1,1] \end{cases}$$

We know that $g(x) \geq 0$ for all $x \in \mathbb{R}$. By Def. II.3.8.4 we know that g is supported on $[-1, 1]$. By Exercise 9.4.7 in Analysis I we know that g is continuous on $[-1, 1]$, thus by Corollary 11.5.2 in Analysis I g is Riemann integrable on $[-1, 1]$. By Ex. II.3.8.2(b) we know that

$$\int_{[-1,1]} g = \int_{[-1,1]} (1-x^2)^N \geq \frac{1}{\sqrt{N}} > 0,$$

so we can define $c = (\int_{[-1,1]} g)^{-1}$ and we have

$$0 < c = \left(\int_{[-1,1]} g\right)^{-1} \leq \sqrt{N}.$$

Now we define $f : \mathbb{R} \to \mathbb{R}$ to be the function $f = cg$. Again we have f is continuous and supported on $[-1, 1]$. Since $c > 0$, we know that $f(x) \geq 0$ for all $x \in \mathbb{R}$. By Def. II.3.8.4 we have

$$\int_{-\infty}^{\infty} f = \int_{[-1,1]} f = \int_{[-1,1]} cg = c\int_{[-1,1]} g = \left(\int_{[-1,1]} g\right)^{-1}\left(\int_{[-1,1]} g\right) = 1.$$

Since

$$\forall x \in [-1,1], \delta \leq |x| \leq 1$$

$$
\begin{aligned}
&\implies 0 \le 1 - x^2 \le 1 - \delta^2 < 1 \\
&\implies 0 \le \sqrt{N}(1 - x^2)^N \le \sqrt{N}(1 - \delta^2)^N < \varepsilon \\
&\implies 0 \le |f(x)| = |cg(x)| \le \left|\sqrt{N}(1 - x^2)^N\right| = \sqrt{N}(1 - x^2)^N < \varepsilon,
\end{aligned}
$$

combine all the proofs above we conclude by Def. II.3.8.6 that f is an (ε, δ)-approximation to the identity. □

Def. II.3.8.9 (Convolution)**.** Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous, compactly supported functions. We define the *convolution* $f * g : \mathbb{R} \to \mathbb{R}$ of f and g to be the function

$$(f * g)(x) := \int_{-\infty}^{\infty} f(y)g(x - y)\, dy.$$

Note. If f and g are continuous and compactly supported, then for each x the function $f(y)g(x - y)$ (thought of as a function of y) is also continuous and compactly supported, so Def. II.3.8.9 makes sense.

Rmk. II.3.8.10. Convolutions play an important role in Fourier analysis and in partial differential equations (PDE), and are also important in physics, engineering, and signal processing.

Prop. II.3.8.11 (Basic properties of convolution)**.** Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$ be continuous, compactly supported functions. Then the following statements are true.

(a) The convolution $f * g$ is also a continuous, compactly supported function.

(b) (Convolution is commutative) We have $f * g = g * f$; in other words

$$
\begin{aligned}
f * g(x) &= \int_{-\infty}^{\infty} f(y)g(x - y)\, dy \\
&= \int_{-\infty}^{\infty} g(y)f(x - y)\, dy \\
&= g * f(x).
\end{aligned}
$$

(c) (Convolution is linear) We have $f * (g + h) = f * g + f * h$. Also, for any real number c, we have $f * (cg) = (cf) * g = c(f * g)$.

Proof. (a) Since f, g are compactly supported, by Def. II.3.8.4 we know that

$$\exists L_f, L_g, U_f, U_g \in \mathbb{R} : \begin{cases} \forall y \in \mathbb{R} \setminus [L_f, U_f], f(y) = 0 \\ \forall y \in \mathbb{R} \setminus [L_g, U_g], g(y) = 0 \end{cases}$$

Note that we can choose $L_f \neq U_f$. Let $L = \min(L_f, L_g)$, let $U = \max(U_f, U_g)$ and let $M = \max(|L|, |U|)$. Then we have

$$\forall y \in \mathbb{R} \setminus [-M, M], \begin{cases} y < -M \leq L \leq L_f \implies f(y) = 0 \\ y < -M \leq L \leq L_g \implies g(y) = 0 \\ y > M \geq U \geq U_f \implies f(y) = 0 \\ y > M \geq U \geq U_g \implies g(y) = 0 \end{cases}$$
$$\implies f(y) = g(y) = 0$$

and

$$\forall y \in \mathbb{R} \setminus [-2M, 2M], (y < -2M \leq -M) \vee (y > 2M \geq M) \implies f(y) = g(y) = 0.$$

Thus, by Def. II.3.8.4 f, g are supported on $[-M, M]$ and $[-2M, 2M]$. Observe that

$$\forall x \in (-\infty, -2M), \forall y \in \mathbb{R}, \begin{cases} x - y < -M \text{ or} \\ x - y \geq -M \end{cases}$$
$$\implies \forall x \in (-\infty, -2M), \forall y \in \mathbb{R}, \begin{cases} x - y < -M \text{ or} \\ -M > x + M \geq y \end{cases}$$
$$\implies \forall x \in (-\infty, -2M), \forall y \in \mathbb{R}, \begin{cases} g(x-y) = 0 & \text{if } x - y < -M \\ f(y) = 0 & \text{if } -M > x + M \geq y \end{cases}$$
$$\implies \forall x \in (-\infty, -2M), \forall y \in \mathbb{R}, f(y)g(x-y) = 0$$

and

$$\forall x \in (2M, +\infty), \forall y \in \mathbb{R}, \begin{cases} x - y > M \text{ or} \\ x - y \leq M \end{cases}$$
$$\implies \forall x \in (2M, +\infty), \forall y \in \mathbb{R}, \begin{cases} x - y > M \text{ or} \\ M < x - M \leq y \end{cases}$$
$$\implies \forall x \in (2M, +\infty), \forall y \in \mathbb{R}, \begin{cases} g(x-y) = 0 & \text{if } x - y > M \\ f(y) = 0 & \text{if } M < x - M \leq y \end{cases}$$
$$\implies \forall x \in (2M, +\infty), \forall y \in \mathbb{R}, f(y)g(x-y) = 0.$$

This means

$$\forall x \in \mathbb{R} \setminus [-2M, 2M], \forall y \in \mathbb{R}, f(y)g(x-y) = 0.$$

For each $x \in \mathbb{R} \setminus [-2M, 2M]$, we define $z_x : \mathbb{R} \to \mathbb{R}$ by setting $z_x(y) = f(y)g(x-y)$. Since z_x is continuous on $\mathbb{R}$, by Def. II.3.8.4 and Def. II.3.8.9 we have

$$\forall x \in \mathbb{R} \setminus [-2M, 2M], \forall y \in \mathbb{R}, z_x(y) = 0$$

$$\begin{aligned}
&\implies \forall x \in \mathbb{R} \setminus [-2M, 2M], \forall y \in \mathbb{R} \setminus [-1,1], z_x(y) = 0 \\
&\implies \forall x \in \mathbb{R} \setminus [-2M, 2M], z_x \text{ is supported on } [-1,1] \\
&\implies \forall x \in \mathbb{R} \setminus [-2M, 2M], \int_{-\infty}^{\infty} z_x = \int_{[-1,1]} z_x = 0 \\
&\implies \forall x \in \mathbb{R} \setminus [-2M, 2M], \int_{-\infty}^{\infty} z_x(y)\, dy = \int_{[-1,1]} z_x(y)\, dy = 0 \\
&\implies \forall x \in \mathbb{R} \setminus [-2M, 2M], (f * g)(x) = \int_{[-1,1]} f(y)g(x-y)\, dy = 0 \\
&\implies f * g \text{ is supported on } [-2M, 2M] \\
&\implies f * g \text{ is compactly supported.}
\end{aligned}$$

Since f, g are compactly supported and continuous on $\mathbb{R}$, by Ex. II.3.8.3 we know that

$$\exists N \in \mathbb{R}^+ : \forall x \in \mathbb{R}, |f(x)| \leq N$$

and

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta \implies |g(x_1) - g(x_2)| < \frac{\varepsilon}{N(U_f - L_f)}.$$

Fix N and one pair of ε and δ. Let $x_0 \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\forall x \in \mathbb{R}, |x - x_0| < \delta \\
&\implies |(f*g)(x) - (f*g)(x_0)| = \left| \int_{-\infty}^{\infty} f(y)g(x-y)\, dy - \int_{-\infty}^{\infty} f(y)g(x_0-y)\, dy \right| \\
&= \left| \int_{[L_f, U_f]} f(y)g(x-y)\, dy - \int_{[L_f, U_f]} f(y)g(x_0-y)\, dy \right| \\
&= \left| \int_{[L_f, U_f]} f(y)\big(g(x-y) - g(x_0-y)\big)\, dy \right| \\
&\leq \left| \int_{[L_f, U_f]} N \frac{\varepsilon}{N(U_f - L_f)}\, dy \right| = \varepsilon.
\end{aligned}$$

Since ε was arbitrary, we know that $f * g$ is continuous at x_0. Since x_0 was arbitrary, we know that $f * g$ is continuous on $\mathbb{R}$. $\square$

Proof. (b) Let $x_0 \in \mathbb{R}$. Since f is compactly supported, we know that

$$\exists L, U \in \mathbb{R} : \forall y \in \mathbb{R} \setminus [L, U], f(y) = 0.$$

Then we have

$$\forall y \in \mathbb{R} \setminus [L, U], f(y) = 0$$

$$\implies \forall y \in \mathbb{R} \setminus [L, U], f(y)g(x_0 - y) = 0.$$

Observe that

$$\begin{aligned}
&\forall y \in \mathbb{R} \setminus [L, U], f(y) = 0 \\
\implies &\forall y \in \mathbb{R} \setminus [-U, -L], f(-y) = 0 \\
\implies &\forall y \in \mathbb{R} \setminus [x_0 - U, x_0 - L], f(x_0 - y) = 0 \\
\implies &\forall y \in \mathbb{R} \setminus [x_0 - U, x_0 - L], g(y)f(x_0 - y) = 0.
\end{aligned}$$

Since f, g are continuous on $\mathbb{R}$, we know that

$$\begin{aligned}
&\forall y_0 \in \mathbb{R}, \begin{cases} f \text{ is continuous at } y_0 \\ g \text{ is continuous at } y_0 \\ f \text{ is continuous at } x_0 - y_0 \\ g \text{ is continuous at } x_0 - y_0 \\ y \mapsto x_0 - y \text{ is continuous at } y_0 \end{cases} \\
\implies &\forall y_0 \in \mathbb{R}, \begin{cases} \lim_{y \to y_0; y \in \mathbb{R}} f(y) = f(y_0) \\ \lim_{y \to y_0; y \in \mathbb{R}} g(y) = g(y_0) \\ \lim_{y \to y_0; y \in \mathbb{R}} f(x_0 - y) = f(x_0 - y_0) \\ \lim_{y \to y_0; y \in \mathbb{R}} g(x_0 - y) = g(x_0 - y_0) \end{cases} \\
\implies &\forall y_0 \in \mathbb{R}, \begin{cases} \lim_{y \to y_0; y \in \mathbb{R}} f(y)g(x_0 - y) = f(y_0)g(x_0 - y_0) \\ \lim_{y \to y_0; y \in \mathbb{R}} g(y)f(x_0 - y) = g(y_0)f(x_0 - y_0) \end{cases}
\end{aligned}$$

This means

$$\begin{aligned}
(f * g)(x_0) &= \int_{-\infty}^{\infty} f(y)g(x_0 - y)\, dy && \text{(by Def. II.3.8.9)} \\
&= \int_{[L,U]} f(y)g(x_0 - y)\, dy; && \text{(by Def. II.3.8.4)} \\
(g * f)(x_0) &= \int_{-\infty}^{\infty} g(y)f(x_0 - y)\, dy && \text{(by Def. II.3.8.9)} \\
&= \int_{[x_0 - U, x_0 - L]} g(y)f(x_0 - y)\, dy; && \text{(by Def. II.3.8.4)}
\end{aligned}$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function $\phi = y \mapsto x_0 - y$. By the formula of changing variable (Exercise 11.10.4 in Analysis I) we have

$$\int_{[L,U]} f(y)g(x_0 - y)\, dy$$

$$
\begin{aligned}
&= \int_{[\phi(x_0-U),\phi(x_0-L)]} f(y)g(x_0-y)\,dy \\
&= -\int_{[x_0-U,x_0-L]} f\big(\phi(y)\big)g\big(x_0-\phi(y)\big)\phi'(y)\,dy \\
&= \int_{[x_0-U,x_0-L]} f(x_0-y)g(y)\,dy \\
&= \int_{[x_0-U,x_0-L]} g(y)f(x_0-y)\,dy.
\end{aligned}
$$

Thus, $(f*g)(x_0)=(g*f)(x_0)$. Since x_0 was arbitrary, we conclude that

$$\forall x\in\mathbb{R}, (f*g)(x)=(g*f)(x).$$

□

Proof. (c) Let $x_0\in\mathbb{R}$. Since g,h are compactly supported, by Def. II.3.8.4 we know that

$$\exists L_g,L_h,U_g,U_h\in\mathbb{R}:\begin{cases}\forall y\in\mathbb{R}\setminus[L_g,U_g], g(y)=0\\ \forall y\in\mathbb{R}\setminus[L_h,U_h], h(y)=0\end{cases}$$

Let $L=\min(L_g,L_h)$ and let $U=\min(U_g,U_h)$. Then we have

$$
\begin{aligned}
&\forall y\in\mathbb{R}\setminus[L,U], \begin{cases} y<L\le L_g & \implies g(y)=0\\ y>U\ge U_g & \implies g(y)=0\\ y<L\le L_h & \implies h(y)=0\\ y>U\ge U_h & \implies h(y)=0\end{cases}\\
&\implies \forall y\in\mathbb{R}\setminus[L,U], g(y)=h(y)=0\\
&\implies \forall y\in\mathbb{R}\setminus[-U,-L], g(-y)=h(-y)=0\\
&\implies \forall y\in\mathbb{R}\setminus[x_0-U,x_0-L], g(x_0-y)=h(x_0-y)=0\\
&\implies \forall y\in\mathbb{R}\setminus[x_0-U,x_0-L], f(y)g(x_0-y)=f(y)h(x_0-y)=0\\
&\implies \forall y\in\mathbb{R}\setminus[x_0-U,x_0-L], f(y)\big(g(x_0-y)+h(x_0-y)\big)=0.
\end{aligned}
$$

Since f,g,h are continuous on $\mathbb{R}$, we know that

$$
\begin{aligned}
&\forall y_0\in\mathbb{R}, \begin{cases} f \text{ is continuous at } y_0\\ g \text{ is continuous at } y_0\\ h \text{ is continuous at } y_0\\ y\mapsto x_0-y \text{ is continuous at } y_0\end{cases}\\
&\implies \forall y_0\in\mathbb{R}, \begin{cases} \lim\limits_{y\to y_0;y\in\mathbb{R}} f(y)=f(y_0)\\ \lim\limits_{y\to y_0;y\in\mathbb{R}} g(x_0-y)=g(x_0-y_0)\\ \lim\limits_{y\to y_0;y\in\mathbb{R}} h(x_0-y)=h(x_0-y_0)\end{cases}
\end{aligned}
$$

$$\implies \forall y_0 \in \mathbb{R}, \begin{cases} \lim\limits_{y \to y_0; y \in \mathbb{R}} f(y)g(x_0 - y) = f(y_0)g(x_0 - y_0) \\ \lim\limits_{y \to y_0; y \in \mathbb{R}} f(y)h(x_0 - y) = f(y_0)h(x_0 - y_0) \end{cases}$$
$$\implies \forall y_0 \in \mathbb{R}, \lim_{y \to y_0; y \in \mathbb{R}} f(y)\big(g(x_0 - y) + h(x_0 - y)\big) = f(y_0)\big(g(x_0 - y_0) + h(x_0 - y_0)\big).$$

Thus, we have

$$\begin{aligned}
&\big(f * (g + h)\big)(x_0) \\
&= \int_{-\infty}^{\infty} f(y)(g + h)(x_0 - y)\, dy && \text{(by Def. II.3.8.9)} \\
&= \int_{-\infty}^{\infty} f(y)\big(g(x_0 - y) + h(x_0 - y)\big)\, dy \\
&= \int_{[x_0 - U, x_0 - L]} f(y)\big(g(x_0 - y) + h(x_0 - y)\big)\, dy && \text{(by Def. II.3.8.4)} \\
&= \int_{[x_0 - U, x_0 - L]} f(y)g(x_0 - y)\, dy + \int_{[x_0 - U, x_0 - L]} f(y)h(x_0 - y)\big)\, dy \\
&= \int_{-\infty}^{\infty} f(y)g(x_0 - y)\, dy + \int_{-\infty}^{\infty} f(y)h(x_0 - y)\, dy && \text{(by Def. II.3.8.4)} \\
&= (f * g)(x_0) + (f * h)(x_0). && \text{(by Def. II.3.8.9)}
\end{aligned}$$

Observe that

$$\forall y \in \mathbb{R} \setminus [x_0 - U, x_0 - L], f(y)g(x_0 - y) = cf(y)g(x_0 - y) = 0.$$

Since f is continuous on $\mathbb{R}$, we know that cf is also continuous on $\mathbb{R}$ and

$$\forall y_0 \in \mathbb{R}, \begin{cases} cf \text{ is continuous at } y_0 \\ g \text{ is continuous at } y_0 \\ y \mapsto x_0 - y \text{ is continuous at } y_0 \end{cases}$$
$$\implies \forall y_0 \in \mathbb{R}, \begin{cases} \lim\limits_{y \to y_0; y \in \mathbb{R}} cf(y) = cf(y_0) \\ \lim\limits_{y \to y_0; y \in \mathbb{R}} g(x_0 - y) = g(x_0 - y_0) \end{cases}$$
$$\implies \forall y_0 \in \mathbb{R}, \Big\{ \lim_{y \to y_0; y \in \mathbb{R}} cf(y)g(x_0 - y) = cf(y_0)g(x_0 - y_0)$$

Thus, we have

$$\begin{aligned}
&\big((cf) * g\big)(x_0) \\
&= \int_{-\infty}^{\infty} (cf)(y)g(x_0 - y)\, dy && \text{(by Def. II.3.8.9)} \\
&= \int_{-\infty}^{\infty} cf(y)g(x_0 - y)\, dy
\end{aligned}$$

$$
\begin{aligned}
&= \int_{[x_0-U,x_0-L]} cf(y)g(x_0-y)\,dy && \text{(by Def. II.3.8.4)}\\
&= c\int_{[x_0-U,x_0-L]} f(y)g(x_0-y)\,dy \\
&= c\int_{-\infty}^{\infty} f(y)g(x_0-y)\,dy && \text{(by Def. II.3.8.4)}\\
&= c(f*g)(x_0). && \text{(by Def. II.3.8.9)}
\end{aligned}
$$

Using similar arguments, we can show that $\big((cg)*f\big)(x_0) = c(g*f)(x_0)$. By Prop. II.3.8.11(b) we thus have

$$\big(f*(cg)\big)(x_0) = \big((cg)*f\big)(x_0) = c(g*f)(x_0) = c(f*g)(x_0) = \big((cf)*g\big)(x_0).$$

Since x_0 was arbitrary, we conclude that

$$\forall x \in \mathbb{R}, \begin{cases} \big(f*(g+h)\big)(x) = (f*g)(x) + (f*h)(x) \\ \big(f*(cg)\big)(x) = \big((cf)*g\big)(x) = c(f*g)(x) \end{cases}$$

□

Rmk. II.3.8.12. There are many other important properties of convolution, for instance it is associative, $(f*g)*h = f*(g*h)$, and it commutes with derivatives, $(f*g)' = f'*g = f*g'$, when f and g are differentiable. The Dirac delta function δ mentioned earlier is an identity for convolution: $f*\delta = \delta*f = f$. These results are slightly harder to prove than the ones in Prop. II.3.8.11, however, and we will not need them in this text.

Lem. II.3.8.13. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function supported on $[0,1]$, and let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function supported on $[-1,1]$ which is a polynomial on $[-1,1]$. Then $f*g$ is a polynomial on $[0,1]$. (Note however that it may be non-polynomial outside of $[0,1]$.)

Proof. Since g is polynomial on $[-1,1]$, we may find an integer $n \geq 0$ and real numbers $c_0, c_1, \ldots, c_n$ such that

$$g(x) = \sum_{j=0}^{n} c_j x^j \text{ for all } x \in [-1,1].$$

On the other hand, for all $x \in [0,1]$, we have

$$f*g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy = \int_{[0,1]} f(y)g(x-y)\,dy$$

since f is supported on $[0,1]$. Since $x \in [0,1]$ and the variable of integration y is also in $[0,1]$, we have $x-y \in [-1,1]$. Thus, we may substitute in our formula for g to obtain

$$f*g(x) = \int_{[0,1]} f(y)\sum_{j=0}^{n} c_j (x-y)^j\,dy.$$

We expand this using the binomial formula (Exercise 7.1.4 in Analysis I) to obtain

$$f * g(x) = \int_{[0,1]} f(y) \sum_{j=0}^{n} c_j \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} x^k (-y)^{j-k}\, dy.$$

We can interchange the two summations (by Corollary 7.1.14 in Analysis I) to obtain

$$f * g(x) = \int_{[0,1]} f(y) \sum_{k=0}^{n} \sum_{j=k}^{n} c_j \frac{j!}{k!(j-k)!} x^k (-y)^{j-k}\, dy.$$

(why did the limits of summation change? It may help to plot j and k on a graph). Now we interchange the k summation with the integral, and observe that x^k is independent of y, to obtain

$$f * g(x) = \sum_{k=0}^{n} x^k \int_{[0,1]} f(y) \sum_{j=k}^{n} c_j \frac{j!}{k!(j-k)!} (-y)^{j-k}\, dy.$$

If we thus define

$$C_k := \int_{[0,1]} f(y) \sum_{j=k}^{n} c_j \frac{j!}{k!(j-k)!} (-y)^{j-k}\, dy$$

for each $k = 0, \ldots, n$, then C_k is a number which is independent of x, and we have

$$f * g(x) = \sum_{k=0}^{n} C_k x^k$$

for all $x \in [0,1]$. Thus, $f * g$ is a polynomial on $[0,1]$. □

Lem. II.3.8.14. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function supported on $[0,1]$, which is bounded by some $M > 0$ (i.e., $|f(x)| \leq M$ for all $x \in \mathbb{R}$), and let $\varepsilon > 0$ and $0 < \delta < 1$ be such that one has $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in \mathbb{R}$ and $|x - y| < \delta$. Let g be any (ε, δ)-approximation to the identity. Then we have

$$|f * g(x) - f(x)| \leq (1 + 4M)\varepsilon$$

for all $x \in [0,1]$.

Proof. Since g is an (ε, δ)-approximation to the identity, by Def. II.3.8.6 we have

- g is supported on $[-1,1]$ and $g(x) \geq 0$ for all $x \in [-1,1]$.
- g is continuous on $\mathbb{R}$ and $\displaystyle\int_{-\infty}^{\infty} g = 1$.
- $|g(x)| \leq \varepsilon$ for all $\delta \leq |x| \leq 1$.

Since f is continuous on $\mathbb{R}$, by Def. II.3.8.9 we have

$$\begin{aligned}
&\forall x \in [0,1], (f * g)(x)\\
&= \int_{-\infty}^{\infty} g(y) f(x-y)\, dy\\
&= \int_{[-1,1]} g(y) f(x-y)\, dy\\
&= \int_{[-1,-\delta]} g(y) f(x-y)\, dy + \int_{[-\delta,\delta]} g(y) f(x-y)\, dy + \int_{[\delta,1]} g(y) f(x-y)\, dy.
\end{aligned}$$

By Ex. II.3.8.6 we have

$$1 - 2\varepsilon \le \int_{[-\delta,\delta]} g = \int_{[-\delta,\delta]} g(y)\, dy \le 1.$$

Since

$$\begin{aligned}
&\begin{cases} \forall x \in \mathbb{R}, |f(x)| \le M \\ \forall y \in \mathbb{R}, \delta \le |y| \le 1 \implies |g(y)| < \varepsilon \end{cases}\\
&\implies \forall x \in \mathbb{R}, \forall \delta \le |y| \le 1, g(y) f(x-y) \le M\varepsilon\\
&\implies \forall x \in \mathbb{R}, \begin{cases} -M\varepsilon(1-\delta) \le \int_{[-1,-\delta]} g(y) f(x-y)\, dy \le M\varepsilon(1-\delta) \\ -M\varepsilon(1-\delta) \le \int_{[\delta,1]} g(y) f(x-y)\, dy \le M\varepsilon(1-\delta) \end{cases}\\
&\implies \forall x \in \mathbb{R}, \begin{cases} -M\varepsilon \le \int_{[-1,-\delta]} g(y) f(x-y)\, dy \le M\varepsilon \\ -M\varepsilon \le \int_{[\delta,1]} g(y) f(x-y)\, dy \le M\varepsilon \end{cases} && (\delta < 1)\\
&\implies \forall x \in \mathbb{R}, \left| \int_{[-1,-\delta]} g(y) f(x-y)\, dy + \int_{[\delta,1]} g(y) f(x-y)\, dy \right| \le 2M\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\forall x \in [0,1], \forall y \in [-\delta,\delta], |(x-y) - x| = |y| < \delta\\
&\implies \forall x \in [0,1], \forall y \in [-\delta,\delta], |f(x-y) - f(x)| < \varepsilon && \text{(by hypothesi}\\
&\implies \forall x \in [0,1], \forall y \in [-\delta,\delta],\\
&\quad |g(y) f(x-y) - g(y) f(x)| \le \varepsilon g(y) && (g(y) \ge 0)\\
&\implies \forall x \in [0,1], \forall y \in [-\delta,\delta],\\
&\quad g(y) f(x) - \varepsilon g(y) \le g(y) f(x-y) \le g(y) f(x) + \varepsilon g(y)
\end{aligned}$$

$$
\begin{aligned}
&\implies \forall x \in [0,1],\\
&\qquad \big(f(x)-\varepsilon\big) \int_{[-\delta,\delta]} g(y)\ dy\\
&\qquad \le \int_{[-\delta,\delta]} g(y) f(x-y)\ dy\\
&\qquad \le \big(f(x)+\varepsilon\big) \int_{[-\delta,\delta]} g(y)\ dy\\
&\implies \forall x \in [0,1],\\
&\qquad \big(f(x)-\varepsilon\big)(1-2\varepsilon) \le \int_{[-\delta,\delta]} g(y) f(x-y)\ dy \le f(x)+\varepsilon && \text{(by Ex. II.3.8}\\
&\implies \forall x \in [0,1],\\
&\qquad -2\varepsilon f(x) - \varepsilon + 2\varepsilon^2 \le \int_{[-\delta,\delta]} g(y) f(x-y)\ dy - f(x) \le \varepsilon\\
&\implies \forall x \in [0,1],\\
&\qquad -2\varepsilon M - \varepsilon + 2\varepsilon^2 \le \int_{[-\delta,\delta]} g(y) f(x-y)\ dy - f(x) \le \varepsilon && (f(x) \le M)\\
&\implies \forall x \in [0,1],\\
&\qquad -2\varepsilon M - \varepsilon \le \int_{[-\delta,\delta]} g(y) f(x-y)\ dy - f(x) \le \varepsilon && (\varepsilon \in \mathbb{R}^+)\\
&\implies \forall x \in [0,1],\\
&\qquad -\varepsilon(2M+1) \le \int_{[-\delta,\delta]} g(y) f(x-y)\ dy - f(x) \le \varepsilon(2M+1) && (2M+1>1)\\
&\implies \forall x \in [0,1], \left| \int_{[-\delta,\delta]} g(y) f(x-y)\ dy - f(x) \right| \le \varepsilon(2M+1),
\end{aligned}
$$

we know that

$$
\begin{aligned}
&\forall x \in [0,1], |(f*g)(x) - f(x)|\\
&= \left| \int_{[-1,-\delta]} g(y) f(x-y)\ dy + \int_{[-\delta,\delta]} g(y) f(x-y)\ dy - f(x) + \int_{[\delta,1]} g(y) f(x-y)\ dy \right|\\
&\le \left| \int_{[-1,-\delta]} g(y) f(x-y)\ dy + \int_{[\delta,1]} g(y) f(x-y)\ dy \right| + \left| \int_{[-\delta,\delta]} g(y) f(x-y)\ dy - f(x) \right|\\
&\le 2M\varepsilon + \varepsilon(2M+1)\\
&= (1+4M)\varepsilon.
\end{aligned}
$$

$\square$

Cor. II.3.8.15 (Weierstrass approximation theorem I)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function supported on $[0, 1]$. Then for every $\varepsilon > 0$, there exists a function $P : \mathbb{R} \to \mathbb{R}$ which is polynomial on $[0, 1]$ and such that $|P(x) - f(x)| \le \varepsilon$ for all $x \in [0, 1]$.

Proof. Let $\varepsilon \in \mathbb{R}^+$. Since f is continuous on $\mathbb{R}$ and supported on $[0, 1]$, by Ex. II.3.8.3 we know that f is bounded by some $M \in \mathbb{R}^+$ and f is uniformly continuous on $\mathbb{R}$. This means

$$\exists \delta \in \mathbb{R}^+ : \forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \frac{\varepsilon}{1 + 4M}.$$

In particular, we can choose some δ such that $0 < \delta < 1$. By Lem. II.3.8.8 we know that there exists a polynomial P on $[-1, 1]$ such that P is an $(\frac{\varepsilon}{1 + 4M}, \delta)$-approximation to the identity. By Def. II.3.8.6 we know that P is continuous on $\mathbb{R}$ and supported on $[-1, 1]$. Since f is continuous on $\mathbb{R}$ and supported on $[0, 1]$, by Lem. II.3.8.13 we know that $f * P$ is a polynomial on $[0, 1]$. Then by Lem. II.3.8.14 we have

$$\forall x \in [0, 1], |(f * P)(x) - f(x)| \le (1 + 4M)\frac{\varepsilon}{1 + 4M} = \varepsilon.$$

Since ε was arbitrary, we conclude that

$$\forall \varepsilon \in \mathbb{R}^+, \exists P \in \mathbb{R}^{\mathbb{R}} : \begin{cases} P \text{ is polynomial on } [0, 1] \\ \forall x \in [0, 1], |P(x) - f(x)| \le \varepsilon \end{cases}$$

□

Lem. II.3.8.16. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function which equals 0 on the boundary of $[0, 1]$, i.e., $f(0) = f(1) = 0$. Let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by setting $F(x) \coloneqq f(x)$ for $x \in [0, 1]$ and $F(x) \coloneqq 0$ for $x \notin [0, 1]$. Then F is also continuous.

Proof. Since f is continuous on $[0, 1]$, we know that f is continuous at 0 and 1. Thus, we have

$$\begin{aligned}
&\lim_{x \to 0; x \in [0,1]} f(x) = f(0) = 0 \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in [0, 1], |x| < \delta \implies |f(x)| < \varepsilon\big) \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in [0, \delta], |f(x)| < \varepsilon\big) \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in [0, \delta], |F(x)| < \varepsilon\big) \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in \mathbb{R}, |x| < \delta \implies |F(x)| < \varepsilon\big) \qquad (F(x) = 0 \text{ if } x < 0) \\
\implies &\lim_{x \to 0; x \in \mathbb{R}} F(x) = F(0) = 0.
\end{aligned}$$

Similarly, we have $\lim_{x \to 1; x \in \mathbb{R}} F(x) = F(1) = 0$. This means F is continuous at 0 and 1.

Since f is continuous on $[0, 1]$, we know that f is continuous on $(0, 1)$. Let $x_0 \in (0, 1)$. Then we have

$$\lim_{x \to x_0; x \in (0,1)} f(x) = f(x_0)$$

$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in (0,1), |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon\big)$$
$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in (0,1), |x - x_0| < \delta \implies |F(x) - F(x_0)| < \varepsilon\big).$$

Now we fix one pair of ε and δ. Let $\delta' = \min(\delta, |x_0 - 0|, |1 - x_0|)$. Then we have $\delta' \in \mathbb{R}^+$ and

$$\begin{aligned}
&\forall x \in \mathbb{R}, |x - x_0| < \delta' \\
\implies &\begin{cases} |x - x_0| < |x_0 - 0| \\ |x - x_0| < |1 - x_0| \end{cases} \\
\implies &\begin{cases} |x - x_0| < x_0 \\ |x - x_0| < 1 - x_0 \end{cases} \\
\implies &\begin{cases} 0 < x < 2x_0 \\ 2x_0 - 1 < x < 1 \end{cases} \\
\implies &\max(0, 2x_0 - 1) < x < \min(2x_0, 1) \\
\implies &0 < x < 1.
\end{aligned}$$

Thus
$$\forall x \in \mathbb{R}, |x - x_0| < \delta' \implies |F(x) - F(x_0)| < \varepsilon.$$
Since ε was arbitrary, we conclude that $\lim\limits_{x \to x_0; x \in \mathbb{R}} F(x) = F(x_0)$. Since x_0 was arbitrary, we conclude that $\lim\limits_{x \to x_0; x \in \mathbb{R}} F(x) = F(x_0)$ for each $x_0 \in (0,1)$.

Let $x_0 \in (-\infty, 0)$ and let $\delta = |0 - x_0|$. Since $F(x) = 0$ for all $x \in (-\infty, 0)$, we have $F(x_0) = 0$ and

$$\begin{aligned}
&\forall x \in \mathbb{R}, |x - x_0| < \delta \\
\implies &|x - x_0| < |0 - x_0| \\
\implies &|x - x_0| < -x_0 \\
\implies &2x_0 < x < 0 \\
\implies &F(x) = 0 \\
\implies &\forall \varepsilon \in \mathbb{R}^+, |F(x) - F(x_0)| = 0 \leq \varepsilon.
\end{aligned}$$

Thus, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in \mathbb{R}, |x - x_0| < \delta \implies |F(x) - F(x_0)| < \varepsilon$$

and $\lim\limits_{x \to x_0; x \in \mathbb{R}} F(x) = F(x_0) = 0$. Since x_0 was arbitrary, we conclude that

$$\forall x_0 \in (-\infty, 0), \lim_{x \to x_0; x \in \mathbb{R}} F(x) = F(x_0) = 0.$$

Using similar arguments, we can show that $\lim_{x \to x_0; x \in \mathbb{R}} F(x) = F(x_0) = 0$ for all $x_0 \in (1, \infty)$. Combine all proofs above we have

$$\forall x_0 \in \mathbb{R}, \lim_{x \to x_0; x \in \mathbb{R}} F(x) = F(x_0) = 0$$

and thus F is continuous on $\mathbb{R}$. □

Rmk. II.3.8.17. The function F obtained in Lem. II.3.8.16 is sometimes known as the *extension of f by zero.*

Cor. II.3.8.18 (Weierstrass approximation theorem II)**.** Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$. Then for every $\varepsilon > 0$ there exists a polynomial $P : [0, 1] \to \mathbb{R}$ such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0, 1]$.

Proof. Using Def. II.3.8.6 we can define an $F : \mathbb{R} \to \mathbb{R}$ such that

$$\forall x \in \mathbb{R}, F(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus [0, 1] \\ f(x) & \text{if } x \in [0, 1] \end{cases}$$

and F is continuous and supported on $[0, 1]$. Then by Cor. II.3.8.15 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists P \in \mathbb{R}^{\mathbb{R}} : \begin{cases} P \text{ is a polynomial on } [0, 1] \\ \forall x \in [0, 1], |P(x) - f(x)| = |P(x) - F(x)| \leq \varepsilon \end{cases}$$

□

Cor. II.3.8.19 (Weierstrass approximation theorem III)**.** Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Then for every $\varepsilon > 0$ there exists a polynomial $P : [0, 1] \to \mathbb{R}$ such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0, 1]$.

Proof. Let $F : [0, 1] \to \mathbb{R}$ denote the function

$$F(x) := f(x) - f(0) - x\big(f(1) - f(0)\big).$$

Observe that F is also continuous, and that $F(0) = F(1) = 0$. By Cor. II.3.8.18, we can thus find a polynomial $Q : [0, 1] \to \mathbb{R}$ such that $|Q(x) - F(x)| \leq \varepsilon$ for all $x \in [0, 1]$. But

$$Q(x) - F(x) = Q(x) + f(0) + x\big(f(1) - f(0)\big) - f(x),$$

so the claim follows by setting P to be the polynomial $P(x) := Q(x) + f(0) + x\big(f(1) - f(0)\big)$. □

Rmk. II.3.8.20. Note that the Weierstrass approximation theorem only works on bounded intervals $[a, b]$; continuous functions on $\mathbb{R}$ cannot be uniformly approximated by polynomials. For instance, the exponential function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := e^x$ (which we shall study rigorously in Section 4.5) cannot be approximated by any polynomial, because exponential functions grow faster than any polynomial (Exercise 4.5.9) and so there is no way one can even make the sup metric between f and a polynomial finite.

Rmk. II.3.8.21. There is a generalization of the Weierstrass approximation theorem to higher dimensions: if K is any compact subset of $\mathbb{R}^n$ (with the Euclidean metric d_{l^2}), and $f : K \to \mathbb{R}$ is a continuous function, then for every $\varepsilon > 0$ there exists a polynomial $P : K \to \mathbb{R}$ of n variables $x_1, \ldots, x_n$ such that $d_\infty(f, P) < \varepsilon$. This general theorem can be proven by a more complicated variant of the arguments here, but we will not do so. (There is in fact an even more general version of this theorem applicable to an arbitrary metric space, known as the *Stone-Weierstrass theorem*, but this is beyond the scope of this text.)

— Exercises —

Ex. II.3.8.1. Prove Lem. II.3.8.5.

Proof. See Lem. II.3.8.5. □

Ex. II.3.8.2.

(a) Prove that for any real number $0 \leq y \leq 1$ and any natural number $n \geq 0$, that $(1-y)^n \geq 1 - ny$.

(b) Show that $\displaystyle\int_{-1}^{1} (1-x^2)^n \, dx \geq \frac{1}{\sqrt{n}}$.

(c) Prove Lem. II.3.8.8.

Proof. (a) For each $n \in \mathbb{N}$, let $P(n)$ be the statement "for each $y \in \mathbb{R}$, if $0 \leq y \leq 1$, then $(1-y)^n \geq 1 - ny$." We induct on n to show that $P(n)$ is true for all $n \in \mathbb{N}$. For $n = 0$, we have

$$\forall y \in \mathbb{R}, 0 \leq y \leq 1 \implies (1-y)^0 = 1 \geq 1 - 0y = 1.$$

Thus, the base case holds. Suppose inductively that $P(n)$ is true for some $n \geq 0$. Then we want to show that $P(n+1)$ is true. Let $y \in \mathbb{R}$ such that $0 \leq y \leq 1$. Then we have

$$\begin{aligned}
(1-y)^{n+1} &= (1-y)^n(1-y) \\
&\geq (1-ny)(1-y) && \text{(by the induction hypothesis)} \\
&= 1 - (n+1)y + ny^2 \\
&\geq 1 - (n+1)y. && (0 \leq y \leq 1)
\end{aligned}$$

Since y was arbitrary, we know that $P(n+1)$ is true and this closes the induction. □

Proof. (b) Let $n \in \mathbb{Z}^+$. Since $f(x) = 1 - x^2$ is continuous and bounded on $[-1, 1]$, by Proposition 9.4.9 and 9.6.7 in Analysis I we know that $f^n(x) = (1-x^2)^n$ is continuous and bounded on $[-1, 1]$. Thus, by Corollary 11.5.2 in Analysis I we know that f^n is Riemann integrable. By Corollary 11.10.3 in Analysis I we have

$$\int_{[-1,1]} f^n = \int_{[-1,1]} f^n \cdot 1 = \int_{[-1,1]} f^n \cdot x' = \int_{[-1,1]} f^n \, dx = \int_{-1}^{1} f^n \, dx.$$

Thus

$$\int_{-1}^{1}(1-x^2)^n\,dx=\int_{[-1,1]}(1-x^2)^n=\int_{[-1,\frac{-1}{\sqrt{n}}]}(1-x^2)^n+\int_{[\frac{-1}{\sqrt{n}},\frac{1}{\sqrt{n}}]}(1-x^2)^n+\int_{[\frac{1}{\sqrt{n}},1]}(1-x^2)^n.$$

Since

$$\forall x\in[-1,1],1\geq|x|\geq\frac{1}{\sqrt{n}}\implies 1\geq x^2\geq\frac{1}{n}\implies 0\leq 1-x^2\leq\frac{n-1}{n},$$

we know that

$$\int_{-1}^{1}(1-x^2)^n\,dx\geq\int_{[\frac{-1}{\sqrt{n}},\frac{1}{\sqrt{n}}]}(1-x^2)^n.$$

By Ex. II.3.8.2(a) we have

$$\int_{-1}^{1}(1-x^2)^n\,dx\geq\int_{[\frac{-1}{\sqrt{n}},\frac{1}{\sqrt{n}}]}(1-x^2)^n\geq\int_{[\frac{-1}{\sqrt{n}},\frac{1}{\sqrt{n}}]}(1-nx^2).$$

Since

$$\begin{aligned}\int_{[\frac{-1}{\sqrt{n}},\frac{1}{\sqrt{n}}]}(1-nx^2)&=\int_{[\frac{-1}{\sqrt{n}},\frac{1}{\sqrt{n}}]}1-n\int_{[\frac{-1}{\sqrt{n}},\frac{1}{\sqrt{n}}]}x^2\\&=\frac{2}{\sqrt{n}}-\frac{n}{3}\left(\frac{1}{n\sqrt{n}}-\frac{-1}{n\sqrt{n}}\right)\\&=\frac{2}{\sqrt{n}}-\frac{2}{3\sqrt{n}}\\&=\frac{4}{3\sqrt{n}}\geq\frac{1}{\sqrt{n}},\end{aligned}$$

we have

$$\int_{-1}^{1}(1-x^2)^n\,dx\geq\frac{1}{\sqrt{n}}.$$

□

Proof. (c) See Lem. II.3.8.8. □

Ex. II.3.8.3. Let $f:\mathbb{R}\to\mathbb{R}$ be a compactly supported, continuous function. Show that f is bounded and uniformly continuous.

Proof. Since f is compactly supported, by Def. II.3.8.4 we know that there exists some $a,b\in\mathbb{R}$ such that

$$\forall x\notin[a,b],f(x)=0.$$

Since $[a,b]$ is closed and bounded in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$, by Thm. II.1.5.7 we know that $\big([a,b], d_{l^1}|_{\mathbb{R}\times\mathbb{R}}\big)$ is compact. Since $\big([a,b], d_{l^1}|_{\mathbb{R}\times\mathbb{R}}\big)$ is compact and f is continuous on $[a,b]$, by Prop. II.2.3.2 we know that f is bounded. Since f is bounded and continuous on $[a,b]$, by Thm. II.2.3.5 f is uniformly continuous on $[a,b]$.

Since f is continuous at a, we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in \mathbb{R}, |x-a| < \delta \implies |f(x) - f(a)| < \varepsilon\big)\\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in \mathbb{R}, x \in (a-\delta, a+\delta) \implies |f(x) - f(a)| < \varepsilon\big)\\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \big(\forall x \in \mathbb{R}, x \in (a-\delta, a) \implies |f(x) - f(a)| = |f(a)| < \varepsilon\big)\\
\implies &\forall \varepsilon \in \mathbb{R}^+, |f(a)| < \varepsilon\\
\implies &f(a) = 0.
\end{aligned}$$

Similarly, we have $f(b) = 0$. If $a = b$, then f is zero function, and we have

$$\forall \varepsilon \in \mathbb{R}^+, \forall \delta \in \mathbb{R}^+, \forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| = 0 < \varepsilon.$$

Thus, f is uniformly continuous on $\mathbb{R}$. Suppose that $a \neq b$. Since f in uniformly continuous on $[a,b]$, by Def. II.2.3.4 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta_1 \in \mathbb{R}^+ : \forall x_1, x_2 \in [a,b], |x_1 - x_2| < \delta_1 \implies |f(x_1) - f(x_2)| < \varepsilon.$$

Now fix one pair of ε and δ_1. Since $\lim\limits_{x\to a; x\in\mathbb{R}} f(x) = f(a) = 0$, we have

$$\exists \delta_2 \in \mathbb{R}^+ : \forall x \in \mathbb{R}, |x-a| < \delta_2 < b-a \implies |f(x) - f(a)| = |f(x)| < \varepsilon.$$

Similarly, we have

$$\exists \delta_3 \in \mathbb{R}^+ : \forall x \in \mathbb{R}, |x-b| < \delta_3 < b-a \implies |f(x) - f(b)| = |f(x)| < \varepsilon.$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. Then we have

$$\begin{aligned}
&\forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta\\
\implies &\begin{cases}
|x_1 - x_2| < \delta_1 \implies |f(x_1) - f(x_2)| < \varepsilon & \text{if } \big(x_1, x_2 \in [a,b]\big)\\
x_1 - x_2 < \delta_2 \implies a \le x_1 < x_2 + \delta_2 & \text{if } \big(x_1 \in [a,b]\big) \wedge \big(x_2 \in (-\infty, a)\big)\\
x_2 - x_1 < \delta_3 \implies x_2 - \delta_3 < x_1 \le b & \text{if } \big(x_1 \in [a,b]\big) \wedge \big(x_2 \in (b, \infty)\big)\\
|f(x_1) - f(x_2)| = 0 < \varepsilon & \text{if } \big(x_1, x_2 \notin [a,b]\big)
\end{cases}\\
\implies &\begin{cases}
|f(x_1) - f(x_2)| < \varepsilon & \text{if } \big(x_1, x_2 \in [a,b]\big)\\
x_1 - a < x_2 - a + \delta_2 < \delta_2 & \text{if } \big(x_1 \in [a,b]\big) \wedge \big(x_2 \in (-\infty, a)\big)\\
\delta_3 > b - x_2 + \delta_3 > b - x_1 & \text{if } \big(x_1 \in [a,b]\big) \wedge \big(x_2 \in (b, \infty)\big)\\
|f(x_1) - f(x_2)| < \varepsilon & \text{if } \big(x_1, x_2 \notin [a,b]\big)
\end{cases}
\end{aligned}$$

$$\implies \begin{cases} |f(x_1) - f(x_2)| < \varepsilon & \text{if } \big(x_1, x_2 \in [a,b]\big) \\ |x_1 - a| < \delta_2 \implies |f(x_1)| < \varepsilon & \text{if } \big(x_1 \in [a,b]\big) \wedge \big(x_2 \in (-\infty, a)\big) \\ |x_1 - b| < \delta_3 \implies |f(x_1)| < \varepsilon & \text{if } \big(x_1 \in [a,b]\big) \wedge \big(x_2 \in (b, \infty)\big) \\ |f(x_1) - f(x_2)| < \varepsilon & \text{if } \big(x_1, x_2 \notin [a,b]\big) \end{cases}$$
$$\implies |f(x_1) - f(x_2)| < \varepsilon.$$

Since ε was arbitrary, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x_1, x_2 \in \mathbb{R}, |x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$$

and f is uniformly continuous on $\mathbb{R}$. □

Ex. II.3.8.4. Prove Prop. II.3.8.11.

Proof. See Prop. II.3.8.11. □

Ex. II.3.8.5. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous, compactly supported functions. Suppose that f is supported on the interval $[0, 1]$, and g is constant on the interval $[0, 2]$ (i.e., there is a real number c such that $g(x) = c$ for all $x \in [0, 2]$). Show that the convolution $f * g$ is constant on the interval $[1, 2]$.

Proof. We have

$$\begin{aligned} \forall x \in [1,2], (f*g)(x) &= \int_{-\infty}^{\infty} f(y) g(x-y)\, dy && \text{(by Def. II.3.8.9)} \\ &= \int_{[0,1]} f(y) g(x-y)\, dy && \text{(by Def. II.3.8.4)} \\ &= \int_{[0,1]} c f(y)\, dy && (x - y \in [0,2]) \\ &= c \int_{[0,1]} f(y)\, dy. \end{aligned}$$

Since $\int_{[0,1]} f(y)\, dy$ is independent of x, we know that $f * g$ is constant on $[1, 2]$. □

Ex. II.3.8.6.

(a) Let g be an (ε, δ) approximation to the identity. Show that $1 - 2\varepsilon \le \int_{[-\delta,\delta]} g \le 1$.

(b) Prove Lem. II.3.8.14.

Proof. (a) By Def. II.3.8.6 we know that

- $\varepsilon \in \mathbb{R}^+$.

- $\delta \in \mathbb{R}^+$ such that $0 < \delta < 1$.
- g is supported on $[-1, 1]$ and $g(x) \geq 0$ for all $x \in [-1, 1]$.
- g is continuous on $\mathbb{R}$ and $\displaystyle\int_{-\infty}^{\infty} g = 1$.
- $|g(x)| \leq \varepsilon$ for each $\delta \leq |x| \leq 1$.

By Def. II.3.8.4 we have

$$\int_{-\infty}^{\infty} g = \int_{[-1,1]} g = 1.$$

Thus

$$\begin{aligned}
&\forall \delta \leq |x| \leq 1, |g(x)| \leq \varepsilon \\
\implies 1 &= \int_{[-1,1]} g \\
&= \int_{[-1,-\delta]} g + \int_{[-\delta,\delta]} g + \int_{[\delta,1]} g \\
&\leq (-\delta + 1)\varepsilon + \int_{[-\delta,\delta]} g + (1-\delta)\varepsilon \\
&= 2\varepsilon(1-\delta) + \int_{[-\delta,\delta]} g \\
&\leq 2\varepsilon + \int_{[-\delta,\delta]} g && (1 - \delta < 1) \\
&\leq 2\varepsilon + \int_{[-1,-\delta]} g + \int_{[-\delta,\delta]} g + \int_{[\delta,1]} g && (g(x) \geq 0 \text{ for all } x \in [-1,1]) \\
&= 2\varepsilon + \int_{[-1,1]} g \\
&= 2\varepsilon + 1 \\
\implies 1 - 2\varepsilon &\leq \int_{[-\delta,\delta]} g \leq 1.
\end{aligned}$$

□

Proof. (b) See Lem. II.3.8.14. □

Ex. II.3.8.7. Prove Cor. II.3.8.15.

Proof. See Cor. II.3.8.15. □

Ex. II.3.8.8. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function, and suppose that $\displaystyle\int_{[0,1]} f(x)x^n\, dx = 0$ for all non-negative integers $n = 0, 1, 2, \ldots$. Show that f must be the zero function $f \equiv 0$.

Proof. Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial with degree n. Then by Def. II.3.8.1 we have

$$\forall x \in \mathbb{R}, P(x) = \sum_{j=0}^{n} c_j x^j.$$

By hypothesis we have

$$\begin{aligned}\int_{[0,1]} f(x)P(x)\ dx &= \int_{[0,1]} f(x) \sum_{j=0}^{n} c_j x^j\ dx \\ &= \sum_{j=0}^{n} c_j \int_{[0,1]} f(x)x^j\ dx \\ &= 0.\end{aligned}$$

Since $[0,1]$ is closed and bounded in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$, by Thm. II.1.5.7 we know that $\big([0,1], d_{l^1}|_{\mathbb{R}\times\mathbb{R}}\big)$ is compact. By Prop. II.2.3.2 we know that f is bounded, i.e., there exists a $M \in \mathbb{R}^+$ such that $|f(x)| \le M$ for all $x \in [0,1]$.

Since f is continuous on $[0,1]$, by Thm. II.3.8.3 we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists P \in \mathbb{R}^{\mathbb{R}} : \begin{cases} P \text{ is a polynomial on } [0,1] \\ d_\infty(P,f) \le \dfrac{\varepsilon}{M} \end{cases}$$

Fix one pair of ε and P. Then we have

$$\begin{aligned}
& d_\infty(P,f) \le \frac{\varepsilon}{M} \\
\implies & \sup_{x\in[0,1]} |P(x) - f(x)| \le \frac{\varepsilon}{M} && \text{(by Def. II.3.4.2)} \\
\implies & \forall x \in [0,1], |P(x) - f(x)| \le \frac{\varepsilon}{M} \\
\implies & \forall x \in [0,1], |f(x)P(x) - f(x)f(x)| \le \frac{\varepsilon|f(x)|}{M} \le \frac{\varepsilon M}{M} \le \varepsilon \\
\implies & \forall x \in [0,1], f(x)P(x) - \varepsilon \le \big(f(x)\big)^2 \le f(x)P(x) + \varepsilon \\
\implies & \forall x \in [0,1], \int_{[0,1]} f(x)P(x) - \varepsilon\ dx = -\varepsilon \\
& \le \int_{[0,1]} \big(f(x)\big)^2\ dx \le \int_{[0,1]} f(x)P(x) + \varepsilon\ dx = \varepsilon && \text{(by hypothesis)} \\
\implies & \forall x \in [0,1], -\varepsilon \le \int_{[0,1]} \big(f(x)\big)^2\ dx \le \varepsilon.
\end{aligned}$$

Since ε was arbitrary, we know that

$$\forall \varepsilon \in \mathbb{R}^+, \left|\int_{[0,1]} \big(f(x)\big)^2\ dx\right| \le \varepsilon \implies \left|\int_{[0,1]} \big(f(x)\big)^2\ dx\right| = \int_{[0,1]} \big(f(x)\big)^2\ dx = 0.$$

Since f is continuous on $[0,1]$ and $\big(f(x)\big)^2 \geq 0$ for all $x \in [0,1]$, by Exercise 11.4.2 in Analysis I we know that

$$\forall x \in [0,1], \big(f(x)\big)^2 = 0.$$

Thus, we have $f(x) = 0$ for all $x \in [0,1]$. □

Ex. II.3.8.9. Prove Lem. II.3.8.16.

Proof. See Lem. II.3.8.16. □

Chapter II.4

Power series

II.4.1 Formal power series

Def. II.4.1.1 (Formal power series)**.** Let a be a real number. A *formal power series centered at a* is any series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

where $c_0, c_1, \ldots$ is a sequence of real numbers (not depending on x); we refer to c_n as the n^{th} *coefficient* of this series. Note that each term $c_n(x-a)^n$ in this series is a function of a real variable x.

Note. We call these power series *formal* because we do not yet assume that these series converge for any x. However, these series are automatically guaranteed to converge when $x = a$. In general, the closer x gets to a, the easier it is for this series to converge.

Def. II.4.1.3 (Radius of convergence)**.** Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a formal power series. We define the *radius of convergence* R of this series to be the quantity

$$R := \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}$$

where we adopt the convention that $\dfrac{1}{0} = +\infty$ and $\dfrac{1}{+\infty} = 0$.

Rmk. II.4.1.4. Each number $|c_n|^{1/n}$ is non-negative, so the limit $\limsup_{n\to\infty} |c_n|^{1/n}$ can take on any value from 0 to $+\infty$, inclusive. Thus, R can also take on any value between 0 and $+\infty$ inclusive (in particular, it is not necessarily a real number). Note that the radius of convergence always exists, even if the sequence $|c_n|^{1/n}$ is not convergent, because the lim sup of any sequence always exists (though it might be $+\infty$ or $-\infty$).

Thm. II.4.1.6. Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a formal power series, and let R be its radius of convergence.

(a) (Divergence outside of the radius of convergence) If $x \in \mathbb{R}$ is such that $|x-a| > R$, then the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is divergent for that value of x.

(b) (Convergence inside the radius of convergence) If $x \in \mathbb{R}$ is such that $|x-a| < R$, then the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is absolutely convergent for that value of x.

For parts (c)-(e) we assume that $R > 0$ (i.e., the series converges at at least one other point than $x = a$). Let $f : (a-R, a+R) \to \mathbb{R}$ be the function $f(x) \coloneqq \sum_{n=0}^{\infty} c_n(x-a)^n$; this function is guaranteed to exist by (b).

(c) (Uniform convergence on compact sets) For any $0 < r < R$, the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges uniformly to f on the compact interval $[a-r, a+r]$. In particular, f is continuous on $(a-R, a+R)$.

(d) (Differentiation of power series) The function f is differentiable on $(a-R, a+R)$, and for any $0 < r < R$, the series $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ converges uniformly to f' on the interval $[a-r, a+r]$.

(e) (Integration of power series) For any closed interval $[y, z]$ contained in $(a-R, a+R)$, we have

$$\int_{[y,z]} f = \sum_{n=0}^{\infty} c_n \frac{(z-a)^{n+1} - (y-a)^{n+1}}{n+1}.$$

Proof. (a)(b) We split into three cases:

- $R = +\infty$. Since $|x-a| \geq 0$, we cannot have $|x-a| > +\infty$. Thus, we only consider the case $|x-a| < +\infty$.

$$\begin{aligned}
& |x-a| < +\infty = \frac{1}{0} && \text{(by Def. II.4.1.3)} \\
\implies & \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = 0 && \text{(by Def. II.4.1.3)} \\
\implies & |x-a| \cdot \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = 0
\end{aligned}$$

$$\implies \limsup_{n\to\infty} |c_n(x-a)^n|^{\frac{1}{n}} = 0 < 1$$

$$\implies \sum_{n=0}^{\infty} c_n(x-a)^n \text{ is absolutely convergent.} \quad \text{(by Theorem 7.5.1 in Analysis I)}$$

- $R = 0$. Since $|x-a| \geq 0$, we cannot have $|x-a| < 0$. Thus, we only consider the case $|x-a| > 0$.

$$|x-a| > 0 = \frac{1}{+\infty} \qquad \text{(by Def. II.4.1.3)}$$

$$\implies \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = +\infty \qquad \text{(by Def. II.4.1.3)}$$

$$\implies \limsup_{n\to\infty} \left(|c_n|^{\frac{1}{n}}|x-a|\right) = +\infty \qquad \text{(proof by contradiction)}$$

$$\implies \limsup_{n\to\infty} |c_n(x-a)^n|^{\frac{1}{n}} = +\infty > 1$$

$$\implies \sum_{n=0}^{\infty} c_n(x-a)^n \text{ is divergent.} \qquad \text{(by Theorem 7.5.1 in Analysis I)}$$

- $R \in \mathbb{R}^+$. First, suppose that $|x-a| > R$. Then we have

$$|x-a| > \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}} \qquad \text{(by Def. II.4.1.3)}$$

$$\implies |x-a| \cdot \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} > 1$$

$$\implies \limsup_{n\to\infty} |c_n(x-a)^n|^{\frac{1}{n}} > 1$$

$$\implies \sum_{n=0}^{\infty} c_n(x-a)^n \text{ is divergent.} \qquad \text{(by Theorem 7.5.1 in Analysis I)}$$

Now suppose that $|x-a| < R$. Then we have

$$|x-a| < \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}} \qquad \text{(by Def. II.4.1.3)}$$

$$\implies |x-a| \cdot \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} < 1$$

$$\implies \limsup_{n\to\infty} |c_n(x-a)^n|^{\frac{1}{n}} < 1$$

$$\implies \sum_{n=0}^{\infty} c_n(x-a)^n \text{ is absolutely convergent.} \quad \text{(by Theorem 7.5.1 in Analysis I)}$$

From all cases above, we conclude that

$$\begin{cases} |x-a| < R \implies \displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n \text{ is absolutely convergent} \\ |x-a| > R \implies \displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n \text{ is divergent} \end{cases}$$

□

Proof. (c) Let $r \in (0, R)$. Since

$$\forall x \in [a-r, a+r] \implies |x-a| < r < R,$$

by Thm. II.4.1.6(b) we know that $\sum_{n=0}^{\infty} c_n(x-a)^n$ is absolutely convergent for all $x \in [a-r, a+r]$. For each $n \in \mathbb{N}$, we define $f_n : [a-r, a+r] \to \mathbb{R}$ by setting $f_n(x) = c_n(x-a)^n$ for all $x \in [a-r, a+r]$. Since

$$\begin{aligned} & r < R \\ \implies & \frac{r}{R} < 1 \\ \implies & r\Big(\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}\Big) = \limsup_{n\to\infty} |c_n r^n|^{\frac{1}{n}} < 1 && \text{(by Def. II.4.1.3)} \\ \implies & \sum_{n=0}^{\infty} c_n r^n \text{ is absolutely convergent} && \text{(by root test)} \end{aligned}$$

and

$$\begin{aligned} & \forall n \in \mathbb{N}, \forall x \in [a-r, a+r], (x-a)^n \leq r^n \\ \implies & \forall n \in \mathbb{N}, \forall x \in [a-r, a+r], c_n(x-a)^n \leq c_n r^n \\ \implies & \forall n \in \mathbb{N}, \|f_n\|_\infty \leq c_n r^n && \text{(by Def. II.3.5.5)} \\ \implies & \sum_{n=0}^{\infty} \|f_n\|_\infty \leq \sum_{n=0}^{\infty} c_n r^n, \end{aligned}$$

by Thm. II.3.5.7 we know that $\left(\sum_{n=0}^{N} f_n\right)_{N=0}^{\infty}$ converges uniformly to some function $g : [a-r, a+r] \to \mathbb{R}$ on $[a-r, a+r]$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$ and g is continuous on $[a-r, a+r]$. But by Def. II.3.5.2 we know that

$$\forall x \in [a-r, a+r], g(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = f(x).$$

Thus, f is continuous on $[a-r, a+r]$. Since r was arbitrary, we conclude that

$$\begin{aligned}&\forall r \in (0, R), \left(x \mapsto \sum_{n=0}^{N} c_n(x-a)^n\right)_{N=0}^{\infty} \text{ converges uniformly to}\\ &f = \left(x \mapsto \sum_{n=0}^{\infty} c_n(x-a)^n\right) \text{ on } [a-r, a+r] \text{ with respect to } d_{l^1}|_{\mathbb{R}\times\mathbb{R}}\\ &\text{and } f \text{ is continuous on } (a-R, a+R).\end{aligned}$$

□

Proof. (d) Let $r \in (0, R)$. For each $n \in \mathbb{Z}^+$, we define $f_n : [a-r, a+r] \to \mathbb{R}$ by setting $f_n(x) = c_n(x-a)^n$ for all $x \in [a-r, a+r]$. Since f_n is polynomial for all $n \in \mathbb{Z}^+$, we know that f_n' is well-defined and

$$\forall n \in \mathbb{Z}^+, \forall x \in [a-r, a+r], f_n'(x) = nc_n(x-a)^{n-1}.$$

Again, f_n' is polynomial and thus is continuous on $[a-r, a+r]$ for all $n \in \mathbb{Z}^+$. By limit laws we have

$$\begin{aligned}&\forall N \in \mathbb{Z}^+, \forall x \in [a-r, a+r], \left(\sum_{n=1}^{N} f_n\right)'(x) = \sum_{n=1}^{N} f_n'(x)\\ \implies &\forall N \in \mathbb{Z}^+, \left(\sum_{n=1}^{N} f_n\right)' = \sum_{n=1}^{N} f_n'.\end{aligned}$$

Since

$$\begin{aligned}&\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \frac{1}{R} && \text{(by Def. II.4.1.3)}\\ \implies &\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} \in \mathbb{R} && (R > 0)\\ \implies &\left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)\left(\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}\right) = \frac{1}{R} && \text{(by Proposition 7.5.4 in Analysis I)}\end{aligned}$$

$$\Longrightarrow \left(\limsup_{n\to\infty} n^{\frac{1}{n}}\right)\left(\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}\right) = \frac{1}{R} \qquad \text{(by Proposition 6.4.12(f) in Analysis I)}$$

$$\Longrightarrow \limsup_{n\to\infty} |nc_n|^{\frac{1}{n}} = \frac{1}{R}$$

$$\Longrightarrow \limsup_{n\to\infty} |nc_n(x-a)^n|^{\frac{1}{n}} = \frac{|x-a|}{R} < 1, \qquad \text{(if } |x-a| < r\text{)}$$

by root test we know that $\sum_{n=1}^{\infty} nc_n(x-a)^n$ is absolutely convergent. Since

$$\sum_{n=1}^{\infty} nc_n(x-a)^n = (x-a)\left(\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}\right),$$

we know that $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ is convergent. For each $n \in \mathbb{Z}^+$, we define $c'_{n-1} = nc_n$. Then by Proposition 7.2.14 in Analysis I we have

$$\forall x \in [a-r, a+r], \sum_{n=1}^{\infty} f'_n(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = \sum_{n=1}^{\infty} c'_{n-1}(x-a)^{n-1} = \sum_{n=0}^{\infty} c'_n(x-a)^n.$$

For each $n \in \mathbb{N}$, we define $g_n : [a-r, a+r] \to \mathbb{R}$ by setting $g_n(x) = c'_n(x-a)^n$ for all $x \in [a-r, a+r]$. By Thm. II.4.1.6(c) we know that

$$\left(\sum_{n=0}^{N} g_n\right)_{N=0}^{\infty} \text{ converges uniformly to some } g : [a-r, a+r] \to \mathbb{R}$$
$$\text{on } [a-r, a+r] \text{ with respect to } d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$$
$$\Longrightarrow \left(\left(\sum_{n=1}^{N} f_n\right)'\right)_{N=1}^{\infty} \text{ converges uniformly to some } g : [a-r, a+r] \to \mathbb{R}$$
$$\text{on } [a-r, a+r] \text{ with respect to } d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$$

By Def. II.3.5.2 we know that

$$\forall x \in [a-r, a+r], g(x) = \sum_{n=0}^{\infty} c'_n(x-a)^n = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = \sum_{n=1}^{\infty} f'_n(x).$$

Since

$$\lim_{N\to\infty} \sum_{n=1}^{N} f_n(a) = \lim_{N\to\infty} \sum_{n=1}^{N} c_n(a-a)^n = \lim_{N\to\infty} 0 = 0,$$

by Thm. II.3.7.1 we have

$$\left(\sum_{n=1}^{N} f_n\right)_{N=1}^{\infty} \text{ converges uniformly to } f$$
$$\text{on } [a-r, a+r] \text{ with respect to } d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$$

and $f' = g$. By Def. II.3.5.2 this means

$$\forall x \in [a-r, a+r], f'(x) = \left(\sum_{n=1}^{\infty} f_n\right)'(x) = \sum_{n=1}^{\infty} nc_n|x-a|^{n-1}.$$

Since r was arbitrary, we conclude that f is differentiable on $(a-R, a+R)$ and

$$\forall r \in (0, R), \left(x \mapsto \sum_{n=1}^{N} nc_n(x-a)^{n-1}\right)_{N=1}^{\infty} \text{ converges uniformly to}$$
$$f' = \left(x \mapsto \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}\right) \text{ on } [a-r, a+r] \text{ with respect to } d_{l^1}|_{\mathbb{R}\times\mathbb{R}}.$$

□

Proof. (e) If $y = z = a$, then we have

$$\int_{[a,a]} f = 0 = \sum_{n=0}^{\infty} c_n \frac{(a-a)^{n+1} - (a-a)^{n+1}}{n+1}.$$

So suppose that $(y \neq a) \vee (z \neq a)$. Without the loss of generality, suppose that $y \neq a$. Let $r = \max(|y-a|, |z-a|)$. Since $[y, z] \subseteq (a-R, a+R)$, we have

$$\begin{aligned}
& a - R < y \leq z < a + R \\
\implies & -R < y - a \leq z - a < R \\
\implies & \begin{cases} 0 < |y-a| < R \\ 0 \leq |z-a| < R \end{cases} \\
\implies & \begin{cases} 0 < |y-a| \leq r < R \\ 0 \leq |z-a| \leq r < R \end{cases} \\
\implies & -R < -r \leq y - a < z - a \leq r < R \\
\implies & [y, z] \subseteq [a-r, a+r].
\end{aligned}$$

By Thm. II.4.1.6(c) we know that

$$\left(x \mapsto \sum_{n=0}^{N} c_n(x-a)^n\right)_{N=0}^{\infty} \text{ converges uniformly to}$$

$$f = \left(x \mapsto \sum_{n=0}^{\infty} c_n (x-a)^n\right) \text{ on } [a-r, a+r] \text{ with respect to } d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$$

and f is continuous on $[a-r, a+r]$.

Thus, we have

$$\left(x \mapsto \sum_{n=0}^{N} c_n (x-a)^n\right)_{N=0}^{\infty} \text{ converges uniformly to}$$
$$f = \left(x \mapsto \sum_{n=0}^{\infty} c_n (x-a)^n\right) \text{ on } [y, z] \text{ with respect to } d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$$

and f is continuous on $[y, z]$.

By Corollary 11.5.2 we know that $\displaystyle\int_{[y,z]} f$ is well-defined. Thus, we have

$$\begin{aligned}
\int_{[y,z]} f &= \int_{[y,z]} f(x)\, dx \\
&= \int_{[y,z]} \left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) dx \\
&= \sum_{n=0}^{\infty} \left(\int_{[y,z]} c_n (x-a)^n\, dx\right) && \text{(by Cor. II.3.6.2)} \\
&= \sum_{n=0}^{\infty} c_n \frac{(z-a)^{n+1} - (y-a)^{n+1}}{n+1}.
\end{aligned}$$

□

Note. Thm. II.4.1.6 (a) and (b) of the above theorem give another way to find the radius of convergence, by using your favorite convergence test to work out the range of x for which the power series converges.

Rmk. II.4.1.8. Thm. II.4.1.6 is silent on what happens when $|x-a| = R$, i.e., at the points $a-R$ and $a+R$. Indeed, one can have either convergence or divergence at those points.

Rmk. II.4.1.9. Note that while Thm. II.4.1.6 assures us that the power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$ will converge pointwise on the interval $(a-R, a+R)$, it need not converge uniformly on that interval (see Ex. II.4.1.2(e)). On the other hand, Thm. II.4.1.6(c) assures us that the power series will converge uniformly on any smaller interval $[a-r, a+r]$. In particular, being uniformly convergent on every closed subinterval of $(a-R, a+R)$ is not enough to guarantee being uniformly convergent on all of $(a-R, a+R)$.

— Exercises —

Ex. II.4.1.1. Prove Thm. II.4.1.6.

Proof. See Thm. II.4.1.6. $\square$

Ex. II.4.1.2. Give examples of a formal power series $\sum_{n=0}^{\infty} c_n x^n$ centered at 0 with radius of convergence 1, which

(a) diverges at both $x = 1$ and $x = -1$;

(b) diverges at $x = 1$ but converges at $x = -1$;

(c) converges at $x = 1$ but diverges at $x = -1$;

(d) converges at both $x = 1$ and $x = -1$.

(e) converges pointwise on $(-1, 1)$, but does not converge uniformly on $(-1, 1)$.

Proof. (a) Let $c_n = 1$ for all $n \in \mathbb{N}$. Then we have

$$\frac{1}{\limsup_{n\to\infty} |c_n|} = \frac{1}{\limsup_{n\to\infty} 1} = \frac{1}{1} = 1.$$

But we know that $\sum_{n=0}^{\infty} 1^n$ and $\sum_{n=0}^{\infty} (-1)^n$ are divergent. $\square$

Proof. (b) Let $c_n = \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Then we have

$$\frac{1}{\limsup_{n\to\infty} |c_n|} = \frac{1}{\limsup_{n\to\infty} \frac{1}{n+1}} = \frac{1}{1} = 1.$$

By Corollary 7.3.7 in Analysis I we know that $\sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges. By Corollary 7.2.12 in Analysis I we know that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ converges. $\square$

Proof. (c) Let $c_n = \frac{(-1)^n}{n+1}$ for all $n \in \mathbb{N}$. Then we have

$$\frac{1}{\limsup_{n\to\infty} |c_n|} = \frac{1}{\limsup_{n\to\infty} \frac{1}{n+1}} = \frac{1}{1} = 1.$$

By Corollary 7.2.12 in Analysis I we know that $\sum_{n=0}^{\infty} \frac{(-1)^n 1^n}{n+1}$ converges. By Corollary 7.3.7 in Analysis I we know that $\sum_{n=0}^{\infty} \frac{(-1)^{2n}}{n+1}$ diverges. $\square$

Proof. (d) Let $c_n = \frac{1}{n^2 - 1/2}$ for all $n \in \mathbb{N}$. Then we have

$$\frac{1}{\limsup_{n\to\infty}|c_n|} = \frac{1}{\limsup_{n\to\infty} \frac{1}{n^2-1/2}} = \frac{1}{1} = 1.$$

By Corollary 7.3.7 in Analysis I we know that $\sum_{n=0}^{\infty} \frac{1}{n^2 - 1/2}$ converges. By Corollary 7.2.12 in Analysis I we know that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 - 1/2}$ converges. $\square$

Proof. (e) Let $c_n = 1$ for all $n \in \mathbb{N}$. Then we have

$$\frac{1}{\limsup_{n\to\infty}|c_n|} = \frac{1}{\limsup_{n\to\infty} 1} = \frac{1}{1} = 1.$$

By Lemma 7.3.3 in Analysis I we know that $\sum_{n=0}^{\infty} x^n$ converges for all $x \in (-1, 1)$. But by E.g. II.3.5.8 we know that $\sum_{n=0}^{\infty} x^n$ does not converge uniformly on $(-1, 1)$. $\square$

II.4.2 Real analytic functions

Def. II.4.2.1 (Real analytic functions)**.** Let E be a subset of $\mathbb{R}$, and let $f : E \to \mathbb{R}$ be a function. If a is an interior point of E, we say that f is *real analytic at* a if there exists an open interval $(a - r, a + r)$ in E for some $r > 0$ such that there exists a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a which has a radius of convergence greater than or equal to r, and which converges to f on $(a - r, a + r)$. If E is an open set, and f is real analytic at every point a of E, we say that f is *real analytic on* E.

E.g. II.4.2.2. Consider the function $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}$ defined by $f(x) := \frac{1}{1-x}$. This function is real analytic at 0 because we have a power series $\sum_{n=0}^{\infty} x^n$ centred at 0 which converges to

$\frac{1}{1-x} = f(x)$ on the interval $(-1,1)$. This function is also real analytic at 2 because we have a power series $\sum_{n=0}^{\infty}(-1)^{n+1}(x-2)^n$ which converges to $\frac{-1}{1-\big(-(x-2)\big)} = \frac{1}{1-x} = f(x)$ on the interval $(1,3)$ (why? use Lemma 7.3.3 in Analysis I). In fact this function is real analytic on all of $\mathbb{R}\setminus\{1\}$; see Ex. II.4.2.2.

Rmk. II.4.2.3. The notion of being real analytic is closely related to another notion, that of being *complex analytic*, but this is a topic for complex analysis, and will not be discussed here.

Note. From Thm. II.4.1.6(c) and (d) we see that if f is real analytic at a point a, then f is both continuous and differentiable on $(a-r, a+r)$ for some $r>0$.

Def. II.4.2.4 (k-times differentiability)**.** Let E be a subset of $\mathbb{R}$ with the property that every element of E is a limit point of E. We say a function $f : E \to \mathbb{R}$ is *once differentiable on* E iff it is differentiable, in particular, $f' : E \to \mathbb{R}$ is also a function on E. More generally, for any $k \geq 2$ we say that $f : E \to \mathbb{R}$ is k *times differentiable on* E, or just k *times differentiable*, iff f is differentiable, and f' is $k-1$ times differentiable. If f is k times differentiable, we define the k^{th} derivative $f^{(k)} : E \to \mathbb{R}$ by the recursive rule $f^{(1)} := f'$, and $f^{(k)} = (f^{(k-1)})'$ for all $k \geq 2$. We also define $f^{(0)} := f$ (this is f differentiated 0 times), and we allow every function to be zero times differentiable (since clearly $f^{(0)}$ exists for every f). A function is said to be *infinitely differentiable* (or *smooth*) iff it is k times differentiable for every $k \geq 0$.

A.Cor. II.4.2.1. For each $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty}\left(\frac{(n+k)!}{n!}\right)^{\frac{1}{n}} = 1.$$

Proof. We induct on k. For $k=0$, we have

$$\lim_{n\to\infty}\left(\frac{(n+0)!}{n!}\right)^{\frac{1}{n}} = \lim_{n\to\infty} 1^{\frac{1}{n}} = \lim_{n\to\infty} 1 = 1.$$

Thus, the base case holds. Suppose inductively that

$$\lim_{n\to\infty}\left(\frac{(n+k)!}{n!}\right)^{\frac{1}{n}} = 1$$

for some $k \geq 0$. We want to show that $k+1$ is also true. Observe that

$$\begin{aligned} &\exists N \in \mathbb{Z}^+ : \forall n \geq N, Nn > k+1 \\ \implies &\exists N \in \mathbb{Z}^+ : \forall n \geq N, (N+1)n > n+k+1 \end{aligned}$$

$$
\begin{aligned}
&\Longrightarrow \exists N \in \mathbb{Z}^+ : \forall n \geq N, Nn > n+k+1 > n \\
&\Longrightarrow \exists N \in \mathbb{Z}^+ : \forall n \geq N, (Nn)^{\frac{1}{n}} > (n+k+1)^{\frac{1}{n}} > n^{\frac{1}{n}}.
\end{aligned}
$$

Now we fix such N. Since

$$
\begin{aligned}
&\begin{cases} \lim\limits_{n\to\infty} N^{\frac{1}{n}} = 1 \\ \lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1 \end{cases} \\
\Longrightarrow\ & \lim_{n\to\infty} (Nn)^{\frac{1}{n}} = 1 \\
\Longrightarrow\ & \lim_{n\to\infty} (n+k+1)^{\frac{1}{n}} = 1, && \text{(by squeeze test)}
\end{aligned}
$$

we have

$$
\begin{aligned}
& \lim_{n\to\infty} \left(\frac{(n+k)!}{n!}\right)^{\frac{1}{n}} = 1 && \text{(by the induction hypothesis)} \\
\Longrightarrow\ & \left(\lim_{n\to\infty} \left(\frac{(n+k)!}{n!}\right)^{\frac{1}{n}}\right)\left(\lim_{n\to\infty} (n+k+1)^{\frac{1}{n}}\right) = 1 \\
\Longrightarrow\ & \lim_{n\to\infty} \left(\left(\frac{(n+k)!}{n!}\right)^{\frac{1}{n}} (n+k+1)^{\frac{1}{n}}\right) = 1 \\
\Longrightarrow\ & \lim_{n\to\infty} \left(\frac{(n+k+1)!}{n!}\right)^{\frac{1}{n}} = 1.
\end{aligned}
$$

This closes the induction. □

Prop. II.4.2.6 (Real analytic functions are k-times differentiable)**.** Let E be a subset of $\mathbb{R}$, let a be an interior point of E, and and let f be a function which is real analytic at a, thus there is an $r > 0$ for which we have the power series expansion

$$
f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n
$$

for all $x \in (a-r, a+r)$. Then for every $k \geq 0$, the function f is k-times differentiable on $(a-r, a+r)$, and for each $k \geq 0$ the k^{th} derivative is given by

$$
f^{(k)}(x) = \sum_{n=0}^{\infty} c_{n+k}(n+1)(n+2)\ldots(n+k)(x-a)^n
$$

$$= \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!}(x-a)^n$$

for all $x \in (a-r, a+r)$.

Proof. Let R be the radius of convergence of f. By Def. II.4.2.1 we know that $r \leq R$ and thus

$$\forall x \in (a-r, a+r), |x-a| < r \leq R.$$

We induct on k. For $k = 0$, by Def. II.4.2.4 we have

$$\forall x \in (a-r, a+r), f^{(0)}(x) = f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_{n+0} \frac{(n+0)!}{n!}(x-a)^n.$$

Thus, the base case holds. Suppose inductively that

$$\forall x \in (a-r, a+r), f^{(k)}(x) = \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!}(x-a)^n$$

for some $k \geq 0$. By A.Cor. II.4.2.1 we know that

$$\limsup_{n\to\infty} |c_{n+k}|^{\frac{1}{n}} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \frac{1}{R}$$

$$\implies \left(\limsup_{n\to\infty} |c_{n+k}|^{\frac{1}{n}} \right) \left(\limsup_{n\to\infty} \left(\frac{(n+k)!}{n!} \right)^{\frac{1}{n}} \right) = \frac{1}{R}$$

$$\implies \limsup_{n\to\infty} \left| c_{n+k} \frac{(n+k)!}{n!} \right|^{\frac{1}{n}} = \frac{1}{R}.$$

Thus, by Thm. II.4.1.6(b) we know that $f^{(k)}(x)$ converges for all $x \in (a-r, a+r)$. Now we define

$$\forall n \in \mathbb{N}, b_n = c_{n+k} \frac{(n+k)!}{n!}.$$

Then by Thm. II.4.1.6(d) we have

$$\forall x \in (a-r, a+r), f^{(k)}(x) = \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!}(x-a)^n = \sum_{n=0}^{\infty} b_n (x-a)^n$$

$$\implies \forall x \in (a-r, a+r), f^{(k+1)}(x) = (f^{(k)})'(x) = \sum_{n=1}^{\infty} n b_n (x-a)^{n-1}$$

$$\implies \forall x \in (a-r, a+r), f^{(k+1)}(x) = \sum_{n=1}^{\infty} n c_{n+k} \frac{(n+k)!}{n!}(x-a)^{n-1}$$

$$\implies \forall x \in (a-r, a+r), f^{(k+1)}(x) = \sum_{n=1}^{\infty} c_{n+k} \frac{(n+k)!}{(n-1)!} (x-a)^{n-1}$$
$$\implies \forall x \in (a-r, a+r), f^{(k+1)}(x) = \sum_{n=0}^{\infty} c_{(n+1)+k} \frac{\big((n+1)+k\big)!}{\big((n+1)-1\big)!} (x-a)^{(n+1)-1}$$
$$\implies \forall x \in (a-r, a+r), f^{(k+1)}(x) = \sum_{n=0}^{\infty} c_{n+k+1} \frac{(n+k+1)!}{n!} (x-a)^{n}$$

and this closes the induction. □

Cor. II.4.2.7 (Real analytic functions are infinitely differentiable)**.** Let E be an open subset of $\mathbb{R}$, and let $f : E \to \mathbb{R}$ be a real analytic function on E. Then f is infinitely differentiable on E. Also, all derivatives of f are also real analytic on E.

Proof. For every point $a \in E$ and $k \geq 0$, we know from Prop. II.4.2.6 that f is k-times differentiable at a (we will have to apply Exercise 10.1.1 in Analysis I k times here). Thus, f is k-times differentiable on E for every $k \geq 0$ and is hence infinitely differentiable. Also, from Prop. II.4.2.6 we see that each derivative $f^{(k)}$ of f has a convergent power series expansion at every $x \in E$ and thus $f^{(k)}$ is real analytic. □

Rmk. II.4.2.9. The converse statement to Cor. II.4.2.7 is not true; there are infinitely differentiable functions which are not real analytic.

Note. Prop. II.4.2.6 has an important corollary (Cor. II.4.2.10), due to Brook Taylor (1685–1731).

Cor. II.4.2.10 (Taylor's formula)**.** Let E be a subset of $\mathbb{R}$, let a be an interior point of E, and let $f : E \to \mathbb{R}$ be a function which is real analytic at a and has the power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for all $x \in (a-r, a+r)$ and some $r > 0$. Then for any integer $k \geq 0$, we have

$$f^{(k)}(a) = k! c_k,$$

where $k! := 1 \times 2 \times \cdots \times k$ (and we adopt the convention that $0! = 1$). In particular, we have Taylor's formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

for all x in $(a-r, a+r)$.

Proof. We have

$$\begin{aligned}\forall k \in \mathbb{N}, f^{(k)}(a) &= \sum_{n=0}^{\infty} c_{n+k}\frac{(n+k)!}{n!}(a-a)^n && \text{(by Prop. II.4.2.6)}\\ &= c_k k! && (0^0 = 1)\end{aligned}$$

and thus

$$\begin{aligned}\forall x \in (a-r, a+r), f(x) &= \sum_{n=0}^{\infty} c_n(x-a)^n \\ &= \sum_{n=0}^{\infty} \frac{c_n n!}{n!}(x-a)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. && \text{(from the proof above)}\end{aligned}$$

□

Note. The power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is sometimes called the *Taylor series* of f around a. Taylor's formula thus asserts that if a function is real analytic, then it is equal to its Taylor series.

Rmk. II.4.2.11. Note that Taylor's formula only works for functions which are real analytic; there are examples of functions which are infinitely differentiable but for which Taylor's theorem fails.

Cor. II.4.2.12 (Uniqueness of power series)**.** Let E be a subset of $\mathbb{R}$, let a be an interior point of E, and let $f : E \to \mathbb{R}$ be a function which is real analytic at a. Suppose that f has two power series expansions

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

and

$$f(x) = \sum_{n=0}^{\infty} d_n(x-a)^n$$

centered at a, each with a non-zero radius of convergence. Then $c_n = d_n$ for all $n \geq 0$.

Proof. By Cor. II.4.2.10, we have $f^{(k)}(a) = k!c_k$ for all $k \geq 0$. But we also have $f^{(k)}(a) = k!d_k$, by similar reasoning. Since $k!$ is never zero, we can cancel it and obtain $c_k = d_k$ for all $k \geq 0$, as desired. □

Rmk. II.4.2.13. While a real analytic function has a unique power series around any given point, it can certainly have different power series at different points. For instance, the function $f(x) := \dfrac{1}{1-x}$, defined on $\mathbb{R} \setminus \{1\}$, has the power series

$$f(x) := \sum_{n=0}^{\infty} x^n$$

around 0, on the interval $(-1, 1)$, but also has the power series

$$\begin{aligned} f(x) &= \frac{1}{1-x} \\ &= \frac{2}{1 - 2(x - \frac{1}{2})} \\ &= \sum_{n=0}^{\infty} 2\left(2\left(x - \frac{1}{2}\right)\right)^n \\ &= \sum_{n=0}^{\infty} 2^{n+1}\left(x - \frac{1}{2}\right)^n \end{aligned}$$

around $1/2$, on the interval $(0, 1)$ (note that the above power series has a radius of convergence of $1/2$, thanks to the root test).

— Exercises —

Ex. II.4.2.1. Let $n \geq 0$ be an integer, let c, a be real numbers, and let f be the function $f(x) := c(x-a)^n$. Show that f is infinitely differentiable, and that $f^{(k)}(x) = c\dfrac{n!}{(n-k)!}(x - a)^{n-k}$ for all integers $0 \leq k \leq n$. What happens when $k > n$?

Proof. For each $n \in \mathbb{N}$, let $P(n)$ be the statement "If $f(x) = c(x-a)^n$, then $f^{(k)}(x) = c\dfrac{n!}{(n-k)!}(x-a)^{n-k}$ for all $0 \leq k \leq n$." We induct on n to show that $P(n)$ is true for all $n \in \mathbb{N}$. For $n = 0$, we have $0 \leq k \leq 0 \implies k = 0$. By Def. II.4.2.4 we have

$$f^{(0)} = f = c(x-a)^0 = c = c\frac{0!}{(0-0)!}(x-a)^{0-0}$$

and Thus, the base case holds. Suppose inductive that $P(n)$ is true for some $n \geq 0$. Then we want to show that $P(n+1)$ is true. Let $f(x) = c(x-a)^{n+1}$. Then we have

$$f'(x) = c(n+1)(x-a)^n.$$

By the induction hypothesis we know that

$$\frac{f'(x)}{n+1} = c(x-a)^n$$

$$\begin{aligned}
&\implies \forall 0 \le k \le n, \left(\frac{f'(x)}{n+1}\right)^{(k)} = c\frac{n!}{(n-k)!}(x-a)^{n-k} \\
&\implies \forall 0 \le k \le n, \big(f'(x)\big)^{(k)} = c\frac{(n+1)!}{(n-k)!}(x-a)^{n-k} \\
&\implies \forall 0 \le k \le n, f(x)^{(k+1)} = c\frac{(n+1)!}{(n-k)!}(x-a)^{n-k} && \text{(by Def. II.4.2.4)} \\
&\implies \forall 1 \le k \le n+1, \\
&\qquad f(x)^{\big((k-1)+1\big)} = c\frac{(n+1)!}{\big(n-(k-1)\big)!}(x-a)^{n-(k-1)} \\
&\implies \forall 1 \le k \le n+1, f(x)^{(k)} = c\frac{(n+1)!}{(n+1-k)!}(x-a)^{n+1-k}
\end{aligned}$$

and we know that

$$f(x)^{(0)} = f(x) = c(x-a)^{n+1} = c\frac{(n+1)!}{(n+1-0)!}(x-a)^{n+1-0}.$$

Thus, we have

$$\forall 0 \le k \le n+1, f(x)^{(k)} = c\frac{(n+1)!}{(n+1-k)!}(x-a)^{n+1-k}$$

and this closes the induction.

Now let $n \in \mathbb{N}$ and let $f(x) = c(x-a)^n$. From the proof above we know that

$$f^{(n)}(x) = c\frac{n!}{(n-n)!}(x-a)^{n-n} = cn!$$

is a constant function. Thus, we have

$$\forall k > n, f^{(k)}(x) = 0$$

and by Def. II.4.2.4 f is infinitely differentiable. □

Ex. II.4.2.2. Show that the function f defined in E.g. II.4.2.2 is real analytic on all of $\mathbb{R} \setminus \{1\}$.

Proof. Let $a \in \mathbb{R} \setminus \{1\}$, let $r = |1-a|$, let $x \in (a-r, a+r)$ and let $c_n = (\dfrac{1}{1-a})^{n+1}$ for all $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
&a - r < x < a + r \\
\implies &|x-a| < r \\
\implies &\left|\frac{x-a}{1-a}\right| < 1
\end{aligned}$$

and by Lemma 7.3.3 in Analysis I we know that

$$\begin{aligned}\sum_{n=0}^{\infty} c_n(x-a)^n &= \sum_{n=0}^{\infty} \left(\frac{1}{1-a}\right)^{n+1}(x-a)^n\\ &= \frac{1}{1-a}\sum_{n=0}^{\infty}\left(\frac{x-a}{1-a}\right)^n\\ &= \frac{1}{1-a}\frac{1}{1-\dfrac{x-a}{1-a}}\\ &= \frac{1}{1-x}.\end{aligned}$$

Since x was arbitrary, we know that $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges to f on $(a-r, a+r)$. By Def. II.4.2.1 we know that f is real analytic at a. Since a was arbitrary, by Def. II.4.2.1 we know that f is real analysis at a for each $a \in \mathbb{R} \setminus \{1\}$. □

Ex. II.4.2.3. Prove Prop. II.4.2.6.

Proof. See Prop. II.4.2.6. □

Ex. II.4.2.4. Use Prop. II.4.2.6 and Ex. II.4.2.1 to prove Cor. II.4.2.10.

Proof. See Cor. II.4.2.10. □

Ex. II.4.2.5. Let a, b be real numbers, and let $n \geq 0$ be an integer. Prove the identity

$$(x-a)^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(b-a)^{n-m}(x-b)^m$$

or any real number x. Explain why this identity is consistent with Taylor's theorem and Ex. II.4.2.1. (Note however that Taylor's theorem cannot be rigorously applied until one verifies Ex. II.4.2.6 below.)

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be the function by setting $f(x) = (x-a)^n$ for all $x \in \mathbb{R}$. By Exercise 7.1.4 in Analysis I we have

$$\begin{aligned}\forall x \in \mathbb{R}, f(x) &= (x-a)^n\\ &= (x-b+b-a)^n\\ &= \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(x-b)^m(b-a)^{n-m}.\end{aligned}$$

If we define

$$\forall m \in \mathbb{N}, c_m = \begin{cases} \dfrac{n!(b-a)^{n-m}}{m!(n-m)!} & \text{if } m \leq n \\ 0 & \text{if } m > n \end{cases}$$

then we have

$$\forall x \in \mathbb{R}, f(x) = \sum_{m=0}^{\infty} c_m (x-b)^m = \sum_{m=0}^{n} c_m (x-b)^m.$$

Thus, for arbitrary $r \in \mathbb{R}^+$, $\sum_{m=0}^{\infty} c_m (x-b)^m$ converges to $f(x)$ for all $x \in (b-r, b+r)$. By Def. II.4.2.1 we know that f is real analytic at b. By Cor. II.4.2.10 we have

$$\forall x \in \mathbb{R}, f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(b)}{m!}(x-b)^m.$$

By Ex. II.4.2.1 we have

$$\forall m \in \mathbb{N}, \forall x \in \mathbb{R}, f^{(m)}(x) = \begin{cases} \dfrac{n!}{(n-m)!}(x-a)^{n-m} & \text{if } 0 \leq m \leq n \\ 0 & \text{if } m > n \end{cases}$$

Thus

$$\begin{aligned} \forall x \in \mathbb{R}, f(x) &= \sum_{m=0}^{\infty} \frac{f^{(m)}(b)}{m!}(x-b)^m \\ &= \sum_{m=0}^{n} \frac{f^{(m)}(b)}{m!}(x-b)^m \\ &= \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(x-b)^m (b-a)^{n-m}. \end{aligned}$$

□

Ex. II.4.2.6. Using Ex. II.4.2.5, show that every polynomial $P(x)$ of one variable is real analytic on $\mathbb{R}$.

Proof. Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial with degree n. First, we show that P is real analytic at 0. By Def. II.3.8.1 we know that

$$\forall x \in \mathbb{R}, P(x) = \sum_{i=0}^{n} c_i x^i$$

where $c_0, \ldots, c_n \in \mathbb{R}$ and $c_n \neq 0$. If we define $c_i = 0$ for all $i > n$, then we have

$$\forall x \in \mathbb{R}, \sum_{i=0}^{\infty} c_i(x-0)^i = \sum_{i=0}^{\infty} c_i x^i = \sum_{i=0}^{n} c_i x^i = P(x).$$

Thus, for arbitrary $r \in \mathbb{R}^+$, $\sum_{i=0}^{\infty} c_i(x-0)^i$ converges to $P(x)$ for all $x \in (-r, r)$. By Def. II.4.2.1 P is real analytic at 0.

Now we show that P is real analytic at $a \in \mathbb{R} \setminus \{0\}$. Since $a \neq 0$, we have

$$\begin{aligned}
\forall x \in \mathbb{R}, P(x) &= \sum_{i=0}^{n} c_i x^i \\
&= \sum_{i=0}^{n} c_i \left(\sum_{m=0}^{i} \frac{i!}{m!(i-m)!} a^{i-m}(x-a)^m \right) && \text{(by Ex. II.4.2.5)} \\
&= \sum_{i=0}^{n} c_i \left(\sum_{m=0}^{i} \frac{i!}{m!(i-m)!} a^{i-m}(x-a)^{m-i} \right)(x-a)^i
\end{aligned}$$

If we define

$$\forall i \in \mathbb{N}, d_i = \begin{cases} c_i \left(\sum_{m=0}^{i} \frac{i!}{m!(i-m)!} a^{i-m}(x-a)^{m-i} \right) & \text{if } 0 \leq i \leq n \\ 0 & \text{if } i > n \end{cases}$$

Then we have

$$\forall x \in \mathbb{R}, \sum_{i=0}^{\infty} d_i(x-a)^i = \sum_{i=0}^{n} d_i(x-a)^i = P(x).$$

Thus, for arbitrary $r \in \mathbb{R}^+$, $\sum_{i=0}^{\infty} d_i(x-a)^i$ converges to $P(x)$ for all $x \in (-r, r)$. By Def. II.4.2.1 P is real analytic at a. Combine the proof above we conclude that P is real analytic on $\mathbb{R}$. Since P was arbitrary, we conclude that polynomials of one variable are real analytic on $\mathbb{R}$. □

Ex. II.4.2.7. Let $m \geq 0$ be a positive integer, and let $0 < x < r$ be real numbers. Use Lemma 7.3.3 in Analysis I to establish the identity

$$\frac{r}{r-x} = \sum_{n=0}^{\infty} x^n r^{-n}$$

for all $x \in (-r, r)$. Using Prop. II.4.2.6, conclude the identity

$$\frac{r}{(r-x)^{m+1}} = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^{n-m} r^{-n}$$

for all integers $m \geq 0$ and $x \in (-r, r)$. Also explain why the series on the right-hand side is absolutely convergent.

Proof. By Lemma 7.3.3 in Analysis I we have

$$\begin{aligned}
&0 < x < r \\
\implies &\frac{x}{r} < 1 \\
\implies &\sum_{n=0}^{\infty} \left(\frac{x}{r}\right)^n = \sum_{n=0}^{\infty} x^n r^{-n} = \frac{1}{1 - \frac{x}{r}} = \frac{r}{r-x}.
\end{aligned}$$

Since

$$x = 0 \implies \sum_{n=0}^{\infty} 0^n r^{-n} = 0^0 r^0 + \sum_{n=1}^{\infty} 0^n r^{-n} = 1 = \frac{r}{r-0},$$

we know that

$$\forall x \in (-r, r), \frac{r}{r-x} = \sum_{n=0}^{\infty} x^n r^{-n}$$

and by Def. II.4.2.1 $x \mapsto \dfrac{r}{r-x}$ is real analytic at 0.

Next we induct on m to show that

$$\forall m \in \mathbb{N}, \forall x \in (-r, r), \left(y \mapsto \frac{r}{r-y}\right)^{(m)}(x) = \frac{m!r}{(r-x)^{m+1}}.$$

For $m = 0$, we have

$$\forall x \in (-r, r), \left(y \mapsto \frac{r}{r-y}\right)^{(0)}(x) = \frac{r}{r-x} = \frac{0!r}{(r-x)^{0+1}}$$

and Thus, the base case holds. Suppose inductively that

$$\forall x \in (-r, r), \left(y \mapsto \frac{r}{r-y}\right)^{(m)}(x) = \frac{m!r}{(r-x)^{m+1}}$$

for some $m \geq 0$. Then by Def. II.4.2.4 we have

$$\forall x \in (-r, r), \left(y \mapsto \frac{r}{r-y}\right)^{(m+1)}(x) = \left(\frac{m!r}{(r-x)^{m+1}}\right)'(x) = \frac{(m+1)!r}{(r-x)^{m+2}}.$$

This closes the induction.

Now we show that

$$\forall m \in \mathbb{N}, \frac{r}{(r-x)^{m+1}} = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^{n-m} r^{-n}.$$

From the proof above we know that

$$\forall m \in \mathbb{N}, \forall x \in (-r, r), \frac{r}{(r-x)^{m+1}} = \frac{1}{m!}\left(y \mapsto \frac{r}{r-y}\right)^{(m)}.$$

By Prop. II.4.2.6 we know that

$$\begin{aligned}\forall m \in \mathbb{N}, \forall x \in (-r, r), \left(y \mapsto \frac{r}{r-y}\right)^{(m)} &\\ &= \sum_{n=0}^{\infty} r^{-(n+m)} \frac{(n+m)!}{n!} x^n \\ &= \sum_{n=m}^{\infty} r^{-n} \frac{n!}{(n-m)!} x^{n-m} \\ &= m! \sum_{n=m}^{\infty} r^{-n} \frac{n!}{m!(n-m)!} x^{n-m}.\end{aligned}$$

Thus, we have

$$\forall m \in \mathbb{N}, \forall x \in (-r, r), \frac{r}{(r-x)^{m+1}} = \sum_{n=m}^{\infty} r^{-n} \frac{n!}{m!(n-m)!} x^{n-m}.$$

Since

$$\begin{aligned}&\forall m \in \mathbb{N}, \lim_{n\to\infty} \left(\frac{(n+m)!}{n!}\right)^{\frac{1}{n}} = 1 \qquad \text{(by A.Cor. II.4.2.1)}\\ \Longrightarrow\ &\forall m \in \mathbb{N}, \lim_{n\to\infty} \left(\frac{n!}{(n-m)!}\right)^{\frac{1}{n-m}} = 1\\ \Longrightarrow\ &\forall m \in \mathbb{N}, \lim_{n\to\infty} \left(\frac{n!}{(n-m)!}\right)^{\frac{1}{n}} = 1\\ \Longrightarrow\ &\forall m \in \mathbb{N}, \lim_{n\to\infty} \left(\frac{n!}{m!(n-m)!}\right)^{\frac{1}{n}} \leq 1,\end{aligned}$$

we have

$$\forall m \in \mathbb{N}, \limsup_{n\to\infty} \left|\left(\frac{n!}{m!(n-m)!}\right)^{\frac{1}{n}}\right| \leq 1$$

$$\implies \forall m \in \mathbb{N}, \limsup_{n\to\infty} \left| \left(\frac{n!}{m!(n-m)!} \right)^{\frac{1}{n}} \frac{x}{r} \right| < 1$$

$$\implies \forall m \in \mathbb{N}, \limsup_{n\to\infty} \left| \left(\frac{n!}{m!(n-m)!} \right)^{\frac{1}{n}} (x^n r^{-n})^{\frac{1}{n}} \right| < 1$$

and by root test

$$\sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^n r^{-n}$$

is absolutely converges, and so does

$$x^{-m} \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^n r^{-n} = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^{n-m} r^{-n}.$$

□

Ex. II.4.2.8. Let E be a subset of $\mathbb{R}$, let a be an interior point of E, and let $f : E \to \mathbb{R}$ be a function which is real analytic at a, and has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

at a which converges on the interval $(a-r, a+r)$. Let $(b-s, b+s)$ be any sub-interval of $(a-r, a+r)$ for some $s > 0$.

(a) Prove that $|a-b| \le r-s$, so, in particular, $|a-b| < r$.

(b) Show that for every $0 < \varepsilon < r$, there exists a $C > 0$ such that $|c_n| \le C(r-\varepsilon)^{-n}$ for all integers $n \ge 0$.

(c) Show that the numbers $d_0, d_1, \dots$ given by the formula

$$d_m := \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} (b-a)^{n-m} c_n \text{ for all integers } m \ge 0$$

are well-defined, in the sense that the above series is absolutely convergent.

(d) Show that for every $0 < \varepsilon < s$ there exists a $C > 0$ such that

$$|d_m| \le C(s-\varepsilon)^{-m}$$

for all integers $m \ge 0$.

(e) Show that the power series $\sum_{m=0}^{\infty} d_m(x-b)^m$ is absolutely convergent for $x \in (b-s, b+s)$ and converges to $f(x)$.

(f) Conclude that f is real analytic at every point in $(a-r, a+r)$.

Proof. (a)

$$\begin{aligned}
& a - r \leq b - s \leq b + s \leq a + r \\
\implies & a - b - r \leq -s \leq s \leq a - b + r \\
\implies & s - r \leq a - b \leq r - s \\
\implies & |a - b| \leq r - s < r. && (s > 0)
\end{aligned}$$

□

Proof. (b) Let $\varepsilon \in (0, r)$. Since

$$\begin{aligned}
& 0 < \varepsilon < r \\
\implies & 0 < r - \varepsilon < r \\
\implies & (a - r + \varepsilon, a + r - \varepsilon) \subseteq (a - r, a + r) \\
\implies & \forall x \in (a, a + r) \setminus (a, a + r - \varepsilon), r - \varepsilon \leq x - a < r \\
\implies & \forall n \in \mathbb{N}, \forall x \in (a, a + r) \setminus (a, a + r - \varepsilon), (r - \varepsilon)^n \leq (x - a)^n < r^n
\end{aligned}$$

and

$$\begin{aligned}
& \forall x \in (a, a + r) \setminus (a, a + r - \varepsilon), \\
& r - \varepsilon < x - a < r \leq \frac{1}{\limsup_{n \to \infty} |c_n|^{\frac{1}{n}}} && \text{(by Def. II.4.2.1)} \\
\implies & \forall x \in (a, a + r) \setminus (a, a + r - \varepsilon), \\
& \sum_{n=0}^{\infty} c_n (x - a)^n \text{ is absolutely convergent} && \text{(by Thm. II.4.1.6(b))} \\
\implies & \forall x \in (a, a + r) \setminus (a, a + r - \varepsilon), \\
& \lim_{n \to \infty} c_n (x - a)^n = 0 && \text{(by Corollary 7.2.6 in Analysis I)} \\
\implies & \forall x \in (a, a + r) \setminus (a, a + r - \varepsilon), \\
& \lim_{n \to \infty} |c_n| (x - a)^n = 0 \\
\implies & \lim_{n \to \infty} |c_n| (r - \varepsilon)^n = 0, && \text{(by squeeze test)}
\end{aligned}$$

we know that

$$\lim_{n \to \infty} |c_n| (r - \varepsilon)^n = 0$$

$$\begin{aligned}&\Longrightarrow \exists N \in \mathbb{Z}^+ : \forall n \geq N, |c_n(r-\varepsilon)^n| < 1\\&\Longrightarrow \exists N \in \mathbb{Z}^+ : \forall n \geq N, |c_n| < (r-\varepsilon)^{-n}.\end{aligned}$$

Now we fix such N. If we define

$$C = 1 + \max\left(|c_0|(r-\varepsilon)^0, \dots, |c_{N-1}|(r-\varepsilon)^{N-1}\right),$$

then we have

$$\begin{aligned}&\begin{cases}\forall n \geq N, |c_n| < (r-\varepsilon)^{-n}\\ \forall 0 \leq n \leq N-1, |c_n| \leq C\end{cases}\\ \Longrightarrow\ &\forall n \geq N, |c_n| \leq C(r-\varepsilon)^{-n}. && (C \geq 1)\end{aligned}$$

Since ε was arbitrary, we conclude that

$$\forall \varepsilon \in (0, r), \exists C \in \mathbb{R}^+ : \forall n \in \mathbb{N}, |c_n| \leq C(r-\varepsilon)^{-n}.$$

□

Proof. (c) By Ex. II.4.2.8(a) we know that $|b-a| = |a-b| < r$, thus

$$|b-a| < r \implies \exists \varepsilon \in \mathbb{R}^+ : \begin{cases}|b-a| + \varepsilon < r\\ \varepsilon < r\end{cases} \implies \exists \varepsilon \in \mathbb{R}^+ : \begin{cases}|b-a| < r-\varepsilon\\ 0 < r-\varepsilon\end{cases}$$

Fix such ε. By Ex. II.4.2.8(b) we know that

$$\exists C \in \mathbb{R}^+ : \forall n \in \mathbb{N}, |c_n| \leq C(r-\varepsilon)^{-n}.$$

Fix such C. Since

$$\begin{aligned}&\forall m \in \mathbb{N}, \frac{r-\varepsilon}{(r-\varepsilon-|b-a|)^{m+1}}\\ &= \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}|b-a|^{n-m}(r-\varepsilon)^{-n} && \text{(by Cor. II.4.2.7)}\\ &= \frac{1}{C}\sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}|b-a|^{n-m}C(r-\varepsilon)^{-n}\\ &\geq \frac{1}{C}\sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}|b-a|^{n-m}|c_n|\\ &\geq \frac{1}{C}\sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}(b-a)^{n-m}c_n && \text{(by comparison test)}\\ &= \frac{1}{C}d_m,\end{aligned}$$

we know that d_m are absolutely convergent for all $m \in \mathbb{N}$. □

Proof. (d) Let $\varepsilon \in (0, s)$. By Ex. II.4.2.8(a) we have

$$\begin{aligned}
&|a-b| \leq r-s \\
\implies &|a-b|+s \leq r \\
\implies &s \leq r \\
\implies &0 < \varepsilon < r \\
\implies &0 < s-\varepsilon \leq r-\varepsilon \\
\implies &\forall n \in \mathbb{N}, 0 < (r-\varepsilon)^{-n} \leq (s-\varepsilon)^{-n}.
\end{aligned}$$

By Ex. II.4.2.8(b) we know that

$$\exists C \in \mathbb{R}^{+} : \forall n \in \mathbb{N}, |c_n| \leq C(r-\varepsilon)^{-n} \leq C(s-\varepsilon)^{-n}.$$

Since

$$\begin{aligned}
&|a-b| \leq r-s < r-\varepsilon \\
\implies &\forall m \in \mathbb{N}, |a-b|^{m} \leq (r-s)^{m} \\
\implies &\forall m \in \mathbb{N}, \forall n \geq m, |a-b|^{n-m} \leq (r-s)^{n-m},
\end{aligned}$$

we know that

$$\begin{aligned}
\forall m \in \mathbb{N}, |d_m| &= \left| \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}(b-a)^{n-m} c_n \right| \\
&\leq \sum_{n=m}^{\infty} \left| \frac{n!}{m!(n-m)!}(b-a)^{n-m} c_n \right| && \text{(by comparison test)} \\
&\leq C \sum_{n=m}^{\infty} \left| \frac{n!}{m!(n-m)!}(b-a)^{n-m} (r-\varepsilon)^{-n} \right| \\
&= C \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}|b-a|^{n-m} (r-\varepsilon)^{-n} \\
&\leq C \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}(r-s)^{n-m} (r-\varepsilon)^{-n} \\
&= C \frac{r-\varepsilon}{\big((r-\varepsilon)-(r-s)\big)^{m+1}} && \text{(by Cor. II.4.2.7)} \\
&= C \frac{r-\varepsilon}{(s-\varepsilon)^{m+1}} \\
&= C \frac{r-\varepsilon}{s-\varepsilon}(s-\varepsilon)^{-m}.
\end{aligned}$$

Thus, by setting $C' = C\dfrac{r-\varepsilon}{s-\varepsilon}$ we have

$$\forall m \in \mathbb{N}, |d_m| \leq C'(s-\varepsilon)^{-m}.$$

Since ε was arbitrary, we conclude that

$$\forall \varepsilon \in (0,s), \exists C \in \mathbb{R}^+ : \forall m \in \mathbb{N}, |d_m| \leq C(s-\varepsilon)^{-m}.$$

□

Proof. (e) Let $x \in (b-s, b+s)$. We have

$$\begin{aligned}
& x \in (b-s, b+s) \\
\implies & 0 \leq |x-b| < s \\
\implies & \exists \varepsilon \in \mathbb{R}^+ : 0 \leq |x-b| < s - \varepsilon \\
\implies & \exists \varepsilon \in \mathbb{R}^+ : \begin{cases} 0 \leq \left|\dfrac{x-b}{s-\varepsilon}\right| < 1 \\ \exists C \in \mathbb{R}^+ : \forall m \in \mathbb{N}, |d_m| \leq C(s-\varepsilon)^{-m} \end{cases} && \text{(by Ex. II.4.2.8(e))}
\end{aligned}$$

Fix such ε and C. Since

$$\begin{aligned}
& \left|\frac{x-b}{s-\varepsilon}\right| < 1 \\
\implies & \sum_{m=0}^{\infty} \left|\frac{x-b}{s-\varepsilon}\right|^m \text{ is absolutely convergent,} && \text{(by Lemma 7.3.3 in Analysis I)}
\end{aligned}$$

we have

$$\begin{aligned}
& \forall m \in \mathbb{N}, |d_m| \leq C(s-\varepsilon)^{-m} \\
\implies & \forall m \in \mathbb{N}, |d_m(x-b)^m| \leq C\left|\frac{x-b}{s-\varepsilon}\right|^m \\
\implies & \sum_{m=0}^{\infty} |d_m(x-b)^m| \leq \sum_{m=0}^{\infty} C\left|\frac{x-b}{s-\varepsilon}\right|^m = C\sum_{m=0}^{\infty}\left|\frac{x-b}{s-\varepsilon}\right|^m \\
\implies & \sum_{m=0}^{\infty} d_m(x-b)^m \text{ is absolutely convergent.} && \text{(by comparison test)}
\end{aligned}$$

Since $\mathbb{N} \times \mathbb{N}$ is countable, we know that $\mathbb{N} \times S$ is also countable for every non-empty subset S of $\mathbb{N}$. Thus

$$\begin{aligned}
& \sum_{m=0}^{\infty} d_m(x-b)^m \\
&= \sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}(b-a)^{n-m} c_n \right)(x-b)^m \\
&= \sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} \frac{c_n n!}{m!(n-m)!}(b-a)^{n-m}(x-b)^m \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{c_n n!}{m!(n-m)!} (b-a)^{n-m} (x-b)^m \right) && \text{(by Fubini's theorem)} \\
&= \sum_{n=0}^{\infty} c_n \left(\sum_{m=0}^{n} \frac{n!}{m!(n-m)!} (b-a)^{n-m} (x-b)^m \right) \\
&= \sum_{n=0}^{\infty} c_n (x-a) && \text{(by Ex. II.4.2.5)} \\
&= f(x).
\end{aligned}
$$

Since x was arbitrary, we conclude that

$$\forall x \in (b-s, b+s), \sum_{m=0}^{\infty} d_m (x-b)^m = f(x) \text{ is absolutely convergent.}$$

□

Proof. (f) Let $b \in (a-r, a+r)$. Since

$$
\begin{aligned}
& |b-a| < r \\
\Longrightarrow\ & \exists s \in \mathbb{R}^+ : |b-a| < r-s \\
\Longrightarrow\ & \exists s \in \mathbb{R}^+ : s-r < b-a < r-s \\
\Longrightarrow\ & \exists s \in \mathbb{R}^+ : a-r < b-s < b+s < a+r,
\end{aligned}
$$

by Ex. II.4.2.8(e) we know that f is real analytic at b. Since b was arbitrary, we conclude that f is real analytic at x for all $x \in (a-r, a+r)$. □

II.4.3 Abel's theorem

Note. Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at a with a radius of convergence $0 < R < \infty$ strictly between 0 and infinity. From Thm. II.4.1.6 we know that the power series converges absolutely whenever $|x-a| < R$, and diverges when $|x-a| > R$. However, at the boundary $|x-a| = R$ the situation is more complicated; the series may either converge or diverge (see Ex. II.4.1.2). However, if the series does converge at the boundary point, then it is reasonably well behaved; in particular, it is continuous at that boundary point.

Thm. II.4.3.1 (Abel's theorem)**.** Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at a with radius of convergence $0 < R < \infty$. If the power series converges at $a+R$, then f is continuous at $a+R$, i.e.

$$\lim_{x \to a+R; x \in (a-R, a+R)} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n R^n.$$

Similarly, if the power series converges at $a-R$, then f is continuous at $a-R$, i.e.

$$\lim_{x\to a-R; x\in(a-R,a+R)} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n(-R)^n.$$

Proof. It will suffice to prove the first claim, i.e., that

$$\lim_{x\to a+R; x\in(a-R,a+R)} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n R^n.$$

whenever the sum $\sum_{n=0}^{\infty} c_n R^n$ converges; the second claim will then follow by replacing c_n by $(-1)^n c_n$ in the above claim. If we make the substitutions $d_n := c_n R^n$ and $y := \frac{x-a}{R}$, then the above claim can be rewritten as

$$\lim_{y\to 1; y\in(-1,1)} \sum_{n=0}^{\infty} d_n y^n = \sum_{n=0}^{\infty} d_n$$

whenever the sum $\sum_{n=0}^{\infty} d_n$ converges.

Write $D := \sum_{n=0}^{\infty} d_n$, and for every $N \geq 0$ write

$$S_N := \left(\sum_{n=0}^{N-1} d_n\right) - D$$

so, in particular, $S_0 = -D$. Then observe that $\lim_{N\to\infty} S_N = 0$, and that $d_n = S_{n+1} - S_n$. Thus, for any $y \in (-1,1)$ we have

$$\sum_{n=0}^{\infty} d_n y^n = \sum_{n=0}^{\infty} (S_{n+1} - S_n) y^n.$$

Applying the summation by parts formula (Lem. II.4.3.2), and noting that $\lim_{n\to\infty} y^n = 0$, we obtain

$$\sum_{n=0}^{\infty} d_n y^n = -S_0 y^0 - \sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n).$$

Observe that $-S_0 y^0 = +D$. Thus, to finish the proof of Abel's theorem, it will suffice to show that

$$\lim_{y\to 1; y\in(-1,1)} \sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n) = 0.$$

Since y converges to 1, we may as well restrict y to $[0,1)$ instead of $(-1,1)$; in particular, we may take y to be positive.

From the triangle inequality for series (Proposition 7.2.9 in Analysis I), we have

$$\begin{aligned}\left|\sum_{n=0}^{\infty} S_{n+1}(y^{n+1}-y^n)\right| &\leq \sum_{n=0}^{\infty}\left|S_{n+1}(y^{n+1}-y^n)\right| \\ &= \sum_{n=0}^{\infty}|S_{n+1}|(y^n-y^{n+1}),\end{aligned}$$

so by the squeeze test (Corollary 6.4.14 in Analysis I) it suffices to show that

$$\lim_{y\to 1; y\in[0,1)} \sum_{n=0}^{\infty}|S_{n+1}|(y^n-y^{n+1}) = 0.$$

The expression $\sum_{n=0}^{\infty}|S_{n+1}|(y^n-y^{n+1})$ is clearly non-negative, so it will suffice to show that

$$\limsup_{y\to 1; y\in[0,1)} \sum_{n=0}^{\infty}|S_{n+1}|(y^n-y^{n+1}) = 0.$$

Let $\varepsilon > 0$. Since S_n converges to 0, there exists an N such that $|S_n| \leq \varepsilon$ for all $n > N$. Thus, we have

$$\sum_{n=0}^{\infty}|S_{n+1}|(y^n-y^{n+1}) \leq \sum_{n=0}^{N}|S_{n+1}|(y^n-y^{n+1}) + \sum_{n=N+1}^{\infty}\varepsilon(y^n-y^{n+1}).$$

The last summation is a telescoping series, which sums to εy^{N+1} (See Lemma 7.2.15 in Analysis I, recalling from Lemma 6.5.2 in Analysis I that $y^n \to 0$ as $n \to \infty$), and thus

$$\sum_{n=0}^{\infty}|S_{n+1}|(y^n-y^{n+1}) \leq \sum_{n=0}^{N}|S_{n+1}|(y^n-y^{n+1}) + \varepsilon y^{N+1}.$$

Now take limits as $y \to 1$. Observe that $y^n - y^{n+1} \to 0$ as $y \to 1$ for every $n \in 0, 1, \ldots, N$. Since we can interchange limits and *finite* sums (Exercise 7.1.5 in Analysis I), we thus have

$$\limsup_{y\to 1; y\in[0,1)} \sum_{n=0}^{\infty}|S_{n+1}|(y^n-y^{n+1}) \leq \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, and thus we must have

$$\limsup_{y\to 1; y\in[0,1)} \sum_{n=0}^{\infty}|S_{n+1}|(y^n-y^{n+1}) = 0$$

since the left-hand side must be non-negative. The claim follows. □

Lem. II.4.3.2 (Summation by parts formula)**.** Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of real numbers which converge to limits A and B respectively, i.e., $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Suppose that the sum $\sum_{n=0}^{\infty}(a_{n+1}-a_n)b_n$ is convergent. Then the sum $\sum_{n=0}^{\infty} a_{n+1}(b_{n+1}-b_n)$ is also convergent, and

$$\sum_{n=0}^{\infty}(a_{n+1}-a_n)b_n = AB - a_0b_0 - \sum_{n=0}^{\infty} a_{n+1}(b_{n+1}-b_n).$$

Proof. Since

$$\begin{aligned}\forall N \in \mathbb{N}, &\sum_{n=0}^{N}(a_{n+1}-a_n)b_n + \sum_{n=0}^{N} a_{n+1}(b_{n+1}-b_n)\\ &= \sum_{n=0}^{N} a_{n+1}(b_n - a_nb_n + a_{n+1}b_{n+1} - a_{n+1}b_n)\\ &= \sum_{n=0}^{N} a_{n+1}(b_{n+1} - a_nb_n)\\ &= a_{N+1}b_{N+1} - a_0b_0\end{aligned}$$

and

$$\begin{aligned}&\begin{cases}\lim_{n\to\infty} a_n = A\\ \lim_{n\to\infty} b_n = B\end{cases}\\ \implies &\lim_{n\to\infty} a_nb_n = AB,\end{aligned}$$

we have

$$\begin{aligned}&\lim_{N\to\infty}\left(\sum_{n=0}^{N}(a_{n+1}-a_n)b_n + \sum_{n=0}^{N} a_{n+1}(b_{n+1}-b_n)\right)\\ &= \lim_{N\to\infty}(a_{N+1}b_{N+1} - a_0b_0)\\ &= \lim_{N\to\infty}(a_{N+1}b_{N+1}) - a_0b_0\\ &= AB - a_0b_0.\end{aligned}$$

Thus

$$\left(\sum_{n=0}^{\infty}(a_{n+1}-a_n)b_n\right) - AB + a_0b_0$$

$$
\begin{aligned}
&= \lim_{N\to\infty}\left(\sum_{n=0}^{N}(a_{n+1}-a_n)b_n\right) - \lim_{N\to\infty}\left(\sum_{n=0}^{N}(a_{n+1}-a_n)b_n + \sum_{n=0}^{N}a_{n+1}(b_{n+1}-b_n)\right)\\
&= \lim_{N\to\infty} -\left(\sum_{n=0}^{N}a_{n+1}(b_{n+1}-b_n)\right)\\
&= -\lim_{N\to\infty}\left(\sum_{n=0}^{N}a_{n+1}(b_{n+1}-b_n)\right)\\
&= -\sum_{n=0}^{\infty}a_{n+1}(b_{n+1}-b_n)
\end{aligned}
$$

and

$$\sum_{n=0}^{\infty}(a_{n+1}-a_n)b_n = AB - a_0b_0 - \sum_{n=0}^{\infty}a_{n+1}(b_{n+1}-b_n).$$

□

Rmk. II.4.3.3. One should compare this formula with the more well-known *integration by parts formula*

$$\int_0^\infty f'(x)g(x)\,dx = f(x)g(x)|_0^\infty - \int_0^\infty f(x)g'(x)\,dx,$$

see Proposition 11.10.1 in Analysis I.

— Exercises —

Ex. II.4.3.1. Prove Lem. II.4.3.2.

Proof. See Lem. II.4.3.2. □

II.4.4 Multiplication of power series

Thm. II.4.4.1. Let $f : (a-r, a+r) \to \mathbb{R}$ and $g : (a-r, a+r) \to \mathbb{R}$ be functions analytic on $(a-r, a+r)$, with power series expansions

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

and

$$g(x) = \sum_{n=0}^{\infty} d_n(x-a)^n$$

respectively. Then $fg : (a-r, a+r) \to \mathbb{R}$ is also analytic on $(a-r, a+r)$, with power series expansion

$$f(x)g(x) = \sum_{n=0}^{\infty} e_n (x-a)^n$$

where $e_n := \sum_{m=0}^{n} c_m d_{n-m}$.

Proof. We have to show that the series $\sum_{n=0}^{\infty} e_n (x-a)^n$ converges to $f(x)g(x)$ for all $x \in (a-r, a+r)$. Now fix x to be any point in $(a-r, a+r)$. By Thm. II.4.1.6, we see that both f and g have radii of convergence at least r. In particular, the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ and $\sum_{n=0}^{\infty} d_n (x-a)^n$ are absolutely convergent. Thus, if we define

$$C := \sum_{n=0}^{\infty} |c_n (x-a)^n|$$

and

$$D := \sum_{n=0}^{\infty} |d_n (x-a)^n|$$

then C and D are both finite.

For any $N \geq 0$, consider the partial sum

$$\sum_{n=0}^{N} \sum_{m=0}^{\infty} |c_m (x-a)^m d_n (x-a)^n|.$$

We can rewrite this as

$$\sum_{n=0}^{N} |d_n (x-a)^n| \sum_{m=0}^{\infty} |c_m (x-a)^m|,$$

which by definition of C is equal to

$$\sum_{n=0}^{N} |d_n (x-a)^n| C,$$

which by definition of D is less than or equal to DC. Thus, the above partial sums are bounded by DC for every N. In particular, the series

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |c_m (x-a)^m d_n (x-a)^n|$$

is convergent, which means that the sum

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m(x-a)^m d_n(x-a)^n$$

is absolutely convergent.

Let us now compute this sum in two ways. First, of all, we can pull the $d_n(x-a)^n$ factor out of the $\sum_{m=0}^{\infty}$ summation, to obtain

$$\sum_{n=0}^{\infty} d_n(x-a)^n \sum_{m=0}^{\infty} c_m(x-a)^m.$$

By our formula for $f(x)$, this is equal to

$$\sum_{n=0}^{\infty} d_n(x-a)^n f(x);$$

by our formula for $g(x)$, this is equal to $f(x)g(x)$. Thus

$$f(x)g(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m(x-a)^m d_n(x-a)^n.$$

Now we compute this sum in a different way. We rewrite it as

$$f(x)g(x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m d_n(x-a)^{n+m}.$$

By Fubini's theorem for series (Theorem 8.2.2 in Analysis I), because the series was absolutely convergent, we may rewrite it as

$$f(x)g(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m d_n(x-a)^{n+m}.$$

Now make the substitution $n' := n+m$, to rewrite this as

$$f(x)g(x) = \sum_{m=0}^{\infty}\sum_{n'=m}^{\infty} c_m d_{n'-m}(x-a)^{n'}.$$

If we adopt the convention that $d_j = 0$ for all negative j, then this is equal to

$$f(x)g(x) = \sum_{m=0}^{\infty}\sum_{n'=0}^{\infty} c_m d_{n'-m}(x-a)^{n'}.$$

Applying Fubini's theorem again, we obtain

$$f(x)g(x) = \sum_{n'=0}^{\infty} \sum_{m=0}^{\infty} c_m d_{n'-m} (x-a)^{n'},$$

which we can rewrite as

$$f(x)g(x) = \sum_{n'=0}^{\infty} (x-a)^{n'} \sum_{m=0}^{\infty} c_m d_{n'-m}.$$

Since d_j was 0 when j is negative, we can rewrite this as

$$f(x)g(x) = \sum_{n'=0}^{\infty} (x-a)^{n'} \sum_{m=0}^{n'} c_m d_{n'-m},$$

which by definition of e is

$$f(x)g(x) = \sum_{n'=0}^{\infty} e_{n'} (x-a)^{n'},$$

as desired. $\square$

Rmk. II.4.4.2. The sequence $(e_n)_{n=0}^{\infty}$ is sometimes referred to as the *convolution* of the sequences $(c_n)_{n=0}^{\infty}$ and $(d_n)_{n=0}^{\infty}$; it is closely related (though not identical) to the notion of convolution introduced in Def. II.3.8.9.

II.4.5 The exponential and logarithm functions

Def. II.4.5.1 (Exponential function)**.** For every real number x, we define the *exponential function* $\exp(x)$ to be the real number

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Thm. II.4.5.2 (Basic properties of exponential)**.**

(a) For every real number x, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent. In particular, $\exp(x)$ exists and is real for every $x \in \mathbb{R}$, the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has an infinite radius of convergence, and exp is a real analytic function on $(-\infty, \infty)$.

(b) exp is differentiable on R, and for every $x \in \mathbb{R}$, $\exp'(x) = \exp(x)$.

(c) exp is continuous on R, and for every interval $[a, b]$, we have $\int_{[a,b]} \exp(x)\ dx = \exp(b) - \exp(a)$.

(d) For every $x, y \in \mathbb{R}$, we have $\exp(x + y) = \exp(x) \exp(y)$.

(e) We have $\exp(0) = 1$. Also, for every $x \in \mathbb{R}$, $\exp(x)$ is positive, and $\exp(-x) = 1/\exp(x)$.

(f) exp is strictly monotone increasing: in other words, if x, y are real numbers, then we have $\exp(y) > \exp(x)$ iff $y > x$.

Proof. (a) If $x = 0$, then we have

$$1 = 1 + 0 = \frac{0^0}{0!} + \sum_{n=1}^{\infty} \frac{0^n}{n!} = \sum_{n=0}^{\infty} \frac{0^n}{n!} = \exp(0).$$

So suppose that $x \in \mathbb{R} \setminus \{0\}$. Since

$$\limsup_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \limsup_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$$
$$\implies \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ is absolutely convergent,} \qquad \text{(by ratio test)}$$

we know that $\exp(x)$ exists for all $x \in \mathbb{R} \setminus \{0\}$. Combine all proofs above we have

$$\forall x \in \mathbb{R}, \begin{cases} \sum_{n=0}^{\infty} \frac{x^n}{n!} \text{ is absolutely convergent} \\ \exp(x) \in \mathbb{R} \end{cases}$$

and by Def. II.4.2.1 exp is real analytic on $(-\infty, \infty)$. □

Proof. (b) By Thm. II.4.5.2(a) we know that exp is real analytic on $\mathbb{R}$, thus by Thm. II.4.1.6(d) we know that exp is differentiable on $\mathbb{R}$ and

$$\forall x \in \mathbb{R}, \exp'(x) = \sum_{n=1}^{\infty} n \frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x).$$

□

Proof. (c) By Thm. II.4.5.2(a) we know that exp is real analytic on $\mathbb{R}$, thus by Thm. II.4.1.6(c) we know that exp is continuous on $\mathbb{R}$. Let $a, b \in \mathbb{R}$ such that $a \le b$. Then by Thm. II.4.5.2(a) we know that $\exp(a)$ and $\exp(b)$ are well-defined. Since $[a, b] \subseteq \mathbb{R}$, by Thm. II.4.1.6(e) we know that exp is Riemann integrable on $[a, b]$ and

$$\int_{[a,b]} \exp = \int_a^b \exp(x)\ dx$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(b-0)^{n+1} - (a-0)^{n+1}}{n+1} && \text{(by Thm. II.4.1.6(e))} \\
&= \sum_{n=0}^{\infty} \frac{b^{n+1} - a^{n+1}}{(n+1)!} \\
&= \sum_{n=1}^{\infty} \frac{b^{n} - a^{n}}{n!} \\
&= \frac{b^0 - a^0}{0!} + \sum_{n=1}^{\infty} \frac{b^{n} - a^{n}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{b^{n} - a^{n}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{b^{n}}{n!} - \sum_{n=0}^{\infty} \frac{a^{n}}{n!} \\
&= \exp(b) - \exp(a). && \text{(by Def. II.4.5.1)}
\end{aligned}
$$

Since a, b was arbitrary, we conclude that

$$\forall [a,b] \subseteq \mathbb{R}, \int_{[a,b]} \exp = \exp(b) - \exp(a).$$

□

Proof. (d) If $x = 0$, then we have

$$\exp(0) = \sum_{n=0}^{\infty} \frac{0^n}{n!} = \frac{0^0}{0!} + \sum_{n=1}^{\infty} \frac{0^n}{n!} = 1$$

and thus

$$\forall y \in \mathbb{R}, \exp(0+y) = \exp(y) = \exp(0)\exp(y).$$

Since addition and multiplication of real numbers are commutative, we have

$$\forall x \in \mathbb{R}, \exp(x+0) = \exp(0+x) = \exp(0)\exp(x) = \exp(x)\exp(0).$$

So suppose that $x, y \in \mathbb{R} \setminus \{0\}$. By Thm. II.4.5.2(a) we know that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\sum_{n=0}^{\infty} \frac{y^n}{n!}$ are absolutely convergent. Thus, we know that

$$X = \sum_{n=0}^{\infty} \left| \frac{x^n}{n!} \right|$$

$$Y = \sum_{n=0}^{\infty} \left| \frac{y^n}{n!} \right|$$

are well-defined. Since

$$\begin{aligned}
\forall N \in \mathbb{N}, \sum_{n=0}^{N} \sum_{m=0}^{\infty} \left| \frac{x^n y^m}{n!m!} \right| &= \sum_{n=0}^{N} \left(\left| \frac{x^n}{n!} \right| \sum_{m=0}^{\infty} \left| \frac{y^m}{m!} \right| \right) \\
&= \sum_{n=0}^{N} \left(\left| \frac{x^n}{n!} \right| Y \right) \\
&= Y \sum_{n=0}^{N} \left| \frac{x^n}{n!} \right| \\
&\leq YX,
\end{aligned}$$

and $(\sum_{n=0}^{N} \sum_{m=0}^{\infty} \left| \frac{x^n y^m}{n!m!} \right|)_{N=0}^{\infty}$ is monotone increasing, we know that $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^n y^m}{n!m!}$ is absolutely convergent. Now we define

$$\forall n \in \mathbb{Z}, c_n = \begin{cases} \dfrac{1}{n!} & \text{if } n \geq 0; \\ 0 & \text{if } n < 0. \end{cases}$$

Then we have

$$\begin{aligned}
&\exp(x)\exp(y) \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^n y^m}{n!m!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n c_m x^n y^m \\
&= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} c_n c_{m-n} x^n y^{m-n} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n c_{m-n} x^n y^{m-n} && (c_{m-n} = 0 \text{ if } m < n) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_n c_{m-n} x^n y^{m-n} && \text{(by Fubini's theorem)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m} c_n c_{m-n} x^n y^{m-n} && (c_{m-n} = 0 \text{ if } n > m)\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{m} \frac{x^n y^{m-n}}{n!(m-n)!}\\
&= \sum_{m=0}^{\infty}\left(\frac{1}{m!}\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}(x^n y^{m-n})\right)\\
&= \sum_{m=0}^{\infty}\frac{(x+y)^m}{m!} && (\text{by Ex. II.4.2.5})\\
&= \exp(x+y). && (\text{by Def. II.4.5.1})
\end{aligned}
$$

Combine all proofs above we conclude that

$$\forall x, y \in \mathbb{R}, \exp(x+y) = \exp(x)\exp(y).$$

□

Proof. (e) We have

$$\exp(0) = \sum_{n=0}^{\infty}\frac{0^n}{n!} = \frac{0^0}{0!} + \sum_{n=1}^{\infty}\frac{0^n}{n!} = 1$$

and

$$\forall x \in \mathbb{R}^+, \exp(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!} = \frac{x^0}{0!} + \sum_{n=1}^{\infty}\frac{x^n}{n!} \geq 1.$$

Since

$$
\begin{aligned}
\forall x \in \mathbb{R}, \exp(0) &= \exp(x-x)\\
&= \exp(x)\exp(-x) && (\text{by Thm. II.4.5.2(d)})\\
&= 1,
\end{aligned}
$$

we know that

$$\forall x \in \mathbb{R}, \exp(-x) = \frac{1}{\exp(x)}.$$

Thus

$$\forall x \in \mathbb{R}^-, \exp(-x) \geq 1 \implies \exp(x) = \frac{1}{\exp(-x)} > 0.$$

Combine all proofs above we conclude that

$$\forall x \in \mathbb{R}, \exp(x) > 0.$$

□

Proof. (f) By Thm. II.4.5.2(b)(e) we know that

$$\forall x \in \mathbb{R}, \exp'(x) = \exp(x) > 0.$$

Thus, by Proposition 10.3.3 in Analysis I we know that exp is strictly monotone increasing. $\square$

Note. One can write the exponential function in a more compact form, introducing famous *Euler's number* $e = 2.71828183\ldots$, also known as the *base of the natural logarithm.*

Def. II.4.5.3 (Euler's number)**.** The number e is defined to be

$$e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots.$$

Prop. II.4.5.4. For every real number x, we have $\exp(x) = e^x$.

Proof. First, we induct on x to show that $e^x = \exp(x)$ for all $x \in \mathbb{N}$. For $x = 0$, we have

$$\begin{aligned} e^0 &= \big(\exp(1)\big)^0 && \text{(by Def. II.4.5.3)} \\ &= 1 && \text{(by Thm. II.4.5.2(a))} \\ &= \exp(0) && \text{(by Thm. II.4.5.2(e))} \end{aligned}$$

and the base case holds. Suppose inductively that $e^x = \exp(x)$ for some $x \geq 0$. Then for $x + 1$, we have

$$\begin{aligned} e^{x+1} &= \exp(1)^{x+1} && \text{(by Def. II.4.5.3)} \\ &= \exp(1)\exp(1)^x && \text{(by Thm. II.4.5.2(a))} \\ &= \exp(1)e^x && \text{(by Def. II.4.5.3)} \\ &= \exp(1)\exp(x) && \text{(by the induction hypothesis)} \\ &= \exp(x+1) && \text{(by Thm. II.4.5.2(d))} \end{aligned}$$

and this closes the induction.

Next we show that $e^x = \exp(x)$ for all $x \in \mathbb{Z}$. Let $x \in \mathbb{Z}^-$. Since $-x \in \mathbb{N}$, we know that

$$\begin{aligned} e^{-x} &= \exp(-x) && \text{(from the proof above)} \\ &= \frac{1}{\exp(x)} && \text{(by Thm. II.4.5.2(e))} \\ &= \big(\exp(x)\big)^{-1} \end{aligned}$$

and thus $e^x = \exp(x)$. Since x was arbitrary, combine the proofs above we conclude that $e^x = \exp(x)$ for all $x \in \mathbb{Z}$.

Next we show that $e^x = \exp(x)$ for all $x \in \mathbb{Q}$. Let $x \in \mathbb{Q}$. Since $x = \dfrac{a}{b}$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$, we know that

$$e^a = \exp(a) \qquad \text{(from the proof above)}$$

$$
\begin{aligned}
&= \exp(\frac{ab}{b}) \\
&= \exp(\sum_{i=1}^{b} \frac{a}{b}) \\
&= \prod_{i=1}^{b} \exp(\frac{a}{b}) && \text{(by Thm. II.4.5.2(d))} \\
&= \exp(\frac{a}{b})^{b} \\
&= \exp(x)^{b}
\end{aligned}
$$

and thus

$$
\begin{aligned}
e^{x} &= e^{\frac{a}{b}} \\
&= \left(e^{a}\right)^{\frac{1}{b}} \\
&= \left(\left(\exp(x)\right)^{b}\right)^{\frac{1}{b}} && \text{(from the proof above)} \\
&= \exp(x).
\end{aligned}
$$

Since x was arbitrary, we conclude that $e^{x} = \exp(x)$ for all $x \in \mathbb{Q}$.

Finally we show that $e^{x} = \exp(x)$ for all $x \in \mathbb{R}$. Let $x \in \mathbb{R}$. Then we know that there exists a Cauchy sequence $(q_n)_{n=1}^{\infty}$ in Q such that $\lim_{n\to\infty} q_n = x$. Thus

$$
\begin{aligned}
e^{x} &= \lim_{n\to\infty} e^{q_n} \\
&= \lim_{n\to\infty} \exp(q_n) && \text{(from the proof above)} \\
&= \exp(x). && \text{(by Thm. II.4.5.2(c))}
\end{aligned}
$$

Since x was arbitrary, we conclude that $e^{x} = \exp(x)$ for all $x \in \mathbb{R}$. $\square$

Note. In light of Def. II.4.5.3 we can and will use e^{x} and $\exp(x)$ interchangeably.

Note. Since $e > 1$, we see that $e^{x} \to +\infty$ as $x \to +\infty$, and $e^{x} \to 0$ as $x \to -\infty$. From this and the intermediate value theorem (Theorem 9.7.1 in Analysis I) we see that the codomain of the function exp is $(0, \infty)$. Since exp is increasing, it is injective, and hence exp is a bijection from $\mathbb{R}$ to $(0, \infty)$, and thus has an inverse from $(0, \infty) \to \mathbb{R}$.

Def. II.4.5.5 (Logarithm)**.** We define the *natural logarithm function*

$$\log : (0, \infty) \to \mathbb{R}$$

(also called ln) to be the inverse of the exponential function. Thus, $\exp\left(\log(x)\right) = x$ and $\log\left(\exp(x)\right) = x$.

Thm. II.4.5.6. arithm properties]

(a) For every $x \in (0, \infty)$, we have $\ln'(x) = \frac{1}{x}$. In particular, by the fundamental theorem of calculus, we have $\int_{[a,b]} \frac{1}{x}\, dx = \ln(b) - \ln(a)$ for any interval $[a, b]$ in $(0, \infty)$.

(b) We have $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y \in (0, \infty)$.

(c) We have $\ln(1) = 0$ and $\ln(1/x) = -\ln(x)$ for all $x \in (0, \infty)$.

(d) For any $x \in (0, \infty)$ and $y \in R$, we have $\ln(x^y) = y \ln(x)$.

(e) For any $x \in (-1, 1)$, we have

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

In particular, ln is analytic at 1, with the power series expansion

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x - 1)^n$$

for $x \in (0, 2)$, with radius of convergence 1.

Proof. (a) Since exp is continuous and strictly monotone increasing, we see that ln is also continuous and strictly monotone increasing (see Proposition 9.8.3 in Analysis I). Since exp is also differentiable, and the derivative is never zero, we see from the inverse function theorem (Theorem 10.4.2 in Analysis I) that ln is also differentiable. Thus, we have

$$\begin{aligned}
&\forall x \in (0, \infty), \exp\big(\ln(x)\big) = x && \text{(by Def. II.4.5.5)}\\
\Longrightarrow\ &\forall x \in (0, \infty), \ln'(x) = \frac{1}{\exp'\big(\ln(x)\big)} && \text{(Theorem 10.4.2 in Analysis I)}\\
\Longrightarrow\ &\forall x \in (0, \infty), \ln'(x) = \frac{1}{\exp\big(\ln(x)\big)} && \text{(by Thm. II.4.5.2(b))}\\
\Longrightarrow\ &\forall x \in (0, \infty), \ln'(x) = \frac{1}{x}. && \text{(by Def. II.4.5.5)}
\end{aligned}$$

Since $\frac{1}{x}$ is continuous on arbitrary interval $[a, b] \subseteq (0, \infty)$, by Corollary 11.5.2 in Analysis I we know that $\frac{1}{x}$ is Riemann integrable on $[a, b]$. By the fundamental theorem of calculus (Theorem 11.9.4) we thus have

$$\int_a^b \frac{1}{x}\, dx = \ln(b) - \ln(a).$$

□

Proof. (b) We have

$$\begin{aligned}
\forall x, y \in (0,\infty), \ln(xy) &= \ln(e^{\ln(x)}e^{\ln(y)}) && \text{(by Def. II.4.5.5)}\\
&= \ln(e^{\ln(x)+\ln(y)}) && \text{(by Thm. II.4.5.2(d))}\\
&= \ln(x) + \ln(y). && \text{(by Def. II.4.5.5)}
\end{aligned}$$

□

Proof. (c) We have

$$\begin{aligned}
\ln(1) &= \ln(e^0) && \text{(by Thm. II.4.5.2(e))}\\
&= 0 && \text{(by Def. II.4.5.5)}
\end{aligned}$$

and

$$\begin{aligned}
&\forall x \in (0,\infty), \ln(\frac{x}{x}) = 0 \text{(from the proof above)}\\
\implies &\forall x \in (0,\infty), \ln(x) + \ln(\frac{1}{x}) = 0 && \text{(by Thm. II.4.5.6(b))}\\
\implies &\forall x \in (0,\infty), \ln(\frac{1}{x}) = -\ln(x).
\end{aligned}$$

□

Proof. (d) Let $x \in (0,\infty)$. We know that $x^y \in (0,\infty)$ for all $y \in \mathbb{R}$, thus by Def. II.4.5.5 $\ln(x^y)$ is well-defined and we have

$$\begin{aligned}
y\ln(x) &= \ln(e^{y\ln(x)}) && \text{(by Def. II.4.5.5)}\\
&= \ln\left((e^{\ln(x)})^y\right)\\
&= \ln(x^y). && \text{(by Def. II.4.5.5)}
\end{aligned}$$

Since x was arbitrary, we conclude that

$$\forall x \in (0,\infty), \forall y \in \mathbb{R}, y\ln(x) = \ln(x^y).$$

□

Proof. (e) Since

$$x \in (-1,1) \iff 1 - x \in (0,2),$$

by Def. II.4.5.5 we know that $\ln(1-x)$ is well-defined. Observe that

$$\begin{aligned}
&\forall x \in (-1,1), \ln'(1-x)\\
&\quad = \left(\ln'(y)|_{y=1-x}\right) \times \left((y \mapsto 1-y)'(x)\right) && \text{(by chain rule)}\\
&\quad = \frac{1}{1-x} \times (-1) && \text{(by Thm. II.4.5.6(a))}
\end{aligned}$$

$$= \frac{-1}{1-x}$$

and

$$\begin{aligned}
& x \in (-1,1) \\
\implies & \forall n \in \mathbb{Z}^+, x^n \in (-1,1) \\
\implies & \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. && \text{(by Lemma 7.3.3 in Analysis I)}
\end{aligned}$$

First, suppose that $x = 0$. Then we have

$$\begin{aligned}
\ln(1-0) &= \ln(1) \\
&= 0 && \text{(by Thm. II.4.5.6(c))} \\
&= -\sum_{n=1}^{\infty} \frac{0^n}{n}.
\end{aligned}$$

Now suppose that $x \in (0,1)$. Then we have

$$\begin{aligned}
& -\sum_{n=0}^{\infty} x^n = \ln'(1-x) \\
\implies & \int_0^x \left(-\sum_{n=0}^{\infty} y^n\right) dy = \int_0^x \ln'(1-y)\, dy && \text{(by Thm. II.4.5.6(a))} \\
\implies & -\sum_{n=0}^{\infty} \frac{x^{n+1} - 0^{n+1}}{n+1} = \int_0^x \ln'(1-y)\, dy && \text{(by Thm. II.4.1.6(e))} \\
\implies & -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \ln(1-x) - \ln(1-0) && \text{(by fundamental theorem of calculus)} \\
\implies & -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \ln(1-x) && \text{(by Thm. II.4.5.6(c))} \\
\implies & -\sum_{n=1}^{\infty} \frac{x^n}{n} = \ln(1-x).
\end{aligned}$$

Now suppose that $x \in (-1,0)$. Then we have

$$\begin{aligned}
& -\sum_{n=0}^{\infty} x^n = \ln'(1-x) \\
\implies & \int_x^0 \left(-\sum_{n=0}^{\infty} y^n\right) dy = \int_x^0 \ln'(1-y)\, dy && \text{(by Thm. II.4.5.6(a))}
\end{aligned}$$

$$\Longrightarrow -\sum_{n=0}^{\infty}\frac{0^{n+1}-x^{n+1}}{n+1}=\int_x^0 \ln'(1-y)\,dy \qquad \text{(by Thm. II.4.1.6(e))}$$

$$\Longrightarrow \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=\ln(1-0)-\ln(1-x) \qquad \text{(by fundamental theorem of calculus)}$$

$$\Longrightarrow \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=-\ln(1-x) \qquad \text{(by Thm. II.4.5.6(c))}$$

$$\Longrightarrow -\sum_{n=1}^{\infty}\frac{x^{n}}{n}=\ln(1-x).$$

Combine all proofs above we conclude that

$$\begin{aligned}
&\forall x\in(-1,1), \ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}\\
\Longrightarrow\ &\forall -x\in(-1,1), \ln\big(1-(-x)\big)=-\sum_{n=1}^{\infty}\frac{(-x)^n}{n}\\
\Longrightarrow\ &\forall -(x-1)\in(-1,1), \ln\Big(1-\big(-(x-1)\big)\Big)=-\sum_{n=1}^{\infty}\frac{\big(-(x-1)\big)^n}{n}\\
\Longrightarrow\ &\forall x\in(0,2), \ln(x)=-\sum_{n=1}^{\infty}\frac{(-1)^n(x-1)^n}{n}\\
\Longrightarrow\ &\forall x\in(0,2), \ln(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n}.
\end{aligned}$$

By Def. II.4.2.1 ln is real analytic at 1 with radius of convergence 1. □

E.g. II.4.5.7. We now give a modest application of Abel's theorem (Thm. II.4.3.1): from the alternating series test we see that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is convergent. By Abel's theorem we thus see that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\lim_{x\to 2;x\in(0,2)}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n=\lim_{x\to 2;x\in(0,2)}\ln(x)=\ln(2),$$

thus we have the formula

$$\ln(2)=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\dots.$$

A.Cor. II.4.5.1. $e^x>x$ for all $x\in\mathbb{R}$.

Proof. We have $e^0 = 1 > 0$. Suppose that $x \in \mathbb{R}^+$. Then we have

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} && \text{(by Def. II.4.5.1)} \\ &= \frac{x^0}{0!} + \frac{x^1}{1!} + \sum_{n=2}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!} \\ &> 1 + x && (x > 0) \\ &> x. \end{aligned}$$

Now suppose that $x \in \mathbb{R}^-$. Then by Thm. II.4.5.2(e) we have $e^x > 0 > x$. Combine all proofs above we conclude that $e^x > x$ for all $x \in \mathbb{R}$. ☐

— Exercises —

Ex. II.4.5.1. Prove Thm. II.4.5.2.

Proof. See Thm. II.4.5.2. ☐

Ex. II.4.5.2. Show that for every integer $n \geq 3$, we have

$$0 < \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{n!}.$$

Conclude that $n!e$ is not an integer for every $n \geq 3$. Deduce from this that e is irrational.

Proof. We first show that $(n+k)! > 2^k n!$ for all $n \geq 3$ and $k \in \mathbb{Z}^+$. We induct on k. For $k = 1$, we have

$$\forall n \geq 3, (n+1)! = (n+1)(n!) \geq 4(n!) > 2^1(n!)$$

and the base case holds. Suppose inductively that $(n+k)! > 2^k n!$ for some $k \geq 1$. Then for $k+1$, we have

$$\begin{aligned} \forall n \geq 3, (n+k+1)! &= (n+k+1)(n+k)! \\ &\geq (4+k)(n+k)! \\ &> 2(n+k)! \\ &> 2(2^k)(n!) && \text{(by the induction hypothesis)} \\ &= 2^{k+1}(n!) \end{aligned}$$

and this closes the induction.

Next we show that

$$\forall n \geq 3, 0 < \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots < \frac{1}{n!}.$$

Since

$$\begin{aligned}
&\forall n \geq 3, \forall k \geq 1, \frac{1}{(n+k)!} < \frac{1}{2^k(n!)} && \text{(from the proof above)} \\
\implies &\forall n \geq 3, \\
&\sum_{k=1}^{\infty} \frac{1}{(n+k)!} \leq \sum_{k=1}^{\infty} \frac{1}{2^k(n!)} = \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2(n!)} \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} && \text{(geometric series)} \\
&= \frac{1}{2(n!)} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{2}{2(n!)} = \frac{1}{n!} \\
\implies &\forall n \geq 3, \sum_{k=1}^{\infty} \frac{1}{(n+k)!} \leq \frac{1}{n!},
\end{aligned}$$

we only need to show that

$$\forall n \geq 3, \sum_{k=1}^{\infty} \frac{1}{(n+k)!} \neq \frac{1}{n!}.$$

So suppose for the sake of contradiction that there exists some $n \geq 3$ such that the identity above holds. Then we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{(n+k)!} = \frac{1}{n!} = \sum_{k=1}^{\infty} \frac{1}{2^k(n!)} \\
\implies &\sum_{k=1}^{\infty} \left(\frac{1}{2^k(n!)} - \frac{1}{(n+k)!} \right) = 0.
\end{aligned}$$

But we know that

$$\forall k \geq 1, \frac{1}{2^k(n!)} - \frac{1}{(n+k)!} > 0 \implies \sum_{k=1}^{\infty} \left(\frac{1}{2^k(n!)} - \frac{1}{(n+k)!} \right) > 0,$$

a contradiction. Thus, we have

$$\forall n \geq 3, \sum_{k=1}^{\infty} \frac{1}{(n+k)!} < \frac{1}{n!}.$$

Now we show that $n!e$ is not an integer for all $n \geq 3$. Since

$$\forall n \geq 3, \sum_{m=n+1}^{\infty} \frac{1}{m!} < \frac{1}{n!}$$

$$\implies \forall n \geq 3, \sum_{m=n+1}^{\infty} \frac{n!}{m!} < 1$$

and

$$\forall n \geq 3, \sum_{m=0}^{n} \frac{n!}{m!} = \sum_{m=0}^{n} (n-m)! \in \mathbb{N},$$

we have

$$\begin{aligned}
\forall n \geq 3, n!e &= n! \sum_{m=0}^{\infty} \frac{1}{m!} && \text{(by Def. II.4.5.3)} \\
&= \sum_{m=0}^{\infty} \frac{n!}{m!} \\
&= \sum_{m=0}^{n} \frac{n!}{m!} + \sum_{m=n+1}^{\infty} \frac{n!}{m!} \\
&= \sum_{m=0}^{n} (n-m)! + \sum_{m=n+1}^{\infty} \frac{n!}{m!} \\
&\in \left(\left(\sum_{m=0}^{n} (n-m)! \right), \left(\sum_{m=0}^{n} (n-m)! + 1 \right) \right).
\end{aligned}$$

Thus, $n!e$ is not an integer for all $n \geq 3$.

Finally we show that e is irrational. Suppose for the sake of contradiction that $e \in \mathbb{Q}$. Then we know that $e = \dfrac{a}{b}$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. But then we have

$$\begin{aligned}
e &= \frac{a}{b} = \frac{3a}{3b} \\
\implies (3b)!e &= \frac{(3a)(3b)!}{3b} = (3a)(3b-1)! \in \mathbb{N},
\end{aligned}$$

a contradiction. Thus, $e \in \mathbb{R} \setminus \mathbb{Q}$. □

Ex. II.4.5.3. Prove Prop. II.4.5.4.

Proof. See Prop. II.4.5.4. □

Ex. II.4.5.4. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by setting $f(x) := \exp(-1/x)$ when $x > 0$, and $f(x) := 0$ when $x \leq 0$. Prove that f is infinitely differentiable, and $f^{(k)}(0) = 0$ for every integer $k \geq 0$, but that f is not real analytic at 0.

Proof. First, we induct on k to show that

$$\forall k \in \mathbb{N}, \forall x \in \mathbb{R}^+, f^{(k)}(x) = P_k(x^{-1}) \exp(-x^{-1}) \text{ where } P_k(x) \text{ is some polynomial.}$$

For $k = 0$, we have

$$\forall x \in \mathbb{R}^+, f^{(0)}(x) = f(x) = \exp(-x^{-1}) = (x^{-1})^0 \exp(-x^{-1}).$$

Thus, the base case holds. Suppose inductively that

$$\forall x \in \mathbb{R}^+, f^{(k)}(x) = P_k(x^{-1}) \exp(-x^{-1}) \text{ where } P_k(x) \text{ is some polynomial.}$$

for some $k \geq 0$. Then for $k + 1$, we have

$$\begin{aligned}
\forall x \in \mathbb{R}^+, & f^{(k+1)}(x) \\
&= (f^{(k)})'(x) \\
&= \left(y \mapsto P_k(y^{-1}) \exp(-x^{-1})\right)'(x) \\
&= P_k'(x^{-1}) \exp(-x^{-1}) + P_k(x^{-1}) \exp(-x^{-1}) x^{-2} \\
&= \left(P_k'(x^{-1}) + P_k(x^{-1}) x^{-2}\right) \exp(-x^{-1})
\end{aligned}$$

and this closes the induction. Thus, f is infinitely differentiable on $\mathbb{R}^+$.

Since $f(x) = 0$ for all $x \in \mathbb{R}^-$, we know that f infinitely differentiable on $\in \mathbb{R}^-$. So we only left to show that f is infinitely differentiable at 0. Again, we induct on k. For $k = 0$, we have $f^{(0)}(0) = f(0) = 0$. Thus, the base case holds. Suppose inductively that $f^{(k)}(0) = 0$ for some $k \geq 0$. Then for $k + 1$, we have

$$\begin{aligned}
& \lim_{x \to 0; x \in \mathbb{R}^+} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} \\
&= \lim_{x \to 0; x \in \mathbb{R}^+} \frac{P_k(x^{-1}) \exp(-x^{-1}) - 0}{x} && \text{(by the induction hypothesis)} \\
&= \lim_{x \to 0; x \in \mathbb{R}^+} \frac{P_k(x^{-1}) x^{-1}}{\exp(x^{-1})} && \text{(by Thm. II.4.5.2(e))} \\
&= \lim_{x \to \infty; x \in \mathbb{R}^+} \frac{P_k(x) x}{\exp(x)} \\
&= 0 && \text{(by Ex. II.4.5.8)} \\
&= \lim_{x \to 0; x \in \mathbb{R}^-} \frac{0 - 0}{x - 0} \\
&= \lim_{x \to 0; x \in \mathbb{R}^-} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0}
\end{aligned}$$

and thus $f^{(k+1)}(0) = 0$. This closes the induction.

Finally we show that f is not real analytic at 0. Suppose for the sake of contradiction that f is real analytic at 0. Then by Cor. II.4.2.10 there exists an $r \in \mathbb{R}^*$ such that

$$\forall x \in (-r, r), f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

But from the proof above we have

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(\frac{r}{2})^n = 0 \neq f(\frac{r}{2}) = \exp(\frac{-r}{2}) > 0.$$

a contradiction. Thus, f is not real analytic at 0. $\square$

Ex. II.4.5.5. Prove Thm. II.4.5.6.

Proof. See Thm. II.4.5.6. $\square$

Ex. II.4.5.6. Prove that the natural logarithm function is real analytic on $(0, +\infty)$.

Proof. Let $a \in \mathbb{R}^+$. By Thm. II.4.5.6(e) we know that ln is real analytic at 1, thus

$$\begin{aligned}
&\forall x \in (0, 2a), \frac{x}{a} \in (0, 2) \\
\implies &\forall x \in (0, 2a), \ln(\frac{x}{a}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{x}{a} - 1\right)^n && \text{(by Thm. II.4.5.6(e))} \\
\implies &\forall x \in (0, 2a), \ln(x) - \ln(a) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{x}{a} - 1\right)^n && \text{(by Thm. II.4.5.6(b)(c))} \\
\implies &\forall x \in (0, 2a), \ln(x) = \ln(a) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(\frac{x}{a} - 1\right)^n \\
&= \ln(a)(x-a)^0 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{na^n}(x-a)^n \\
\implies &\ln \text{ is real analytic at } a \text{ with radius of convergence } a. && \text{(by Def. II.4.2.1)}
\end{aligned}$$

Since a was arbitrary, we conclude that ln is real analytic on $\mathbb{R}^+$. $\square$

Ex. II.4.5.7. Let $f : \mathbb{R} \to (0, \infty)$ be a positive, real analytic function such that $f'(x) = f(x)$ for all $x \in \mathbb{R}$. Show that $f(x) = Ce^x$ for some positive constant C; justify your reasoning.

Proof. Since f is real analytic on $\mathbb{R}$, we have

$$\begin{aligned}
\forall x \in \mathbb{R}, f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n && \text{(by Cor. II.4.2.10)} \\
&= \sum_{n=0}^{\infty} \frac{f(0)}{n!}x^n && \text{(by hypothesis)} \\
&= f(0)\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \\
&= f(0)e^x. && \text{(by Def. II.4.5.1)}
\end{aligned}$$

$\square$

Ex. II.4.5.8. Let $m > 0$ be an integer. Show that

$$\lim_{x\to+\infty} \frac{e^x}{x^m} = +\infty.$$

Proof. Let $m \in \mathbb{Z}^+$. Observe that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n+1} = 0 \\
\implies &\lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 \\
\implies &\lim_{n\to\infty} \frac{n}{n+1} = 1 \\
\implies &\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^m = 1^m = 1 \\
\implies &\lim_{n\to\infty} e\left(\frac{n}{n+1}\right)^m = e \\
\implies &\lim_{n\to\infty} \frac{e^{n+1}}{e^n}\left(\frac{n}{n+1}\right)^m = e
\end{aligned}$$

and

$$e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \sum_{n=2}^{\infty} \frac{1}{n!} > 2.$$

We know that

$$\begin{aligned}
&\exists N \in \mathbb{Z}^+ : \forall n \geq N, \left|\frac{e^{n+1}}{e^n}\left(\frac{n}{n+1}\right)^m - e\right| < \frac{e}{2} && (e > 2) \\
\implies &\exists N \in \mathbb{Z}^+ : \forall n \geq N, \left|\frac{e^{n+1}}{(n+1)^m} - \frac{e^{n+1}}{n^m}\right| < \frac{e^{n+1}}{2n^m} \\
\implies &\exists N \in \mathbb{Z}^+ : \forall n \geq N, \frac{e^{n+1}}{2n^m} < \frac{e^{n+1}}{(n+1)^m} < \frac{3e^{n+1}}{2n^m} \\
\implies &\exists N \in \mathbb{Z}^+ : \forall n \geq N, \frac{e^n}{n^m} < \frac{e^{n+1}}{2n^m} < \frac{e^{n+1}}{(n+1)^m} && (e > 2) \\
\implies &\exists N \in \mathbb{Z}^+ : (\frac{e^n}{n^m})_{n=N}^{\infty} \text{ is monotone increasing sequence.}
\end{aligned}$$

Fix such N. Now we show that $\sup(\frac{e^n}{n^m})_{n=N}^{\infty} = +\infty$. Since

$$\begin{aligned}
&\forall n \geq N, \frac{e^{n+1}}{(n+1)^m} > \frac{e^{n+1}}{2n^m} \\
\implies &\forall n \geq N, \frac{e^{n+1}}{(n+1)^m} > \left(\frac{e}{2} - 1 + 1\right)\frac{e^n}{n^m}
\end{aligned}$$

$$\implies \forall n \geq N, \frac{e^{n+1}}{(n+1)^m} - \frac{e^n}{n^m} > \left(\frac{e}{2} - 1\right)\frac{e^n}{n^m}$$
$$\implies \forall n \geq N, \frac{e^{n+1}}{(n+1)^m} - \frac{e^n}{n^m} > \left(\frac{e}{2} - 1\right)\frac{e^N}{N^m} > 0, \qquad \text{(monotone increasing)}$$

we know that

$$\begin{aligned} \forall n \geq N, &\frac{e^{n+1}}{(n+1)^m} - \frac{e^N}{N^m} \\ &= \sum_{j=N}^{n}\left(\frac{e^{j+1}}{(j+1)^m} - \frac{e^j}{j^m}\right) && \text{(telescope series)} \\ &> \sum_{j=N}^{n}\left(\left(\frac{e}{2} - 1\right)\frac{e^N}{N^m}\right) && \text{(from the proof above)} \\ &= (n-N)\left(\left(\frac{e}{2} - 1\right)\frac{e^N}{N^m}\right) \end{aligned}$$

and

$$\begin{aligned} &\forall n \geq N, \frac{e^{n+1}}{(n+1)^m} - \frac{e^N}{N^m} > (n-N)\left(\left(\frac{e}{2} - 1\right)\frac{e^N}{N^m}\right) \\ \implies &\forall n \geq N, \frac{e^{n+1}}{(n+1)^m} > (n-N)\left(\left(\frac{e}{2} - 1\right)\frac{e^N}{N^m}\right) + \frac{e^N}{N^m} \\ \implies &\forall n \geq N, \frac{e^{n+1}}{(n+1)^m} > (n-N)\left(\left(\frac{e}{2} - 1\right)\frac{e^N}{N^m}\right) + \left(\frac{e}{2} - 1\right)\frac{e^N}{N^m} \qquad (e < 3) \\ \implies &\forall n \geq N, \frac{e^{n+1}}{(n+1)^m} > (n+1-N)\left(\left(\frac{e}{2} - 1\right)\frac{e^N}{N^m}\right) \end{aligned}$$

Since $\dfrac{e^N}{N^m} > 0$, by Archimedean property we know that

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists K \in \mathbb{Z}^+ : K\left(\left(\frac{e}{2} - 1\right)\frac{e^N}{N^m}\right) > \varepsilon \\ \implies &\forall \varepsilon \in \mathbb{R}^+, \exists K \in \mathbb{Z}^+ : \frac{e^{N+K}}{(N+K)^m} > \varepsilon \\ \implies &\forall \varepsilon \in \mathbb{R}^+, \exists K \in \mathbb{Z}^+ : \frac{e^K}{K^m} > \varepsilon \\ \implies &\forall \varepsilon \in \mathbb{R}^+, \exists K \in \mathbb{Z}^+ : \forall n \geq K, \frac{e^n}{n^m} > \varepsilon \qquad \text{(monotone increasing)} \end{aligned}$$

$$\implies \lim_{n\to\infty} \frac{e^n}{n^m} = +\infty.$$

Since m was arbitrary, we conclude that

$$\forall m \in \mathbb{Z}^+, \lim_{n\to\infty} \frac{e^n}{n^m} = +\infty.$$

□

Ex. II.4.5.9. Let $P(x)$ be a polynomial, and let $c > 0$. Show that there exists a real number $N > 0$ such that $e^{cx} > |P(x)|$ for all $x > N$; thus an exponentially growing function, no matter how small the growth rate c, will eventually overtake any given polynomial $P(x)$, no matter how large.

Proof. By Def. II.3.8.1 we know that

$$\forall x \in \mathbb{R}, P(x) = \sum_{i=0}^{m} a_i x^i$$

for some $m \in \mathbb{N}$. Fix such m. Since m is finite, $a = \max_{i=0}^{m} |a_i|$ is well-defined. Let $b = \lceil a \rceil + 1 \in \mathbb{Z}^+$. By Ex. II.4.5.8 we have

$$\begin{aligned}
&\forall 0 \le i \le m, \exists N_i \in \mathbb{Z}^+ : \forall n \ge N_i, \frac{e^n}{n^i} > 1 \\
\implies &\forall 0 \le i \le m, \exists N_i \in \mathbb{Z}^+ : \forall n \ge N_i, \frac{e^{bn}}{(bn)^i} > 1 \\
\implies &\forall 0 \le i \le m, \exists N_i \in \mathbb{Z}^+ : \forall n \ge N_i, e^{bn} > (bn)^i.
\end{aligned}$$

Let $K = \max_{i=0}^{m}(N_i) + 1$. Then we have

$$\begin{aligned}
&\forall n \ge K, e^{bn} > (bn)^m \\
\implies &\forall x \in (K, +\infty), e^{bx} > (bx)^m > (ax)^m \\
\implies &\forall x \in (K, +\infty), \\
&e^{b(m+1)x} > \big(b(m+1)x\big)^m > \big(a(m+1)x\big)^m \qquad \text{(by Archimedean property)} \\
\implies &\forall x \in (K, +\infty), \\
&e^{b(m+1)x} > \big(a(m+1)x\big)^m = \sum_{i=0}^{m} (ax)^m \ge |P(x)|.
\end{aligned}$$

By setting $N = \dfrac{bK(m+1)}{c}$ we are done. □

Ex. II.4.5.10. Let $f : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ be the exponential function $f(x, y) := x^y$. Show that f is continuous.

Proof. Let $d = d_{l^1}|_{(\mathbb{R}^+\times\mathbb{R})\times(\mathbb{R}^+\times\mathbb{R})}$. We have

$$\begin{aligned}\forall (x,y)\in\mathbb{R}^+\times\mathbb{R}, f(x,y) &= x^y\\ &= \exp\big(\ln(x^y)\big) && \text{(by Def. II.4.5.5)}\\ &= \exp(y\ln(x)). && \text{(by Thm. II.4.5.6(d))}\end{aligned}$$

Let $(x_n,y_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^+\times\mathbb{R}$ such that

$$\lim_{n\to\infty} d\big((x_n,y_n),(x,y)\big) = 0.$$

By Proposition 9.8.3 in Analysis I and Def. II.4.5.5 we know that ln is continuous on $\mathbb{R}^+$. Thus, we have

$$\begin{aligned}& \lim_{n\to\infty}\ln(x_n) = \ln(x) && \text{(by Thm. II.2.1.4(a)(b))}\\ \Longrightarrow\ & \lim_{n\to\infty} y_n\ln(x_n) = y\ln(x) && \text{(by Lem. II.2.2.2)}\\ \Longrightarrow\ & \lim_{n\to\infty}\exp\big(y_n\ln(x_n)\big) = \exp\big(y\ln(x)\big). && \text{(by Thm. II.4.5.2(c))}\end{aligned}$$

Since $(x_n,y_n)_{n=1}^{\infty}$ was arbitrary, by Thm. II.2.1.4(a)(b) we conclude that f is continuous at (x,y) from $(\mathbb{R}^+\times\mathbb{R},d)$ to $(\mathbb{R},d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Since x,y were arbitrary, we conclude that f is continuous on $\mathbb{R}^+\times\mathbb{R}$ from $(\mathbb{R}^+\times\mathbb{R},d)$ to $(\mathbb{R},d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. □

II.4.6 A digression on complex numbers

Def. II.4.6.2 (Formal definition of complex numbers)**.** A *complex number* is any pair of the form (a,b), where a,b are real numbers. Two complex numbers (a,b), (c,d) are said to be equal iff $a=c$ and $b=d$. The set of all complex numbers is denoted $\mathbb{C}$.

A.Cor. II.4.6.1. Def. II.4.6.2 is reflexive, symmetry and transitive.

Proof. Let $(a_1,b_1),(a_2,b_2),(a_3,b_3)$ be complex numbers. By Def. II.4.6.2 we know that $a_1,a_2,a_3,b_1,b_2,b_3\in\mathbb{R}$. Since

$$\begin{aligned}& (a_1=a_1)\wedge(b_1=b_1) && (a_1,b_1\in\mathbb{R})\\ \Longrightarrow\ & (a_1,b_1)=(a_1,b_1), && \text{(by Def. II.4.6.2)}\end{aligned}$$

we know that Def. II.4.6.2 is reflexive. Since

$$\begin{aligned}& (a_1,b_1)=(a_2,b_2)\\ \Longleftrightarrow\ & (a_1=a_2)\wedge(b_1=b_2) && \text{(by Def. II.4.6.2)}\\ \Longleftrightarrow\ & (a_2=a_1)\wedge(b_2=b_1) && (a_1,a_2,b_1,b_2\in\mathbb{R})\\ \Longleftrightarrow\ & (a_2,b_2)=(a_1,b_1), && \text{(by Def. II.4.6.2)}\end{aligned}$$

we know that Def. II.4.6.2 is symmetry. Since

$$
\begin{aligned}
&\big((a_1, b_1) = (a_2, b_2)\big) \wedge \big((a_2, b_2) = (a_3, b_3)\big) \\
\iff &(a_1 = a_2) \wedge (b_1 = b_2) \wedge (a_2 = a_3) \wedge (b_2 = b_3) && \text{(by Def. II.4.6.2)} \\
\iff &(a_1 = a_3) \wedge (b_1 = b_3) && (a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}) \\
\iff &(a_1, b_1) = (a_3, b_3), && \text{(by Def. II.4.6.2)}
\end{aligned}
$$

we know that Def. II.4.6.2 is transitive. □

Note. At this stage the complex numbers $\mathbb{C}$ are indistinguishable from the Cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (also known as the *Cartesian plane*). However, we will introduce a number of operations on the complex numbers, notably that of *complex multiplication*, which are not normally placed on the Cartesian plane $\mathbb{R}^2$. Thus, one can think of the complex number system $\mathbb{C}$ as the Cartesian plane $\mathbb{R}^2$ equipped with a number of additional structures.

Def. II.4.6.3 (Complex addition, negation, and zero)**.** If $z = (a, b)$ and $w = (c, d)$ are two complex numbers, we define their *sum* $z + w$ to be the complex number $z + w := (a + c, b + d)$. We also define the *negation* $-z$ of z to be the complex number $-z := (-a, -b)$. We also define the *complex zero* $0_{\mathbb{C}}$ to be the complex number $0_{\mathbb{C}} = (0, 0)$.

A.Cor. II.4.6.2. If $w, w', z, z' \in \mathbb{C}$ and $w = w'$ and $z = z'$, then $w + z = w' + z'$ and $-w = -w'$.

Proof. Let $w = (a, b), w' = (a', b'), z = (c, d), z' = (c', d')$. By Def. II.4.6.2 we know that $a, a', b, b', c, c', d, d' \in \mathbb{R}$. By A.Cor. II.4.6.1 we know that

$$
\begin{aligned}
a &= a'; \\
b &= b'; \\
c &= c'; \\
d &= d'.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
w + z &= (a + c, b + d) && \text{(by Def. II.4.6.3)} \\
&= (a' + c', b' + d') && (a, a', b, b', c, c', d, d' \in \mathbb{R}) \\
&= w' + z' && \text{(by Def. II.4.6.3)}
\end{aligned}
$$

and

$$
\begin{aligned}
-w &= (-a, -b) && \text{(by Def. II.4.6.3)} \\
&= (-a', -b') && (a, a', b, b' \in \mathbb{R}) \\
&= -w'. && \text{(by Def. II.4.6.3)}
\end{aligned}
$$

□

Lem. II.4.6.4 (The complex numbers are an additive group)**.** If z_1, z_2, z_3 are complex numbers, then we have the commutative property $z_1 + z_2 = z_2 + z_1$, the associative property $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, the identity property $z_1 + 0_{\mathbb{C}} = 0_{\mathbb{C}} + z_1 = z_1$, and the inverse property $z_1 + (-z_1) = (-z_1) + z_1 = 0_{\mathbb{C}}$.

Proof. Let $z_1 = (a, b), z_2 = (c, d), z_3 = (e, f)$. By Def. II.4.6.2 we know that $a, b, c, d, e, f \in \mathbb{R}$. Since

$$\begin{aligned} z_1 + z_2 &= (a + c, b + d) && \text{(by Def. II.4.6.3)} \\ &= (c + a, d + b) && (a, b, c, d \in \mathbb{R}) \\ &= z_2 + z_1, && \text{(by Def. II.4.6.3)} \end{aligned}$$

we know that the addition operation in Def. II.4.6.3 is commutative. Since

$$\begin{aligned} (z_1 + z_2) + z_3 &= (a + c, b + d) + z_3 && \text{(by Def. II.4.6.3)} \\ &= \big((a + c) + e, (b + d) + f\big) && \text{(by Def. II.4.6.3)} \\ &= \big(a + (c + e), b + (d + f)\big) && (a, b, c, d, e, f \in \mathbb{R}) \\ &= z_1 + (c + e, d + f) && \text{(by Def. II.4.6.3)} \\ &= z_1 + (z_2 + z_3), && \text{(by Def. II.4.6.3)} \end{aligned}$$

we know that the addition operation in Def. II.4.6.3 is associative. Since

$$\begin{aligned} 0_{\mathbb{C}} + z_1 &= z_1 + 0_{\mathbb{C}} && \text{(from the proof above)} \\ &= (a + 0, b + 0) && \text{(by Def. II.4.6.3)} \\ &= (a, b) && (a, b, 0 \in \mathbb{R}) \\ &= z_1 \end{aligned}$$

and

$$\begin{aligned} (-z_1) + z_1 &= z_1 + (-z_1) && \text{(from the proof above)} \\ &= \big(a + (-a), b + (-b)\big) && \text{(by Def. II.4.6.3)} \\ &= (0, 0) && (a, b \in \mathbb{R}) \\ &= 0_{\mathbb{C}}, && \text{(by Def. II.4.6.3)} \end{aligned}$$

we know that $0_{\mathbb{C}}$ is the additive identity in $\mathbb{C}$. □

Def. II.4.6.5 (Complex multiplication)**.** If $z = (a, b)$ and $w = (c, d)$ are complex numbers, then we define their *product* zw to be the complex number $zw := (ac - bd, ad + bc)$. We also introduce the *complex identity* $1_{\mathbb{C}} := (1, 0)$.

A.Cor. II.4.6.3. If $w, w', z, z' \in \mathbb{C}$ and $w = w'$ and $z = z'$, then $wz = w'z'$.

Proof. Let $w = (a, b), w' = (a', b'), z = (c, d), z' = (c', d')$. By Def. II.4.6.2 we know that $a, a', b, b', c, c', d, d' \in \mathbb{R}$. By A.Cor. II.4.6.1 we know that

$$\begin{aligned} a &= a'; \\ b &= b'; \\ c &= c'; \\ d &= d'. \end{aligned}$$

Then we have

$$\begin{aligned} wz &= (ac - bd, ad + bc) && \text{(by Def. II.4.6.5)} \\ &= (a'c' - b'd', a'd' + b'c') && (a, a', b, b', c, c', d, d' \in \mathbb{R}) \\ &= w'z'. && \text{(by Def. II.4.6.5)} \end{aligned}$$

□

Lem. II.4.6.6. If z_1, z_2, z_3 are complex numbers, then we have the commutative property $z_1z_2 = z_2z_1$, the associative property $(z_1z_2)z_3 = z_1(z_2z_3)$, the identity property $z_1 1_{\mathbb{C}} = 1_{\mathbb{C}} z_1 = z_1$, and the distributivity properties $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$ and $(z_2 + z_3)z_1 = z_2z_1 + z_3z_1$.

Proof. Let $z_1 = (a, b), z_2 = (c, d), z_3 = (e, f)$. By Def. II.4.6.2 we know that $a, b, c, d, e, f \in \mathbb{R}$. Since

$$\begin{aligned} z_1z_2 &= (ac - bd, ad + bc) && \text{(by Def. II.4.6.5)} \\ &= (ca - db, da + cb) && (a, b, c, d \in \mathbb{R}) \\ &= z_2z_1, && \text{(by Def. II.4.6.5)} \end{aligned}$$

we know that the multiplication operation in Def. II.4.6.5 is commutative. Since

$$\begin{aligned} &(z_1z_2)z_3 \\ &= (ac - bd, ad + bc)z_3 && \text{(by Def. II.4.6.5)} \\ &= \big((ac - bd)e - (ad + bc)f, (ac - bd)f + (ad + bc)e\big) && \text{(by Def. II.4.6.5)} \\ &= \big(a(ce - df) - b(cf + de), a(cf + de) + b(ce - df)\big) && (a, b, c, d, e, f \in \mathbb{R}) \\ &= z_1(ce - df, cf + de) && \text{(by Def. II.4.6.5)} \\ &= z_1(z_2z_3), && \text{(by Def. II.4.6.5)} \end{aligned}$$

we know that the multiplication operation in Def. II.4.6.5 is associative. Since

$$\begin{aligned} 1_{\mathbb{C}}z_1 &= z_1 1_{\mathbb{C}} && \text{(from the proof above)} \\ &= (a1 - b0, a0 + b1) && \text{(by Def. II.4.6.5)} \\ &= (a, b) && (a, b, 1 \in \mathbb{R}) \end{aligned}$$

$$= z_1,$$

we know that $1_{\mathbb{C}}$ is the multiplicative identity in $\mathbb{C}$. Since

$$\begin{aligned}
z_1(z_2+z_3) &= z_1(c+e, d+f) && \text{(by Def. II.4.6} \\
&= \big(a(c+e)-b(d+f), a(d+f)+b(c+e)\big) && \text{(by Def. II.4.6} \\
&= \big((ac-bd)+(ae-bf), (ad+bc)+(af+be)\big) && (a,b,c,d,e,f \in \mathbb{R}) \\
&= (ac-bd, ad+bc)+(ae-bf, af+be) && \text{(by Def. II.4.6} \\
&= z_1z_2+z_1z_3 && \text{(by Def. II.4.6}
\end{aligned}$$

and

$$\begin{aligned}
(z_2+z_3)z_1 &= z_1(z_2+z_3) && \text{(from the proof above)} \\
&= z_1z_2+z_1z_3 && \text{(from the proof above)} \\
&= z_2z_1+z_3z_1, && \text{(from the proof above)}
\end{aligned}$$

we know that the multiplication operation in Def. II.4.6.5 and the addition operation in Def. II.4.6.3 are distributive. □

Note. Lem. II.4.6.6 can also be stated more succinctly, as the assertion that $\mathbb{C}$ is a commutative ring. As is usual, we now write $z-w$ as shorthand for $z+(-w)$.

Note. We now identify the real numbers $\mathbb{R}$ with a subset of the complex numbers $\mathbb{C}$ by identifying any real number x with the complex number $(x, 0)$, thus $x \equiv (x, 0)$. Note that this identification is consistent with equality (thus $x = y$ iff $(x, 0) = (y, 0)$), with addition ($x_1 + x_2 = x_3$ iff $(x_1, 0) + (x_2, 0) = (x_3, 0)$), with negation ($x = -y$ iff $(x, 0) = -(y, 0)$), and multiplication ($x_1x_2 = x_3$ iff $(x_1, 0)(x_2, 0) = (x_3, 0)$), so we will no longer need to distinguish between "real addition" and "complex addition," and similarly for equality, negation, and multiplication. Note also that $0 \equiv 0_{\mathbb{C}}$ and $1 \equiv 1_{\mathbb{C}}$, so we can now drop the $\mathbb{C}$ subscripts from the zero 0 and the identity 1.

Note. We now define i to be the complex number $i := (0, 1)$.

Lem. II.4.6.7. Every complex number $z \in \mathbb{C}$ can be written as $z = a + bi$ for exactly one pair a, b of real numbers. Also, we have $i^2 = -1$, and $-z = (-1)z$.

Proof. Let $z = (a, b)$. We have

$$\begin{aligned}
a+bi &= (a,0)+(b,0)\times(0,1) \\
&= (a,0)+(0,b) && \text{(by Def. II.4.6.5)} \\
&= (a,b) && \text{(by Def. II.4.6.3)} \\
&= z.
\end{aligned}$$

If $z' = (a', b')$ such that $z = z'$, then we have

$$\begin{aligned}
z' &= (a', b') \\
&= (a', 0) + (0, b') && \text{(by Def. II.4.6.3)} \\
&= (a', 0) + (b', 0) \times (0, 1) && \text{(by Def. II.4.6.5)} \\
&= a' + b'i \\
&= a + bi. && \text{(by A.Cor. II.4.6.1)}
\end{aligned}$$

Now we show that $i \times i = -1$.

$$\begin{aligned}
i \times i &= (0, 1) \times (0, 1) \\
&= (0^2 - 1^2, 0 \times 1 + 1 \times 0) && \text{(by Def. II.4.6.5)} \\
&= (-1, 0) \\
&= -(1, 0) && \text{(by Def. II.4.6.3)} \\
&= -1.
\end{aligned}$$

Finally we show that $-z = (-1)z$.

$$\begin{aligned}
-z &= -(a, b) \\
&= (-a, -b) && \text{(by Def. II.4.6.3)} \\
&= (-1, 0) \times (a, b) && \text{(by Def. II.4.6.5)} \\
&= (-1)z.
\end{aligned}$$

□

Note. Because Lem. II.4.6.7, we will now refer to complex numbers in the more usual notation $a + bi$, and discard the formal notation (a, b) henceforth.

Def. II.4.6.8 (Real and imaginary parts)**.** If z is a complex number with the representation $z = a + bi$ for some real numbers a, b, we shall call a the *real part* of z and denote $\Re(z) := a$, and call b the *imaginary part* of z and denote $\Im(z) := b$. In general $z = \Re(z) + i\Im(z)$. Note that z is real iff $\Im(z) = 0$. We say that z is *imaginary* iff $\Re(z) = 0$. 0 is both real and imaginary. We define the *complex conjugate* $\overline{z}$ of z to be the complex number $\overline{z} := \Re(z) - i\Im(z)$.

A.Cor. II.4.6.4. If $z, z' \in \mathbb{C}$ such that $z = z'$, then $\Re(z) = \Re(z')$, $\Im(z) = \Im(z')$ and $\overline{z} = \overline{z'}$.

Proof. Let $z = a + bi$ and $z' = a' + b'i$. By Lem. II.4.6.7 we know that $a = a'$ and $b = b'$. Thus, by Def. II.4.6.8 we have

$$\begin{aligned}
\Re(z) &= a = a' = \Re(z') \\
\Im(z) &= b = b' = \Im(z') \\
\overline{z} &= a - bi = a' - b'i = \overline{z'}
\end{aligned}$$

□

Lem. II.4.6.9 (Complex conjugation is an involution)**.** Let z, w be complex numbers, then $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{-z} = -\overline{z}$, and $\overline{zw} = \overline{z}\,\overline{w}$. Also $\overline{\overline{z}} = z$. Finally, we have $\overline{z} = \overline{w}$ iff $z = w$, and $\overline{z} = z$ iff z is real.

Proof. First, we show that $\overline{z+w} = \overline{z} + \overline{w}$.

$$\begin{aligned}
\overline{z+w} &= \overline{\Re(z) + i\Im(z) + \Re(w) + i\Im(w)} && \text{(by Def. II.4.6.8)}\\
&= \overline{\big(\Re(z) + \Re(w)\big) + \big(i\Im(z) + i\Im(w)\big)} && \text{(by Lem. II.4.6.4)}\\
&= \overline{\big(\Re(z) + \Re(w)\big) + i\big(\Im(z) + \Im(w)\big)} && \text{(by Lem. II.4.6.6)}\\
&= \big(\Re(z) + \Re(w)\big) - i\big(\Im(z) + \Im(w)\big) && \text{(by Def. II.4.6.8)}\\
&= \big(\Re(z) + \Re(w)\big) + (-1)i\big(\Im(z) + \Im(w)\big) && \text{(by Lem. II.4.6.7)}\\
&= \big(\Re(z) + \Re(w)\big) + \big((-1)i\Im(z) + (-1)i\Im(w)\big) && \text{(by Lem. II.4.6.6)}\\
&= \big(\Re(z) + (-1)i\Im(z)\big) + \big(\Re(w) + (-1)i\Im(w)\big) && \text{(by Lem. II.4.6.4)}\\
&= \big(\Re(z) - i\Im(z)\big) + \big(\Re(w) - i\Im(w)\big) && \text{(by Lem. II.4.6.7)}\\
&= \overline{\Re(z) + i\Im(z)} + \overline{\Re(w) + i\Im(w)} && \text{(by Def. II.4.6.8)}\\
&= \overline{z} + \overline{w}. && \text{(by Def. II.4.6.8)}
\end{aligned}$$

Next we show that $\overline{-z} = -\overline{z}$.

$$\begin{aligned}
\overline{-z} &= \overline{-\Re(z) - i\Im(z)} && \text{(by Def. II.4.6.8)}\\
&= \overline{(-1)\Re(z) + (-1)i\Im(z)} && \text{(by Lem. II.4.6.7)}\\
&= (-1)\Re(z) - (-1)i\Im(z) && \text{(by Def. II.4.6.8)}\\
&= (-1)\big(\Re(z) - i\Im(z)\big) && \text{(by Lem. II.4.6.6)}\\
&= (-1)\overline{\Re(z) + i\Im(z)} && \text{(by Def. II.4.6.8)}\\
&= (-1)\overline{z} && \text{(by Def. II.4.6.8)}\\
&= -\overline{z}. && \text{(by Lem. II.4.6.7)}
\end{aligned}$$

Next we show that $\overline{zw} = \overline{z}\,\overline{w}$.

$$\begin{aligned}
\overline{zw} &= \overline{\big(\Re(z) + i\Im(z)\big) \times \big(\Re(w) + i\Im(w)\big)} && \text{(by Def. II.4.6.8)}\\
&= \overline{\Re(z)\Re(w) - \Im(z)\Im(w) + i\big(\Re(z)\Im(w) + \Im(z)\Re(w)\big)} && \text{(by Def. II.4.6.5)}\\
&= \Re(z)\Re(w) - \Im(z)\Im(w) - i\big(\Re(z)\Im(w) + \Im(z)\Re(w)\big) && \text{(by Def. II.4.6.8)}\\
&= \big(\Re(z) - i\Im(z)\big) \times \big(\Re(w) - i\Im(w)\big) && \text{(by Def. II.4.6.5)}\\
&= \overline{\Re(z) + i\Im(z)}\;\overline{\Re(w) + i\Im(w)} && \text{(by Def. II.4.6.8)}\\
&= \overline{z}\,\overline{w}. && \text{(by Def. II.4.6.5)}
\end{aligned}$$

Next we show that $\overline{\overline{z}} = z$.

$$\overline{\overline{z}} = \overline{\overline{\Re(z) + i\Im(z)}} \qquad \text{(by Def. II.4.6.8)}$$

$$
\begin{aligned}
&= \overline{\Re(z) - i\Im(z)} && \text{(by Def. II.4.6.8)}\\
&= \overline{\Re(z) + (-1)i\Im(z)} && \text{(by Lem. II.4.6.7)}\\
&= \overline{\Re(z) + i(-1)\Im(z)} && \text{(by Lem. II.4.6.6)}\\
&= \Re(z) - i(-1)\Im(z) && \text{(by Def. II.4.6.8)}\\
&= \Re(z) + (-1)i(-1)\Im(z) && \text{(by Lem. II.4.6.7)}\\
&= \Re(z) + (-1)(-1)i\Im(z) && \text{(by Lem. II.4.6.7)}\\
&= \Re(z) + i\Im(z) && \text{(by Def. II.4.6.5)}\\
&= z. && \text{(by Def. II.4.6.8)}
\end{aligned}
$$

Next we show that $\overline{z} = \overline{w} \iff z = w$.

$$
\begin{aligned}
&\overline{z} = \overline{w}\\
\implies &\overline{\overline{z}} = \overline{\overline{w}} && \text{(by A.Cor. II.4.6.4)}\\
\implies &z = w && \text{(from the proof above)}\\
\implies &\overline{z} = \overline{w}. && \text{(by A.Cor. II.4.6.4)}
\end{aligned}
$$

Finally we show that $\overline{z} = z \iff \Im(z) = 0$.

$$
\begin{aligned}
&\overline{z} = z\\
\iff &\overline{\Re(z) + i\Im(z)} = \Re(z) + i\Im(z) && \text{(by Def. II.4.6.8)}\\
\iff &\Re(z) - i\Im(z) = \Re(z) + i\Im(z) && \text{(by Def. II.4.6.8)}\\
\iff &\Re(z) + (-1)i\Im(z) = \Re(z) + i\Im(z) && \text{(by Lem. II.4.6.7)}\\
\iff &\Re(z) + i(-1)\Im(z) = \Re(z) + i\Im(z) && \text{(by Lem. II.4.6.6)}\\
\iff &\big(\Re(z) = \Re(z)\big) \wedge \big((-1)\Im(z) = \Im(z)\big) && \text{(by Def. II.4.6.2)}\\
\iff &(-1)\Im(z) = \Im(z)\\
\iff &\Im(z) = 0. && (\Im(z) \in \mathbb{R})
\end{aligned}
$$

□

Note. We cannot extend the definition of absolute value directly to the complex numbers, as most complex numbers are neither positive nor negative. (For instance, we do not classify i as either a positive or negative number)

Def. II.4.6.10 (Complex absolute value)**.** If $z = a + bi$ is a complex number, we define the *absolute value* $|z|$ of z to be the real number $|z| := \sqrt{a^2 + b^2} = (a^2 + b^2)^{1/2}$.

Note. From Exercise 5.6.3 in Analysis I we see that Def. II.4.6.10 generalizes the notion of real absolute value.

Lem. II.4.6.11 (Properties of complex absolute value)**.** Let z, w be complex numbers. Then $|z|$ is a non-negative real number, and $|z| = 0$ iff $z = 0$. Also we have the identity $z\overline{z} = |z|^2$, and so $|z| = \sqrt{z\overline{z}}$. As a consequence we have $|zw| = |z||w|$ and $|\overline{z}| = |z|$. Finally, we have the inequalities

$$-|z| \le \Re(z) \le |z|; \quad -|z| \le \Im(z) \le |z|; \quad |z| \le |\Re(z)| + |\Im(z)|$$

as well as the triangle inequality $|z + w| \le |z| + |w|$.

Proof. We have

$$\begin{aligned}
|z| &= |\Re(z) + i\Im(z)| && \text{(by Def. II.4.6.8)} \\
&= \sqrt{\big(\Re(z)\big)^2 + \big(\Im(z)\big)^2} && \text{(by Def. II.4.6.10)} \\
&\ge 0 && (\Re(z), \Im(z) \in \mathbb{R})
\end{aligned}$$

and

$$\begin{aligned}
& |z| = 0 \\
\iff & |\Re(z) + i\Im(z)| = 0 && \text{(by Def. II.4.6.8)} \\
\iff & \sqrt{\big(\Re(z)\big)^2 + \big(\Im(z)\big)^2} = 0 && \text{(by Def. II.4.6.10)} \\
\iff & \big(\Re(z) = 0\big) \wedge \big(\Im(z) = 0\big) && (\Re(z), \Im(z) \in \mathbb{R}) \\
\iff & z = 0. && \text{(by Def. II.4.6.2)}
\end{aligned}$$

Since

$$\begin{aligned}
z\overline{z} &= \big(\Re(z) + i\Im(z)\big) \times \big(\overline{\Re(z) + i\Im(z)}\big) && \text{(by Def. II.4.} \\
&= \big(\Re(z) + i\Im(z)\big) \times \big(\Re(z) - i\Im(z)\big) && \text{(by Def. II.4.} \\
&= \big(\Re(z) + i\Im(z)\big) \times \big(\Re(z) + (-1)i\Im(z)\big) && \text{(by Lem. II.4} \\
&= \big(\Re(z) + i\Im(z)\big) \times \big(\Re(z) + i(-1)\Im(z)\big) && \text{(by Lem. II.4} \\
&= \big(\Re(z)\big)^2 - (-1)\big(\Im(z)\big)^2 + i\big((-1)\Re(z)\Im(z) + \Re(z)\Im(z)\big) && \text{(by Def. II.4.} \\
&= \big(\Re(z)\big)^2 + \big(\Im(z)\big)^2 && (\Re(z), \Im(z) \in \mathbb{R}) \\
&= \left(\sqrt{\big(\Re(z)\big)^2 + \big(\Im(z)\big)^2}\right)^2 && \big(\Re(z)\big)^2 + \big(\Im(z)\big)^2 \\
&= |z|^2, && \text{(by Def. II.4.}
\end{aligned}$$

we know that $|z| = \sqrt{|z|^2} = \sqrt{z\overline{z}}$. Thus

$$\begin{aligned}
|z||w| &= \sqrt{z\overline{z}}\sqrt{w\overline{w}} && \text{(from the proof above)} \\
&= \sqrt{z\overline{z}w\overline{w}} && (z\overline{z}, w\overline{w} \in \mathbb{R})
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{zw\overline{z}\ \overline{w}} && \text{(by Lem. II.4.6.6)} \\
&= \sqrt{zw\overline{zw}} && \text{(by Lem. II.4.6.9)} \\
&= |zw| && \text{(from the proof above)}
\end{aligned}
$$

and

$$
\begin{aligned}
|\overline{z}| &= \sqrt{\overline{z}\ \overline{\overline{z}}} && \text{(from the proof above)} \\
&= \sqrt{\overline{z}z} && \text{(by Lem. II.4.6.9)} \\
&= \sqrt{z\overline{z}} && \text{(by Lem. II.4.6.6)} \\
&= |z|. && \text{(from the proof above)}
\end{aligned}
$$

Since

$$
\begin{aligned}
&\begin{cases}
|\Re(z)| = \sqrt{|\Re(z)|^2} \le \sqrt{\big(\Re(z)\big)^2 + \big(\Im(z)\big)^2} \\
|\Im(z)| = \sqrt{|\Im(z)|^2} \le \sqrt{\big(\Re(z)\big)^2 + \big(\Im(z)\big)^2} \\
\big(|\Re(z)| + |\Im(z)|\big)^2 \ge \big(\Re(z)\big)^2 + \big(\Im(z)\big)^2
\end{cases} && (\Re(z), \Im(z) \in \mathbb{R}) \\
\Longrightarrow &\begin{cases}
|\Re(z)| \le |z| \\
|\Im(z)| \le |z| \\
|\Re(z)| + |\Im(z)| \ge \sqrt{\big(\Re(z)\big)^2 + \big(\Im(z)\big)^2} = |z|
\end{cases} && \text{(by Def. II.4.6.10)} \\
\Longrightarrow &\begin{cases}
-|z| \le \Re(z) \le |z| \\
-|z| \le \Im(z) \le |z| \\
|\Re(z)| + |\Im(z)| \ge |z|
\end{cases}
\end{aligned}
$$

we know that

$$
\begin{aligned}
\Re(z\overline{w}) &\le |z\overline{w}| && \text{(from the proof above)} \\
&= |z||\overline{w}| && \text{(from the proof above)} \\
&= |z||w|. && \text{(from the proof above)}
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\overline{z\overline{w}} = \overline{z}\ \overline{\overline{w}} = \overline{z}w && \text{(by Lem. II.4.6.9)} \\
\Longrightarrow\ &\Re(z\overline{w}) = \frac{z\overline{w} + \overline{z}w}{2} && \text{(by Ex. II.4.6.5)} \\
\Longrightarrow\ &\frac{z\overline{w} + \overline{z}w}{2} \le |z||w| && \text{(from the proof above)} \\
\Longrightarrow\ &z\overline{w} + \overline{z}w \le 2|z||w| \\
\Longrightarrow\ &|z|^2 + z\overline{w} + \overline{z}w + |w|^2 \le |z|^2 + 2|z||w| + |w|^2
\end{aligned}
$$

$$
\begin{aligned}
&\implies |z|^2 + z\overline{w} + \overline{z}w + |w|^2 \le (|z| + |w|)^2 \\
&\implies z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w} \le (|z| + |w|)^2 && \text{(from the proof above} \\
&\implies (z+w)(\overline{z} + \overline{w}) \le (|z| + |w|)^2 && \text{(by Lem. II.4.6.6)} \\
&\implies (z+w)(\overline{z+w}) \le (|z| + |w|)^2 && \text{(by Lem. II.4.6.9)} \\
&\implies |z+w|^2 \le (|z| + |w|)^2 && \text{(from the proof above} \\
&\implies |z+w| \le |z| + |w|. && (|z+w|, |z|, |w| \in \mathbb{R})
\end{aligned}
$$

□

Def. II.4.6.12 (Complex reciprocal)**.** If z is a non-zero complex number, we define the *reciprocal* z^{-1} of z to be the complex number $z^{-1} := |z|^{-2}\overline{z}$ (note that $|z|^{-2}$ is well-defined as a positive real number because $|z|$ is positive real, thanks to Lem. II.4.6.11). If z is zero, $z = 0$, we leave the reciprocal 0^{-1} undefined.

A.Cor. II.4.6.5. If $z, w \in \mathbb{C}$ such that $z = w \neq 0$, then $z^{-1} = w^{-1}$.

Proof.

$$
\begin{aligned}
& z = w \\
\implies & \overline{z} = \overline{w} && \text{(by A.Cor. II.4.6.4)} \\
\implies & z\overline{z} = w\overline{w} && \text{(by A.Cor. II.4.6.3)} \\
\implies & |z| = |w| && \text{(by Lem. II.4.6.11)} \\
\implies & |z|^{-2} = |w|^{-2} && (z = w \neq 0) \\
\implies & |z|^{-2}\overline{z} = |w|^{-2}\overline{w} && \text{(by A.Cor. II.4.6.3)} \\
\implies & z^{-1} = w^{-1}. && \text{(by Def. II.4.6.12)}
\end{aligned}
$$

□

Note. From the Def. II.4.6.12 and Lem. II.4.6.11, we see that

$$zz^{-1} = z^{-1}z = |z|^{-2}\overline{z}z = |z|^{-2}|z|^2 = 1,$$

and so z^{-1} is indeed the reciprocal of z. We can thus define a notion of quotient z/w for any two complex numbers z, w with $w \neq 0$ in the usual manner by the formula $z/w := zw^{-1}$.

Lem. II.4.6.13. If we define $d(z, w) = |z - w|$, then the complex numbers $\mathbb{C}$ with the distance d form a metric space. If $(z_n)_{n=1}^{\infty}$ is a sequence of complex numbers, and z is another complex number, then we have $\lim_{n\to\infty} z_n = z$ in this metric space iff $\lim_{n\to\infty} \Re(z_n) = \Re(z)$ and $\lim_{n\to\infty} \Im(z_n) = \Im(z)$.

Proof. First, we show that $(\mathbb{C}, d)$ is a metric space. Since

$$\forall z \in \mathbb{C}, d(z, z) = |z - z| = |0| = 0,$$

we know that $(\mathbb{C}, d)$ is a reflexive. Since

$$\begin{aligned}
&\forall w, z \in \mathbb{C}, w \neq z \\
\implies& \big(\Re(w) \neq \Re(z)\big) \vee \big(\Im(w) \neq \Im(z)\big) && \text{(by Def. II.4.6.2)} \\
\implies& \Big(\big(\Re(w) - \Re(z)\big)^2 > 0\Big) \vee \Big(\big(\Im(w) - \Im(z)\big)^2 > 0\Big) \\
\implies& \sqrt{\big(\Re(w) - \Re(z)\big)^2 + \big(\Im(w) - \Im(z)\big)^2} > 0 \\
\implies& d(w, z) = |w - z| > 0, && \text{(by Def. II.4.6.10)}
\end{aligned}$$

we know that $(\mathbb{C}, d)$ is a positive. Since

$$\begin{aligned}
\forall w, z \in \mathbb{C}, d(w, z) &= |w - z| \\
&= |-1||w - z| \\
&= |(-1)(w - z)| && \text{(by Lem. II.4.6.11)} \\
&= |z - w| && \text{(by Lem. II.4.6.4)} \\
&= d(z, w),
\end{aligned}$$

we know that $(\mathbb{C}, d)$ is a symmetry. Since

$$\begin{aligned}
\forall x, y, z \in \mathbb{C}, d(x, z) &= |x - z| \\
&= |x - y + y - z| \\
&\leq |x - y| + |y - z| && \text{(by Lem. II.4.6.11)} \\
&= d(x, y) + d(y, z),
\end{aligned}$$

we know that $(\mathbb{C}, d)$ is a transitive. Combine all proofs above we conclude by Def. II.1.1.2 that $(\mathbb{C}, d)$ is a metric space.

Now we show that

$$d - \lim_{n\to\infty} z_n = z \iff \begin{cases} \lim_{n\to\infty} \Re(z_n) = \Re(z) \\ \lim_{n\to\infty} \Im(z_n) = \Im(z) \end{cases}$$

This is true since

$$\begin{aligned}
&d - \lim_{n\to\infty} z_n = z \\
\implies& \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, d(z_n, z) < \varepsilon \\
\implies& \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, |z_n - z| < \varepsilon
\end{aligned}$$

$$
\begin{aligned}
&\implies \forall\varepsilon\in\mathbb{R}^+,\exists N\in\mathbb{Z}^+:\forall n\geq N, \sqrt{\big(\Re(z_n)-\Re(z)\big)^2+\big(\Im(z_n)-\Im(z)\big)^2}<\varepsilon\\
&\implies \forall\varepsilon\in\mathbb{R}^+,\exists N\in\mathbb{Z}^+:\forall n\geq N, \begin{cases}\sqrt{\big(\Re(z_n)-\Re(z)\big)^2}<\varepsilon\\ \sqrt{\big(\Im(z_n)-\Im(z)\big)^2}<\varepsilon\end{cases}\\
&\implies \forall\varepsilon\in\mathbb{R}^+,\exists N\in\mathbb{Z}^+:\forall n\geq N, \begin{cases}|\Re(z_n)-\Re(z)|<\varepsilon\\ |\Im(z_n)-\Im(z)|<\varepsilon\end{cases}\\
&\implies \begin{cases}\lim\limits_{n\to\infty}\Re(z_n)=\Re(z)\\ \lim\limits_{n\to\infty}\Im(z_n)=\Im(z)\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&\begin{cases}\lim\limits_{n\to\infty}\Re(z_n)=\Re(z)\\ \lim\limits_{n\to\infty}\Im(z_n)=\Im(z)\end{cases}\\
&\implies \forall\varepsilon\in\mathbb{R}^+,\exists N\in\mathbb{Z}^+:\forall n\geq N, \begin{cases}|\Re(z_n)-\Re(z)|<\dfrac{\varepsilon}{2}\\ |\Im(z_n)-\Im(z)|<\dfrac{\varepsilon}{2}\end{cases}\\
&\implies \forall\varepsilon\in\mathbb{R}^+,\exists N\in\mathbb{Z}^+:\forall n\geq N, \begin{cases}\big(\Re(z_n)-\Re(z)\big)^2<\dfrac{\varepsilon^2}{4}\\ \big(\Im(z_n)-\Im(z)\big)^2<\dfrac{\varepsilon^2}{4}\end{cases}\\
&\implies \forall\varepsilon\in\mathbb{R}^+,\exists N\in\mathbb{Z}^+:\forall n\geq N, \big(\Re(z_n)-\Re(z)\big)^2+\big(\Im(z_n)-\Im(z)\big)^2<\frac{\varepsilon^2}{2}\\
&\implies \forall\varepsilon\in\mathbb{R}^+,\exists N\in\mathbb{Z}^+:\forall n\geq N, \sqrt{\big(\Re(z_n)-\Re(z)\big)^2+\big(\Im(z_n)-\Im(z)\big)^2}<\frac{\varepsilon}{\sqrt{2}}<\varepsilon\\
&\implies \forall\varepsilon\in\mathbb{R}^+,\exists N\in\mathbb{Z}^+:\forall n\geq N, \big|\Re(z_n)-\Re(z)+i\big(\Im(z_n)-\Im(z)\big)\big|<\varepsilon\\
&\implies \forall\varepsilon\in\mathbb{R}^+,\exists N\in\mathbb{Z}^+:\forall n\geq N, |z_n-z|<\varepsilon\\
&\implies \forall\varepsilon\in\mathbb{R}^+,\exists N\in\mathbb{Z}^+:\forall n\geq N, d(z_n,z)<\varepsilon\\
&\implies d-\lim_{n\to\infty} z_n=z.
\end{aligned}
$$

□

Lem. II.4.6.14 (Complex limit laws)**.** Let $(z_n)_{n=1}^\infty$ and $(w_n)_{n=1}^\infty$ be convergent sequences of complex numbers, and let c be a complex number. Then the sequences $(z_n+w_n)_{n=1}^\infty$, $(z_n-w_n)_{n=1}^\infty$, $(cz_n)_{n=1}^\infty$, $(z_nw_n)_{n=1}^\infty$, and $(\overline{z_n})_{n=1}^\infty$ are also convergent, with

$$
\begin{aligned}
\lim_{n\to\infty} z_n+w_n &= \lim_{n\to\infty} z_n+\lim_{n\to\infty} w_n\\
\lim_{n\to\infty} z_n-w_n &= \lim_{n\to\infty} z_n-\lim_{n\to\infty} w_n
\end{aligned}
$$

$$\lim_{n\to\infty} cz_n = c \lim_{n\to\infty} z_n$$
$$\lim_{n\to\infty} z_n w_n = \left(\lim_{n\to\infty} z_n\right)\left(\lim_{n\to\infty} w_n\right)$$
$$\lim_{n\to\infty} \overline{z_n} = \overline{\lim_{n\to\infty} z_n}$$

Also, if the w_n are all non-zero and $\lim_{n\to\infty} w_n$ is also non-zero, then $(z_n/w_n)_{n=1}^{\infty}$ is also a convergent sequence, with

$$\lim_{n\to\infty} z_n/w_n = \left(\lim_{n\to\infty} z_n\right)/\left(\lim_{n\to\infty} w_n\right).$$

Proof. Let d be the metric in Def. II.4.6.10. Suppose that $d-\lim_{n\to\infty} w_n = w$ and $d-\lim_{n\to\infty} z_n = z$ for some $z, w \in \mathbb{C}$. By Lem. II.4.6.13 this means

$$\lim_{n\to\infty} \Re(w_n) = \Re(w);$$
$$\lim_{n\to\infty} \Im(w_n) = \Im(w);$$
$$\lim_{n\to\infty} \Re(z_n) = \Re(z);$$
$$\lim_{n\to\infty} \Im(z_n) = \Im(z).$$

Then we have

$$\begin{aligned}
&\Re(z+w) \\
&= \Re(z) + \Re(w) && \text{(by Def. II.4.6.3)} \\
&= \lim_{n\to\infty} \big(\Re(z_n) + \Re(w_n)\big) && \text{(by limit laws)} \\
&= \lim_{n\to\infty} \Re\Big(\Re(z_n) + \Re(w_n) + i\big(\Im(z_n) + \Im(w_n)\big)\Big) && \text{(by Def. II.4.6.8)} \\
&= \lim_{n\to\infty} \Re(z_n + w_n); && \text{(by Def. II.4.6.3)} \\
&\Im(z+w) \\
&= \Im(z) + \Im(w) && \text{(by Def. II.4.6.3)} \\
&= \lim_{n\to\infty} \big(\Im(z_n) + \Im(w_n)\big) && \text{(by limit laws)} \\
&= \lim_{n\to\infty} \Im\Big(\Re(z_n) + \Re(w_n) + i\big(\Im(z_n) + \Im(w_n)\big)\Big) && \text{(by Def. II.4.6.8)} \\
&= \lim_{n\to\infty} \Im(z_n + w_n); && \text{(by Def. II.4.6.3)} \\
&\Re(z-w) \\
&= \Re(z) - \Re(w) && \text{(by Def. II.4.6.3)} \\
&= \lim_{n\to\infty} \big(\Re(z_n) - \Re(w_n)\big) && \text{(by limit laws)} \\
&= \lim_{n\to\infty} \Re\Big(\Re(z_n) - \Re(w_n) + i\big(\Im(z_n) - \Im(w_n)\big)\Big) && \text{(by Def. II.4.6.8)}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Re(z_n - w_n); && \text{(by Def. II.4.6.3)}\\
&\Im(z-w)\\
&= \Im(z) - \Im(w) && \text{(by Def. II.4.6.3)}\\
&= \lim_{n\to\infty} \big(\Im(z_n) - \Im(w_n)\big) && \text{(by limit laws)}\\
&= \lim_{n\to\infty} \Im\Big(\Re(z_n) - \Re(w_n) + i\big(\Im(z_n) - \Im(w_n)\big)\Big) && \text{(by Def. II.4.6.8)}\\
&= \lim_{n\to\infty} \Im(z_n - w_n). && \text{(by Def. II.4.6.3)}
\end{aligned}
$$

Thus, by Lem. II.4.6.13 we have

$$
\begin{aligned}
&\begin{cases}
\Re(z+w) = \lim_{n\to\infty} \Re(z_n + w_n);\\
\Im(z+w) = \lim_{n\to\infty} \Im(z_n + w_n);\\
\Re(z-w) = \lim_{n\to\infty} \Re(z_n - w_n);\\
\Im(z-w) = \lim_{n\to\infty} \Im(z_n - w_n);
\end{cases}\\
\iff &\begin{cases}
d-\lim_{n\to\infty}(z_n + w_n) = z + w = \big(d-\lim_{n\to\infty} z_n\big) + \big(d-\lim_{n\to\infty} w_n\big);\\
d-\lim_{n\to\infty}(z_n - w_n) = z - w = \big(d-\lim_{n\to\infty} z_n\big) - \big(d-\lim_{n\to\infty} w_n\big).
\end{cases}
\end{aligned}
$$

By Def. II.4.6.5 we know that

$$\forall n \in \mathbb{Z}^+, z_n w_n = \Re(z_n)\Re(w_n) - \Im(z_n)\Im(w_n) + i\big(\Re(z_n)\Im(w_n) + \Im(z_n)\Re(w_n)\big)$$

and

$$zw = \Re(z)\Re(w) - \Im(z)\Im(w) + i\big(\Re(z)\Im(w) + \Im(z)\Re(w)\big).$$

Since

$$
\begin{aligned}
&\Re(zw)\\
&= \Re(z)\Re(w) - \Im(z)\Im(w) && \text{(by Def. II.4.6.8)}\\
&= \lim_{n\to\infty} \big(\Re(z_n)\Re(w_n) - \Im(z_n)\Im(w_n)\big) && \text{(by limit laws)}\\
&= \lim_{n\to\infty} \Re(z_n w_n); && \text{(by Def. II.4.6.8)}\\
&\Im(zw)\\
&= \Re(z)\Im(w) + \Im(z)\Re(w) && \text{(by Def. II.4.6.8)}\\
&= \lim_{n\to\infty} \big(\Re(z_n)\Im(w_n) + \Im(z_n)\Re(w_n)\big) && \text{(by limit laws)}\\
&= \lim_{n\to\infty} \Im(z_n w_n), && \text{(by Def. II.4.6.8)}
\end{aligned}
$$

by Lem. II.4.6.13 we have

$$\begin{cases} \Re(zw) = \lim_{n\to\infty} \Re(z_n w_n); \\ \Im(zw) = \lim_{n\to\infty} \Im(z_n w_n); \end{cases}$$
$$\iff d-\lim_{n\to\infty}(z_n w_n) = zw = \big(d-\lim_{n\to\infty} z_n\big)\big(d-\lim_{n\to\infty} w_n\big).$$

Since

$$\forall c \in \mathbb{C}, \forall \varepsilon \in \mathbb{R}^+, \forall n \in \mathbb{Z}^+, d(c,c) = |c-c| = 0 < \varepsilon$$
$$\implies \forall c \in \mathbb{C}, \lim_{n\to\infty} c = c,$$

from the proof above we have

$$\forall c \in \mathbb{C}, d-\lim_{n\to\infty}(cz_n) = cz = c\big(d-\lim_{n\to\infty} z_n\big).$$

By Def. II.4.6.8 we know that

$$\begin{aligned} &\forall n \in \mathbb{Z}^+, z_n = \Re(z_n) + i\Im(z_n); \\ &\forall n \in \mathbb{Z}^+, \overline{z_n} = \Re(z_n) - i\Im(z_n); \\ &z = \Re(z) + i\Im(z); \\ &\overline{z} = \Re(z) - i\Im(z). \end{aligned}$$

Since

$$\begin{aligned} \Re(\overline{z}) &= \Re(z) && \text{(by Def. II.4.6.8)} \\ &= \lim_{n\to\infty} \Re(z_n) \\ &= \lim_{n\to\infty} \Re(\overline{z_n}); && \text{(by Def. II.4.6.8)} \\ \Im(\overline{z}) &= -\Im(z) && \text{(by Def. II.4.6.8)} \\ &= \lim_{n\to\infty} -\Im(z_n) && \text{(by limit laws)} \\ &= \lim_{n\to\infty} \Im(\overline{z_n}), && \text{(by Def. II.4.6.8)} \end{aligned}$$

by Lem. II.4.6.13 we have

$$\begin{cases} \Re(\overline{z}) = \lim_{n\to\infty} \Re(\overline{z_n}); \\ \Im(\overline{z}) = \lim_{n\to\infty} \Im(\overline{z_n}); \end{cases}$$
$$\iff d-\lim_{n\to\infty} \overline{z_n} = \overline{z} = \overline{d-\lim_{n\to\infty} z_n}.$$

Now suppose that $w \neq 0$ and $w_n \neq 0$ for all $n \in \mathbb{Z}^+$. Since

$$\frac{z}{w}$$

$$
\begin{aligned}
&= zw^{-1} \\
&= z|w|^{-2}\overline{w} && \text{(by Def. II.4.6.12)} \\
&= |w|^{-2}z\overline{w} && \text{(by Lem. II.4.6.6)} \\
&= \frac{z\overline{w}}{\big(\Re(w)\big)^2 + \big(\Im(w)\big)^2} && \text{(by Def. II.4.6.10)} \\
&= \frac{\Re(z)\Re(w) + \Im(z)\Im(w) - i\big(\Re(z)\Im(w) - \Im(z)\Re(w)\big)}{\big(\Re(w)\big)^2 + \big(\Im(w)\big)^2} && \text{(by Def. II.4.6.5)}
\end{aligned}
$$

and

$$\forall n \in \mathbb{Z}^+, \frac{z_n}{w_n} = \frac{\Re(z_n)\Re(w_n) + \Im(z_n)\Im(w_n)}{\big(\Re(w_n)\big)^2 + \big(\Im(w_n)\big)^2} - i\frac{\big(\Re(z_n)\Im(w_n) - \Im(z_n)\Re(w_n)\big)}{\big(\Re(w_n)\big)^2 + \big(\Im(w_n)\big)^2},$$

we know that

$$
\begin{aligned}
\Re(\frac{z}{w}) &= \frac{\Re(z)\Re(w) + \Im(z)\Im(w)}{\big(\Re(w)\big)^2 + \big(\Im(w)\big)^2} && \text{(by Def. II.4.6.8)} \\
&= \lim_{n\to\infty} \frac{\Re(z_n)\Re(w_n) + \Im(z_n)\Im(w_n)}{\big(\Re(w_n)\big)^2 + \big(\Im(w_n)\big)^2} && \text{(by limit laws)} \\
&= \lim_{n\to\infty} \Re(\frac{z_n}{w_n}); && \text{(by Def. II.4.6.8)} \\
\Im(\frac{z}{w}) &= \frac{-\Re(z)\Im(w) + \Im(z)\Re(w)}{\big(\Re(w)\big)^2 + \big(\Im(w)\big)^2} && \text{(by Def. II.4.6.8)} \\
&= \lim_{n\to\infty} \frac{-\Re(z_n)\Im(w_n) + \Im(z_n)\Re(w_n)}{\big(\Re(w_n)\big)^2 + \big(\Im(w_n)\big)^2} && \text{(by limit laws)} \\
&= \lim_{n\to\infty} \Im(\frac{z_n}{w_n}). && \text{(by Def. II.4.6.8)}
\end{aligned}
$$

Thus, by Lem. II.4.6.13 we have

$$
\begin{aligned}
&\begin{cases} \Re(\frac{z}{w}) = \lim_{n\to\infty} \Re(\frac{z_n}{w_n}); \\ \Im(\frac{z}{w}) = \lim_{n\to\infty} \Im(\frac{z_n}{w_n}); \end{cases} \\
\iff & d - \lim_{n\to\infty} \frac{z_n}{w_n} = \frac{z}{w} = \frac{d - \lim_{n\to\infty} z_n}{d - \lim_{n\to\infty} w_n}.
\end{aligned}
$$

□

Note. Observe that the real and complex number systems are in fact quite similar; they both obey similar laws of arithmetic, and they have similar structure as metric spaces. Indeed many of the results in this textbook that were proven for real-valued functions, are also valid for complex-valued functions, simply by replacing "real" with "complex" in the proofs but otherwise leaving all the other details of the proof unchanged. Alternatively, one can always split a complex-valued function f into real and imaginary parts $\Re(f)$, $\Im(f)$, thus $f = \Re(f) + i\Im(f)$, and then deduce results for the complex-valued function f from the corresponding results for the real-valued functions $\Re(f)$, $\Im(f)$. For instance, the theory of pointwise and uniform convergence from Ch. II.3, or the theory of power series from this chapter, extends without any difficulty to complex-valued functions. In particular, we can define the complex exponential function in exactly the same manner as for real numbers.

A.Cor. II.4.6.6 (Complex series)**.** Let d be the metric in Def. II.4.6.10 and let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$. If $d-\lim_{N\to\infty}\sum_{n=0}^{N} a_n \in \mathbb{C}$, we define

$$\sum_{n=0}^{\infty} a_n = d - \lim_{N\to\infty}\sum_{n=0}^{N} a_n.$$

Then $\sum_{n=0}^{\infty} a_n$ converges iff both $\sum_{n=0}^{\infty}\Re(a_n)$ and $\sum_{n=0}^{\infty}\Im(a_n)$ converges, and we have

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty}\Re(a_n) + i\left(\sum_{n=0}^{\infty}\Im(a_n)\right).$$

Proof. First, suppose that $\sum_{n=0}^{\infty}\Re(a_n)$ and $\sum_{n=0}^{\infty}\Im(a_n)$ converges. Then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Re(a_n) + i\left(\sum_{n=0}^{\infty}\Im(a_n)\right)\\
&= d-\lim_{N\to\infty}\sum_{n=0}^{N}\Re(a_n) + i\left(d-\lim_{N\to\infty}\sum_{n=0}^{N}\Im(a_n)\right)\\
&= d-\lim_{N\to\infty}\left(\sum_{n=0}^{N}\Re(a_n) + i\left(\sum_{n=0}^{N}\Im(a_n)\right)\right) && \text{(by Lem. II.4.6.14)}\\
&= d-\lim_{N\to\infty}\left(\sum_{n=0}^{N}\Re(a_n) + \sum_{n=0}^{N}\big(i\Im(a_n)\big)\right) && \text{(by Lem. II.4.6.6)}\\
&= d-\lim_{N\to\infty}\sum_{n=0}^{N}\big(\Re(a_n) + i\Im(a_n)\big) && \text{(by Lem. II.4.6.4)}\\
&= d-\lim_{N\to\infty}\sum_{n=0}^{N} a_n && \text{(by Def. II.4.6.8)}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} a_n.$$

Now suppose that $\sum_{n=0}^{\infty} a_n$ converges. Then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n \\
&= \Re\left(\sum_{n=0}^{\infty} a_n\right) + i\Im\left(\sum_{n=0}^{\infty} a_n\right) && \text{(by Def. II.4.6.8)} \\
&= \Re\left(d - \lim_{N\to\infty} \sum_{n=0}^{N} a_n\right) + i\Im\left(d - \lim_{N\to\infty} \sum_{n=0}^{N} a_n\right) \\
&= \left(d - \lim_{N\to\infty} \Re\left(\sum_{n=0}^{N} a_n\right)\right) + i\left(d - \lim_{N\to\infty} \Im\left(\sum_{n=0}^{N} a_n\right)\right) && \text{(by Lem. II.4.6.13)} \\
&= \left(d - \lim_{N\to\infty} \sum_{n=0}^{N} \Re(a_n)\right) + i\left(d - \lim_{N\to\infty} \sum_{n=0}^{N} \Im(a_n)\right) && \text{(by Def. II.4.6.3)} \\
&= \sum_{n=0}^{\infty} \Re(a_n) + i\left(\sum_{n=0}^{\infty} \Im(a_n)\right).
\end{aligned}$$

□

A.Cor. II.4.6.7 (Zero test of complex series)**.** Let d be the metric in Def. II.4.6.10 and let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$. Suppose that $\sum_{n=0}^{\infty} a_n \in \mathbb{C}$. Then $d - \lim_{n\to\infty} a_n = 0$.

Proof. By A.Cor. II.4.6.6 we have

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \Re(a_n) + i\left(\sum_{n=0}^{\infty} \Im(a_n)\right).$$

By Def. II.4.6.8 we know that $\sum_{n=0}^{\infty} \Re(a_n) \in \mathbb{R}$ and $\sum_{n=0}^{\infty} \Im(a_n) \in \mathbb{R}$. By zero test of real series (Corollary 7.2.6 in Analysis I) this means

$$\lim_{n\to\infty} \Re(a_n) = 0;$$
$$\lim_{n\to\infty} \Im(a_n) = 0.$$

Then we have

$$\begin{aligned}
&\begin{cases} \lim_{n\to\infty} \Re(a_n) = 0 = \Re(d - \lim_{n\to\infty} a_n) \\ \lim_{n\to\infty} \Im(a_n) = 0 = \Im(d - \lim_{n\to\infty} a_n) \end{cases} && \text{(by Lem. II.4.6.13)} \\
\implies & d - \lim_{n\to\infty} a_n = \Re(d - \lim_{n\to\infty} a_n) + i\Im(d - \lim_{n\to\infty} a_n) = 0. && \text{(by Def. II.4.6.8)}
\end{aligned}$$

□

A.Cor. II.4.6.8 (Absolutely convergent of complex series)**.** Let d be the metric in Def. II.4.6.10 and let $(a_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{C}$. We say that $\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\sum_{n=0}^{\infty}|a_n| \in \mathbb{R}$. Then we have

$$\sum_{n=0}^{\infty}|a_n| \in \mathbb{R} \implies \sum_{n=0}^{\infty} a_n \in \mathbb{C}.$$

Proof. By Def. II.4.6.10 we know that $|a_n| \in \mathbb{R}$ for all $n \in \mathbb{N}$, thus $\sum_{n=0}^{\infty}|a_n|$ is well-defined. Let $N_1, N_2 \in \mathbb{N}$. Since

$$\sum_{n=0}^{\infty}|a_n| = \lim_{N\to\infty} \sum_{n=0}^{N}|a_n|,$$

we know that the sequence $(\sum_{n=0}^{N}|a_n|)_{N=0}^{\infty}$ is a Cauchy sequence in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Then we have

$$\begin{aligned}
& \forall \varepsilon \in \mathbb{R}^+, \exists M \in \mathbb{N} : \forall N_1, N_2 \geq M, \left| \sum_{n=0}^{N_1}|a_n| - \sum_{n=0}^{N_2}|a_n| \right| < \varepsilon \\
\implies & \forall \varepsilon \in \mathbb{R}^+, \exists M \in \mathbb{N} : \forall N_1, N_2 \geq M, \\
& \begin{cases} \sum_{n=N_1+1}^{N_2} |a_n| = \left| \sum_{n=N_1+1}^{N_2} |a_n| \right| < \varepsilon & \text{if } N_1 \leq N_2 \\ \sum_{n=N_2+1}^{N_1} |a_n| = \left| \sum_{n=N_2+1}^{N_1} |a_n| \right| < \varepsilon & \text{if } N_1 > N_2 \end{cases} \\
\implies & \forall \varepsilon \in \mathbb{R}^+, \exists M \in \mathbb{N} : \forall N_1, N_2 \geq M,
\end{aligned}$$

$$\begin{cases} \left|\sum_{n=N_1+1}^{N_2} a_n\right| \le \sum_{n=N_1+1}^{N_2} |a_n| < \varepsilon & \text{if } N_1 \le N_2 \\ \left|\sum_{n=N_2+1}^{N_1} a_n\right| \le \sum_{n=N_2+1}^{N_1} |a_n| < \varepsilon & \text{if } N_1 > N_2 \end{cases} \qquad \text{(by Lem. II.4.6.11)}$$

$$\implies \forall \varepsilon \in \mathbb{R}^+, \exists M \in \mathbb{N} : \forall N_1, N_2 \ge M, \left|\sum_{n=0}^{N_1} a_n - \sum_{n=0}^{N_2} a_n\right| < \varepsilon.$$

This means $(\sum_{n=0}^{N} a_n)_{N=0}^{\infty}$ is a Cauchy sequence in $(\mathbb{C}, d)$. By Ex. II.4.6.10 we know that $(\mathbb{C}, d)$ is complete, thus $\sum_{n=0}^{\infty} a_n$ converges in $\mathbb{C}$ with respect to d. ☐

A.Cor. II.4.6.9 (Cauchy Product)**.** Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$ be sequences in $\mathbb{C}$. Suppose that $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ are absolutely convergent. Then we have

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} (a_k b_{n-k})\right).$$

Proof. Let d be the metric in Def. II.4.6.10. Let $A = \sum_{n=0}^{\infty} a_n$ and $B = \sum_{n=0}^{\infty} b_n$. By A.Cor. II.4.6.8 we know that A, B are well-defined. We define partial sums

$$\forall N \in \mathbb{N}, A_N = \sum_{n=0}^{N} a_n$$

$$\forall N \in \mathbb{N}, B_N = \sum_{n=0}^{N} b_n$$

$$\forall N \in \mathbb{N}, C_N = \sum_{n=0}^{N} \sum_{k=0}^{n} a_k b_{n-k}$$

By A.Cor. II.4.6.6 we know that $d - \lim_{N\to\infty} (B_N - B) = 0$. Thus, if we define

$$\forall N \in \mathbb{N}, \beta_N = B_N - B,$$

then $d - \lim_{N\to\infty} \beta_N = 0$. Observe that

$$\forall N \in \mathbb{N}, C_N = \sum_{n=0}^{N} \sum_{k=0}^{n} (a_k b_{n-k})$$

$$
\begin{aligned}
&= \sum_{(n,k)\in\mathbb{N}^2 : n+k\le N} (a_n b_k) \\
&= \sum_{n=0}^{N} \sum_{k=0}^{N-n} (a_n b_k) \\
&= \sum_{n=0}^{N} a_n \left(\sum_{k=0}^{N-n} b_k \right) && \text{(by Lem. II.4.6.6)} \\
&= \sum_{n=0}^{N} (a_n B_{N-n}) \\
&= \sum_{n=0}^{N} \big(a_n (B_{N-n} - B + B)\big) && \text{(by Lem. II.4.6.4)} \\
&= \sum_{n=0}^{N} \big(a_n (\beta_{N-n} + B)\big) && \text{(by Lem. II.4.6.6)} \\
&= \sum_{n=0}^{N} (a_n \beta_{N-n} + a_n B) && \text{(by Lem. II.4.6.6)} \\
&= \sum_{n=0}^{N} (a_n \beta_{N-n}) + \sum_{n=0}^{N} (a_n B) && \text{(by Lem. II.4.6.4)} \\
&= \sum_{n=0}^{N} (a_n \beta_{N-n}) + \left(\sum_{n=0}^{N} a_n \right) B && \text{(by Lem. II.4.6.6)} \\
&= \sum_{n=0}^{N} (a_n \beta_{N-n}) + A_N B.
\end{aligned}
$$

By A.Cor. II.4.6.6 we want to show that

$$AB = \lim_{N\to\infty} C_N.$$

Since

$$AB = (\lim_{N\to\infty} A_N)B = \lim_{N\to\infty} (A_N B),$$

if we define

$$\forall N \in \mathbb{N}, \gamma_N = \sum_{n=0}^{N} (a_n \beta_{N-n}),$$

then it suffices to show that

$$\lim_{N\to\infty} \gamma_N = \lim_{N\to\infty} (C_N - A_N B) = 0.$$

Let $\varepsilon \in \mathbb{R}^+$ and let $\alpha = \sum_{n=0}^{\infty} |a_n|$. By hypothesis we know that $\sum_{n=0}^{\infty} a_n$ is absolutely convergent, so α is well-defined. Since $d - \lim_{N\to\infty} \beta_N = 0$, we know that

$$\exists M_1 \in \mathbb{N} : \forall N \geq M_1, |\beta_N - 0| = |\beta_N| < \frac{\varepsilon}{2\alpha}.$$

Fix such M_1. Since $\sum_{n=0}^{\infty} a_n \in \mathbb{C}$, by A.Cor. II.4.6.7 we know that $d - \lim_{N\to\infty} a_N = 0$. Thus, we have

$$\exists M_2 \in \mathbb{N} : \forall N \geq M_2, |a_N - 0| = |a_N| < \frac{\varepsilon}{2M_1(\max_{k=0}^{M_1} |\beta_k|)}.$$

Let $M = M_1 + M_2$ and choose one $N \geq M$. Then we have

$$\begin{aligned}
|\gamma_N| &= \left| \sum_{n=0}^{N} (a_n \beta_{N-n}) \right| = \left| \sum_{n=0}^{N} (a_{N-n} \beta_n) \right| \\
&\leq \sum_{n=0}^{N} |a_{N-n}\beta_n| = \sum_{n=0}^{N} (|a_{N-n}||\beta_n|) && \text{(by Lem. II.4.6.11)} \\
&= \sum_{n=0}^{M_1} (|a_{N-n}||\beta_n|) + \sum_{n=M_1+1}^{N} (|a_{N-n}||\beta_n|). && \text{(by Lem. II.4.6.4)}
\end{aligned}$$

Since

$$\begin{aligned}
& 0 \leq n \leq M_1 \\
\iff & 0 \geq -n \geq -M_1 \\
\iff & N \geq N - n \geq N - M_1 \geq M_1 + M_2 - M_1 = M_2,
\end{aligned}$$

we know that

$$\begin{aligned}
\sum_{n=0}^{M_1} (|a_{N-n}||\beta_n|) &\leq \sum_{n=0}^{M_1} \frac{\varepsilon |\beta_n|}{2M_1(\max_{k=0}^{M_1} |\beta_k|)} \\
&= \frac{\varepsilon}{2M_1} \left(\sum_{n=0}^{M_1} \frac{|\beta_n|}{\max_{k=0}^{M_1} |\beta_k|} \right) \\
&\leq \frac{\varepsilon}{2M_1} \left(\sum_{n=0}^{M_1} \frac{\max_{k=0}^{M_1} |\beta_k|}{\max_{k=0}^{M_1} |\beta_k|} \right) \\
&= \frac{\varepsilon}{2M_1} M_1 \\
&= \frac{\varepsilon}{2}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|\gamma_N| &\leq \sum_{n=0}^{M_1}(|a_{N-n}||\beta_n|) + \sum_{n=M_1+1}^{N}(|a_{N-n}||\beta_n|) \\
&\leq \frac{\varepsilon}{2} + \sum_{n=M_1+1}^{N}(|a_{N-n}||\beta_n|) \\
&\leq \frac{\varepsilon}{2} + \sum_{n=M_1+1}^{N}(|a_{N-n}|\frac{\varepsilon}{2\alpha}) && \text{(by the definition of } M_1\text{)} \\
&= \frac{\varepsilon}{2} + \left(\sum_{n=M_1+1}^{N}|a_{N-n}|\right)\frac{\varepsilon}{2\alpha} \\
&\leq \frac{\varepsilon}{2} + \alpha\frac{\varepsilon}{2\alpha} && (\sum_{n=0}^{\infty}|a_n| \text{ is monotone increasing}) \\
&= \varepsilon.
\end{aligned}$$

Since N was arbitrary, we have

$$\forall N \geq M, |\gamma_N| \leq \varepsilon.$$

Since ε was arbitrary, we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists M \in \mathbb{N} : \forall N \geq M, |\gamma_N| \leq \varepsilon$$

and thus $d - \lim_{N\to\infty} \gamma_N = 0$, as desired. □

Def. II.4.6.15 (Complex exponential)**.** If z is a complex number, we define the function $\exp(z)$ by the formula

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Note. Inspired by Prop. II.4.5.4, we shall use $\exp(z)$ and e^z interchangeably. It is also possible to define a^z for complex z and real $a > 0$, but we will not need to do so in this text.

— Exercises —

Ex. II.4.6.1. Prove Lem. II.4.6.4.

Proof. See Lem. II.4.6.4. □

Ex. II.4.6.2. Prove Lem. II.4.6.6.

Proof. See Lem. II.4.6.6. □

Ex. II.4.6.3. Prove Lem. II.4.6.7.

Proof. See Lem. II.4.6.7. □

Ex. II.4.6.4. Prove Lem. II.4.6.9.

Proof. See Lem. II.4.6.9. □

Ex. II.4.6.5. If z is a complex number, show that $\Re(z) = \dfrac{z+\overline{z}}{2}$ and $\Im(z) = \dfrac{z-\overline{z}}{2i}$.

Proof. We have

$$\begin{aligned}
\frac{z+\overline{z}}{2} &= \frac{\Re(z)+i\Im(z)+\overline{\Re(z)+i\Im(z)}}{2} && \text{(by Def. II.4.6.8)}\\
&= \frac{\Re(z)+i\Im(z)+\Re(z)-i\Im(z)}{2} && \text{(by Def. II.4.6.8)}\\
&= \frac{\Re(z)+\Re(z)+i\Im(z)-i\Im(z)}{2} && \text{(by Lem. II.4.6.4)}\\
&= \frac{2\Re(z)}{2} && \text{(by Lem. II.4.6.4)}\\
&= \Re(z) && (\Re(z)\in\mathbb{R})
\end{aligned}$$

and

$$\begin{aligned}
\frac{z-\overline{z}}{2i} &= \frac{\Re(z)+i\Im(z)-\overline{\Re(z)+i\Im(z)}}{2i} && \text{(by Def. II.4.6.8)}\\
&= \frac{\Re(z)+i\Im(z)+\overline{-\big(\Re(z)+i\Im(z)\big)}}{2i} && \text{(by Lem. II.4.6.9)}\\
&= \frac{\Re(z)+i\Im(z)+\overline{-\Re(z)-i\Im(z))}}{2i} && \text{(by Def. II.4.6.3)}\\
&= \frac{\Re(z)+i\Im(z)-\Re(z)+i\Im(z)}{2i} && \text{(by Def. II.4.6.8)}\\
&= \frac{\Re(z)-\Re(z)+i\Im(z)+i\Im(z)}{2i} && \text{(by Lem. II.4.6.4)}\\
&= \frac{2i\Im(z)}{2i} && \text{(by Lem. II.4.6.4)}\\
&= \Im(z). && \text{(by Def. II.4.6.12)}
\end{aligned}$$

□

Ex. II.4.6.6. Prove Lem. II.4.6.11.

Proof. See Lem. II.4.6.11. □

Ex. II.4.6.7. Show that if z, w are complex numbers with $w \neq 0$, then $|z/w| = |z|/|w|$.

Proof.

$$
\begin{aligned}
\left|\frac{z}{w}\right| &= \left|zw^{-1}\right| \\
&= \left|z|w|^{-2}\overline{w}\right| && \text{(by Def. II.4.6.12)} \\
&= |z||w|^{-2}|\overline{w}| && \text{(by Lem. II.4.6.11)} \\
&= |z||w|^{-2}|w| && \text{(by Lem. II.4.6.11)} \\
&= |z||w|^{-1} && (|z|, |w| \in \mathbb{R}) \\
&= \frac{|z|}{|w|}.
\end{aligned}
$$

□

Ex. II.4.6.8. Let z, w be non-zero complex numbers. Show that $|z+w| = |z| + |w|$ iff there exists a positive real number $c > 0$ such that $z = cw$.

Proof. Since

$$
\begin{aligned}
|z+w| &= \left|\Re(z) + \Re(w) + i\big(\Im(z) + \Im(w)\big)\right| && \text{(by Def. II.4.6.8)} \\
&= \sqrt{\big(\Re(z) + \Re(w)\big)^2 + \big(\Im(z) + \Im(w)\big)^2} && \text{(by Def. II.4.6.10)}
\end{aligned}
$$

and

$$
\begin{aligned}
|z| + |w| &= |\Re(z) + i\Im(z)| + |\Re(w) + i\Im(w)| && \text{(by Def. II.4.6.8)} \\
&= \sqrt{\big(\Re(z)\big)^2 + \big(\Im(z)\big)^2} + \sqrt{\big(\Re(w)\big)^2 + \big(\Im(w)\big)^2}, && \text{(by Def. II.4.6.10)}
\end{aligned}
$$

we know that

$$
\begin{aligned}
& |z+w| = |z| + |w| \\
\iff & |z+w|^2 = |z|^2 + 2|z||w| + |w|^2 \\
\iff & \big(\Re(z) + \Re(w)\big)^2 + \big(\Im(z) + \Im(w)\big)^2 \\
& = \big(\Re(z)\big)^2 + \big(\Im(z)\big)^2 + 2|z||w| + \big(\Re(w)\big)^2 + \big(\Im(w)\big)^2 \\
\iff & \Re(z)\Re(w) + \Im(z)\Im(w) = |z||w| \\
\iff & \big(\Re(z)\Re(w)\big)^2 + 2\Re(z)\Re(w)\Im(z)\Im(w) + \big(\Im(z)\Im(w)\big)^2 \\
& = \big(\Re(z)\Re(w)\big)^2 + \big(\Re(z)\Im(w)\big)^2 + \big(\Im(z)\Re(w)\big)^2 + \big(\Im(z)\Im(w)\big)^2 \\
\iff & 2\Re(z)\Re(w)\Im(z)\Im(w) = \big(\Re(z)\Im(w)\big)^2 + \big(\Im(z)\Re(w)\big)^2 \\
\iff & \big(\Re(z)\Im(w)\big)^2 - 2\Re(z)\Re(w)\Im(z)\Im(w) + \big(\Im(z)\Re(w)\big)^2 = 0 \\
\iff & \big(\Re(z)\Im(w) - \Im(z)\Re(w)\big)^2 = 0
\end{aligned}
$$

$$\begin{aligned}&\iff \Re(z)\Im(w) - \Im(z)\Re(w) = 0\\&\iff \Re(z)\Im(w) = \Im(z)\Re(w).\end{aligned}$$

By hypothesis we know that $w \neq 0 \neq z$. Thus, we can split into two cases:

- $\Re(z) \neq 0$ and $\Re(w) \neq 0$. Then we have

$$\begin{aligned}&|z+w| = |z| + |w|\\ \iff& \Re(z)\Im(w) = \Im(z)\Re(w)\\ \iff& \frac{\Im(z)}{\Re(z)} = \frac{\Im(w)}{\Re(w)}\\ \iff& \exists c \in \mathbb{R}^+ : z = cw.\end{aligned}$$

- $\Re(z) \neq 0$ and $\Im(w) \neq 0$. Then we have

$$\begin{aligned}&|z+w| = |z| + |w|\\ \iff& \Re(z)\Im(w) = \Im(z)\Re(w) \neq 0\\ \iff& \frac{\Im(z)}{\Re(z)} = \frac{\Im(w)}{\Re(w)}\\ \iff& \exists c \in \mathbb{R}^+ : z = cw.\end{aligned}$$

- $\Im(z) \neq 0$ and $\Re(w) \neq 0$. Then we have

$$\begin{aligned}&|z+w| = |z| + |w|\\ \iff& \Re(z)\Im(w) = \Im(z)\Re(w) \neq 0\\ \iff& \frac{\Im(z)}{\Re(z)} = \frac{\Im(w)}{\Re(w)}\\ \iff& \exists c \in \mathbb{R}^+ : z = cw.\end{aligned}$$

- $\Im(z) \neq 0$ and $\Im(w) \neq 0$. Then we have

$$\begin{aligned}&|z+w| = |z| + |w|\\ \iff& \Re(z)\Im(w) = \Im(z)\Re(w)\\ \iff& \frac{\Re(z)}{\Im(z)} = \frac{\Re(w)}{\Im(w)}\\ \iff& \exists c \in \mathbb{R}^+ : z = cw.\end{aligned}$$

From all cases above, we conclude that

$$\begin{aligned}&|z+w| = |z| + |w|\\ \iff& \Re(z)\Im(w) = \Im(z)\Re(w)\\ \iff& \exists c \in \mathbb{R}^+ : z = cw.\end{aligned}$$

$\square$

Ex. II.4.6.9. Prove Lem. II.4.6.13.

Proof. See Lem. II.4.6.13. □

Ex. II.4.6.10. Show that the complex numbers $\mathbb{C}$ (with the usual metric d) form a complete metric space.

Proof. Suppose that $(\mathbb{C}, d)$ is not complete. By Def. II.1.4.10 we know that there is a Cauchy sequence $(a_n)_{n=1}^{\infty}$ in $(\mathbb{C}, d)$ which does not converge. Let $x \in \mathbb{C}$. Then we have

$$\exists \varepsilon \in \mathbb{R}^+ : \forall N_1 \in \mathbb{Z}^+, \exists n \geq N_1 : d(a_n, x) \geq \varepsilon.$$

Fix such ε. Let $i, j \in \mathbb{Z}^+$. Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathbb{C}, d)$, by Def. II.1.4.6 we know that

$$\exists N_2 \in \mathbb{Z}^+ : \forall i, j \geq N_2, d(a_i, a_j) < \varepsilon.$$

Let $N = \max(N_1, N_2)$. Then we have

$$\begin{aligned} &\exists n \geq N : \begin{cases} d(a_n, x) \geq \varepsilon \\ \forall i, j \geq n, d(a_i, a_j) < \varepsilon \end{cases} \\ \implies &\exists n \geq N : \forall i, j \geq n, 2\varepsilon \leq d(a_i, x) + d(a_j, x) \leq d(a_i, a_j) \leq \varepsilon \\ \implies &2\varepsilon \leq \varepsilon. \end{aligned}$$

But $\varepsilon \in \mathbb{R}^+$, a contradiction. Thus, such Cauchy sequence does not exists, and $(\mathbb{C}, d)$ must be complete. □

Ex. II.4.6.11. Let $f : \mathbb{R}^2 \to \mathbb{C}$ be the map $f(a, b) := a + bi$. Show that f is a bijection, and that f and f^{-1} are both continuous maps.

Proof. First, we show that f is injective. Let $a, b, c, d \in \mathbb{R}$ such that $f(a, b) = f(c, d)$. Then we have

$$\begin{aligned} &f(a, b) = f(c, d) && \text{(by hypothesis)} \\ \implies &a + bi = c + di && \text{(by the definition of } f\text{)} \\ \implies &(a = c) \wedge (b = d) && \text{(by Def. II.4.6.2)} \\ \implies &(a, b) = (c, d). \end{aligned}$$

Thus, f is injective.

Next we show that f is surjective. Let $x \in \mathbb{C}$. Then by Def. II.4.6.2 we know that there exist some $a, b \in \mathbb{R}$ such that $x = a + bi$. By the definition of f we know that $f(a, b) = x$. Since x was arbitrary, we know that f is surjective. Since f is both injective and surjective, we know that f is bijective, thus f^{-1} is well-defined.

Next we show that f is a continuous map. Let $(a,b) \in \mathbb{R}^2$. Let $d_2 = d_{l^1}|_{\mathbb{R}^2 \times \mathbb{R}^2}$ and let d be the metric defined in Def. II.4.6.10. We want to show that f is continuous at (a,b) from $(\mathbb{R}^2, d_2)$ to $(\mathbb{C}, d)$. By Def. II.2.1.1 we need to show that

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall (x,y) \in \mathbb{R}^2, d_2\big((x,y),(a,b)\big) < \delta \implies d\big(f(x,y), f(a,b)\big) < \varepsilon.$$

Now fix one ε. Then we have

$$\begin{aligned} & \forall (x,y) \in \mathbb{R}^2, d\big(f(x,y), f(a,b)\big) < \varepsilon \\ \implies & \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon && \text{(by Def. II.4.6.10)} \\ \implies & \sqrt{2}\sqrt{(x-a)^2 + (y-b)^2} < \sqrt{2}\varepsilon \\ \implies & |x-a| + |y-b| \le \sqrt{2}\sqrt{(x-a)^2 + (y-b)^2} < \sqrt{2}\varepsilon && \text{(by Ex. II.1.1.8)} \\ \implies & d_2\big((x,y),(a,b)\big) \le \sqrt{2}\varepsilon. \end{aligned}$$

By setting $\delta = \sqrt{2}\varepsilon$ we are done. Since (a,b) was arbitrary, we know that f is continuous on $\mathbb{R}^2$ from $(\mathbb{R}^2, d_2)$ to $(\mathbb{C}, d)$.

Finally we show that f^{-1} is a continuous map. Let $x \in \mathbb{C}$. We want to show that f^{-1} is continuous at x from $(\mathbb{C}, d)$ to $(\mathbb{R}^2, d_2)$. By Def. II.2.1.1 we need to show that

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall y \in \mathbb{C}, d\big(x,y\big) < \delta \implies d_2\big(f^{-1}(x), f^{-1}(y)\big) < \varepsilon.$$

Now fix one ε. Then we have

$$\begin{aligned} & \forall y \in \mathbb{C}, d_2\big(f^{-1}(x), f^{-1}(y)\big) < \varepsilon \\ \implies & d_2\Big(\big(\Re(x), \Im(x)\big), \big(\Re(y), \Im(y)\big)\Big) < \varepsilon && \text{(by Def. II.4.6.8)} \\ \implies & |\Re(x) - \Re(y)| + |\Im(x) - \Im(y)| < \varepsilon \\ \implies & \sqrt{\big(\Re(x) - \Re(y)\big)^2 + \big(\Im(x) - \Im(y)\big)^2} < \varepsilon && \text{(by Ex. II.1.1.8)} \\ \implies & d(x,y) < \varepsilon. \end{aligned}$$

By setting $\delta = \varepsilon$ we are done. Since x was arbitrary, we know that f^{-1} is continuous on $\mathbb{C}$ from $(\mathbb{C}, d)$ to $(\mathbb{R}^2, d_2)$. □

Ex. II.4.6.12. Show that the complex numbers $\mathbb{C}$ (with the usual metric d) form a connected metric space.

Proof. We first show that (C, d) is path-connected. By Ex. II.2.4.7 we want to show that for every $x, y \in \mathbb{C}$, there exists a continuous function $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) = x$ and $\gamma(1) = y$. So let $x, y \in \mathbb{C}$. We define $\gamma : [0,1] \to \mathbb{C}$ as follow:

$$\forall z \in [0,1], \gamma(z) = (1-z)x + zy.$$

Obviously $\gamma(0) = x$ and $\gamma(1) = y$. Let $d_1 = d_{l^1}|_{[0,1]\times[0,1]}$. Now we show that γ is continuous on $[0,1]$ from $([0,1], d_1)$ to $(\mathbb{C}, d)$. Let $f : \mathbb{R}^2 \to \mathbb{C}$ be the function in Ex. II.4.6.11. Then we have

$$\forall z \in [0,1], \gamma(z) = f\big((1-z)\Re(x) + z\Re(y), (1-z)\Im(x) + z\Im(y)\big).$$

By Ex. II.4.6.11 we know that f is continuous on $\mathbb{R}^2$ from $(\mathbb{R}^2, d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{C}, d)$. Thus, by Rmk. II.2.1.3 we know that γ is continuous on $[0,1]$ from $([0,1], d_1)$ to $(\mathbb{C}, d)$. Since x, y were arbitrary, by Ex. II.2.4.7 we know that $(\mathbb{C}, d)$ is path-connected, and we conclude that $(\mathbb{C}, d)$ is connected. □

Ex. II.4.6.13. Let E be a subset of $\mathbb{C}$. Show that E is compact iff E is closed and bounded. In particular, show that $\mathbb{C}$ is not compact.

Proof. Let d be the metric defined in Def. II.4.6.10. By Cor. II.1.5.6 we know that $(E, d|_{E\times E})$ is compact implies $(E, d|_{E\times E})$ is closed and bounded. So we only need to show that $(E, d|_{E\times E})$ is closed and bounded implies $(E, d|_{E\times E})$ is compact. Suppose that $(E, d|_{E\times E})$ is closed and bounded. Let $d_2 = d_{l^1}|_{\mathbb{R}^2\times\mathbb{R}^2}$ and let $f : \mathbb{R}^2 \to \mathbb{C}$ be the function defined in Ex. II.4.6.11. Then we know that f^{-1} is well-defined and both f, f^{-1} are continuous. Since $(E, d|_{E\times E})$ is closed, by Thm. II.2.1.5(a)(d) we know that $\big(f^{-1}(E), d_2|_{f^{-1}(E)\times f^{-1}(E)}\big)$ is closed. Since $(E, d|_{E\times E})$ is bounded, we know that

$$\begin{aligned}
&\forall z \in \mathbb{C}, \exists r \in \mathbb{R}^+ : E \subseteq B_{(\mathbb{C},d)}(z, r) && \text{(by Def. II.1.5.3)}\\
\implies &\exists r \in \mathbb{R}^+ : E \subseteq B_{(\mathbb{C},d)}(0, r)\\
\implies &\exists r \in \mathbb{R}^+ : f^{-1}(E) \subseteq f^{-1}\big(B_{(\mathbb{C},d)}(0, r)\big)\\
\implies &\exists r \in \mathbb{R}^+ : \forall x \in E, \sqrt{\big(\Re(x)\big)^2 + \big(\Im(x)\big)^2} < r && \text{(by Def. II.4.6.10)}\\
\implies &\exists r \in \mathbb{R}^+ : \forall x \in E, \big(\Re(x)\big)^2 + \big(\Im(x)\big)^2 < r^2\\
\implies &\exists r \in \mathbb{R}^+ : \forall x \in E, \begin{cases} \big(\Re(x)\big)^2 < r^2 \\ \big(\Im(x)\big)^2 < r^2 \end{cases}\\
\implies &\exists r \in \mathbb{R}^+ : \forall x \in E, \begin{cases} |\Re(x)| < r \\ |\Im(x)| < r \end{cases}\\
\implies &\exists r \in \mathbb{R}^+ : f^{-1}(E) \subseteq B_{(\mathbb{R}^2,d_2)}(0, r).
\end{aligned}$$

Thus, $\big(f^{-1}(E), d_2|_{f^{-1}(E)\times f^{-1}(E)}\big)$ is bounded. Since $\big(f^{-1}(E), d_2|_{f^{-1}(E)\times f^{-1}(E)}\big)$ is closed and bounded, by Thm. II.1.5.7 we know that $\big(f^{-1}(E), d_2|_{f^{-1}(E)\times f^{-1}(E)}\big)$ is compact. Since $E = f\big(f^{-1}(E)\big)$, by Thm. II.2.3.1 we know that $(E, d|_{E\times E})$ is compact. Thus, we conclude that $(E, d|_{E\times E})$ is compact iff $(E, d|_{E\times E})$ is closed and bounded.

Since $\mathbb{R} \subseteq \mathbb{C}$ and $(\mathbb{R}, d|_{\mathbb{R}\times\mathbb{R}})$ is not compact, we conclude that $(\mathbb{C}, d)$ is not compact. □

Ex. II.4.6.14. Prove Lem. II.4.6.14.

Proof. See Lem. II.4.6.14. □

Ex. II.4.6.15. The purpose of this exercise is to explain why we do not try to organize the complex numbers into positive and negative parts. Suppose that there was a notion of a "positive complex number" and a "negative complex number" which obeyed the following reasonable axioms (cf. Proposition 4.2.9 in Analysis I):

- (Trichotomy) For every complex number z, exactly one of the following statements is true: z is positive, z is negative, z is zero.
- (Negation) If z is a positive complex number, then $-z$ is negative. If z is a negative complex number, then $-z$ is positive.
- (Additivity) If z and w are positive complex numbers, then $z + w$ is also positive.
- (Multiplicativity) If z and w are positive complex numbers, then zw is also positive.

Show that these four axioms are inconsistent, i.e., one can use these axioms to deduce a contradiction.

Proof. First, we show that 1 is positive. Obviously $1 \neq 0$. So 1 can only be positive or negative. Suppose for the sake of contradiction that 1 is negative. Then by axiom above we know that -1 is positive. But by axiom we know that $(-1)(-1) = 1$ is positive, a contradiction. Thus, 1 must be positive.

Since 1 is positive, we know from axiom that -1 is negative. Obviously $i \neq 0$. So i can only be positive or negative. Now we split into two cases:

- If i is positive, then by axiom we know that $i^2 = -1$ is positive. But this contradicts the fact that -1 is negative.
- If i is negative, then by axiom we know that $-i$ is positive. Again by axiom we know that $(-i)(-i) = -1$ is positive. But this contradicts the fact that -1 is negative.

From all cases above, we derived contradictions. Thus, the axioms are inconsistent. □

Ex. II.4.6.16. Prove the ratio test for complex series, and use it to show that the series used to define the complex exponential is absolutely convergent. Then prove that $\exp(z + w) = \exp(z)\exp(w)$ for all complex numbers z, w.

Proof. Let d be the metric in Def. II.4.6.10. First, we proof the ratio test of complex series. Let $(a_n)_{n=m}^{\infty}$ be a sequence in $\mathbb{C} \setminus \{0\}$ and let $b_n = |a_n|$ for all $n \in \{k \in \mathbb{N} : k \geq m\}$. Then we know that $(b_n)_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$. Observe that

$$\forall n \geq m, \frac{|b_{n+1}|}{|b_n|} = \frac{||a_{n+1}||}{||a_n||} \frac{|b_{n+1}|}{|b_n|} = \frac{|a_{n+1}|}{|a_n|}.$$

By ratio test of real series we now split into three cases.

- $\limsup_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} < 1$. Then the series $\sum_{n=0}^{\infty} b_n$ is absolutely convergent. But we have

$$\limsup_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} < 1 \implies \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

  and

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} |a_n|.$$

  By A.Cor. II.4.6.8 we know that $\sum_{n=0}^{\infty} |a_n| \in \mathbb{R}$ implies $\sum_{n=0}^{\infty} a_n \in \mathbb{C}$.

- $\liminf_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} > 1$. Then the series $\sum_{n=0}^{\infty} b_n$ is divergent. Since

$$\begin{aligned} & \liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1 \\ \implies & \lim_{n\to\infty} |a_n| \neq 0 \\ \implies & d - \lim_{n\to\infty} a_n \neq 0, \end{aligned}$$

  by A.Cor. II.4.6.7 we know that $\sum_{n=0}^{\infty} a_n$ is divergent.

- The remaining cases. As in real series we cannot assert any conclusion.

Thus, we conclude that

- If $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.
- If $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then the series $\sum_{n=0}^{\infty} a_n$ is divergent.
- In the remaining cases, we cannot assert any conclusion.

Next we show that Def. II.4.6.15 is absolutely convergent. Let $z \in \mathbb{C}$. Suppose that $z = 0$. Then we have

$$\begin{aligned} & |z| = 0 && \text{(by Lem. II.4.6.11)} \\ \implies & \exp(0) = \sum_{n=0}^{\infty} \frac{0^n}{n!} && \text{(by Def. II.4.6.15)} \end{aligned}$$

$$\Longrightarrow \exp(0) = 1. \qquad \text{(by Thm. II.4.5.2(e))}$$

Thus, complex exponential is absolutely convergent at 0. Now suppose that $z \neq 0$. By Def. II.4.6.10 we know that $|z| \in \mathbb{R}$. Since

$$\begin{aligned}
\forall n \in \mathbb{N}, \frac{\left|\frac{z^{n+1}}{(n+1)!}\right|}{\left|\frac{z^n}{n!}\right|} &= \frac{|z^{n+1}||n!|}{|z^n||(n+1)!|} && \text{(by Ex. II.4.6.7)} \\
&= \frac{|z|^{n+1}|n!|}{|z|^n|(n+1)!|} && \text{(by Lem. II.4.6.11)} \\
&= \frac{|z|}{n+1},
\end{aligned}$$

we know that

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\left|\frac{z^{n+1}}{(n+1)!}\right|}{\left|\frac{z^n}{n!}\right|} &= \limsup_{n\to\infty} \frac{|z|}{n+1} \\
&= |z| \limsup_{n\to\infty} \frac{1}{n+1} \\
&= |z| 0 = 0 < 1.
\end{aligned}$$

By ratio test above we know that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is absolutely convergent.

Finally we show that $\exp(z+w) = \exp(z)\exp(w)$ for any $z, w \in \mathbb{C}$.

$$\begin{aligned}
&\exp(z)\exp(w) \\
&= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) && \text{(by Def. II.4.6.15)} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \left(\frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}\right)\right) && \text{(by A.Cor. II.4.6.9)} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \sum_{k=0}^{n} \left(\frac{n!}{k!(n-k)!} z^k w^{n-k}\right)\right) \\
&= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} && \text{(by Ex. II.4.2.5)} \\
&= \exp(z+w). && \text{(by Def. II.4.6.15)}
\end{aligned}$$

□

II.4.7 Trigonometric functions

Note. There are several other useful special functions in mathematics, such as the hyperbolic trigonometric functions and hypergeometric functions, the gamma and zeta functions, and elliptic functions, but they occur more rarely than trigonometric functions.

Note. Trigonometric functions are often defined using geometric concepts, notably those of circles, triangles, and angles. However, it is also possible to define them using more analytic concepts, and in particular, the (complex) exponential function.

Def. II.4.7.1 (Trigonometric functions)**.** If z is a complex number, then we define

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2}$$

and

$$\sin(z) := \frac{e^{iz} - e^{-iz}}{2i}.$$

We refer to cos and sin as the *cosine* and *sine* functions respectively.

Note. Def. II.4.7.1 were discovered by Leonhard Euler (1707–1783) in 1748, who recognized the link between the complex exponential and the trigonometric functions. Since we have defined the sine and cosine for complex numbers z, we automatically have defined them also for real numbers x. In fact in most applications one is only interested in the trigonometric functions when applied to real numbers.

A.Cor. II.4.7.1. From Def. II.4.6.15, we have

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} + \frac{z^4}{4!} + \dots$$

and

$$e^{-iz} = 1 - iz - \frac{z^2}{2!} + \frac{iz^3}{3!} + \frac{z^4}{4!} - \dots$$

and so from the above formulae we have

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

and

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

In particular, $\cos(x)$ and $\sin(x)$ are always real whenever x is real. From the ratio test we see that the two power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ are absolutely convergent for every x, thus $\sin(x)$ and $\cos(x)$ are real analytic at 0 with an infinite radius of convergence. From Ex. II.4.2.8 we thus see that the sine and cosine functions are real analytic on all of $\mathbb{R}$. (They are also complex analytic on all of $\mathbb{C}$, but we will not pursue this matter in this text.) In particular, the sine and cosine functions are continuous and differentiable (see Prop. II.4.2.6).

Thm. II.4.7.2 (Trigonometric identities)**.** Let x, y be real numbers.

(a) We have $\big(\sin(x)\big)^2 + \big(\cos(x)\big)^2 = 1$. In particular, we have $\sin(x) \in [-1, 1]$ and $\cos(x) \in [-1, 1]$ for all $x \in \mathbb{R}$.

(b) We have $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$.

(c) We have $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.

(d) We have $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.

(e) We have $\sin(0) = 0$ and $\cos(0) = 1$.

(f) We have $e^{ix} = \cos(x) + i\sin(x)$ and $e^{-ix} = \cos(x) - i\sin(x)$. In particular, $\cos(x) = \Re(e^{ix})$ and $\sin(x) = \Im(e^{ix})$.

Proof. (a) Let $x \in \mathbb{R}$. Then we have

$$
\begin{aligned}
&\big(\sin(x)\big)^2 + \big(\cos(x)\big)^2 \\
&= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^2 + \left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 && \text{(by Def. II.4.7.1)} \\
&= \frac{e^{ix}e^{ix} - 2e^{ix}e^{-ix} + e^{-ix}e^{-ix}}{-4} + \frac{e^{ix}e^{ix} + 2e^{ix}e^{-ix} + e^{-ix}e^{-ix}}{4} && \text{(by Def. II.4.6.5)} \\
&= \frac{4e^{ix}e^{-ix}}{4} && \text{(by Lem. II.4.6.4)} \\
&= e^{ix}e^{-ix} \\
&= e^{ix-ix} && \text{(by Ex. II.4.6.16)} \\
&= e^0 \\
&= 1. && \text{(by Thm. II.4.5.2(d))}
\end{aligned}
$$

By A.Cor. II.4.7.1 we know that $\sin(x), \cos(x) \in \mathbb{R}$ when $x \in \mathbb{R}$. Thus, we have

$$
\begin{aligned}
&\begin{cases} \big(\sin(x)\big)^2 \le 1 \\ \big(\cos(x)\big)^2 \le 1 \end{cases} \\
\implies &\begin{cases} \sin(x) \in [-1, 1] \\ \cos(x) \in [-1, 1] \end{cases}
\end{aligned}
$$

$\square$

Proof. (b) Let $x \in \mathbb{R}$. By A.Cor. II.4.7.1 we know that $\sin'$ and $\cos'$ are well-defined. Then we have

$$\begin{aligned}
&\sin'(x)\\
&= \left(y \mapsto \frac{e^{iy} - e^{-iy}}{2i}\right)'(x) && \text{(by Def. II.4.7.1)}\\
&= \left(y \mapsto \frac{\sum_{n=0}^{\infty} \frac{(iy)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iy)^n}{n!}}{2i}\right)'(x) && \text{(by Def. II.4.6.15)}\\
&= \left(y \mapsto \sum_{n=0}^{\infty} \frac{(iy)^n - (-iy)^n}{(2i)(n!)}\right)'(x) && \text{(by Lem. II.4.6.14)}\\
&= \left(y \mapsto \sum_{n=0}^{\infty} \frac{i^n y^n - (-1)^n i^n y^n}{(2i)(n!)}\right)'(x) && \text{(by Lem. II.4.6.6)}\\
&= \left(y \mapsto \sum_{n=0}^{\infty} \left(\frac{\big(1-(-1)^n\big)i^n}{(2i)(n!)} y^n\right)\right)'(x) && \text{(by Lem. II.4.6.6)}\\
&= \left(y \mapsto \sum_{n=1}^{\infty} \left(\frac{n\big(1-(-1)^n\big)i^n}{(2i)(n!)} y^{n-1}\right)\right)(x) && \text{(by Thm. II.4.1.6(d))}\\
&= \left(y \mapsto \sum_{n=1}^{\infty} \left(\frac{\big(1-(-1)^n\big)i^n}{(2i)(n-1)!} y^{n-1}\right)\right)(x)\\
&= \left(y \mapsto 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \dots\right)(x)\\
&= \left(y \mapsto \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!}\right)(x)\\
&= \cos(x) && \text{(by A.Cor. II.4.7.1)}
\end{aligned}$$

and

$$\begin{aligned}
&\cos'(x)\\
&= \left(y \mapsto \frac{e^{iy} + e^{-iy}}{2}\right)'(x) && \text{(by Def. II.4.7.1)}\\
&= \left(y \mapsto \frac{\sum_{n=0}^{\infty} \frac{(iy)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-iy)^n}{n!}}{2}\right)'(x) && \text{(by Def. II.4.6.15)}\\
&= \left(y \mapsto \sum_{n=0}^{\infty} \frac{(iy)^n + (-iy)^n}{2(n!)}\right)'(x) && \text{(by Lem. II.4.6.14)}\\
&= \left(y \mapsto \sum_{n=0}^{\infty} \frac{i^n y^n + (-1)^n i^n y^n}{2(n!)}\right)'(x) && \text{(by Lem. II.4.6.6)}
\end{aligned}$$

$$
\begin{aligned}
&= \left(y \mapsto \sum_{n=0}^{\infty} \left(\frac{\bigl(1+(-1)^n\bigr) i^n}{2(n!)} y^n \right) \right)'(x) && \text{(by Lem. II.4.6.6)}\\
&= \left(y \mapsto \sum_{n=1}^{\infty} \left(\frac{n\bigl(1+(-1)^n\bigr) i^n}{2(n!)} y^{n-1} \right) \right)(x) && \text{(by Thm. II.4.1.6(d))}\\
&= \left(y \mapsto \sum_{n=1}^{\infty} \left(\frac{\bigl(1+(-1)^n\bigr) i^n}{2(n-1)!} y^{n-1} \right) \right)(x)\\
&= \left(y \mapsto -y + \frac{y^3}{3!} - \frac{y^5}{5!} + \frac{y^7}{7!} + \dots \right)(x)\\
&= \left(y \mapsto \sum_{n=0}^{\infty} \frac{(-1)^{n+1} y^{2n+1}}{(2n+1)!} \right)(x)\\
&= \left(y \mapsto -\left(\sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} \right) \right)(x) && \text{(by Lem. II.4.6.14)}\\
&= \bigl(y \mapsto -\sin(y)\bigr)(x) && \text{(by A.Cor. II.4.7.1)}\\
&= -\sin(x).
\end{aligned}
$$

□

Proof. (c) Let $x \in \mathbb{R}$. Then we have

$$
\begin{aligned}
\sin(-x) &= \frac{e^{i(-x)} - e^{-i(-x)}}{2i} && \text{(by Def. II.4.7.1)}\\
&= \frac{e^{-ix} - e^{ix}}{2i} && \text{(by Lem. II.4.6.6)}\\
&= -\frac{e^{ix} - e^{-ix}}{2i} && \text{(by Lem. II.4.6.6)}\\
&= -\sin(x) && \text{(by Def. II.4.7.1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\cos(-x) &= \frac{e^{i(-x)} + e^{-i(-x)}}{2} && \text{(by Def. II.4.7.1)}\\
&= \frac{e^{-ix} + e^{ix}}{2} && \text{(by Lem. II.4.6.6)}\\
&= \frac{e^{ix} + e^{-ix}}{2} && \text{(by Lem. II.4.6.4)}\\
&= \cos(x). && \text{(by Def. II.4.7.1)}
\end{aligned}
$$

□

Proof. (d) Let $x, y \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\sin(x)\cos(y) + \cos(x)\sin(y) \\
&= \frac{e^{ix} - e^{-ix}}{2i}\frac{e^{iy} + e^{-iy}}{2} + \frac{e^{ix} + e^{-ix}}{2}\frac{e^{iy} - e^{-iy}}{2i} && \text{(by Def. II.4.7.1)} \\
&= \frac{e^{ix}e^{iy} + e^{ix}e^{-iy} - e^{-ix}e^{iy} - e^{-ix}e^{-iy}}{4i} && \text{(by Lem. II.4.6.6)} \\
&\quad + \frac{e^{ix}e^{iy} - e^{ix}e^{-iy} + e^{-ix}e^{iy} - e^{-ix}e^{-iy}}{4i} \\
&= \frac{2e^{ix}e^{iy} - 2e^{-ix}e^{-iy}}{4i} && \text{(by Lem. II.4.6.4)} \\
&= \frac{e^{ix}e^{iy} - e^{-ix}e^{-iy}}{2i} && \text{(by Def. II.4.6.12)} \\
&= \frac{e^{ix+iy} - e^{-ix-iy}}{2i} && \text{(by Ex. II.4.6.16)} \\
&= \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} && \text{(by Lem. II.4.6.6)} \\
&= \sin(x+y) && \text{(by Def. II.4.7.1)}
\end{aligned}$$

and

$$\begin{aligned}
&\cos(x)\cos(y) - \sin(x)\sin(y) \\
&= \frac{e^{ix} + e^{-ix}}{2}\frac{e^{iy} + e^{-iy}}{2} - \frac{e^{ix} - e^{-ix}}{2i}\frac{e^{iy} - e^{-iy}}{2i} && \text{(by Def. II.4.7.1)} \\
&= \frac{e^{ix}e^{iy} + e^{ix}e^{-iy} + e^{-ix}e^{iy} + e^{-ix}e^{-iy}}{4} && \text{(by Lem. II.4.6.6)} \\
&\quad + \frac{e^{ix}e^{iy} - e^{ix}e^{-iy} - e^{-ix}e^{iy} + e^{-ix}e^{-iy}}{4} \\
&= \frac{2e^{ix}e^{iy} + 2e^{-ix}e^{-iy}}{4} && \text{(by Lem. II.4.6.4)} \\
&= \frac{e^{ix}e^{iy} + e^{-ix}e^{-iy}}{2} && \text{(by Def. II.4.6.12)} \\
&= \frac{e^{ix+iy} + e^{-ix-iy}}{2} && \text{(by Ex. II.4.6.16)} \\
&= \frac{e^{i(x+y)} + e^{-i(x+y)}}{2} && \text{(by Lem. II.4.6.6)} \\
&= \cos(x+y). && \text{(by Def. II.4.7.1)}
\end{aligned}$$

□

Proof. (e) We have

$$\sin(-0) = -\sin(0) \qquad \text{(by Thm. II.4.7.2(c))}$$

$$\implies 2\sin(0) = 0$$
$$\implies \sin(0) = 0$$

and

$$\begin{aligned}
\cos(0) &= \frac{e^{i0} + e^{-i0}}{2} && \text{(by Def. II.4.7.1)} \\
&= \frac{e^{0} + e^{0}}{2} && \text{(by Def. II.4.6.5)} \\
&= e^{0} = 1. && \text{(by Thm. II.4.5.2(e))}
\end{aligned}$$

□

Proof. (f) Let $x \in \mathbb{R}$. Then we have

$$\begin{aligned}
&\cos(x) + i\sin(x) \\
&= \frac{e^{ix} + e^{-ix}}{2} + i\frac{e^{ix} - e^{-ix}}{2i} && \text{(by Def. II.4.7.1)} \\
&= \frac{e^{ix} + e^{-ix}}{2} + \frac{e^{ix} - e^{-ix}}{2} && \text{(by Def. II.4.6.12)} \\
&= e^{ix} && \text{(by Lem. II.4.6.4)}
\end{aligned}$$

and

$$\begin{aligned}
&\cos(x) - i\sin(x) \\
&= \frac{e^{ix} + e^{-ix}}{2} - i\frac{e^{ix} - e^{-ix}}{2i} && \text{(by Def. II.4.7.1)} \\
&= \frac{e^{ix} + e^{-ix}}{2} - \frac{e^{ix} - e^{-ix}}{2} && \text{(by Def. II.4.6.12)} \\
&= e^{-ix}. && \text{(by Lem. II.4.6.4)}
\end{aligned}$$

By Thm. II.4.7.2(a) we know that $\sin(x), \cos(x) \in \mathbb{R}$. Thus, we have

$$\Re(e^{ix}) = \Re\big(\cos(x) + i\sin(x)\big) = \cos(x)$$

and

$$\Im(e^{ix}) = \Im\big(\cos(x) + i\sin(x)\big) = \sin(x).$$

□

Lem. II.4.7.3. There exists a positive number x such that $\sin(x)$ is equal to 0.

Proof. Suppose for the sake of contradiction that $\sin(x) \neq 0$ for all $x \in (0, \infty)$. Observe that this would also imply that $\cos(x) \neq 0$ for all $x \in (0, \infty)$, since if $\cos(x) = 0$ then $\sin(2x) = 0$ by Thm. II.4.7.2(d). Since $\cos(0) = 1$, this implies by the intermediate value theorem (Theorem 9.7.1 in Analysis I) that $\cos(x) > 0$ for all $x > 0$ (since by A.Cor. II.4.7.1 we know that cos is continuous on $\mathbb{R}$ and by Thm. II.4.7.2(a) this means $\cos(x) \in (0, 1]$). Also, since $\sin(0) = 0$ and $\sin'(0) = 1 > 0$, we see that sin is increasing near 0, hence is positive to the right of 0. By the intermediate value theorem again we conclude that $\sin(x) > 0$ for all $x > 0$ (otherwise sin would have a zero on $(0, +\infty)$).

In particular, if we define the cotangent function $\cot(x) := \cos(x)/\sin(x)$, then $\cot(x)$ would be positive and differentiable on all of $(0, +\infty)$. From the quotient rule (Theorem 10.1.13(h) in Analysis I) and Thm. II.4.7.2 we see that the derivative of $\cot(x)$ is

$$\begin{aligned}
\cot'(x) &= \frac{\cos'(x)\sin(x) - \cos(x)\sin'(x)}{\big(\sin(x)\big)^2} \\
&= \frac{-\big(\sin(x)\big)^2 - \big(\cos(x)\big)^2}{\big(\sin(x)\big)^2} && \text{(by Thm. II.4.7.2(b))} \\
&= \frac{-1}{\big(\sin(x)\big)^2}. && \text{(by Thm. II.4.7.2(a))}
\end{aligned}$$

In particular, we have $\cot'(x) \leq -1$ for all $x > 0$. By the fundamental theorem of calculus (Theorem 11.9.1 in Analysis I) this implies that

$$\begin{aligned}
&\int_x^{x+s} \cot'(t)\, dt \leq \int_x^{x+s} -1\, dt \\
\implies &\cot(x+s) - \cot(x) \leq -s \\
\implies &\cot(x+s) \leq \cot(x) - s
\end{aligned}$$

for all $x > 0$ and $s > 0$. Now fix one $x > 0$ and let $s = \cot(x)$. Since $s > 0$, we know that $x + s + 1 > 0$, and thus $\cot(x + s + 1) > 0$. But

$$\cot(x + s + 1) \leq \cot(x) - (s + 1) = \cot(x) - \cot(x) - 1 < 0,$$

a contradiction. Thus, by letting $s \to \infty$ we see that this contradicts our assertion that cot is positive on $(0, \infty)$. □

Def. II.4.7.4. We define π to be the number

$$\pi := \inf\{x \in (0, +\infty) : \sin(x) = 0\}.$$

A.Cor. II.4.7.2. The following statements are true.

(a) π is well-defined.

(b) $\pi > 0$.

(c) $\sin(\pi) = 0$.

(d) $\sin(x) > 0$ for all $x \in (0, \pi)$.

(e) cos is strictly decreasing on $(0, \pi)$.

(f) $\cos(\pi) = -1$.

(g) $e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1$.

Proof. Let E be the set $E \coloneqq \{x \in (0, +\infty) : \sin(x) = 0\}$, i.e., E is the set of roots of sin on $(0, +\infty)$. By Lem. II.4.7.3, E is non-empty. Since $\sin'(0) > 0$, there exists a $c > 0$ such that $E \subseteq [c, +\infty)$ (see Ex. II.4.7.2). Also, since sin is continuous in $[c, +\infty)$, E is closed in $[c, +\infty)$ (since $\sin(E) = \{0\} = [0, 0]$ is closed in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$, by Thm. II.2.1.5(d) we know that $E = \sin^{-1}(\{0\})$ is closed in $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$). Since $[c, +\infty)$ is closed in $\mathbb{R}$, we conclude that E is closed in $\mathbb{R}$. Thus, E contains all its adherent points, and thus contains $\inf(E) = \pi$ (see Definition 9.1.8 and 9.1.10 in Analysis I).

We have $\pi \in E \subseteq [c, +\infty)$ (so, in particular, $\pi > 0$) and $\sin(\pi) = 0$. By Def. II.4.7.4, sin cannot have any zeroes in $(0, \pi)$, and so, in particular, must be positive on $(0, \pi)$ (cf. the arguments in Lem. II.4.7.3 using the intermediate value theorem). Since $\cos'(x) = -\sin(x)$, we thus conclude that $\cos(x)$ is strictly decreasing on $(0, \pi)$. Since $\cos(0) = 1$, this implies in particular, that $\cos(\pi) < 1$; since $\big(\sin(\pi)\big)^2 + \big(\cos(\pi)\big)^2 = 1$ and $\sin(\pi) = 0$, we thus conclude that $\cos(\pi) = -1$. Thus, we conclude by Thm. II.4.7.2(f) that $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$. ☐

Thm. II.4.7.5. iodicity of trigonometric functions] Let x be a real number.

(a) We have $\cos(x + \pi) = -\cos(x)$ and $\sin(x + \pi) = -\sin(x)$. In particular, we have $\cos(x + 2\pi) = \cos(x)$ and $\sin(x + 2\pi) = \sin(x)$, i.e., sin and cos are periodic with period 2π.

(b) We have $\sin(x) = 0$ iff x/π is an integer.

(c) We have $\cos(x) = 0$ iff x/π is an integer plus $1/2$.

Proof. (a) We have

$$\begin{aligned}
\cos(x + \pi) &= \cos(x)\cos(\pi) - \sin(x)\sin(\pi) && \text{(by Thm. II.4.7.2(d))}\\
&= \cos(x)(-1) - \sin(x)0 && \text{(by A.Cor. II.4.7.2(b)(f))}\\
&= -\cos(x)
\end{aligned}$$

and

$$\sin(x + \pi) = \sin(x)\cos(\pi) + \cos(x)\sin(\pi) \qquad \text{(by Thm. II.4.7.2(d))}$$

$$
\begin{aligned}
&= \sin(x)(-1) - \cos(x)0 && \text{(by A.Cor. II.4.7.2(b)(f))} \\
&= -\sin(x).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\cos(x+2\pi) &= \cos(x+\pi+\pi) \\
&= -\cos(x+\pi) && \text{(from the proof above)} \\
&= -\big(-\cos(x)\big) && \text{(from the proof above)} \\
&= \cos(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\sin(x+2\pi) &= \sin(x+\pi+\pi) \\
&= -\sin(x+\pi) && \text{(from the proof above)} \\
&= -\big(-\sin(x)\big) && \text{(from the proof above)} \\
&= \sin(x).
\end{aligned}
$$

□

Proof. (b) First, we show that $\frac{x}{\pi} \in \mathbb{Z}$ implies $\sin(x) = 0$. Let $P(n)$ be the statement "$n = \frac{x}{\pi}$ implies $\sin(x) = 0$." We use induction to show that $P(n)$ is true for all $n \in \mathbb{N}$. For $n = 0$, we have

$$
\begin{aligned}
& 0 = \frac{x}{\pi} \\
\Longrightarrow\ & x = 0 \\
\Longrightarrow\ & \sin(x) = \sin(0) = 0. && \text{(by Thm. II.4.7.2(e))}
\end{aligned}
$$

Thus, the base holds. Suppose inductively that $P(n)$ is true for some $n \geq 0$. Then for $n+1$, we have

$$
\begin{aligned}
& n+1 = \frac{x}{\pi} \\
\Longrightarrow\ & x = n\pi + \pi \\
\Longrightarrow\ & \sin(x) = \sin(n\pi+\pi) = -\sin(n\pi) && \text{(by Def. II.4.7.4(a))} \\
\Longrightarrow\ & \sin(x) = -\sin(n\pi) = 0. && \text{(by the induction hypothesis)}
\end{aligned}
$$

This closes the induction. Thus, $P(n)$ is true for all $n \in \mathbb{N}$. Since

$$
\begin{aligned}
& \forall n \in \mathbb{Z}^-, n = \frac{x}{\pi} \\
\Longrightarrow\ & -n = -\frac{x}{\pi} \in \mathbb{Z}^+ \\
\Longrightarrow\ & \sin(-x) = \sin(-n\pi) = 0 && \text{(from the proof above)}
\end{aligned}
$$

$$\begin{aligned}
&\implies -\sin(x) = 0 && \text{(by Thm. II.4.7.2(c))}\\
&\implies \sin(x) = 0,
\end{aligned}$$

we know that $P(n)$ is true for all $n \in \mathbb{Z}$.

Now we show that $\sin(x) = 0$ implies $\dfrac{x}{\pi}$ is an integer. Let $x \in \mathbb{R}$. Suppose that $\sin(x) = 0$. Now we split into three cases:

- $x = 0$. By A.Cor. II.4.7.2(b) we know that $\pi > 0$, thus we know that $\dfrac{0}{\pi} = 0 \in \mathbb{N}$ and by Thm. II.4.7.2(e) we have $\sin(0) = 0$.

- $x \in \mathbb{R}^+$. Since $\pi > 0$, by Archimedean property we know that

$$\begin{aligned}
&\exists n \in \mathbb{N} : n\pi \leq x < (n+1)\pi\\
\implies &\exists n \in \mathbb{N} : 0 \leq x - n\pi < \pi.
\end{aligned}$$

  Fix such n. By A.Cor. II.4.7.2(d) we know that $\sin(y) > 0$ for all $y \in (0, \pi)$. Thus, we must have $x - n\pi = 0$. This means $\dfrac{x}{\pi} = n \in \mathbb{N}$.

- $x \in \mathbb{R}^-$. Then we have

$$\begin{aligned}
&\sin(x) = 0\\
\implies &-\big(-\sin(x)\big) = 0\\
\implies &-\sin(-x) = 0 && \text{(by Thm. II.4.7.2(e))}\\
\implies &\sin(-x) = 0\\
\implies &\frac{-x}{\pi} \in \mathbb{N} && \text{(from the proof above)}\\
\implies &\frac{x}{\pi} \in \mathbb{Z}.
\end{aligned}$$

From all cases above, we conclude that

$$\sin(x) = 0 \implies \frac{x}{\pi} \in \mathbb{Z}.$$

□

Proof. (c) By A.Cor. II.4.7.2(d) we know that $\sin(\frac{\pi}{2}) > 0$. Since

$$\begin{aligned}
0 &= \sin(\pi) && \text{(by A.Cor. II.4.7.2(c))}\\
&= \sin(\frac{\pi}{2} + \frac{\pi}{2})\\
&= \sin(\frac{\pi}{2})\cos(\frac{\pi}{2}) + \cos(\frac{\pi}{2})\sin(\frac{\pi}{2}) && \text{(by Thm. II.4.7.2(d))}
\end{aligned}$$

$$= 2\sin(\frac{\pi}{2})\cos(\frac{\pi}{2}),$$

we know that $\cos(\frac{\pi}{2}) = 0$. Since

$$\begin{aligned}
&\big(\sin(\frac{\pi}{2})\big)^2 + \big(\cos(\frac{\pi}{2})\big)^2 = 1 && \text{(by Thm. II.4.7.2(a))}\\
\implies &\big(\sin(\frac{\pi}{2})\big)^2 = 1 && \text{(from the proof above)}\\
\implies &\sin(\frac{\pi}{2}) = 1, && \text{(by A.Cor. II.4.7.2(d))}
\end{aligned}$$

We know that

$$\begin{aligned}
\forall x \in \mathbb{R}, \cos(x) &= \cos(x + \frac{\pi}{2} - \frac{\pi}{2})\\
&= \cos(x + \frac{\pi}{2})\cos(-\frac{\pi}{2}) - \sin(x + \frac{\pi}{2})\sin(-\frac{\pi}{2}) && \text{(by Thm. II.4.7.2(d))}\\
&= \cos(x + \frac{\pi}{2})\cos(\frac{\pi}{2}) + \sin(x + \frac{\pi}{2})\sin(\frac{\pi}{2}) && \text{(by Thm. II.4.7.2(c))}\\
&= \cos(x + \frac{\pi}{2}) \times 0 + \sin(x + \frac{\pi}{2}) \times 1 && \text{(from the proof above)}\\
&= \sin(x + \frac{\pi}{2}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\forall x \in \mathbb{R}, \left(\sin(x + \frac{\pi}{2}) = 0 \iff \frac{x + \frac{\pi}{2}}{\pi} \in \mathbb{Z}\right) && \text{(by Thm. II.4.7.5(b))}\\
\iff &\forall x \in \mathbb{R}, \left(\sin(x + \frac{\pi}{2}) = 0 \iff \frac{x}{\pi} + \frac{1}{2} \in \mathbb{Z}\right)\\
\iff &\forall x \in \mathbb{R}, \left(\cos(x) = 0 \iff \frac{x}{\pi} + \frac{1}{2} \in \mathbb{Z}\right) && \text{(from the proof above)}\\
\iff &\forall x \in \mathbb{R}, \left(\cos(x) = 0 \iff \exists n \in \mathbb{Z} : n = \frac{x}{\pi} + \frac{1}{2}\right)\\
\iff &\forall x \in \mathbb{R}, \left(\cos(x) = 0 \iff \exists n \in \mathbb{Z} : n + 1 = \frac{x}{\pi} + \frac{1}{2}\right)\\
\iff &\forall x \in \mathbb{R}, \left(\cos(x) = 0 \iff \exists n \in \mathbb{Z} : n + \frac{1}{2} = \frac{x}{\pi}\right).
\end{aligned}$$

$\square$

— Exercises —

Ex. II.4.7.1. Prove Thm. II.4.7.2.

Proof. See Thm. II.4.7.2. □

Ex. II.4.7.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is differentiable at x_0, with $f(x_0) = 0$ and $f'(x_0) \neq 0$. Show that there exists a $c > 0$ such that $f(y)$ is non-zero whenever $0 < |x_0 - y| < c$. Conclude in particular, that there exists a $c > 0$ such that $\sin(x) \neq 0$ for all $0 < x < c$.

Proof. Since

$$\begin{aligned}
&f'(x_0) \neq 0 \\
\implies &\lim_{y \to x_0; y \in \mathbb{R}\setminus\{x_0\}} \frac{f(y) - f(x_0)}{y - x_0} = \lim_{y \to x_0; y \in \mathbb{R}\setminus\{x_0\}} \frac{f(y)}{y - x_0} \neq 0 \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall y \in \mathbb{R} \setminus \{x_0\}, \left(|y - x_0| < \delta \implies \left| \frac{f(y)}{y - x_0} - f'(x_0) \right| < \varepsilon \right) \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall y \in \mathbb{R}, \left(0 < |y - x_0| < \delta \implies \left| \frac{f(y)}{y - x_0} - f'(x_0) \right| < \varepsilon \right) \\
\implies &\exists \delta \in \mathbb{R}^+ : \forall y \in \mathbb{R}, \left(0 < |y - x_0| < \delta \implies \left| \frac{f(y)}{y - x_0} - f'(x_0) \right| < \frac{|f'(x_0)|}{2} \right) \\
\implies &\exists \delta \in \mathbb{R}^+ : \forall y \in \mathbb{R}, \\
&\left(0 < |y - x_0| < \delta \implies f'(x_0) - \frac{|f'(x_0)|}{2} < \frac{f(y)}{y - x_0} < f'(x_0) + \frac{|f'(x_0)|}{2} \right),
\end{aligned}$$

by setting $c = \delta$ we know that

$$\begin{aligned}
&0 < |x - x_0| < c \\
\implies &\begin{cases} \frac{3f'(x_0)}{2} < \frac{f(y)}{y - x_0} < \frac{f'(x_0)}{2} < 0 & \text{if} f'(x_0) < 0 \\ 0 < \frac{f'(x_0)}{2} < \frac{f(y)}{y - x_0} < \frac{3f'(x_0)}{2} & \text{if} f'(x_0) > 0 \end{cases} \\
\implies &f(y) \neq 0. && (y - x_0 \neq 0)
\end{aligned}$$

By Thm. II.4.7.2(b)(e) we know that $\sin'(0) = \cos(0) = 1$, thus we conclude that

$$\exists c \in \mathbb{R}^+ : \forall x \in (0, c), \sin(x) \neq 0.$$

□

Ex. II.4.7.3. Prove Thm. II.4.7.5.

Proof. See Thm. II.4.7.5. □

Ex. II.4.7.4. Let x, y be real numbers such that $x^2 + y^2 = 1$. Show that there is exactly one real number $\theta \in (-\pi, \pi]$ such that $x = \sin(\theta)$ and $y = \cos(\theta)$.

Proof. Observe that

$$\begin{aligned} & x^2 + y^2 = 1 \\ \Longrightarrow\ & x, y \in [-1, 1]. \end{aligned}$$

We split into three cases:

- $x = 0$. Then we know that $y = \pm 1$. Since

  $$\begin{aligned} & \theta \in (-\pi, \pi] \\ \Longrightarrow\ & \frac{\theta}{\pi} \in (-1, 1] \\ \Longrightarrow\ & \left(\sin(\theta) = 0 \iff \frac{\theta}{\pi} = \{0, 1\} \right) && \text{(by Thm. II.4.7.5(b))} \\ \Longrightarrow\ & \left(\sin(\theta) = 0 \iff \theta = \{0, \pi\} \right) \end{aligned}$$

  and

  $$\begin{aligned} & \cos(0) = 1; && \text{(by Thm. II.4.7.2(e))} \\ & \cos(\pi) = -1, && \text{(by A.Cor. II.4.7.2(f))} \end{aligned}$$

  we know that

  $$\begin{aligned} (x, y) = (0, 1) = \big(\sin(0), \cos(0)\big) &\iff \theta = 0; \\ (x, y) = (0, -1) = \big(\sin(\pi), \cos(\pi)\big) &\iff \theta = \pi. \end{aligned}$$

- $x \in (0, 1]$. Then we have

  $$x^2 + y^2 = 1 \implies y = \pm\sqrt{1 - x^2} \in (-1, 1).$$

  Since

  $$\begin{aligned} & z \in (0, \pi) \\ \Longrightarrow\ & \sin(z) > 0 && \text{(by A.Cor. II.4.7.2(d))} \\ \Longrightarrow\ & -\sin(z) < 0 \\ \Longrightarrow\ & \sin(-z) < 0, && \text{(by Thm. II.4.7.2(c))} \end{aligned}$$

  we know that

  $$\forall z \in (-\pi, 0), \sin(z) < 0.$$

  Thus

  $$\sin(\theta) = x > 0$$

$$\implies \theta \in (0, \pi).$$

Since $\sin(\frac{\pi}{2}) = 1$ (cf. the proof of Thm. II.4.7.5(c)) and sin is continuous on $\mathbb{R}$ (by A.Cor. II.4.7.1), by intermediate value theorem we know that

$$\begin{aligned} &\exists \theta_1 \in (0, \frac{\pi}{2}] : \sin(\theta_1) = x && \text{since } \sin\big((0, \frac{\pi}{2}]\big) \subseteq (0, 1] \\ &\exists \theta_2 \in [\frac{\pi}{2}, \pi) : \sin(\theta_2) = x && \text{since } \sin\big([\frac{\pi}{2}, \pi)\big) \subseteq (0, 1] \end{aligned}$$

Since $\cos(0) = 1$ (by Thm. II.4.7.2(e)) and $\cos(\frac{\pi}{2}) = 0$ (cf. the proof of Thm. II.4.7.5(c)), we know that

$$\begin{aligned} &\cos \text{ is strictly decreasing on } (0, \pi) && \text{(by A.Cor. II.4.7.2(e))} \\ \implies &\cos\big((0, \frac{\pi}{2}]\big) \subseteq [0, 1). \end{aligned}$$

Using similar arguments, we can show that $\cos\big([\frac{\pi}{2}, \pi)\big) \subseteq (-1, 0]$. Thus, we have

$$\begin{aligned} \big(x \in (0, 1]\big) \wedge \big(y \in [0, 1)\big) &\implies \exists \theta_1 \in (0, \frac{\pi}{2}] : \big(\sin(\theta_1) = x\big) \wedge \big(\cos(\theta_1) = y\big); \\ \big(x \in (0, 1]\big) \wedge \big(y \in (-1, 0]\big) &\implies \exists \theta_2 \in [\frac{\pi}{2}, \pi) : \big(\sin(\theta_2) = x\big) \wedge \big(\cos(\theta_2) = y\big). \end{aligned}$$

But cos is strictly decreasing on $(0, \pi)$ implies the choices of θ_1 and θ_2 are unique. And we conclude that

$$\forall x \in (0, 1], \exists!\ \theta \in (0, \pi) : \begin{cases} \sin(\theta) = x \\ \cos(\theta) = y \\ x^2 + y^2 = 1 \end{cases}$$

- $x \in [-1, 0)$. Then we have

$$\begin{aligned} &-x \in (0, 1] \\ \implies &\exists!\ \theta \in (0, \pi) : \begin{cases} \sin(\theta) = -x \\ \cos(\theta) = y \\ (-x)^2 + y^2 = x^2 + y^2 = 1 \end{cases} && \text{(from the proof above)} \\ \implies &\exists!\ \theta \in (0, \pi) : \begin{cases} \sin(-\theta) = -\sin(\theta) = x \\ \cos(-\theta) = \cos(\theta) = y \\ x^2 + y^2 = 1 \end{cases} && \text{(by Thm. II.4.7.2(c))} \\ \implies &\exists!\ \theta \in (-\pi, 0) : \begin{cases} \sin(\theta) = x \\ \cos(\theta) = y \\ x^2 + y^2 = 1 \end{cases} \end{aligned}$$

From all cases above, we conclude that

$$\exists!\ \theta \in (-\pi, \pi] : \begin{cases} \sin(\theta) = x \\ \cos(\theta) = y \\ x^2 + y^2 = 1 \end{cases}$$

$\square$

Ex. II.4.7.5. Show that if $r, s > 0$ are positive real numbers, and θ, α are real numbers such that $re^{i\theta} = se^{i\alpha}$, then $r = s$ and $\theta = \alpha + 2\pi k$ for some integer k.

Proof. By A.Cor. II.4.7.2(g) we know that

$$\begin{aligned} re^{i\theta} &= r\big(\cos(\theta) + i\sin(\theta)\big); \\ se^{i\alpha} &= s\big(\cos(\alpha) + i\sin(\alpha)\big). \end{aligned}$$

Since

$$\begin{aligned} & re^{i\theta} = se^{i\alpha} \\ \implies & r\big(\cos(\theta) + i\sin(\theta)\big) = s\big(\cos(\alpha) + i\sin(\alpha)\big) \\ \implies & r\cos(\theta) - s\cos(\alpha) + i\big(r\sin(\theta) - s\sin(\alpha)\big) = 0 \\ \implies & \begin{cases} r\cos(\theta) - s\cos(\alpha) = 0 \\ r\sin(\theta) - s\sin(\alpha) = 0 \end{cases} \\ \implies & \begin{cases} r\cos(\theta) = s\cos(\alpha) \\ r\sin(\theta) = s\sin(\alpha) \end{cases} \end{aligned}$$

we know that

$$\begin{aligned} & (r\sin(\theta))^2 + (r\cos(\theta))^2 = (s\sin(\alpha))^2 + (s\cos(\alpha))^2 \\ \implies & r^2\Big(\big(\sin(\theta)\big)^2 + \big(\cos(\theta)\big)^2\Big) = s^2\Big(\big(\sin(\alpha)\big)^2 + \big(\cos(\alpha)\big)^2\Big) \\ \implies & r^2 = s^2 && \text{(by Thm. II.4.7.2(a} \\ \implies & r = s. && (r, s \in \mathbb{R}^+) \end{aligned}$$

Thus, we have

$$\begin{aligned} & \begin{cases} r\cos(\theta) = s\cos(\alpha) \\ r\sin(\theta) = s\sin(\alpha) \end{cases} \\ \implies & \begin{cases} \cos(\theta) = \cos(\alpha) \\ \sin(\theta) = \sin(\alpha) \end{cases} && (r = s \in \mathbb{R}^+) \end{aligned}$$

and

$$\begin{aligned}
\sin(\theta-\alpha) &= \sin(\theta)\cos(-\alpha)+\cos(\theta)\sin(-\alpha) && \text{(by Thm. II.4.7.2(d))}\\
&= \sin(\theta)\cos(\alpha)-\cos(\theta)\sin(\alpha) && \text{(by Thm. II.4.7.2(c))}\\
&= \sin(\alpha)\cos(\alpha)-\cos(\alpha)\sin(\alpha) && \text{(from the proof above)}\\
&= 0.
\end{aligned}$$

By Thm. II.4.7.5(b) we know that

$$\begin{aligned}
&\sin(\theta-\alpha)=0\\
\iff &\frac{\theta-\alpha}{\pi}\in\mathbb{Z}\\
\iff &\exists k\in\mathbb{Z}: k=\frac{\theta-\alpha}{\pi}\\
\iff &\exists k\in\mathbb{Z}: k\pi=\theta-\alpha\\
\iff &\exists k\in\mathbb{Z}: \theta=\alpha+k\pi.
\end{aligned}$$

By Thm. II.4.7.5 we know that for any $\alpha\in\mathbb{R}$, $\cos(\alpha+k\pi)=\cos(\alpha)$ when k is even. So we only need to show that for any $\alpha\in\mathbb{R}$, $\cos(\alpha+k\pi)\neq\cos(\alpha)$ when k is odd. Suppose for the sake of contradiction that for any $\alpha\in\mathbb{R}$, $\cos(\alpha+k\pi)=\cos(\alpha)$ when k is odd. Let $k=2n+1$ for some $n\in\mathbb{Z}$. Then we have

$$\begin{aligned}
\cos(\theta) &= \cos(\alpha+k\pi)\\
&= \cos\big(\alpha+(2n+1)\pi\big)\\
&= \cos(\alpha+2n\pi+\pi)\\
&= -\cos(\alpha+2n\pi) && \text{(by Thm. II.4.7.5(a))}\\
&= -\cos(\alpha) && \text{(by Thm. II.4.7.5(a))}\\
&= -\cos(\theta). && \text{(from the proof above)}
\end{aligned}$$

This means $\cos(\theta)=0$. But by Thm. II.4.7.5(c) we know that

$$\begin{aligned}
&\exists m\in\mathbb{Z}: m+\frac{1}{2}=\frac{\theta}{\pi}=\frac{\alpha}{\pi}+k\\
\implies &\exists m\in\mathbb{Z}: m-k+\frac{1}{2}=\frac{\alpha}{\pi}\notin\mathbb{Z}.
\end{aligned}$$

Thus, when $\alpha=\pi$ we derive contradiction. We conclude that

$$\forall k\in\mathbb{Z},\theta=\alpha+2k\pi.$$

□

Ex. II.4.7.6. Let z be a non-zero complex number. Using Ex. II.4.7.4, show that there is exactly one pair of real numbers r,θ such that $r>0$, $\theta\in(-\pi,\pi]$, and $z=re^{i\theta}$. (This is sometimes known as the *standard polar representation* of z.)

Proof. Observe that

$$
\begin{aligned}
& z \neq 0 \\
\implies & |z| > 0 && \text{(by Lem. II.4.6.11)} \\
\implies & \frac{z}{|z|} \in \mathbb{C} \\
\implies & \left|\frac{z}{|z|}\right| = \frac{|z|}{||z||} = \frac{|z|}{|z|} = 1 && \text{(by Ex. II.4.6.7)} \\
\implies & \big(\Re(\frac{z}{|z|})\big)^2 + \big(\Im(\frac{z}{|z|})\big)^2 = 1 && \text{(by Def. II.4.6.10)} \\
\implies & \exists!\ \theta \in (-\pi, \pi] : \begin{cases} \cos(\theta) = \Re(\frac{z}{|z|}) \\ \sin(\theta) = \Im(\frac{z}{|z|}) \end{cases} && \text{(by Ex. II.4.7.4)} \\
\implies & \exists!\ \theta \in (-\pi, \pi] : \frac{z}{|z|} = \Re(\frac{z}{|z|}) + i\Im(\frac{z}{|z|}) = \cos(\theta) + i\sin(\theta) && \text{(by Def. II.4.6.8)} \\
\implies & \exists!\ \theta \in (-\pi, \pi] : \frac{z}{|z|} = e^{i\theta} && \text{(by Thm. II.4.7.2(f))} \\
\implies & \exists!\ \theta \in (-\pi, \pi] : z = |z|e^{i\theta}. && \text{(by Thm. II.4.7.2(f))}
\end{aligned}
$$

By setting $r = |z|$ we are done. □

Ex. II.4.7.7. For any real number θ and integer n, prove the *de Moivre identities*

$$\cos(n\theta) = \Re\Big(\big(\cos(\theta) + i\sin(\theta)\big)^n\Big); \quad \sin(n\theta) = \Im\Big(\big(\cos(\theta) + i\sin(\theta)\big)^n\Big).$$

Proof. By Thm. II.4.7.2(a) we know that $\big(\cos(\theta)\big)^2 + \big(\sin(\theta)\big)^2 = 1$, thus we cannot have $\cos(\theta) = 0$ and $\sin(\theta) = 0$ at the same time. By Def. II.4.6.12 this means $\big(\cos(\theta)+i\sin(\theta)\big)^{-1}$ is well-defined and

$$\big(\cos(\theta) + i\sin(\theta)\big)^0 = \big(\cos(\theta) + i\sin(\theta)\big)\big(\cos(\theta) + i\sin(\theta)\big)^{-1} = 1.$$

First, suppose that $n = 0$. Then we have

$$
\begin{aligned}
\cos(0\theta) &= \cos(0) \\
&= 1 && \text{(by Thm. II.4.7.2(e))} \\
&= \Re(1) && \text{(by Def. II.4.6.8)} \\
&= \Re\Big(\big(\cos(\theta) + i\sin(\theta)\big)^0\Big) && \text{(by Def. II.4.6.12)}
\end{aligned}
$$

and

$$\sin(0\theta) = \sin(0)$$

$$
\begin{aligned}
&= 0 && \text{(by Thm. II.4.7.2(e))}\\
&= \Im(1) && \text{(by Def. II.4.6.8)}\\
&= \Im\Big(\big(\cos(\theta) + i\sin(\theta)\big)^0\Big) && \text{(by Def. II.4.6.12)}
\end{aligned}
$$

Next suppose that $n \in \mathbb{Z}^+$. Since

$$
\begin{aligned}
\big(\cos(\theta) + i\sin(\theta)\big)^n &= (e^{i\theta})^n && \text{(by Thm. II.4.7.2(f))}\\
&= e^{ni\theta} && \text{(by Ex. II.4.6.16)}\\
&= e^{in\theta}, && \text{(by Lem. II.4.6.6)}
\end{aligned}
$$

we know that

$$
\begin{aligned}
\cos(n\theta) &= \Re(e^{in\theta}) && \text{(by Thm. II.4.7.2(f))}\\
&= \Re\Big(\big(\cos(\theta) + i\sin(\theta)\big)^n\Big) && \text{(from the proof above)}
\end{aligned}
$$

and

$$
\begin{aligned}
\sin(n\theta) &= \Im(e^{in\theta}) && \text{(by Thm. II.4.7.2(f))}\\
&= \Im\Big(\big(\cos(\theta) + i\sin(\theta)\big)^n\Big). && \text{(from the proof above)}
\end{aligned}
$$

Finally suppose that $n \in \mathbb{Z}^-$. Let $k \in \mathbb{Z}^+$ such that $-k = n$. Since

$$
\begin{aligned}
&\big(\cos(\theta) + i\sin(\theta)\big)^{-1}\\
&= |\cos(\theta) + i\sin(\theta)|^{-2}\big(\overline{\cos(\theta) + i\sin(\theta)}\big) && \text{(by Def. II.4.6.12)}\\
&= \Big(\big(\cos(\theta)\big)^2 + \big(\sin(\theta)\big)^2\Big)\big(\overline{\cos(\theta) + i\sin(\theta)}\big) && \text{(by Def. II.4.6.10)}\\
&= \overline{\cos(\theta) + i\sin(\theta)} && \text{(by Thm. II.4.7.2(a))}\\
&= \cos(\theta) - i\sin(\theta)\\
&= e^{-i\theta}, && \text{(by Thm. II.4.7.2(f))}
\end{aligned}
$$

we know that

$$
\begin{aligned}
&\big(\cos(\theta) + i\sin(\theta)\big)^n\\
&= \big(\cos(\theta) + i\sin(\theta)\big)^{-k}\\
&= \Big(\big(\cos(\theta) + i\sin(\theta)\big)^{-1}\Big)^k && \text{(by Def. II.4.6.12)}\\
&= (e^{-i\theta})^k && \text{(from the proof above)}\\
&= e^{k(-i\theta)} && \text{(by Ex. II.4.6.16)}\\
&= e^{-ik\theta} && \text{(by Lem. II.4.6.6)}
\end{aligned}
$$

$$
\begin{aligned}
&= \cos(k\theta) - i\sin(k\theta) && \text{(by Thm. II.4.7.2(f))}\\
&= \cos(-k\theta) + i\sin(-k\theta) && \text{(by Thm. II.4.7.2(c))}\\
&= \cos(n\theta) + i\sin(n\theta).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\cos(n\theta) &= \Re\big(\cos(n\theta) + i\sin(n\theta)\big) && \text{(by Def. II.4.6.8)}\\
&= \Re\Big(\big(\cos(\theta) + i\sin(\theta)\big)^n\Big) && \text{(from the proof above)}
\end{aligned}
$$

and

$$
\begin{aligned}
\sin(n\theta) &= \Im\big(\cos(n\theta) + i\sin(n\theta)\big) && \text{(by Def. II.4.6.8)}\\
&= \Im\Big(\big(\cos(\theta) + i\sin(\theta)\big)^n\Big). && \text{(from the proof above)}
\end{aligned}
$$

□

Ex. II.4.7.8. Let $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ be the tangent function $\tan(x) \coloneqq \sin(x)/\cos(x)$. Show that tan is differentiable and monotone increasing, with

$$\frac{d}{dx}\tan(x) = 1 + \big(\tan(x)\big)^2,$$

and that $\lim_{x\to\pi/2} \tan(x) = +\infty$ and $\lim_{x\to-\pi/2} \tan(x) = -\infty$. Conclude that tan is in fact a bijection from $(-\pi/2, \pi/2) \to \mathbb{R}$, and thus has an inverse function $\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$ (this function is called the *arctangent function*). Show that $\tan^{-1}$ is differentiable and $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.

Proof. By Thm. II.4.7.5(c) we know that $\cos(-\frac{\pi}{2}) = \cos(\frac{\pi}{2}) = 0$ and $\cos(x) \neq 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, $\tan(x)$ is well-defined on $(-\frac{\pi}{2}, \frac{\pi}{2})$. By A.Cor. II.4.7.1 we know that sin and cos are differentiable on $\mathbb{R}$, thus by Theorem 10.1.13(h) in Analysis I we know that tan is differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$. In particular, for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$
\begin{aligned}
\tan'(x) &= \frac{\sin'(x)\cos(x) - \sin(x)\cos'(x)}{\big(\cos(x)\big)^2}\\
&= \frac{\big(\cos(x)\big)^2 + \big(\sin(x)\big)^2}{\big(\cos(x)\big)^2} && \text{(by Thm. II.4.7.2(b))}\\
&= 1 + \big(\tan(x)\big)^2.
\end{aligned}
$$

Since $\tan'(x) > 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, by Proposition 10.3.3 we know that tan is strictly monotone increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Since $\sin(\frac{\pi}{2}) = 1$ (cf. the proof of Thm. II.4.7.5(c)), by Thm. II.4.7.2(b) we know that $\sin(-\frac{\pi}{2}) = -1$. Since sin and cos are continuous on $\mathbb{R}$, we know that

$$\lim_{x \to \frac{\pi}{2}} \sin(x) = \sin(\frac{\pi}{2}) = 1$$

$$\lim_{x \to -\frac{\pi}{2}} \sin(x) = \sin(-\frac{\pi}{2}) = -1$$

$$\lim_{x \to \frac{\pi}{2}} \cos(x) = \cos(\frac{\pi}{2}) = 0$$

$$\lim_{x \to -\frac{\pi}{2}} \cos(x) = \cos(-\frac{\pi}{2}) = 0$$

Since tan is monotone increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\tan(0) = \dfrac{\sin(0)}{\cos(0)} = 0$, we know that $\tan(x) > 0$ for all $x \in (0, \frac{\pi}{2})$ and $\tan(x) < 0$ for all $x \in (-\frac{\pi}{2})$. Suppose for the sake of contradiction that $\lim\limits_{x \to \frac{\pi}{2}} \tan(x) \in \mathbb{R}$. Then we have

$$\begin{cases} \forall x \in (0, \frac{\pi}{2}), \tan(x) > 0 \\ \tan \text{ is strictly monotone increasing on } (-\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$$

$$\implies \lim_{x \to \frac{\pi}{2}} \tan(x) \in \mathbb{R}^+$$

$$\implies \frac{\lim_{x \to \frac{\pi}{2}} \tan(x)}{\sin(\frac{\pi}{2})} = \frac{\lim_{x \to \frac{\pi}{2}} \tan(x)}{\lim_{x \to \frac{\pi}{2}} \sin(x)} = \lim_{x \to \frac{\pi}{2}} \frac{\tan(x)}{\sin(x)} = \lim_{x \to \frac{\pi}{2}} \frac{1}{\cos(x)} \in \mathbb{R}^+.$$

But this contradicts the fact that $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1}{\cos(x)} = +\infty$. Thus, we know that

$$\lim_{x \to \frac{\pi}{2}} \tan(x) \in \{-\infty, \infty\}.$$

But strictly monotone increasing implies

$$\lim_{x \to \frac{\pi}{2}} \tan(x) = +\infty.$$

Using similar arguments, we can show that

$$\lim_{x\to -\frac{\pi}{2}} \tan(x) = -\infty.$$

This means tan is a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$. Thus, $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is well-defined.

Since $\tan'(x) > 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, by inverse function theorem (Theorem 10.4.2 in Analysis I) we have

$$\begin{aligned}
&\forall x \in (-\frac{\pi}{2}, \frac{\pi}{2}), \tan(x) = y \\
\implies &(\tan^{-1})'(y) = \frac{1}{\tan'(x)} = \frac{1}{1 + \big(\tan(x)\big)^2} = \frac{1}{1+y^2}.
\end{aligned}$$

□

Ex. II.4.7.9. Recall the arctangent function $\tan^{-1}$ from Ex. II.4.7.8. By modifying the proof of Thm. II.4.5.6(e), establish the identity

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

for all $x \in (-1, 1)$. Using Abel's theorem (Thm. II.4.3.1) to extend this identity to the case $x = 1$, conclude in particular the identity

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

(Note that the series converges by the alternating series test, Proposition 7.2.12 in Analysis I.) Conclude in particular that $4 - \frac{4}{3} < \pi < 4$. (One can of course compute $\pi = 3.1415926\ldots$ to much higher accuracy, though if one wishes to do so it is advisable to use a different formula than the one above, which converges very slowly.)

Proof. By Ex. II.4.7.9, we know that $\tan^{-1}(x)$ is well-defined for all $x \in \mathbb{R}$, in particular, $\tan(x)$ is well-defined for all $x \in (-1, 1)$. First, suppose that $x = 0$. By Ex. II.4.7.9, we know that tan is bijective from $(-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, we have

$$\begin{aligned}
\tan^{-1}(0) &= \tan^{-1}\left(\frac{\sin(0)}{\cos(0)}\right) && \text{(by Thm. II.4.7.2(e))} \\
&= \tan^{-1}\big(\tan(0)\big) && \text{(by Ex. II.4.7.9)} \\
&= 0
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{2n+1}.$$

Now suppose that $x \in (-1, 1)$. Observe that

$$x \in (-1,1) \implies x^2 \in (-1,1) \implies -x^2 \in (-1,1).$$

Since

$$\begin{aligned}
(\tan^{-1})'(x) &= \frac{1}{1+x^2} && \text{(by Ex. II.4.7.8)} \\
&= \frac{1}{1-(-x^2)} && (-x^2 \in (-1,1)) \\
&= \sum_{n=0}^{\infty} (-x^2)^n && \text{(geometric series)} \\
&= \sum_{n=0}^{\infty} \left((-1)^n x^{2n}\right),
\end{aligned}$$

we know that

$$\begin{aligned}
&\tan^{-1}(x) \\
&= \tan^{-1}(x) - \tan^{-1}(0) && (\tan^{-1}(0) = 0) \\
&= \int_0^x (\tan^{-1})'(y)\, dy && \text{(by fundamental theorem of calculus)} \\
&= \int_0^x \left(\sum_{n=0}^{\infty} \left((-1)^n y^{2n}\right) \right) dy && \text{(by Thm. II.4.1.6(c)(e))} \\
&= \sum_{n=0}^{\infty} \left(\int_0^x \left((-1)^n y^{2n}\right) \right) dy && \text{(by Cor. II.3.6.2)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (x^{2n+1} - 0^{2n+1})}{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.
\end{aligned}$$

Since the sequence $(\frac{1}{2n+1})_{n=0}^{\infty}$ is monotone decreasing, we know that the following series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{2n+1};$$

$$-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1}$$

are convergent. Thus, by Abel's theorem (Thm. II.4.3.1) we know that

$$\forall x\in[-1,1], \tan^{-1}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}.$$

Since

$$\begin{aligned}
0 &= \cos(\frac{\pi}{2}) && \text{(by Thm. II.4.7.5(c))}\\
&= \cos(\frac{\pi}{4}+\frac{\pi}{4})\\
&= \cos(\frac{\pi}{4})\cos(\frac{\pi}{4})-\sin(\frac{\pi}{4})\sin(\frac{\pi}{4}) && \text{(by Thm. II.4.7.2(d))}
\end{aligned}$$

we know that

$$\begin{aligned}
&\big(\sin(\frac{\pi}{4})\big)^2=\big(\cos(\frac{\pi}{4})\big)^2\\
\implies &\left(\frac{\sin(\frac{\pi}{4})}{\cos(\frac{\pi}{4})}\right)^2=\big(\tan(\frac{\pi}{4})\big)^2=1 && \text{(by Ex. II.4.7.8)}\\
\implies &\tan(\frac{\pi}{4})=1. && \big(\tan\big((0,\frac{\pi}{2})\big)\subseteq\mathbb{R}^+\big)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\frac{\pi}{4}=\tan^{-1}\big(\tan(\frac{\pi}{4})\big)=\tan^{-1}(1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} && \text{(from the proof above)}\\
\implies &\pi=4\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\right).
\end{aligned}$$

This means

$$\begin{aligned}
\pi &= 4\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\right)\\
&= 4+4\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\right)\\
&= 4+4\left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2(n+1)+1}\right)\\
&= 4+4\left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+3}\right)
\end{aligned}$$

$$\begin{aligned}
&= 4 + 4\left(\sum_{n=0}^{\infty}\left(\frac{-1}{2(2n)+3} + \frac{1}{2(2n+1)+3}\right)\right) && \text{(grouping each two terms)}\\
&= 4 + 4\left(\sum_{n=0}^{\infty}\left(\frac{-1}{4n+3} + \frac{1}{4n+5}\right)\right)\\
&= 4 + 4\left(\sum_{n=0}^{\infty}\left(\frac{-2}{(4n+3)(4n+5)}\right)\right)\\
&= 4 - 4\left(\sum_{n=0}^{\infty}\left(\frac{2}{(4n+3)(4n+5)}\right)\right)\\
&< 4
\end{aligned}$$

and

$$\begin{aligned}
\pi &= 4\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\right)\\
&= 4\left(\sum_{n=0}^{\infty}\left(\frac{1}{2(2n)+1} - \frac{1}{2(2n+1)+1}\right)\right) && \text{(grouping each two terms)}\\
&= 4\left(\sum_{n=0}^{\infty}\left(\frac{1}{4n+1} - \frac{1}{4n+3}\right)\right)\\
&= 4\left(\sum_{n=0}^{\infty}\frac{2}{(4n+1)(4n+3)}\right)\\
&= \frac{8}{3} + 4\left(\sum_{n=1}^{\infty}\frac{2}{(4n+1)(4n+3)}\right)\\
&= 4 - \frac{4}{3} + 4\left(\sum_{n=1}^{\infty}\frac{2}{(4n+1)(4n+3)}\right)\\
&> 4 - \frac{4}{3}.
\end{aligned}$$

Thus, we conclude that $\pi \in (\frac{4}{3}, 4)$. $\square$

Ex. II.4.7.10. Let $f : \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) \coloneqq \sum_{n=1}^{\infty}\left(4^{-n}\cos(32^n \pi x)\right).$$

(a) Show that this series is uniformly convergent, and that f is continuous.

(b) Show that for every integer j and every integer $m \geq 1$, we have

$$\left| f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right) \right| \geq 4^{-m}.$$

(c) Using (b), show that for every real number x_0, the function f is not differentiable at x_0.

(d) Explain briefly why the result in (c) does not contradict Cor. II.3.7.3.

Proof. (a) Since

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\| 4^{-n} \cos(32^n \pi x) \right\|_{\infty} \\
&= \sum_{n=1}^{\infty} \sup\left\{ \left| 4^{-n} \cos(32^n \pi x) \right| : x \in \mathbb{R} \right\} && \text{(by Def. II.3.5.5)} \\
&= \sum_{n=1}^{\infty} 4^{-n} \\
&= \frac{1}{1 - \frac{1}{4}} && \text{(by geometric series)}
\end{aligned}$$

By Weierstrass M-test (Thm. II.3.5.7) we know that $\sum_{n=1}^{\infty} \left(4^{-n} \cos(32^n \pi x)\right)$ converges uniformly to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. □

Proof. (b) First, we show that

$$\forall x, y \in \mathbb{R}, |\cos(x) - \cos(y)| \leq |x - y|.$$

Fix one pair of x, y and without the loss of generality suppose that $y \leq x$. By Thm. II.4.7.2(a)(b) we know that sin is continuous and bounded on $[y, x]$, thus by Corollary 11.5.2 in Analysis I we know that sin is Riemann integrable on $[y, x]$. Then we have

$$\begin{aligned}
&\int_y^x -1\, dz \leq \int_y^x \sin(z)\, dz \leq \int_y^x 1\, dz && \text{(by Thm. II.4.7.2(a))} \\
\implies &-(x - y) \leq -\big(\cos(x) - \cos(y)\big) \leq x - y && \text{(by Thm. II.4.7.2(b))} \\
\implies &|\cos(x) - \cos(y)| \leq |x - y|.
\end{aligned}$$

The case $x \leq y$ can be proven similarly.

Define

$$\forall (j, m, n) \in \mathbb{Z} \times \mathbb{Z}^+ \times \mathbb{Z}^+, a_n^{(j,m)} = 4^{-n}\left(\cos\left(\frac{32^n \pi (j+1)}{32^m}\right) - \cos\left(\frac{32^n \pi j}{32^m}\right)\right).$$

Then we have

$$
\begin{aligned}
&\left|f\left(\frac{j+1}{32^m}\right)-f\left(\frac{j}{32^m}\right)\right| \\
&=\left|\sum_{n=1}^{\infty}\left(4^{-n}\cos\left(\frac{32^n\pi(j+1)}{32^m}\right)\right)-\sum_{n=1}^{\infty}\left(4^{-n}\cos\left(\frac{32^n\pi j}{32^m}\right)\right)\right| \\
&=\left|\sum_{n=1}^{\infty}\left(4^{-n}\cos\left(\frac{32^n\pi(j+1)}{32^m}\right)-4^{-n}\cos\left(\frac{32^n\pi j}{32^m}\right)\right)\right| \\
&=\left|\sum_{n=1}^{\infty}a_n^{(j,m)}\right|.
\end{aligned}
$$

Now we split into three cases:

- If $n<m$, then we have

$$
\begin{aligned}
\left|a_n^{(j,m)}\right| &= \left|4^{-n}\left(\cos\left(\frac{32^n\pi(j+1)}{32^m}\right)-\cos\left(\frac{32^n\pi j}{32^m}\right)\right)\right| \\
&\le \left|4^{-n}\left(\frac{32^n\pi(j+1)}{32^m}-\frac{32^n\pi j}{32^m}\right)\right| && \text{(from the proof above)} \\
&= 4^{-n}\frac{32^n\pi}{32^m}=\frac{8^n\pi}{32^m}=\frac{8^n\pi}{8^m4^m}=\frac{\pi}{8^{m-n}4^m} \\
&\le \frac{4}{8^{m-n}4^m} && \text{(by Ex. II.4.7.9)} \\
&\le \frac{4^{m-n}}{8^{m-n}4^m} && (m>n) \\
&= \frac{4^{m-n}}{4^{m-n}2^{m-n}4^m}=2^{n-m}4^{-m}.
\end{aligned}
$$

- If $n=m$, then we have

$$
\begin{aligned}
&\left|a_m^{(j,m)}\right| \\
&=\left|4^{-m}\Big(\cos\left(\pi(j+1)\right)-\cos(\pi j)\Big)\right| \\
&=\left|4^{-m}\big(-2\cos(\pi j)\big)\right| && \text{(by Thm. II.4.7.5(a))} \\
&=2\cdot 4^{-m}. && \text{(by A.Cor. II.4.7.2(f))}
\end{aligned}
$$

- If $n>m$, then we have

$$
\begin{aligned}
&a_n^{(j,m)} \\
&=4^{-n}\Big(\cos\left(32^{n-m}\pi(j+1)\right)-\cos(32^{n-m}\pi j)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= 4^{-n}\Big(\cos(32^{n-m}\pi j + 32^{n-m}\pi) - \cos(32^{n-m}\pi j)\Big) \\
&= 4^{-n}\Big(\cos(32^{n-m}\pi j) - \cos(32^{n-m}\pi j)\Big) && \text{(by Thm. II.4.7.5(a))} \\
&= 0.
\end{aligned}
$$

From all cases above, we conclude that

$$
\begin{aligned}
&\left| f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right) \right| \\
&= \left| \sum_{n=1}^{\infty} a_n^{(j,m)} \right| \\
&= \left| \sum_{n=1}^{m-1} a_n^{(j,m)} + a_m^{(j,m)} + \sum_{n=m+1}^{\infty} a_n^{(j,m)} \right| \\
&= \left| \sum_{n=1}^{m-1} a_n^{(j,m)} + a_m^{(j,m)} \right| && \text{(from the proof above)} \\
&\geq \left| a_m^{(j,m)} \right| - \left| \sum_{n=1}^{m-1} a_n^{(j,m)} \right| && (|x+y| \geq |x| - |y|) \\
&\geq \left| a_m^{(j,m)} \right| - \sum_{n=1}^{m-1} \left| a_n^{(j,m)} \right| \\
&\geq 2 \cdot 4^{-m} - \sum_{n=1}^{m-1} (2^{n-m} 4^{-m}) && \text{(from the proof above)} \\
&= 2 \cdot 4^{-m} - 4^{-m} \left(\sum_{n=1}^{m-1} 2^{n-m} \right) \\
&\geq 2 \cdot 4^{-m} - 4^{-m} \left(\sum_{n=1}^{\infty} 2^{-n} \right) \\
&= 2 \cdot 4^{-m} - 4^{-m} \left(-1 + \sum_{n=0}^{\infty} 2^{-n} \right) \\
&= 2 \cdot 4^{-m} - 4^{-m} \left(-1 + \frac{1}{1-\frac{1}{2}} \right) && \text{(geometric series)} \\
&= 2 \cdot 4^{-m} - 4^{-m} = 4^{-m}.
\end{aligned}
$$

□

Proof. (c) Suppose for the sake of contradiction that there exists a $x_0 \in \mathbb{R}$ such that

$f'(x_0) \in \mathbb{R}$. Then by Proposition 10.1.7 in Analysis I we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \\
&\forall x \in \mathbb{R}, |x - x_0| < \delta \implies \left|\big(f(x) - f(x_0)\big) - f'(x_0)(x - x_0)\right| < \varepsilon|x - x_0|.
\end{aligned}$$

In particular, we have

$$\begin{aligned}
&\exists \delta \in \mathbb{R}^+ : \forall x \in \mathbb{R}, |x - x_0| < \delta \\
\implies &\left|f(x) - f(x_0) - f'(x_0)(x - x_0)\right| < |x - x_0| \\
\implies &|f(x) - f(x_0)| - \left|f'(x_0)(x - x_0)\right| < |x - x_0| && (|a| - |b| \le |a - b|) \\
\implies &|f(x) - f(x_0)| < \left|f'(x_0)(x - x_0)\right| + |x - x_0| \\
\implies &|f(x) - f(x_0)| < \big(\left|f'(x_0)\right| + 1\big)|x - x_0|.
\end{aligned}$$

Fix such δ. Since $\lim_{m\to\infty} 32^m = +\infty$, we know that

$$\exists m_1 \in \mathbb{Z}^+ : 32^{m_1} > \frac{1}{\delta} \implies \exists m_1 \in \mathbb{Z}^+ : \frac{1}{32^{m_1}} < \delta.$$

Similarly, since $\lim_{m\to\infty} 8^m = +\infty$, we know that

$$\exists m_2 \in \mathbb{Z}^+ : 8^{m_2} > \left|f'(x_0)\right| + 1.$$

Let $m = \max(m_1, m_2)$. By Exercise 5.4.3 in Analysis I we know that

$$\begin{aligned}
&32^m x_0 \in \mathbb{R} \\
\implies &\exists j \in \mathbb{Z} : j \le 32^m x_0 < j + 1 \\
\implies &\exists j \in \mathbb{Z} : \frac{j}{32^m} \le x_0 < \frac{j+1}{32^m}.
\end{aligned}$$

Fix such j. Then we have

$$\begin{aligned}
&\frac{j+1}{32^m} - \frac{j}{32^m} = \frac{1}{32^m} < \delta \\
\implies &\begin{cases} \dfrac{j+1}{32^m} - x_0 < \delta \\ x_0 - \dfrac{j}{32^m} < \delta \end{cases} \\
\implies &\begin{cases} \left|f\left(\dfrac{j+1}{32^m}\right) - f(x_0)\right| < \left(\left|f'(x_0)\right| + 1\right)\left(\dfrac{j+1}{32^m} - x_0\right) \\ \left|f\left(\dfrac{j}{32^m}\right) - f(x_0)\right| < \left(\left|f'(x_0)\right| + 1\right)\left(x_0 - \dfrac{j}{32^m}\right) \end{cases} && \text{(from the proof above)} \\
\implies &4^{-m} \le \left|f\left(\frac{j+1}{32^m}\right) - f\left(\frac{j}{32^m}\right)\right| && \text{(by Ex. II.4.7.10)}
\end{aligned}$$

$$
\begin{aligned}
&\le \left|f\left(\frac{j+1}{32^m}\right) - f(x_0)\right| + \left|f\left(\frac{j}{32^m}\right) - f(x_0)\right| \\
&< \frac{|f'(x_0)| + 1}{32^m} \\
\Longrightarrow\ & 8^m < \left|f'(x_0)\right| + 1.
\end{aligned}
$$

But this contradicts the fact that $8^m > \left|f'(x_0)\right| + 1$. Thus, such x_0 does not exist and f is not differentiable on $\mathbb{R}$. □

Proof. (d) Let $f_n(x) = 4^{-n}\cos(32^n\pi x)$ for all $n \in \mathbb{Z}^+$. By Thm. II.4.7.2(b) we know that f_n is differentiable on $\mathbb{R}$ and by chain rule we have

$$\forall n \in \mathbb{Z}^+, f_n'(x) = 4^{-n}\big(-\sin(32^n\pi x)\big)(32^n\pi) = -8^n\pi\sin(32^n\pi x).$$

Since

$$
\begin{aligned}
\sum_{n=1}^{\infty}\left\|f_n'\right\|_\infty &= \sum_{n=1}^{\infty}\sup\left\{\left|f_n'(x)\right| : x \in \mathbb{R}\right\} && \text{(by Def. II.3.5.5)} \\
&= \sum_{n=1}^{\infty} 8^n\pi \\
&= +\infty, && \text{(by ratio test)}
\end{aligned}
$$

the condition in Cor. II.3.7.3 is not satisfied. Thus, this does not contradict to Cor. II.3.7.3. □

Chapter II.5

Fourier series

Note. Power series are already immensely useful, especially when dealing with special functions such as the exponential and trigonometric functions discussed earlier. However, there are some circumstances where power series are not so useful, because one has to deal with functions (e.g., $\sqrt{x}$) which are not real analytic, and so do not have power series.

Note. Fortunately, there is another type of series expansion, known as *Fourier series*, which is also a very powerful tool in analysis (though used for slightly different purposes). Instead of analyzing compactly supported functions, it instead analyzes *periodic functions*; instead of decomposing into polynomials, it decomposes into *trigonometric polynomials*. Roughly speaking, the theory of Fourier series asserts that just about every periodic function can be decomposed as an (infinite) sum of sines and cosines.

Rmk. II.5.0.1. Jean-Baptiste Fourier (1768–1830) was, among other things, an administrator accompanying Napoleon on his invasion of Egypt, and then a Prefect in France during Napoleons reign. After the Napoleonic wars, he returned to mathematics. He introduced Fourier series in an important 1807 paper in which he used them to solve what is now known as the heat equation. At the time, the claim that every periodic function could be expressed as a sum of sines and cosines was extremely controversial, even such leading mathematicians as Euler declared that it was impossible. Nevertheless, Fourier managed to show that this was indeed the case, although the proof was not completely rigorous and was not totally accepted for almost another hundred years.

Note. For instance, the convergence of Fourier series is usually not uniform (i.e., not in the L^∞ metric), but instead we have convergence in a different metric, the L^2-metric. We will need to use complex numbers heavily in our theory, while they played only a tangential rôle in power series.

Note. The theory of Fourier series (and of related topics such as Fourier integrals and the Laplace transform) is vast, and deserves an entire course in itself. It has many, many applications, most directly to differential equations, signal processing, electrical engineering, physics, and analysis, but also to algebra and number theory.

II.5.1 Periodic functions

Def. II.5.1.1. Let $L > 0$ be a real number. A function $f : \mathbb{R} \to \mathbb{C}$ is periodic with period L, or L-periodic, if we have $f(x + L) = f(x)$ for every real number x.

E.g. II.5.1.2. The real-valued functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$ are 2π-periodic, as is the complex-valued function $f(x) = e^{ix}$. These functions are also 4π-periodic, 6π-periodic, etc. The function $f(x) = x$, however, is not periodic. The constant function $f(x) = 1$ is L-periodic for every L.

Rmk. II.5.1.3. If a function f is L-periodic, then we have $f(x + kL) = f(x)$ for every integer k (why? Use induction for the positive k, and then use a substitution to convert the positive k result to a negative k result. The $k = 0$ case is of course trivial). In particular, if a function f is 1-periodic, then we have $f(x + k) = f(x)$ for every $k \in \mathbb{Z}$. Because of this, 1-periodic functions are sometimes also called $\mathbb{Z}$-periodic (and L-periodic functions called $L\mathbb{Z}$-periodic).

E.g. II.5.1.4. For any integer n, the functions $x \mapsto \cos(2\pi nx)$, $x \mapsto \sin(2\pi nx)$, and $x \mapsto e^{2\pi inx}$ are all $\mathbb{Z}$-periodic. Another example of a $\mathbb{Z}$-periodic function is the function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(x) := 1$ when $x \in [n, n + \frac{1}{2})$ for some integer n, and $f(x) := 0$ when $x \in [n + \frac{1}{2}, n + 1)$ for some integer n. This function is an example of a *square wave*.

Note. In order to completely specify a $\mathbb{Z}$-periodic function $f : \mathbb{R} \to \mathbb{C}$, one only needs to specify its values on the interval $[0, 1)$, since this will determine the values of f everywhere else. This is because every real number x can be written in the form $x = k + y$ where k is an integer (called the *integer part* of x, and sometimes denoted $[x]$) and $y \in [0, 1)$ (this is called the *fractional part* of x, and sometimes denoted $\{x\}$). Because of this, sometimes when we wish to describe a $\mathbb{Z}$-periodic function f we just describe what it does on the interval $[0, 1)$, and then say that it is *extended periodically* to all of $\mathbb{R}$. This means that we define $f(x)$ for any real number x by setting $f(x) := f(y)$, where we have decomposed $x = k + y$ as discussed above. (One can in fact replace the interval $[0, 1)$ by any other half-open interval of length 1, but we will not do so here.)

Note. The space of complex-valued continuous $\mathbb{Z}$-periodic functions is denoted

$$C(\mathbb{R}/\mathbb{Z}; \mathbb{C}).$$

(The notation $\mathbb{R}/\mathbb{Z}$ comes from algebra, and denotes the quotient group of the additive group $\mathbb{R}$ by the additive group $\mathbb{Z}$; more information in this can be found in any algebra text.) By "continuous" we mean continuous at all points on $\mathbb{R}$; merely being continuous on an interval such as $[0, 1]$ will not suffice, as there may be a discontinuity between the left and right limits at 1 (or at any other integer). Thus, for instance, the functions $x \mapsto \sin(2\pi nx)$, $x \mapsto \cos(2\pi nx)$, and $x \mapsto e^{2\pi inx}$ are all elements of $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, as are the constant functions, however the square wave function in E.g. II.5.1.4 is not in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ because it is not continuous at every integer. Also the function $\sin(x)$ would also not qualify to be in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ since it is not $\mathbb{Z}$-periodic.

Lem. II.5.1.5 (Basic properties of $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$).

(a) (Boundedness) If $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, then f is bounded (i.e., there exists a real number $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$).

(b) (Vector space and algebra properties) If $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, then the functions $f + g$, $f - g$, and fg are also in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. Also, if c is any complex number, then the function cf is also in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.

(c) (Closure under uniform limits) If $(f_n)_{n=1}^{\infty}$ is a sequence of functions in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ which converges uniformly to another function $f : \mathbb{R} \to \mathbb{C}$, then f is also in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.

Proof. (a) Since $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, by Def. II.5.1.1 we have

$$\{f(x) : x \in \mathbb{R}\} = \{f(x) : x \in [0,1)\} = \{f(x) : x \in [0,1]\}.$$

So it suffices to show that $\{f(x) : x \in [0,1]\}$ is bounded. Let $d_{\mathbb{R}} = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$ and let $d_{\mathbb{C}}$ be the metric in Def. II.4.6.10. Since $[0,1]$ is closed and bounded in $(\mathbb{R}, d_{\mathbb{R}})$, by Heine-Borel theorem (Thm. II.1.5.7) we know that $([0,1], d_{\mathbb{R}}|_{[0,1]\times[0,1]})$ is compact. Since f is continuous on $[0,1]$, by Thm. II.2.3.1 we know that $\big(f([0,1]), d_{\mathbb{C}}|_{f([0,1])\times f([0,1])}\big)$ is also compact. By Cor. II.1.5.6 we know that compactness implies boundness, thus we have

$$\begin{aligned}
&\forall z \in \mathbb{C}, \exists r \in \mathbb{R}^+ : f([0,1]) \subseteq B_{(\mathbb{C},d_{\mathbb{C}})}(z,r) && \text{(by Def. II.1.5.3)}\\
\implies& \exists r \in \mathbb{R}^+ : f([0,1]) \subseteq B_{(\mathbb{C},d_{\mathbb{C}})}(1,r) \\
\implies& \exists r \in \mathbb{R}^+ : \forall y \in f([0,1]), |y-1| < r && \text{(by Def. II.1.2.1)}\\
\implies& \exists r \in \mathbb{R}^+ : \forall y \in f([0,1]), \\
& |y| = |y-1+1| \leq |y-1| + 1 < r + 1. && \text{(by Lem. II.4.6.11)}
\end{aligned}$$

By setting $M = r + 1$ we are done. □

Proof. (b) We have

$$\begin{aligned}
(f+g)(x+1) &= f(x+1) + g(x+1) \\
&= f(x) + g(x) && \text{(by Def. II.5.1.1)}\\
&= (f+g)(x) \\
(f-g)(x+1) &= f(x+1) - g(x+1) \\
&= f(x) - g(x) && \text{(by Def. II.5.1.1)}\\
&= (f-g)(x) \\
(fg)(x+1) &= f(x+1)g(x+1)
\end{aligned}$$

$$
\begin{aligned}
&= f(x)g(x) && \text{(by Def. II.5.1.1)}\\
&= (fg)(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\forall c \in \mathbb{C}, (cf)(x+1) &= cf(x+1)\\
&= cf(x) && \text{(by Def. II.5.1.1)}\\
&= (cf)(x).
\end{aligned}
$$

□

Proof. (c) Let $d_{\mathbb{R}} = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$ and let $d_{\mathbb{C}}$ be the metric in Def. II.4.6.10. Since $(f_n)_{n=0}^{\infty}$ converges uniformly to f on $\mathbb{C}$ with respect to $d_{\mathbb{C}}$, by Cor. II.3.3.2 we know that f is continuous from $(\mathbb{R}, d_{\mathbb{R}})$ to $(\mathbb{C}, d_{\mathbb{C}})$. Suppose for the sake of contradiction that $f \notin C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. By Def. II.5.1.1 this means

$$\exists x \in \mathbb{R} : f(x+1) \neq f(x).$$

By Ex. II.3.2.2 we know that $(f_n)_{n=0}^{\infty}$ converges pointwise to f on $\mathbb{C}$ with respect to $d_{\mathbb{C}}$, thus by Def. II.3.2.1 we have

$$
\begin{aligned}
&\begin{cases} d - \lim\limits_{n\to\infty} f_n(x) = f(x) \\ d - \lim\limits_{n\to\infty} f_n(x+1) = f(x+1) \end{cases}\\
\implies &\forall \varepsilon \in \mathbb{R}, \exists N \in \mathbb{Z}^+ : \forall n \geq N,\\
&\begin{cases} |f_n(x) - f(x)| < \dfrac{\varepsilon}{2} \\ |f_n(x+1) - f(x+1)| < \dfrac{\varepsilon}{2} \end{cases}\\
\implies &\forall \varepsilon \in \mathbb{R}, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \begin{cases} |f_n(x) - f(x)| < \dfrac{\varepsilon}{2} \\ |f_n(x) - f(x+1)| < \dfrac{\varepsilon}{2} \end{cases} && \text{(by Def. II.5.1.1)}\\
\implies &\forall \varepsilon \in \mathbb{R}, \exists N \in \mathbb{Z}^+ : \forall n \geq N,\\
&|f(x) - f(x+1)| \leq |f(x) - f_n(x)| + |f_n(x) - f(x+1)|\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\\
\implies &\forall \varepsilon \in \mathbb{R}^+, |f(x) - f(x+1)| < \varepsilon\\
\implies &f(x) = f(x+1).
\end{aligned}
$$

But this contradicts the definition of x. Thus, $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. □

Note. One can make $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ into a metric space by re-introducing the now familiar sup-norm metric

$$d_\infty(f, g) = \sup_{x\in\mathbb{R}} |f(x) - g(x)| = \sup_{x\in[0,1)} |f(x) - g(x)|$$

of uniform convergence.

A.Cor. II.5.1.1 (modular operation)**.** Let $n \in \mathbb{Z}^+$. Define $\mathrm{mod}_n : \mathbb{R} \to \mathbb{R}$ as follow:

$$\forall x \in \mathbb{R}, \mathrm{mod}_n(x) = x - \left[\frac{x}{n}\right] n.$$

Then $\mathrm{mod}(\mathbb{R}) \subseteq [0, n)$ and mod is n-periodic. We often use $x \mod n$ instead of $\mathrm{mod}_n(x)$.

Proof. Since

$$\begin{aligned}
&\forall x \in \mathbb{R}, \left[\frac{x}{n}\right] \leq \frac{x}{n} < \left[\frac{x}{n}\right] + 1 && \text{(by Def. II.5.1.1)} \\
\implies &\left[\frac{x}{n}\right] n \leq x < \left[\frac{x}{n}\right] n + n && (n \in \mathbb{Z}^+) \\
\implies &0 \leq x - \left[\frac{x}{n}\right] n < n,
\end{aligned}$$

we know that $\mathrm{mod}_n(x) \subseteq [0, n)$. Since

$$\begin{aligned}
\forall x \in \mathbb{R}, \mathrm{mod}_n(x+n) &= x + n - \left[\frac{x+n}{n}\right] n \\
&= x + n - \left(\left[\frac{x}{n}\right] + 1\right) n && \text{(by Ex. II.5.1.1)} \\
&= x + n - \left[\frac{x}{n}\right] n - n \\
&= x - \left[\frac{x}{n}\right] n \\
&= \mathrm{mod}_n(x),
\end{aligned}$$

by Def. II.5.1.1 we know that mod_n is n-periodic. □

— Exercises —

Ex. II.5.1.1. Show that every real number x can be written in exactly one way in the form $x = k + y$, where k is an integer and $y \in [0, 1)$.

Proof. By Exercise 5.4.3 we know that

$$\forall x \in \mathbb{R}, \exists!\ k \in \mathbb{Z} : k \leq x < k + 1.$$

Thus, by setting $y = x - k$ we have $x = y + k$ and $y \in [0, 1)$. □

Ex. II.5.1.2. Prove Lem. II.5.1.5.

Proof. See Lem. II.5.1.5. □

Ex. II.5.1.3. Show that $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ with the sup-norm metric d_∞ is a metric space. Furthermore, show that this metric space is complete.

Proof. First, we show that $\big(C(\mathbb{R}/\mathbb{Z};\mathbb{C}), d_\infty\big)$ is a metric space. Since

$$\forall f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}), d_\infty(f,f) = \sup_{x\in[0,1)} |f(x) - f(x)| = 0,$$

we know that d_∞ satisfied Def. II.1.1.2(a). Since

$$\begin{aligned}
&\forall f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}), f \neq g \\
\implies &\exists x \in \mathbb{R} : f(x) \neq g(x) \\
\implies &\exists x \in [0,1) : f(x) \neq g(x) && \text{(by Def. II.5.1.1)} \\
\implies &0 < \sup_{x\in[0,1)} |f(x) - g(x)| < +\infty && \text{(by Lem. II.5.1.5(a))} \\
\implies &0 < d_\infty(f,g) < +\infty,
\end{aligned}$$

we know that d_∞ satisfied Def. II.1.1.2(b). Since

$$\begin{aligned}
\forall f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}), d_\infty(f,g) &= \sup_{x\in[0,1)} |f(x) - g(x)| \\
&= \sup_{x\in[0,1)} |g(x) - f(x)| && \text{(by Def. II.4.6.10)} \\
&= d_\infty(g,f),
\end{aligned}$$

we know that d_∞ satisfied Def. II.1.1.2(c). Since

$$\begin{aligned}
&\forall f, g, h \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}), d_\infty(f,h) \\
&= \sup_{x\in[0,1)} |f(x) - h(x)| \\
&= \sup_{x\in[0,1)} |f(x) - g(x) + g(x) - h(x)| \\
&\leq \sup_{x\in[0,1)} \big(|f(x) - g(x)| + |g(x) - h(x)|\big) && \text{(by Lem. II.4.6.11)} \\
&= d_\infty(f,g) + d_\infty(g,h),
\end{aligned}$$

we know that d_∞ satisfied Def. II.1.1.2(d). From all proofs above, we conclude by Def. II.1.1.2 that $\big(C(\mathbb{R}/\mathbb{Z};\mathbb{C}), d_\infty\big)$ is a metric space.

Now we show that $\big(C(\mathbb{R}/\mathbb{Z};\mathbb{C}), d_\infty\big)$ is complete. Let $d_\mathbb{R} = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$ and let $d_\mathbb{C}$ be the metric in Def. II.4.6.10. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $\big(C(\mathbb{R}/\mathbb{Z};\mathbb{C}), d_\infty\big)$ and let $n_1, n_2 \in \mathbb{Z}^+$. Then by Def. II.1.4.6 we have

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n_1, n_2 \geq N, d_\infty(f_{n_1}, f_{n_2}) < \varepsilon$$

$$\begin{aligned}
&\implies \forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n_1, n_2 \geq N, \\
&\qquad |f_{n_1}(x) - f_{n_2}(x)| \leq \sup_{y \in \mathbb{R}} |f_{n_1}(y) - f_{n_2}(y)| = d_\infty(f_{n_1}, f_{n_2}) < \varepsilon \\
&\implies \forall x \in \mathbb{R}, \big(f_n(x)\big)_{n=1}^\infty \text{ is a Cauchy sequence in } (\mathbb{C}, d_{\mathbb{C}}).
\end{aligned}$$

Since $(\mathbb{C}, d_{\mathbb{C}})$ is complete (by Ex. II.4.6.10), we know that

$$\forall x \in \mathbb{C}, \lim_{n \to \infty} f_n(x) \in \mathbb{C}.$$

Thus, we can define $f : \mathbb{R} \to \mathbb{C}$ as follow:

$$\forall x \in \mathbb{R}, f(x) = \lim_{n \to \infty} f_n(x).$$

And we have

$$\begin{aligned}
&\forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \\
&\begin{cases} |f(x) - f_n(x)| < \dfrac{\varepsilon}{2} \\ |f(x+1) - f_n(x+1)| = |f(x+1) - f_n(x)| < \dfrac{\varepsilon}{2} \end{cases} \qquad (f_n \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})) \\
\implies &\forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \\
&|f(x) - f(x+1)| \leq |f(x) - f_n(x)| + |f(x+1) - f_n(x)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \\
\implies &\forall x \in \mathbb{R}, \forall \varepsilon \in \mathbb{R}^+, |f(x) - f(x+1)| < \varepsilon \\
\implies &\forall x \in \mathbb{R}, f(x) = f(x+1) \\
\implies &f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C}).
\end{aligned}$$

Since $(f_n)_{n=1}^\infty$ was arbitrary, by Def. II.1.4.10 we know that $\big(C(\mathbb{R}/\mathbb{Z}; \mathbb{C}), d_\infty\big)$ is complete. □

II.5.2 Inner products on periodic functions

Def. II.5.2.1 (Inner product)**.** If $f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, we define the *inner product* $\langle f, g \rangle$ to be the quantity

$$\langle f, g \rangle = \int_{[0,1]} f(x)\overline{g(x)}\, dx.$$

Rmk. II.5.2.2. In order to integrate a complex-valued function over real variables, we use the definition that

$$\int_{[a,b]} f(x)\, dx := \int_{[a,b]} \Re\big(f(x)\big)\, dx + i \int_{[a,b]} \Im\big(f(x)\big)\, dx;$$

i.e., we integrate the real and imaginary parts of the function separately. It is easy to verify that all the standard rules of calculus (integration by parts, fundamental theorem of calculus, substitution, etc.) still hold when the functions are complex-valued instead of real-valued.

Proof. Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, let $d_{\mathbb{R}} = d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$ and let $d_{\mathbb{C}}$ be the metric in Def. II.4.6.10. Let $x_0 \in \mathbb{R}$ and let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} a_n = x_0$. Since f is continuous on $\mathbb{R}$ from $(\mathbb{R}, d_{\mathbb{R}})$ to $(\mathbb{C}, d_{\mathbb{C}})$, we know that

$$\begin{aligned} &\lim_{n\to\infty} f(a_n) = f(x_0) && \text{(by Thm. II.2.1.4)} \\ \implies &\begin{cases} \lim_{n\to\infty} \Re\big(f\big(a_n\big)\big) = \Re\big(f(x_0)\big) \\ \lim_{n\to\infty} \Im\big(f\big(a_n\big)\big) = \Im\big(f(x_0)\big) \end{cases} && \text{(by Lem. II.4.6.13)} \end{aligned}$$

Since $(a_n)_{n=0}^{\infty}$ was arbitrary, by Thm. II.2.1.4 we know that $\Re \circ f$ is continuous at x_0 from $(\mathbb{R}, d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$. Since x_0 was arbitrary, by Thm. II.2.1.5 we know that $\Re \circ f$ is continuous on $\mathbb{R}$ from $(\mathbb{R}, d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$. Using similar arguments, we can show that $\Im \circ f$ is continuous on $\mathbb{R}$ from $(\mathbb{R}, d_{\mathbb{R}})$ to $(\mathbb{R}, d_{\mathbb{R}})$. Since

$$\begin{aligned} \forall x \in \mathbb{R}, \Re(f(x+1)) = \Re(f(x)) &\implies \Re \circ f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}); \\ \forall x \in \mathbb{R}, \Im(f(x+1)) = \Im(f(x)) &\implies \Im \circ f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}), \end{aligned}$$

by Lem. II.5.1.5(a) we know that both $\Re \circ f$ and $\Im \circ f$ are bounded in $(\mathbb{C}, d_{\mathbb{C}})$. In particular, by Def. II.4.6.8 we know that $(\Re \circ f)(\mathbb{R}) \subseteq \mathbb{R}$ and $(\Im \circ f)(\mathbb{R}) \subseteq \mathbb{R}$. Thus, both $\Re \circ f$ and $\Im \circ f$ are bounded in $(\mathbb{R}, d_{\mathbb{R}})$. Since $\Re \circ f$ and $\Im \circ f$ are continuous and bounded on $[0, 1]$, by Corollary 11.5.2 in Analysis I we know that $\Re \circ f$ and $\Im \circ f$ are Riemann integrable on $[0, 1]$. Thus

$$\int_{[0,1]} f(x)\, dx = \int_{[0,1]} \Re\big(f(x)\big)\, dx + i\left(\int_{[0,1]} \Im\big(f(x)\big)\, dx\right) \in \mathbb{C}$$

is well-defined. The same argument holds on arbitrary closed interval $[a, b]$ since $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. ☐

E.g. II.5.2.3. Let f be the constant function $f(x) := 1$, and let $g(x)$ be the function $g(x) := e^{2\pi i x}$. Then we have

$$\begin{aligned} \langle f, g\rangle &= \int_{[0,1]} 1\overline{e^{2\pi i x}}\, dx \\ &= \int_{[0,1]} e^{-2\pi i x}\, dx \\ &= \frac{e^{-2\pi i x}}{-2\pi i}\Big|_{x=0}^{x=1} \\ &= \frac{e^{-2\pi i} - e^0}{-2\pi i} \\ &= \frac{1-1}{-2\pi i} \\ &= 0. \end{aligned}$$

Rmk. II.5.2.4. In general, the inner product $\langle f, g\rangle$ will be a complex number. (Note that $f(x)\overline{g(x)}$ will be Riemann integrable since both functions are bounded and continuous.)

Note. Roughly speaking, the inner product $\langle f, g\rangle$ is to the space $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ what the dot product $x \cdot y$ is to Euclidean spaces such as $\mathbb{R}^n$. A more in-depth study of inner products on vector spaces can be found in any linear algebra text but is beyond the scope of this text.

Lem. II.5.2.5. Let $f, g, h \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.

(a) (Hermitian property) We have $\langle g, f\rangle = \overline{\langle f, g\rangle}$.

(b) (Positivity) We have $\langle f, f\rangle \geq 0$. Furthermore, we have $\langle f, f\rangle = 0$ iff $f = 0$ (i.e., $f(x) = 0$ for all $x \in \mathbb{R}$).

(c) (Linearity in the first variable) We have $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$. For any complex number c, we have $\langle cf, g\rangle = c\langle f, g\rangle$.

(d) (Antilinearity in the second variable) We have $\langle f, g + h\rangle = \langle f, g\rangle + \langle f, h\rangle$. For any complex number c, we have $\langle f, cg\rangle = c\langle f, g\rangle$.

Proof. (a) We have

$$\begin{aligned}
\overline{\langle f, g\rangle} &= \overline{\int_{[0,1]} f(x)\overline{g(x)}\,dx} && \text{(by Def. II.5.2.1)}\\
&= \overline{\int_{[0,1]} \Re\big(f(x)\overline{g(x)}\big)\,dx + i\left(\int_{[0,1]} \Im\big(f(x)\overline{g(x)}\big)\,dx\right)} && \text{(by Rmk. II.5.2.2)}\\
&= \int_{[0,1]} \Re\big(f(x)\overline{g(x)}\big)\,dx - i\left(\int_{[0,1]} \Im\big(f(x)\overline{g(x)}\big)\,dx\right) && \text{(by Def. II.4.6.8)}\\
&= \int_{[0,1]} \Re\big(f(x)\overline{g(x)}\big)\,dx + i\left(-\int_{[0,1]} \Im\big(f(x)\overline{g(x)}\big)\,dx\right) && \text{(by Lem. II.4.6.6)}\\
&= \int_{[0,1]} \Re\big(f(x)\overline{g(x)}\big)\,dx + i\left(\int_{[0,1]} -\Im\big(f(x)\overline{g(x)}\big)\,dx\right) && \\
&= \int_{[0,1]} \Re\big(\overline{f(x)\overline{g(x)}}\big)\,dx + i\left(\int_{[0,1]} \Im\big(\overline{f(x)\overline{g(x)}}\big)\,dx\right) && \text{(by Def. II.4.6.8)}\\
&= \int_{[0,1]} \Re\big(\overline{f(x)}g(x)\big)\,dx + i\left(\int_{[0,1]} \Im\big(\overline{f(x)}g(x)\big)\,dx\right) && \text{(by Lem. II.4.6.9)}\\
&= \int_{[0,1]} \Re\big(g(x)\overline{f(x)}\big)\,dx + i\left(\int_{[0,1]} \Im\big(g(x)\overline{f(x)}\big)\,dx\right) && \text{(by Lem. II.4.6.6)}\\
&= \int_{[0,1]} g(x)\overline{f(x)}\,dx && \text{(by Rmk. II.5.2.2)}\\
&= \langle g, f\rangle. && \text{(by Def. II.5.2.1)}
\end{aligned}$$

□

Proof. (b) We have

$$\begin{aligned}
\langle f, f\rangle &= \int_{[0,1]} f(x)\overline{f(x)}\,dx && \text{(by Def. II.5.2.1)}\\
&= \int_{[0,1]} |f(x)|^2\,dx && \text{(by Lem. II.4.6.11)}\\
&\geq \int_{[0,1]} 0\,dx && \text{(by Theorem 11.4.1(d) in Analysis I)}\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&\int_{[0,1]} |f(x)|^2\,dx = 0\\
\iff&\ \forall x \in [0,1], |f(x)|^2 = 0 && \text{(by Exercise 11.4.2 in Analysis I)}\\
\iff&\ \forall x \in [0,1], |f(x)| = 0\\
\iff&\ \forall x \in [0,1], f(x) = 0. && \text{(by Lem. II.4.6.11)}
\end{aligned}$$

□

Proof. (c) We have

$$\begin{aligned}
&\langle f+g, h\rangle\\
&= \int_{[0,1]} (f+g)(x)\overline{h(x)}\,dx && \text{(by Def. II.5.2.1}\\
&= \int_{[0,1]} \Re\big((f+g)(x)\overline{h(x)}\big)\,dx && \text{(by Rmk. II.5.2}\\
&\quad + i\left(\int_{[0,1]} \Im\big((f+g)(x)\overline{h(x)}\big)\,dx\right)\\
&= \int_{[0,1]} \Re\big(f(x)\overline{h(x)} + g(x)\overline{h(x)}\big)\,dx\\
&\quad + i\left(\int_{[0,1]} \Im\big(f(x)\overline{h(x)} + g(x)\overline{h(x)}\big)\,dx\right)\\
&= \int_{[0,1]} \Re\big(f(x)\overline{h(x)}\big) + \Re\big(g(x)\overline{h(x)}\big)\,dx && \text{(by Def. II.4.6.8}\\
&\quad + i\left(\int_{[0,1]} \Im\big(f(x)\overline{h(x)}\big) + \Im\big(g(x)\overline{h(x)}\big)\,dx\right)\\
&= \int_{[0,1]} \Re\big(f(x)\overline{h(x)}\big)\,dx + \int_{[0,1]} \Re\big(g(x)\overline{h(x)}\big)\,dx && (f\overline{h}, g\overline{h} \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}))
\end{aligned}$$

$$+ i\left(\int_{[0,1]} \Im\big(f(x)\overline{h(x)}\big)\,dx + \int_{[0,1]} \Im\big(g(x)\overline{h(x)}\big)\,dx \right)$$
$$= \int_{[0,1]} f(x)\overline{h(x)}\,dx + \int_{[0,1]} g(x)\overline{h(x)}\,dx \qquad \text{(by Rmk. II.5.2}$$
$$= \langle f,h\rangle + \langle g,h\rangle \qquad \text{(by Def. II.5.2.1}$$

and

$$\langle cf, g\rangle$$
$$= \int_{[0,1]} (cf)(x)\overline{g(x)}\,dx \qquad \text{(by Def. II.5.2}$$
$$= \int_{[0,1]} \Re\big((cf)(x)\overline{g(x)}\big)\,dx + i\left(\int_{[0,1]} \Im\big((cf)(x)\overline{g(x)}\big)\,dx \right) \qquad \text{(by Rmk. II.5}$$
$$= \int_{[0,1]} \Re\big(cf(x)\overline{g(x)}\big)\,dx + i\left(\int_{[0,1]} \Im\big(cf(x)\overline{g(x)}\big)\,dx \right)$$
$$= \int_{[0,1]} \Re(c)\Re\big(f(x)\overline{g(x)}\big) - \Im(c)\Im\big(f(x)\overline{g(x)}\big)\,dx \qquad \text{(by Def. II.4.6}$$
$$+ i\left(\int_{[0,1]} \Re(c)\Im\big(f(x)\overline{g(x)}\big) + \Im(c)\Re\big(f(x)\overline{g(x)}\big)\,dx \right)$$
$$= \Re(c)\left(\int_{[0,1]} \Re\big(f(x)\overline{g(x)}\big)\,dx \right) \qquad (f\overline{g} \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}))$$
$$- \Im(c)\left(\int_{[0,1]} \Im\big(f(x)\overline{g(x)}\big)\,dx \right)$$
$$+ i\Re(c)\left(\int_{[0,1]} \Im\big(f(x)\overline{g(x)}\big)\,dx \right)$$
$$+ i\Im(c)\left(\int_{[0,1]} \Re\big(f(x)\overline{g(x)}\big)\,dx \right)$$
$$= \Re(c)\left(\int_{[0,1]} \Re\big(f(x)\overline{g(x)}\big)\,dx + i\int_{[0,1]} \Im\big(f(x)\overline{g(x)}\big)\,dx \right) \qquad \text{(by Lem. II.4.}$$
$$+ i\Im(c)\left(\int_{[0,1]} \Re\big(f(x)\overline{g(x)}\big)\,dx + i\int_{[0,1]} \Im\big(f(x)\overline{g(x)}\big)\,dx \right) \qquad \text{(by Def. II.4.6}$$
$$= \Re(c)\int_{[0,1]} f(x)\overline{g(x)}\,dx + i\Im(c)\int_{[0,1]} f(x)\overline{g(x)}\,dx \qquad \text{(by Rmk. II.5}$$
$$= \Re(c)\langle f,g\rangle + i\Im(c)\langle f,g\rangle \qquad \text{(by Def. II.5.2}$$
$$= \big(\Re(c) + i\Im(c)\big)\langle f,g\rangle \qquad \text{(by Lem. II.4.}$$
$$= c\langle f,g\rangle. \qquad \text{(by Def. II.4.6}$$

□

Proof. (d) We have

$$
\begin{aligned}
&\langle f, g+h\rangle \\
&= \int_{[0,1]} f(x)\overline{(g+h)(x)}\,dx && \text{(by Def. II.5.2.1} \\
&= \int_{[0,1]} \Re\big(f(x)\overline{(g+h)(x)}\big)\,dx \\
&\quad + i\bigg(\int_{[0,1]} \Im\big(f(x)\overline{(g+h)(x)}\big)\,dx\bigg) && \text{(by Rmk. II.5.2} \\
&= \int_{[0,1]} \Re\big(f(x)\overline{g(x)} + f(x)\overline{h(x)}\big)\,dx && \text{(by Lem. II.4.6.} \\
&\quad + i\bigg(\int_{[0,1]} \Im\big(f(x)\overline{g(x)} + f(x)\overline{h(x)}\big)\,dx\bigg) \\
&= \int_{[0,1]} \Re\big(f(x)\overline{g(x)}\big) + \Re\big(f(x)\overline{h(x)}\big)\,dx && \text{(by Def. II.4.6.8} \\
&\quad + i\bigg(\int_{[0,1]} \Im\big(f(x)\overline{g(x)}\big) + \Im\big(f(x)\overline{h(x)}\big)\,dx\bigg) \\
&= \int_{[0,1]} \Re\big(f(x)\overline{g(x)}\big)\,dx + \int_{[0,1]} \Re\big(f(x)\overline{h(x)}\big)\,dx && (f\overline{g}, f\overline{h} \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})) \\
&\quad + i\bigg(\int_{[0,1]} \Im\big(f(x)\overline{g(x)}\big)\,dx + \int_{[0,1]} \Im\big(f(x)\overline{h(x)}\big)\,dx\bigg) \\
&= \int_{[0,1]} f(x)\overline{g(x)}\,dx + \int_{[0,1]} f(x)\overline{h(x)}\,dx && \text{(by Rmk. II.5.2} \\
&= \langle f, g\rangle + \langle f, h\rangle && \text{(by Def. II.5.2.1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle f, cg\rangle \\
&= \int_{[0,1]} (x)\overline{(cg)(x)}\,dx && \text{(by Def. II.5.2.1)} \\
&= \int_{[0,1]} \Re\big(f(x)\overline{(cg)(x)}\big)\,dx + i\bigg(\int_{[0,1]} \Im\big(f(x)\overline{(cg)(x)}\big)\,dx\bigg) && \text{(by Rmk. II.5.2.2)} \\
&= \int_{[0,1]} \Re\big(\overline{c}f(x)\overline{g(x)}\big)\,dx + i\bigg(\int_{[0,1]} \Im\big(\overline{c}f(x)\overline{g(x)}\big)\,dx\bigg) && \text{(by Lem. II.4.6.9)} \\
&= \int_{[0,1]} \Re\big((\overline{c}f)(x)\overline{g(x)}\big)\,dx + i\bigg(\int_{[0,1]} \Im\big((\overline{c}f)(x)\overline{g(x)}\big)\,dx\bigg) \\
&= \int_{[0,1]} (\overline{c}f)(x)\overline{g(x)}\,dx && \text{(by Rmk. II.5.2.2)}
\end{aligned}
$$

$$
\begin{aligned}
&= \langle \bar{c} f, g \rangle && \text{(by Def. II.5.2.1)} \\
&= \bar{c} \langle f, g \rangle . && \text{(by Lem. II.5.2.5(c))}
\end{aligned}
$$

□

A.Cor. II.5.2.1. From the positivity property (Lem. II.5.2.5(b)), it makes sense to define the L^2 norm $\|f\|_2$ of a function $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ by the formula

$$\|f\|_2 \coloneqq \sqrt{\langle f, f\rangle} = \left(\int_{[0,1]} f(x)\overline{f(x)}\,dx\right)^{1/2} = \left(\int_{[0,1]} |f(x)|^2\,dx\right)^{1/2}.$$

Thus, $\|f\|_2 \geq 0$ for all f. The norm $\|f\|_2$ is sometimes called the *root mean square* of f.

Note. This L^2 norm is related to, but is distinct from, the L^∞ norm

$$\|f\|_\infty \coloneqq \sup_{x\in\mathbb{R}} |f(x)|.$$

In general, the best one can say is that $0 \leq \|f\|_2 \leq \|f\|_\infty$.

Lem. II.5.2.7. Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.

(a) (Non-degeneracy) We have $\|f\|_2 = 0$ iff $f = 0$.

(b) (Cauchy-Schwarz inequality) We have $|\langle f, g\rangle| \leq \|f\|_2 \|g\|_2$.

(c) (Triangle inequality) We have $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

(d) (Pythagoras' theorem) If $\langle f, g\rangle = 0$, then $\|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$.

(e) (Homogeneity) We have $\|cf\|_2 = |c|\|f\|_2$ for all $c \in \mathbb{C}$.

Proof. (a) We have

$$
\begin{aligned}
& \|f\|_2 = 0 \\
\iff & \sqrt{\langle f, f\rangle} = 0 && \text{(by A.Cor. II.5.2.1)} \\
\iff & \langle f, f\rangle = 0 \\
\iff & f = 0. && \text{(by Lem. II.5.2.5(b))}
\end{aligned}
$$

□

Proof. (b) If g is zero function on $[0,1]$, then we have

$$|\langle f, g\rangle| = \left|\int_{[0,1]} f(x)\overline{g(x)}\,dx\right| \qquad \text{(by Def. II.5.2.1)}$$

$$\begin{aligned}
&= \left|\int_{[0,1]} f(x)\cdot 0\,dx\right| \\
&= \left|\int_{[0,1]} 0\,dx\right| \\
&= 0 \\
&= \|f\|_2\|g\|_2. && \text{(by Lem. II.5.2.7(a))}
\end{aligned}$$

So suppose that g is not zero function on $[0,1]$. Observe that

$$\begin{aligned}
&\|g\|_2 \in \mathbb{R} && \text{(by A.Cor. II.5.2.1)} \\
\Longrightarrow\ &\|g\|_2^2 \in \mathbb{R} \\
\Longrightarrow\ &\|g\|_2^2\cdot f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}) && \text{(by Lem. II.5.1.5(b))} \\
\Longrightarrow\ &\|g\|_2^2\cdot f - \langle f,g\rangle\cdot g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}). && \text{(by Lem. II.5.1.5(b))}
\end{aligned}$$

If we let $h = \|g\|_2^2\cdot f - \langle f,g\rangle\cdot g$, then by A.Cor. II.5.2.1 we know that $\langle h,h\rangle$ is well-defined. Thus, we have

$$\begin{aligned}
&\langle h,h\rangle \\
&= \left\langle \|g\|_2^2\cdot f - \langle f,g\rangle\cdot g, \|g\|_2^2\cdot f - \langle f,g\rangle\cdot g\right\rangle \\
&= \left\langle \|g\|_2^2\cdot f, \|g\|_2^2\cdot f - \langle f,g\rangle\cdot g\right\rangle && \text{(by Lem. II.5.2.5(c))} \\
&\quad + \left\langle -\langle f,g\rangle\cdot g, \|g\|_2^2\cdot f - \langle f,g\rangle\cdot g\right\rangle \\
&= \left\langle \|g\|_2^2\cdot f, \|g\|_2^2\cdot f\right\rangle + \left\langle \|g\|_2^2\cdot f, -\langle f,g\rangle\cdot g\right\rangle && \text{(by Lem. II.5.2.5(d))} \\
&\quad + \left\langle -\langle f,g\rangle\cdot g, \|g\|_2^2\cdot f\right\rangle + \langle -\langle f,g\rangle\cdot g, -\langle f,g\rangle\cdot g\rangle \\
&= \|g\|_2^2\left\langle f, \|g\|_2^2\cdot f\right\rangle + \|g\|_2^2\langle f, -\langle f,g\rangle\cdot g\rangle && \text{(by Lem. II.5.2.5(c))} \\
&\quad - \langle f,g\rangle\left\langle g, \|g\|_2^2\cdot f\right\rangle - \langle f,g\rangle\langle g, -\langle f,g\rangle\cdot g\rangle \\
&= \|g\|_2^2\overline{\|g\|_2^2}\langle f,f\rangle + \|g\|_2^2\overline{-\langle f,g\rangle}\langle f,g\rangle && \text{(by Lem. II.5.2.5(d))} \\
&\quad - \langle f,g\rangle\overline{\|g\|_2^2}\langle g,f\rangle - \langle f,g\rangle\overline{-\langle f,g\rangle}\langle g,g\rangle \\
&= \|g\|_2^4\langle f,f\rangle - \|g\|_2^2\overline{\langle f,g\rangle}\langle f,g\rangle && \text{(by Lem. II.4.6.9)} \\
&\quad - \langle f,g\rangle\|g\|_2^2\langle g,f\rangle + \langle f,g\rangle\overline{\langle f,g\rangle}\langle g,g\rangle \\
&= \|g\|_2^4\langle f,f\rangle - \|g\|_2^2\overline{\langle f,g\rangle}\langle f,g\rangle \\
&\quad - \langle f,g\rangle\|g\|_2^2\overline{\langle f,g\rangle} + \langle f,g\rangle\overline{\langle f,g\rangle}\langle g,g\rangle && \text{(by Lem. II.5.2.5(a))} \\
&= \|g\|_2^4\langle f,f\rangle - 2\|g\|_2^2|\langle f,g\rangle|^2 + |\langle f,g\rangle|^2\langle g,g\rangle && \text{(by Def. II.4.6.10)}
\end{aligned}$$

$$
\begin{aligned}
&= \|g\|_2^4\|f\|_2^2 - 2\|g\|_2^2|\langle f,g\rangle|^2 + |\langle f,g\rangle|^2\|g\|_2^2 && \text{(by A.Cor. II.5.2.1)}\\
&= \|g\|_2^4\|f\|_2^2 - \|g\|_2^2|\langle f,g\rangle|^2
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle h,h\rangle \geq 0 && \text{(by Lem. II.5.2.5(b))}\\
\Longrightarrow\ &\|g\|_2^4\|f\|_2^2 - \|g\|_2^2|\langle f,g\rangle|^2 \geq 0\\
\Longrightarrow\ &\|g\|_2^2\|f\|_2^2 - |\langle f,g\rangle|^2 \geq 0 && \text{(by Lem. II.5.2.7(a))}\\
\Longrightarrow\ &\|g\|_2\|f\|_2 \geq |\langle f,g\rangle|. && \text{(by Lem. II.5.2.5(b))}
\end{aligned}
$$

□

Proof. (c) We have

$$
\begin{aligned}
\|f+g\|_2^2 &= \langle f+g, f+g\rangle && \text{(by A.Cor. II.5.2.1)}\\
&= \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle && \text{(by Lem. II.5.2.5(c)(d))}\\
&= \|f\|_2^2 + \langle f,g\rangle + \langle g,f\rangle + \|g\|_2^2 && \text{(by A.Cor. II.5.2.1)}\\
&\leq \|f\|_2^2 + 2\|f\|_2\|g\|_2 + \|g\|_2^2 && \text{(by Lem. II.5.2.7(b))}\\
&= \big(\|f\|_2 + \|g\|_2\big)^2.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|f+g\|_2^2 \leq (\|f\|_2 + \|g\|_2)^2\\
\Longrightarrow\ &\|f+g\|_2 \leq \|f\|_2 + \|g\|_2. && \text{(by Lem. II.5.2.5(b))}
\end{aligned}
$$

□

Proof. (d) We have

$$
\begin{aligned}
\|f+g\|_2^2 &= \langle f+g, f+g\rangle && \text{(by A.Cor. II.5.2.1)}\\
&= \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle && \text{(by Lem. II.5.2.5(c)(d))}\\
&= \langle f,f\rangle + \langle f,g\rangle + \overline{\langle f,g\rangle} + \langle g,g\rangle && \text{(by Lem. II.5.2.5(a))}\\
&= \langle f,f\rangle + \langle g,g\rangle && \text{(by hypothesis)}\\
&= \|f\|_2^2 + \|g\|_2^2. && \text{(by A.Cor. II.5.2.1)}
\end{aligned}
$$

□

Proof. (e) We have

$$
\begin{aligned}
\|cf\|_2 &= \sqrt{\langle cf, cf\rangle} && \text{(by A.Cor. II.5.2.1)}\\
&= \sqrt{c\bar{c}\langle f,f\rangle} && \text{(by Lem. II.5.2.5(c)(d))}
\end{aligned}
$$

$$
\begin{aligned}
&= \sqrt{|c|^2 \langle f, f\rangle} && \text{(by Lem. II.4.6.11)}\\
&= |c|\sqrt{\langle f, f\rangle}\\
&= |c|\|f\|_2. && \text{(by A.Cor. II.5.2.1)}
\end{aligned}
$$

□

Note. In light of Pythagoras' theorem, we sometimes say that f and g are *orthogonal* iff $\langle f, g\rangle = 0$.

A.Cor. II.5.2.2. We can now define the L^2 metric d_{L^2} on $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ by defining

$$d_{L^2}(f,g) := \|f-g\|_2 = \left(\int_{[0,1]} |f(x)-g(x)|^2\, dx\right)^{1/2}.$$

Rmk. II.5.2.8. One can verify that d_{L^2} is indeed a metric. Indeed, the L^2 metric is very similar to the l^2 metric on Euclidean spaces $\mathbb{R}^n$, which is why the notation is deliberately chosen to be similar; you should compare the two metrics yourself to see the analogy.

Note. A sequence f_n of functions in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ will *converge in the L^2 metric* to $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ if $d_{L^2}(f_n, f) \to 0$ as $n \to \infty$, or in other words that

$$\lim_{n\to\infty} \int_{[0,1]} |f_n(x) - f(x)|^2\, dx = 0.$$

Rmk. II.5.2.9. The notion of convergence in L^2 metric is different from that of uniform or pointwise convergence.

Rmk. II.5.2.10. The L^2 metric is not as well-behaved as the L_∞ metric. For instance, it turns out the space $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is not complete in the L^2 metric, despite being complete in the L_∞ metric.

— Exercises —

Ex. II.5.2.1. Prove Lem. II.5.2.5.

Proof. See Lem. II.5.2.5. □

Ex. II.5.2.2. Prove Lem. II.5.2.7.

Proof. See Lem. II.5.2.7. □

Ex. II.5.2.3. If $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is a non-zero function, show that $0 < \|f\|_2 \le \|f\|_\infty$. Conversely, if $0 < A \le B$ are real numbers, show that there exists a non-zero function $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ such that $\|f\|_2 = A$ and $\|f\|_\infty = B$.

Proof. First, we show that $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ and $f \neq 0$ implies $0 < \|f\|_2 \leq \|f\|_\infty$. By Lem. II.5.2.7(a) we know that $0 < \|f\|_2$. Thus, we only need to show that $\|f\|_2 \leq \|f\|_\infty$. By Lem. II.5.1.5(a) we know that f is bounded, thus by Def. II.3.5.5

$$\|f\|_\infty = \sup_{y\in\mathbb{R}} |f(x)| = \sup_{y\in[0,1]} |f(x)| \in \mathbb{R}^+ \cup \{0\}.$$

Since

$$\begin{aligned}
\|f\|_2^2 &= \int_{[0,1]} |f(x)|^2 \, dx && \text{(by A.Cor. II.5.2.1)}\\
&\leq \int_{[0,1]} \big(\sup_{y\in[0,1]} |f(y)|\big)^2 \, dx\\
&= \big(\sup_{y\in[0,1]} |f(y)|\big)^2\\
&= \|f\|_\infty^2, && \text{(by Def. II.3.5.5)}
\end{aligned}$$

we know that

$$\|f\|_2^2 \leq \|f\|_\infty^2 \implies \|f\|_2 \leq \|f\|_\infty.$$

Now we show that for arbitrary $A, B \in \mathbb{R}$, we have

$$0 < A \leq B \implies \exists f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}) : \begin{cases} f \neq 0 \\ \|f\|_2 = A \\ \|f\|_\infty = B \end{cases}$$

So let $A, B \in \mathbb{R}$ such that $0 < A \leq B$. We want to find some $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ such that

$$\begin{aligned}
A^2 &= \|f\|_2^2 = \int_{[0,1]} |f(x)|^2 \, dx;\\
B^2 &= \|f\|_\infty^2 = \big(\sup_{x\in[0,1]} |f(x)|\big)^2.
\end{aligned}$$

In particular, we want our f to look like

$$\forall x \in [0,1], f(x) = \sqrt{c + dg(x)},$$

where $c, d \in \mathbb{R}^+$ and $g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ such that $g(\mathbb{R}) \subseteq \mathbb{R}^+ \cup \{0\}$. So we are trying to solve the following equations:

$$\begin{aligned}
A^2 &= \int_{[0,1]} \left|\sqrt{c + dg(x)}\right|^2 \, dx\\
&= \int_{[0,1]} c + dg(x) \, dx
\end{aligned}$$

$$
\begin{aligned}
&= c + d\int_{[0,1]} g(x)\ dx;\\
B^2 &= \Big(\sup_{x\in[0,1]}\Big|\sqrt{c+dg(x)}\Big|\Big)^2\\
&= \sup_{x\in[0,1]}\Big|\sqrt{c+dg(x)}\Big|^2\\
&= \sup_{x\in[0,1]}\big(c+dg(x)\big)\\
&= c + d\big(\sup_{x\in[0,1]} g(x)\big).
\end{aligned}
$$

By setting

$$c = \frac{A^2}{2};$$

$$d = \frac{1}{2};$$

$$\forall x\in[0,1], g(x) = \begin{cases} \dfrac{(2B^2-A^2)^2}{A^2}x & \text{if } x\in[0,\dfrac{A^2}{2B^2-A^2}) \\ \dfrac{-(2B^2-A^2)^2}{A^2}x + 2(2B^2-A^2) & \text{if } x\in[\dfrac{A^2}{2B^2-A^2},\dfrac{2A^2}{2B^2-A^2}) \\ 0 & \text{if } x\in[\dfrac{2A^2}{2B^2-A^2},1] \end{cases},$$

we have

$$
\begin{aligned}
&\int_{[0,1]} g(x)\ dx\\
&= \int_{[0,\frac{A^2}{2B^2-A^2}]}\frac{(2B^2-A^2)^2}{A^2}x\ dx + \int_{[\frac{A^2}{2B^2-A^2},\frac{2A^2}{2B^2-A^2}]}\frac{-(2B^2-A^2)^2}{A^2}x + 2(2B^2-A^2)\ d\\
&= \frac{(2B^2-A^2)^2}{A^2}\left(\frac{x^2}{2}\Big|_{x=0}^{x=\frac{A^2}{2B^2-A^2}}\right) - \frac{(2B^2-A^2)^2}{A^2}\left(\frac{x^2}{2}\Big|_{x=\frac{A^2}{2B^2-A^2}}^{x=\frac{2A^2}{2B^2-A^2}}\right)\\
&\quad + 2(2B^2-A^2)\left(\frac{2A^2}{2B^2-A^2} - \frac{A^2}{2B^2-A^2}\right)\\
&= \frac{A^2}{2} - 2A^2 + \frac{A^2}{2} + 2A^2\\
&= A^2
\end{aligned}
$$

and

$$\sup_{[0,1]} g(x) = \frac{(2B^2-A^2)^2}{A^2}\frac{A^2}{2B^2-A^2}$$

$$= 2B^2 - A^2.$$

Thus

$$\begin{aligned} c + d\int_{[0,1]} g(x)\ dx &= \frac{A^2}{2} + \frac{A^2}{2} \\ &= A^2; \\ c + d\big(\sup_{x\in[0,1]} g(x)\big) &= \frac{A^2}{2} + \frac{2B^2 - A^2}{2} \\ &= B^2. \end{aligned}$$

Note that the idea behind the definition of g is we try to build a triangle in the interval $[0, 1]$ with height equals to $2B^2 - A^2$ (this explains the result of supremum), and we want that triangle's area equals to A^2 (this explains the result of integration). One can easily show that by extended g periodically with period 1 we know that $g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. □

Ex. II.5.2.4. Prove that the d_{L^2} metric on $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ does indeed turn $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ into a metric space.

Proof. Let $f, g, h \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. Since

$$\begin{aligned} d_{L^2}(f, f) &= \|f - f\|_2 && \text{(by A.Cor. II.5.2.2)} \\ &= \|0\|_2 && \text{(by Lem. II.5.2.5(b))} \\ &= 0, && \text{(by Lem. II.5.2.7(a))} \end{aligned}$$

we know that $\big(C(\mathbb{R}/\mathbb{Z}; \mathbb{C}), d_{L^2}\big)$ satisfies Def. II.1.1.2(a). Since

$$\begin{aligned} & f \neq g \\ \Longrightarrow\ & f - g \neq 0 && \text{(by Lem. II.5.2.5(b))} \\ \Longrightarrow\ & \|f - g\|_2 > 0 && \text{(by Lem. II.5.2.7(a))} \\ \Longrightarrow\ & d_{L^2}(f, g) > 0, && \text{(by A.Cor. II.5.2.2)} \end{aligned}$$

we know that $\big(C(\mathbb{R}/\mathbb{Z}; \mathbb{C}), d_{L^2}\big)$ satisfies Def. II.1.1.2(b). Since

$$\begin{aligned} d_{L^2}(f, g) &= \|f - g\|_2 && \text{(by A.Cor. II.5.2.2)} \\ &= \sqrt{\langle f - g, f - g\rangle} && \text{(by A.Cor. II.5.2.1)} \\ &= \sqrt{\langle g - f, g - f\rangle} && \text{(by Lem. II.5.2.5(c)(d))} \\ &= \|g - f\|_2 && \text{(by A.Cor. II.5.2.1)} \\ &= d_{L^2}(g, f), && \text{(by A.Cor. II.5.2.2)} \end{aligned}$$

we know that $\big(C(\mathbb{R}/\mathbb{Z}; \mathbb{C}), d_{L^2}\big)$ satisfies Def. II.1.1.2(c). Since

$$d_{L^2}(f, g) + d_{L^2}(g, h) = \|f - g\|_2 + \|g - h\|_2 \qquad \text{(by A.Cor. II.5.2.2)}$$

$$\begin{aligned}
&\geq \|f-g+g-h\|_2 && \text{(by Lem. II.5.2.7(c))}\\
&= \|f-h\|_2\\
&= d_{L^2}(f,h), && \text{(by A.Cor. II.5.2.2)}
\end{aligned}$$

we know that $\big(C(\mathbb{R}/\mathbb{Z};\mathbb{C}), d_{L^2}\big)$ satisfies Def. II.1.1.2(d). From all proofs above, we conclude by Def. II.1.1.2 that $\big(C(\mathbb{R}/\mathbb{Z};\mathbb{C}), d_{L^2}\big)$ is a metric space. □

Ex. II.5.2.5. Find a sequence of continuous periodic functions which converge in L^2 to a discontinuous periodic function.

Proof. By E.g. II.5.1.4 we can define a $\mathbb{Z}$-periodic square wave function $f:\mathbb{R}\to\mathbb{C}$ as follow:

$$\forall x\in\mathbb{R}, f(x) = \begin{cases} 1 & \text{if } x\in[n, n+\frac{1}{2}) \text{ for some } n\in\mathbb{Z} \\ 0 & \text{if } x\in[n+\frac{1}{2}, n+1) \text{ for some } n\in\mathbb{Z}\end{cases}$$

Note that f is 1-periodic but f is not continuous on $\mathbb{R}$. Let $\mathbb{N}_{\geq 10} = \{n\in\mathbb{N} : n\geq 10\}$. For each $k\in\mathbb{N}_{\geq 10}$, we define $f_k : [0,1)\to\mathbb{C}$ to be the function:

$$\forall x\in[0,1), f_k(x) = \begin{cases} kx & \text{if } x\in[0,\frac{1}{k}) \\ 1 & \text{if } x\in[\frac{1}{k}, \frac{1}{2}-\frac{1}{k}) \\ -kx+\frac{k}{2} & \text{if } x\in[\frac{1}{2}-\frac{1}{k}, \frac{1}{2}) \\ 0 & \text{if } x\in[0+\frac{1}{2}, 1)\end{cases}$$

If we extended f_k periodically with period 1, then $f_k\in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ for all $k\in\mathbb{N}_{\geq 10}$. Note that the choice of 10 is to make sure $\frac{1}{k} < \frac{1}{2}-\frac{1}{k} < \frac{1}{2}$. Now we show that $(f_k)_{k=10}^{\infty}$ converges to f on $[0,1)$ with respect to d_{L^2}. In particular, we want to show that

$$\begin{aligned}
&\lim_{k\to\infty} d_{L^2}(f_k, f) = 0\\
\iff &\lim_{k\to\infty}\left(\int_{[0,1]} |f_k(x)-f(x)|^2\,dx\right)^{1/2} = 0 && \text{(by A.Cor. II.5.2.2)}\\
\iff &\lim_{k\to\infty}\int_{[0,1]} |f_k(x)-f(x)|^2\,dx = 0.
\end{aligned}$$

Since for each $k\in\mathbb{N}_{\geq 10}$, we have

$$\int_{[0,1]} |f_k(x)-f(x)|^2\,dx$$

$$
\begin{aligned}
&= \int_{[0,\frac{1}{k}]} (1-kx)^2 \, dx + \int_{[\frac{1}{2}-\frac{1}{k},\frac{1}{2}]} \left(1-\frac{k}{2}+kx\right)^2 dx \\
&= \int_{[0,\frac{1}{k}]} 1-2kx+k^2x^2 \, dx + \int_{[\frac{1}{2}-\frac{1}{k},\frac{1}{2}]} 1-k+\frac{k^2}{4}+2kx-k^2x+k^2x^2 \, dx \\
&= \frac{1}{k} - 2k\left(\frac{x^2}{2}\Big|_{x=0}^{x=\frac{1}{k}}\right) + k^2\left(\frac{x^3}{3}\Big|_{x=0}^{x=\frac{1}{k}}\right) \\
&\quad + \frac{1}{k}\left(1-k+\frac{k^2}{4}\right) + (2k-k^2)\left(\frac{x^2}{2}\Big|_{x=\frac{1}{2}-\frac{1}{k}}^{x=\frac{1}{2}}\right) + k^2\left(\frac{x^3}{3}\Big|_{x=\frac{1}{2}-\frac{1}{k}}^{x=\frac{1}{2}}\right) \\
&= \frac{1}{k} - \frac{1}{k} + \frac{1}{3k} + \frac{1}{k} - 1 + \frac{k}{4} + \frac{2k-k^2}{2}\left(\frac{1}{k}-\frac{1}{k^2}\right) + \frac{k^2}{3}\left(\frac{3}{4k}-\frac{3}{2k^2}+\frac{1}{k^3}\right) \\
&= \frac{2}{3k},
\end{aligned}
$$

we know that

$$\lim_{k\to\infty} \int_{[0,1]} |f_k(x)-f(x)|^2 \, dx = \lim_{k\to\infty} \frac{2}{3k} = 0.$$

Thus, $(f_k)_{k=10}^{\infty}$ converges to f on $[0,1)$ with respect to d_{L^2}. Since f and f_k are 1-periodic for all $k \in \mathbb{N}_{\geq 10}$, we know that $(f_k)_{k=10}^{\infty}$ converges to f on $\mathbb{R}$ with respect to d_{L^2}. $\square$

Ex. II.5.2.6. Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.

(a) Show that if f_n converges uniformly to f, then f_n also converges to f in the L^2 metric.

(b) Give an example where f_n converges to f in the L^2 metric, but does not converge to f uniformly.

(c) Give an example where f_n converges to f in the L^2 metric, but does not converge to f pointwise.

(d) Give an example where f_n converges to f pointwise, but does not converge to f in the L^2 metric.

Proof. (a) We have

$$
\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \forall x \in \mathbb{R}, && \text{(by Def. II.3.2.7)} \\
&|f_n(x)-f(x)| < \frac{\varepsilon^{\frac{1}{2}}}{2} \\
\implies &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \forall x \in [0,1],
\end{aligned}
$$

$$
\begin{aligned}
& |f_n(x) - f(x)| < \frac{\varepsilon\frac{1}{2}}{2} \\
\implies & \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \forall x \in [0,1], \\
& |f_n(x) - f(x)|^2 < \frac{\varepsilon}{4} \\
\implies & \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \\
& \int_{[0,1]} |f_n(x) - f(x)|^2 \, dx \leq \int_{[0,1]} \frac{\varepsilon}{4} \, dx = \frac{\varepsilon}{4} < \varepsilon \\
\implies & \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \\
& d_{L^2}(f_n, f) < \varepsilon && \text{(by A.Cor. II.5.2.2)} \\
\implies & d_{L^2} - \lim_{n\to\infty} f_n = f. && \text{(by Def. II.1.1.14)}
\end{aligned}
$$

□

Proof. (b) Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ such that $f = 0$ and let $\mathbb{N}_{\geq 2} = \{n \in \mathbb{N} : n \geq 2\}$. For all $n \in \mathbb{N}_{\geq 2}$, we define $f_n \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ as follow:

$$
\forall x \in [0,1), f_n(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2} - \frac{1}{n^3}) \\ n^4 x + n - \frac{n^4}{2} & \text{if } x \in [\frac{1}{2} - \frac{1}{n^3}, \frac{1}{2}) \\ -n^4 x + n + \frac{n^4}{2} & \text{if } x \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n^3}) \\ 0 & \text{if } x \in [\frac{1}{2} + \frac{1}{n^3}, 1) \end{cases}
$$

Since for all $n \in \mathbb{N}_{\geq 2}$, we have

$$
\begin{aligned}
& \int_{[0,1]} |f_n(x) - f(x)|^2 \, dx \\
= & \int_{[\frac{1}{2}-\frac{1}{n^3}, \frac{1}{2}]} (n^4 x + n - \frac{n^4}{2})^2 \, dx + \int_{[\frac{1}{2}, \frac{1}{2}+\frac{1}{n^3}]} (-n^4 x + n + \frac{n^4}{2})^2 \, dx \\
= & \int_{[\frac{1}{2}-\frac{1}{n^3}, \frac{1}{2}]} n^8 x^2 + (2n^5 - n^8)x + n^2 - n^5 + \frac{n^8}{4} \, dx \\
& + \int_{[\frac{1}{2}, \frac{1}{2}+\frac{1}{n^3}]} n^8 x^2 + (-2n^5 - n^8)x + n^2 + n^5 + \frac{n^8}{4} \, dx \\
= & n^8 \left(\frac{x^3}{3} \Big|_{x=\frac{1}{2}-\frac{1}{n^3}}^{x=\frac{1}{2}+\frac{1}{n^3}} \right) + (2n^5 - n^8) \left(\frac{x^2}{2} \Big|_{x=\frac{1}{2}-\frac{1}{n^3}}^{x=\frac{1}{2}} \right) + (-2n^5 - n^8) \left(\frac{x^2}{2} \Big|_{x=\frac{1}{2}}^{x=\frac{1}{2}+\frac{1}{n^3}} \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(n^2-n^5+\frac{n^8}{4}\right)\frac{1}{n^3}+\left(n^2+n^5+\frac{n^8}{4}\right)\frac{1}{n^3}\\
&=\frac{n^5}{2}+\frac{2}{3n}+n^2-\frac{n^5}{2}-\frac{1}{n}+\frac{n^2}{2}-n^2-\frac{n^5}{2}-\frac{1}{n}-\frac{n^2}{2}+\frac{2}{n}+\frac{n^5}{2}\\
&=\frac{2}{3n},
\end{aligned}
$$

we know that

$$
\begin{aligned}
\lim_{n\to\infty} d_{L^2}(f_n,f) &= \lim_{n\to\infty}\int_{[0,1]}|f_n(x)-f(x)|^2\,dx && \text{(by A.Cor. II.5.2.2)}\\
&=\lim_{n\to\infty}\frac{2}{3n}\\
&=0.
\end{aligned}
$$

Thus, by Def. II.1.1.14 we have

$$d_{L^2}-\lim_{n\to\infty} f_n = f.$$

But for all $n\in\mathbb{N}_{\geq 2}$, we have

$$
\begin{aligned}
&f_n(\frac{1}{2})=n\\
\implies &\left|f_n(\frac{1}{2})-f(\frac{1}{2})\right|=\left|f_n(\frac{1}{2})\right|\geq n>1\\
\implies &\exists\varepsilon\in\mathbb{R}^+:\forall n\geq\mathbb{N}_{\geq 2},\exists x\in[0,1):|f_n(x)-f(x)|>\varepsilon.
\end{aligned}
$$

Thus, by Def. II.3.2.7 $(f_n)_{n=2}^{\infty}$ does not converges uniformly to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. □

Proof. (c) Using the definition of f, f_n in Ex. II.5.2.6(b), we see that $(f_n)_{n=2}^{\infty}$ does not converges pointwise to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. □

Proof. (d) Let $f\in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ such that $f=0$ and let $\mathbb{N}_{\geq 2}=\{n\in\mathbb{N}:n\geq 2\}$. For all $n\in\mathbb{N}_{\geq 2}$, we define $f_n\in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ as follow:

$$
\forall x\in[0,1), f_n(x)=\begin{cases}
2n^2x & \text{if } x\in[0,\frac{1}{2n})\\
-2n^2x+2n & \text{if } x\in[\frac{1}{2n},\frac{1}{n})\\
0 & \text{if } x\in[\frac{1}{n},1)
\end{cases}
$$

Since

$$\lim_{n\to\infty}\frac{1}{n}=0$$

$$\begin{aligned}
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, \frac{1}{n} < x \\
&\implies \forall x \in (0, \frac{1}{2}), \exists N \in \mathbb{Z}^+ : \forall n \geq N, \frac{1}{n} < x \\
&\implies \forall x \in (0, \frac{1}{2}), \exists N \in \mathbb{Z}^+ : \forall n \geq N, f(\frac{1}{n}) = f_n(x) = 0 \\
&\implies \forall x \in (0, \frac{1}{2}), \lim_{n\to\infty} f_n(x) = 0 = f(x)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} f_n(0) = 0 = f(0) \\
\forall x \in [\frac{1}{2}, 1), \lim_{n\to\infty} f_n(x) = 0 = f(x)
\end{aligned}$$

by Def. II.3.2.1 we know that $(f_n)_{n=2}^{\infty}$ converges pointwise to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{R}\times\mathbb{R}}$. But

$$\begin{aligned}
&\int_{[0,1]} |f_n(x) - f(x)|^2 \, dx \\
&= \int_{[0,\frac{1}{2n}]} (2n^2x)^2 \, dx + \int_{[\frac{1}{2n},\frac{1}{n}]} (-2n^2x + 2n)^2 \, dx \\
&= \int_{[0,\frac{1}{2n}]} 4n^4x^2 \, dx + \int_{[\frac{1}{2n},\frac{1}{n}]} 4n^4x^2 - 4n^3x + 4n^2 \, dx \\
&= 4n^4\left(\frac{x^3}{3}\Big|_{x=0}^{x=\frac{1}{n}}\right) - 4n^3\left(\frac{x^2}{2}\Big|_{x=\frac{1}{2n}}^{x=\frac{1}{n}}\right) + 4n^2\frac{1}{2n} \\
&= \frac{4n}{3} - \frac{3n}{2} + 2n \\
&= \frac{11n}{6}
\end{aligned}$$

implies

$$\begin{aligned}
\lim_{n\to\infty} d_{L^2}(f_n, f) &= \lim_{n\to\infty} \int_{[0,1]} |f_n(x) - f(x)|^2 \, dx && \text{(by A.Cor. II.5.2.2)} \\
&= \lim_{n\to\infty} \frac{11n}{6} \\
&= +\infty.
\end{aligned}$$

Thus, $(f_n)_{n=2}^{\infty}$ does not converges to f with respect to d_{L^2}. $\square$

II.5.3 Trigonometric polynomials

Note. We now define the concept of a *trigonometric polynomial.* Just as polynomials are combinations of the functions x^n (sometimes called *monomials*), trigonometric polynomials are combinations of the functions $e^{2\pi inx}$ (sometimes called *characters*).

Def. II.5.3.1 (Characters)**.** For every integer n, we let $e_n \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ denote the function

$$e_n(x) := e^{2\pi inx}.$$

This is sometimes referred to as the *character with frequency n.*

Def. II.5.3.2 (Trigonometric polynomials)**.** A function f in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is said to be a *trigonometric polynomial* if we can write $f = \sum_{n=-N}^{N} c_n e_n$ for some integer $N \geq 0$ and some complex numbers $(c_n)_{n=-N}^{N}$.

E.g. II.5.3.4. For any integer n, the function $\cos(2\pi nx)$ is a trigonometric polynomial, since

$$\cos(2\pi nx) = \frac{e^{2\pi nx} + e^{-2\pi nx}}{2} = \frac{1}{2}e_{-n} + \frac{1}{2}e_n.$$

Similarly, the function $\sin(2\pi nx) = \frac{-1}{2i}e_{-n} + \frac{1}{2i}e_n$ is a trigonometric polynomial. In fact, any linear combination of sines and cosines is also a trigonometric polynomial.

Lem. II.5.3.5 (Characters are an orthonormal system)**.** For any integers n and m, we have $\langle e_n, e_m\rangle = 1$ when $n = m$ and $\langle e_n, e_m\rangle = 0$ when $n \neq m$. Also, we have $\|e_n\|_2 = 1$.

Proof. Let $n, m \in \mathbb{Z}$. Observe that

$$\begin{aligned}
\langle e_n, e_m\rangle &= \int_{[0,1]} e_n(x)\overline{e_m(x)}\,dx && \text{(by Def. II.5.2.1)}\\
&= \int_{[0,1]} e^{2\pi inx}\overline{e^{2\pi imx}}\,dx && \text{(by Def. II.5.3.1)}\\
&= \int_{[0,1]} e^{2\pi inx}e^{-2\pi imx}\,dx && \text{(by Thm. II.4.7.2(c)(f))}\\
&= \int_{[0,1]} e^{2\pi inx-2\pi imx}\,dx && \text{(by Ex. II.4.6.16)}\\
&= \int_{[0,1]} e^{2\pi i(n-m)x}\,dx.
\end{aligned}$$

If $n = m$, then we have

$$\langle e_n, e_n\rangle = \int_{[0,1]} e^{2\pi i(n-n)x}\,dx$$

$$\begin{aligned}
&= \int_{[0,1]} e^0 \, dx \\
&= \int_{[0,1]} 1 \, dx && \text{(by Thm. II.4.5.2(e))} \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\|e_n\|_2 &= \sqrt{\langle e_n, e_n \rangle} && \text{(by A.Cor. II.5.2.1)} \\
&= \sqrt{1} = 1.
\end{aligned}$$

If $n \neq m$, then we have

$$\begin{aligned}
&\langle e_n, e_m \rangle \\
&= \int_{[0,1]} e^{2\pi i(n-m)x} \, dx \\
&= \int_{[0,1]} \cos\big(2\pi(n-m)x\big) + i \sin\big(2\pi(n-m)x\big) \, dx && \text{(by Thm. II.4.7.2(f))} \\
&= \int_{[0,1]} \cos\big(2\pi(n-m)x\big) \, dx && \text{(by Rmk. II.5.2.2)} \\
&\quad + i \int_{[0,1]} \sin\big(2\pi(n-m)x\big) \, dx \\
&= \left(\frac{\sin\big(2\pi(n-m)x\big)}{2\pi(n-m)} \Big|_{x=0}^{x=1} \right) && \text{(by Thm. II.4.7.2(b))} \\
&\quad + i \left(\frac{-\cos\big(2\pi(n-m)x\big)}{2\pi(n-m)} \Big|_{x=0}^{x=1} \right) \\
&= 0 - 0 && \text{(by A.Cor. II.4.7.2(c))} \\
&\quad + i \left(\frac{-(-1) + (-1)}{2\pi(n-m)} \right) && \text{(by A.Cor. II.4.7.2(f))} \\
&= 0.
\end{aligned}$$

□

Cor. II.5.3.6. Let $f = \sum_{n=-N}^{N} c_n e_n$ be a trigonometric polynomial. Then we have the formula

$$c_n = \langle f, e_n \rangle$$

for all integers $-N \leq n \leq N$. Also, we have $0 = \langle f, e_n \rangle$ whenever $n > N$ or $n < -N$. Also, we have the identity

$$\|f\|_2^2 = \sum_{n=-N}^{N} |c_n|^2.$$

Proof. Let $m \in \mathbb{N}$. Then we have

$$\begin{aligned}
\langle f, e_m \rangle &= \left\langle \sum_{n=-N}^{N} (c_n e_n), e_m \right\rangle && \text{(by hypothesis)} \\
&= \sum_{n=-N}^{N} \langle c_n e_n, e_m \rangle && \text{(by Lem. II.5.2.5(c))} \\
&= \sum_{n=-N}^{N} \big(c_n \langle e_n, e_m \rangle\big) && \text{(by Lem. II.5.2.5(c))} \\
&= \begin{cases} c_m & \text{if } -N \le m \le N \\ 0 & \text{if } m > N \text{ or } m < -N \end{cases} && \text{(by Lem. II.5.3.5)}
\end{aligned}$$

and

$$\begin{aligned}
\|f\|_2^2 &= \langle f, f \rangle && \text{(by A.Cor. II.5.2.1)} \\
&= \left\langle f, \sum_{n=-N}^{N} (c_n e_n) \right\rangle && \text{(by hypothesis)} \\
&= \sum_{n=-N}^{N} \langle f, c_n e_n \rangle && \text{(by Lem. II.5.2.5(d))} \\
&= \sum_{n=-N}^{N} \big(\overline{c_n} \langle f, e_n \rangle\big) && \text{(by Lem. II.5.2.5(d))} \\
&= \sum_{n=-N}^{N} \big(\overline{c_n} c_n\big) && \text{(from the proof above)} \\
&= \sum_{n=-N}^{N} |c_n|^2. && \text{(by Lem. II.4.6.11)}
\end{aligned}$$

□

Def. II.5.3.7 (Fourier transform)**.** For any function $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, and any integer $n \in \mathbb{Z}$, we define the n^{th} *Fourier coefficient of* f, denoted $\hat{f}(n)$, by the formula

$$\hat{f}(n) := \langle f, e_n \rangle = \int_{[0,1]} f(x) e^{-2\pi i n x}\, dx.$$

The function $\hat{f} : \mathbb{Z} \to \mathbb{C}$ is called the *Fourier transform* of f.

A.Cor. II.5.3.1. From Cor. II.5.3.6, we see that whenever

$$f = \sum_{n=-N}^{N} c_n e_n$$

is a trigonometric polynomial, we have

$$f = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n$$

and in particular, we have the *Fourier inversion formula*

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$$

or in other words

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

The right-hand side is referred to as the *Fourier series* of f. Also, from the second identity of Cor. II.5.3.6 we have the *Plancherel formula*

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2.$$

Rmk. II.5.3.8. We stress that at present we have only proven the Fourier inversion and Plancherel formulae in the case when f is a trigonometric polynomial. Note that in this case that the Fourier coefficients $\hat{f}(n)$ are mostly zero (indeed, they can only be non-zero when $-N \le n \le N$), and so this infinite sum is really just a finite sum in disguise. In particular, there are no issues about what sense the above series converge in; they both converge pointwise, uniformly, and in L^2 metric, since they are just finite sums.

Note. In the next few sections we will extend the Fourier inversion and Plancherel formulae to general functions in $C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, not just trigonometric polynomials. (It is also possible to extend the formula to discontinuous functions such as the square wave, but we will not do so here.) To do this we will need a version of the Weierstrass approximation theorem, this time requiring that a continuous periodic function be approximated uniformly by *trigonometric* polynomials. Just as convolutions were used in the proof of the polynomial Weierstrass approximation theorem, we will also need a notion of convolution tailored for periodic functions.

— Exercises —

Ex. II.5.3.1. Show that the sum or product of any two trigonometric polynomials is again a trigonometric polynomial.

Proof. Let $f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$ such that

$$\exists N \in \mathbb{N} : \big((c_n)_{n=-N}^{N} \text{ is in } \mathbb{C}\big) \wedge \left(f = \sum_{n=-N}^{N} c_n e_n\right);$$

$$\exists M \in \mathbb{N} : \big((d_n)_{n=-M}^{M} \text{ is in } \mathbb{C}\big) \wedge \left(g = \sum_{n=-M}^{M} d_n e_n\right).$$

Without the loss of generality suppose that $N \leq M$. Then we have

$$\begin{aligned} f + g &= \sum_{n=-N}^{N} (c_n e_n) + \sum_{n=-M}^{M} (d_n e_n) \\ &= \sum_{n=-M}^{M} (a_n e_n) \end{aligned}$$

where

$$a_n = \begin{cases} c_n + d_n & \text{if } -N \leq n \leq N \\ d_n & \text{if } (-M \leq n < -N) \vee (N < n \leq M) \end{cases}$$

For fg, we induct on M to show that fg is trigonometric polynomial. For $M = 0$, we have

$$\begin{aligned} fg &= \left(\sum_{n=-N}^{N} (c_n e_n)\right)(d_0 e^0) \\ &= \left(\sum_{n=-N}^{N} (c_n e_n)\right) d_0 && \text{(by Thm. II.4.5.2(e))} \\ &= \sum_{n=-N}^{N} (c_n d_0 e_n). \end{aligned}$$

Clearly, fg is trigonometric polynomial and Thus, the base case holds. Suppose inductively that fg is trigonometric polynomial for some $M \geq 0$. Then for $M + 1$, we have

$$\begin{aligned} fg &= \left(\sum_{n=-N}^{N} (c_n e_n)\right)\left(\sum_{m=-(M+1)}^{M+1} (d_m e_m)\right) \\ &= \left(\sum_{n=-N}^{N} (c_n e_n)\right)\left(\sum_{m=-M}^{M} (d_m e_m) + d_{-M-1} e_{-M-1} + d_{M+1} e_{M+1}\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{n=-N}^{N}(c_n e_n)\right)\left(\sum_{m=-M}^{M}(d_m e_m)\right) + \left(\sum_{n=-N}^{N}(c_n e_n)\right)(d_{-M-1}e_{-M-1}) \\
&\quad + \left(\sum_{n=-N}^{N}(c_n e_n)\right)(d_{M+1}e_{M+1}) \\
&= \left(\sum_{n=-N}^{N}(c_n e_n)\right)\left(\sum_{m=-M}^{M}(d_m e_m)\right) + \sum_{n=-N}^{N}(c_n d_{-M-1}e_{n-M-1}) \\
&\quad + \sum_{n=-N}^{N}(c_n d_{M+1}e_{n+M+1}) \\
&= \left(\sum_{n=-N}^{N}(c_n e_n)\right)\left(\sum_{m=-M}^{M}(d_m e_m)\right) + \sum_{n=-N-M-1}^{N-M-1}(c_{n+M+1}d_{-M-1}e_n) \\
&\quad + \sum_{n=-N+M+1}^{N+M+1}(c_{n-M-1}d_{M+1}e_n).
\end{aligned}
$$

By setting

$$
a_n = \begin{cases} c_{n+M+1}d_{-M-1} & \text{if } -N-M-1 \le n \le N-M-1 \\ 0 & \text{if } N-M-1 < n \le N+M+1 \end{cases}
$$

$$
b_n = \begin{cases} c_{n-M-1}d_{M+1} & \text{if } -N+M+1 \le n \le N+M+1 \\ 0 & \text{if } -N-M-1 \le n < -N+M+1 \end{cases}
$$

we have

$$
fg = \left(\sum_{n=-N}^{N}(c_n e_n)\right)\left(\sum_{m=-M}^{M}(d_m e_m)\right) + \sum_{n=-N-M-1}^{N+M+1}(a_n e_n) + \sum_{n=-N-M-1}^{N+M+1}(b_n e_n).
$$

By the induction hypothesis we know that $\left(\sum_{n=-N}^{N}(c_n e_n)\right)\left(\sum_{m=-M}^{M}(d_m e_m)\right)$ is trigonometric polynomial. Thus, from the proof above we know that fg is trigonometric polynomial, and this closes the induction. □

Ex. II.5.3.2. Prove Lem. II.5.3.5.

Proof. See Lem. II.5.3.5. □

Ex. II.5.3.3. Prove Cor. II.5.3.6.

Proof. See Cor. II.5.3.6. □

II.5.4 Periodic convolutions

Thm. II.5.4.1. Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, and let $\varepsilon > 0$. Then there exists a trigonometric polynomial P such that $\|f - P\|_\infty \leq \varepsilon$.

Proof. Let f be any element of $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$; we know that f is bounded (by Lem. II.5.1.5(a)), so that we have some $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$.

Let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous (by Thm. II.2.3.5), there exists a $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| \leq \delta$. Now use Lem. II.5.4.6 to find a trigonometric polynomial P which is a (ε, δ) approximation to the identity. Then $f * P$ is also a trigonometric polynomial (by A.Cor. II.5.4.1). We now estimate $\|f - f * P\|_\infty$.

Let x be any real number. We have

$$\begin{aligned}
&|f(x) - f * P(x)| \\
&= |f(x) - P * f(x)| && \text{(by Lem. II.5.4.4(a)(b))} \\
&= \left| f(x) - \int_{[0,1]} f(x-y)P(y)\, dy \right| && \text{(by Def. II.5.4.2)} \\
&= \left| \int_{[0,1]} f(x)P(y)\, dy - \int_{[0,1]} f(x-y)P(y)\, dy \right| && \text{(by Def. II.5.4.5(a))} \\
&= \left| \int_{[0,1]} \big(f(x) - f(x-y)\big)P(y)\, dy \right| \\
&\leq \int_{[0,1]} |f(x) - f(x-y)|P(y)\, dy. && \text{(by Rmk. II.5.2.2)}
\end{aligned}$$

The right-hand side can be split as

$$\int_{[0,\delta]} |f(x) - f(x-y)|P(y)\, dy + \int_{[\delta,1-\delta]} |f(x) - f(x-y)|P(y)\, dy + \int_{[1-\delta,1]} |f(x) - f(x-y)|P(y)\, dy$$

which we can bound from above by

$$\begin{aligned}
&\leq \int_{[0,\delta]} \varepsilon P(y)\, dy + \int_{[\delta,1-\delta]} 2M\varepsilon\, dy + \int_{[1-\delta,1]} |f(x-1) - f(x-y)|P(y)\, dy \\
&\leq \int_{[0,\delta]} \varepsilon P(y)\, dy + \int_{[\delta,1-\delta]} 2M\varepsilon\, dy + \int_{[1-\delta,1]} \varepsilon P(y)\, dy \\
&\leq \varepsilon + 2M\varepsilon + \varepsilon \\
&= (2M+2)\varepsilon.
\end{aligned}$$

Thus, we have $\|f - f * P\|_\infty \leq (2M+2)\varepsilon$. Since M is fixed and ε was arbitrary, we can thus make $f * P$ arbitrarily close to f in sup norm, which proves the periodic Weierstrass approximation theorem. $\square$

Note. Thm. II.5.4.1 asserts that any continuous periodic function can be uniformly approximated by trigonometric polynomials. To put it another way, if we let

$$P(\mathbb{R}/\mathbb{Z};\mathbb{C})$$

denote the space of all trigonometric polynomials, then the closure of $P(\mathbb{R}/\mathbb{Z};\mathbb{C})$ in the L^∞ metric is $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.

Note. It is possible to prove this theorem directly from the Weierstrass approximation theorem for polynomials (Thm. II.3.8.3), and both theorems are a special case of a much more general theorem known as the *Stone-Weierstrass theorem*, which we will not discuss here. However we shall instead prove this theorem from scratch, in order to introduce a couple of interesting notions, notably that of periodic convolution. The proof here, though, should strongly remind you of the arguments used to prove Thm. II.3.8.3.

Def. II.5.4.2 (Periodic convolution)**.** Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. Then we define the periodic convolution $f * g : \mathbb{R} \to \mathbb{C}$ of f and g by the formula

$$f * g(x) := \int_{[0,1]} f(y)g(x-y)\,dy$$

Rmk. II.5.4.3. Note that Def. II.5.4.2 is slightly different from the convolution for compactly supported functions defined in Def. II.3.8.9, because we are only integrating over $[0,1]$ and not on all of $\mathbb{R}$. Thus, in principle we have given the symbol $f * g$ two conflicting meanings. However, in practice there will be no confusion, because it is not possible for a non-zero function to both be periodic and compactly supported.

Lem. II.5.4.4 (Basic properties of periodic convolution)**.** Let $f, g, h \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.

(a) (Closure) The convolution $f * g$ is continuous and $\mathbb{Z}$-periodic. In other words, $f * g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$.

(b) (Commutativity) We have $f * g = g * f$.

(c) (Bilinearity) We have $f * (g + h) = f * g + f * h$ and $(f + g) * h = f * h + g * h$. For any complex number c, we have $c(f * g) = (cf) * g = f * (cg)$.

Proof. (a) By Rmk. II.5.2.2 we know that $f * g$ is continuous on $\mathbb{R}$. Since

$$\begin{aligned}
\forall x \in \mathbb{R}, f * g(x+1) &= \int_{[0,1]} f(y)g(x+1-y)\,dy && \text{(by Def. II.5.4.2)}\\
&= \int_{[0,1]} f(y)g(x-y)\,dy && (g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C}))\\
&= f * g(x), && \text{(by Def. II.5.4.2)}
\end{aligned}$$

we know that $f * g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. $\square$

Proof. (b) Let $x \in \mathbb{R}$ and let $\phi : [x-1, x] \mapsto [0, 1]$ be the function $\phi(y) = x - y$. Then we have

$$
\begin{aligned}
&f * g(x) \\
&= \int_{[0,1]} f(y)g(x-y)\, dy && \text{(by Def. II.5.4.2)} \\
&= \int_{\left[\phi(x),\phi(x-1)\right]} f(y)g(x-y)\, dy \\
&= -\int_{[x-1,x]} f\big(\phi(y)\big)g\big(x-\phi(y)\big)\phi'(y)\, dy && \text{(by Exercise 11.10.4 in Analysis I)} \\
&= \int_{[x-1,x]} f(x-y)g(y)\, dy \\
&= \int_{[x-1,x]} g(y)f(x-y)\, dy.
\end{aligned}
$$

Let $[x]$ be the integer defined in Ex. II.5.1.1. Then we have

$$
\begin{aligned}
&[x] \le x < [x] + 1 \\
\Longrightarrow\ &[x] - 1 \le x - 1 < [x]
\end{aligned}
$$

and

$$
\begin{aligned}
&f * g(x) \\
&= \int_{[x-1,x]} g(y)f(x-y)\, dy \\
&= \int_{\left[x-1,[x]\right]} g(y)f(x-y)\, dy + \int_{\left[[x],x\right]} g(y)f(x-y)\, dy \\
&= \int_{\left[[x],x\right]} g(y)f(x-y)\, dy + \int_{\left[x-1,[x]\right]} g(y)f(x-y)\, dy \\
&= \int_{\left[[x],x\right]} g(y)f(x-y)\, dy \\
&\quad + \int_{\left[x-1+1,[x]+1\right]} g(y-1)f(x-y-1)\, dy \\
&= \int_{\left[[x],x\right]} g(y)f(x-y)\, dy + \int_{\left[x,[x]+1\right]} g(y)f(x-y)\, dy \quad (f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})) \\
&= \int_{\left[[x],[x]+1\right]} g(y)f(x-y)\, dy
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\left[[x]-[x],[x]+1-[x]\right]} g(y+[x]) f(x-y+[x])\, dy \\
&= \int_{[0,1]} g(y) f(x-y)\, dy && (f, g \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})) \\
&= g * f(x). && \text{(by Def. II.5.4.2)}
\end{aligned}
$$

Since x was arbitrary, we conclude that $f * g = g * f$. □

Proof. (c) By Lem. II.5.1.5(b) we know that $f+g, g+h, cf, cg \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$. Thus, $f*(g+h), (f+g)*h, (cf)*g, f*(cg)$ are well-defined. Let $x \in \mathbb{R}$. Then we have

$$
\begin{aligned}
&\big(f * (g+h)\big)(x) \\
&= \int_{[0,1]} f(y) \cdot (g+h)(x-y)\, dy && \text{(by Def. II.5.4.2)} \\
&= \int_{[0,1]} f(y) \cdot \big(g(x-y) + h(x-y)\big)\, dy \\
&= \int_{[0,1]} f(y) g(x-y) + f(y) h(x-y)\, dy \\
&= \int_{[0,1]} f(y) g(x-y)\, dy + \int_{[0,1]} f(y) h(x-y)\, dy && \text{(cf the proof of Lem. II.5.2.5(c))} \\
&= (f*g)(x) + (f*h)(x) && \text{(by Def. II.5.4.2)} \\
&= (f*g + f*h)(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\big((cf) * g\big)(x) &= \int_{[0,1]} (cf)(y) \cdot g(x-y)\, dy && \text{(by Def. II.5.4.2)} \\
&= \int_{[0,1]} cf(y) g(x-y)\, dy \\
&= c \int_{[0,1]} f(y) g(x-y)\, dy && \text{(cf the proof of Lem. II.5.2.5(c))} \\
&= c(f*g)(x). && \text{(by Def. II.5.4.2)}
\end{aligned}
$$

Since x was arbitrary, we conclude that $f*(g+h) = f*g + f*h$ and $(cf)*g = c(f*g)$. This implies

$$
\begin{aligned}
(f+g)*h &= h*(f+g) && \text{(by Lem. II.5.4.4(b))} \\
&= h*f + h*g && \text{(from the proof above)} \\
&= f*h + g*h && \text{(by Lem. II.5.4.4(b))}
\end{aligned}
$$

and

$$\begin{aligned}
f * (cg) &= (cg) * f && \text{(by Lem. II.5.4.4(b))} \\
&= c(g * f) && \text{(from the proof above)} \\
&= c(f * g). && \text{(by Lem. II.5.4.4(b))}
\end{aligned}$$

□

A.Cor. II.5.4.1. Now we observe an interesting identity: for any $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ and any integer n, we have

$$f * e_n = \hat{f}(n) e_n.$$

To prove this, we compute

$$\begin{aligned}
f * e_n(x) &= \int_{[0,1]} f(y) e_n(x-y)\, dy && \text{(by Def. II.5.4.2)} \\
&= \int_{[0,1]} f(y) e^{2\pi i n(x-y)}\, dy && \text{(by Def. II.5.3.1)} \\
&= e^{2\pi i n x} \int_{[0,1]} f(y) e^{-2\pi i n y}\, dy && \text{(cf the proof of Lem. II.5.2.5(c))} \\
&= \langle f, e_n \rangle e^{2\pi i n x} && \text{(by Def. II.5.2.1)} \\
&= \hat{f}(n) e^{2\pi i n x} && \text{(by Def. II.5.3.7)} \\
&= \hat{f}(n) e_n && \text{(by Def. II.5.3.1)}
\end{aligned}$$

as desired. More generally, we see from Lem. II.5.4.4(c) that for any trigonometric polynomial $P = \sum_{n=-N}^{N} c_n e_n$, we have

$$f * P = \sum_{n=-N}^{N} c_n (f * e_n) = \sum_{n=-N}^{N} \hat{f}(n) c_n e_n.$$

Thus, the periodic convolution of any function in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ with a trigonometric polynomial, is again a trigonometric polynomial. (Compare with Lem. II.3.8.13.)

Def. II.5.4.5 (Periodic approximation to the identity)**.** Let $\varepsilon > 0$ and $0 < \delta < 1/2$. A function $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ is said to be a *periodic* (ε, δ) *approximation to the identity* if the following properties are true:

(a) $f(x) \geq 0$ for all $x \in \mathbb{R}$, and $\int_{[0,1]} f(x)\, dx = 1$.

(b) We have $f(x) < \varepsilon$ for all $\delta \leq |x| \leq 1 - \delta$.

A.Cor. II.5.4.2 (Fejér kernel)**.** Let $N \geq 1$ be an integer. Then we have

$$\sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e_n = \frac{1}{N} \left| \sum_{n=0}^{N-1} e_n \right|^2.$$

Proof. First, we claim that

$$\sum_{n=0}^{2N-2} (N - |n - N + 1|) \cdot e_n = \left(\sum_{n=0}^{N-1} e_n \right)^2.$$

We proof the claim by induction on N and we start with $N = 1$. For $N = 1$, we have

$$\begin{aligned}
\sum_{n=0}^{0} (1 - |n - 1 + 1|) \cdot e_n &= \sum_{n=0}^{0} (1 - |n|) e_n \\
&= 1 e_0 \\
&= 1 && \text{(by Thm. II.4.5.2(e))} \\
&= e_0^2 && \text{(by Thm. II.4.5.2(e))} \\
&= \left(\sum_{n=0}^{0} e_n \right)^2.
\end{aligned}$$

Thus, the base case holds. Suppose inductively that the claim is true for some $N \geq 1$. Then for $N + 1$, we want to show that

$$\sum_{n=0}^{2(N+1)-2} \big(N + 1 - |n - (N + 1) + 1|\big) \cdot e_n = \sum_{n=0}^{2N} (N + 1 - |n - N|) \cdot e_n = \left(\sum_{n=0}^{N} e_n \right)^2.$$

This is true since

$$\begin{aligned}
&\left(\sum_{n=0}^{N} e_n \right)^2 \\
&= \left(\left(\sum_{n=0}^{N-1} e_n \right) + e_N \right)^2 && \text{(by Lem. II.4.6.4)} \\
&= \left(\sum_{n=0}^{N-1} e_n \right)^2 + 2 e_N \left(\sum_{n=0}^{N-1} e_n \right) + e_N^2 && \text{(by Lem. II.4.6.6)} \\
&= \sum_{n=0}^{2N-2} (N - |n - N + 1|) \cdot e_n + 2 e_N \left(\sum_{n=0}^{N-1} e_n \right) + e_N^2 && \text{(by the induction hypothesis)} \\
&= \sum_{n=0}^{N-1} (N - |n - N + 1|) \cdot e_n + \sum_{n=N}^{2N-2} (N - |n - N + 1|) \cdot e_n && \text{(by Lem. II.4.6.4)}
\end{aligned}$$

$$
\begin{aligned}
&+ 2e_N\left(\sum_{n=0}^{N-1} e_n\right) + e_N^2 \\
&= \sum_{n=0}^{N-1}(n+1)\cdot e_n + \sum_{n=N}^{2N-2}(2N-n-1)\cdot e_n \\
&\quad + 2e_N\left(\sum_{n=0}^{N-1} e_n\right) + e_N^2 \\
&= \sum_{n=0}^{N-1} ne_n + \sum_{n=0}^{N-1} e_n + \sum_{n=N}^{2N-2}(2N-n)\cdot e_n - \sum_{n=N}^{2N-2} e_n && \text{(by Lem. II.4.6.6)} \\
&\quad + 2\left(\sum_{n=0}^{N-1} e_N e_n\right) + e_N^2 && \text{(by Lem. II.4.6.6)} \\
&= \sum_{n=0}^{N-1} ne_n + \sum_{n=N}^{2N-2}(2N-n)\cdot e_n && \text{(by Lem. II.4.6.4)} \\
&\quad + \sum_{n=0}^{N-1} e_n - \sum_{n=N}^{2N-2} e_n + 2\left(\sum_{n=0}^{N-1} e_N e_n\right) + e_N^2 \\
&= \sum_{n=0}^{N-1}(N-|n-N|)\cdot e_n + \sum_{n=N}^{2N-2}(N-|n-N|)\cdot e_n \\
&\quad + \sum_{n=0}^{N-1} e_n - \sum_{n=N}^{2N-2} e_n + 2\left(\sum_{n=0}^{N-1} e_N e_n\right) + e_N^2
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{2N-2}(N-|n-N|)\cdot e_n && \text{(conti. from above)} \\
&\quad + \sum_{n=0}^{N-1} e_n - \sum_{n=N}^{2N-2} e_n + 2\left(\sum_{n=0}^{N-1} e_N e_n\right) + e_N^2 \\
&= \sum_{n=0}^{2N-2}(N+1-|n-N|)\cdot e_n - \sum_{n=0}^{2N-2} e_n && \text{(by Lem. II.4.6.6)} \\
&\quad + \sum_{n=0}^{N-1} e_n - \sum_{n=N}^{2N-2} e_n + 2\left(\sum_{n=0}^{N-1} e_N e_n\right) + e_N^2 \\
&= \sum_{n=0}^{2N-2}(N+1-|n-N|)\cdot e_n && \text{(by Lem. II.4.6.6)}
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 2\left(\sum_{n=N}^{2N-2} e_n\right) + 2\left(\sum_{n=0}^{N-1} e_N e_n\right) + e_N^2 \\
&= \sum_{n=0}^{2N}(N+1-|n-N|)\cdot e_n - 2e_{2N-1} - e_{2N} && \text{(by Lem. II.4.6.6)} \\
&\quad - 2\left(\sum_{n=N}^{2N-2} e_n\right) + 2\left(\sum_{n=0}^{N-1} e_N e_n\right) + e_N^2 \\
&= \sum_{n=0}^{2N}(N+1-|n-N|)\cdot e_n - 2e_{2N-1} - e_{2N} \\
&\quad - 2\left(\sum_{n=N}^{2N-2} e_n\right) + 2\left(\sum_{n=0}^{N-1} e_{n+N}\right) + e_{2N} && \text{(by Ex. II.4.6.16)} \\
&= \sum_{n=0}^{2N}(N+1-|n-N|)\cdot e_n && \text{(by Lem. II.4.6.4)} \\
&\quad - 2\left(\sum_{n=N}^{2N-1} e_n\right) + 2\left(\sum_{n=0}^{N-1} e_{n+N}\right) \\
&= \sum_{n=0}^{2N}(N+1-|n-N|)\cdot e_n \\
&\quad - 2\left(\sum_{n=N}^{2N-1} e_n\right) + 2\left(\sum_{n=N}^{2N-1} e_n\right) \\
&= \sum_{n=0}^{2N}(N+1-|n-N|)\cdot e_n.
\end{aligned}
$$

This closes the induction.

Using the claim above we have

$$
\begin{aligned}
&\sum_{n=-N}^{N}\left(1-\frac{|n|}{N}\right)e_n \\
&= \frac{1}{N}\sum_{n=-N}^{N}(N-|n|)e_n && \text{(by Lem. II.4.6.6)} \\
&= \frac{1}{N}\sum_{n=-(N-1)}^{N-1}(N-|n|)\cdot e_n \\
&= \frac{1}{N}\sum_{n=0}^{2N-2}(N-|n-N+1|)\cdot e_{n-N+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{n=0}^{2N-2}(N-|n-N+1|)\cdot e_n\cdot e_{-N+1} && \text{(by Ex. II.4.6.16)}\\
&= \frac{e_{-N+1}}{N}\sum_{n=0}^{2N-2}(N-|n-N+1|)\cdot e_n && \text{(by Lem. II.4.6.6)}\\
&= \frac{e_{-N+1}}{N}\left(\sum_{n=0}^{N-1} e_n\right)^2 && \text{(from the claim above)}\\
&= \frac{e_{-N+1}}{N}\left(\sum_{n=0}^{N-1} e_n\right)\left(\sum_{n=0}^{N-1} e_n\right)\\
&= \frac{1}{N}\left(\sum_{n=0}^{N-1} e_n\right)\left(\sum_{n=0}^{N-1} e_{-N+1}e_n\right) && \text{(by Lem. II.4.6.6)}\\
&= \frac{1}{N}\left(\sum_{n=0}^{N-1} e_n\right)\left(\sum_{n=0}^{N-1} e_{n-N+1}\right) && \text{(by Ex. II.4.6.16)}\\
&= \frac{1}{N}\left(\sum_{n=0}^{N-1} e_n\right)\left(\sum_{n=0}^{N-1} e_{-n}\right) && \text{(by Lem. II.4.6.4)}\\
&= \frac{1}{N}\left(\sum_{n=0}^{N-1} e_n\right)\left(\sum_{n=0}^{N-1} \overline{e_n}\right) && \text{(by Def. II.4.6.15)}\\
&= \frac{1}{N}\left(\sum_{n=0}^{N-1} e_n\right)\left(\overline{\sum_{n=0}^{N-1} e_n}\right) && \text{(by Lem. II.4.6.9)}\\
&= \frac{1}{N}\left|\sum_{n=0}^{N-1} e_n\right|^2. && \text{(by Lem. II.4.6.11)}
\end{aligned}
$$

□

Lem. II.5.4.6. For every $\varepsilon > 0$ and $0 < \delta < 1/2$, there exists a trigonometric polynomial P which is an (ε, δ) approximation to the identity.

Proof. Let $N \geq 1$ be an integer. We define the *Fejér kernel* F_N to be the function

$$F_N = \sum_{n=-N}^{N}\left(1-\frac{|n|}{N}\right)e_n.$$

Clearly, F_N is a trigonometric polynomial. We observe the identity

$$F_N = \frac{1}{N}\left|\sum_{n=0}^{N-1} e_n\right|^2$$

by A.Cor. II.5.4.2. But from the geometric series formula (Lemma 7.3.3 in Analysis I) we have

$$\begin{aligned}
\sum_{n=0}^{N-1} e_n(x) &= \sum_{n=0}^{N-1} \big(e_1(x)\big)^n && \text{(by Ex. II.4.6.16)}\\
&= \frac{\big(e_1(x)\big)^N - 1}{e_1(x) - 1} && \text{(geometric series)}\\
&= \frac{\big(e_1(x)\big)^N - e_0(x)}{e_1(x) - e_0(x)} && \text{(by Thm. II.4.5.2(e))}\\
&= \frac{e_N(x) - e_0(x)}{e_1(x) - e_0(x)} && \text{(by Ex. II.4.6.16)}\\
&= \frac{e^{2\pi i N x} - e^0}{e^{2\pi i x} - e^0} && \text{(by Def. II.5.3.1)}\\
&= \frac{e^{\pi i N x}e^{\pi i N x} - e^{\pi i N x}e^{-\pi i N x}}{e^{\pi i x}e^{\pi i x} - e^{\pi i x}e^{-\pi i x}} && \text{(by Ex. II.4.6.16)}\\
&= \frac{e^{\pi i N x}\big(e^{\pi i N x} - e^{-\pi i N x}\big)}{e^{\pi i x}\big(e^{\pi i x} - e^{-\pi i x}\big)} && \text{(by Lem. II.4.6.6)}\\
&= \frac{e^{\pi i (N-1) x}\big(e^{\pi i N x} - e^{-\pi i N x}\big)}{e^{\pi i x} - e^{-\pi i x}} && \text{(by Def. II.4.6.12)}\\
&= \frac{2i e^{\pi i (N-1) x}\sin(\pi N x)}{2i\sin(\pi x)} && \text{(by Def. II.4.7.1)}\\
&= \frac{e^{\pi i (N-1) x}\sin(\pi N x)}{\sin(\pi x)} && \text{(by Def. II.4.6.12)}
\end{aligned}$$

when x is not an integer, and hence we have the formula

$$\begin{aligned}
F_N(x) &= \frac{1}{N}\left|\sum_{n=0}^{N-1} e_n(x)\right|^2 && \text{(by A.Cor. II.5.4.2)}\\
&= \frac{1}{N}\left|\frac{e^{\pi i (N-1) x}\sin(\pi N x)}{\sin(\pi x)}\right|^2 && \text{(from the proof above)}\\
&= \frac{\left|e^{\pi i (N-1) x}\right|^2 |\sin(\pi N x)|^2}{N|\sin(\pi x)|^2} && \text{(by Ex. II.4.6.7)}\\
&= \frac{\left|e^{\pi i (N-1) x}\right|^2 \big(\sin(\pi N x)\big)^2}{N\big(\sin(\pi x)\big)^2} && (x \in \mathbb{R})\\
&= \frac{e^{\pi i (N-1) x}e^{-\pi i (N-1) x} \big(\sin(\pi N x)\big)^2}{N\big(\sin(\pi x)\big)^2} && \text{(by Lem. II.4.6.11)}
\end{aligned}$$

$$= \frac{e^0\big(\sin(\pi N x)\big)^2}{N\big(\sin(\pi x)\big)^2} \qquad \text{(by Ex. II.4.6.16)}$$
$$= \frac{\big(\sin(\pi N x)\big)^2}{N\big(\sin(\pi x)\big)^2}. \qquad \text{(by Thm. II.4.5.2(e))}$$

When x is an integer, the geometric series formula does not apply, but one has $F_N(x) = N$ in that case, as one can see by direct computation. In either case we see that $F_N(x) \geq 0$ for any x. Also, we have

$$\begin{aligned}
&\int_{[0,1]} F_N(x)\,dx \\
&= \int_{[0,1]} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e_n(x)\,dx \\
&= \sum_{n=-N}^{N} \left(\left(1 - \frac{|n|}{N}\right) \int_{[0,1]} e_n(x)\,dx\right) && \text{(by Rmk. II.5.2.2)} \\
&= \sum_{n=-N}^{N} \left(\left(1 - \frac{|n|}{N}\right) \int_{[0,1]} e_{n-1}(x) e_1(x)\,dx\right) && \text{(by Ex. II.4.6.16)} \\
&= \sum_{n=-N}^{N} \left(\left(1 - \frac{|n|}{N}\right) \int_{[0,1]} e_{n-1}(x)\overline{e_{-1}(x)}\,dx\right) && \text{(by Def. II.4.6.15)} \\
&= \sum_{n=-N}^{N} \left(\left(1 - \frac{|n|}{N}\right) \langle e_{n-1}, e_{-1}\rangle\right) && \text{(by Def. II.5.2.1)} \\
&= \left(1 - \frac{|0|}{N}\right) 1 && \text{(by Lem. II.5.3.5)} \\
&= 1.
\end{aligned}$$

Finally, since $\sin(\pi N x) \leq 1$, we have

$$F_N(x) \leq \frac{1}{N\big(\sin(\pi x)\big)^2} \leq \frac{1}{N\big(\sin(\pi\delta)\big)^2}$$

whenever $\delta < |x| < 1 - \delta$ (this is because sin is increasing on $[0, \pi/2]$ and decreasing on $[\pi/2, \pi]$). Thus, by choosing N large enough, we can make $F_N(x) \leq \varepsilon$ for all $\delta < |x| < 1 - \delta$. Note that since

$$\big(\sin(\pi|x|)\big)^2 = \begin{cases} \big(\sin(\pi x)\big)^2 & \text{if } x \geq 0 \\ \big(\sin(\pi - x)\big)^2 & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} \big(\sin(\pi x)\big)^2 & \text{if } x \geq 0 \\ \big(-\sin(\pi x)\big)^2 & \text{if } x < 0 \end{cases} \qquad \text{(by Thm. II.4.7.2(c))}$$
$$= \big(\sin(\pi x)\big)^2$$

and

$$\begin{cases} \delta < |x| \leq \frac{1}{2} & \text{if } |x| \leq \frac{1}{2} \\ \frac{1}{2} < |x| < 1-\delta & \text{if } |x| > \frac{1}{2} \end{cases}$$
$$\Longrightarrow \begin{cases} \pi\delta < \pi|x| \leq \frac{\pi}{2} & \text{if } |x| \leq \frac{1}{2} \\ \frac{\pi}{2} < \pi|x| < \pi - \pi\delta & \text{if } |x| > \frac{1}{2} \end{cases}$$
$$\Longrightarrow \begin{cases} \sin(\pi\delta) < \sin(\pi|x|) \leq \sin(\frac{\pi}{2}) & \text{if } |x| \leq \frac{1}{2} \\ \sin(\frac{\pi}{2}) > \sin(\pi|x|) > \sin(\pi - \pi\delta) & \text{if } |x| > \frac{1}{2} \end{cases}$$
$$\Longrightarrow \begin{cases} \sin(\pi\delta) < \sin(\pi|x|) \leq \sin(\frac{\pi}{2}) & \text{if } |x| \leq \frac{1}{2} \\ \sin(\frac{\pi}{2}) > \sin(\pi|x|) > -\sin(-\pi\delta) & \text{if } |x| > \frac{1}{2} \end{cases} \qquad \text{(by Thm. II.4.7.5(a))}$$
$$\Longrightarrow \begin{cases} \sin(\pi\delta) < \sin(\pi|x|) \leq \sin(\frac{\pi}{2}) & \text{if } |x| \leq \frac{1}{2} \\ \sin(\frac{\pi}{2}) > \sin(\pi|x|) > \sin(\pi\delta) & \text{if } |x| > \frac{1}{2} \end{cases} \qquad \text{(by Thm. II.4.7.2(c))}$$
$$\Longrightarrow \sin(\pi\delta) < \sin(\pi|x|),$$

we have

$$0 < \big(\sin(\pi\delta)\big)^2 < \big(\sin(\pi|x|)\big)^2 \qquad \text{(by A.Cor. II.4.7.2(d))}$$
$$\Longrightarrow 0 < \big(\sin(\pi\delta)\big)^2 < \big(\sin(\pi x)\big)^2 \qquad \text{(from the proof above)}$$
$$\Longrightarrow 0 < \frac{1}{\big(\sin(\pi x)\big)^2} < \frac{1}{\big(\sin(\pi\delta)\big)^2}.$$

□

— Exercises —

Ex. II.5.4.1. Show that if $f : \mathbb{R} \to \mathbb{C}$ is both compactly supported and $\mathbb{Z}$-periodic, then it is identically zero.

Proof. Since f is compactly supported, by Def. II.3.8.4 we know that

$$\exists L, U \in \mathbb{R} : \forall x \in \mathbb{R} \setminus [L, U], f(x) = 0.$$

Fix such L, U. Let $[U - L]$ be the integer defined in Ex. II.5.1.1. Then we have

$$\begin{aligned}
&\forall x \in [L, U], L \leq x \\
\implies& L + [U - L] \leq x + [U - L] \\
\implies& U = L + U - L < L + [U - L] + 1 \leq x + [U - L] + 1 && \text{(by Ex. II.5.1.1)} \\
\implies& f(x) = f(x + [U - L] + 1) = 0. && (f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C}))
\end{aligned}$$

Thus, $f = 0$. □

Ex. II.5.4.2. Prove Lem. II.5.4.4.

Proof. See Lem. II.5.4.4. □

Ex. II.5.4.3. Fill in the gaps marked in Lem. II.5.4.6.

Proof. See Lem. II.5.4.6. □

II.5.5 The Fourier and Plancherel theorems

Thm. II.5.5.1 (Fourier theorem)**.** For any $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, the series $\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$ converges in L^2 metric to f. In other words, we have

$$\lim_{N \to \infty} \left\| f - \sum_{n=-N}^{N} \hat{f}(n) e_n \right\|_2 = 0.$$

Proof. Let $\varepsilon > 0$. We have to show that there exists an N_0 such that

$$\left\| f - \sum_{n=-N}^{N} \hat{f}(n) e_n \right\|_2 \leq \varepsilon$$

for all $N \geq N_0$.

By the Weierstrass approximation theorem (Thm. II.5.4.1), we can find a trigonometric polynomial $P = \sum_{n=-N_0}^{N_0} c_n e_n$ such that $\|f - P\|_\infty \leq \varepsilon$, for some $N_0 > 0$. In particular, we have $\|f - P\|_2 \leq \varepsilon$ (Ex. II.5.2.3).

Now let $N > N_0$, and let $F_N \coloneqq \sum_{n=-N}^{N} \hat{f}(n) e_n$. We claim that $\|f - F_N\|_2 \leq \varepsilon$. First, observe that for any $|m| \leq N$, we have

$$\langle f - F_N, e_m \rangle = \langle f, e_m \rangle - \sum_{n=-N}^{N} \hat{f}(n) \langle e_n, e_m \rangle = \hat{f}(m) - \hat{f}(m) = 0,$$

where we have used Lem. II.5.3.5 and Lem. II.5.2.5. In particular, we have

$$\langle f - F_N, F_N - P\rangle = 0$$

since we can write $F_N - P$ as a linear combination of the e_m for which $|m| \leq N$. By Pythagoras' theorem (Lem. II.5.2.7(d)) we therefore have

$$\|f - P\|_2^2 = \|f - F_N\|_2^2 + \|F_N - P\|_2^2$$

and in particular

$$\|f - F_N\|_2 \leq \|f - P\|_2 \leq \varepsilon$$

as desired. □

Rmk. II.5.5.2. Note that we have only obtained convergence of the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ to f in the L^2 metric. One may ask whether one has convergence in the uniform or pointwise sense as well, but it turns out (perhaps somewhat surprisingly) that the answer is no to both of those questions. However, if one assumes that the function f is not only continuous, but is also differentiable, then one can recover pointwise convergence; if one assumes continuously differentiable, then one gets uniform convergence as well. These results are beyond the scope of this text and will not be proven here. However, we will prove one theorem about when one can improve the L^2 convergence to uniform convergence.

Thm. II.5.5.3. Let $f \in C(\mathbb{R}/\mathbb{Z}; \mathbb{C})$, and suppose that the series $\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|$ is absolutely convergent. Then the series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ converges uniformly to f. In other words, we have

$$\lim_{N\to\infty} \left\| f - \sum_{n=-N}^{N} \hat{f}(n)e_n \right\|_{\infty} = 0.$$

Proof. By the Weierstrass M-test (Thm. II.3.5.7), we see that $\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ converges to some function F, which by Lem. II.5.1.5(c) is also continuous and $\mathbb{Z}$-periodic. (Strictly speaking, the Weierstrass M-test was phrased for series from $n = 1$ to $n = +\infty$, but also works for series from $n = -\infty$ to $n = +\infty$; this can be seen by splitting the doubly infinite series into two pieces.) Thus

$$\lim_{N\to\infty} \left\| F - \sum_{n=-N}^{N} \hat{f}(n)e_n \right\|_{\infty} = 0$$

which implies that

$$\lim_{N\to\infty} \left\| F - \sum_{n=-N}^{N} \hat{f}(n) e_n \right\|_2 = 0$$

since the L^2 norm is always less than or equal to the L^∞ norm (Ex. II.5.2.3). But the sequence $\sum_{n=-N}^{N} \hat{f}(n)e_n$ is already converging in L^2 metric to f by the Fourier theorem (Thm. II.5.5.1), so can only converge in L^2 metric to F if $F = f$ (cf. Prop. II.1.1.20). Thus, $F = f$, and so we have

$$\lim_{N\to\infty} \left\| f - \sum_{n=-N}^{N} \hat{f}(n) e_n \right\|_\infty = 0$$

as desired. □

Thm. II.5.5.4. ncherel theorem] For any $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, the series

$$\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2$$

is absolutely convergent, and

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2.$$

Proof. Let $\varepsilon > 0$. By the Fourier theorem (Thm. II.5.5.1) we know that

$$\left\| f - \sum_{n=-N}^{N} \hat{f}(n) e_n \right\|_2 \le \varepsilon$$

if N is large enough (depending on ε). In particular, by the triangle inequality (Lem. II.5.2.7(c)(e) this implies that

$$\|f\|_2 - \varepsilon \le \left\| \sum_{n=-N}^{N} \hat{f}(n) e_n \right\|_2 \le \|f\|_2 + \varepsilon.$$

On the other hand, by Cor. II.5.3.6 we have

$$\left\| \sum_{n=-N}^{N} \hat{f}(n) e_n \right\|_2 = \left(\sum_{n=-N}^{N} \left|\hat{f}(n)\right|^2 \right)^{1/2}$$

and hence

$$(\|f\|_2 - \varepsilon)^2 \le \sum_{n=-N}^{N} \left|\hat{f}(n)\right|^2 \le (\|f\|_2 + \varepsilon)^2.$$

Taking lim sup, we obtain

$$(\|f\|_2 - \varepsilon)^2 \leq \limsup_{N\to\infty} \sum_{n=-N}^{N} \left|\hat{f}(n)\right|^2 \leq (\|f\|_2 + \varepsilon)^2.$$

Since ε was arbitrary, we thus obtain by the squeeze test that

$$\limsup_{N\to\infty} \sum_{n=-N}^{N} \left|\hat{f}(n)\right|^2 = \|f\|_2^2$$

and the claim follows. □

Note. Thm. II.5.5.4 is also known as *Parseval's theorem.*

— Exercises —

Ex. II.5.5.1. Let f be a function in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, and define the *trigonometric Fourier coefficients* a_n, b_n for $n = 0, 1, 2, 3, \ldots$ by

$$a_n = 2\int_{[0,1]} f(x)\cos(2\pi nx)\; dx; \quad b_n = 2\int_{[0,1]} f(x)\sin(2\pi nx)\; dx.$$

(a) Show that the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \big(a_n\cos(2\pi nx) + b_n\sin(2\pi nx)\big)$$

converges in L_2 metric to f.

(b) Show that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, then the above series actually converges uniformly to f, and not just in L_2 metric.

Proof. (a) Observe that for all $n \in \mathbb{Z}$, we have

$$\begin{aligned}
&\hat{f}(n)\\
&= \int_{[0,1]} f(x)e^{-2\pi inx}\; dx && \text{(by Def. II.5.3.7)}\\
&= \int_{[0,1]} f(x)\big(\cos(2\pi nx) - i\sin(2\pi nx)\big)\; dx && \text{(by Thm. II.4.7.2(f))}\\
&= \int_{[0,1]} f(x)\cos(2\pi nx)\; dx - i\int_{[0,1]} f(x)\sin(2\pi nx)\; dx && \text{(by Rmk. II.5.2.2)}
\end{aligned}$$

$$= \frac{1}{2}\left(2\int_{[0,1]} f(x)\cos(2\pi nx)\,dx - 2i\int_{[0,1]} f(x)\sin(2\pi nx)\,dx\right)$$
$$= \frac{1}{2}(a_n - ib_n)$$

and

$$\begin{aligned}
&\hat{f}(-n)\\
&= \frac{1}{2}(a_{-n} - ib_{-n}) && \text{(from the proof above)}\\
&= \frac{1}{2}\left(2\int_{[0,1]} f(x)\cos(-2\pi nx)\,dx\right.\\
&\quad \left. - 2i\int_{[0,1]} f(x)\sin(-2\pi nx)\,dx\right)\\
&= \frac{1}{2}\left(2\int_{[0,1]} f(x)\cos(2\pi nx)\,dx\right.\\
&\quad \left. - 2i\int_{[0,1]} -f(x)\sin(2\pi nx)\,dx\right) && \text{(by Thm. II.4.7.2(c))}\\
&= \frac{1}{2}\left(2\int_{[0,1]} f(x)\cos(2\pi nx)\,dx + 2i\int_{[0,1]} f(x)\sin(2\pi nx)\,dx\right) && \text{(by Rmk. II.5.2.2)}\\
&= \frac{1}{2}(a_n + ib_n).
\end{aligned}$$

By Fourier theroem (Thm. II.5.5.1) we know that

$$\lim_{N\to\infty}\left\| f - \sum_{n=-N}^{N} \hat{f}(n)e_n \right\|_2 = 0.$$

Since for all $N \in \mathbb{Z}^+$, we have

$$\begin{aligned}
&\sum_{n=-N}^{N} \hat{f}(n)e_n\\
&= \hat{f}(0)e_0 + \sum_{n=1}^{N} \hat{f}(n)e_n + \sum_{n=-N}^{-1} \hat{f}(n)e_n\\
&= \hat{f}(0)e_0 + \sum_{n=1}^{N} \hat{f}(n)e_n + \sum_{n=1}^{N} \hat{f}(-n)e_{-n}\\
&= \frac{(a_0 - ib_0)e_0}{2} + \sum_{n=1}^{N} \frac{(a_n - ib_n)e_n}{2} + \sum_{n=1}^{N} \frac{(a_n + ib_n)e_{-n}}{2} && \text{(from the proof above)}
\end{aligned}$$

$$= \frac{a_0 e_0}{2} + \sum_{n=1}^{N} \frac{(a_n - ib_n)e_n}{2} + \sum_{n=1}^{N} \frac{(a_n + ib_n)e_{-n}}{2} \qquad \text{(by Thm. II.4.7.2(e))}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{N} \frac{(a_n - ib_n)e_n}{2} + \sum_{n=1}^{N} \frac{(a_n + ib_n)e_{-n}}{2} \qquad \text{(by Thm. II.4.5.2(e))}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{N} \frac{a_n(e_n + e_{-n}) - ib_n(e_n - e_{-n})}{2} \qquad \text{(by Lem. II.4.6.6)}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(2\pi nx) + b_n \sin(2\pi nx), \qquad \text{(by Def. II.4.7.1)}$$

we know that

$$\lim_{N\to\infty} \left\| f - \left(\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(2\pi nx) + b_n \sin(2\pi nx) \right) \right\|_2 = 0.$$

Thus, by Def. II.1.1.14 we have

$$\begin{aligned} &d_{L^2} - \lim_{N\to\infty} \left(\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(2\pi nx) + b_n \sin(2\pi nx) \right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(2\pi nx) + b_n \sin(2\pi nx) \\ &= f. \end{aligned}$$

□

Proof. (b) Observe that

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \qquad \text{(by hypothesis)}$$

$$= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} |a_n| + \lim_{N\to\infty} \sum_{n=1}^{N} |b_n| \qquad \text{(by A.Cor. II.4.6.6)}$$

$$= \frac{a_0}{2} + \lim_{N\to\infty} \sum_{n=1}^{N} |a_n| + |b_n| \qquad \text{(by Lem. II.4.6.14)}$$

$$= \frac{a_0}{2} + 2\left(\lim_{N\to\infty} \sum_{n=1}^{N} \frac{|a_n| + |b_n|}{2} \right) \qquad \text{(by Lem. II.4.6.14)}$$

$$= \frac{a_0}{2} + 2\sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{2} \qquad \text{(by A.Cor. II.4.6.6)}$$

$$
\begin{aligned}
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{2} + \sum_{n=1}^{\infty} \frac{|a_n| + |b_n|}{2} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{|a_n| + |ib_n|}{2} + \sum_{n=1}^{\infty} \frac{|a_n| + |-ib_n|}{2} && \text{(by Lem. II.4.6.11)} \\
&\geq \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{|a_n + ib_n|}{2} + \sum_{n=1}^{\infty} \frac{|a_n - ib_n|}{2} && \text{(by Lem. II.4.6.11)} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{|a_{-n} - ib_{-n}|}{2} + \sum_{n=1}^{\infty} \frac{|a_n - ib_n|}{2} && \text{(by Thm. II.4.7.2(c))} \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{|a_{-n} - ib_{-n}|}{2} + \sum_{n=1}^{\infty} \frac{|a_n - ib_n|}{2} && \text{(by Thm. II.4.7.2(e))} \\
&= \lim_{N\to\infty} \frac{a_0}{2} + \sum_{n=1}^{N} \frac{|a_{-n} - ib_{-n}|}{2} + \sum_{n=1}^{N} \frac{|a_n - ib_n|}{2} && \text{(by Lem. II.4.6.14)} \\
&= \lim_{N\to\infty} \frac{a_0}{2} + \sum_{n=-N}^{-1} \frac{|a_n - ib_n|}{2} + \sum_{n=1}^{N} \frac{|a_n - ib_n|}{2} \\
&= \lim_{N\to\infty} \frac{a_0 - ib_0}{2} + \sum_{n=-N}^{-1} \frac{|a_n - ib_n|}{2} + \sum_{n=1}^{N} \frac{|a_n - ib_n|}{2} && \text{(by Thm. II.4.7.2(e))} \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{|a_{-n} - ib_{-n}|}{2} \\
&= \sum_{n=-\infty}^{\infty} \frac{|a_{-n} - ib_{-n}|}{2}.
\end{aligned}
$$

Since

$$\hat{f}(n) = \frac{1}{2}(a_n - ib_n)$$

for all $n \in \mathbb{Z}$ (cf. the proof of Ex. II.5.5.1(a)), we know that

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right| &= \sum_{n=-\infty}^{\infty} \left|\frac{a_n - ib_n}{2}\right| \\
&= \sum_{n=-\infty}^{\infty} \frac{|a_n - ib_n|}{2} && \text{(by Ex. II.4.6.7)} \\
&\leq \frac{a_0}{2} + \sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n|. && \text{(from the proof above)}
\end{aligned}
$$

Thus, $\sum_{n=-\infty}^{\infty}\left|\hat{f}(n)\right|$ is absolutely convergent. By Thm. II.5.5.3 we know that

$$\lim_{N\to\infty}\left\|f-\sum_{n=-N}^{N}\hat{f}(n)e_n\right\|_\infty=0.$$

and $\sum_{n=-\infty}^{\infty}\hat{f}(n)e_n$ converges uniformly to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{C}\times\mathbb{C}}$. In particular, we have (by Ex. II.5.2.3 and squeeze test)

$$\lim_{N\to\infty}\left\|f-\sum_{n=-N}^{N}\hat{f}(n)e_n\right\|_2=0.$$

By Ex. II.5.5.1(a) we know that

$$\lim_{N\to\infty}\left\|f-\left(\frac{a_0}{2}+\sum_{n=1}^{N}a_n\cos(2\pi nx)+b_n\sin(2\pi nx)\right)\right\|_2=0.$$

Thus, by Prop. II.1.1.20 we have

$$\lim_{N\to\infty}\left\|f-\left(\frac{a_0}{2}+\sum_{n=1}^{N}a_n\cos(2\pi nx)+b_n\sin(2\pi nx)\right)\right\|_\infty=0.$$

and $\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(2\pi nx)+b_n\sin(2\pi nx)$ converges uniformly to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{C}\times\mathbb{C}}$. □

Ex. II.5.5.2. Let $f(x)$ be the function defined by $f(x)=(1-2x)^2$ when $x\in[0,1)$, and extended to be $\mathbb{Z}$-periodic for the rest of the real line.

(a) Using Ex. II.5.5.1, show that the series

$$\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{\pi^2n^2}\cos(2\pi nx)$$

converges uniformly to f .

(b) Conclude that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

(c) Conclude that $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$.

Proof. (a) For each $n \in \mathbb{N}$, we define a_n, b_n as in Ex. II.5.5.1. Observe that for all $n \in \mathbb{Z}^+$, we have

$$\begin{aligned}
a_n &= 2\int_{[0,1]} (1-2x)^2 \cos(2\pi n x)\ dx && \text{(by Ex. II.5.5.1)}\\
&= \frac{2}{2\pi n}\int_{[0,1]} (1-2x)^2 \sin'(2\pi n x)\ dx && \text{(by Thm. II.4.7.2(b))}\\
&= \frac{1}{\pi n}\Bigg(\left((1-2x)^2 \sin(2\pi n x)\right)\Big|_{x=0}^{x=1} && \text{(by Proposition 11.10.1)}\\
&\quad - \int_{[0,1]} -4(1-2x)\sin(2\pi n x)\ dx\Bigg)\\
&= \frac{4}{\pi n}\int_{[0,1]} (1-2x)\sin(2\pi n x)\ dx && \text{(by Thm. II.4.7.2(e))}\\
&= \frac{-4}{2\pi^2 n^2}\int_{[0,1]} (1-2x)\cos'(2\pi n x)\ dx && \text{(by Thm. II.4.7.2(b))}\\
&= \frac{-2}{\pi^2 n^2}\Bigg(\left((1-2x)\cos(2\pi n x)\right)\Big|_{x=0}^{x=1}\\
&\quad - \int_{[0,1]} -2\cos(2\pi n x)\ dx\Bigg) && \text{(by Proposition 11.10.1)}\\
&= \frac{-2}{\pi^2 n^2}\left(-2 + 2\int_{[0,1]} \cos(2\pi n x)\ dx\right) && \text{(by Thm. II.4.7.2(e))}\\
&= \frac{-2}{\pi^2 n^2}\left(-2 + \frac{2\sin(2\pi n x)}{2\pi n}\Big|_{x=0}^{x=1}\right) && \text{(by Thm. II.4.7.2(b))}\\
&= \frac{4}{\pi^2 n^2} && \text{(by Thm. II.4.7.2(e))}
\end{aligned}$$

and

$$\begin{aligned}
b_n &= 2\int_{[0,1]} (1-2x)^2 \sin(2\pi n x)\ dx && \text{(by Ex. II.5.5.1)}\\
&= \frac{-2}{2\pi n}\int_{[0,1]} (1-2x)^2 \cos'(2\pi n x)\ dx && \text{(by Thm. II.4.7.2(b))}\\
&= \frac{-1}{\pi n}\Bigg(\left((1-2x)^2 \cos(2\pi n x)\right)\Big|_{x=0}^{x=1} && \text{(by Proposition 11.10.1)}\\
&\quad - \int_{[0,1]} -4(1-2x)\cos(2\pi n x)\ dx\Bigg)\\
&= \frac{-4}{\pi n}\int_{[0,1]} (1-2x)\cos(2\pi n x)\ dx && \text{(by Thm. II.4.7.2(e))}\\
&= \frac{-4}{2\pi^2 n^2}\int_{[0,1]} (1-2x)\sin'(2\pi n x)\ dx && \text{(by Thm. II.4.7.2(b))}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-2}{\pi^2 n^2}\Bigg(\big((1-2x)\sin(2\pi n x)\big)\big|_{x=0}^{x=1} \\
&\quad - \int_{[0,1]} -2\sin(2\pi n x)\, dx \Bigg) && \text{(by Proposition 11.10.1)} \\
&= \frac{-4}{\pi^2 n^2} \int_{[0,1]} \sin(2\pi n x)\, dx && \text{(by Thm. II.4.7.2(e))} \\
&= \frac{-4}{\pi^2 n^2} \frac{-\cos(2\pi n x)}{2\pi n}\Big|_{x=0}^{x=1} && \text{(by Thm. II.4.7.2(b))} \\
&= 0. && \text{(by Thm. II.4.7.2(e))}
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \\
\sum_{n=1}^{\infty} |b_n| &= \sum_{n=1}^{\infty} 0
\end{aligned}
$$

are absolutely convergent (by Corollary 7.3.7 in Analysis I), by Ex. II.5.5.1(b) we know that the series

$$
\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(2\pi n x) + b_n \sin(2\pi n x))
$$

converges uniformly to f on $\mathbb{R}$ with respect to $d_{l^1}|_{\mathbb{C}\times\mathbb{C}}$, and

$$
\begin{aligned}
&\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(2\pi n x) + b_n \sin(2\pi n x)) \\
&= \int_{[0,1]} (1-2x)^2 \cos(0)\, dx + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x) && \text{(from the proof above)} \\
&= \int_{[0,1]} (1-2x)^2\, dx + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x) && \text{(by Thm. II.4.7.2(e))} \\
&= 1 - \left(2x^2|_{x=0}^{x=1}\right) + \left(\frac{4x^3}{3}|_{x=0}^{x=1}\right) + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x) \\
&= 1 - 2 + \frac{4}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x) \\
&= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} \cos(2\pi n x).
\end{aligned}
$$

□

Proof. (b) We have

$$
\begin{aligned}
(1-2\cdot 0)^2 &= \frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{\pi^2n^2}\cos(2\pi n\cdot 0) && \text{(by Ex. II.5.5.2(a))}\\
\implies 1 &= \frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{\pi^2n^2} && \text{(by Thm. II.4.7.2(e))}\\
\implies \frac{2}{3} &= \sum_{n=1}^{\infty}\frac{4}{\pi^2n^2}\\
\implies \frac{\pi^2}{6} &= \sum_{n=1}^{\infty}\frac{1}{n^2}.
\end{aligned}
$$

□

Proof. (c) By Ex. II.5.5.2(a) we know that

$$f(x)=(1-2x)^2=\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{\pi^2n^2}\cos(2\pi nx)$$

and the series on the right hand side converges uniformly to f. Observe that for each $m\in\mathbb{Z}$, we have

$$
\begin{aligned}
&\hat{f}(m)\\
&=\int_{[0,1]}\left(\frac{1}{3}+\sum_{n=1}^{\infty}\frac{4}{\pi^2n^2}\cos(2\pi nx)\right)e^{-2\pi imx}\,dx && \text{(by Def. II.5.3.7)}\\
&=\int_{[0,1]}\frac{e^{-2\pi imx}}{3}\,dx+\int_{[0,1]}\sum_{n=1}^{\infty}\frac{4e^{-2\pi imx}}{\pi^2n^2}\cos(2\pi nx)\,dx && \text{(by Rmk. II.5.2.2)}\\
&=\int_{[0,1]}\frac{e^{-2\pi imx}}{3}\,dx+\sum_{n=1}^{\infty}\int_{[0,1]}\frac{4e^{-2\pi imx}}{\pi^2n^2}\cos(2\pi nx)\,dx && \text{(by Cor. II.3.6.2)}\\
&=\int_{[0,1]}\frac{e^{-2\pi imx}}{3}\,dx+\sum_{n=1}^{\infty}\int_{[0,1]}\frac{2e^{-2\pi imx}(e^{2\pi inx}+e^{-2\pi inx})}{\pi^2n^2}\,dx && \text{(by Def. II.4.7.1)}\\
&=\int_{[0,1]}\frac{e^{-2\pi imx}}{3}\,dx+\sum_{n=1}^{\infty}\int_{[0,1]}\frac{2(e^{2\pi i(n-m)x}+e^{-2\pi i(n+m)x})}{\pi^2n^2}\,dx. && \text{(by Ex. II.4.6.16)}
\end{aligned}
$$

Now we split into two cases:

- If $m=0$, then we have

$$\hat{f}(0)=\int_{[0,1]}\frac{1}{3}\,dx+\sum_{n=1}^{\infty}\int_{[0,1]}\frac{2(e^{2\pi inx}+e^{-2\pi inx})}{\pi^2n^2}\,dx \qquad \text{(by Thm. II.4.5.2(e))}$$

$$
\begin{aligned}
&= \frac{1}{3} + \sum_{n=1}^{\infty} \left(\frac{2}{\pi^2 n^2} \int_{[0,1]} e^{2\pi i n x} + e^{-2\pi i n x}\, dx \right) \\
&= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} (0+0) && \text{(by Lem. II.5.3.5)} \\
&= \frac{1}{3}.
\end{aligned}
$$

- If $m \neq 0$, then we have

$$
\begin{aligned}
&\hat{f}(m) \\
&= 0 + \sum_{n=1}^{\infty} \int_{[0,1]} \frac{2(e^{2\pi i (n-m)x} + e^{-2\pi i (n+m)x})}{\pi^2 n^2}\, dx && \text{(by Lem. II.5.3.5)} \\
&= \sum_{n=1}^{\infty} \left(\frac{2}{\pi^2 n^2} \left(\int_{[0,1]} e^{2\pi i (n-m)x} + e^{-2\pi i (n+m)x}\, dx \right) \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{2}{\pi^2 n^2} \left(\langle e_n, e_m \rangle + \langle e_{-n}, e_m \rangle \right) \right) && \text{(by Def. II.5.2.1)} \\
&= \frac{2}{\pi^2 m^2}. && \text{(by Lem. II.5.3.5)}
\end{aligned}
$$

From all cases above, we have

$$
\begin{aligned}
&\|f\|_2^2 = \sum_{n=-\infty}^{\infty} \left| \hat{f}(n) \right|^2 && \text{(by Thm. II.5.5.4)} \\
\Longrightarrow\ & \int_{[0,1]} (1-2x)^4\, dx = \sum_{n=-\infty}^{\infty} \left| \hat{f}(n) \right|^2 \\
\Longrightarrow\ & \int_{[0,1]} 1 - 8x + 24x^2 - 32x^3 + 16x^4\, dx = \sum_{n=-\infty}^{\infty} \left| \hat{f}(n) \right|^2 \\
\Longrightarrow\ & 1 - 4(x^2|_{x=0}^{x=1}) + 8(x^3|_{x=0}^{x=1}) - 8(x^4|_{x=0}^{x=1}) + \frac{16}{5}(x^5|_{x=0}^{x=1}) \\
&= \sum_{n=-\infty}^{\infty} \left| \hat{f}(n) \right|^2 \\
\Longrightarrow\ & \frac{1}{5} = \frac{1}{9} + \sum_{n=1}^{\infty} \left| \frac{2}{\pi^2 n^2} \right|^2 + \sum_{n=1}^{\infty} \left| \frac{2}{\pi^2 (-n)^2} \right|^2 && \text{(from the proof above)} \\
\Longrightarrow\ & \frac{4}{45} = 2 \sum_{n=1}^{\infty} \left| \frac{2}{\pi^2 n^2} \right|^2
\end{aligned}
$$

$$\implies \frac{4}{45} = 8\sum_{n=1}^{\infty} \frac{1}{\pi^4 n^4}$$
$$\implies \frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

$\square$

Ex. II.5.5.3. If $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ and P is a trigonometric polynomial, show that

$$\widehat{f*P}(n) = \hat{f}(n)c_n = \hat{f}(n)\hat{P}(n)$$

for all integers n. More generally, if $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, show that

$$\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n)$$

for all integers n. (A fancy way of saying this is that the Fourier transform *intertwines* convolution and multiplication.)

Proof. Let $P = \sum_{n=-N}^{N} c_n e_n$ for some $N \in \mathbb{Z}^+$ and some $(c_n)_{n=-N}^{N}$ in $\mathbb{C}$. By A.Cor. II.5.4.1 we know that

$$f * P = \sum_{n=-N}^{N} \hat{f}(n)c_n e_n.$$

Thus, we have

$$\begin{aligned}
\forall m \in \mathbb{Z}, \widehat{f*P}(m) &= \langle f*P, e_m\rangle && \text{(by Def. II.5.3.7)}\\
&= \sum_{n=-N}^{N} \hat{f}(n)c_n\langle e_n, e_m\rangle && \text{(by Lem. II.5.2.5(c))}\\
&= \begin{cases} \hat{f}(m)c_m & \text{if } n = m\\ 0 & \text{if } n \neq m \end{cases} && \text{(by Lem. II.5.3.5)}\\
&= \hat{f}(m)\sum_{n=-N}^{N} c_n\langle e_n, e_m\rangle && \text{(by Lem. II.5.3.5)}\\
&= \hat{f}(m)\left\langle \sum_{n=-N}^{N} c_n e_n, e_m\right\rangle && \text{(by Lem. II.5.2.5(c))}\\
&= \hat{f}(m)\hat{P}(m). && \text{(by Def. II.5.3.7)}
\end{aligned}$$

Now we show that $\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n)$ for all $n \in \mathbb{Z}$. In particular, we want to show that

$$\forall \varepsilon \in \mathbb{R}^+, \left|\widehat{f*g}(n) - \hat{f}(n)\hat{g}(n)\right| \leq \varepsilon.$$

Let $\varepsilon \in \mathbb{R}^+$. Since $g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, by Thm. II.5.4.1 we know that there exists a trigonometric polynomial P such that

$$\|g - P\|_\infty \le \varepsilon.$$

Fix such P. Since $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, by Thm. II.5.5.4 we know that $\sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\right|^2 \in \mathbb{R}$, thus

$$\exists M \in \mathbb{R}^+ : \forall n \in \mathbb{Z}, \left|\hat{f}(n)\right| \le M.$$

Fix such M. Then for all $n \in \mathbb{Z}$, we have

$$\begin{aligned}
\left|\widehat{f*g}(n) - \widehat{f*P}(n)\right| &= |\langle f*g, e_n\rangle - \langle f*P, e_n\rangle| && \text{(by Def. II.5.3.7)}\\
&= |\langle f*g - f*P, e_n\rangle| && \text{(by Lem. II.5.2.5(c))}\\
&= |\langle f*(g-P), e_n\rangle| && \text{(by Lem. II.5.4.4(c))}\\
&\le \|f*(g-P)\|_2 \|e_n\|_2 && \text{(by Lem. II.5.2.7(b))}\\
&= \|f*(g-P)\|_2. && \text{(by Lem. II.5.3.5)}
\end{aligned}$$

Need some helps. □

Ex. II.5.5.4. Let $f \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$ be a function which is differentiable, and whose derivative f' is also continuous. Show that f' also lies in $C(\mathbb{R}/\mathbb{Z};\mathbb{C})$, and that $\hat{f}'(n) = 2\pi i n \hat{f}(n)$ for all integers n. Here the derivative of a complex-valued function is defined in exactly the same fashion as for real-valued functions.

Ex. II.5.5.5. Let $f, g \in C(\mathbb{R}/\mathbb{Z};\mathbb{C})$. Prove the *Parseval identity*

$$\Re\left(\int_0^1 f(x)\overline{g(x)}\,dx\right) = \Re\left(\sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{(g)}(n)}\right).$$

Then conclude that the real parts can be removed, thus

$$\int_0^1 f(x)\overline{g(x)}\,dx = \sum_{n\in\mathbb{Z}} \hat{f}(n)\overline{\hat{(g)}(n)}.$$

Ex. II.5.5.6. In this exercise we shall develop the theory of Fourier series for functions of any fixed period L.

Let $L > 0$, and let $f : \mathbb{R} \to \mathbb{C}$ be a complex-valued function which is continuous and L-periodic. Define the numbers c_n for every integer n by

$$c_n := \frac{1}{L}\int_{[0,L]} f(x)e^{-2\pi i n x/L}\,dx.$$

(a) Show that the series

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L}$$

converges in L_2 metric to f. More precisely, show that

$$\lim_{N\to\infty} \int_{[0,L]} \left| f(x) - \sum_{n=-N}^{N} c_n e^{2\pi inx/L} \right|^2 dx = 0.$$

(b) If the series $\displaystyle\sum_{n=-\infty}^{\infty} |c_n|$ is absolutely convergent, show that

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L}$$

converges uniformly to f.

(c) Show that

$$\frac{1}{L} \int_{[0,L]} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Chapter II.6

Several variable differential calculus

II.6.1 Linear transformations

Def. II.6.1.1 (Row vector)**.** Let $n \geq 1$ be an integer. We refer to elements of $\mathbb{R}^n$ as *n-dimensional row vectors.* A typical n-dimensional row vector may take the form $x = (x_1, x_2, \ldots, x_n)$, which we abbreviate as $(x_i)_{1\leq i\leq n}$; the quantities $x_1, x_2, \ldots, x_n$ are of course real numbers. If $(x_i)_{1\leq i\leq n}$ and $(y_i)_{1\leq i\leq n}$ are n-dimensional row vectors, we can define their vector sum by

$$(x_i)_{1\leq i\leq n} + (y_i)_{1\leq i\leq n} = (x_i + y_i)_{1\leq i\leq n},$$

and also if $c \in \mathbb{R}$ is any scalar, we can define the scalar product $c(x_i)_{1\leq i\leq n}$ by

$$c(x_i)_{1\leq i\leq n} \coloneqq (cx_i)_{1\leq i\leq n}.$$

Of course one has similar operations on $\mathbb{R}^m$ as well. However, if $n \neq m$, then we do not define any operation of vector addition between vectors in $\mathbb{R}^n$ and vectors in $\mathbb{R}^m$. We also refer to the vector $(0, \ldots, 0)$ in $\mathbb{R}^n$ as the *zero vector* and also denote it by 0. (Strictly speaking, we should denote the zero vector of $\mathbb{R}^n$ by $0_{\mathbb{R}^n}$, as they are technically distinct from each other and from the number zero, but we shall not take care to make this distinction.) We abbreviate $(-1)x$ as $-x$.

Lem. II.6.1.2 ($\mathbb{R}^n$ is a vector space)**.** Let x, y, z be vectors in $\mathbb{R}^n$, and let c, d be real numbers. Then we have the commutativity property $x + y = y + x$, the additive associativity property $(x + y) + z = x + (y + z)$, the additive identity property $x + 0 = 0 + x = x$, the additive inverse property $x + (-x) = (-x) + x = 0$, the multiplicative associativity property $(cd)x = c(dx)$, the distributivity properties $c(x + y) = cx + cy$ and $(c + d)x = cx + dx$, and the multiplicative identity property $1x = x$.

Proof. First, we show the commutative property.

$$x + y = (x_i)_{1\leq i\leq n} + (y_i)_{1\leq i\leq n} \qquad \text{(by Def. II.6.1.1)}$$

$$
\begin{aligned}
&= (x_i + y_i)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= (y_i + x_i)_{1 \le i \le n} && (x_i, y_i \in \mathbb{R}) \\
&= (y_i)_{1 \le i \le n} + (x_i)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= y + x. && \text{(by Def. II.6.1.1)}
\end{aligned}
$$

Next we show the additive associativity property.

$$
\begin{aligned}
(x + y) + z &= \big((x_i)_{1 \le i \le n} + (y_i)_{1 \le i \le n}\big) + (z_i)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= (x_i + y_i)_{1 \le i \le n} + (z_i)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= \big((x_i + y_i) + z_i\big)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= \big(x_i + (y_i + z_i)\big)_{1 \le i \le n} && (x_i, y_i, z_i \in \mathbb{R}) \\
&= (x_i)_{1 \le i \le n} + (y_i + z_i)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= (x_i)_{1 \le i \le n} + \big((y_i)_{1 \le i \le n} + (z_i)_{1 \le i \le n}\big) && \text{(by Def. II.6.1.1)} \\
&= x + (y + z). && \text{(by Def. II.6.1.1)}
\end{aligned}
$$

Next we show that $0_{\mathbb{R}^n}$ is the additive identity.

$$
\begin{aligned}
x + 0_{\mathbb{R}^n} &= 0_{\mathbb{R}^n} + x && \text{(from the proof above)} \\
&= (0)_{1 \le i \le n} + (x_i)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= (0 + x_i)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= (x_i)_{1 \le i \le n} && (x_i \in \mathbb{R}) \\
&= x. && \text{(by Def. II.6.1.1)}
\end{aligned}
$$

Next we show that every $-x$ is the additive inverse of x.

$$
\begin{aligned}
x + (-x) &= (-x) + x && \text{(from the proof above)} \\
&= (-1)(x) + x && \text{(by Def. II.6.1.1)} \\
&= (-1)(x_i)_{1 \le i \le n} + (x_i)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= \big((-1)x_i\big)_{1 \le i \le n} + (x_i)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= \big((-1)x_i + x_i\big)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= (0)_{1 \le i \le n} && (x_i \in \mathbb{R}) \\
&= 0_{\mathbb{R}^n}. && \text{(by Def. II.6.1.1)}
\end{aligned}
$$

Next we show that multiplicative associativity property.

$$
\begin{aligned}
(cd)x &= (cd)(x_i)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= \big((cd)x_i\big)_{1 \le i \le n} && \text{(by Def. II.6.1.1)} \\
&= \big(c(dx_i)\big)_{1 \le i \le n} && (c, d, x_i \in \mathbb{R})
\end{aligned}
$$

$$
\begin{aligned}
&= c(dx_i)_{1\le i\le n} && \text{(by Def. II.6.1.1)}\\
&= c(dx). && \text{(by Def. II.6.1.1)}
\end{aligned}
$$

Next we show that distributivity property.

$$
\begin{aligned}
c(x+y) &= c\big((x_i)_{1\le i\le n} + (y_i)_{1\le i\le n}\big) && \text{(by Def. II.6.1.1)}\\
&= c(x_i+y_i)_{1\le i\le n} && \text{(by Def. II.6.1.1)}\\
&= \big(c(x_i+y_i)\big)_{1\le i\le n} && \text{(by Def. II.6.1.1)}\\
&= (cx_i+cy_i)_{1\le i\le n} && (c, x_i, y_i \in \mathbb{R})\\
&= (cx_i)_{1\le i\le n} + (cy_i)_{1\le i\le n} && \text{(by Def. II.6.1.1)}\\
&= c(x_i)_{1\le i\le n} + c(y_i)_{1\le i\le n} && \text{(by Def. II.6.1.1)}\\
&= cx+cy. && \text{(by Def. II.6.1.1)}
\end{aligned}
$$

$$
\begin{aligned}
(c+d)x &= (c+d)(x_i)_{1\le i\le n} && \text{(by Def. II.6.1.1)}\\
&= \big((c+d)x_i\big)_{1\le i\le n} && \text{(by Def. II.6.1.1)}\\
&= (cx_i+dx_i)_{1\le i\le n} && (c, d, x_i \in \mathbb{R})\\
&= (cx_i)_{1\le i\le n} + (dx_i)_{1\le i\le n} && \text{(by Def. II.6.1.1)}\\
&= c(x_i)_{1\le i\le n} + d(x_i)_{1\le i\le n} && \text{(by Def. II.6.1.1)}\\
&= cx+dx. && \text{(by Def. II.6.1.1)}
\end{aligned}
$$

Finally we show that 1 is the multiplicative identity.

$$
\begin{aligned}
1x &= 1(x_i)_{1\le i\le n} && \text{(by Def. II.6.1.1)}\\
&= (1x_i)_{1\le i\le n} && \text{(by Def. II.6.1.1)}\\
&= (x_i)_{1\le i\le n} && (x_i \in \mathbb{R})\\
&= x. && \text{(by Def. II.6.1.1)}
\end{aligned}
$$

□

Def. II.6.1.3 (Transpose)**.** If $(x_i)_{1\le i\le n} = (x_1, x_2, \ldots, x_n)$ is an n-dimensional row vector, we can define its *transpose* $(x_i)_{1\le i\le n}^\top$ by

$$
(x_i)_{1\le i\le n}^\top = (x_1, x_2, \ldots, x_n)^\top := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.
$$

We refer to objects such as $(x_i)_{1\le i\le n}^\top$ as *n-dimensional column vectors.*

Rmk. II.6.1.4. There is no functional difference between a row vector and a column vector (e.g., one can add and scalar multiply column vectors just as well as we can row vectors), however we shall (rather annoyingly) need to transpose our row vectors into column vectors in order to be consistent with the conventions of matrix multiplication, which we will see later. Note that we view row vectors and column vectors as residing in different spaces; thus for instance we will not define the sum of a row vector with a column vector, even when they have the same number of elements.

Def. II.6.1.5 (Standard basis row vectors)**.** We identify n special vectors in $\mathbb{R}^n$, the *standard basis row vectors* $e_1, \ldots, e_n$. For each $1 \leq j \leq n$, e_j is the vector which has 0 in all entries except for the j^{th} entry, which is equal to 1.

Note. If $x = (x_i)_{1 \leq i \leq n}$ is a vector in $\mathbb{R}^n$, then

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = \sum_{j=1}^{n} x_j e_j,$$

or in other words every vector in $\mathbb{R}^n$ is a *linear combination* of the standard basis vectors $e_1, \ldots, e_n$. (The notation $\sum_{j=1}^{n} x_j e_j$ is unambiguous because the operation of vector addition is both commutative and associative). Of course, just as every row vector is a linear combination of standard basis row vectors, every column vector is a linear combination of standard basis column vectors:

$$x^\top = x_1 e_1^\top + x_2 e_2^\top + \cdots + x_n e_n^\top = \sum_{j=1}^{n} x_j e_j^\top.$$

Def. II.6.1.6 (Linear transformations)**.** A *linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ is any function from one Euclidean space $\mathbb{R}^n$ to another $\mathbb{R}^m$ which obeys the following two axioms:

(a) (Additivity) For every $x, x' \in \mathbb{R}^n$, we have $T(x + x') = T(x) + T(x')$.

(b) (Homogeneity) For every $x \in \mathbb{R}^n$ and every $c \in \mathbb{R}$, we have $T(cx) = cT(x)$.

Def. II.6.1.10 (Matrices)**.** An $m \times n$ matrix is an object A of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix};$$

we shall abbreviate this as

$$A = (a_{ij})_{1 \leq i \leq m; 1 \leq j \leq n}.$$

In particular, n-dimensional row vectors are $1 \times n$ matrices, while n-dimensional column vectors are $n \times 1$ matrices.

Def. II.6.1.11 (Matrix product)**.** Given an $m \times n$ matrix A and an $n \times p$ matrix B, we can define the matrix product AB to be the $m \times p$ matrix defined as

$$(a_{ij})_{1 \le i \le m; 1 \le j \le n}(b_{jk})_{1 \le j \le n; 1 \le k \le p} := \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right)_{1 \le i \le m; 1 \le k \le p}.$$

In particular, if $x^\top = (x_i)_{1 \le i \le n}^\top$ is an n-dimensional column vector, and

$$A = (a_{ij})_{1 \le i \le m; 1 \le j \le n}$$

is an $m \times n$ matrix, then $Ax^\top$ is an m-dimensional column vector:

$$Ax^\top = \left(\sum_{j=1}^{n} a_{ij} x_j \right)_{1 \le i \le m}^\top.$$

A.Cor. II.6.1.1. We now relate matrices to linear transformations. If A is an $m \times n$ matrix, we can define the transformation $L_A : \mathbb{R}^n \to \mathbb{R}^m$ by the formula

$$\big(L_A(x)\big)^\top = Ax^\top.$$

More generally, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

then we have

$$L_A\big((x_j)_{1 \le j \le n}\big) = \left(\sum_{j=1}^{n} a_{ij} x_j \right)_{1 \le i \le m}.$$

For any $m \times n$ matrix A, the transformation L_A is automatically linear; one can easily verify that $L_A(x + y) = L_A(x) + L_A(y)$ and $L_A(cx) = cL_A(x)$ for any n-dimensional row vectors x, y and any scalar c.

Proof. We have

$$\begin{aligned} L_A(x+y) &= L_A\big((x_j)_{1 \le j \le n} + (y_j)_{1 \le j \le n}\big) && \text{(by Def. II.6.1.1)} \\ &= L_A\big((x_j + y_j)_{1 \le j \le n}\big) && \text{(by Def. II.6.1.1)} \\ &= \left(\sum_{j=1}^{n} a_{ij}(x_j + y_j) \right)_{1 \le i \le m} \\ &= \left(\sum_{j=1}^{n} a_{ij} x_j + \sum_{j=1}^{n} a_{ij} y_j \right)_{1 \le i \le m} \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{j=1}^{n} a_{ij}x_j\right)_{1\le i\le m} + \left(\sum_{j=1}^{n} a_{ij}y_j)\right)_{1\le i\le m} && \text{(by Def. II.6.1.1)}\\
&= L_A\big((x_j)_{1\le j\le n}\big) + L_A\big((y_j)_{1\le j\le n}\big)\\
&= L_A(x) + L_A(y) && \text{(by Def. II.6.1.1)}
\end{aligned}
$$

and

$$
\begin{aligned}
L_A(cx) &= L_A\big(c(x_j)_{1\le j\le n}\big) && \text{(by Def. II.6.1.1)}\\
&= L_A\big((cx_j)_{1\le j\le n}\big) && \text{(by Def. II.6.1.1)}\\
&= \left(\sum_{j=1}^{n} a_{ij}(cx_j)\right)_{1\le i\le m}\\
&= \left(c\sum_{j=1}^{n} a_{ij}x_j\right)_{1\le i\le m}\\
&= c\left(\sum_{j=1}^{n} a_{ij}x_j\right)_{1\le i\le m} && \text{(by Def. II.6.1.1)}\\
&= cL_A\big((x_j)_{1\le j\le n}\big)\\
&= cL_A(x). && \text{(by Def. II.6.1.1)}
\end{aligned}
$$

Thus, by Def. II.6.1.6 L_A is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. □

Lem. II.6.1.13. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists exactly one $m \times n$ matrix A such that $T = L_A$.

Proof. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Let $e_1, e_2, \dots, e_n$ be the standard basis row vectors of $\mathbb{R}^n$. Then $T(e_1), T(e_2), \dots, T(e_n)$ are vectors in $\mathbb{R}^m$. For each $1 \le j \le n$, we write $T(e_j)$ in co-ordinates as

$$T(e_j) = (a_{1j}, a_{2j}, \dots, a_{mj}) = (a_{ij})_{1\le i\le m},$$

i.e., we define a_{ij} to be the i^{th} component of $T(e_j)$. Then for any n- dimensional row vector $x = (x_1, \dots, x_n)$, we have

$$T(x) = T\left(\sum_{j=1}^{n} x_j e_j\right),$$

which (since T is linear) is equal to

$$
\begin{aligned}
&= \sum_{j=1}^{n} T(x_j e_j)\\
&= \sum_{j=1}^{n} x_j T(e_j)
\end{aligned}
$$

$$= \sum_{j=1}^{n} x_j (a_{ij})_{1\le i\le m}$$
$$= \sum_{j=1}^{n} (a_{ij}x_j)_{1\le i\le m}$$
$$= \left(\sum_{j=1}^{n} a_{ij}x_j\right)_{1\le i\le m}.$$

But if we let A be the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

then the previous vector is precisely $L_A(x)$. Thus, $T(x) = L_A(x)$ for all n-dimensional vectors x, and thus $T = L_A$.

Now we show that A is unique, i.e., there does not exist any other matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

for which T is equal to L_B. Suppose for the sake of contradiction that we could find such a matrix B which was different from A. Then we would have $L_A = L_B$. In particular, we have $L_A(e_j) = L_B(e_j)$ for every $1 \le j \le n$. But from the definition of L_A we see that

$$L_A(e_j) = (a_{ij})_{1\le i\le m}$$

and

$$L_B(e_j) = (b_{ij})_{1\le i\le m}$$

and thus we have $a_{ij} = b_{ij}$ for every $1 \le i \le m$ and $1 \le j \le n$, thus A and B are equal, a contradiction. □

Rmk. II.6.1.14. Lem. II.6.1.13 establishes a one-to-one correspondence between linear transformations and matrices, and is one of the fundamental reasons why matrices are so important in linear algebra. One may ask then why we bother dealing with linear transformations at all, and why we don't just work with matrices all the time. The reason is that sometimes one does not want to work with the standard basis $e_1, \dots, e_n$, but instead wants to use some other basis. In that case, the correspondence between linear transformations and matrices changes, and so it is still important to keep the notions of linear transformation and matrix distinct. More discussion on this somewhat subtle issue can be found in any linear algebra text.

Rmk. II.6.1.15. If $T = L_A$, then A is sometimes called the *matrix representation* of T, and is sometimes denoted $A = [T]$. We shall avoid this notation here, however.

Note. The composition $T \circ S$ of two linear transformations T, S is again a linear transformation (Ex. II.6.1.2). It is customary in linear algebra to abbreviate such compositions $T \circ S$ of linear transformations by droppinng the $\circ$ symbol, thus $T \circ S = TS$.

Lem. II.6.1.16. Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix. Then $L_A L_B = L_{AB}$.

Proof. Note that $L_A L_B = L_A \circ L_B$, and we will work with the notation $L_A \circ L_B$ instead. By Def. II.6.1.11 AB is well-defined. Let $C = AB = (c_{ik})_{1 \le i \le m; 1 \le k \le p}$. Then by Def. II.6.1.11 we have

$$(c_{ik})_{1 \le i \le m; 1 \le k \le p} = \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right)_{1 \le i \le m; 1 \le k \le p}.$$

Let $x \in \mathbb{R}^p$. Then we have

$$\begin{aligned}
&(L_A \circ L_B)(x) \\
&= L_A\big(L_B(x)\big) \\
&= L_A\Big(L_B\big((x_k)_{1 \le k \le p}\big)\Big) && \text{(by Def. II.6.1.1)} \\
&= L_A\left(\left(\sum_{k=1}^{p} b_{jk} x_k \right)_{1 \le j \le n} \right) && \text{(by A.Cor. II.6.1.1)} \\
&= \left(\sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{p} b_{jk} x_k \right) \right)_{1 \le i \le m} && \text{(by A.Cor. II.6.1.1)} \\
&= \left(\sum_{j=1}^{n} \left(\sum_{k=1}^{p} a_{ij} b_{jk} x_k \right) \right)_{1 \le i \le m} \\
&= \left(\sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij} b_{jk} x_k \right) \right)_{1 \le i \le m} \\
&= \left(\sum_{k=1}^{p} \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right) x_k \right)_{1 \le i \le m} \\
&= \left(\sum_{k=1}^{p} c_{ik} x_k \right)_{1 \le i \le m} && \text{(by Def. II.6.1.11)} \\
&= L_C\big((x_k)_{1 \le k \le p}\big) && \text{(by A.Cor. II.6.1.1)} \\
&= L_C(x) && \text{(by Def. II.6.1.1)} \\
&= L_{AB}(x).
\end{aligned}$$

Since x was arbitrary, we have $L_A \circ L_B = L_{AB}$. $\square$

— Exercises —

Ex. II.6.1.1. Prove Lem. II.6.1.2.

Proof. See Lem. II.6.1.2. □

Ex. II.6.1.2. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and $S : \mathbb{R}^p \to \mathbb{R}^n$ is a linear transformation, show that the composition $T \circ S : \mathbb{R}^p \to \mathbb{R}^m$ of the two transforms, defined by $T \circ S(x) \coloneqq T\big(S(x)\big)$, is also a linear transformation.

Proof. Let $x, y \in \mathbb{R}^p$ and let $c \in \mathbb{R}$. Then we have

$$\begin{aligned}(T \circ S)(x+y) &= T\big(S(x+y)\big) \\ &= T\big(S(x)+S(y)\big) && \text{(by Def. II.6.1.6)} \\ &= T\big(S(x)\big)+T\big(S(y)\big) && \text{(by Def. II.6.1.6)} \\ &= (T \circ S)(x)+(T \circ S)(y)\end{aligned}$$

and

$$\begin{aligned}(T \circ S)(cx) &= T\big(S(cx)\big) \\ &= T\big(cS(x)\big) && \text{(by Def. II.6.1.6)} \\ &= cT\big(S(x)\big) && \text{(by Def. II.6.1.6)} \\ &= c(T \circ S)(x).\end{aligned}$$

Thus, by Def. II.6.1.6 we know that $T \circ S$ is a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^m$. □

Ex. II.6.1.3. Prove Lem. II.6.1.16.

Proof. See Lem. II.6.1.16. □

Ex. II.6.1.4. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that there exists a number $M > 0$ such that $\|T(x)\| \le M\|x\|$ for all $x \in \mathbb{R}^n$. Conclude in particular that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is continuous.

Proof. Since T is a linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, by Lem. II.6.1.13 we know that there exists an $m \times n$ matrix such that

$$\forall x \in \mathbb{R}^n, \big(T(x)\big)^\top = Ax^\top.$$

If we write

$$A = (a_{ij})_{1 \le i \le m; 1 \le j \le n},$$

then

$$M = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|$$

is well-defined.

Let $x \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
\|T(x)\| &= \left\| \left(\sum_{j=1}^{n} a_{ij} x_j \right)_{1 \le i \le m} \right\| && \text{(by Def. II.6.1.11)} \\
&\le \left| \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j \right)_{1 \le i \le m} \right| && \text{(by Ex. II.1.1.8)} \\
&\le \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij} x_j| \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}||x_j| \\
&= \sum_{j=1}^{n} \sum_{i=1}^{m} |a_{ij}||x_j| \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} |a_{ij}| \right) |x_j| \\
&\le \sum_{j=1}^{n} M |x_j| \\
&= M \sum_{j=1}^{n} |x_j| \\
&\le M\sqrt{n}\|x\|. && \text{(by Ex. II.1.1.8)}
\end{aligned}$$

Since x was arbitrary, we have $\|T(x)\| \le M\sqrt{n}\|x\|$ for all $x \in \mathbb{R}^n$.

Now we show that T is continuous on $\mathbb{R}^n$ from $(\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})$ to $(\mathbb{R}^m, d_{l^1}|_{\mathbb{R}^m \times \mathbb{R}^m})$. From the proof above we know that there exists a $M \in \mathbb{R}^+$ such that $\|T(x)\| \le M\|x\|$ for all $x \in \mathbb{R}^n$. Fix such M. Let $x_0 \in \mathbb{R}^n$. Since

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \forall x \in \mathbb{R}^n, \|x - x_0\| < \frac{\varepsilon}{M} \\
\implies &M\|x - x_0\| < \varepsilon \\
\implies &\|T(x - x_0)\| \le M\|x - x_0\| < \varepsilon \\
\implies &\|T(x) - T(x_0)\| \le M\|x - x_0\| < \varepsilon, && \text{(by Def. II.6.1.6)}
\end{aligned}$$

by setting $\delta = \dfrac{\varepsilon}{M}$ we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in \mathbb{R}^n, \|x - x_0\| < \delta \implies \|T(x) - T(x_0)\| < \varepsilon.$$

Since x_0 was arbitrary, by Def. II.2.1.1 this means T is continuous on $\mathbb{R}^n$ from $(\mathbb{R}^n, d_{l^1}|_{\mathbb{R}^n \times \mathbb{R}^n})$ to $(\mathbb{R}^m, d_{l^1}|_{\mathbb{R}^m \times \mathbb{R}^m})$. $\square$

II.6.2 Derivatives in several variable calculus

Note. In single variable calculus, when one wants to differentiate a function $f : E \to \mathbb{R}$ at a point x_0, where E is a subset of $\mathbb{R}$ and x_0 is a limit point of E, this is given by

$$f'(x_0) := \lim_{x \to x_0; x \in E \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}.$$

One could try to mimic this definition in the several variable case $f : E \to \mathbb{R}^m$, where E is now a subset of $\mathbb{R}^n$, however we encounter a difficulty in this case: the quantity $f(x) - f(x_0)$ will live in $\mathbb{R}^m$, and $x - x_0$ lives in $\mathbb{R}^n$, and we do not know how to divide an m-dimensional vector by an n-dimensional vector.

To get around this problem, we first rewrite the concept of derivative (in one dimension) in a way which does not involve division of vectors. Instead, we view differentiability at a point x_0 as an assertion that a function f is "approximately linear" near x_0.

Lem. II.6.2.1. Let E be a subset of $\mathbb{R}$, $f : E \to \mathbb{R}$ be a function, $L \in \mathbb{R}$, and let x_0 be a limit point of E. Then the following two statements are equivalent.

(a) f is differentiable at x_0, and $f'(x_0) = L$.

(b) We have

$$\lim_{x \to x_0; x \in E \setminus \{x_0\}} \frac{\left|f(x) - \big(f(x_0) + L(x - x_0)\big)\right|}{|x - x_0|} = 0.$$

Proof. We have

$$\begin{aligned}
&\lim_{x \to x_0; x \in E \setminus \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = L \\
\iff &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \\
&\left(|x - x_0| < \delta \implies \left| \frac{f(x) - f(x_0)}{x - x_0} - L \right| < \varepsilon \right) \\
\iff &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \\
&\left(|x - x_0| < \delta \implies \left| \frac{f(x) - f(x_0) - L(x - x_0)}{x - x_0} \right| < \varepsilon \right) \\
\iff &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \\
&\left(|x - x_0| < \delta \implies \left| \frac{f(x) - \big(f(x_0) + L(x - x_0)\big)}{x - x_0} \right| < \varepsilon \right) \\
\iff &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \\
&\left(|x - x_0| < \delta \implies \left| \frac{\left|f(x) - \big(f(x_0) + L(x - x_0)\big)\right|}{|x - x_0|} \right| < \varepsilon \right)
\end{aligned}$$

$$\begin{aligned}
&\iff \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \\
&\qquad \left(|x - x_0| < \delta \implies \left| \frac{\left|f(x) - \left(f(x_0) + L(x - x_0)\right)\right|}{|x - x_0|} - 0 \right| < \varepsilon \right) \\
&\iff \lim_{x \to x_0; x \in E \setminus \{x_0\}} \frac{\left|f(x) - \left(f(x_0) + L(x - x_0)\right)\right|}{|x - x_0|} = 0.
\end{aligned}$$

□

Note. In light of Lem. II.6.2.1, we see that the derivative $f'(x_0)$ can be interpreted as the number L for which $\left|f(x) - \left(f(x_0) + L(x - x_0)\right)\right|$ is small, in the sense that it tends to zero as x tends to x_0, even if we divide out by the very small number $|x - x_0|$. More informally, the derivative is the quantity L such that we have the approximation $f(x) - f(x_0) \approx L(x - x_0)$.

This does not seem too different from the usual notion of differentiation, but the point is that we are no longer explicitly dividing by $x - x_0$. (We are still dividing by $|x - x_0|$, but this will turn out to be OK.) When we move to the several variable case $f : E \to \mathbb{R}^m$, where $E \subseteq \mathbb{R}^n$, we shall still want the derivative to be some quantity L such that $f(x) - f(x_0) \approx L(x - x_0)$. However, since $f(x) - f(x_0)$ is now an m-dimensional vector and $x - x_0$ is an n-dimensional vector, we no longer want L to be a scalar; we want it to be a linear transformation.

Def. II.6.2.2 (Differentiability)**.** Let E be a subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$ be a function, $x_0 \in E$ be a limit point, and let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. We say that f is *differentiable at* x_0 *with derivative* L if we have

$$\lim_{x \to x_0; x \in E \setminus \{x_0\}} \frac{\left\|f(x) - \left(f(x_0) + L(x - x_0)\right)\right\|}{\|x - x_0\|} = 0.$$

Here $\|x\|$ is the length of x (as measured in the l^2 metric)

$$\|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}.$$

Lem. II.6.2.4 (Uniqueness of derivatives)**.** Let E be a subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$ be a function, $x_0 \in E$ be an *interior point* of E, and let $L_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $L_2 : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations. Suppose that f is differentiable at x_0 with derivative L_1, and also differentiable at x_0 with derivative L_2. Then $L_1 = L_2$.

Proof. Suppose for the sake of contradiction that $L_1 \neq L_2$. Then there exists a $v \in \mathbb{R}^n$ such that $L_1(v) \neq L_2(v)$. Fix such v. We know that $v \neq 0_{\mathbb{R}^n}$ since if $v = 0_{\mathbb{R}^n}$, then we would have

$$L_1(0_{\mathbb{R}^n}) = L_1(0 \cdot 0_{\mathbb{R}^n}) = 0L_1(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m} = 0L_2(0_{\mathbb{R}^n}) = L_2(0 \cdot 0_{\mathbb{R}^n}) = L_2(0_{\mathbb{R}^n}).$$

Observe that

$$\forall t \in \mathbb{R}, \|x_0 + tv - x_0\| = \sqrt{\sum_{i=1}^{n} \left((x_0 + tv - x_0)_i\right)^2} \qquad \text{(by E.g. II.1.1.6)}$$

$$= \sqrt{\sum_{i=1}^{n}(tv_i)^2} \qquad \text{(by Def. II.6.1.1)}$$

$$= |t|\sqrt{\sum_{i=1}^{n}(v_i)^2}.$$

Since $\lim_{t\to 0; t\neq 0} |t| = 0$, we have

$$\lim_{t\to 0; t\neq 0} \|x_0 + tv - x_0\| = 0$$

which means

$$\forall \varepsilon_t \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall t \in \mathbb{R} \setminus \{0\}, |t| < \delta \implies \|x_0 + tv - x_0\| < \varepsilon_t.$$

Since x_0 is an interior point, by Def. II.1.2.5 we know that

$$\exists t \in \mathbb{R} \setminus \{0\} : x_0 + tv \in E.$$

Thus, we have

$$\forall \varepsilon_t \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall t \in \mathbb{R} \setminus \{0\}, |t| < \delta \implies \begin{cases} \|x_0 + tv - x_0\| < \varepsilon_t; \\ x_0 + tv \in E. \end{cases}$$

Since f is differentiable at x_0 with derivative L_1, by Def. II.6.2.2 we know that

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists \delta_1 \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \\ &\|x - x_0\| < \delta_1 \implies \frac{\left\| f(x) - \big(f(x_0) + L_1(x - x_0)\big) \right\|}{\|x - x_0\|} < \varepsilon. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists \delta_2 \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \\ &\|x - x_0\| < \delta_2 \implies \frac{\left\| f(x) - \big(f(x_0) + L_2(x - x_0)\big) \right\|}{\|x - x_0\|} < \varepsilon. \end{aligned}$$

Let $\varepsilon_t = \min(\delta_1, \delta_2)$. Then we have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall t \in \mathbb{R} \setminus \{x_0\}, |t| < \delta \\ &\implies \begin{cases} \|x_0 + tv - x_0\| < \varepsilon_t \\ x_0 + tv \in E \end{cases} \end{aligned}$$

$$\Longrightarrow \begin{cases} \dfrac{\left\|f(x_0+tv)-\big(f(x_0)+L_1(x_0+tv-x_0)\big)\right\|}{\|x_0+tv-x_0\|} < \varepsilon \\ \dfrac{\left\|f(x_0+tv)-\big(f(x_0)+L_2(x_0+tv-x_0)\big)\right\|}{\|x_0+tv-x_0\|} < \varepsilon \end{cases}$$

$$\Longrightarrow \begin{cases} \dfrac{\left\|f(x_0+tv)-\big(f(x_0)+L_1(tv)\big)\right\|}{\|tv\|} < \varepsilon \\ \dfrac{\left\|f(x_0+tv)-\big(f(x_0)+L_2(tv)\big)\right\|}{\|tv\|} < \varepsilon \end{cases}$$

and thus

$$\lim_{t\to 0;t\neq 0,x_0+tv\in E} \frac{\left\|f(x_0+tv)-\big(f(x_0)+L_1(tv)\big)\right\|}{\|tv\|} = 0;$$
$$\lim_{t\to 0;t\neq 0,x_0+tv\in E} \frac{\left\|f(x_0+tv)-\big(f(x_0)+L_2(tv)\big)\right\|}{\|tv\|} = 0.$$

By limit laws we have

$$\lim_{t\to 0;t\neq 0,x_0+tv\in E} \frac{\left\|f(x_0+tv)-\big(f(x_0)+L_1(tv)\big)\right\| + \left\|f(x_0+tv)-\big(f(x_0)+L_2(tv)\big)\right\|}{\|tv\|}$$
$$= 0.$$

By E.g. II.1.1.6 and Def. II.1.1.2 we know that

$$\begin{aligned} &\left\|f(x_0+tv)-\big(f(x_0)+L_1(tv)\big)\right\| + \left\|f(x_0+tv)-\big(f(x_0)+L_2(tv)\big)\right\| \\ &= \left\|f(x_0+tv)-\big(f(x_0)+L_1(tv)\big)\right\| + \left\|\big(f(x_0)+L_2(tv)\big)-f(x_0+tv)\right\| \\ &\geq \left\|f(x_0+tv)-\big(f(x_0)+L_1(tv)\big)+\big(f(x_0)+L_2(tv)\big)-f(x_0+tv)\right\| \\ &= \|L_2(tv)-L_1(tv)\| \\ &\geq 0. \end{aligned}$$

Thus, by squeeze test we have

$$\lim_{t\to 0;t\neq 0,x_0+tv\in E} \frac{\|L_2(tv)-L_1(tv)\|}{\|tv\|} = 0$$

which implies

$$\begin{aligned} &\lim_{t\to 0;t\neq 0,x_0+tv\in E} \frac{\|L_2(tv)-L_1(tv)\|}{\|tv\|} \\ &= \lim_{t\to 0;t\neq 0,x_0+tv\in E} \frac{\|tL_2(v)-tL_1(v)\|}{\|tv\|} && \text{(by Def. II.6.1.6)} \\ &= \lim_{t\to 0;t\neq 0,x_0+tv\in E} \frac{|t|\cdot\|L_2(v)-L_1(v)\|}{|t|\cdot\|v\|} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to 0; t\neq 0, x_0+tv\in E} \frac{\|L_2(v)-L_1(v)\|}{\|v\|} \\
&= \frac{\|L_2(v)-L_1(v)\|}{\|v\|} \\
&= 0.
\end{aligned}
$$

But $v \neq 0$ implies $\|v\| \neq 0$ (by Def. II.1.1.2(b)), so we must have $L_2(v) = L_1(v)$, a contradiction. Thus, $L_1 = L_2$. □

Note. Because of Lem. II.6.2.4, we can now talk about *the* derivative of f at interior points x_0, and we will denote this derivative by $f'(x_0)$. Thus, $f'(x_0)$ is the unique linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that

$$\lim_{x\to x_0; x\in E\setminus\{x_0\}} \frac{\|f(x)-(f(x_0)+f'(x_0)(x-x_0))\|}{\|x-x_0\|} = 0.$$

Informally, this means that the derivative $f'(x_0))$ is the linear transformation such that we have

$$f(x) - f(x_0) \approx f'(x_0)(x-x_0)$$

or equivalently

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$$

(this is known as Newton's approximation; compare with Proposition 10.1.7 in Analysis I).

Note. Another consequence of Lem. II.6.2.4 is that if you know that $f(x) = g(x)$ for all $x \in E$, and f, g are differentiable at x_0, then you also know that $f'(x_0) = g'(x_0)$ at every *interior* point of E. However, this is not necessarily true if x_0 is a boundary point of E; for instance, if E is just a single point $E = \{x_0\}$, merely knowing that $f(x_0) = g(x_0)$ does not imply that $f'(x_0) = g'(x_0)$. We will not deal with these boundary issues here, and only compute derivatives on the interior of the domain.

Note. We will sometimes refer to f' as the *total derivative* of f, to distinguish this concept from that of partial and directional derivatives below. The total derivative f is also closely related to the *derivative matrix* Df, which we shall define in the next section.

— Exercises —

Ex. II.6.2.1. Prove Lem. II.6.2.1.

Proof. See Lem. II.6.2.1. □

Ex. II.6.2.2. Prove Lem. II.6.2.4.

Proof. See Lem. II.6.2.4. □

II.6.3 Partial and directional derivatives

Def. II.6.3.1 (Directional derivative)**.** Let E be a subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$ be a function, let x_0 be an interior point of E, and let v be a vector in $\mathbb{R}^n$. If the limit

$$\lim_{t\to 0; t>0, x_0+tv\in E} \frac{f(x_0+tv)-f(x_0)}{t}$$

exists, we say that f is *differentiable in the direction v at x_0*, and we denote the above limit by $D_v f(x_0)$:

$$D_v f(x_0) := \lim_{t\to 0; t>0, x_0+tv\in E} \frac{f(x_0+tv)-f(x_0)}{t}.$$

Rmk. II.6.3.2. One should compare Def. II.6.3.1 with Def. II.6.2.2. Note that we are dividing by a scalar t, rather than a vector, so this definition makes sense, and $D_v f(x_0)$ will be a vector in $\mathbb{R}^m$. It is sometimes possible to also define directional derivatives on the boundary of E, if the vector v is pointing in an "inward" direction (this generalizes the notion of left derivatives and right derivatives from single variable calculus); but we will not pursue these matters here.

E.g. II.6.3.3. If $f : \mathbb{R} \to \mathbb{R}$ is a function, then $D_{+1} f(x)$ is the same as the right derivative of $f(x)$ (if it exists), and similarly $D_{-1} f(x)$ is the same as the negative of the left derivative of $f(x)$ (if it exists).

Proof. We have

$$\begin{aligned} D_{+1} f(x) &= \lim_{t\to 0; t>0} \frac{f(x_0+t)-f(x_0)}{t} && \text{(by Def. II.6.3.1)} \\ &= \lim_{x\to x_0; x>x_0} \frac{f\big(x_0+(x-x_0)\big)-f(x_0)}{x-x_0} \\ &= \lim_{x\to x_0; x>x_0} \frac{f(x)-f(x_0)}{x-x_0} \\ &= f(x_0+) \end{aligned}$$

and

$$\begin{aligned} D_{-1} f(x) &= \lim_{t\to 0; t>0} \frac{f(x_0-t)-f(x_0)}{t} && \text{(by Def. II.6.3.1)} \\ &= \lim_{x\to x_0; x<x_0} \frac{f\big(x_0-(x_0-x)\big)-f(x_0)}{x_0-x} \\ &= \lim_{x\to x_0; x<x_0} \frac{f(x)-f(x_0)}{x_0-x} \\ &= -\lim_{x\to x_0; x<x_0} \frac{f(x)-f(x_0)}{x-x_0} \\ &= -f(x_0-). \end{aligned}$$

□

Lem. II.6.3.5. Let E be a subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$ be a function, x_0 be an interior point of E, and let v be a vector in $\mathbb{R}^n$. If f is differentiable at x_0, then f is also differentiable in the direction v at x_0, and

$$D_v f(x_0) = f'(x_0)(v).$$

Proof. Since f is differentiable at x_0, by Def. II.6.2.2 we know that $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ exists and $f'(x_0)$ is a linear transformation. If $v = 0_{\mathbb{R}^n}$, then we have

$$\begin{aligned}
&f'(x_0)(0_{\mathbb{R}^n}) \\
&= 0_{\mathbb{R}^m} && \text{(cf. the proof of Lem. II.6.2.4)} \\
&= \lim_{t\to 0; t>0, x_0+t0_{\mathbb{R}^n}\in E} \frac{f(x_0 + t0_{\mathbb{R}^n}) - f(x_0)}{t} \\
&= D_{0_{\mathbb{R}^n}} f(x_0). && \text{(by Def. II.6.3.1)}
\end{aligned}$$

So suppose that $v \neq 0_{\mathbb{R}^n}$. Then we have

$$\begin{aligned}
&\lim_{x\to x_0; x\in E\setminus\{x_0\}} \frac{\|f(x) - f(x_0) - f'(x_0)(x - x_0)\|}{\|x - x_0\|} = 0 \\
\implies &\lim_{t\to 0; t>0, x_0+tv\in E\setminus\{x_0\}} \frac{\|f(x_0 + tv) - f(x_0) - f'(x_0)(x_0 + tv - x_0)\|}{\|x_0 + tv - x_0\|} = 0 \\
\implies &\lim_{t\to 0; t>0, x_0+tv\in E\setminus\{x_0\}} \frac{\|f(x_0 + tv) - f(x_0) - f'(x_0)(tv)\|}{\|tv\|} = 0 \\
\implies &\lim_{t\to 0; t>0, x_0+tv\in E\setminus\{x_0\}} \frac{\left\|t\frac{f(x_0 + tv) - f(x_0)}{t} - f'(x_0)(tv)\right\|}{\|tv\|} = 0 \\
\implies &\lim_{t\to 0; t>0, x_0+tv\in E\setminus\{x_0\}} \frac{\left\|t\frac{f(x_0 + tv) - f(x_0)}{t} - tf'(x_0)(v)\right\|}{\|tv\|} = 0 \\
\implies &\lim_{t\to 0; t>0, x_0+tv\in E\setminus\{x_0\}} \frac{t\left\|\frac{f(x_0 + tv) - f(x_0)}{t} - f'(x_0)(v)\right\|}{t\|v\|} = 0 \\
\implies &\lim_{t\to 0; t>0, x_0+tv\in E\setminus\{x_0\}} \frac{\left\|\frac{f(x_0 + tv) - f(x_0)}{t} - f'(x_0)(v)\right\|}{\|v\|} = 0 \\
\implies &\lim_{t\to 0; t>0, x_0+tv\in E\setminus\{x_0\}} \left\|\frac{f(x_0 + tv) - f(x_0)}{t} - f'(x_0)(v)\right\| = 0 \\
\implies &\lim_{t\to 0; t>0, x_0+tv\in E\setminus\{x_0\}} \frac{f(x_0 + tv) - f(x_0)}{t} = f'(x_0)(v) \\
\implies &D_v f(x_0) = f'(x_0)(v).
\end{aligned}$$

Thus, we conclude that

$$f'(x_0) \text{ exists} \implies \forall v \in \mathbb{R}^n, D_v f(x_0) = f'(x_0)(v).$$

□

Rmk. II.6.3.6. One consequence of Lem. II.6.3.5 is that total differentiability implies directional differentiability. However, the converse is not true; see Ex. II.6.3.3.

Def. II.6.3.7 (Partial derivative)**.** Let E be a subset of $\mathbb{R}^n$, let $f : E \to \mathbb{R}^m$ be a function, let x_0 be an interior point of E, and let $1 \leq j \leq n$. Then the *partial derivative of f with respect to the x_j variable* at x_0, denoted $\frac{\partial f}{\partial x_j}(x_0)$, is defined by

$$\frac{\partial f}{\partial x_j}(x_0) := \lim_{t\to 0; t\neq 0, x_0+te_j\in E} \frac{f(x_0+te_j)-f(x_0)}{t} = \frac{d}{dt} f(x_0+te_j)|_{t=0}$$

provided of course that the limit exists. (If the limit does not exist, we leave $\frac{\partial f}{\partial x_j}(x_0)$ undefined).

A.Cor. II.6.3.1. Informally, the partial derivative can be obtained by holding all the variables other than x_j fixed, and then applying the single-variable calculus derivative in the x_j variable. Note that if f takes values in $\mathbb{R}^m$, then so will $\frac{\partial f}{\partial x_j}$. Indeed, if we write f in components as $f = (f_1, \ldots, f_m)$, it is easy to see (by Prop. II.1.1.18) that

$$\frac{\partial f}{\partial x_j}(x_0) = \left(\frac{\partial f_1}{\partial x_j}(x_0), \ldots, \frac{\partial f_m}{\partial x_j}(x_0)\right)$$

i.e., to differentiate a vector-valued function one just has to differentiate each of the components separately.

Note. We sometimes replace the variables x_j in $\frac{\partial f}{\partial x_j}$ with other symbols. One should caution however that one should only relabel the variables if it is absolutely clear which symbol refers to the first variable, which symbol refers to the second variable, etc.; otherwise one may become unintentionally confused. The operation of total differentiation $\frac{d}{dx}$ is not the same as that of partial differentiation $\frac{\partial}{\partial x}$.

A.Cor. II.6.3.2. From Lem. II.6.3.5 (and Proposition 9.5.3 from Analysis I), we know that if a function is differentiable at a point x_0, then all the partial derivatives $\frac{\partial f}{\partial x_j}$ exists at x_0, and that

$$\frac{\partial f}{\partial x_j}(x_0) = D_{e_j} f(x_0) = -D_{-e_j} f(x_0) = f'(x_0)(e_j).$$

Also, if $v = (v_1, \ldots, v_n) = \sum_{j=1}^{n} v_j e_j$, then we have

$$D_v f(x_0) = f'(x_0)\Big(\sum_{j=1}^{n} v_j e_j\Big) = \sum_{j=1}^{n} v_j f'(x_0)(e_j)$$

(since $f'(x_0)$ is linear) and thus

$$D_v f(x_0) = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0).$$

Thus, one can write directional derivatives in terms of partial derivatives, *provided that* the function is actually differentiable at that point.

Note. Just because the partial derivatives exist at a point x_0, we cannot conclude that the function is differentiable there (Ex. II.6.3.3). However, if we know that the partial derivatives not only exist, but are continuous, then we can in fact conclude differentiability, thanks to the Thm. II.6.3.8

Thm. II.6.3.8. Let E be a subset of $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$ be a function, F be a subset of E, and x_0 be an interior point of F. If all the partial derivatives $\frac{\partial f}{\partial x_j}$ exist on F and are continuous at x_0, then f is differentiable at x_0, and the linear transformation $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$f'(x_0)\big((v_j)_{1\le j\le n}\big) = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0).$$

Proof. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation

$$L\big((v_j)_{1\le j\le n}\big) := \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0).$$

We have to prove that

$$\lim_{x\to x_0; x\in E\setminus\{x_0\}} \frac{\big\|f(x) - \big(f(x_0) + L(x - x_0)\big)\big\|}{\|x - x_0\|} = 0.$$

Let $\varepsilon > 0$. It will suffice to find a radius $\delta > 0$ such that

$$\frac{\big\|f(x) - \big(f(x_0) + L(x - x_0)\big)\big\|}{\|x - x_0\|} \le \varepsilon$$

for all $x \in B_{(\mathbb{R}^n, d_{l^2})}(x_0, \delta) \setminus \{x_0\}$. Equivalently, we wish to show that

$$\|f(x) - f(x_0) - L(x - x_0)\| \le \varepsilon \|x - x_0\|$$

for all $x \in B_{(\mathbb{R}^n,d_{l^2})}(x_0,\delta) \setminus \{x_0\}$.

Because x_0 is an interior point of F, there exists a ball $B_{(\mathbb{R}^n,d_{l^2})}(x_0,r)$ which is contained inside F. Because each partial derivative $\dfrac{\partial f}{\partial x_j}$ exists on F and is continuous at x_0, there thus exists an $0 < \delta_j < r$ such that $\left\|\dfrac{\partial f}{\partial x_j}(x) - \dfrac{\partial f}{\partial x_j}(x_0)\right\| \leq \dfrac{\varepsilon}{nm}$ for every $x \in B_{(\mathbb{R}^n,d_{l^2})}(x_0,\delta_j)$. If we take $\delta = \min(\delta_1,\ldots,\delta_n)$, then we thus have $\left\|\dfrac{\partial f}{\partial x_j}(x) - \dfrac{\partial f}{\partial x_j}(x_0)\right\| \leq \dfrac{\varepsilon}{nm}$ for every $x \in B_{(\mathbb{R}^n,d_{l^2})}(x_0,\delta)$ and every $1 \leq j \leq n$.

Let $x \in B_{(\mathbb{R}^n,d_{l^2})}(x_0,\delta)$. We write $x = x_0 + v_1e_1 + v_2e_2 + \cdots + v_ne_n$ for some scalars $v_1,\ldots,v_n$. Note that

$$\|x - x_0\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

and in particular, we have $|v_j| \leq \|x - x_0\|$ for all $1 \leq j \leq n$. Our task is to show that

$$\left\|f(x_0 + v_1e_1 + \cdots + v_ne_n) - f(x_0) - \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0)\right\| \leq \varepsilon\|x - x_0\|.$$

Write f in components as $f = (f_1, f_2, \ldots, f_m)$ (so each f_i is a function from E to $\mathbb{R}$). From the mean value theorem in the x_1 variable, we see that

$$f_i(x_0 + v_1e_1) - f_i(x_0) = \frac{\partial f_i}{\partial x_1}(x_0 + t_ie_1)v_1$$

for some t_i between 0 and v_1. This is done as follow: If $v_1 = 0$, then by setting $t_i = 0$ we have

$$f_i(x_0 + 0e_1) - f_i(x_0) = 0_{\mathbb{R}^m} = \frac{\partial f_i}{\partial x_1}(x_0 + 0e_1) \cdot 0.$$

So suppose that $0 < v_1$. First, observe that for any $y \in F$, we have

$$\begin{aligned}
&\frac{\partial f}{\partial x_1}(y) \in \mathbb{R}^m \\
\implies &\lim_{t\to 0; t\neq 0, y+te_1\in F} \frac{f(y + te_1) - f(y)}{t} \in \mathbb{R}^m && \text{(by Def. II.6.3.7)} \\
\implies &\forall 1 \leq i \leq m, \\
&\lim_{t\to 0; t\neq 0, y+te_1\in F} \frac{f_i(y + te_1) - f_i(y)}{t} \in \mathbb{R}. && \text{(by Prop. II.1.1.18)}
\end{aligned}$$

Since $B_{(\mathbb{R}^n,l^2)}(x_0,\delta) \subseteq F$, by the definition of v_1 we know that

$$\begin{aligned}
&\forall v \in [0, v_1], x_0 + ve_1 \in B_{(\mathbb{R}^n,l^2)}(x_0,\delta) \subseteq F \\
\implies &\forall 1 \leq i \leq m, \lim_{t\to 0; t\neq 0, x_0+(v+t)e_1\in F} \frac{f_i(x_0 + (v+t)e_1) - f_i(x_0 + ve_1)}{t} \in \mathbb{R}.
\end{aligned}$$

If we define $g_i : [0, v_1] \to \mathbb{R}$ for all $1 \le i \le m$ as follow:

$$\forall v \in [0, v_1], g_i(v) = f_i(x_0 + ve_1),$$

then we know that

$$\begin{aligned}
&\lim_{t\to 0; t\neq 0, x_0+(v+t)e_1 \in F} \frac{f_i(x_0 + (v+t)e_1) - f_i(x_0 + ve_1)}{t} \in \mathbb{R} \\
\implies &\lim_{t\to 0; t\neq 0, v+t\in[0,v_1]} \frac{g_i(v+t) - g_i(v)}{t} \in \mathbb{R} \\
\implies &\lim_{w\to v; w\in[0,v_1]\setminus\{v\}} \frac{g_i(w) - g_i(v)}{w - v} \in \mathbb{R} \\
\implies &g_i'(v) \in \mathbb{R}
\end{aligned}$$

for every $1 \le i \le m$ and every $v \in [0, v_1]$. Thus, by mean value theorem we know that

$$\begin{aligned}
&\exists t_i \in (0, v_1) : g_i'(t_i) = \frac{g_i(v_1) - g_i(0)}{v_1 - 0} \\
\implies &\exists t_i \in (0, v_1) : \lim_{t\to t_i; t\in[0,v_1]\setminus\{t_i\}} \frac{g_i(t) - g_i(t_i)}{t - t_i} = \frac{g_i(v_1) - g_i(0)}{v_1 - 0} \\
\implies &\exists t_i \in (0, v_1) : \lim_{t\to 0; t\neq 0, t+t_i\in[0,v_1]\setminus\{t_i\}} \frac{g_i(t + t_i) - g_i(t_i)}{t} = \frac{g_i(v_1) - g_i(0)}{v_1 - 0} \\
\implies &\exists t_i \in (0, v_1) : \\
&\lim_{t\to 0; t\neq 0, t+t_i\in[0,v_1]\setminus\{t_i\}} \frac{f_i(x_0 + (t + t_i)e_1) - f_i(x_0 + t_ie_1)}{t} = \frac{f_i(x_0 + v_1e_1) - f_i(x_0)}{v_1} \\
\implies &\exists t_i \in (0, v_1) : \frac{\partial f_i}{\partial x_1}(x_0 + t_ie_1) = \frac{f_i(x_0 + v_1e_1) - f_i(x_0)}{v_1} \\
\implies &\exists t_i \in (0, v_1) : \frac{\partial f_i}{\partial x_1}(x_0 + t_ie_1)v_1 = f_i(x_0 + v_1e_1) - f_i(x_0).
\end{aligned}$$

The case $v_1 < 0$ can be proven similarly. But we have

$$\left|\frac{\partial f_i}{\partial x_j}(x_0 + t_ie_1) - \frac{\partial f_i}{\partial x_j}(x_0)\right| \le \left\|\frac{\partial f}{\partial x_j}(x_0 + t_ie_1) - \frac{\partial f}{\partial x_j}(x_0)\right\| \le \frac{\varepsilon}{nm}$$

and hence

$$\left|f_i(x_0 + v_1e_1) - f_i(x_0) - \frac{\partial f_i}{\partial x_1}(x_0)v_1\right| \le \frac{\varepsilon|v_1|}{nm}.$$

Summing this over all $1 \le i \le m$ (and noting that $\|(y_1, \dots, y_m)\| \le |y_1| + \cdots + |y_m|$ from the triangle inequality) we obtain

$$\left\|f(x_0 + v_1e_1) - f(x_0) - \frac{\partial f}{\partial x_1}(x_0)v_1\right\| \le \frac{\varepsilon|v_1|}{n};$$

since $|v_1| \leq \|x - x_0\|$, we thus have

$$\left\| f(x_0 + v_1 e_1) - f(x_0) - \frac{\partial f}{\partial x_1}(x_0) v_1 \right\| \leq \frac{\varepsilon \|x - x_0\|}{n}.$$

A similar argument gives

$$\left\| f(x_0 + v_1 e_1 + v_2 e_2) - f(x_0 + v_1 e_1) - \frac{\partial f}{\partial x_2}(x_0) v_2 \right\| \leq \frac{\varepsilon \|x - x_0\|}{n}$$

and so forth up to

$$\left\| f(x_0 + v_1 e_1 + \cdots + v_n e_n) - f(x_0 + v_1 e_1 + \cdots + v_{n-1} e_{n-1}) - \frac{\partial f}{\partial x_n}(x_0) v_n \right\| \\ \leq \frac{\varepsilon \|x - x_0\|}{n}.$$

If we sum these n inequalities and use the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$, we obtain a telescoping series which simplifies to

$$\left\| f(x_0 + v_1 e_1 + \cdots + v_n e_n) - f(x_0) - \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x_0) v_j \right\| \leq \varepsilon \|x - x_0\|$$

as desired. □

A.Cor. II.6.3.3. From Thm. II.6.3.8 and Lem. II.6.3.5 we see that if the partial derivatives of a function $f : E \to \mathbb{R}^m$ exist and are continuous on some set F, then all the directional derivatives also exist at every interior point x_0 of F, and we have the formula

$$D_{(v_1,\ldots,v_n)} f(x_0) = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0).$$

In particular, if $f : E \to \mathbb{R}$ is a real-valued function, and we define the *gradient* $\nabla f(x_0)$ of f at x_0 to be the n-dimensional row vector

$$\nabla f(x_0) := \left(\frac{\partial f}{\partial x_1}(x_0), \ldots, \frac{\partial f}{\partial x_n}(x_0) \right),$$

then we have the familiar formula

$$D_v f(x_0) = v \cdot \nabla f(x_0)$$

whenever x_0 is in the interior of the region where the gradient exists and is continuous.

A.Cor. II.6.3.4. More generally, if $f : E \to \mathbb{R}^m$ is a function taking values in $\mathbb{R}^m$, with $f = (f_1, \dots, f_m)$, and x_0 is in the interior of the region where the partial derivatives of f exist and are continuous, then we have from Thm. II.6.3.8 that

$$f'(x_0)\big((v_j)_{1\le j\le n}\big) = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0) = \left(\sum_{j=1}^{n} v_j \frac{\partial f_i}{\partial x_j}(x_0)\right)_{1\le i\le m},$$

which we can rewrite as

$$L_{Df(x_0)}\big((v_j)_{1\le j\le n}\big)$$

where $Df(x_0)$ is the $m \times n$ matrix

$$\begin{aligned} Df(x_0) &:= \left(\frac{\partial f_i}{\partial x_j}(x_0)\right)_{1\le i\le m; 1\le j\le n} \\ &= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \frac{\partial f_1}{\partial x_2}(x_0) & \dots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) & \frac{\partial f_2}{\partial x_2}(x_0) & \dots & \frac{\partial f_2}{\partial x_n}(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \frac{\partial f_m}{\partial x_2}(x_0) & \dots & \frac{\partial f_m}{\partial x_n}(x_0) \end{pmatrix}. \end{aligned}$$

Thus, we have

$$\big(D_v f(x_0)\big)^\top = \big(f'(x_0)(v)\big)^\top = Df(x_0)v^\top.$$

The matrix $Df(x_0)$ is sometimes also called the *derivative matrix* or *differential matrix* of f at x_0, and is closely related to the total derivative $f'(x_0)$. One can also write Df as

$$Df(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0)^\top, \frac{\partial f}{\partial x_2}(x_0)^\top, \dots, \frac{\partial f}{\partial x_n}(x_0)^\top\right),$$

i.e., each of the columns of $Df(x_0)$ is one of the partial derivatives of f, expressed as a column vector. Or one could write

$$Df(x_0) = \begin{pmatrix} \nabla f_1(x_0) \\ \nabla f_2(x_0) \\ \vdots \\ \nabla f_m(x_0) \end{pmatrix}$$

i.e., the rows of $Df(x_0)$ are the gradient of various components of f. In particular, if f is scalar-valued (i.e., $m = 1$), then Df is the same as ∇f.

— Exercises —

Ex. II.6.3.1. Prove Lem. II.6.3.5.

Proof. See Lem. II.6.3.5. □

Ex. II.6.3.2. Let E be a subset of $\mathbb{R}^n$, let $f : E \to \mathbb{R}^m$ be a function, let x_0 be an interior point of E, and let $1 \leq j \leq n$. Show that $\frac{\partial f}{\partial x_j}(x_0)$ exists iff $D_{e_j} f(x_0)$ and $D_{-e_j} f(x_0)$ exist and are negatives of each other (thus $D_{e_j} f(x_0) = -D_{-e_j} f(x_0)$); furthermore, one has $\frac{\partial f}{\partial x_j}(x_0) = D_{e_j} f(x_0)$ in this case.

Proof. We have

$$\begin{aligned}
&\frac{\partial f}{\partial x_j}(x_0) \\
&= \lim_{t\to 0; t\neq 0, x_0+te_j\in E} \frac{f(x_0+te_j)-f(x_0)}{t} && \text{(by Def. II.6.3.7)} \\
&= \begin{cases} \displaystyle\lim_{t\to 0; t>0, x_0+te_j\in E} \frac{f(x_0+te_j)-f(x_0)}{t} \\ \displaystyle\lim_{t\to 0; t<0, x_0+te_j\in E} \frac{f(x_0+te_j)-f(x_0)}{t} \end{cases} && \text{(by Proposition 9.5.3 in Analysis I)} \\
&= \begin{cases} \displaystyle\lim_{t\to 0; t>0, x_0+te_j\in E} \frac{f(x_0+te_j)-f(x_0)}{t} \\ \displaystyle\lim_{t\to 0; t>0, x_0+te_j\in E} \frac{f(x_0-te_j)-f(x_0)}{-t} \end{cases} \\
&= \begin{cases} \displaystyle\lim_{t\to 0; t>0, x_0+te_j\in E} \frac{f(x_0+te_j)-f(x_0)}{t} \\ \displaystyle -\lim_{t\to 0; t>0, x_0+te_j\in E} \frac{f\big(x_0+t(-e_j)\big)-f(x_0)}{t} \end{cases} \\
&= \begin{cases} D_{e_j} f(x_0) \\ -D_{-e_j} f(x_0) \end{cases}. && \text{(by Def. II.6.3.1)}
\end{aligned}$$

□

Ex. II.6.3.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) := \frac{x^3}{x^2+y^2}$ when $(x, y) \neq (0, 0)$, and $f(0, 0) := 0$. Show that f is not differentiable at $(0, 0)$, despite being differentiable in every direction $v \in \mathbb{R}^2$ at $(0, 0)$. Explain why this does not contradict Thm. II.6.3.8.

Proof. First, we show that f is differentiable in every direction $v \in \mathbb{R}^2$ at $(0, 0)$. Since

$$\forall v \in \mathbb{R} \setminus \{(0,0)\}, \lim_{t\to 0; t>0, (0,0)+tv\in\mathbb{R}^2} \frac{f\big((0,0)+tv\big)-f(0,0)}{t}$$

$$
\begin{aligned}
&= \lim_{t\to 0; t>0, (0,0)+tv\in\mathbb{R}^2} \frac{f(tv)}{t} \\
&= \lim_{t\to 0; t>0, (0,0)+tv\in\mathbb{R}^2} \frac{t^3v_1^3}{(t^2v_1^2+t^2v_2^2)t} \\
&= \lim_{t\to 0; t>0, (0,0)+tv\in\mathbb{R}^2} \frac{v_1^3}{v_1^2+v_2^2} \\
&= \frac{v_1^3}{v_1^2+v_2^2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{t\to 0; t>0, (0,0)+t(0,0)\in\mathbb{R}^2} \frac{f\big((0,0)+t(0,0)\big)-f(0,0)}{t} \\
&= \lim_{t\to 0; t>0, (0,0)+t(0,0)\in\mathbb{R}^2} 0 \\
&= 0,
\end{aligned}
$$

by Def. II.6.3.1 we know that f is differentiable in every direction $v\in\mathbb{R}^2$ at $(0,0)$.

Next we show that f is not differentiable at $(0,0)$. Suppose for the sake of contradiction that f is differentiable at $(0,0)$. Then by A.Cor. II.6.3.2 we know that

$$\forall v\in\mathbb{R}^2, D_vf(0,0) = f'(0,0)(v) = \sum_{i=1}^{2} v_i f'(0,0)(e_i).$$

But

$$f'(0,0)(1,1) = \frac{1^3}{1^2+1^2} = \frac{1}{2}$$

is not equal to

$$\sum_{i=1}^{2} 1f'(0,0)(e_i) = f'(0,0)\big((1,0)\big) + f'(0,0)\big((0,1)\big) = \frac{1^3}{1^2+0^2} + \frac{0^3}{0^2+1^2} = 1,$$

a contradiction. Thus, f is not differentiable at $(0,0)$.

Now we show that this does not contradict Thm. II.6.3.8. We claim that $\dfrac{\partial f}{\partial x}$ is not continuous at $(0,0)$. Since

$$D_{e_1}f(0,0) = \frac{1^3}{1^2+0^2} = 1 = -\frac{(-1)^3}{(-1)^2+0^2} = -D_{-e_1}f(0,0),$$

by Ex. II.6.3.2 we know that $\dfrac{\partial f}{\partial x}(0,0) = 1$. But for each $(x_0,y_0)\in\mathbb{R}^2\setminus\{(0,0)\}$, we have

$$= \frac{\partial f}{\partial x}(x_0,y_0)$$

$$
\begin{aligned}
&= \lim_{t\to 0;t\neq 0,(x_0,y_0)+t(1,0)\in\mathbb{R}^2} \frac{f\big((x_0,y_0)+t(1,0)\big)-f(x_0,y_0)}{t}\\
&= \lim_{t\to 0;t\neq 0,(x_0,y_0)+t(1,0)\in\mathbb{R}^2} \frac{f(x_0+t,y_0)-f(x_0,y_0)}{t}\\
&= \lim_{t\to 0;t\neq 0,(x_0,y_0)+t(1,0)\in\mathbb{R}^2} \frac{\dfrac{(x_0+t)^3}{(x_0+t)^2+y_0^2}-\dfrac{x_0^3}{x_0^2+y_0^2}}{t}\\
&= \lim_{t\to 0;t\neq 0,(x_0,y_0)+t(1,0)\in\mathbb{R}^2} \frac{(x_0+t)^3(x_0^2+y_0^2)-x_0^3\big((x_0+t)^2+y_0^2\big)}{t\big((x_0+t)^2+y_0^2\big)(x_0^2+y_0^2)}\\
&= \lim_{t\to 0;t\neq 0,(x_0,y_0)+t(1,0)\in\mathbb{R}^2} \frac{(x_0^3+3x_0^2t+3x_0t^2+t^3)(x_0^2+y_0^2)-x_0^3(x_0^2+2tx_0+t^2+y_0^2)}{t(x_0^2+2tx_0+t^2+y_0^2)(x_0^2+y_0^2)}\\
&= \lim_{t\to 0;t\neq 0,(x_0,y_0)+t(1,0)\in\mathbb{R}^2} \frac{(3x_0^2+3x_0t+t^2)(x_0^2+y_0^2)-x_0^3(2x_0+t)}{(x_0^2+2tx_0+t^2+y_0^2)(x_0^2+y_0^2)}\\
&= \lim_{t\to 0;t\neq 0,(x_0,y_0)+t(1,0)\in\mathbb{R}^2} \frac{x_0^4+2tx_0^3+t^2x_0^2+3x_0^2y_0^2+3tx_0y_0^2+t^2y_0^2}{x_0^4+2tx_0^3+t^2x_0^2+2x_0^2y_0^2+2tx_0y_0^2+t^2y_0^2+y_0^4}\\
&= \frac{x_0^4+3x_0^2y_0^2}{x_0^4+2x_0^2y_0^2+y_0^4}.
\end{aligned}
$$

Thus, we see that $(x_0,y_0)\to(0,0)$ implies $\dfrac{\partial f}{\partial x}(x_0,y_0)\not\to 1$, which means $\dfrac{\partial f}{\partial x}$ is not continuous at $(0,0)$. □

Ex. II.6.3.4. Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a differentiable function such that $f'(x)=0$ for all $x\in\mathbb{R}^n$. Show that f is constant. For a tougher challenge, replace the domain $\mathbb{R}^n$ by an open connected subset Ω of $\mathbb{R}^n$.

Proof. First, we show the case when the domain of f is $\mathbb{R}^n$. By A.Cor. II.6.3.2 we know that

$$
\begin{aligned}
&\forall x_0\in\mathbb{R}^n,\forall y\in\mathbb{R}^n, f'(x_0)(y)=\sum_{j=1}^n y_j\frac{\partial f}{\partial x_j}(x_0)=0_{\mathbb{R}^m}\\
\implies &\forall x_0\in\mathbb{R}^n,\forall 1\le j\le n, \frac{\partial f}{\partial x_j}(x_0)=0_{\mathbb{R}^m}.
\end{aligned}
$$

Let $y\in\mathbb{R}^n$. Since

$$y=\sum_{j=1}^n y_je_j,$$

by mean value theorem (cf. the proof of Thm. II.6.3.8) we know that

$$\exists t_i\in(y_1,0)\cup(0,y_1):$$

$$f_i(0_{\mathbb{R}^n} + y_1 e_1) - f_i(0_{\mathbb{R}^n}) = \frac{\partial f_i}{\partial x_1}(0_{\mathbb{R}^n} + t_i e_1) y_1 = 0_{\mathbb{R}^m}$$

for all $1 \le i \le m$. Similar arguments show that

$$f_i(0_{\mathbb{R}^n} + y_1 e_1 + y_2 e_2) - f_i(0_{\mathbb{R}^n} + y_1 e_1) = 0_{\mathbb{R}^m}$$

and

$$f_i(0_{\mathbb{R}^n} + \sum_{j=1}^{n} y_j e_j) - f_i(0_{\mathbb{R}^n} + \sum_{j=1}^{n-1} y_j e_j) = 0_{\mathbb{R}^m}.$$

Summing all n terms above we obtain a telescoping series

$$f_i(0_{\mathbb{R}^n} + \sum_{j=1}^{n} y_j e_j) - f_i(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}$$

which means

$$f_i(y) - f_i(0_{\mathbb{R}^n}) = 0_{\mathbb{R}^m}.$$

Thus, we have $f(y) = f(0_{\mathbb{R}^n})$. Since y was arbitrary, we conclude that f is constant on $\mathbb{R}^n$.

Now we show the case when Ω, the domain of f, is an open connected subset of $\mathbb{R}^n$. By A.Cor. II.6.3.2 we know that

$$\forall x_0 \in \Omega, \forall y \in \Omega, f'(x_0)(y) = \sum_{j=1}^{n} y_j \frac{\partial f}{\partial x_j}(x_0) = 0_{\mathbb{R}^m}$$
$$\implies \forall x_0 \in \Omega, \forall 1 \le j \le n, \frac{\partial f}{\partial x_j}(x_0) = 0.$$

Let $x_0 \in \Omega$ and let $d = d_{l^2}|_{\Omega \times \Omega}$. Since (Ω, d) is open, by Prop. II.1.2.15(a) we know that

$$\exists \delta \in \mathbb{R}^+ : B_{(\Omega,d)}(x_0, \delta) \subseteq \Omega.$$

We now claim that f is constant on $B_{(\Omega,d)}(x_0, \delta)$. Let $y \in B_{(\Omega,d)}(x_0, \delta)$. We write $y = x_0 + v_1 e_1 + \cdots + v_n e_n$. By mean value theorem (cf. the proof of Thm. II.6.3.8) we know that

$$\exists t_i \in (v_1, 0) \cup (0, v_1) :$$
$$f_i(x_0 + v_1 e_1) - f_i(x_0) = \frac{\partial f_i}{\partial x_1}(x_0 + t_i e_1) v_1 = 0_{\mathbb{R}^m}$$

for all $1 \le i \le m$. Similar arguments show that

$$f_i(x_0 + v_1 e_1 + v_2 e_2) - f_i(x_0 + v_1 e_1) = 0_{\mathbb{R}^m}$$

and

$$f_i(x_0 + \sum_{j=1}^{n} v_j e_j) - f_i(x_0 + \sum_{j=1}^{n-1} v_j e_j) = 0_{\mathbb{R}^m}.$$

Summing all n terms above we obtain a telescoping series

$$f_i(x_0 + \sum_{j=1}^{n} v_j e_j) - f_i(x_0) = 0_{\mathbb{R}^m}$$

which means

$$f_i(y) - f_i(x_0) = 0_{\mathbb{R}^m}.$$

Thus, we have $f(y) = f(x_0)$. Since y was arbitrary, we conclude that f is constant on $B_{(\Omega,d)}(x_0, \delta)$. Since x_0 was arbitrary, we conclude that f is constant on every open ball of Ω. But by Def. II.2.4.1 we know that (Ω, d) is connected implies

$$\forall x_0, y_0 \in \Omega, \exists \delta_1, \delta_2 \in \mathbb{R}^+ : \begin{cases} B_{(\Omega,d)}(x_0, \delta_1) \subseteq \Omega \\ B_{(\Omega,d)}(y_0, \delta_2) \subseteq \Omega \\ B_{(\Omega,d)}(x_0, \delta_1) \cap B_{(\Omega,d)}(y_0, \delta_2) \neq \emptyset \end{cases}$$
$$\implies \forall z \in B_{(\Omega,d)}(x_0, \delta_1) \cap B_{(\Omega,d)}(y_0, \delta_2), f(x_0) = f(z) = f(y_0).$$

Thus, f is constant on Ω. □

II.6.4 The several variable calculus chain rule

Thm. II.6.4.1 (Several variable calculus chain rule)**.** Let E be a subset of $\mathbb{R}^n$, and let F be a subset of $\mathbb{R}^m$. Let $f : E \to F$ be a function, and let $g : F \to \mathbb{R}^p$ be another function. Let x_0 be a point in the interior of E. Suppose that f is differentiable at x_0, and that $f(x_0)$ is in the interior of F. Suppose also that g is differentiable at $f(x_0)$. Then $g \circ f : E \to \mathbb{R}^p$ is also differentiable at x_0, and we have the formula

$$(g \circ f)'(x_0) = g'\big(f(x_0)\big) \circ f'(x_0).$$

Proof. By Def. II.6.2.2 we want to show that

$$\lim_{x \to x_0; x \in E \setminus \{x_0\}} \frac{\left\| (g \circ f)(x) - (g \circ f)(x_0) - \Big(g'\big(f(x_0)\big) \circ f'(x_0)\Big)(x - x_0) \right\|}{\|x - x_0\|} = 0.$$

Equivalently, we want to show that

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \|x - x_0\| < \delta$$
$$\implies \frac{\left\| (g \circ f)(x) - (g \circ f)(x_0) - \Big(g'\big(f(x_0)\big) \circ f'(x_0)\Big)(x - x_0) \right\|}{\|x - x_0\|} < \varepsilon$$
$$\implies \left\| (g \circ f)(x) - (g \circ f)(x_0) - \Big(g'\big(f(x_0)\big) \circ f'(x_0)\Big)(x - x_0) \right\| < \varepsilon \|x - x_0\|.$$

Since $f'(x_0)$ exists, we know that

$$\lim_{x\to x_0; x\in E\setminus\{x_0\}} \frac{\|f(x)-f(x_0)-f'(x_0)(x-x_0)\|}{\|x-x_0\|} = 0.$$

Equivalently, we know that

$$\begin{aligned}
&\forall \varepsilon_f \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \|x-x_0\| < \delta \\
\Longrightarrow\ & \frac{\|f(x)-f(x_0)-f'(x_0)(x-x_0)\|}{\|x-x_0\|} < \varepsilon_f \\
\Longrightarrow\ & \big\|f(x)-f(x_0)-f'(x_0)(x-x_0)\big\| < \varepsilon_f\|x-x_0\| \\
\Longrightarrow\ & \|f(x)-f(x_0)\| < \varepsilon_f\|x-x_0\| + \big\|f'(x_0)(x-x_0)\big\|.
\end{aligned}$$

Since $g'\big(f(x_0)\big)$ exists, we know that

$$\lim_{y\to f(x_0); x\in F\setminus\{f(x_0)\}} \frac{\big\|g(y)-g\big(f(x_0)\big)-g'\big(f(x_0)\big)\big(y-f(x_0)\big)\big\|}{\|y-f(x_0)\|} = 0.$$

Equivalently, we know that

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists \delta_g \in \mathbb{R}^+ : \forall y \in F \setminus \{f(x_0)\}, \|y-f(x_0)\| < \delta_g \\
\Longrightarrow\ & \frac{\big\|g(y)-g\big(f(x_0)\big)-g'\big(f(x_0)\big)\big(y-f(x_0)\big)\big\|}{\|y-f(x_0)\|} < \varepsilon \\
\Longrightarrow\ & \big\|g(y)-g\big(f(x_0)\big)-g'\big(f(x_0)\big)\big(y-f(x_0)\big)\big\| < \varepsilon\|y-f(x_0)\|.
\end{aligned}$$

Fix one pair of ε and δ_g. Then we have

$$\begin{aligned}
&\exists \delta \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \|x-x_0\| < \delta \\
\Longrightarrow\ & \begin{cases} \|f(x)-f(x_0)\| < \varepsilon\|x-x_0\| + \big\|f'(x_0)(x-x_0)\big\|; \\ \|f(x)-f(x_0)\| < \delta_g; \end{cases} \\
\Longrightarrow\ & \begin{cases} \|f(x)-f(x_0)\| < \varepsilon\|x-x_0\| + \big\|f'(x_0)(x-x_0)\big\|; \\ \big\|g\big(f(x)\big)-g\big(f(x_0)\big)-g'\big(f(x_0)\big)\big(f(x)-f(x_0)\big)\big\| < \varepsilon\|f(x)-f(x_0)\|; \end{cases} \\
\Longrightarrow\ & \big\|g\big(f(x)\big)-g\big(f(x_0)\big)-g'\big(f(x_0)\big)\big(f(x)-f(x_0)\big)\big\| \\
& < \varepsilon^2\|x-x_0\| + \varepsilon\big\|f'(x_0)(x-x_0)\big\| \\
\Longrightarrow\ & \Big\|g\big(f(x)\big)-g\big(f(x_0)\big)-\Big(g'\big(f(x_0)\big)\circ f'(x_0)\Big)(x-x_0)\Big\| \\
& < \varepsilon^2\|x-x_0\| + \varepsilon\big\|f'(x_0)(x-x_0)\big\| \\
& \quad + \Big\|g'\big(f(x_0)\big)\big(f(x)-f(x_0)\big)-\Big(g'\big(f(x_0)\big)\circ f'(x_0)\Big)(x-x_0)\Big\|.
\end{aligned}$$

Since $g'\big(f(x_0)\big)$ is a linear transformation, by Def. II.6.1.6 we know that

$$\Big\|g'\big(f(x_0)\big)\big(f(x)-f(x_0)\big)-\Big(g'\big(f(x_0)\big)\circ f'(x_0)\Big)(x-x_0)\Big\|$$

$$= \left\| g'\big(f(x_0)\big)\big(f(x) - f(x_0) - f'(x_0)(x - x_0)\big) \right\|.$$

By Ex. II.6.1.4 we know that

$$\begin{aligned} \exists M \in \mathbb{R}^+ : & \left\| g'\big(f(x_0)\big)\big(f(x) - f(x_0) - f'(x_0)(x - x_0)\big) \right\| \\ & \leq M \left\| f(x) - f(x_0) - f'(x_0)(x - x_0) \right\| \\ & \leq M\varepsilon \|x - x_0\|. \end{aligned}$$

Fix such M. Then we have

$$\begin{aligned} & \exists \delta \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \|x - x_0\| < \delta \\ \implies & \left\| g\big(f(x)\big) - g\big(f(x_0)\big) - \Big(g'\big(f(x_0)\big) \circ f'(x_0)\Big)(x - x_0) \right\| \\ & < \varepsilon^2 \|x - x_0\| + \varepsilon \left\| f'(x_0)(x - x_0) \right\| + M\varepsilon \|x - x_0\|. \end{aligned}$$

Since ε was arbitrary, we conclude that

$$\begin{aligned} & \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \|x - x_0\| < \delta \\ \implies & \left\| g\big(f(x)\big) - g\big(f(x_0)\big) - \Big(g'\big(f(x_0)\big) \circ f'(x_0)\Big)(x - x_0) \right\| < \varepsilon. \end{aligned}$$

$\square$

Note. As a corollary of the chain rule and Lem. II.6.1.16 (and Lem. II.6.1.13), we see that

$$D(g \circ f)(x_0) = Dg\big(f(x_0)\big) \cdot Df(x_0);$$

i.e., we can write the chain rule in terms of matrices and matrix multiplication, instead of in terms of linear transformations and composition.

E.g. II.6.4.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be differentiable functions. We form the combined function $h : \mathbb{R}^n \to \mathbb{R}^2$ by defining $h(x) := \big(f(x), g(x)\big)$. Now let $k : \mathbb{R}^2 \to \mathbb{R}$ be the multiplication function $k(a, b) := ab$.

We first show that

$$Dh(x_0) = \begin{pmatrix} \nabla f(x_0) \\ \nabla g(x_0) \end{pmatrix}.$$

Let $x_0 \in \mathbb{R}^n$. Since

$$\begin{aligned} & \left\| h(x) - h(x_0) - (x - x_0) Dh(x_0)^\top \right\| \\ &= \left\| \big(f(x), g(x)\big) - \big(f(x_0), g(x_0)\big) - \big((x - x_0)\nabla f(x_0)^\top, (x - x_0)\nabla g(x_0)^\top\big) \right\| \\ &= \left\| \big(f(x) - f(x_0) - (x - x_0)\nabla f(x_0)^\top, g(x) - g(x_0) - (x - x_0)\nabla g(x_0)^\top\big) \right\| \\ &\leq \left\| f(x) - f(x_0) - (x - x_0)\nabla f(x_0)^\top \right\| + \left\| g(x) - g(x_0) - (x - x_0)\nabla g(x_0)^\top \right\| \end{aligned}$$

(note that the last line follow by Ex. II.1.1.8), by squeeze test we know that

$$\begin{cases} \lim_{x\to x_0; x\in\mathbb{R}^n\setminus\{x_0\}} \dfrac{\|f(x)-f(x_0)-f'(x_0)(x-x_0)\|}{\|x-x_0\|} = 0 \\ \lim_{x\to x_0; x\in\mathbb{R}^n\setminus\{x_0\}} \dfrac{\|g(x)-g(x_0)-g'(x_0)(x-x_0)\|}{\|x-x_0\|} = 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \lim_{x\to x_0; x\in\mathbb{R}^n\setminus\{x_0\}} \dfrac{\left\|f(x)-f(x_0)-(x-x_0)\nabla f(x_0)^\top\right\|}{\|x-x_0\|} = 0 \\ \lim_{x\to x_0; x\in\mathbb{R}^n\setminus\{x_0\}} \dfrac{\left\|g(x)-g(x_0)-(x-x_0)\nabla g(x_0)^\top\right\|}{\|x-x_0\|} = 0 \end{cases}$$

$$\Longrightarrow \lim_{x\to x_0; x\in\mathbb{R}^n\setminus\{x_0\}} \frac{\left\|h(x)-h(x_0)-(x-x_0)Dh(x_0)^\top\right\|}{\|x-x_0\|} = 0.$$

Since x_0 was arbitrary, we conclude that the identity is true.

Now we show that

$$Dk(a,b) = (b,a).$$

Let $(a,b)\in\mathbb{R}^2$. Observe that for any $(x,y)\in\mathbb{R}^2\setminus\{(a,b)\}$, we have

$$\begin{aligned} &\frac{\left\|k(x,y)-k(a,b)-\big((x,y)-(a,b)\big)(b,a)^\top\right\|}{\|(x,y)-(a,b)\|} \\ &= \frac{\|xy-ab-xb-ay+ab+ba\|}{\|(x-a,y-b)\|} \\ &= \frac{\|(x-a)(y-b)\|}{\|(x-a,y-b)\|} \\ &= \sqrt{\frac{(x-a)^2(y-b)^2}{(x-a)^2+(y-b)^2}} \\ &\le \sqrt{\frac{2(x-a)^2(y-b)^2}{(x-a)^2+(y-b)^2}} \\ &\le \sqrt{\frac{(x-a)^4+2(x-a)^2(y-b)^2+(y-b)^4}{(x-a)^2+(y-b)^2}} \\ &= \sqrt{\frac{\big((x-a)^2+(y-b)^2\big)^2}{(x-a)^2+(y-b)^2}} \\ &= \sqrt{(x-a)^2+(y-b)^2}. \end{aligned}$$

Since

$$\lim_{(x,y)\to(a,b);(x,y)\in\mathbb{R}^2\setminus\{(a,b)\}} (x-a)^2+(y-b)^2 = 0$$

$$\Longrightarrow \lim_{(x,y)\to(a,b);(x,y)\in\mathbb{R}^2\setminus\{(a,b)\}} \sqrt{(x-a)^2+(y-b)^2} = 0$$

$$\Longrightarrow \lim_{(x,y)\to(a,b);(x,y)\in\mathbb{R}^2\setminus\{(a,b)\}} \frac{\left\|k(x,y)-k(a,b)-\big((x,y)-(a,b)\big)(b,a)^\top\right\|}{\|(x,y)-(a,b)\|} = 0$$

(note that the last line follow by squeeze test), by Def. II.6.2.2 we know that the identity is true.

By the chain rule, we thus see that

$$D(k\circ h)(x_0) = \big(g(x_0), f(x_0)\big)\begin{pmatrix}\nabla f(x_0)\\ \nabla g(x_0)\end{pmatrix} = g(x_0)\nabla f(x_0) + f(x_0)\nabla g(x_0).$$

But $k\circ h = fg$, and $D(fg) = \nabla(fg)$. We have thus proven the *product rule*

$$\nabla(fg) = g\nabla f + f\nabla g.$$

A similar argument gives the sum rule $\nabla(f+g) = \nabla f + \nabla g$, or the difference rule $\nabla(f-g) = \nabla f - \nabla g$, as well as the quotient rule (Ex. II.6.4.4).

Note. We do record one further useful application of the chain rule. Let $T:\mathbb{R}^n\to\mathbb{R}^m$ be a linear transformation. From Ex. II.6.4.1 we observe that T is continuously differentiable at every point, and in fact $T'(x) = T$ for every x. (This equation may look a little strange, but perhaps it is easier to swallow if you view it in the form $\dfrac{d}{dx}(Tx) = T$.) Thus, for any differentiable function $f:E\to\mathbb{R}^n$, we see that $Tf:E\to\mathbb{R}^m$ is also differentiable, and hence by the chain rule

$$(Tf)'(x_0) = T\big(f'(x_0)\big).$$

This is a generalization of the single-variable calculus rule $(cf)' = c(f')$ for constant scalars c.

A.Cor. II.6.4.1. If $f:\mathbb{R}^n\to\mathbb{R}^m$ is some differentiable function, and $x_j:\mathbb{R}\to\mathbb{R}$ are differentiable functions for each $j = 1,\dots n$, then

$$\frac{d}{dt}f\big(x_1(t), x_2(t),\dots,x_n(t)\big) = \sum_{j=1}^n x_j'(t)\frac{\partial f}{\partial x_j}\big(x_1(t), x_2(t),\dots,x_n(t)\big).$$

Proof. Let $h:\mathbb{R}\to\mathbb{R}^n$ be the function

$$\forall t\in\mathbb{R}, h(t) = \big(x_1(t),\dots,x_n(t)\big).$$

We claim that h is differentiable on $\mathbb{R}$, and

$$\forall t\in\mathbb{R}, Dh(t) = \big(x_1'(t),\dots,x_n'(t)\big)^\top.$$

Let $t_0 \in \mathbb{R}$. Since

$$\begin{aligned}
&\frac{\left\|h(t)-h(t_0)-(t-t_0)D_h(t_0)^\top\right\|}{\|t-t_0\|} \\
&= \frac{\left\|\big(x_1(t),\dots,x_n(t)\big)-\big(x_1(t_0),\dots,x_n(t_0)\big)-(t-t_0)\big(x_1'(t),\dots,x_n'(t)\big)^\top\right\|}{|t-t_0|} \\
&= \frac{\left\|\big(x_1(t)-x_1(t_0)-x_1'(t_0)(t-t_0),\dots,x_n(t)-x_n(t_0)-x_n'(t_0)(t-t_0)\big)\right\|}{|t-t_0|} \\
&\le \frac{\sum_{i=1}^n\left|\big(x_i(t)-x_i(t_0)-x_i'(t_0)(t-t_0)\big)\right|}{|t-t_0|} \\
&= \sum_{i=1}^n \frac{\left|\big(x_i(t)-x_i(t_0)-x_i'(t_0)(t-t_0)\big)\right|}{|t-t_0|},
\end{aligned}$$

we know that

$$\begin{aligned}
&\forall 1\le i\le n, \lim_{t\to t_0; t\in\mathbb{R}\setminus\{t_0\}} \frac{|x_i(t)-x_i(t_0)-x_i'(t_0)(x-x_0)|}{|t-t_0|}=0 \\
\Longrightarrow & \lim_{t\to t_0; t\in\mathbb{R}\setminus\{t_0\}} \sum_{i=1}^n \frac{|x_i(t)-x_i(t_0)-x_i'(t_0)(x-x_0)|}{|t-t_0|}=0 && \text{(by limit laws)} \\
\Longrightarrow & \lim_{t\to t_0; t\in\mathbb{R}\setminus\{t_0\}} \frac{\left\|h(t)-h(t_0)-(t-t_0)D_h(t_0)^\top\right\|}{\|t-t_0\|}. && \text{(by squeeze test)}
\end{aligned}$$

Since t_0 was arbitrary, by Def. II.6.2.2 we know that

$$\forall t\in\mathbb{R}, Dh(t)=\big(x_1'(t),\dots,x_n'(t)\big)^\top.$$

Using chain rule we have

$$\begin{aligned}
\forall t\in\mathbb{R},& D(f\circ h)(t) \\
&= Df\big(h(t)\big)\cdot Dh(t) && \text{(by Thm. II.6.4.1)} \\
&= Df\big(x_1(t),\dots,x_n(t)\big)\cdot\big(x_1'(t),\dots,x_n'(t)\big)^\top \\
&= D_{\left(x_1'(t),\dots,x_n'(t)\right)} f\big(x_1(t),\dots,x_n(t)\big) && \text{(by Lem. II.6.3.5)} \\
&= \sum_{j=1}^n x_j'(t)\frac{\partial f}{\partial x_j}\big(x_1(t),\dots,x_n(t)\big). && \text{(by A.Cor. II.6.3.2)}
\end{aligned}$$

$\square$

— Exercises —

Ex. II.6.4.1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that T is continuously differentiable at every point, and in fact $T'(x) = T$ for every x. What is DT?

Proof. Let $x_0 \in \mathbb{R}^n$. Since

$$\begin{aligned}
&\lim_{x\to x_0; x\in\mathbb{R}^n\setminus\{x_0\}} \frac{\|T(x) - T(x_0) - T(x - x_0)\|}{\|x - x_0\|} \\
&= \lim_{x\to x_0; x\in\mathbb{R}^n\setminus\{x_0\}} \frac{\|T(x) - T(x_0) - T(x) - T(x_0)\|}{\|x - x_0\|} && \text{(by Def. II.6.1.6)} \\
&= \lim_{x\to x_0; x\in\mathbb{R}^n\setminus\{x_0\}} \frac{\|0_{\mathbb{R}^m}\|}{\|x - x_0\|} \\
&= 0
\end{aligned}$$

and x_0 was arbitrary, we conclude by Def. II.6.2.2 that

$$\forall x \in \mathbb{R}^n, T'(x) = T.$$

By Lem. II.6.1.13 we know that

$$\forall x \in \mathbb{R}^n, DT(x) = \big(T(e_1)^\top, \dots, T(e_n)^\top\big).$$

$\square$

Ex. II.6.4.2. Let E be a subset of $\mathbb{R}^n$. Prove that if a function $f : E \to \mathbb{R}^m$ is differentiable at an interior point x_0 of E, then it is also continuous at x_0.

Proof. By Def. II.6.2.2 we have

$$\begin{aligned}
&\forall \varepsilon \in \mathbb{R}^+, \exists \delta_1 \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \\
&\quad \left(\|x - x_0\| < \delta_1 \implies \frac{\|f(x) - f(x_0) - f'(x_0)(x - x_0)\|}{\|x - x_0\|} < \frac{\varepsilon}{2}\right) \\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists \delta_1 \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \\
&\quad \left(\|x - x_0\| < \delta_1 \implies \big\|f(x) - f(x_0) - f'(x_0)(x - x_0)\big\| < \frac{\varepsilon}{2}\|x - x_0\|\right) \\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists \delta_1 \in \mathbb{R}^+ : \forall x \in E \setminus \{x_0\}, \\
&\quad \left(\|x - x_0\| < \delta_1 \implies \|f(x) - f(x_0)\| < \frac{\varepsilon}{2}\|x - x_0\| + \big\|f'(x_0)(x - x_0)\big\|\right) \\
&\implies \forall \varepsilon \in \mathbb{R}^+, \exists \delta_1 \in \mathbb{R}^+ : \forall x \in E, \\
&\quad \left(\|x - x_0\| < \delta_1 \implies \|f(x) - f(x_0)\| < \frac{\varepsilon}{2}\|x - x_0\| + \big\|f'(x_0)(x - x_0)\big\|\right).
\end{aligned}$$

By Ex. II.6.1.4 we know that every linear transformation is continuous. Thus, we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta_2 \in \mathbb{R}^+ : \forall x \in E,$$

$$\begin{aligned}
&\quad\left(\|x-x_0\|<\delta_2 \implies \left\|f'(x_0)(x)-f'(x_0)(x_0)\right\|<\frac{\varepsilon}{2}\right)\\
&\implies \forall\varepsilon\in\mathbb{R}^+,\exists\delta_2\in\mathbb{R}^+:\forall x\in E,\\
&\quad\left(\|x-x_0\|<\delta_2 \implies \left\|f'(x_0)(x-x_0)\right\|<\frac{\varepsilon}{2}\right). \qquad \text{(by Def. II.6.1.6)}
\end{aligned}$$

Let $\delta=\min(\delta_1,\delta_2,1)$. Then we have

$$\begin{aligned}
&\forall\varepsilon\in\mathbb{R}^+,\exists\delta\in\mathbb{R}^+:\forall x\in E,\|x-x_0\|<\delta\\
&\implies \|f(x)-f(x_0)\|<\frac{\varepsilon}{2}\|x-x_0\|+\left\|f'(x_0)(x-x_0)\right\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

Thus, f is continuous at x_0 from $(E,d_{l^2}|_{E\times E})$ to $(\mathbb{R}^m,d_{l^2}|_{\mathbb{R}^m\times\mathbb{R}^m})$. □

Ex. II.6.4.3. Prove Thm. II.6.4.1.

Proof. See Thm. II.6.4.1. □

Ex. II.6.4.4. State and prove some version of the quotient rule for functions of several variables (i.e., functions of the form $f:E\to\mathbb{R}$ for some subset E of $\mathbb{R}^n$). In other words, state a rule which gives a formula for the gradient of f/g; compare your answer with Theorem 10.1.13(h) in Analysis I. Be sure to make clear what all your assumptions are.

Proof. Let $E\subseteq\mathbb{R}^n$ and let x_0 be an interior point of E. Let $f:E\to\mathbb{R}$ and $g:E\to\mathbb{R}$ be functions where $g(x)\neq 0$ for all $x\in E$. If f,g are differentiable at x_0, then f/g is differentiable at x_0, and

$$\nabla\left(\frac{f}{g}\right)(x_0)=\frac{g(x_0)\nabla f(x_0)-f(x_0)\nabla g(x_0)}{\big(g(x_0)\big)^2}.$$

If f,g are differentiable on E, then f/g is differentiable on E, and

$$\nabla\left(\frac{f}{g}\right)=\frac{g\nabla f-f\nabla g}{g^2}.$$

Let $k:\mathbb{R}\times(\mathbb{R}\setminus\{0\})\to\mathbb{R}$ be the function

$$\forall(a,b)\in\mathbb{R}\times(\mathbb{R}\setminus\{0\}),k(a,b)=\frac{a}{b}.$$

We claim that k is differentiable on $\mathbb{R}\times(\mathbb{R}\setminus\{0\})$ and

$$\forall(a,b)\in\mathbb{R}\times(\mathbb{R}\setminus\{0\}),Dk(a,b)=\left(\frac{1}{b},\frac{-a}{b^2}\right).$$

Let $(a,b) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$. Since for each $(x,y) \in \big(\mathbb{R} \times (\mathbb{R} \setminus \{0\})\big) \setminus \{(a,b)\}$, we have

$$\begin{aligned}
&\frac{\left\| k(x,y) - k(a,b) - \big((x,y)-(a,b)\big)\left(\frac{1}{b}, \frac{-a}{b^2}\right)^\top \right\|}{\|(x,y)-(a,b)\|} \\
&= \frac{\left\| \frac{x}{y} - \frac{a}{b} - \frac{x-a}{b} + \frac{a(y-b)}{b^2} \right\|}{\|(x-a, y-b)\|} \\
&= \frac{\left\| \frac{b^2x - bxy + ay^2 - aby}{b^2y} \right\|}{\|(x-a, y-b)\|} \\
&= \frac{\left\| \frac{(ay-bx)(y-b)}{b^2y} \right\|}{\|(x-a, y-b)\|} \\
&= \sqrt{\frac{(ay-bx)^2(y-b)^2}{b^4y^2\big((x-a)^2+(y-b)^2\big)}} \\
&\leq \sqrt{\frac{(ay-bx)^2\big((x-a)^2+(y-b)^2\big)}{b^4y^2\big((x-a)^2+(y-b)^2\big)}} \\
&= \sqrt{\frac{(ay-bx)^2}{b^4y^2}},
\end{aligned}$$

we know that

$$\begin{aligned}
&\lim_{(x,y)\to(a,b);(x,y)\in\big(\mathbb{R}\times(\mathbb{R}\setminus\{0\})\big)\setminus\{(a,b)\}} (ay-bx)^2 = 0 \\
\implies &\lim_{(x,y)\to(a,b);(x,y)\in\big(\mathbb{R}\times(\mathbb{R}\setminus\{0\})\big)\setminus\{(a,b)\}} \frac{(ay-bx)^2}{b^4y^2} = 0 \\
\implies &\lim_{(x,y)\to(a,b);(x,y)\in\big(\mathbb{R}\times(\mathbb{R}\setminus\{0\})\big)\setminus\{(a,b)\}} \sqrt{\frac{(ay-bx)^2}{b^4y^2}} = 0 \\
\implies &\lim_{(x,y)\to(a,b);(x,y)\in\big(\mathbb{R}\times(\mathbb{R}\setminus\{0\})\big)\setminus\{(a,b)\}} \frac{\left\| k(x,y) - k(a,b) - \big((x,y)-(a,b)\big)\left(\frac{1}{b}, \frac{-a}{b^2}\right)^\top \right\|}{\|(x,y)-(a,b)\|} \\
&= 0.
\end{aligned}$$

(note that the last line was done by squeeze test) Since (a,b) was arbitrary, by Def. II.6.2.2 we conclude that

$$\forall (a,b) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}), Dk(a,b) = \left(\frac{1}{b}, \frac{-a}{b^2}\right).$$

Let h be the function in E.g. II.6.4.2. Using chain rule we have

$$\begin{aligned}
D(k\circ h)(x_0) &= Dk\big(h(x_0)\big)\cdot Dh(x_0) && \text{(by Thm. II.6.4.1)}\\
&= Dk\big(f(x_0),g(x_0)\big)\cdot\begin{pmatrix}\nabla f(x_0)\\ \nabla g(x_0)\end{pmatrix} && \text{(by E.g. II.6.4.2)}\\
&= \left(\frac{1}{g(x_0)},\frac{-f(x_0)}{\big(g(x_0)\big)^2}\right)\cdot\begin{pmatrix}\nabla f(x_0)\\ \nabla g(x_0)\end{pmatrix}\\
&= \frac{\nabla f(x_0)}{g(x_0)}-\frac{f(x_0)\nabla g(x_0)}{\big(g(x_0)\big)^2}\\
&= \frac{g(x_0)\nabla f(x_0)-f(x_0)\nabla g(x_0)}{\big(g(x_0)\big)^2}.
\end{aligned}$$

But $k\circ h = f/g$. Thus, we have

$$\nabla\left(\frac{f}{g}\right)(x_0)=\frac{g(x_0)\nabla f(x_0)-f(x_0)\nabla g(x_0)}{\big(g(x_0)\big)^2}.$$

□

Ex. II.6.4.5. Let $\vec{x}:\mathbb{R}\to\mathbb{R}^3$ be a differentiable function, and let $r:\mathbb{R}\to\mathbb{R}$ be the function $r(t):=\|\vec{x}(t)\|$, where $\|\vec{x}\|$ denotes the length of $\vec{x}$ as measured in the usual l^2 metric. Let t_0 be a real number. Show that if $r(t_0)\neq 0$, then r is differentiable at t_0, and

$$r'(t_0)=\frac{\vec{x}'(t_0)\cdot\vec{x}(t_0)}{r(t_0)}.$$

Proof. Let $f:\mathbb{R}^3\to\mathbb{R}$ be the function

$$\forall x\in\mathbb{R}^3, f(x)=\|x\|.$$

Since

$$\forall y\in\mathbb{R}^3\setminus\{0_{\mathbb{R}^3}\},\forall 1\le i\le 3,\frac{\partial f}{\partial x_i}(y)=\frac{y_i}{\|y\|},$$

and $\dfrac{\partial f}{\partial x_i}$ is continuous on $\mathbb{R}^3\setminus\{0_{\mathbb{R}^3}\}$, by Thm. II.6.3.8 we know that

$$\begin{aligned}
\forall y\in\mathbb{R}^3\setminus\{0_{\mathbb{R}^3}\}, Df(y) &= \nabla f(y)\\
&= \left(\frac{\partial f}{\partial x_1}(y),\frac{\partial f}{\partial x_2}(y),\frac{\partial f}{\partial x_3}(y)\right)\\
&= \left(\frac{y_1}{\|y\|},\frac{y_2}{\|y\|},\frac{y_3}{\|y\|}\right)\\
&= \frac{y}{\|y\|}.
\end{aligned}$$

By chain rule (Thm. II.6.4.1) we know that

$$\begin{aligned}&\forall t \in \mathbb{R}, \vec{x}(t) \neq 0\\ \implies &D(f \circ \vec{x})(t) = Df\big(\vec{x}(t)\big) \cdot D\vec{x}(t) = \frac{\vec{x}(t)}{\|\vec{x}(t)\|} \cdot \vec{x}'(t)^\top.\end{aligned}$$

Since $r = f \circ \vec{x}$, we know that

$$\begin{aligned}&\forall t \in \mathbb{R}, \vec{x}(t) \neq 0\\ \implies &\|t\| \neq 0\\ \implies &r(t) \neq 0\\ \implies &Dr(t) = \frac{\vec{x}(t) \cdot \vec{x}'(t)^\top}{r(t)}.\end{aligned}$$

□

II.6.5 Double derivatives and Clairaut's theorem

Def. II.6.5.1 (Twice continuous differentiability)**.** Let E be an open subset of $\mathbb{R}^n$, and let $f : E \to \mathbb{R}^m$ be a function. We say that f is *continuously differentiable* if the partial derivatives $\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}$ exist and are continuous on E. We say that f is *twice continuously differentiable* if it is continuously differentiable, and the partial derivatives $\dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_n}$ are themselves continuously differentiable.

Rmk. II.6.5.2. Continuously differentiable functions are sometimes called C^1 functions; twice continuously differentiable functions are sometimes called C^2 functions. One can also define C^3, C^4, etc. but we shall not do so here.

Thm. II.6.5.4 (Clairaut's theorem)**.** Let E be an open subset of $\mathbb{R}^n$, and let $f : E \to \mathbb{R}^m$ be a twice continuously differentiable function on E. Then we have $\dfrac{\partial}{\partial x_j}\dfrac{\partial f}{\partial x_i}(x_0) = \dfrac{\partial}{\partial x_i}\dfrac{\partial f}{\partial x_j}(x_0)$ for all $1 \leq i, j \leq n$.

Proof. By working with one component of f at a time we can assume that $m = 1$. The claim is trivial if $i = j$, so we shall assume that $i \neq j$. We shall prove the theorem for $x_0 = 0$; the general case is similar. (Actually, once one proves Clairaut's theorem for $x_0 = 0$, one can immediately obtain it for general x_0 by applying the theorem with $f(x)$ replaced by $f(x + x_0)$.)

Let a be the number $a \coloneqq \dfrac{\partial}{\partial x_j}\dfrac{\partial f}{\partial x_i}(0)$, and a' denote the quantity $a' \coloneqq \dfrac{\partial}{\partial x_i}\dfrac{\partial f}{\partial x_j}(0)$. Our task is to show that $a' = a$.

Let $\varepsilon > 0$. Because the double derivatives of f are continuous, we can find a $\delta > 0$ such that

$$\left|\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i}(x) - a\right| \leq \varepsilon$$

and

$$\left|\frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j}(x) - a'\right| \leq \varepsilon$$

whenever $\|x\| \leq 2\delta$.

Now we consider the quantity

$$X := f(\delta e_i + \delta e_j) - f(\delta e_i) - f(\delta e_j) + f(0).$$

From the fundamental theorem of calculus in the e_i variable, we have

$$f(\delta e_i + \delta e_j) - f(\delta e_j) = \int_0^\delta \frac{\partial f}{\partial x_i}(x_i e_i + \delta e_j)\, dx_i$$

and

$$f(\delta e_i) - f(0) = \int_0^\delta \frac{\partial f}{\partial x_i}(x_i e_i)\, dx_i$$

and hence

$$X = \int_0^\delta \left(\frac{\partial f}{\partial x_i}(x_i e_i + \delta e_j) - \frac{\partial f}{\partial x_i}(x_i e_i)\right) dx_i.$$

But by the mean value theorem, for each x_i we have

$$\frac{\partial f}{\partial x_i}(x_i e_i + \delta e_j) - \frac{\partial f}{\partial x_i}(x_i e_i) = \delta \frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i}(x_i e_i + x_j e_j)$$

for some $0 \leq x_j \leq \delta$. By our construction of δ, we thus have

$$\left|\frac{\partial f}{\partial x_i}(x_i e_i + \delta e_j) - \frac{\partial f}{\partial x_i}(x_i e_i) - \delta a\right| \leq \varepsilon\delta.$$

Integrating this from 0 to δ, we thus obtain

$$\left|X - \delta^2 a\right| \leq \varepsilon\delta^2.$$

We can run the same argument with the rôle of i and j reversed (note that X is symmetric in i and j), to obtain

$$\left|X - \delta^2 a'\right| \leq \varepsilon\delta^2.$$

From the triangle inequality we thus obtain

$$\left|\delta^2 a - \delta^2 a'\right| \leq 2\varepsilon\delta^2,$$

and thus

$$\left|a - a'\right| \leq 2\varepsilon.$$

But this is true for all $\varepsilon > 0$, and a and a' do not depend on ε, and so we must have $a = a'$, as desired. $\square$

— Exercises —

Ex. II.6.5.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x,y) := \dfrac{xy^3}{x^2+y^2}$ when $(x,y) \neq (0,0)$, and $f(0,0) := 0$. Show that f is continuously differentiable, and the double derivatives $\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}$ exist, but are not equal to each other at $(0,0)$. Explain why this does not contradict Clairaut's theorem.

Proof. Let $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$. We have

$$\begin{aligned}
&\frac{\partial f}{\partial x}(a,b)\\
&= \lim_{t\to 0; t\neq 0} \frac{f\big((a,b)+t(1,0)\big) - f(a,b)}{t} && \text{(by Def. II.6.3.7)}\\
&= \lim_{t\to 0; t\neq 0} \frac{f(a+t,b) - f(a,b)}{t}\\
&= \lim_{t\to 0; t\neq 0} \frac{\dfrac{(a+t)b^3}{(a+t)^2+b^2} - \dfrac{ab^3}{a^2+b^2}}{t}\\
&= \lim_{t\to 0; t\neq 0} \frac{b^3(a+t)(a^2+b^2) - ab^3\big((a+t)^2+b^2\big)}{t\big((a+t)^2+b^2\big)(a^2+b^2)}\\
&= \lim_{t\to 0; t\neq 0} \frac{tb^5 - ta^2b^3 - t^2ab^3}{t\big((a+t)^2+b^2\big)(a^2+b^2)}\\
&= \lim_{t\to 0; t\neq 0} \frac{b^5 - a^2b^3 - tab^3}{\big((a+t)^2+b^2\big)(a^2+b^2)}\\
&= \frac{b^5 - a^2b^3}{(a^2+b^2)^2}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial f}{\partial y}(a,b)\\
&= \lim_{t\to 0; t\neq 0} \frac{f\big((a,b)+t(0,1)\big) - f(a,b)}{t} && \text{(by Def. II.6.3.7)}\\
&= \lim_{t\to 0; t\neq 0} \frac{f(a,b+t) - f(a,b)}{t}\\
&= \lim_{t\to 0; t\neq 0} \frac{\dfrac{a(b+t)^3}{a^2+(b+t)^2} - \dfrac{ab^3}{a^2+b^2}}{t}\\
&= \lim_{t\to 0; t\neq 0} \frac{a(b+t)^3(a^2+b^2) - ab^3\big(a^2+(b+t)^2\big)}{t\big(a^2+(b+t)^2\big)(a^2+b^2)}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{t\to 0; t\neq 0} \frac{(3b^2t+3bt^2+t^3)(a^3+ab^2)-ab^3(2bt+t^2)}{t\big(a^2+(b+t)^2\big)(a^2+b^2)} \\
&= \lim_{t\to 0; t\neq 0} \frac{(3b^2+3bt+t^2)(a^3+ab^2)-ab^3(2b+t)}{\big(a^2+(b+t)^2\big)(a^2+b^2)} \\
&= \frac{3a^3b^2+ab^4}{(a^2+b^2)^2}.
\end{aligned}
$$

Since (a,b) was arbitrary, by Lem. II.2.2.2 we know that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$ from $(\mathbb{R}^2, d_{l^2}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Observe that

$$
\begin{aligned}
\left|\frac{b^5-a^2b^3}{(a^2+b^2)^2}\right| &= \frac{\big|b(b^4-a^2b^2)\big|}{(a^2+b^2)^2} \\
&= \frac{|b|\big|b^4-a^2b^2\big|}{(a^2+b^2)^2} \\
&\leq \frac{|b|(b^4+2a^2b^2+a^4)}{(a^2+b^2)^2} \\
&= \frac{|b|(a^2+b^2)^2}{(a^2+b^2)^2} \\
&= |b|
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{3a^3b^2+ab^4}{(a^2+b^2)^2}\right| &= \frac{\big|a(2a^2b^2+b^4)+a(a^2b^2)\big|}{(a^2+b^2)^2} \\
&\leq \frac{|a|\big|2a^2b^2+b^4\big|+|a|\big|a^2b^2\big|}{(a^2+b^2)^2} \\
&\leq \frac{|a|(a^4+2a^2b^2+b^4)+|a|(a^4+2a^2b^2+b^4)}{(a^2+b^2)^2} \\
&= \frac{|a|(a^2+b^2)^2+|a|(a^2+b^2)^2}{(a^2+b^2)^2} \\
&= 2|a|.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\lim_{(a,b)\to(0,0);(a,b)\neq(0,0)} |b| = 0 \\
\implies &\lim_{(a,b)\to(0,0);(a,b)\neq(0,0)} \left|\frac{b^5-a^2b^3}{(a^2+b^2)^2}\right| = 0 \qquad \text{(by squeeze test)} \\
\implies &\lim_{(a,b)\to(0,0);(a,b)\neq(0,0)} \frac{b^5-a^2b^3}{(a^2+b^2)^2} = 0
\end{aligned}
$$

$$\implies \lim_{(a,b)\to(0,0);(a,b)\neq(0,0)} \frac{\partial f}{\partial x}(a,b) = 0$$

and

$$\begin{aligned}
&\lim_{(a,b)\to(0,0);(a,b)\neq(0,0)} 2|a| = 0 \\
\implies &\lim_{(a,b)\to(0,0);(a,b)\neq(0,0)} \left|\frac{3a^3b^2+ab^4}{(a^2+b^2)^2}\right| = 0 && \text{(by squeeze test)} \\
\implies &\lim_{(a,b)\to(0,0);(a,b)\neq(0,0)} \frac{3a^3b^2+ab^4}{(a^2+b^2)^2} = 0 \\
\implies &\lim_{(a,b)\to(0,0);(a,b)\neq(0,0)} \frac{\partial f}{\partial y}(a,b) = 0.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{t\to0;t\neq0} \frac{f\big((0,0)+t(1,0)\big)-f(0,0)}{t} && \text{(by Def. II.6.3.7)} \\
&= \lim_{t\to0;t\neq0} \frac{\frac{t0^3}{t^2+0^2}-0}{t} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial f}{\partial y}(0,0) &= \lim_{t\to0;t\neq0} \frac{f\big((0,0)+t(0,1)\big)-f(0,0)}{t} && \text{(by Def. II.6.3.7)} \\
&= \lim_{t\to0;t\neq0} \frac{\frac{0t^3}{0^2+t^2}-0}{t} \\
&= 0,
\end{aligned}$$

by Def. II.2.1.1 we know that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous at $(0,0)$ from $(\mathbb{R}^2, d_{l^2}|_{\mathbb{R}^2\times\mathbb{R}^2})$ to $(\mathbb{R}, d_{l^1}|_{\mathbb{R}\times\mathbb{R}})$. Combine the proof above we conclude by Def. II.6.5.1 that f is continuously differentiable on $\mathbb{R}^2$.

Let $(a,b)\in\mathbb{R}^2\setminus\{(0,0)\}$. Observe that

$$\begin{aligned}
&\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(a,b) \\
&= \lim_{t\to0;t\neq0} \frac{\frac{\partial f}{\partial x}\big((a,b)+t(0,1)\big)-\frac{\partial f}{\partial x}(a,b)}{t}
\end{aligned}$$

$$= \lim_{t\to 0; t\neq 0} \frac{\dfrac{\partial f}{\partial x}(a, b+t) - \dfrac{\partial f}{\partial x}(a,b)}{t}$$
$$= \lim_{t\to 0; t\neq 0} \frac{\dfrac{(b+t)^5 - a^2(b+t)^3}{\big(a^2+(b+t)^2\big)^2} - \dfrac{b^5 - a^2b^3}{(a^2+b^2)^2}}{t}$$
$$= \lim_{t\to 0; t\neq 0} \frac{\big((b+t)^5 - a^2(b+t)^3\big)(a^2+b^2)^2 - (b^5 - a^2b^3)\big(a^2+(b+t)^2\big)^2}{t\big(a^2+(b+t)^2\big)^2(a^2+b^2)^2}$$
$$= \frac{-3a^6b^2 + 3a^4b^4 + 7a^2b^6 + b^8}{(a^2+b^2)^4}$$

and

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(a,b)$$
$$= \lim_{t\to 0; t\neq 0} \frac{\dfrac{\partial f}{\partial y}\big((a,b)+t(1,0)\big) - \dfrac{\partial f}{\partial y}(a,b)}{t}$$
$$= \lim_{t\to 0; t\neq 0} \frac{\dfrac{\partial f}{\partial y}(a+t, b) - \dfrac{\partial f}{\partial y}(a,b)}{t}$$
$$= \lim_{t\to 0; t\neq 0} \frac{\dfrac{3(a+t)^3b^2 + (a+t)b^4}{\big((a+t)^2+b^2\big)^2} - \dfrac{3a^3b^2 + ab^4}{(a^2+b^2)^2}}{t}$$
$$= \lim_{t\to 0; t\neq 0} \frac{\big(3(a+t)^3b^2 + (a+t)b^4\big)(a^2+b^2)^2 - (3a^3b^2 + ab^4)\big((a+t)^2+b^2\big)^2}{t\big((a+t)^2+b^2\big)^2(a^2+b^2)^2}$$
$$= \frac{-3a^6b^2 + 3a^4b^4 + 7a^2b^6 + b^8}{(a^2+b^2)^4}.$$

Thus, by Def. II.6.3.7 $\dfrac{\partial}{\partial y}\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial}{\partial x}\dfrac{\partial f}{\partial y}$ exist for all $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Since

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0,0)$$
$$= \lim_{t\to 0; t\neq 0} \frac{\dfrac{\partial f}{\partial x}\big((0,0) - t(0,1)\big) - \dfrac{\partial f}{\partial x}(0,0)}{t}$$
$$= \lim_{t\to 0; t\neq 0} \frac{\dfrac{t^5 - 0^2t^3}{(0^2+t^2)^2} - 0}{t}$$

$$
\begin{aligned}
&= \lim_{t\to 0; t\neq 0} \frac{t^5}{t^5} \\
&= 1
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(0,0) \\
&= \lim_{t\to 0; t\neq 0} \frac{\frac{\partial f}{\partial y}\big((0,0)-t(1,0)\big) - \frac{\partial f}{\partial y}(0,0)}{t} \\
&= \lim_{t\to 0; t\neq 0} \frac{\frac{3t^3 0^2 + t0^4}{(t^2+0^2)^2} - 0}{t} \\
&= 0,
\end{aligned}
$$

we know that both $\frac{\partial}{\partial y}\frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial x}\frac{\partial f}{\partial y}$ exist at $(0,0)$ but are not equal to each other. This does not contradict to Thm. II.6.5.4 since both $\frac{\partial}{\partial y}\frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial x}\frac{\partial f}{\partial y}$ are not continuous at $(0,0)$. □

II.6.6 The contraction mapping theorem

Def. II.6.6.1 (Contraction)**.** Let (X,d) be a metric space, and let $f : X \to X$ be a map. We say that f is a *contraction* if we have $d\big(f(x), f(y)\big) \le d(x,y)$ for all $x, y \in X$. We say that f is a *strict contraction* if there exists a constant $0 < c < 1$ such that $d\big(f(x), f(y)\big) \le cd(x,y)$ for all $x, y \in X$; we call c the *contraction constant* of f.

E.g. II.6.6.2. The map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x+1$ is a contraction but not a strict contraction. The map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x/2$ is a strict contraction. The map $f : [0,1] \to [0,1]$ defined by $f(x) := x - x^2$ is a contraction but not a strict contraction.

Proof. Since

$$\forall x, y \in \mathbb{R}, |(x+1)-(y+1)| = |x-y| \le |x-y|,$$

by Def. II.6.6.1 we know that $x \mapsto x+1$ is a contraction. Suppose for the sake of contradiction that $x \mapsto x+1$ is a strict contraction. Then there exists a $c \in (0,1)$ such that

$$\forall x, y \in \mathbb{R}, |(x+1)-(y+1)| \le c|x-y|.$$

But we see that when $x \neq y$, we have

$$|x-y| \le c|x-y| \implies 1 \le c,$$

which contradict to the fact that $c \in (0,1)$. Thus, $x \mapsto x+1$ is not a strict contraction.

Since
$$\forall x, y \in \mathbb{R}, \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x-y| \le \frac{1}{2}|x-y|,$$
by Def. II.6.6.1 we know that $x \mapsto \dfrac{x}{2}$ is a strict contraction.

Since
$$\begin{aligned}
&\forall x, y \in [0,1], \begin{cases} -1 \le -x \le 0 \\ -1 \le -y \le 0 \end{cases} \\
\implies& -2 \le -x-y \le 0 \\
\implies& -1 \le 1-x-y \le 1 \\
\implies& 0 \le |1-x-y| \le 1,
\end{aligned}$$
we know that
$$\begin{aligned}
\forall x, y \in [0,1], \left|(x-x^2)-(y-y^2)\right| &= \left|x-y-(x^2-y^2)\right| \\
&= |x-y-(x-y)(x+y)| \\
&= |(x-y)(1-x-y)| \\
&= |x-y||1-x-y| \\
&\le |x-y|.
\end{aligned}$$
Thus, by Def. II.6.6.1 we know that $x \mapsto x-x^2$ is a contraction on $[0,1]$. Suppose for the sake of contradiction that $x \mapsto x - x^2$ is a strict contraction on $[0,1]$. Then there exists a $c \in (0,1)$ such that
$$\forall x, y \in \mathbb{R}, \left|(x-x^2)-(y-y^2)\right| \le c|x-y|.$$
But when $(x,y) = (\dfrac{1-c}{2}, 0)$ we have
$$\begin{aligned}
\left|(x-x^2)-(y-y^2)\right| &= \left|\frac{1-c}{2} - \left(\frac{1-c}{2}\right)^2 - (0-0^2)\right| \\
&= \frac{1-c}{2}\left|1 - \frac{1-c}{2}\right| \\
&= \frac{1-c}{2}\frac{1+c}{2} \\
&> \frac{1-c}{2}\frac{c+c}{2} && (c \in (0,1)) \\
&= c\frac{1-c}{2} \\
&= c\left|\frac{1-c}{2} - 0\right|
\end{aligned}$$

$$= c|x - y|,$$

a contraction. Thus, $x \mapsto x - x^2$ is not a strict contraction on $[0, 1]$. $\square$

Def. II.6.6.3 (Fixed points)**.** Let $f : X \to X$ be a map, and $x \in X$. We say that x is a *fixed point* of f if $f(x) = x$.

Note. Contractions do not necessarily have any fixed points; for instance, the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$ does not. However, it turns out that *strict* contractions always do, at least when X is complete.

Thm. II.6.6.4 (Contraction mapping theorem)**.** Let (X, d) be a metric space, and let $f : X \to X$ be a strict contraction. Then f can have at most one fixed point. Moreover, if we also assume that X is non-empty and complete, then f has exactly one fixed point.

Proof. We first show that f can have at most one fixed point. Suppose for the sake of contradiction that f has two fixed point $x_1, x_2 \in X$. Since f is a strict contraction, by Def. II.6.6.1 we know that there exists a $c \in (0, 1)$ such that

$$d\big(f(x_1), f(x_2)\big) \leq cd(x_1, x_2).$$

Since x_1, x_2 are fixed points, by Def. II.6.6.3 we know that

$$d(x_1, x_2) \leq cd(x_1, x_2)$$

which implies $1 \leq c$, a contradiction. Thus, f can have at most one fixed point.

Now we show that when $X \neq \emptyset$ and (X, d) is complete, f has exactly one fixed point. Let $x_0 \in X$. Since f is a strict contraction, by Def. II.6.6.1 we know that there exists a $c \in (0, 1)$ such that

$$\forall x, y \in X, d\big(f(x), f(y)\big) \leq cd(x, y).$$

Now we define a sequence $(x_n)_{n=1}^{\infty}$ as follow:

$$\forall n \in \mathbb{Z}^+, x_n = f(x_{n-1}).$$

We claim that

$$\forall n \in \mathbb{Z}^+, d(x_{n+1}, x_n) \leq c^n d(x_1, x_0).$$

We induct on n to proof the claim. For $n = 0$, we have

$$d(x_1, d_0) \leq c^0 d(x_1, x_0) = d(x_1, x_0).$$

Thus, the base case holds. Suppose inductively that $d(x_{n+1}, x_n) \leq c^n d(x_1, x_0)$ for some $n \geq 0$. Then for $n + 1$, we have

$$\begin{aligned} d(x_{n+2}, x_{n+1}) &= d\big(f(x_{n+1}), f(x_n)\big) && \text{(by the definition of } (x_n)_{n=1}^{\infty}\text{)} \\ &\leq cd(x_{n+1}, x_n) && \text{(by Def. II.6.6.1)} \end{aligned}$$

$$\leq c \cdot c^n d(x_1, x_0)$$
$$= c^{n+1} d(x_1, x_0).$$

This closes the induction.

Next we claim that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in (X, d). Let $\varepsilon \in \mathbb{R}^+$. Observe that

$$\begin{aligned}
& c \in (0,1) \\
\implies & \lim_{n\to\infty} c^n = 0 \\
\implies & \exists N \in \mathbb{Z}^+ : \forall n \geq N, c^n < \frac{\varepsilon(1-c)}{1+d(x_1,x_0)} \\
\implies & \exists N \in \mathbb{Z}^+ : \forall n \geq N, \frac{d(x_1,x_0)}{1-c} \leq \frac{1+d(x_1,x_0)}{1-c} < \frac{\varepsilon}{c^n}.
\end{aligned}$$

Fix such N. Let $n, m \geq N$ and without the loss of generality suppose that $n \leq m$. Since

$$\begin{aligned}
d(x_n, x_m) &\leq \sum_{i=n}^{m} d(x_i, x_{i+1}) && \text{(by Def. II.1.1.2(d))} \\
&= \sum_{i=n}^{m} d(x_{i+1}, x_i) && \text{(by Def. II.1.1.2(c))} \\
&\leq \sum_{i=n}^{m} c^i d(x_1, x_0) && \text{(from the claim above)} \\
&= d(x_1, x_0)\left(\sum_{i=n}^{m} c^i\right) \\
&= d(x_1, x_0)c^n\left(\sum_{i=0}^{m-n} c^i\right) \\
&\leq d(x_1, x_0)c^n\left(\sum_{i=0}^{\infty} c^i\right) && (c \in (0,1)) \\
&= \frac{d(x_1, x_0)c^n}{1-c} && \text{(by geometric series)} \\
&< \frac{\varepsilon c^n}{c^n} && (n \geq N) \\
&= \varepsilon
\end{aligned}$$

and ε was arbitrary, we know that

$$\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n, m \geq N, d(x_n, x_m) \leq \varepsilon.$$

By Def. II.1.4.6 this means $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in (X, d). By hypothesis we know that (X, d) is complete, thus by Def. II.1.4.10 there exists a $y_0 \in X$ such that $\lim_{n\to\infty} d(y_0, x_n) = 0$.

Now we claim that y_0 is the fixed point of f. Suppose for the sake of contradiction that y_0 is not the fixed point of f. By Def. II.6.6.3 this means $f(y_0) \neq y_0$, in other words, $d\big(f(y_0), y_0\big) > 0$. Since $(x_n)_{n=1}^{\infty}$ converges to y_0 in X with respect to d, we know that

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \exists N \in \mathbb{Z}^+ : \forall n \geq N, d(y_0, x_n) < \varepsilon \\ \implies &\exists N \in \mathbb{Z}^+ : \forall n \geq N, d(y_0, x_n) < \frac{d\big(f(y_0), y_0\big)}{2}. \end{aligned}$$

Fix such N. But then we have

$$\begin{aligned} &\begin{cases} d(y_0, x_N) < \dfrac{d\big(f(y_0), y_0\big)}{2} \\ d(y_0, x_{N+1}) < \dfrac{d\big(f(y_0), y_0\big)}{2} \end{cases} \\ \implies &\begin{cases} cd(y_0, x_N) < c\dfrac{d\big(f(y_0), y_0\big)}{2} \\ d(y_0, x_{N+1}) < \dfrac{d\big(f(y_0), y_0\big)}{2} \end{cases} \\ \implies &\begin{cases} d\big(f(y_0), f(x_N)\big) < cd(y_0, x_N) < c\dfrac{d\big(f(y_0), y_0\big)}{2} \\ d(y_0, x_{N+1}) < \dfrac{d\big(f(y_0), y_0\big)}{2} \end{cases} \\ \implies &\begin{cases} d\big(f(y_0), x_{N+1}\big) < cd(y_0, x_N) < c\dfrac{d\big(f(y_0), y_0\big)}{2} \\ d(y_0, x_{N+1}) < \dfrac{d\big(f(y_0), y_0\big)}{2} \end{cases} \\ \implies &d\big(f(y_0), y_0\big) \leq d\big(f(y_0), x_{N+1}\big) + d(y_0, x_{N+1}) \\ &< (c+1)\frac{d\big(f(y_0), y_0\big)}{2} < d\big(f(y_0), y_0\big), \qquad (c \in (0,1)) \end{aligned}$$

a contraction. Thus, y_0 is the fixed point of f. $\square$

Rmk. II.6.6.5. The contraction mapping theorem is one example of a *fixed point theorem* - a theorem which guarantees, assuming certain conditions, that a map will have a fixed point. There are a number of other fixed point theorems which are also useful. One amusing one is the so-called *hairy ball theorem*, which (among other things) states that any continuous map $f : S^2 \to S^2$ from the sphere $S^2 := \big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\big\}$ to itself, must contain either a fixed point, or an anti-fixed point (a point $x \in S^2$ such that $f(x) = -x$). A proof of this theorem can be found in any topology text; it is beyond the scope of this text.

Note. Basically, Lem. II.6.6.6 says that any map f on a ball which is a "small" perturbation (since g is a strict contraction) of the identity map (since $f(x) = x + g(x)$), remains one-to-one and cannot create any internal holes in the ball (there is a smaller ball contained in the origin ball such that every element in the smaller ball can be mapped by f).

Lem. II.6.6.6. Let $B(0,r)$ be a ball in $\mathbb{R}^n$ centered at the origin, and let $g : B(0,r) \to \mathbb{R}^n$ be a map such that $g(0) = 0$ and

$$\|g(x) - g(y)\| \leq \frac{1}{2}\|x - y\|$$

for all $x, y \in B(0,r)$ (here $\|x\|$ denotes the length of x in $\mathbb{R}^n$). Then the function $f : B(0,r) \to \mathbb{R}^n$ defined by $f(x) := x + g(x)$ is one-to-one, and furthermore the image $f\big(B(0,r)\big)$ of this map contains the ball $B(0,r/2)$.

Proof. We first show that f is one-to-one. Suppose for the sake of contradiction that we had two different points $x, y \in B(0,r)$ such that $f(x) = f(y)$. But then we would have $x + g(x) = y + g(y)$, and hence

$$\|g(x) - g(y)\| = \|x - y\|.$$

The only way this can be consistent with our hypothesis $\|g(x) - g(y)\| \leq \frac{1}{2}\|x - y\|$ is if $\|x - y\| = 0$, i.e., if $x = y$, a contradiction. Thus, f is one-to-one.

Now we show that $f\big(B(0,r)\big)$ contains $B(0,r/2)$. Let y be any point in $B(0,r/2)$; our objective is to find a point $x \in B(0,r)$ such that $f(x) = y$, or in other words that $x = y - g(x)$. So the problem is now to find a fixed point of the map $x \mapsto y - g(x)$.

Let $F : B(0,r) \to B(0,r)$ denote the function $F(x) := y - g(x)$. Observe that if $x \in B(0,r)$, then

$$\|F(x)\| \leq \|y\| + \|g(x)\| \leq \frac{r}{2} + \|g(x) - g(0)\| \leq \frac{r}{2} + \frac{1}{2}\|x - 0\| < \frac{r}{2} + \frac{r}{2} = r,$$

so F does indeed map $B(0,r)$ to itself. The same argument shows that for a sufficiently small $\varepsilon > 0$, F maps the closed ball $\overline{B(0,r-\varepsilon)}$ to itself. Also, for any x, x' in $B(0,r)$ we have

$$\big\|F(x) - F(x')\big\| = \big\|g(x') - g(x)\big\| \leq \frac{1}{2}\big\|x' - x\big\|$$

so F is a strict contraction on $B(0,r)$, and hence on the complete space $\overline{B(0,r-\varepsilon)}$ (see Thm. II.1.5.7 and Prop. II.1.5.5). By the contraction mapping theorem, F has a fixed point, i.e., there exists an x such that $x = y - g(x)$. But this means that $f(x) = y$, as desired. ☐

— Exercises —

Ex. II.6.6.1. Let $f : [a,b] \to [a,b]$ be a differentiable function of one variable such that $\big|f'(x)\big| \leq 1$ for all $x \in [a,b]$. Prove that f is a contraction. If in addition $\big|f'(x)\big| < 1$ for all $x \in [a,b]$ and f' is continuous, show that f is a strict contraction.

Proof. First, we show that $\big|f'(x)\big| \leq 1$ for all $x \in [a,b]$ implies f is a contraction. Let $x, y \in [a,b]$. If $x = y$, then we have

$$|f(x) - f(y)| = 0 \leq 0 = |x - y|.$$

So suppose that $x \neq y$. By mean value theorem (Corollary 10.2.9 in Analysis I) we know that

$$\begin{aligned} &\exists c \in (x,y) \cup (y,x) : \frac{f(x)-f(y)}{x-y} = f'(c) \\ \implies &\exists c \in (x,y) \cup (y,x) : \left|\frac{f(x)-f(y)}{x-y}\right| = \left|f'(c)\right| \leq 1 \\ \implies &|f(x)-f(y)| \leq |x-y|. \end{aligned}$$

Thus, by Def. II.6.6.1 f is a contraction.

Now we show that $\left|f'(x)\right| < 1$ for all $x \in [a,b]$ and f' is continuous implies f is a strict contraction. Since f' is continuous, by Proposition 9.6.7 in Analysis I we know that

$$\begin{aligned} &\exists x_{\min}, x_{\max} \in [a,b] : \forall x \in [a,b], f'(x_{\min}) \leq f'(x) \leq f'(x_{\max}) \\ \implies &\exists x_{\min}, x_{\max} \in [a,b] : \forall x \in [a,b], \left|f'(x)\right| \leq \max\left(\left|f'(x_{\min})\right|, \left|f'(x_{\max})\right|\right) < 1. \end{aligned}$$

Let $x, y \in [a,b]$. If $x = y$, then we have

$$\begin{aligned} |f(x)-f(y)| &= 0 \\ &\leq 0 \\ &= \max\left(\left|f'(x_{\min})\right|, \left|f'(x_{\max})\right|\right) \cdot 0 \\ &= \max\left(\left|f'(x_{\min})\right|, \left|f'(x_{\max})\right|\right) \cdot |x-y|. \end{aligned}$$

So suppose that $x \neq y$. By mean value theorem (Corollary 10.2.9) we know that

$$\begin{aligned} &\exists c \in (x,y) \cup (y,x) : \frac{f(x)-f(y)}{x-y} = f'(c) \\ \implies &\exists c \in (x,y) \cup (y,x) : \left|\frac{f(x)-f(y)}{x-y}\right| = \left|f'(c)\right| < \max\left(\left|f'(x_{\min})\right|, \left|f'(x_{\max})\right|\right) \\ \implies &|f(x)-f(y)| \leq \max\left(\left|f'(x_{\min})\right|, \left|f'(x_{\max})\right|\right) \cdot |x-y|. \end{aligned}$$

Thus, by Def. II.6.6.1 f is a strict contraction. □

Ex. II.6.6.2. Show that if $f : [a,b] \to \mathbb{R}$ is differentiable and is a contraction, then $\left|f'(x)\right| \leq 1$.

Proof. We have

$$\begin{aligned} &\forall x_0 \in [a,b], \forall x \in [a,b] \setminus \{x_0\}, |f(x)-f(x_0)| \leq |x-x_0| \qquad \text{(by Def. II.6.6.1)} \\ \implies &\forall x_0 \in [a,b], \forall x \in [a,b] \setminus \{x_0\}, \left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq 1 \\ \implies &\forall x_0 \in [a,b], \lim_{x \to x_0; x \in [a,b] \setminus \{x_0\}} \left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq 1 \\ \implies &\forall x_0 \in [a,b], f'(x_0) \leq 1. \end{aligned}$$

□

Ex. II.6.6.3. Give an example of a function $f : [a, b] \to \mathbb{R}$ which is continuously differentiable and such that $|f(x) - f(y)| < |x - y|$ for all distinct $x, y \in [a, b]$, but such that $\left|f'(x)\right| = 1$ for at least one value of $x \in [a, b]$.

Proof. Let $f : [0, 0.5] \to \mathbb{R}$ be the function

$$\forall x \in [0, 0.5], f(x) = x^2.$$

Let $x, y \in [0, 0.5]$ such that $x \neq y$. Since $x \neq y$, we know that $x + y < 1$. Then we have

$$\begin{aligned} |f(x) - f(y)| &= \left|x^2 - y^2\right| \\ &= |(x - y)(x + y)| \\ &= |x - y|(x + y) \\ &< |x - y|. && (x + y < 1) \end{aligned}$$

Since $f'(x) = 2x$, we know that $\left|f'(0.5)\right| = |1| = 1$. $\square$

Ex. II.6.6.4. Given an example of a function $f : [a, b] \to \mathbb{R}$ which is a strict contraction but which is not differentiable for at least one point x in $[a, b]$.

Proof. Let $f : [0, 1] \to \mathbb{R}$ be the function

$$\forall x \in [0, 1], f(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0.5, 1] \\ \frac{x}{3} & \text{if } x \in [0, 0.5) \end{cases}.$$

Observe that

$$\lim_{x \to 0.5+} f(x) = \frac{0.5}{2} \neq \frac{0.5}{3} = \lim_{x \to 0.5-} f(x).$$

Thus, f is not continuous at 0.5 and by Proposition 10.1.10 in Analysis I f is not differentiable at 0.5. Since

$$\forall x, y \in [0, 1], \begin{cases} |f(x) - f(y)| = \frac{1}{2}|x - y| & \text{if } x, y \in [0.5, 1] \\ |f(x) - f(y)| = \frac{1}{3}|x - y| & \text{if } x, y \in [0, 0.5) \\ |f(x) - f(y)| = \frac{x}{2} - \frac{y}{3} & \text{if } x \in [0.5, 1] \land y \in [0, 0.5) \\ |f(x) - f(y)| = \frac{y}{2} - \frac{x}{3} & \text{if } x \in [0, 0.5) \land y \in [0.5, 1] \end{cases}$$

$$\implies \begin{cases} |f(x) - f(y)| \leq \frac{1}{2}|x - y| & \text{if } x, y \in [0.5, 1] \\ |f(x) - f(y)| = \frac{1}{3}|x - y| \leq \frac{1}{2}|x - y| & \text{if } x, y \in [0, 0.5) \\ |f(x) - f(y)| < \frac{1}{3}(x - y) \leq \frac{1}{2}|x - y| & \text{if } x \in [0.5, 1] \land y \in [0, 0.5) \\ |f(x) - f(y)| < \frac{1}{3}(y - x) \leq \frac{1}{2}|x - y| & \text{if } x \in [0, 0.5) \land y \in [0.5, 1] \end{cases}$$

$$\implies |f(x) - f(y)| \leq \frac{1}{2}|x - y|,$$

by Def. II.6.6.1 we know that f is a strict contraction. □

Ex. II.6.6.5. Verify the claims in E.g. II.6.6.2.

Proof. See E.g. II.6.6.2. □

Ex. II.6.6.6. Show that every contraction on a metric space X is necessarily continuous.

Proof. Let (X, d) be a metric space, let $f : X \to X$ be a contraction of X and let $x_0 \in X$. We have

$$\begin{aligned} &\forall \varepsilon \in \mathbb{R}^+, \forall x \in X, d(x, x_0) < \varepsilon \\ \implies &d\big(f(x), f(x_0)\big) \leq d(x, x_0) < \varepsilon. && \text{(by Def. II.6.6.1)} \end{aligned}$$

By setting $\delta = \varepsilon$ we see that

$$\forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+ : \forall x \in X, d(x, x_0) < \delta \implies d\big(f(x), f(x_0)\big) < \varepsilon.$$

Since x_0 was arbitrary, by Def. II.2.1.1 this means f is continuous on X from (X, d) to (X, d). □

Ex. II.6.6.7. Prove Thm. II.6.6.4.

Proof. See Thm. II.6.6.4. □

Ex. II.6.6.8. Let (X, d) be a complete metric space, and let $f : X \to X$ and $g : X \to X$ be two strict contractions on X with contraction coefficients c and c' respectively. From Thm. II.6.6.4 we know that f has some fixed point x_0, and g has some fixed point y_0. Suppose we know that there is an $\varepsilon > 0$ such that $d\big(f(x), g(x)\big) \leq \varepsilon$ for all $x \in X$ (i.e., f and g are within ε of each other in the uniform metric). Show that $d(x_0, y_0) \leq \varepsilon / \big(1 - \min(c, c')\big)$. Thus, nearby contractions have nearby fixed points.

Proof. We have

$$\begin{aligned} d(x_0, y_0) &= d\big(f(x_0), g(y_0)\big) && \text{(by Def. II.6.6.3)} \\ &\leq d\big(f(x_0), g(x_0)\big) + d\big(g(x_0), g(y_0)\big) && \text{(by Def. II.1.1.2(d))} \\ &\leq \varepsilon + d\big(g(x_0), g(y_0)\big) && \text{(by hypothesis)} \\ &\leq \varepsilon + c' d(x_0, y_0) && \text{(by Def. II.6.6.1)} \end{aligned}$$

and

$$\begin{aligned} d(x_0, y_0) &= d\big(f(x_0), g(y_0)\big) && \text{(by Def. II.6.6.3)} \\ &\leq d\big(f(x_0), f(y_0)\big) + d\big(f(y_0), g(y_0)\big) && \text{(by Def. II.1.1.2(d))} \end{aligned}$$

$$
\begin{aligned}
&\leq d\big(f(x_0), f(y_0)\big) + \varepsilon && \text{(by hypothesis)} \\
&\leq cd(x_0, y_0) + \varepsilon. && \text{(by Def. II.6.6.1)}
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\begin{cases} d(x_0, y_0) \leq \varepsilon + \max(c, c') \cdot d(x_0, y_0) \\ d(x_0, y_0) \leq \varepsilon + \min(c, c') \cdot d(x_0, y_0) \end{cases} \\
\implies &\begin{cases} \big(1 - \max(c, c')\big) d(x_0, y_0) \leq \varepsilon \\ \big(1 - \min(c, c')\big) d(x_0, y_0) \leq \varepsilon \end{cases} \\
\implies &\begin{cases} d(x_0, y_0) \leq \dfrac{\varepsilon}{1 - \max(c, c')} \\ d(x_0, y_0) \leq \dfrac{\varepsilon}{1 - \min(c, c')} \end{cases}.
\end{aligned}
$$

□

II.6.7 The inverse function theorem in several variable calculus

Note. We recall the inverse function theorem in single variable calculus (Theorem 10.4.2 in Analysis I), which asserts that if a function $f : \mathbb{R} \to \mathbb{R}$ is invertible, differentiable, and $f'(x_0)$ is non-zero, then f^{-1} is differentiable at $f(x_0)$, and

$$(f^{-1})'\big(f(x_0)\big) = \frac{1}{f'(x_0)}.$$

In fact, one can say something even when f' is not invertible, as long as we know that f is *continuously* differentiable. If $f'(x_0)$ is non-zero, then $f'(x_0)$ must be either strictly positive or strictly negative, which implies (since we are assuming f' to be continuous) that $f'(x)$ is either strictly positive for x near x_0, or strictly negative for x near x_0. In particular, f must be either strictly increasing near x_0, or strictly decreasing near x_0. In either case, f will become invertible if we restrict the domain and codomain of f to be sufficiently close to x_0 and to $f(x_0)$ respectively. (The technical terminology for this is that f is *locally invertible near* x_0.)

Lem. II.6.7.1. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation which is also invertible. Then the inverse transformation $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is also linear.

Proof. Let $x, y \in \mathbb{R}^n$ and let $c \in \mathbb{R}$. We have

$$
\begin{aligned}
T^{-1}(x + y) &= T^{-1}\Big(T\big(T^{-1}(x)\big) + T\big(T^{-1}(y)\big)\Big) \\
&= T^{-1}\Big(T\big(T^{-1}(x) + T^{-1}(y)\big)\Big) && \text{(by Def. II.6.1.6)} \\
&= T^{-1}(x) + T^{-1}(y)
\end{aligned}
$$

and

$$\begin{aligned} T^{-1}(cx) &= T^{-1}\Big(cT\big(T^{-1}(x)\big)\Big) \\ &= T^{-1}\Big(T\big(cT^{-1}(x)\big)\Big) && \text{(by Def. II.6.1.6)} \\ &= cT^{-1}(x). \end{aligned}$$

Thus, by Def. II.6.1.6 T^{-1} is a linear transformation. □

Thm. II.6.7.2 (Inverse function theorem)**.** Let E be an open subset of $\mathbb{R}^n$, and let $f : E \to \mathbb{R}^n$ be a function which is continuously differentiable on E. Suppose $x_0 \in E$ is such that the linear transformation $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then there exists an open set U in E containing x_0, and an open set V in $\mathbb{R}^n$ containing $f(x_0)$, such that f is a bijection from U to V. In particular, there is an inverse map $f^{-1} : V \to U$. Furthermore, this inverse map is differentiable at $f(x_0)$, and

$$(f^{-1})'\big(f(x_0)\big) = \big(f'(x_0)\big)^{-1}.$$

Proof. We first observe that once we know the inverse map f^{-1} is differentiable, the formula $(f^{-1})'\big(f(x_0)\big) = \big(f'(x_0)\big)^{-1}$ is automatic. This comes from starting with the identity

$$I = f^{-1} \circ f$$

on U, where $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map $I(x) := x$, and then differentiating both sides using the chain rule at x_0 to obtain

$$I'(x_0) = (f^{-1})'\big(f(x_0)\big) \circ f'(x_0).$$

Since $I'(x_0) = I$, we thus have $(f^{-1})'\big(f(x_0)\big) = \big(f'(x_0)\big)^{-1}$ as desired.

We remark that this argument shows that if $f'(x_0)$ is *not* invertible, then there is no way that an inverse f^{-1} can exist and be differentiable at $f(x_0)$.

Next, we observe that it suffices to prove the theorem under the additional assumption $f(x_0) = 0$. The general case then follows from the special case by replacing f by a new function $\tilde{f}(x) := f(x) - f(x_0)$ and then applying the special case to $\tilde{f}$ (note that V will have to shift by $f(x_0)$). Note that if $V_f = \{y \in \mathbb{R}^n : y - f(x_0) \in V\}$, then

$$\begin{cases} \tilde{f} : U \to V \\ \tilde{f}^{-1} : V \to U \end{cases} \implies \begin{cases} f : U \to V_f \\ f^{-1} : V_f \to U \end{cases}$$

(one can show that f is bijective using proof by contradiction) and thus

$$\begin{aligned} &\forall x \in U, f(x) = y \\ \implies & f^{-1}(y) = x = \tilde{f}^{-1}\big(\tilde{f}(x)\big) = \tilde{f}^{-1}\big(f(x) - f(x_0)\big) = \tilde{f}^{-1}\big(y - f(x_0)\big). \end{aligned}$$

Henceforth we will always assume $f(x_0) = 0$.

In a similar manner, one can make the assumption $x_0 = 0$. The general case then follows from this case by replacing f by a new function $\tilde{f}(x) := f(x+x_0)$ and applying the special case to $\tilde{f}$ (note that E and U will have to shift by x_0). Note that if $U_f = \{x \in E : x - x_0 \in U\}$, then

$$\begin{cases} \tilde{f} : U \to V \\ \tilde{f}^{-1} : V \to U \end{cases} \implies \begin{cases} f : U_f \to V \\ f^{-1} : V \to U_f \end{cases}$$

(one can show that f is bijective using proof by contradiction) and thus

$$\begin{aligned} &\forall x \in U, \tilde{f}(x) = f(x + x_0) = y \\ \implies &f^{-1}(y) = x + x_0 = \tilde{f}^{-1}\big(\tilde{f}(x)\big) + x_0 = \tilde{f}^{-1}\big(f(x + x_0)\big) + x_0 = \tilde{f}^{-1}(y) + x_0. \end{aligned}$$

Henceforth we will always assume $x_0 = 0$. Thus, we now have that $f(0) = 0$ and that $f'(0)$ is invertible.

Finally, one can assume that $f'(0) = I$, where $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity transformation $I(x) = x$. The general case then follows from this case by replacing f with a new function $\tilde{f} : E \to \mathbb{R}^n$ defined by $\tilde{f}(x) := \big(f'(0)\big)^{-1}\big(f(x)\big)$, and applying the special case to this case. Note from Lem. II.6.7.1 that $\big(f'(0)\big)^{-1}$ is a linear transformation. In particular, we note that $\tilde{f}(0) = 0$ and that

$$\begin{aligned} \tilde{f}'(0) &= \Big(\big(f'(0)\big)^{-1}\Big)'\big(f(0)\big) \circ f'(0) && \text{(by Thm. II.6.4.1)} \\ &= \big(f'(0)\big)^{-1} \circ f'(0) && \text{(by Ex. II.6.4.1)} \\ &= I, \end{aligned}$$

so by the special case of the inverse function theorem we know that there exists an open set U' containing 0, and an open set V' containing 0, such that $\tilde{f}$ is a bijection from U' to V', and that $\tilde{f}^{-1} : V' \to U'$ is differentiable at 0 with derivative I. But we have

$$\begin{aligned} &\tilde{f}(x) = \big(f'(0)\big)^{-1}\big(f(x)\big) \\ \implies &f'(0)\big(\tilde{f}(x)\big) = f'(0)\Big(\big(f'(0)\big)^{-1}\big(f(x)\big)\Big) \\ \implies &f(x) = f'(0)\big(\tilde{f}(x)\big), \end{aligned}$$

and hence f is a bijection from U' to $f'(0)(V')$ (note that $f'(0)$ is also a bijection). Since $f'(0)$ and its inverse are both continuous, $f'(0)(V')$ is open (see Thm. II.2.1.5(a)(c)), and it certainly contains 0. Now consider the inverse function $f^{-1} : f'(0)(V') \to U'$. Note that

$$\begin{aligned} &f = f'(0) \circ \tilde{f} \\ \implies &f^{-1} = \tilde{f}^{-1} \circ \big(f'(0)\big)^{-1} \\ \implies &\forall y \in f'(0)(V'), f^{-1}(y) = \tilde{f}^{-1}\Big(\big(f'(0)\big)^{-1}(y)\Big). \end{aligned}$$

In particular, we see that f^{-1} is differentiable at 0.

So all we have to do now is prove the inverse function theorem in the special case, when $x_0 = 0$, $f(x_0) = 0$, and $f'(x_0) = I$. Let $g : E \to \mathbb{R}^n$ denote the function $g(x) = f(x) - x$. Then $g(0) = 0$ and $g'(0) = 0$. In particular

$$\frac{\partial g}{\partial x_j}(0) = 0$$

for $j = 1, \ldots, n$. Since g is continuously differentiable, there thus exists a ball $B(0, r)$ in E such that

$$\left\| \frac{\partial g}{\partial x_j}(x) \right\| \leq \frac{1}{2n^2}$$

for all $x \in B(0, r)$. (There is nothing particularly special about $\frac{1}{2n^2}$, we just need a nice small number here.) In particular, for any $x \in B(0, r)$ and $v = (v_1, \ldots, v_n)$ we have

$$\begin{aligned}
\|D_v g(x)\| &= \left\| \sum_{j=1}^{n} v_j \frac{\partial g}{\partial x_j}(x) \right\| \\
&\leq \sum_{j=1}^{n} |v_j| \left\| \frac{\partial g}{\partial x_j}(x) \right\| \\
&\leq \sum_{j=1}^{n} \|v\| \frac{1}{2n^2} \\
&\leq \frac{1}{2n} \|v\|.
\end{aligned}$$

But now for any $x, y \in B(0, r)$, we have by the fundamental theorem of calculus

$$\begin{aligned}
g(y) - g(x) &= g\big(x + t(y - x)\big)\Big|_{t=0}^{t=1} \\
&= \int_0^1 \frac{d}{dt} g\big(x + t(y - x)\big)\, dt \\
&= \int_0^1 D_{y-x} g\big(x + t(y - x)\big)\, dt
\end{aligned}$$

where the integral of a vector-valued function is defined by integrating each component separately. By the previous remark, the vectors $D_{y-x} g\big(x + t(y - x)\big)$ have a magnitude of at most $\frac{1}{2n}\|y - x\|$. Thus, every component of these vectors has magnitude at most $\frac{1}{2n}\|y - x\|$. Thus, every component of $g(y) - g(x)$ has magnitude at most $\frac{1}{2n}\|y - x\|$, and hence $g(y) - g(x)$ itself has magnitude at most $\frac{1}{2}\|y - x\|$ (actually, it will be substantially less than this, but this bound will be enough for our purposes). In other words, g is a contraction. By Lem. II.6.6.6, the map $f = g + I$ is thus one-to-one on $B(0, r)$, and the image $f\big(B(0, r)\big)$ contains $B(0, \frac{r}{2})$. In particular, we have an inverse map $f^{-1} : B(0, \frac{r}{2}) \to B(0, r)$ defined on $B(0, \frac{r}{2})$.

Applying the contraction bound with $y = 0$ we obtain, in particular, that

$$\|g(x)\| \leq \frac{1}{2}\|x\|$$

for all $x \in B(0, r)$, and so by the triangle inequality

$$\frac{1}{2}\|x\| \leq \|f(x)\| \leq \frac{3}{2}\|x\|$$

for all $x \in B(0, r)$.

Now we set $V := B(0, \frac{r}{2})$ and $U := f^{-1}(V) \cap B(0, r)$. Then by construction f is a bijection from U to V. V is clearly open, and U is also open since f is continuous. (Notice that if a set is open relative to $B(0, r)$, then it is open in $\mathbb{R}^n$ as well.) Now we want to show that $f^{-1} : V \to U$ is differentiable at 0 with derivative $I^{-1} = I$. In other words, we wish to show that

$$\lim_{x \to 0; x \in V \setminus \{0\}} \frac{\big\|f^{-1}(x) - f^{-1}(0) - I(x - 0)\big\|}{\|x\|} = 0.$$

Since $f(0) = 0$, we have $f^{-1}(0) = 0$, and the above simplifies to

$$\lim_{x \to 0; x \in V \setminus \{0\}} \frac{\big\|f^{-1}(x) - x\big\|}{\|x\|} = 0.$$

Let $(x_n)_{n=1}^{\infty}$ be any sequence in $V \setminus \{0\}$ that converges to 0. By Prop. II.3.1.5(b), it suffices to show that

$$\lim_{n \to \infty} \frac{\big\|f^{-1}(x_n) - x_n\big\|}{\|x_n\|} = 0.$$

Write $y_n := f^{-1}(x_n)$. Then $y_n \in B(0, r)$ and $x_n = f(y_n)$. In particular, we have

$$\frac{1}{2}\|y_n\| \leq \|x_n\| \leq \frac{3}{2}\|y_n\|$$

and so since $\|x_n\|$ goes to 0, $\|y_n\|$ goes to 0 also, and their ratio remains bounded. It will thus suffice to show that

$$\lim_{n \to \infty} \frac{\|y_n - f(y_n)\|}{\|y_n\|} = 0.$$

But since y_n is going to 0, and f is differentiable at 0, we have

$$\lim_{n \to \infty} \frac{\|f(y_n) - f(0) - f'(0)(y_n - 0)\|}{\|y_n\|} = 0$$

as desired (since $f(0) = 0$ and $f'(0) = I$). $\square$

Note. The inverse function theorem gives a useful criterion for when a function is (locally) invertible at a point x_0 - all we need is for its derivative $f'(x_0)$ to be invertible (and then we even get further information, for instance we can compute the derivative of f^{-1} at $f(x_0)$). Of course, this begs the question of how one can tell whether the linear transformation $f'(x_0)$ is invertible or not. Recall that we have $f'(x_0) = L_{Df(x_0)}$, so by Lem. II.6.1.13 and II.6.1.16 we see that the linear transformation $f'(x_0)$ is invertible iff the matrix $Df(x_0)$ is. There are many ways to check whether a matrix such as $Df(x_0)$ is invertible; for instance, one can use determinants, or alternatively Gaussian elimination methods. We will not pursue this matter here, but refer the reader to any linear algebra text.

Note. If $f'(x_0)$ exists but is non-invertible, then the inverse function theorem does not apply. In such a situation it is not possible for f^{-1} to exist and be differentiable at $f(x_0)$; this was remarked in the proof of Thm. II.6.7.2. But it is still possible for f to be invertible. For instance, the single-variable function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is invertible despite $f'(0)$ not being invertible.

— Exercises —

Ex. II.6.7.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) := x + x^2 \sin(1/x^4)$ for $x \neq 0$ and $f(0) := 0$. Show that f is differentiable and $f'(0) = 1$, but f is not increasing on any open set containing 0.

Proof. Let $x \in \mathbb{R} \setminus \{0\}$. Then we have

$$\begin{aligned}
&(x \mapsto x^{-4})' = (-4)x^{-5} \\
\implies &\big(x \mapsto \sin(x^{-4})\big)' = (-4)x^{-5} \cos(x^{-4}) \\
\implies &\big(x \mapsto x^2 \sin(x^{-4})\big)' = 2x \sin(x^{-4}) - 4x^{-3} \cos(x^{-4}) \\
\implies &f'(x) = 1 + 2x \sin(x^{-4}) - 4x^{-3} \cos(x^{-4}).
\end{aligned}$$

Observe that

$$\forall x \in \mathbb{R} \setminus \{0\}, \big|x \sin(x^{-4})\big| = |x|\big|\sin(x^{-4})\big| \leq |x| \cdot 1.$$

Thus, we have

$$\lim_{x \to 0; x \in \mathbb{R}\setminus\{0\}} x = 0 \implies \lim_{x \to 0; x \in \mathbb{R}\setminus\{0\}} x \sin(x^{-4}) = 0$$

and by squeeze test

$$\begin{aligned}
&\lim_{x \to 0; x \in \mathbb{R}\setminus\{0\}} \frac{f(x) - f(0)}{x - 0} \\
= &\lim_{x \to 0; x \in \mathbb{R}\setminus\{0\}} \frac{x + x^2 \sin(x^{-4})}{x} \\
= &\lim_{x \to 0; x \in \mathbb{R}\setminus\{0\}} 1 + x \sin(x^{-4})
\end{aligned}$$

$$= 1.$$

We conclude that f is differentiable on $\mathbb{R}$ and $f'(0) = 1$.

Let E be an open set in $\mathbb{R}$ containing 0. By Prop. II.1.2.15(a) we know that

$$\exists r \in \mathbb{R}^+ : B(0,r) \subseteq E \implies (-r,r) \subseteq E.$$

Fix such r. Since

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} = 0 \\
\implies &\lim_{n\to\infty} \frac{1}{2n\pi} = 0 \\
\implies &\lim_{n\to\infty} \sqrt[4]{\frac{1}{2n\pi}} = 0 \\
\implies &\exists N \in \mathbb{Z}^+ : \forall n \geq N, \sqrt[4]{\frac{1}{2n\pi}} < r,
\end{aligned}$$

by fixing such N we know that

$$\begin{aligned}
&\sqrt[4]{\frac{1}{2N\pi}} \in (-r,r) \subseteq E \\
\implies f'\left(\sqrt[4]{\frac{1}{2N\pi}}\right) &= 1 + 2\sqrt[4]{\frac{1}{2N\pi}}\sin(2N\pi) - 4(2N\pi)^{\frac{3}{4}}\cos(2N\pi) \\
&= 1 - 4(2N\pi)^{\frac{3}{4}} \leq 1 - 4 = -3 < 0.
\end{aligned}$$

Thus, f is not increasing at $\sqrt[4]{\frac{1}{2N\pi}}$, and not increasing on E. □

Ex. II.6.7.2. Prove Lem. II.6.7.1.

Proof. See Lem. II.6.7.1. □

Ex. II.6.7.3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function such that $f'(x)$ is an invertible linear transformation for every $x \in \mathbb{R}^n$. Show that whenever V is an open set in $\mathbb{R}^n$, that $f(V)$ is also open.

Proof. Let $d = d_{l^2}|_{\mathbb{R}^n \times \mathbb{R}^n}$, let V be an open set in $(\mathbb{R}^n, d)$ and let $y \in f(V)$. We know that there exists a $x \in V$ such that $f(x) = y$. Since f is continuously differentiable on $\mathbb{R}^n$ and $f'(x)$ is invertible, by inverse function theorem (Thm. II.6.7.2) we know that

$$\exists U, W \subseteq \mathbb{R}^n : \begin{cases} U, W \text{ are open sets in } (\mathbb{R}^n, d) \\ x \in U \\ y \in W \\ f : U \to W \text{ is a bijection} \\ (f^{-1})'(y) = (f^{-1})'\big(f(x)\big) = \big(f'(x)\big)^{-1} \end{cases}.$$

Fix such U, W. Since V is open in $(\mathbb{R}^n, d)$, we know that there exists a $r \in \mathbb{R}^+$ such that $B_{(\mathbb{R}^n,d)}(x,r) \subseteq V$. Fix such r. By Prop. II.1.2.15(f) we know that $U \cap B_{(\mathbb{R}^n,d)}(x,r)$ is open in $(\mathbb{R}^n, d)$. Since $U \cap B_{(\mathbb{R}^n,d)}(x,r) \subseteq U$, we know that f is a bijection from $U \cap B_{(\mathbb{R}^n,d)}(x,r)$ to $f\big(U \cap B_{(\mathbb{R}^n,d)}(x,r)\big)$. By hypotheses we know that f^{-1} is differentiable on $f\big(U \cap B_{(\mathbb{R}^n,d)}(x,r)\big)$, by Ex. II.6.4.2 we know that f^{-1} is continuous $f\big(U \cap B_{(\mathbb{R}^n,d)}(x,r)\big)$. Thus, by Thm. II.2.1.5(a)(c) we know that $f\big(U \cap B_{(\mathbb{R}^n,d)}(x,r)\big)$ is open in $(\mathbb{R}, d)$. Now we have

$$
\begin{aligned}
& x \in U \cap B_{(\mathbb{R}^n,d)}(x,r)\big) \\
\implies & y \in f\big(U \cap B_{(\mathbb{R}^n,d)}(x,r)\big) \\
\implies & \exists r' \in \mathbb{R}^+ : B_{(\mathbb{R}^n,d)}(y,r') \subseteq f\big(U \cap B_{(\mathbb{R}^n,d)}(x,r)\big) && \text{(by Prop. II.1.2.} \\
\implies & \exists r' \in \mathbb{R}^+ : B_{(\mathbb{R}^n,d)}(y,r') \subseteq f(V). && (U \cap B_{(\mathbb{R}^n,d)}(x,r) \subseteq V)
\end{aligned}
$$

Since y was arbitrary, we know that $f(V)$ is open in $(\mathbb{R}^n, d)$. Since V was arbitrary, we know that if V is an open set in $(\mathbb{R}^n, d)$, then $f(V)$ is also an open set in $(\mathbb{R}^n, d)$. □

Ex. II.6.7.4. Let the notation and hypotheses be as in Thm. II.6.7.2. Show that, after shrinking the open sets U, V if necessary (while still having $x_0 \in U$, $f(x_0) \in V$ of course), the derivative map $f'(x)$ is invertible for all $x \in U$, and that the inverse map f^{-1} is differentiable at every point of V with $(f^{-1})'\big(f(x)\big) = \big(f'(x)\big)^{-1}$ for all $x \in U$. Finally, show that f^{-1} is continuously differentiable on V.

II.6.8 The implicit function theorem

Note. Any function $g : \mathbb{R}^n \to \mathbb{R}$ gives rise to a graph $\big\{\big(x, g(x)\big) : x \in \mathbb{R}^n\big\}$ in $\mathbb{R}^{n+1}$, which in general looks like some sort of n-dimensional surface in $\mathbb{R}^{n+1}$ (the technical term for this is a *hypersurface*). Conversely, one may ask which hypersurfaces are actually graphs of some function, and whether that function is continuous or differentiable.

Note. If the hypersurface is given geometrically, then one can again invoke the vertical line test to work out whether it is a graph or not. But what if the hypersurface is given algebraically, or more generally, the hypersurface is given as some function? In this case, it is still possible to say whether the hypersurface is a graph, locally at least, by means of the *implicit function theorem.*

Thm. II.6.8.1 (Implicit function theorem)**.** Let E be an open subset of $\mathbb{R}^n$, let $f : E \to \mathbb{R}$ be continuously differentiable, and let $y = (y_1, \dots, y_n)$ be a point in E such that $f(y) = 0$ and $\dfrac{\partial f}{\partial x_n}(y) \neq 0$. Then there exists an open subset U of $\mathbb{R}^{n-1}$ containing $(y_1, \dots, y_{n-1})$, an open subset V of E containing y, and a function $g : U \to \mathbb{R}$ such that $g(y_1, \dots, y_{n-1}) = y_n$, and

$$\{(x_1, \dots, x_n) \in V : f(x_1, \dots, x_n) = 0\}$$

$$= \{(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})) : (x_1, \dots, x_{n-1}) \in U\}.$$

In other words, the set $\{x \in V : f(x) = 0\}$ is a graph of a function over U. Moreover, g is differentiable at $(y_1, \dots, y_{n-1})$, and we have

$$\frac{\partial g}{\partial x_j}(y_1, \dots, y_{n-1}) = -\frac{\partial f}{\partial x_j}(y) / \frac{\partial f}{\partial x_n}(y) \tag{6.1}$$

for all $1 \leq j \leq n-1$.

Proof. This theorem looks somewhat fearsome, but actually it is a fairly quick consequence of the inverse function theorem. Let $F : E \to \mathbb{R}^n$ be the function

$$F(x_1, \dots, x_n) := (x_1, \dots, x_{n-1}, f(x_1, \dots, x_n)).$$

This function is continuously differentiable. Also note that

$$F(y) = (y_1, \dots, y_{n-1}, 0)$$

and

$$DF(y) = \left(\frac{\partial F}{\partial x_1}(y)^\top, \frac{\partial F}{\partial x_2}(y)^\top, \dots, \frac{\partial F}{\partial x_n}(y)^\top\right)$$
$$= \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ \frac{\partial f}{\partial x_1}(y) & \frac{\partial f}{\partial x_2}(y) & \dots & \frac{\partial f}{\partial x_{n-1}}(y) & \frac{\partial f}{\partial x_n}(y) \end{pmatrix}.$$

Since $\frac{\partial f}{\partial x_n}(y)$ is assumed by hypothesis to be non-zero, this matrix is invertible; this can be seen either by computing the determinant, or using row reduction, or by computing the inverse explicitly, which is

$$DF(y)^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -\frac{\partial f}{\partial x_1}(y)/a & -\frac{\partial f}{\partial x_2}(y)/a & \dots & -\frac{\partial f}{\partial x_{n-1}}(y)/a & 1/a \end{pmatrix},$$

where we have written $a = \frac{\partial f}{\partial x_n}(y)$ for short. Thus, the inverse function theorem (Thm. II.6.7.2) applies, and we can find an open set V in E containing y, and an open set W in $\mathbb{R}^n$ containing $F(y) = (y_1, \dots, y_{n-1}, 0)$, such that F is a bijection from V to W, and that F^{-1} is differentiable at $(y_1, \dots, y_{n-1}, 0)$.

Let us write F^{-1} in co-ordinates as

$$F^{-1}(x) = \big(h_1(x), h_2(x), \dots, h_n(x)\big)$$

where $x \in W$. Since $F\big(F^{-1}(x)\big) = x$, we have $h_j(x_1, \dots, x_n) = x_j$ for all $1 \le j \le n-1$ and $x \in W$, and

$$f\big(x_1, \dots, x_{n-1}, h_n(x_1, \dots, x_n)\big) = x_n.$$

Also, h_n is differentiable at $(y_1, \dots, y_{n-1}, 0)$ since F^{-1} is.

Now we set $U := \big\{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : (x_1, \dots, x_{n-1}, 0) \in W\big\}$. Note that U is open and contains $(y_1, \dots, y_{n-1})$. Now we define $g : U \to \mathbb{R}$ by $g(x_1, \dots, x_{n-1}) := h_n(x_1, \dots, x_{n-1}, 0)$. Then g is differentiable at $(y_1, \dots, y_{n-1})$. Now we prove that

$$\begin{aligned} &\{(x_1, \dots, x_n) \in V : f(x_1, \dots, x_n) = 0\} \\ &= \big\{\big(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})\big) : (x_1, \dots, x_{n-1}) \in U\big\}. \end{aligned}$$

First, suppose that $(x_1, \dots, x_n) \in V$ and $f(x_1, \dots, x_n) = 0$. Then we have

$$F(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, 0),$$

which lies in W. Thus, $(x_1, \dots, x_{n-1})$ lies in U. Applying F^{-1}, we see that

$$(x_1, \dots, x_n) = F^{-1}(x_1, \dots, x_{n-1}, 0).$$

In particular, $x_n = h_n(x_1, \dots, x_{n-1}, 0)$, and hence $x_n = g(x_1, \dots, x_{n-1})$. Thus, every element of the left-hand set lies in the right-hand set. The reverse inclusion comes by reversing all the above steps and is left to the reader.

Finally, we show the formula for the partial derivatives of g. From the preceding discussion we have

$$f\big(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})\big) = 0$$

for all $(x_1, \dots, x_{n-1}) \in U$. Since g is differentiable at $(y_1, \dots, y_{n-1})$, and f is differentiable at $\big(y_1, \dots, y_{n-1}, g(y_1, \dots, y_{n-1})\big) = y$, we may use the chain rule, differentiating in x_j, to obtain

$$\frac{\partial f}{\partial x_j}(y) + \frac{\partial f}{\partial x_n}(y)\frac{\partial g}{\partial x_j}(y_1, \dots, y_{n-1}) = 0$$

and the claim follows by simple algebra. □

Rmk. II.6.8.2. Eq. (6.1) is sometimes derived using *implicit differentiation*. Basically, the point is that if you know that

$$f(x_1, \dots, x_n) = 0$$

then (as long as $\dfrac{\partial f}{\partial x_n} \neq 0$) the variable x_n is "implicitly" defined in terms of the other $n-1$ variables, and one can differentiate the above identity in, say, the x_j direction using the chain rule to obtain

$$\frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial x_j} = 0$$

which is Eq. (6.1) in disguise (we are using g to represent the implicit function defining x_n in terms of $x_1, \ldots, x_n$). Thus, the implicit function theorem allows one to define a dependence implicitly, by means of a constraint rather than by a direct formula of the form $x_n = g(x_1, \ldots, x_{n-1})$.

Note. In the implicit function theorem, if the derivative $\dfrac{\partial f}{\partial x_n}$ equals zero at some point, then it is unlikely that the set $\{x \in \mathbb{R}^n : f(x) = 0\}$ can be written as a graph of the x_n variable in terms of the other $n-1$ variables near that point. However, if some other derivative $\dfrac{\partial f}{\partial x_j}$ is non-zero, then it would be possible to write the x_j variable in terms of the other $n-1$ variables, by a variant of the implicit function theorem. Thus, as long as the gradient ∇f is not entirely zero, one can write this set $\{x \in \mathbb{R}^n : f(x) = 0\}$ as a graph of *some* variable x_j in terms of the other $n-1$ variables. (The circle $\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 1 = 0\right\}$ is a good example of this; it is not a graph of y in terms of x, or x in terms of y, but near every point it is one of the two. And this is because the gradient of $x^2 + y^2 - 1$ is never zero on the circle.) However, if ∇f does vanish at some point x_0, then we say that f has a *critical point* at x_0 and the behavior there is much more complicated. For instance, the set $\left\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 0\right\}$ has a critical point at $(0, 0)$ and there the set does not look like a graph of any sort (it is the union of two lines).

Rmk. II.6.8.3. Sets which look like graphs of continuous functions at every point have a name, they are called *manifolds*. Thus, $\{x \in \mathbb{R}^n : f(x) = 0\}$ will be a manifold if it contains no critical points of f. The theory of manifolds is very important in modern geometry (especially differential geometry and algebraic geometry), but we will not discuss it here as it is a graduate level topic.

— Exercises —

Ex. II.6.8.1. Let the notation and hypotheses be as in Thm. II.6.8.1. Show that, after shrinking the open sets U, V if necessary, that the function g becomes continuously differentiable on all of U, and Eq. (6.1) holds at all points of U.