The other Tamano's Theorem and Glicksberg's Theorem.

Friday, 4-27-'07

Fact 1 Let X, Y be Tychonoff spaces. If π_X is z-closed, then $X \times Y$ is C^*-embedded in $X \times \beta Y$.

Fact 2 Let X be a Tychonoff space. If X is pseudocompact, then every locally finite family of non-empty open subsets of X is finite.

Tamano's Theorem. Let X, Y be Tychonoff spaces. If $X \times Y$ is pseudocompact, then the projection map $\pi_X : X \times Y \to X$, is z-closed.

Proof. Let Z be a zero-set in $X \times Y$. Suppose that $\pi_X[Z]$ is not closed in X. Let $p \in \overline{\pi_X[Z]}^X \setminus \pi_X[Z]$.

Since Z is a zero-set in $X \times Y$, $Z = f^{\leftarrow}(0)$ for some $f \in C^*(X \times Y)$. Define $h : X \times Y \to \mathbb{R}$ such that $h(x, y) = \frac{f(x,y)}{f(p,y)}$. So, $h\left[\{p\} \times Y\right] \subseteq \{1\}$ and $Z = h \leftarrow (0)$.

We will show that there are open sets U_n, V_n in X, and W_n in Y for $n < \omega$ such that for $m < \omega$, the following hold:

1. $p \in U_m$
2. $(V_m \times W_m) \cap Z \neq \emptyset$
3. $h\left[V_m \times W_m\right] \subseteq [0, \frac{1}{3})$
4. $h\left[U_m \times W_m\right] \subseteq (\frac{2}{3}, 1]$
5. $U_{m+1} \cup V_{m+1} \subseteq U_m$

First, pick $(x_1, y_1) \in Z$ and open sets $U_1, V_1 \in \tau(X)$ and $W_1 \in \tau(Y)$ such that $p \in U_1, x_1 \in V_1, y_1 \in W_1$, and $h\left[V_1 \times W_1\right] \subseteq [0, \frac{1}{3})$ and $h\left[U_1 \times W_1\right] \subseteq (\frac{2}{3}, 1]$. This can be done because h is continuous, $h(x_1, y_1) = 0$, and $h(p, y_1) = 1$.

Now, $U_1 \cap \pi_X[Z] \neq \emptyset$ because $x_1 \in U_1 \in \tau(X)$, and $x_1 \in \overline{\pi_X[Z]}^X$. So there is some $(x_2, y_2) \in Z$ such that $x_2 \in U_1$. Find open neighborhoods U_2 of p, V_2 of x_2, and W_2 of y_2 such that $h\left[V_2 \times W_2\right] \subseteq [0, \frac{1}{3}), h\left[U_2 \times W_2\right] \subseteq (\frac{2}{3}, 1]$, and $U_2 \cup V_2 \subseteq U_1$. Continue by induction.

The family $D = \{V_n \times W_n : n < \omega\}$ is pairwise disjoint because the V_n's are pairwise disjoint by our construction. If D is locally finite, then by **Fact 2**, D is finite. But D is infinite by our definition, so D cannot be locally finite. Then, there exists $(q, r) \in X \times Y$ with the property that for every neighborhood $R \times T$ of (q, r), $A = \{n \in \mathbb{N} : (V_n \times W_n) \cap (R \times T) \neq \emptyset\}$ is infinite.

On one hand, we have $(q, r) \in \overline{\bigcup\{V_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}$. Then,

$$h(q, r) \in h\left[\overline{\bigcup\{V_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}\right]$$

$$\subseteq \overline{h\left[\bigcup\{V_m \times W_m : m \in \mathbb{N}\}\right]}^{\mathbb{R}} \subseteq \overline{[0, \frac{1}{3})}^{\mathbb{R}} = [0, \frac{1}{3}].$$

On the other hand, if n and $n + k$ in A where $n, k \in \mathbb{N}$, then $V_{n+k} \subseteq U_{n+k-1} \subseteq \cdots \subseteq U_n$ by the way we constructed V_n's and U_n's. Since $(R \times T) \cap (V_{n+k} \times W_{n+k}) \neq \emptyset, (R \times T) \cap (U_n \times W_n) \neq \emptyset$ as well.

So, $(q, r) \in \overline{\bigcup\{U_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}$. Then,

$$h(q, r) \in h\left[\overline{\bigcup\{U_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}\right]$$

$$\subseteq \overline{h\left[\bigcup\{U_m \times W_m : m \in \mathbb{N}\}\right]}^{\mathbb{R}} \subseteq \overline{(\frac{2}{3}, 1]}^{\mathbb{R}} = [\frac{1}{3}, 1].$$

This is a contradiction, so $\pi_X[Z]$ must be closed in X.

Glicksberg's Theorem: Let $X \times Y$ be Tychonoff spaces. If $X \times Y$ is pseudocompact, then $\beta(X \times Y) = \beta X \times \beta Y$.

Proof. By **Tamano's Theorem**, the projection map $\pi_X : X \times Y \to X$ is z-closed. By **Fact 1**, $X \times Y$ is C^*-embedded in $X \times \beta Y$. Since X is pseudocompact and βY is compact, $X \times \beta Y$ is pseudocompact. Using **Tamano's Theorem**, **Fact 1** again, and by symmetry, $X \times \beta Y$ is C^*-embedded in $\beta X \times \beta Y$. By the transitivity of C^*-embedding, $X \times Y$ is C^*-embedded in $\beta X \times \beta Y$. I.e, $\beta(X \times Y) = \beta X \times \beta Y$.