FACT 1. If X is pseudocompact and Y is compact, then $X \times Y$ is pseudocompact.

Proof. Let $f : X \times Y \to \mathbf{R}$. As Y is compact, $f[\{x\} \times Y]$ is closed and bounded in $\mathbf{R}$ for all $x \in X$. We can define $g : X \to \mathbf{R}$ as

$$g(x) = max\{f(x, y) : y \in Y\}.$$

Fix $x_0 \in X$, we will show that g is continuous at x_0. Let $\epsilon > 0$.

By our definition of $g(x_0)$, there exists some $y_0 \in Y$ such that $g(x_0) = f(x_0, y_0)$. Let $r = f(x_0, y_0)$. Now, define the sets U_y's and V_y's as follows:

For each $y \in Y$:
If $f(x_0, y) \in (r - \epsilon, r + \epsilon)$, we can get $V_y \in \tau(Y)$ and $U_y \in \tau(X)$ such that $y_0 \notin V_y$ and $f[U_y \times V_y] \subseteq (r - \epsilon, r + \epsilon)$.
If $f(x_0, y) \notin (r - \epsilon, r + \epsilon)$, then combining with $f(x_0, y) \leq max\{f(x_0, y) : y \in Y\} = f(x_0, y_0) = r$, we must have $f(x_0, y) \leq r - \epsilon$. Hence, we can get $V_y \in \tau(Y)$ and $U_y \in \tau(X)$ such that $f[U_y \times V_y] \subseteq (-\infty, r)$.

The family $\{V_y : y \in Y\}$ as defined aboved is an open cover of Y. By compactness, there exists $\{V_i : 1 \leq i \leq n\} \subseteq \{V_y : y \in Y\}$ such that $\bigcup\{V_i : 1 \leq i \leq n\} = Y$. Corresponding to $\{V_i : 1 \leq i \leq n\}$, we have the set $\{U_i : 1 \leq i \leq n\}$. Let $U = \bigcap\{U_i : 1 \leq i \leq n\} \cap U_{y_0}$, now U is an open set containing x_0.

Pick any $x \in U$.

On one hand, we have $max\{f(x, y) : y \in Y\} < r + \epsilon$ because
$\{f(x, y) : y \in Y\} = \{f(x, y) : y \in \bigcup\{V_i : 1 \leq i \leq n\}\}$
$= \bigcup \{f[\{x\} \times V_i] : 1 \leq i \leq n\} \subseteq \bigcup\{f[U_i \times V_i] : 1 \leq i \leq n\} \subseteq (-\infty, r + \epsilon)$.

One the other hand, we have $max\{f(x, y) : y \in Y\} > r - \epsilon$. That is because $f(x, y_0) \in f[U \times \{y_0\}] \subseteq f[U_{y_0} \times \{y_0\}] \subseteq f[U_{y_0} \times V_{y_0}] \subseteq (r - \epsilon, r + \epsilon)$.

We have now $r - \epsilon < max\{f(x, y) : y \in Y\} < r + \epsilon$ for all $x \in U$. Hence, $g[U] \subseteq (r - \epsilon, r + \epsilon)$, so g is continuous on X. As X is pseudocompact, g must be bounded. Therefore, f must be bounded as well. Thus, $X \times Y$ is pseudocompact.