FACT 2. Let X be a pseudocompact space. Let $\tau > \omega$ be a regular cardinal and denote by $T(\tau)$ the space of all ordinal numbers less than τ. Then, $X \times T(\tau)$ is pseudocompact.

Proof. Let $f : X \times T(\tau) : \mathbf{R}$ be continuous.

For each $x \in X$, f is bounded on $\{x\} \times T(\tau)$ by Lemma 1.
There exists $\kappa_x < \tau$ such that f is constant on $\{x\} \times [\kappa_x, \tau)$.
As $cf(\tau) > \omega, \{\kappa_x : x \in X\}$ has sup, say $\beta = sup_{x \in X}\{\kappa_x : x \in X\}$. Now, $f\,[X \times [0, \kappa + 1]]$ is bounded because $X \times [0, \kappa + 1]$ is pseudocompact by FACT 1.

For $\alpha \geq \beta, f(x, \alpha) = f(x, \beta)$. Thus, $f\,[X \times [\beta, \tau)] = f\,[X \times \{\beta\}] \subseteq f\,[X \times [0, \beta + 1]]$ which is bounded.

Hence, $f\,[X \times T(\tau)]$ is bounded.

Lemma 1 For each $x \in X$, f is bounded on $\{x\} \times T(\tau)$.

Proof. Define $g : T(\tau) \to \mathbf{R}$ as

$$g(\gamma) = f(x', \gamma)$$

for all $\gamma \in T(\tau)$.

If f is unbounded, then there exists an $x' \in X$ such that $f\,[\{x'\} \times T(\tau)]$ is unbounded in $\mathbf{R}$. Thus, g is unbounded as well. We will define $\{\alpha_i, i < \omega\} \subseteq T(\tau)$ by induction:

`Step 1.` Since g is unbounded, we can find $\alpha_1 \in T(\tau)$ such that $g(\alpha_1) \geq 1$.

`Step N.` Since $[0, \alpha_{n-1}]$ is compact in $T(\tau)$ and g is continuous, $g\,[[0, \alpha_{n-1}]]$ must be bounded in $\mathbf{R}$. But since g is unbounded, $g\,[(\alpha_{n-1}, \tau)]$ must be unbounded in $\mathbf{R}$. So there exists $\alpha_n \in (\alpha_{n-1}, \tau)$ such that $g(\alpha_n) \geq n$.

Having defined $\alpha_i \in T(\tau)$ for all $i < \omega$, let $\beta = sup\{\alpha_i : i < \omega\}$. Such β exists in $T(\tau)$ because $cf(\tau) > \omega$. As g is continuous, we have

$$g(\beta) = lim_{i<\omega} g(\alpha_i)$$

. This can't happen because the sequence $\{g(\alpha_i) : i < \omega\}$ diverges to infinity. Thus, for each $x \in X$, f must be bounded on $\{x\} \times T(\tau)$.

Lemma 2. If the function g is as defined in Lemma 1, then g is constant on $[\kappa, \tau)$ for some $\kappa < \tau$.

By our Lemma 3 below, $[\alpha, \tau)$ is countable compact for all $\alpha < \tau$. This is because if $A = \{x_1, x_2, ...\}$ is a countably infinite subset of $[\alpha, \tau)$, then let

$\beta = \sup\{x_1, x_2, ...\}$. This β exists because $cf(\tau) > \omega$. So A has an accumulation point, namely β.

Since g is continuous, $g\,[[\alpha, \tau)]$ is countably compact. In metric spaces, countably compact is equivalent to compact because metric spaces are Lindelöff. Hence, $g\,[\alpha, \tau)]$ is compact for all $\alpha < \tau$. Thus, there exists $p \in \bigcap_{\alpha<\tau} f\,[\alpha, \tau)]$. Suppose that there exists $q \in \bigcap_{\alpha<\tau} g\,[\alpha, \tau)]$ also.

. There exists $\alpha_0 \in [0, \tau)$ such that $g(\alpha_0) = p$. Then there exists $\alpha_1 \in [\alpha_0 + 1, \tau)$ such that $g(\alpha_1) = q$. Then there exists $\alpha_2 \in [\alpha_1 + 1, \tau)$ such that $g(\alpha_2) = p$. We continue this process by induction. We have now:

$$p = g(\alpha_0) = g(\alpha_2) = g(\alpha_4 = \cdots$$
$$q = g(\alpha_1) = g(\alpha_3) = g(\alpha_5 = \cdots$$

Let $\beta = sup\{\alpha_n : n < \omega\}$, which exists because $cf(\tau) > \omega$. By continuity of g, $g(\beta) = lim_{n<\omega} g(\alpha_n)$. Thus, $p = qq$.

Now, since $\{p\} = \bigcap_{\alpha<\tau} f\,[\alpha, \tau)]$, For all $n < \omega$, there exists α_n such that $g\,[[\alpha_n, \tau)] \subseteq (p - \frac{1}{n}, p + \frac{1}{n})$. Let $\kappa = sup_{n<\omega} \alpha_n$. So, we have $g\,[[\kappa, \tau)] \subseteq \bigcap_{n<\omega} (p - \frac{1}{n}, p + \frac{1}{n}) = \{p\}$.

Lemma 3 For every Hausdorff spaces X, the following statements are equivalent:

1. The space X is countably compact.
2. For every decreasing sequence $F_1 \supset F_2 \supset \cdots$ of non-empty closed subsets of X, the intersection $\bigcap_{i=1}^{\infty} F_i$ is non-empty.
3. Every countably infinite subset of X has an accumulation point.

Proof.

$1 \Rightarrow 2$: Let $F_1 \supset F_2 \supset \cdots$ be non-empty closed subsets of X. If $\bigcup_{i=1}^{\infty} F_i = \emptyset$, then $X \backslash F_i : 1 \leq i \leq \infty$ would be an countable open cover of X, so there is a finite subcover $\{X \backslash F_i' : 1 \leq i \leq n\} \subseteq \{X \backslash F_i : 1 \leq i \leq \infty\}$ such that $\bigcup \{X \backslash F_i' : 1 \leq i \leq n\} = X$. Now, because the F_i's are decreasing, WLOG, $F_1' \supset F_2' \supset \cdots \supset F_n'$. So, $\bigcup \{X \backslash F_i' : 1 \leq i \leq n\} = X \backslash F_n'$. Contradiction.

$2 \Rightarrow 1$: By way of contradiction, suppose that X is not countably compact. Let $\{U_i \tau(X) : 1 \leq i \leq \infty\}$ be a countable cover of X that does not yield an open subcover. For each $1 \leq n \leq \infty$, define $F_n = X \backslash \bigcup \{U_i : 1 \leq i \leq n\}$. For each n, F_n is non-empty because if it is, then $\{U_i \tau(X) : 1 \leq i \leq n\}$ would be a finite subcover, which contradicts that X is not countably compact. Thus, we have $F_1 \supset F_2 \supset \cdots$ and each F_n is a non-empty closed subset of X.

Now, by our assumption, the intersection $\bigcap_{i=1}^{\infty} F_i$ is non-empty. So there exists some $x \in \bigcap_{i=1}^{\infty} F_i$. So $x \in F_i$ for all $1 \leq i \leq \infty$. That means $x \notin U_i$ for all $1 \leq i \leq \infty$, contradicting that $\{U_i : 1 \leq i \leq \infty\}$ is a cover of X.

$1 \Rightarrow 3$: By way of contradiction, suppose we have a countably infinite subset $\{x_i \in X : 1 \leq i \leq \infty\}$ with no accumulation point. Then every point in $\{x_i \in X : 1 \leq i \leq \infty\}$ is an isolated point. So $\{\{x_i\} \in \tau(X) : 1 \leq i \leq \infty\}$ is an countable open cover that yields no finite subcover, contradicting that X is countably compact.

$3 \Rightarrow 1$ By way of contradiction, suppose that $\{U_i \tau(X) : 1 \leq i \leq \infty\}$ is a countable cover of X which does not yield an open subcover. Then, by the equivalent of 1 and 2, there exists a decreasing sequence $F_1 \supset F_2 \cdots$ of non-empty closed subsets of X such that $\bigcup_{i=1}^{\infty} F_i = \emptyset$. We define the set $A = \{x_1, x_2, ...\}$ such that $x_i \in F_i$ for each $1 \leq i \leq \infty$. If A is finite, then by pigeon-hole principle, there must be some $x_j \in A$ such that x_j belongs to infinitely many F_i's, and since F_i's are decreasing, x_j' would have to be in all F_i's. Contradicting $\bigcap_{i=1}^{\infty} F_i = \emptyset$. Hence, A is an infinite set. By our assumption, A has an accumulation point. However, for every $x \in X$, there exists an i such that $x \notin F_i$. Now, $U = X \backslash F_i$ is an open set that contains x, and U does not contain an point of the set $\{x_i, x_{i+1}, x_{i+2} ...\}$. Now, $V = \{x\} \cup X \backslash \{x_1, x_2, ..., x_{i-1}\}$ is an open set that contains x. Hence, we have $x \in (U \cap V) \in \tau(X)$ and $(U \cap V) \cap A = \{x\}$. Thus x is an isolated point with repect to A. Since x was arbitrary, we get that A has no accumulation point, contradiction. Lemma 3 is proved.