Buzjakova's Theorem. A pseudocompact space X condenses onto a compact space if and only if the space $X \times T(|\beta X|^+ + 1)$ condenses onto a normal space.

Proof.

($\Rightarrow$:) Let X be a pseudocompact space that condenses onto a compact space K. So there exists $f : X \to K$ such that f is one-to-one, onto, and continuous. Define $g : X \times T(|\beta X|^+ + 1) \to K \times T(|\beta X|^+ + 1)$ by $g(x, \alpha) = (f(x), \alpha)$. Then, g is one-to-one, onto, and continuous. Hence $X \times T(|\beta X|^+ + 1)$ condenses onto a normal space.

($\Rightarrow$:) By **Fact 3.4**, we know that $X \times T(|\beta X|^+)$ condenses onto $X \times T(|\beta X|^+ + 1)$. Since $X \times T(|\beta X|^+ + 1)$ condenses onto some normal space Z, the space $X \times T(\beta X|^+)$ must condenses onto Z, too. So, there exists $f : X \times T(|\beta X|^+) \to Z$, where f is ono-to-one, onto, and continuous. Let
$C_1 = \{x \in \beta X \backslash X : |\bar{f}[\{x\} \times T(|\beta X|^+)] \cap Z| = |\beta X|^+\}$, and
$C_2 = \{x \in \beta X \backslash X : \left|\bar{f}[\{x\} \times T(|\beta X|^+)] \cap Z\right| < |\beta X|^+$ and for any $\alpha \in T(|\beta X|^+)$, there exists $\alpha_1 \in T(|\beta X|^+)$ such that $\alpha_1 > \alpha$ and $\bar{f}(x, \alpha_1) \in Z\}$.

Let $T = \bigcap\{T_x : x \in C_1\}$, where T_x was defined in **Lemma 3.10**. By **Fact 3.3,** T is closed in $T(\tau)$ and $|T| = |\beta X|^+$.

Let $x \in \overline{C_2}^{\beta X}$. Since $\overline{C_2}^{\beta X} \cap C_1$, by **Lemma 3.11**, $x \notin C_1$. Thus, $\left|\bar{f}\,[\{x\} \times T(|\beta X|^+)] \cap Z\right| < |\beta X|^+$. Let $\alpha_x = \sup\{\alpha \in T(|\beta X|^+) : f(y, \alpha) \in \bar{f}\,[\{x\} \times T(|\beta X|^+)] \cap Z$ for some $y \in X\}$. Since $\left|\bar{f}\,[\{x\} \times T(|\beta X|^+)] \cap Z\right| < |\beta X|^+$, such α_x exists in $T(|\beta X|^+)$.

Now, for any $\alpha_1 > \alpha_x$, we have either $\bar{f}(x, \alpha_1) \in \beta Z \backslash Z$ or $\bar{f}(x, \alpha_1) \in Z$. If $\bar{f}(x, \alpha_1) \in Z$, we must have $\bar{f}(x, \alpha_1) = f(x, \alpha_2)$ because f is onto. Because $\alpha_1 > \alpha_x$, and by our definition of α_x, $\alpha_1 > \alpha_2$ must hold.

Let $\alpha^* = \sup\{\alpha_x : x \in \overline{C_2}^{\beta X}\}$. Such α^* exists because $\left|\overline{C_2}^{\beta X}\right| \leq |\beta X| < cf(|\beta X|^+)$. For any $\alpha > \alpha^*, \alpha \in T$,

$$f\,[X \times \{\alpha\}] \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right] = \emptyset$$

holds.

This is because if $z \in f\,[X \times \{\alpha\}] \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$, we can write $z = \bar{f}(x, \alpha)$ for some $x \in \overline{C_2}^{\beta X}$. From the definition of α^* and the fact that $\alpha > \alpha^*$, we have either $\bar{f}(x, \alpha) = f(y, \alpha_2)$ where $y \in X$ and $\alpha_2 < \alpha$, or, we have $\bar{f}(x, \alpha) \in \beta Z \backslash Z$. We reach contradiction in the first case because f is one-to-one, so $\bar{f} = f(y, \alpha_2) \neq f(y', \alpha)$ for any $y' \in X$. We reach contradiction again in the second case because since $z \in f\,[X \times \{\alpha\}]\,, z \in Z$. So $z = \bar{f}(x, \alpha) \in \beta Z \backslash Z$ is a contradiction.

We have two cases:

CASE I. For all $\alpha \in T, \alpha > \alpha^*$, the set $f\left[X \times \{\alpha\}\right] \cup \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$is not compact.

Let

$$C_3 = (\beta X \backslash X) \backslash (C_1 \cup \overline{C_2}^{\beta X}).$$

C_3 is nonempty because if it is , then $\beta X \backslash X = C_1 \cup \overline{C_2}^{\beta X}$. We will show that $f\left[\beta X \times \{\alpha\}\right] \subseteq f\left[X \times \{\alpha\}\right] \cup \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$. Let $x \in \beta X \backslash X$, if $x \in C_1$, then $\bar{f}(x, \alpha) = f(y_x, \alpha)$ for some $y_x \in X$ because $\alpha \in T \subseteq T_x$. Hence $\bar{f}(x, \alpha) \in f\left[X \times \{\alpha\}\right]$. If $x \in \overline{C_2}^{\beta X}$, then $\bar{f}(x, \alpha) \in \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$. The other inclusion is trivial. So now, we have

$$\bar{f}\left[\beta X \times \{\alpha\}\right] = f\left[X \times \{\alpha\}\right] \cup \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right].$$

However, as $\bar{f}$ is continuous as $\beta X \times \{\alpha\}$ is compact, $\bar{f}\left[\beta X \times \{\alpha\}\right]$ is compact. But by our assumption, $f\left[X \times \{\alpha\}\right] \cup \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$ is not compact, contradiction.

For each $x \in C_3$, there exist $\alpha_x \in T(|\beta X|^+)$ such that for $\alpha > \alpha_x$, $\bar{f}(x, \alpha) \in \beta Z \backslash Z$ holds. This follows from the definition of C_1 and C_2. Let $\beta^* = \sup\{\alpha_x : x \in C_3\}$. Such β^* exists because $|C_3| < cf(|\beta X|^+)$.

Let

$$\gamma^* = max\{\alpha^*, \beta^*\}.$$

Fix an arbitrary $\lambda \in T, \lambda > \gamma^*$.

The set $f\left[X \times \{\lambda\}\right]$ is closed in Z. Suppose not, by **Lemma 3.9**, there exists

$$(x, \lambda) \in \overline{X \times \{\lambda\}}^{\beta X \times T(|\beta X|^+ + 1)} \backslash X \times \{\lambda\}$$

such that

$$\bar{f}(x, \lambda) \in \overline{f\left[X \times \{\lambda\}\right]}^{Z} \backslash f\left[X \times \{\lambda\}\right].$$

So, $x \in \beta X \backslash X$. If $x \in C_1$, then since $\lambda \in T \subseteq T_x$, there exists $y_x \in X$ such that $\bar{f}(x, \lambda) = f(y_x, \lambda) \in f\left[X \times \{\lambda\}\right]$, contradiction. If $x \in C_2$, then either $\bar{f}(x, \lambda) = f(y, \lambda_2)$ and $\lambda > \lambda_2$, or $\bar{f}(x, \lambda) \in \beta Z \backslash Z$. Contradiction in the first case because $\bar{f}(x, \lambda) =\notin Z$. Contradiction in both cases. If $x \in C_3$, then $\bar{f}(x, \lambda) \in \beta Z \backslash Z$, contradiction. Therefore, $f\left[X \times \{\lambda\}\right]$ is closed in Z.

By **Fact 3.5,** there exists a system $D = \{D_\alpha\}$ satisfying the first 4 conditions:

1. For each α, the set D_α is non-empty and closed in $f\left[X \times \{\lambda\}\right]$.
2. For $\alpha > \beta, D_\alpha \subseteq D_\beta$, and if β is a limit ordinal, then $D_\beta = \bigcap\{D_\alpha : \alpha < \beta\}$.

3. $\bigcap\{D_\alpha\} = \emptyset$.
4. $\overline{D_1}^{\beta Z} \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\lambda\}\right] = \emptyset$.

   Since f is one-to-one, by condition 4, $f^{-1}[D_\alpha] \subseteq X \times \{\lambda\}$ for any α.
   Define a system $A = \{A_\alpha\}$ of subsets of X such that
   $f^{-1}[D_\alpha] = A_\alpha \times \{\lambda\}$. A satisfies the following conditions:

5. For each α, the set A_α is closed in X.
   `Proof- Since` D_α `is closed in` $f[X \times \{\lambda\}]$ `and` $f[X \times \{\lambda\}]$ `is closed in` Z, D_α `must be closed` Z `as well. As` f `is continuous,` $f^{-1}[D_\alpha]$ `must be closed in` $X \times T(|\beta X|^+)$. `But` $f^{-1}[D_\alpha] = A_\alpha \times \{\lambda\}$, `so it means that` A_α `is closed in` X.

6. $\bigcap\{A_\alpha\} = \emptyset$.
   `Proof- By our definition of` A_α, `we have` $f^{-1}[\bigcap\{D_\alpha\}] = \bigcap\left\{f^{-1}[D_\alpha]\right\} = \bigcap\{A_\alpha\}\} = \bigcap\{A_\alpha\}\times\{\lambda\}$. `Since` $\bigcap\{D_\alpha\} = \emptyset$ `by condition 3, we must have` $\bigcap\{A_\alpha\}\emptyset$.

7. For $\alpha > \beta, A_\alpha \subseteq A_\beta$ and if β is a limit ordinal, then $A_\beta = \bigcap\{A_\alpha\}$.
   `Proof- If` $\alpha > \beta, D_\alpha \subseteq D_\beta$ `holds by condition 2. So,` $f^{-1}[D_\alpha] \subset f^{-1}[D_\beta] \Rightarrow A_\alpha \times \{\lambda\} \subseteq A_\beta \times \{\lambda\{\lambda\} \Rightarrow A_\lambda \subseteq A_\beta$. `If` β `is a limit ordinal, then by condition 2,` $D_\beta = \bigcap\{D_\alpha\}$. `So` $A_\beta\times\{\lambda\} = f^{-1}[D_\beta] = f^{-1}[\bigcap\{D_\alpha\}] = \bigcap f^{-1}[D_\alpha] = \bigcap\{A_\alpha \times \{\lambda\}\}$. `Thus,` $A_\beta = \bigcap\{A_\alpha\}$.

8. $\overline{A_1}^{\beta X} \cap \overline{C_2}^{\beta X} = \emptyset$.
   `Proof- Suppose there exists` $x \in \overline{A_1}^{\beta X}\cap\overline{C_2}^{\beta X}$. `Since` $x \in \overline{A_1}^{\beta X}, \bar{f}(x,\lambda) \in \bar{f}\left[\overline{A_1}^{\beta X} \times \{\lambda\}\right] \subseteq \overline{\bar{f}[A_1 \times \{\lambda\}]}^{\beta Z} = \overline{f[A_1 \times \{\lambda\}]}^{\beta Z} = \overline{D_1}^{\beta Z}$. `On the other hand, since` $x \in \overline{C_2}^{\beta X}$, $\bar{f} \in \bar{f}\left[\overline{C_2}^{\beta X} \times \{\lambda\}\right]$. `Therefore,` $\bar{f}(x,\lambda) \in \overline{D_1}^{\beta Z} \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\lambda\}\right]$. `This contradicts condition 4.`

9. If $x \in \bigcap\overline{A_\alpha}^{\beta X}$, then $x \in C_3$.
   `Proof- Since` $\beta X\backslash X$ `is partitioned in to` $C_1, \overline{C_2}^{\beta X}$, `and` C_3, `we need to show that` $x \notin C_1$ `and` $x \notin \overline{C_2}^{\beta X}$. `Suppose that` $x \in C_1$, `then` $\bar{f}(x,\lambda) = f(y_x,\lambda)$ `for some` $y_x \in X$. `Since`

10. $f[A_\alpha \times \{\alpha\}]$ is closed in Z for each $\gamma > \gamma^*, \gamma \in T$.
    `Proof- By Lemma 3.9, there` $x \in \overline{A_\alpha}\backslash A_\alpha$ `such that` $\bar{f}(x,\gamma) \in \overline{f[A_\alpha \times \{\alpha\}]}^{Z}\backslash f[A_\alpha \times \{\alpha\}]$

`holds. Since` $\overline{A_1}^{\beta X} \cap \overline{C_2}^{\beta X} = \emptyset, x \notin \overline{C_2}^{\beta X}$`. Also, if` $x \in C_3$`, then` $\bar{f}(x,\gamma) = f(x,\gamma')$ `for some` $\gamma' > \gamma^*$`, contradictin, so` $x \notin C_3$`. Thus,` $x \in C_1$`. But` $f\left[A_\alpha \times \{\lambda\}\right] = D_\alpha$ `closed in` Z`. So` $\bar{f}(x,\lambda) \in f\left[A_\alpha \times \{\lambda\}\right]$`.` `As` $x \in C_1, \bar{f}(x,\lambda) = f(y_x,\lambda)$`, so we have` $\bar{f}(x,\gamma) \in f\left[A_\alpha \times \{\gamma\}\right]$`. Contradiction.`

Let $|A| = \gamma$. Choose a closed subset $G = \{\gamma_\alpha : \alpha \leq \gamma\}$ of T such that $\gamma_1 > \gamma^*$ and for $\alpha > \beta, \gamma_\alpha > \alpha_\beta$. Define

$$B_1 = \bigcup \{A_\alpha \times \{\gamma_\alpha\}\}$$

$$B_2 = A_1 \times \{\gamma_\gamma\}$$

B_1 is closed in $X \times T(|\beta X|^+)$ because $\bigcap\{A_\alpha\} = \emptyset$ by condition 6. B_2 is closd in $X \times T(|\beta X|^+)$ because A_1 is closed by condition 5 and by our choice of G. B_1 and B_2 are disjoint because B_1 doesn't contain any element with the second coordinate equal to γ_γ.

Now, as each $\overline{A_\alpha}^{\beta X} \neq \emptyset$, $x \in \bigcap\left\{\overline{A_\alpha}^{\beta X}\right\}$. Thus $(x,\gamma_\gamma) \in \overline{B_2}^{\beta X \times T(|\beta X|^+ +1)}$. On the other hand, we have $(x,\gamma_\gamma) \in \overline{B_1}^{\beta X \times T(|\beta X|^+ +1)}$. Thus, $(x,\gamma_\gamma) \in \overline{B_1}^{\beta X \times T(|\beta X|^+ +1)} \cap \overline{B_2}^{\beta X \times T(|\beta X|^+ +1)}$. By **Fact 3.6**, the sets B_1 and B_2 are not completely separated in $X \times T(|\beta X|^+)$.

Let us consider the $f\left[B_1\right]$ and $f\left[B_2\right]$. Since f is one-to-one, we have $f\left[B_1\right] \cap f\left[B_2\right] = \emptyset$. By condition 10, $f\left[B_2\right]$ is closed in Z. We shall prove that $f\left[B_1\right]$ is closed in Z. Assume the contrary. Then, by **Lemma 3.9**, there exists $(x,\gamma_\alpha) \in \overline{B_1}^{\beta X \times T(|\beta X|^+ +1)} \backslash B_1$ such that $\bar{f}(x,\gamma_\alpha) \in \overline{f\left[B_1\right]}^Z \backslash f\left[B_1\right]$.

Case 1. $\alpha < \gamma$:
Since $(x,\gamma_\alpha) \in \overline{B_1}^{\beta X \times T(|\beta X|^+ +1)}, (x,\gamma_\alpha) \in \overline{A_\alpha \times \{\gamma_\alpha\}}^{\beta X \times T(|\beta X|^+ +1)}$. So $\bar{f}(x,\gamma_\alpha) \in \bar{f}\left[\overline{A_\alpha \times \{\gamma_\alpha\}}^{\beta X \times T(|\beta X|^+ +1)}\right] \subset \overline{\bar{f}\left[A_\alpha \times \{\gamma_\alpha\}\right]}^{\beta Z} = \overline{f\left[A_\alpha \times \{\gamma_\alpha\}\right]}^{\beta Z}$.

By 10, $f\left[A_\alpha \times \{\gamma_\alpha\}\right]$ is closed in Z. So either $\bar{f}(x,\gamma_\alpha) \notin Z$ or $\bar{f}(x,\gamma_\alpha) \in f\left[A_\alpha \times \{\gamma_\alpha\}\right]$. If $\bar{f}(x,\gamma_\alpha) \notin Z$, then $\bar{f}(x,\gamma_\alpha) \notin \overline{f\left[B_1\right]}^Z$, contradiction. If $\bar{f}(x,\gamma_\alpha) \in f\left[A_\alpha \times \{\gamma_\alpha\}\right]$, then $\bar{f}(x,\gamma_\alpha) \in f\left[B_1\right]$,contradiction.

Case 2. $\alpha = \gamma$:
The only way (x,γ_γ) qualifies to be in $\overline{B_1}^{\beta \times T(|\beta X|^+ +1)}$ but not in B_1 is if $x \in \bigcap\left\{\overline{A_\alpha}^{\beta X}\right\}$. By condition 9, $x \in C_3$. But since $\gamma_\gamma > \gamma^* \geq \beta^*, \bar{f}(x,\gamma_\gamma) \in \beta Z \backslash Z$. This contradicts that $\bar{f}(x,\gamma_\gamma) \in \overline{f\left[B_1\right]}^Z$.

Hence, $f\left[B_1\right]$ is closed in Z.

Now, the sets $f[B_1]$ and $f[B_2]$ and closed and disjoint in Z. Since Z is normal, by Urysohn's Lemma, there exists a continuous function $g : Z \to [0,1]$ such that $g[f[B_1]] \subseteq \{0\}$ and $g[f[B_2]] \subseteq \{1\}$. Let $h = g \circ f$. Then B_1 and B_2 are completely separated by the continuous function h. This is a contradiction, so **CASE I** cannot happen.

CASE II There exists an ordinal $\alpha \in T, \alpha > \alpha^*$ such that the set $f[X \times \{\alpha\}] \cup \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$ is compact.

Since $f[X \times \{\alpha\}] \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right] = \emptyset$ and the set $\bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$ is compact, the set $f[X \times \{\alpha\}]$ is locally compact. This is because for any $z \in f[X \times \{\alpha\}], z \notin \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$. By Z being normal and hence regular, there exists a closed nhood N_z of z such that $z \in N_z \subseteq \beta Z \backslash \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$. As βZ is compact, N_z is a compact nhood of z. Hence, $f[X \times \{\alpha\}]$ is locally compact.

By **Fact 3.7**, $f[X \times \{\alpha\}]$ condenses onto a compact space. Hence, $X \times \{\alpha\}$ condenses onto a compact space as well. Since X is homeomorphic to $X \times \{\alpha\}, X$ condenses onto a compact space, too. The theorem is proved.