Properties of Pseudocompact Space Condensation

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A thesis presented for the degree
of
Master of Arts

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August 2007

Contents

Introduction

The existence of a compactification is a characteristic property of a Tychonoff space, and one can reasonably expect that the Stone-$\check{C}$ech compactification, which is the largest of all compactifications, plays an important role in the theory of Tychonoff spaces.

In 1960, Tamano showed that a Tychonoff space X is paracompact iff $X \times \beta X$ is normal. This is known is the Tamano's Theorem. Around the same time, Glicksberg showed that for Tychonoff spaces X, Y, if $X \times Y$ is pseudocompact, then $\beta(X \times Y) = \beta X \times \beta Y$. In 1997, Buzjakova used Glicksberg's Theorem, among other things, to establish a criterion that a pseudocompact space condenses onto a compact space. Buzjakova's Theorem states that a Tychonoff, pseudocompact space X condenses onto a compact space if and only if the space $X \times T(|\beta X|^+ + 1)$ condenses onto a normal space.

It is interesting to note that Buzjakova's Theorem can be interpreted as a condensation version of the Tamano's Theorem for the pseudocompact case, and that both theorems demonstrate how some properties of a Tychonoff space X can be characterized by the properties of the Stone-$\check{C}$ech compactification βX.

In the following chapters, we will prove Tamano's Theorem, Glicksberg's Theorem, some facts and lemmas used in Buzjakova's Theorem, and finally, the proof of Buzjakova's Theorem.

1 Tamano's Theorem

In Tamano's 1960 paper, he proved that if X is Tychonoff and $X \times \beta X$ normal, then X is paracompact. The converse was well known before Tamano's paper. The following lemmas will be used to prove the converse, that if X is Tychonoff and paracompact, then $X \times \beta X$ is normal.

1.1 Lemma.

The product of a paracompact space with a compact Hausdorff space is paracompact.

Proof. Let X be paracompact, Y compact, and let $\mathcal{U}$ be an open cover of $X \times Y$. For fixed $x \in X$, as $\{x\} \times Y$ is compact in $X \times Y$, a finite number of elements of $\mathcal{U}$, say $U^x_{\alpha_1}, \ldots, U^x_{\alpha_{n_x}}$, cover $\{x\} \times Y$. Pick an open nhood V_x of x in X such that $V_x \times Y \subseteq \bigcup_{i=1}^{n_x} U^x_{\alpha_i}$.

The sets V_x, as x ranges through X, form an open cover of X. By the paracompactness of X, let $\mathcal{V}$ be an open locally finite refinement of the sets V_x. For each $V \in \mathcal{V}, V \subseteq V_x$ for some V_x. Let

$$\mathcal{W}_V = \left\{(V \times Y) \cap U^x_{\alpha_i} : 1 \leq i \leq n_x\right\},$$

and let

$$\mathcal{R} = \bigcup \{\mathcal{W}_V : V \in \mathcal{V}\}.$$

Since $\mathcal{W}_V \subseteq \mathcal{U}$ for each $V \in \mathcal{V}$, $\mathcal{R}$ is a refinement of $\mathcal{U}$. For each $x \in X$, W_V is a cover for $\{x\} \times Y$ for some $V \in \mathcal{V}$. Thus, $\mathcal{R}$ is a cover of $X \times Y$. Lastly, $\mathcal{R}$ is locally finite because given $(x, y) \in X \times Y$, there is a neighborhood U_x of x which meets only finitely many V's in $\mathcal{V}$ because $\mathcal{V}$ is locally finite. Then the neighborhood $U_x \times Y$ of (x, y) can then only meet only finitely many sets of $\mathcal{R}$. Hence, $X \times Y$ is paracompact. $\square$

1.2 Lemma.

Every paracompact space is normal.

Proof. We first establish regularity. Suppose A is a closed set in a paracompact space X and $x \notin A$. For each $y \in A$, as X is Hausdorff, we can find an open set V_y containing y such that $x \notin \overline{V_y}$. Then the sets $V_y, y \in A$, together with the set $X \backslash A$, form an open cover of X. Let $\mathcal{W}$ be an open locally finite

refinement and let
$$V = \bigcup \{W \in \mathcal{W} : W \cap A \neq \emptyset\}.$$

Then V is an open set containing A. Now,
$$\overline{V} = \bigcup \{\overline{W} \in \mathcal{W} : W \cap A \neq \emptyset\}$$
holds because:
$\supseteq$: If $z \in \bigcup\{\overline{W} \in \mathcal{W} : W \cap A \neq \emptyset\}$, then $z \in \overline{W}$ for some $W \in \mathcal{W}$. Since $W \subseteq V$, we have $z \in \overline{W} \subseteq \overline{V}$.
$\subseteq$: If $z \in \overline{V}$, then there exists a net $\{w_\alpha\} \subseteq V = \bigcup\{W \in \mathcal{W} : W \cap A \neq \emptyset\}$ converging to z. Since $\{W \in \mathcal{W} : W \cap A \neq \emptyset\}$ is locally finite, the tail of $\{w_\alpha\}$ must be contained in finitely many W's, say $\{W_1, \ldots W_n\}$. So, $z \in \overline{W_1 \cup \cdots \cup W_n} = \overline{W_1} \cup \cdots \cup \overline{W_n}$. Thus, $z \in \overline{W_k}$ for some $k \in \{1, \ldots, n\}$.

Since $x \notin \overline{V_y}$ for each $y \in A$ and $\overline{W} \subseteq \overline{V_y}$ for each $T \in \{W \in \mathcal{W} : W \cap A \neq \emptyset\}$, $x \notin \overline{T}$ for each $T \in \{W \in \mathcal{W} : W \cap A \neq \emptyset\}$. So $x \notin \overline{V}$. Regularity is established.

To establish normality, suppose A and B are disjoint closed sets in X. For each $y \in A$, by regularity, we can find an open set V_y such that $y \in V_y$ and $\overline{V_y} \cap B = \emptyset$. Then proceed exactly as before, we can produce an open set V such that $A \subseteq V$ and $\overline{V} \cap B = \emptyset$. Thus X is normal. $\square$

1.3 Theorem.

(Tamano's Theorem) Let X be a Tychonoff space. Then $X \times \beta X$ is normal iff X is paracompact.

Proof of $\Leftarrow$: Since X is paracompact and βX is compact T_2, by **Lemma 1.1**, $X \times \beta X$ is paracompact. By **Lemma 1.2**, paracompact implies normal and so $X \times \beta X$ is normal.

Proof of $\Rightarrow$: (Based on Engelking) We will prove a slighty stronger version that for any compactification cX of X, if $X \times cX$ is normal, then X is paracompact.

Let cX be a compactification of X such that $X \times cX$ is normal. Let $\{U_a : a \in A\}$ be an open cover of X. We will show that it has an open locally finite refinement.

X is a subspace of cX, so for each $a \in A$, there exists V_a open in cX such that $V_a \cap X = U_a$. Let $F = cX \setminus \bigcup\{V_a : a \in A\}$. We can assume that F is nonempty. Since if $F = \emptyset$, then $cX = \bigcup\{V_a : a \in A\}$, and since cX is compact, we can find a finite subcover $\{V_{a_i} : 1 \leq i \leq n\} \subseteq \{V_a : a \in A\}$. Then, $\{U_{a_i} : 1 \leq i \leq n\}$ is an open locally finite refinement of $\{U_a : a \in A\}$.

Let $\triangle = \{(x, x) : x \in X\}$. Both $X \times F$ and $\triangle$ are closed in $X \times cX$. Since $X \times cX$ is normal, by Urysohn's Lemma, there is a continuous function $f\colon X \times cX \to [0, 1]$ with $f[\triangle] \subseteq \{0\}$ and $f[X \times F] \subseteq \{1\}$.

Define $d : X \times X \to \mathbb{R}$
such that

$$d(x, y) = \sup \{|f(x, z) - f(y, z)| : z \in X\}$$

for all $(x, y) \in X \times X$. Now, for all $x, y, w \in X$, we have:

1. $d(x, x) = \sup_{z\in X} |f(x, z) - f(x, z)| = 0$
2. $d(x, y) = \sup_{z\in X} |f(x, z) - f(y, z)| = \sup_{z\in X} |f(y, z) - f(x, z)| = d(y, x)$
3. $d(x, w) = \sup_{z\in X} |f(x, z) - f(w, z)|$
   $= \sup_{z\in X} |f(x, z) - f(y, z) + f(y, z) - f(w, z)|$
   $\leq \sup_{z\in X} |f(x, z) - f(y, z)| + \sup_{z\in X} |f(y, z) - f(w, z)|$
   $= d(x, y) + d(y, w)$

Thus d is a pseudometric on X. Let $\tau_d(X)$ denote the set of open sets in the topology induced by the pseudometric d.

For each $B(x_0, \epsilon) \in \tau_d$, pick any point $x' \in B(x_0, \epsilon)$. Let $\epsilon' = \epsilon - d(x_0, x')$. The set $\Gamma = \{G \times H \subseteq X \times cX : G \times H$ is open in $X \times cX, x' \in G$, and $diam\,(f\,[G \times H]) < \epsilon'\}$ is an open cover of $\{x'\} \times cX$. To show this, pick any point $(x', y) \in \{x'\} \times cX$. Let $c = f(x', y) \in [0, 1]$. Since f is continuous and the set $E = (c - \frac{\epsilon'}{2}, c + \frac{\epsilon'}{2}) \cap [0, 1]$ is open in $[0, 1]$, $f^{\leftarrow}[E]$ must be open in $X \times cX$. Since $f^{\leftarrow}[E]$ is an open set that contains (x', y) in $X \times cX$, there exist $G_b \in \tau(X)$ containing x', and $H_b \in \tau(cX)$ containing y such that $G_b \times H_b \subseteq f^{\leftarrow}[E]$. Thus $G_b \times H_b$ is an element of Γ. Since (x', y) was arbitrarily chosen from $\{x'\} \times cX$, we conclude that Γ is an open cover of $\{x'\} \times cX$.

Γ being an open cover of $\{x'\} \times cX$ means that $\{H : G \times H \in \Gamma\}$ is an open cover of cX. Since cX is compact, there is an finite subcover $\{H_i : 1 \leq i \leq n\}$. Corresponding to $\{H_i : 1 \leq i \leq n\}$ is the set $\{G_i : 1 \leq i \leq n\}$. Where for each $i \in \{1 \ldots n\}$, we have $f[G_i \times H_i] \subseteq (c_i - \frac{\epsilon'}{2}, c_i + \frac{\epsilon'}{2})$ for some $c_i \in (0, 1)$. Pick any $z \in cX = \bigcup\{H_i : 1 \leq i \leq n\}$. For some $1 \leq k \leq n$, $z \in H_k$. Then $f[G_k \times \{z\}] \subseteq f[G_k \times H_k] \subseteq (c_k - \frac{\epsilon'}{2}, c_k + \frac{\epsilon'}{2})$.

Let $S = \bigcap\{G_i : 1 \leq i \leq n\} \subseteq G_k$. Then

$$f[S \times \{z\}] \subseteq f[G_k \times \{z\}] \subseteq (c_k - \frac{\epsilon'}{2}, c_k + \frac{\epsilon'}{2}).$$

For all $x, y \in S, |f(x, z) - f(y, z)| < \epsilon'$, and because this inequality holds true for all $z \in cX$, $d(x, y) = \sup\limits_{z\in cX} |f(x, z) - f(y, z)| \leq \epsilon'$. Note that $x' \in$

$\bigcap\{G_i : 1 \leq i \leq n\} = S \in \tau(X)$ and that $d(x, y) \leq \epsilon'$ for all $x, y \in S$. We have $x' \in S \subseteq B(x', \epsilon') \subseteq B(x_0, \epsilon)$.

Thus, for each $x' \in B(x_0, \epsilon)$, we can find $S \in \tau(X)$ such that $x' \in S \subseteq B(x_0, \epsilon)$. So, $B(x_0, \epsilon) \in \tau_d(X)$ and $\tau_d(X) \subseteq \tau(X)$.

By Stone's Theorem, a pseudo-metrizable space is paracompact. So X is paracompact with respect to the pseudo-metrizable topology τ_d. For an open cover $\left\{B(x, \frac{9}{10}) : x \in X\right\}$, there is an open locally finite refinement, $\{W_t : t \in T\}$. Since $\tau_d(X) \subseteq \tau(X)$, $\{W_t : t \in T\} \subseteq \tau(X)$.

Pick any $x_0 \in X$ and $x' \in B(x_0, \frac{9}{10})$. We have $f(x_0, x') = |f(x_0, x') - 0| = |f(x_0, x') - f(x', x')| \leq \sup_{z \in X} |f(x_0, z) - f(x_0, z)| = d(x_0, x') < \frac{9}{10}$. So, $f\left[\{x_0\} \times B(x_0, \frac{9}{10})\right] \subseteq [0, \frac{9}{10})$. By continuity, $f\left[\{x_0\} \times \overline{B(x_0, \frac{9}{10})}^{cX}\right] \subseteq [0, \frac{9}{10}]$. So, $\overline{B(x_0, \frac{9}{10})}^{cX} \cap F = \emptyset$ because $f\left[X \times F\right] \subseteq \{1\}$.

As $\{W_t : t \in T\}$ refines $\{B(x_0, \frac{9}{10}) : x_0 \in X\}$ and $\overline{B(x_0, \frac{9}{10})}^{cX} \cap F = \emptyset$ for all $x_0 \in X$, $\overline{W_t}^{cX} \cap F = \emptyset$ for every $t \in T$. Then, for each $t \in T$, $\overline{W_t}^{cX} \subseteq cX \backslash F = \bigcup\{V_a : a \in A\}$. Since $\overline{W_t}^{cX}$ is compact in cX, there exists a finite subcover $\left\{V_j^t : 1 \leq j \leq m_t\right\} \subseteq \{V_a : a \in A\}$ such that $\overline{W_t}^{cX} \subseteq \bigcup\left\{V_j^t : 1 \leq j \leq m\right\}$. We have:

$$X \cap \overline{W_t}^{cX} \subseteq X \cap (\bigcup\left\{V_j^t : 1 \leq j \leq m\right\}).$$

$$\text{Thus, } X \cap W_t \subseteq X \cap \overline{W_t}^{cX} \subseteq \bigcup\left\{U_j^t : 1 \leq j \leq m\right\}.$$

The set $\{W_t \cap U_j^t : t \in T, 1 \leq j \leq m_t\} \subseteq \tau(X)$ is the desired locally finite open cover of X which refines $\{U_a : a \in A\}$. Hence X is paracompact. $\square$

CHAPTER II

2 Glicksberg's Theorem

In this chapter, we will prove Glickberg's Theorem. The full version of Glicksberg's Theorem actually states that if the Cartesian product $\Pi_{s\in S}X_s$ is pseudocompact, then $\beta\left(\Pi_{s\in S}X_s\right) = \Pi_{s\in S}\beta X_s$. However, since we need only the finite version of it in the proof of Buzjakova's Theorem, we will prove the finite version.

We need some preliminary facts before we prove two important lemmas - **Lemma 2.6** and **Lemma 2.7**. The proof of Glicksberg's Theorem follows immediately from these two lemmas.

2.1 Fact.

Let Y be an extension of a space X, let Z be a regular space, and let $f : X \to Z$ be continuous. The following are equivalent:

1. There exists a continuous function $F : Y \to Z$ such that $F|_X = f$.
2. For each $y \in Y$, the filter $\mathcal{F}_y = \{A \subseteq Z : A \supseteq f[U] \text{ for some } U \in O^y\}$ converges(where $O^y = \{W \cap X : W \text{ is open in } Y \text{ and } y \in W\}$).

Proof of 1 $\Rightarrow$ 2: Suppose F exists and $y \in Y$. We will show $\mathcal{F}_y$ converges to $F(y)$. Let W be an open neighborhood of $F(y)$ in Z. By continuity, there is an open neighborhood U of y such that $F[U] \subseteq W$, where U is open in Y. Thus, $f\left[U \cap X\right] = F\left[U \cap X\right] \subseteq W$ and $f\left[U \cap X\right] \in \mathcal{F}_y$. Thus, $\mathcal{F}_y$ converges to $F(y)$.

Proof of 2 $\Rightarrow$ 1: Suppose for each $y \in Y, \mathcal{F}_y$ converges to some point. As Z is regular and hence Hausdorff, $\mathcal{F}_y$ converges to an unique point which we denote by $F(y)$. Thus, we have just defined a function $F : Y \to X$.

If $x \in X$ and W is an open neighborhood of $f(x)$, there is an open set U of X with $x \in U$ and $f[U] \subseteq W$. If V is an open set in Y such that $V \cap X = U$, then $f\left[V \cap X\right] \in \mathcal{F}_x$. Thus, $\mathcal{F}_x$ converges to $f(x)$ for all $x \in X$. So, $F(x) = f(x)$ for all $x \in X$, i.e., $F|_X = f$.

To show F is continuous, let $y \in Y$ and let W be an open neighborhood of $F(y)$. As Z is regular, there is an open subset V of Z such that $F(y) \in V \subseteq \overline{V}^Z \subseteq W$. Since $\mathcal{F}_y$ converges to $F(y)$, there is an open set U of Y such that $y \in U$ and $f[U \cap X] \subseteq V$.

Let $p \in U$. We will show that $F(p) \in \overline{V}^Z$. Let T be an open set of Z containing $F(p)$. From the definition of $\mathcal{F}_p$, there is an open set $R \in \tau(Y)$ containing p such that $R \subseteq U$ and $f[R \cap X] \subseteq T$. As X is dense in Y, $R \cap X \neq \emptyset$. Hence $f[R \cap X] \neq \emptyset$. Since $f[U \cap X] \subseteq V$ and $R \subseteq U$, we have $f[R \cap X] \subseteq V$. Thus $f[R \cap X] \subseteq T \cap V$. As $T \cap V \neq \emptyset$, and T was an arbitrary open set containing $F(p)$, $F(p) \in \overline{V}^Z$.

Since for every $p \in U$, $F(p) \in \overline{V}^Z \subseteq W$, we conclude that $F[U] \subseteq W$. Thus F is continuous. $\square$

2.2 Fact.

Let Y be an extension of a space X and let Z be a regular space. Let $g : Y \to Z$ be such that for each $y \in Y, g|_{X \cup \{y\}}$ is continuous. Then g is continuous.

Proof: Let

$$O_1^y = \{W \cap X : W \text{ is open in } Y \text{ and } y \in W\}, \text{ and}$$
$$O_2^y = \{W \cap X : W \text{ is open in } X \cup \{y\} \text{ and } y \in W\}.$$

We have $O_1^y = O_2^y$ because:

($\subseteq$:) Let $W \cap X \in O_1^y$. W is open in Y and $y \in W$. Since $X \cup \{y\}$ is the subspace of Y, $W \cap (X \cup \{y\})$ is open in $X \cup \{y\}$. Also, $y \in W \cap (X \cup \{y\})$. So, $W \cap X = (W \cap (X \cup \{y\})) \cap X \in O_2^y$.

($\supseteq$:) Let $W \cap X \in O_2^y$. Since W is open in $X \cup \{y\}$, there exists $V \in \tau(Y)$ such that $W = V \cap (X \cup \{y\})$. Since $y \in W$ and thus $y \in V$, we have $V \cap X \in O_1^y$. As $W \cap X = V \cap X$, $W \cap X \in O_1^y$.

Since $g|_{X \cup \{y\}}$ is continuous, then by **Fact 2.1**, the filter $\{A \subseteq Z : A \supseteq g|_X[U] \text{ for some } U \in O_2^y\}$ converges to $g(y)$. Since $O_1^y = O_2^y$, we have $\{A \subseteq Z : A \supseteq g|_X[U] \text{ for some } U \in O_1^y\}$ converging to $g(y)$ as well. By the other direction of **Fact 2.1**, g is continuous. $\square$

2.3 Fact.

The following are equivalent for a dense subspace S of the Tychonoff space X:

1. S is C^*-embedded in X.
2. If Z_1 and Z_2 are disjoint zero-sets of S, then $\overline{Z_1}^X \cap \overline{Z_2}^X = \emptyset$.

Proof of 1 $\Rightarrow$ 2: By the transitivity of C^*-embedding, S is C^*-embedded and dense in βX. Thus βS is equivalent to βX. Let Z_1 and Z_2 are disjoint zero-sets of S. Since disjoint zero-sets in S have disjoint closures in $\beta S \equiv_S \beta X$, $\overline{Z_1}^{\beta X} \cap \overline{Z_2}^{\beta X} = \emptyset$. Hence $\overline{Z_1}^X \cap \overline{Z_2}^X = \emptyset$.

Proof of 2 $\Rightarrow$ 1: It suffices to show that S is C^*-embedded in $S \cup \{p\}$ for each $p \in X \backslash S$ by **Fact 2.2**.

So it remains to be shown that for each $p \in X \backslash S$, S is C^*-embedded in $S \cup \{p\}$. Let $p \in X \backslash S$, and let $C(p)$ denote the collection of closed neighborhoods of p in $S \cup \{p\}$. If $f \in C^*(S)$, for each $A \in C(p)$, the $\overline{f[A \cap X]}^{\mathbb{R}}$ is a compact nonempty subset of $\mathbb{R}$ and the set $\left\{\overline{f[A \cap S]}^{\mathbb{R}} : A \in C(p)\right\}$ has the finite intersection property. Thus, $\bigcap \left\{\overline{f[A \cap S]}^{\mathbb{R}} : A \in C(p)\right\} \neq \emptyset$. Note that if $s \in \bigcap \left\{\overline{f[A \cap S]}^{\mathbb{R}} : A \in C(p)\right\}$ and $\epsilon > 0$, then $p \in \overline{f^{\leftarrow}[[s-\epsilon, s+\epsilon]]}^{S \cup \{p\}}$. This is because if $A \in C(p)$, then $(s-\epsilon, s+\epsilon) \cap f[A \cap S] \neq \emptyset$ and so $A \cap f^{\leftarrow}[(s-\epsilon, s+\epsilon)] \neq \emptyset$.

Choose $r \in \bigcap \left\{\overline{f[A \cap S]}^{\mathbb{R}} : A \in C(p)\right\}$ and define $F : S \cup \{p\} \to \mathbb{R}$ as

$$F|_S = f \text{ and } F(p) = r.$$

Since f is continuous, F is continuous at each point of S. We must show that F is continuous at p. Let $\epsilon > 0$ be given. We claim that there exists $A_0 \in C(p)$ such that $f[A_0 \cap S] \subseteq (r-\epsilon, r+\epsilon)$. For if this were not the case, then $\overline{f[A \cap S]}^{\mathbb{R}} \backslash (r - \frac{3\epsilon}{4}, r + \frac{3\epsilon}{4})$ is a nonempty compact subset of $\mathbb{R}$ for each $A \in C(p)$. As $\left\{\overline{f[A \cap S]}^{\mathbb{R}} \backslash (r - \frac{3\epsilon}{4}, r + \frac{3\epsilon}{4}) : A \in C(p)\right\}$ has the finite intersection property, there exists $s \in \bigcap \left\{\overline{f[A \cap S]}^{\mathbb{R}} \backslash (r - \frac{3\epsilon}{4}, r + \frac{3\epsilon}{4}) : A \in C(p)\right\}$. As noted in the previous paragraph, it follows that

$$p \in \overline{f^{\leftarrow}\left[[s - \frac{\epsilon}{4}, s + \frac{\epsilon}{4}]\right]}^{S \cup \{p\}}.$$

On the other hand, since $r \in \bigcap \left\{\overline{f[A \cap S]}^{\mathbb{R}} : A \in C(p)\right\}$, we have

$$p \in \overline{f^{\leftarrow}\left[[r - \frac{\epsilon}{4}, r + \frac{\epsilon}{4}]\right]}^{S \cup \{p\}}.$$

As $f^{\leftarrow}\left[\left[s-\frac{\epsilon}{4}, s+\frac{\epsilon}{4}\right]\right]$ and $f^{\leftarrow}\left[\left[r-\frac{\epsilon}{4}, r+\frac{\epsilon}{4}\right]\right]$ are disjoint zero-sets of S, this is a contradiction to our hypothesis that if Z_1 and Z_2 are disjoint zero-sets of S, then $\overline{Z_1}^X \cap \overline{Z_2}^X = \emptyset$.

Thus, there exists $A_0 \in C(p)$ such that $f\left[A_0 \cap S\right] \subseteq (r-\epsilon, r+\epsilon)$. Thus $F[A_0] \subseteq (r-\epsilon, r+\epsilon)$ and F is continuous at p. As f was arbitrarily chosen from $C^*(S)$, it follows that S is C^*-embedded in $S \cup \{p\}$. □

Definition. Let X, Y be spaces, and let f be a function from X to Y. If $f[Z]$ is closed in Y for any zero-set Z of X, then f is called **z-closed**.

2.4 Fact.

Let X and Y be Tychonoff spaces, and $\pi_X : X \times Y \to X$ be the projection map. If π_X is z-closed, Z is a zero-set in $X \times Y$, and $(x,p) \in \overline{Z}^{X \times \beta Y}$, then $(x,p) \in \overline{Z \cap (\{x\} \times Y)}^{X \times \beta Y}$.

Proof: Assume that $(x,p) \notin \overline{Z \cap (\{x\} \times Y)}^{X \times \beta Y}$. Since $X \times \beta Y$ is Tychonoff, there exists a continuous function $f : X \times \beta Y \to [0,1]$ such that $f\left[\overline{Z \cap (\{x\} \times Y)}^{X \times \beta Y}\right] \subseteq \{1\}$ and $f[U] \subseteq \{0\}$, where U is some neighborhood of (x,p).

Let $Z_f = f^{\leftarrow}(0)$. So $(x,p) \in int(Z_f)$. We have $(x,p) \in \overline{Z \cap Z_f}^{X \times \beta Y}$, and so

$$x \in \pi_X\left[\overline{Z \cap Z_f}^{X \times \beta Y}\right] \subseteq \overline{\pi_X\left[Z \cap Z_f\right]}^X \subseteq \pi_X\left[Z \cap Z_f\right].$$

On the other hand, since $\overline{Z \cap (\{x\} \times Y)}^{X \times \beta Y} \cap Z_f = \emptyset$, we have $Z \cap (\{x\} \times Y) \cap Z_f = \emptyset$. Now, as $x \in \pi_X\left[Z \cap Z_f\right]$, then $(x,y) \in Z \cap Z_f \neq \emptyset$ for some $y \in Y$, hence $Z \cap Z_f \cap (\{x\} \times Y) \neq \emptyset$, contradiction. Thus, $(x,p) \in \overline{Z \cap (\{x\} \times Y)}^{X \times \beta Y}$. □

2.5 Fact.

Let X be a Tychonoff space. If X is pseudocompact, then every locally finite family of nonempty open subsets of X is finite.

Proof. By way of contradiction, suppose that there exists a locally finite family $\mathcal{F} = \{U_i \in \tau(X) : U_i \neq \emptyset, i \in \mathbb{N}\}$ of nonempty open sets. Since each U_i is nonempty, choose a point $x_i \in U_i$ for each $i \in \mathbb{N}$. Since X is a Tychonoff space, there exists continuous functions $f_i : X \to [0, i]$ such that $f_i(x_i) = i$ and $f_i[X \backslash U_i] \subseteq \{0\}$ for each $i \in \mathbb{N}$.

Define the function

$$f : X \to \mathbb{R} \text{ as } f(x) = \Sigma_{i=1}^{\infty} |f_i(x)|.$$

To show that f is continuous, pick $x_0 \in X$ and an open set V of $\mathbb{R}$ containing $f(x_0)$. We can assume that $V = (f(x_0) - \frac{1}{m}, f(x_0) + \frac{1}{m})$ for some $m \in \mathbb{N}$. Since $\mathcal{F}$ is locally finite, there exists an open set $U_0 \in \tau(X)$ containing x_0 such that U_0 meets $\mathcal{F}$ only finitely many times. So we have $\{a_i\}_{i=1}^{n} \subset \mathbb{N}$ such that $U_0 \cap U_{a_i} \neq \emptyset$ for $i \leq n$, i.e., $U_0 \subseteq \bigcup \{U_{a_i} : i \leq n\}$.

As $f|_{U_0} = \Sigma_{i=1}^{n} f_{a_i}|_{U_0}$, f is continuous on U_0. By the pasting theorem, f is continuous on X. However, since $f(x_i) \geq i$ for all $i \in \mathbb{N}$, f is not bounded. This contradicts the pseudocompactness of X. $\square$

2.6 Lemma.

Let X, Y be Tychonoff spaces. If $X \times Y$ is pseudocompact, then the projection map $\pi_X : X \times Y \to X$, is z-closed.

Proof. Let Z be a zero-set in $X \times Y$. Suppose that $\pi_X[Z]$ is not closed in X. Let $p \in \overline{\pi_X[Z]}^X \backslash \pi_X[Z]$.

Since Z is a zero-set in $X \times Y$, $Z = f^{\leftarrow}(0)$ for some $f \in C^*(X \times Y)$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) = \frac{f(x,y)}{f(p,y)}$. So, $h[\{p\} \times Y] \subseteq \{1\}$ and $Z = h^{\leftarrow}(0)$. Without loss of generality, we can assume that the range of h is $[0, 1]$.

We will show that there are open sets U_n, V_n in X, and W_n in Y for $n < \omega$ such that for $m < \omega$, the following hold:

1. $p \in U_m$
2. $(V_m \times W_m) \cap Z \neq \emptyset$
3. $h[V_m \times W_m] \subseteq [0, \frac{1}{3})$
4. $h[U_m \times W_m] \subseteq (\frac{2}{3}, 1]$

5. $U_{m+1} \cup V_{m+1} \subseteq U_m$

First, pick $(x_1, y_1) \in Z$ and open sets $U_1, V_1 \in \tau(X)$ and $W_1 \in \tau(Y)$ such that $p \in U_1, x_1 \in V_1, y_1 \in W_1$, and $h[V_1 \times W_1] \subseteq [0, \frac{1}{3})$ and $h[U_1 \times W_1] \subseteq (\frac{2}{3}, 1]$. This can be done because h is continuous, $h(x_1, y_1) = 0$, and $h(p, y_1) = 1$.

Now, $U_1 \cap \pi_X[Z] \neq \emptyset$ because $x_1 \in U_1 \in \tau(X)$, and $x_1 \in \overline{\pi_X[Z]}^X$. So there is some $(x_2, y_2) \in Z$ such that $x_2 \in U_1$. Find open neighborhoods U_2 of p, V_2 of x_2, and W_2 of y_2 such that $h[V_2 \times W_2] \subseteq [0, \frac{1}{3}), h[U_2 \times W_2] \subseteq (\frac{2}{3}, 1]$, and $U_2 \cup V_2 \subseteq U_1$. Continue by induction.

The family $D = \{V_n \times W_n : n < \omega\}$ is pairwise disjoint because the V_n's are pairwise disjoint by our construction. If D is locally finite, then by **Fact 2.5**, D is finite. But D is infinite by our definition, so D cannot be locally finite. Then, there exists $(q, r) \in X \times Y$ with the property that for every neighborhood $R \times T$ of (q, r), $A = \{n \in \mathbb{N} : (V_n \times W_n) \cap (R \times T) \neq \emptyset\}$ is infinite.

On one hand, we have $(q, r) \in \overline{\bigcup \{V_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}$. Then,

$$h(q, r) \in h\left[\overline{\bigcup \{V_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}\right]$$

$$\subseteq \overline{h\left[\bigcup \{V_m \times W_m : m \in \mathbb{N}\}\right]}^{\mathbb{R}} \subseteq \overline{[0, \frac{1}{3})}^{\mathbb{R}} = [0, \frac{1}{3}].$$

On the other hand, if n and $n + k$ in A where $n, k \in \mathbb{N}$, then $V_{n+k} \subseteq U_{n+k-1} \subseteq \cdots \subseteq U_n$ by the way we constructed V_n's and U_n's. Since $(R \times T) \cap (V_{n+k} \times W_{n+k}) \neq \emptyset, (R \times T) \cap (U_n \times W_n) \neq \emptyset$ as well.

So, $(q, r) \in \overline{\bigcup \{U_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}$. Then,

$$h(q, r) \in h\left[\overline{\bigcup \{U_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}\right]$$

$$\subseteq \overline{h\left[\bigcup \{U_m \times W_m : m \in \mathbb{N}\}\right]}^{\mathbb{R}} \subseteq \overline{(\frac{2}{3}, 1]}^{\mathbb{R}} = [\frac{2}{3}, 1].$$

This is a contradiction, so $\pi_X[Z]$ must be closed in X. $\square$

2.7 Lemma.

Let X, Y be Tychonoff spaces. If π_X is z-closed, then $X \times Y$ is C^*-embedded in $X \times \beta Y$.

Proof: By **Fact 2.3**, it suffices to show that if Z_1 and Z_2 are disjoint zero-sets of $X \times Y$, then $\overline{Z_1}^{X\times\beta Y} \cap \overline{Z_2}^{X\times\beta Y} = \emptyset$.

Assume there is some point $(x,p) \in \overline{Z_1}^{X\times\beta Y} \cap \overline{Z_2}^{X\times\beta Y}$, where $x \in X$ and $p \in \beta Y \backslash Y$. By **Fact 2.4**,

$$(x,p) \in \overline{Z_1 \cap (\{x\} \times Y)}^{\{x\}\times\beta Y} \cap \overline{Z_2 \cap (\{x\} \times Y)}^{\{x\}\times\beta Y}.$$

Now, $Z_1 \cap (\{x\} \times Y)$ and $Z_2 \cap (\{x\} \times Y)$ are disjoint zero-sets in $\{x\} \times Y$. Since $\{x\} \times Y$ is C^*-embedded in $\{x\} \times \beta Y$, then, by the other direction of **Fact 2.3**,

$$\overline{Z_1 \cap (\{x\} \times Y)}^{\{x\}\times\beta Y} \cap \overline{Z_2 \cap (\{x\} \times Y)}^{\{x\}\times\beta Y} = \emptyset,$$

a contradiction, so $\overline{Z_1}^{X\times\beta Y} \cap \overline{Z_2}^{X\times\beta Y} = \emptyset$. □

2.8 Lemma.

Let Y be an extension of the space X. If X is pseudocompact, so is Y.

Proof. Let $f \in C(Y)$. Since X is pseudocompact, $f|_X$ is bounded. There is $n \in \mathbb{N}$ such that $f[X] \subseteq [-n,n]$. Now, $f[Y] = f\left[\overline{X}^Y\right] \subseteq \overline{f[X]}^{\mathbb{R}} \subseteq \overline{[-n,n]}^{\mathbb{R}} = [-n,n]$. Hence, f is bounded an Y is pseudocompact. □

2.9 Glicksberg's Theorem.

Let $X \times Y$ be Tychonoff spaces. If $X \times Y$ is pseudocompact, then $\beta(X \times Y) = \beta X \times \beta Y$.

Proof. By **Lemma 2.6**, the projection map $\pi_X : X \times Y \to X$ is z-closed. By **Lemma 2.7**, $X \times Y$ is C^*-embedded in $X \times \beta Y$. By **Lemma 2.8**, the extension $X \times \beta Y$ of $X \times Y$ is pseudocompact. Using **Lemma 2.6**, **Lemma 2.7** again, and by symmetry, $X \times \beta Y$ is C^*-embedded in $\beta X \times \beta Y$. By the transitivity of C^*-embedding, $X \times Y$ is C^*-embedded in $\beta X \times \beta Y$, i.e., $\beta(X \times Y) = \beta X \times \beta Y$. □

3 Topological Facts

In this chapter, we will prove various facts that may be of use in the proof of Buzjakova's Theorem.

Fact 3.1 If X is pseudocompact and Y is compact, then $X \times Y$ is pseudocompact.

Proof. Let $f : X \times Y \to \mathbb{R}$. As $\{x\} \times Y$ is compact, $f[\{x\} \times Y]$ is closed and bounded in $\mathbb{R}$ for all $x \in X$. We can define $g : X \to \mathbb{R}$ as

$$g(x) = \max\{f(x, y) : y \in Y\}.$$

Fix $x_0 \in X$, we will show that g is continuous at x_0. Let $\epsilon > 0$.

By our definition of g and since Y is compact, there exists some $y_0 \in Y$ such that $g(x_0) = f(x_0, y_0)$. Let $r = f(x_0, y_0)$. Now, define the sets U_y's and V_y's as follows:

For each $y \in Y$:

If $f(x_0, y) \in (r - \epsilon, r + \epsilon)$, we can find $V_y \in \tau(Y)$ and $U_y \in \tau(X)$ such that $x_0 \in U_y, y_0 \in V_y$ and $f[U_y \times V_y] \subseteq (r - \epsilon, r + \epsilon)$. In particular, since $f(x_0, y_0) \in (r - \epsilon, r + \epsilon)$, $x_0 \in U_{y_0} \in \tau(X)$, $y_0 \in V_{y_0} \in \tau(Y)$, and $f[U_{y_0} \times V_{y_0}] \subseteq (r - \epsilon, r + \epsilon)$

If $f(x_0, y) \notin (r - \epsilon, r + \epsilon)$, then since $f(x_0, y) \leq \max\{f(x_0, y) : y \in Y\} = r$, we must have $f(x_0, y) \leq r - \epsilon$. Hence, we can get $V_y \in \tau(Y)$ and $U_y \in \tau(X)$ such that $x_0 \in U_y, y_0 \in V_y$, and $f[U_y \times V_y] \subseteq (-\infty, r)$.

The family $\{V_y : y \in Y\}$ as defined aboved is an open cover of Y. By compactness, there exists $\{V_i : 1 \leq i \leq n\} \subseteq \{V_y : y \in Y\}$ such that $\bigcup\{V_i : 1 \leq i \leq n\} = Y$. Corresponding to $\{V_i : 1 \leq i \leq n\}$, we have the set $\{U_i : 1 \leq i \leq n\}$. Let $U = \bigcap\{U_i : 1 \leq i \leq n\} \cap U_{y_0}$, now U is an open set containing x_0.

For any $x \in U$, we have $\max\{f(x, y) : y \in Y\} < r + \epsilon$ because:

$$\{f(x, y) : y \in Y\} = \left\{f(x, y) : y \in \bigcup\{V_i : 1 \leq i \leq n\}\right\}$$

$$
\begin{aligned}
&= \bigcup \{f[\{x\} \times V_i] : 1 \le i \le n\} \\
&\subseteq \bigcup \{f[U_i \times V_i] : 1 \le i \le n\} \\
&\subseteq (-\infty, r+\epsilon).
\end{aligned}
$$

One the other hand, we have $\max\{f(x,y) : y \in Y\} > r - \epsilon$ because:

$$\max\{f(x,y) : y \in Y\} \ge f(x, y_0), \text{and}$$

$$f(x, y_0) \in f[U \times \{y_0\}] \subseteq f[U_{y_0} \times \{y_0\}] \subseteq f[U_{y_0} \times V_{y_0}] \subseteq (r-\epsilon, r+\epsilon).$$

We have now $r - \epsilon < \max\{f(x,y) : y \in Y\} < r + \epsilon$ for all $x \in U$. Hence, $g[U] \subseteq (r-\epsilon, r+\epsilon)$, so g is continuous on X. As X is pseudocompact, g must be bounded. Therefore, f must be bounded as well. Thus, $X \times Y$ is pseudocompact. □

Definition. A space X is *countably compact* iff each countable open cover of X has a finite subcover.

Fact 3.2 For every Hausdorff spaces X, the following statements are equivalent:

1. The space X is countably compact.
2. For every decreasing sequence $F_1 \supseteq F_2 \supseteq \cdots$ of nonempty closed subsets of X, the intersection $\bigcap_{i=1}^{\infty} F_i$ is nonempty.
3. Every countably infinite subset of X has an accumulation point.

Proof.

1⇒2: Let $F_1 \supseteq F_2 \supseteq \cdots$ be nonempty closed subsets of X. If $\bigcap_{i=1}^{\infty} F_i = \emptyset$, then $\{X \backslash F_i : i \in \mathbb{N}\}$ would be an countable open cover of X, so there is a finite subcover $\{X \backslash F_i' : 1 \le i \le n\} \subseteq \{X \backslash F_i : i \in \mathbb{N}\}$ such that $\bigcup\{X \backslash F_i' : 1 \le i \le n\} = X$. Now, because the F_i's are decreasing, without loss of generality, $F_1' \supseteq F_2' \supseteq \cdots \supseteq F_n'$. So, $X = \bigcup\{X \backslash F_i' : 1 \le i \le n\} = X \backslash F_n'$ and $F_n' = \emptyset$, a contradiction.

2⇒1: By way of contradiction, suppose that X is not countably compact. Let $\{U_i \in \tau(X) : i \in \mathbb{N}\}$ be a countable cover of X without an finite subcover. For each $n \in \mathbb{N}$, define $F_n = X \backslash \bigcup\{U_i : 1 \le i \le n\}$. For each n, F_n is nonempty because if it is, then $\{U_i : i \in \mathbb{N}\}$ would be a finite subcover of X, a contradiction. Thus, we have $F_1 \supseteq F_2 \supseteq \cdots$ and each F_n is a nonempty closed subset of X.

Now, by our assumption, the intersection $\bigcap_{i \in \mathbb{N}} F_i$ is nonempty. So there exists some $x \in \bigcap_{i \in \mathbb{N}} F_i$. So $x \in F_i$ for all $i \in \mathbb{N}$. That means $x \notin U_i$ for all $i \in \mathbb{N}$, contradicting that $\{U_i : i \in \mathbb{N}\}$ is a cover of X.

1⇒3: By way of contradiction, suppose we have a countably infinite subset $A = \{x_i \in X : i \in \mathbb{N}\}$ with no accumulation point in X. Then A is closed in X and every point in A is an isolated point with respect to A. For each $x_i \in A$, there is $x_i \in U_{x_i} \in \tau(X)$ such that $U_{x_i} \cap A = \{x_i\}$. So $\{X \backslash A\} \cup \{U_{x_i} \in \tau(X) : i \in \mathbb{N}\}$ is an countable open cover of X that yields no finite subcover, contradicting that X is countably compact.

3⇒1: By way of contradiction, suppose that $\{U_i \in \tau(X) : i \in \mathbb{N}\}$ is a countable cover of X with no finite subcover. Then, by the equivalence of **1** and **2**, there exists a decreasing sequence $F_1 \supset F_2 \cdots$ of nonempty closed subsets of X such that $\bigcap_{i \in \mathbb{N}} F_i = \emptyset$. We define the set $A = \{x_n : n \in \mathbb{N}\}$ by $x_i \in F_i$ for each $i \in \mathbb{N}$. If A is finite, there must be some $x_j \in A$ such that x_j belongs to infinitely many F_i's, and since F_i's are decreasing, x_j would have to be in all F_i's. Contradicting $\bigcap_{i \in \mathbb{N}} F_i = \emptyset$. Hence, A is an infinite set. By our assumption, A has an accumulation point. Let x be an accumulation point of A.

Since $\bigcap_{i \in \mathbb{N}} F_i = \emptyset$, there exists an i such that $x \notin F_i$. Now, $U = X \backslash F_i$ is an open set that contains x, and U does not contain any point of the set $\{x_j : j \geq i\} \subseteq F_i$. That is, $U \cap A \subseteq \{x_1, \ldots, x_i\}$ and x is not an accumulation point of A, a contradiction. □

Definition. For an ordinal τ, let $T(\tau)$ denote the space of ordinals less than τ.

Fact 3.3 Let τ be uncountable regular cardinal. Let $g : T(\tau) \to \mathbb{R}$ be continuous. Then g is constant on $[\kappa, \tau)$ for some $\kappa < \tau$.

Proof. Let $\alpha < \tau$. If $A = \{\alpha_n : n \in \mathbb{N}\}$ is a countably infinite subset of $[\alpha, \tau)$, then since τ is uncountable regular, $\beta = \sup\{\alpha_n : n \in \mathbb{N}\} < \tau$ and $A \subseteq [\alpha, \beta]$, a compact subset. So A has an accumulation point. Thus $[\alpha, \tau)$ must be countably compact by **Fact 3.2**.

Since g is continuous, $g\,[[\alpha, \tau)]$ is a countably compact subset of $\mathbb{R}$. In metric spaces, countably compact is equivalent to compact because metric spaces are Lindelöf. Hence, $g[[\alpha,\tau)]$ is compact for all $\alpha < \tau$. Thus, there exists $p \in \bigcap_{\alpha<\tau} g[[\alpha,\tau)]$. To show that p is unique, suppose that there exists $q \in \bigcap_{\alpha<\tau} g[[\alpha,\tau)]$.

There exists some $\alpha_0 \in [0,\tau)$ such that $g(\alpha_0) = p$. As $q \in g[[\alpha_0+1,\tau)]$, there exists $\alpha_1 \in [\alpha_0+1,\tau)$ such that $g(\alpha_1) = q$. As $p \in g[[\alpha_1+1,\tau)]$, there exists $\alpha_2 \in [\alpha_1+1,\tau)$ such that $g(\alpha_2) = p$. We continue this process by induction. We have now:

$$p = g(\alpha_0) = g(\alpha_2) = g(\alpha_4) = \cdots$$

$$q = g(\alpha_1) = g(\alpha_3) = g(\alpha_5) = \cdots$$

Let $\beta = \sup\{\alpha_n : n < \omega\}$, which exists because $cf(\tau) > \omega$. By the continuity of g, $g(\beta) = \lim_{n<\omega} g(\alpha_n)$. Thus,

$$p = \lim_{n<\omega} g(\alpha_{2n}) = g(\beta) = \lim_{n<\omega} g(\alpha_{2n+1}) = q$$

So, $\bigcap_{\alpha<\tau} g[[\alpha,\tau)] = \{p\}$. Since $\mathbb{R}$ is locally compact, for each $n < \omega$, we can find some $\gamma_n \in T(\tau)$ such that $g[[\gamma_n,\tau)] \subseteq (p - \frac{1}{n}, p + \frac{1}{n})$. Let $\kappa = \sup_{n<\omega} \gamma_n$. So, we have

$$g[[\kappa,\tau)] \subseteq \bigcap_{n<\omega} (p - \frac{1}{n}, p + \frac{1}{n}) = \{p\}.$$

□

Fact 3.4 Let τ be an uncountable regular cardinal. The space $T(\tau)$ is pseudocompact.

Proof. Let $g : T(\tau) \to \mathbb{R}$ be a continuous function. By **Fact 3.3**, there exists $\kappa < \tau$ such that $g[[\kappa,\tau)] = \{r\}$ for some $r \in \mathbb{R}$. As $[0,\kappa+1]$ is compact, g is bounded on $[0,\kappa+1]$. Thus, $g[T(\tau)] = g[[0,\kappa+1]] \cup \{r\}$ is bounded. □

Fact 3.5 Let X be a pseudocompact Tychonoff space and $\tau = |\beta X|^+$. Then, $X \times T(\tau)$ is pseudocompact.

Proof. Let $f : X \times T(\tau) \to \mathbb{R}$ be continuous. By **Fact 3.4** , the ordinal space $T(\tau)$ is pseudocompact.

By **Fact 3.3**, for each $x \in X$, there exists $\kappa_x < \tau$ such that f is constant on $\{x\} \times [\kappa_x, \tau)$. As $cf(\tau) > |X|$, $\kappa = \sup_{x\in X}\{\kappa_x : x \in X\} < \tau$. Now, $f[X \times [0,\kappa+1]]$ is bounded because $X \times [0,\kappa+1]$ is pseudocompact by **Fact**

3.1. For $\alpha \geq \kappa, f(x,\alpha) = f(x,\beta)$. Thus, $f[X \times [\kappa,\tau)] = f[X \times \{\kappa\}]$ which is bounded because X is pseudocompact. The boundedness of $f[X \times [0,\kappa+1]]$ and $f[X \times [\kappa,\tau)]$ gives us that $f[X \times T(\tau)]$ is bounded. Hence, $X \times T(\tau)$ is pseudocompact. $\square$

Fact 3.6 Let τ be an uncountable regular cardinal. Let $T(\tau)$ be the space of all ordinal numbers less than τ. Let A_α be a closed, unbounded subset of $T(\tau)$. Let $\gamma \in T(\tau)$. Then, $\bigcap\{A_\alpha : \alpha < \gamma\}$ is closed, unbounded and $|\bigcap\{A_\alpha : \alpha < \gamma\}| = \tau$.

Proof.
We will construct the set $\{p_\alpha : \alpha < \tau\}$ by transfinite induction.

`Step 1.`
Pick any element $a_{1,1} \in A_1$, we can find some element $a_{1,2} \in A_2$ such that $a_{1,2} > a_{1,1}$ because A_2 is unbounded. Then, by continuing this process, we can define $a_{1,n}$ in the same way, for all $n < \omega$. For all $\alpha < \gamma$, If α is a successor ordinal, then since A_α is unbounded, we can find some $a_{1,\alpha} \in A_\alpha$ such that $a_{1,\alpha} > a_{1,\alpha-1}$. If α is a limit ordinal, then let $\beta = \sup_{\kappa<\alpha}\{a_{1,\kappa}\}$, which exists because $\alpha < cf(\tau)$. Now, since A_α is unbounded, we can find some $a_{1,\alpha} \in A_\alpha$ such that $a_{1,\alpha} > \beta$.

Thus, we have defined the set $\{a_{1,\alpha} : \alpha < \gamma\}$. Let $\beta_1 = \sup\{a_{1,\alpha} : \alpha < \gamma\}$, which exists because $\gamma < cf(\tau)$.

`Step N.` Let $a_{n,1} \in A_1$ be such that $a_{n,1} > \beta_{n-1}$. Let $a_{n,2} \in A_2$ be such that $a_{n,2} > a_{n,1}$. Now continuing the same way as in Step 1, we can define $a_{n,\alpha}$ for all $\alpha < \gamma$. Let $\beta_n = \sup\{a_{n,\alpha} : \alpha < \gamma\}$.

So, we have contructed the set $\{a_{n,\alpha} : n < \omega, \alpha < \gamma\}$.

For all $\alpha < \gamma, \lim_{n<\omega} a_{n,\alpha} \in A_\alpha$ because A_α is closed. Moreover, if $\alpha, \alpha' < \gamma$, then $\lim_{n<\omega} a_{n,\alpha} = \lim_{n<\omega} a_{n,\alpha'}$. So if we define $p_1 = \lim_{n<\omega} a_{n,\alpha}$ for some $\alpha < \gamma$, then $p_1 \in \bigcap\{A_\alpha : \alpha < \gamma\}$.

For all $\alpha < \tau$, if α is an isolated ordinal, then we start from $p_{\alpha-1} \in A_1$ in Step 1 again, and define p_α the same way as we did for p_1. If α is a limit ordinal, then we let $p_\alpha = \sup\{p_\kappa : \kappa < \alpha\}$. This exists because $\alpha < cf(\tau)$.

We've finished contruction of the set $\{p_\alpha : \alpha < \tau\} \subseteq T(\tau)$. From the way we contructed it, this set is closed, unbounded and its cardinality is τ. $\square$

Fact 3.7 Let X be a Tychonoff space and $|X| > \aleph_0$. Let $\tau = |\beta X|^+$. Then,

$T(\tau)$ can be condensed onto $T(\tau+1)$. Moreover, for any space $X, X \times T(\tau)$ condenses onto $X \times T(\tau+1)$.

Proof. Define $g : T(\tau) \to T(\tau+1)$ by $g(0) = \tau$, $g(\alpha) = \alpha - 1$ for all $0 < \alpha < \omega$, and $g(\alpha) = \alpha$ for $\omega \leq \alpha < \tau$. Now, g is one-to-one and onto. Note that g is continuous at ω because if $(\beta, \omega]$ is an open set containing $g(\omega)$,then $(\beta+1, \omega]$ is an open set such that $g\,[(\beta+1, \omega]] \subseteq (\beta, \omega]$, and g is continuous on all $\alpha < \omega$ because $\{\alpha\} \in T(\tau)$; finally, g is continuous on all $\alpha > \omega$ because $g|_{(\omega,\tau)}$ is the identity function. Thus, $T(\tau)$ can be condensed onto $T(\tau+1)$.

Now, define $h : X \times T(\tau) \to X \times T(\tau+1)$ by $h(x, \alpha) = (x, g(\alpha))$. Since g is one-to-one, onto, and continuous, then, h must also be one-to-one, onto, and continuous. $\square$

Fact 3.8 Let Z be a Tychonoff space. Let A be a closed subset of Z, and B be a compact subset of βZ such that A and B are disjoint. Moreover, the set $A \cup B$ is not compact in βZ. Then, there exists a system $D = \{D_\alpha\}_{\alpha<l}$ satisfying the following conditions:

1. For each α, the set D_α is nonempty and closed in A.
2. For $\alpha > \beta, D_\alpha \subseteq D_\beta$ and if β is a limit ordinal number, then $D_\beta = \bigcap\{D_\alpha : \alpha < \beta\}$.
3. $\bigcap\{D_\alpha : \alpha < l\} = \emptyset$.
4. $\overline{D_1}^{\beta Z} \cap B = \emptyset$.

Proof.
Since $A \cup B$ is not compact, there is an open cover $\mathcal{C} \subset \tau(\beta Z)$ of $A \cup B$ that has no finite subcover. Since B is compact and $\mathcal{C}$ covers B, there is a finite subcover $\{C_i : 1 \leq i \leq n\} \subseteq \mathcal{C}$ such that $B \subseteq \bigcup\{C_i : 1 \leq i \leq n\}$.

Let $E = A \backslash \bigcup\{C_i : 1 \leq i \leq n\}$. E is closed in A. E is nonempty because otherwise $\{C_i : 1 \leq i \leq n\}$ covers A as well as B, a contradiction. Furthermore, E is not compact. If E is compact, we can get a finite subcover $\{C'_i : 1 \leq i \leq n'\}$ from $\mathcal{C}$. Then, $\{C'_i : 1 \leq i \leq n'\} \cup \{C_i : 1 \leq i \leq n\}$ is a finite subcover that covers $A \cup B$, a contradiction.

As E is not compact, we can find an open cover $\mathcal{F} \subseteq \tau(\beta Z)$ such that no finite subcover of $\mathcal{F}$ covers E. without loss of generality, we can assume that $|\mathcal{F}| = L(E)$, the Lindelöf number of E. We can well-order $\mathcal{F}$, so $\mathcal{F} = \{F_\alpha : \alpha < L(E)\}$. Define $D_\alpha = E \backslash \bigcup\{F_\gamma : \gamma < \alpha\}$ for each $\alpha < L(E)$. We shall verify that D satisfies all four condictions:

1. For each α, the set D_α is nonempty and closed in A.
   `Proof-` Each D_α is nonempty because if $D_\alpha = \emptyset$ for some α, then $E \backslash \bigcup \{F_\gamma : \gamma < \alpha\} =$ and so $E \subseteq \{F_\gamma : \gamma < \alpha\}$. However, since $\alpha < L(E)$, we have a contradiction. So D_α is nonempty. Moreover, D_α is closed in A because it is closed in E, and E is closed in A.

2. For $\alpha > \beta, D_\alpha \subseteq D_\beta$ and if $\beta < L(E)$ is a limit ordinal number, then $D_\beta = \bigcap \{D_\alpha : \alpha < \beta\}$.
   `Proof-` By the way we defined the $D_\alpha{}'s$, $D_\alpha \subseteq D_\beta$ if $\alpha > \beta$. If β a limit ordinal number, and if $D_\beta \neq \bigcap \{D_\alpha : \alpha < \beta\}$, then we replace D_β with the set $\bigcap \{D_\alpha : \alpha < \beta\}$, which is nonempty and
   closed in A. So now, $D_\beta = \bigcap \{D_\alpha : \alpha < \beta\}$.

3. $\bigcap \{D_\alpha : \alpha < L(E)\} = \emptyset$.
   `Proof-` This is true because $\bigcap \{D_\alpha\} = E \backslash \bigcup \{F_\gamma : \gamma < L(E)\} = \emptyset$.

4. $\overline{E}^{\beta Z} \cap B = \emptyset$.
   `Since` $\overline{E}^{\beta Z} \cap B = \emptyset$`, and` $D_1 \subseteq E$`, then` $\overline{D_1}^{\beta Z} \cap B = \emptyset$.

□

Fact 3.9 Let X be a Tychonoff space. If B_1, B_2 are subsets of X such that $\overline{B_1}^{\beta X} \cap \overline{B_2}^{\beta X} \neq \emptyset$, then B_1 and B_2 are not completely separated in X.

Proof. Suppose B_1 and B_2 are completely separated in X. Then there exists a continuous function function $f : X \to [0,1]$ such that $f[B_1] \subseteq \{0\}$ and $f[B_2] \subseteq \{1\}$. Let $\bar{f} : \beta X \to [0,1]$ be the extension of f. The sets $\bar{f}^{\leftarrow}(0)$ and $\bar{f}^{\leftarrow}(1)$ are closed in βX such that $\overline{B_1}^{\beta X} \subseteq \bar{f}^{\leftarrow}(0)$ and $\overline{B_2}^{\beta X} \subseteq \bar{f}^{\leftarrow}(1)$. Since $\bar{f}^{\leftarrow}(0) \cap \bar{f}^{\leftarrow}(1) = \emptyset$, we have $\overline{B_1}^{\beta X} \cap \overline{B_2}^{\beta X} = \emptyset$, a contradiction. □

Fact 3.10 Let X be a Tychonoff space. If X is locally compact, then X can be condensed onto a compact space.

Proof. Let $X \cup \{\infty\}$ be the one-point compactification of X. Pick any $x_0 \in X$. Let K be the space $X \cup \{\infty\}$ with the point ∞ identified with x_0. In K, the open sets containing x_0 are of the form $U_{x_0} \cup V_\infty$, where U_{x_0} is any open set containing x_0 in X, and V_∞ is any open set containing ∞ in $X \cup \{\infty\}$. For $x \in K \backslash \{x_0\}$, the open sets containing x in K are same as the open sets containing x in X.

K is compact with the topology we've just defined. Let $f : X \times K$ be the identity map. So, f is one-to-one and onto. Let $U_{x_0} \cup V_\infty$ be an open set containing $f(x_0) = x_0$, then U_{x_0} is an open set in X such that $f[U_{x_0}] = U_{x_0} \subset U_{x_0} \cup V_\infty$. So f is continuous on x_0, as well as on other points of X. Hence, X can be condensed onto K. $\square$

CHAPTER IV

Buzjakova's Theorem

In this chapter, we will prove the Buzjakova's Theorem. Before proving the theorem, we will need to make a remark and proving three lemmas. The last two lemmas, **Lemma 4.3** and **Lemma 4.4**, are essential in the proof of Buzjakova's Theorem.

The following remark is a corollary of Glicksberg's Theorem, and it will be used throughout the proof of Buzjakova's Theorem without being explicitly referenced.

Remark 4.1 Let X be a pseudocompact space and $|X| \geq \aleph_0$. Then

1. $\beta(X \times T(|\beta X|^+ + 1)) = \beta X \times T(|\beta X|^+ + 1)$
2. $\beta(X \times T(|\beta X|^+)) = \beta X \times T(|\beta X|^+ + 1)$

Proof. Since X is pseudocompact and $T(|\beta X|^+ + 1)$ is compact, by **Fact 3.1**, $X \times T(|\beta X|^+ + 1)$ is pseudocompact. By Glicksberg's Theorem, $\beta(X \times T(|\beta X|^+ + 1)) = \beta X \times \beta T(|\beta X|^+ + 1) = \beta X \times T(|\beta X|^+ + 1)$. This proves the first statement.

By **Fact 3.5**, $X \times T(|\beta X|^+)$ is pseudocompact. By Glicksberg's Theorem, $\beta(X \times T(|\beta X|^+)) = \beta X \times \beta T(|\beta X|^+) = \beta X \times T(|\beta X|^+ + 1)$. This proves the second statement. $\square$

Lemma 4.2
Let X, Y be Tychonoff spaces and cX, cY be T_2 compactifications of X, Y. Let f be a continuous function from X onto Y, $\bar{f} : cX \to cY$ be the continuous extension of f. Let A be a closed subset of X. Then we have:
(i) If $f[A]$ is not closed in Y then there exists an element $x \in \overline{A}^{cX} \setminus A$ such that $\bar{f}(x) \in \overline{f[A]}^{Y} \setminus f[A]$.
(ii) If $f[A]$ is closed in Y then for any element $x \in \overline{A}^{cX}$ for which $\bar{f}(x) \in Y$

holds, we have $\bar{f}(x) \in f[A]$ holds.

Before we prove *(i)* and *(ii)*, we note that $\overline{f[A]}^{cY} = \bar{f}\left[\overline{A}^{cX}\right]$ as $\bar{f}$ is a closed function and $\bar{f}\left[\overline{A}^{cX}\right] = \overline{\bar{f}[A]}^{cY} = \overline{f[A]}^{cY}$.

Proof of (i):
By the above, $\overline{f[A]}^{cY} = \bar{f}\left[\overline{A}^{cX}\right]$. So, $\overline{f[A]}^{Y} \backslash f[A] = \overline{f[A]}^{cY} \cap Y \backslash f[A] = \bar{f}\left[\overline{A}^{cX}\right] \cap Y \backslash f[A]$.

As $f[A]$ is not closed in Y, we have $\bar{f}\left[\overline{A}^{cX}\right] \cap Y \backslash f[A] \neq \emptyset$. So there exists $x \in \overline{A}^{cX}$ such that $\bar{f}(x) \in \bar{f}\left[\overline{A}^{cX}\right] \cap Y \backslash f[A] = \overline{f[A]}^{cY} \cap Y \backslash f[A] = \overline{f[A]}^{Y} \backslash f[A]$, as required.

Proof of (ii):
By the above, we have $\bar{f}\left[\overline{A}^{cX}\right] = \overline{f[A]}^{cY}$. So, $\bar{f}\left[\overline{A}^{cX}\right] \cap Y = \overline{f[A]}^{cY} \cap Y$. As $f[A]$ is closed in Y, so $\overline{f[A]}^{cY} \cap Y = \overline{f[A]}^{Y} = f[A]$. So now we have $\bar{f}\left[\overline{A}^{cX}\right] \cap Y = f[A]$, and it gives us that if $x \in \overline{A}^{cX}$ and if $\bar{f}(x) \in Y$, then $\bar{f}(x) \in f[A]$, as required. $\square$

Lemma 4.3 Let X be a pseudocompact space and f be a continuous mapping of $X \times T(|\beta X|^{+})$ onto a Tychonoff space Z. By **Remark 4.1**, there is a continuous function $\bar{f} : \beta X \times T(|\beta X|^{+} + 1) \to \beta Z$ that extends f. Let $C_1 = \{x \in \beta X \backslash X : |\bar{f}[\{x\} \times T(|\beta X|^{+})] \cap Z| = |\beta X|^{+}\}$. If $x \in C_1$, there exists an element $y_x \in X$ and a subset $T_x \subseteq T(|\beta X|^{+})$ satisfying the following conditions:

1. $|T_x| = |\beta X|^{+}$
2. The set T_x is closed in $T(|\beta X|^{+})$.
3. For any ordinal number $\alpha \in T_x, \bar{f}(x, \alpha) = f(y_x, \alpha)$ holds.

Proof:
Let $x \in C_1$. Since f is onto, we can write $Z = \bigcup\{f[\{y\} \times T(|\beta X|^{+})] : y \in X\}$. Then,

$$\bar{f}[\{x\} \times T(|\beta X|^{+})] \cap Z$$

$$= \bar{f}\left[\{x\} \times T(|\beta X|^+)\right] \cap \left(\bigcup \left\{f[\{y\} \times T(|\beta X|^+)] : y \in X\right\}\right)$$

$$= \bigcup \left\{\bar{f}\left[\{x\} \times T(|\beta X|^+)\right] \cap f\left[\{y\} \times T(|\beta X|^+)\right] : y \in X\right\}.$$

As $|\beta X|^+$ is regular, $|X| < |\beta X|^+ = cf(|\beta X|^+)$. So, at lease one of the terms of our union has cardinality $|\beta X|^+$.
Let that term be:

$$\bar{f}\left[\{x\} \times T(|\beta X|^+)\right] \cap f\left[\{y_x\} \times T(|\beta X|^+)\right].$$

Fix this y_x. We will now construct the set $T_x = \{\beta_\alpha : \alpha < |\beta X|^+\}$ by transfinite induction:

`Step 1.`
Pick two ordinals, α_1 and α^1 from $T(|\beta X|^+)$ such that $\bar{f}(x, \alpha_1) = f(y_x, \alpha^1)$. This can be done because $\bar{f}[\{x\} \times T(|\beta X|^+)] \cap f[\{y_x\} \times T(|\beta X|^+)]$ is nonempty.

`Induction Hypothesis.`
For each $k < n$, we can find two ordinals α_k and α^k from $T(|\beta X|^+)$, satisfying the following conditions:
`a)` $\bar{f}(x, \alpha_k) = f(y_x, \alpha^k)$.
`b)` $\alpha_k > \max\{\alpha_{k-1}, \alpha^{k-1}\}$.
`c)` $\alpha^k > \max\{\alpha_{k-1}, \alpha^{k-1}\}$.

`Step N.`
Let $\alpha = \max\{\alpha_{n-1} + 1, \alpha^{n-1}\}$. Now,

$$\begin{aligned}
&|\beta X|^+ \\
&= \left|\bar{f}\left[\{x\} \times T(|\beta X|^+)\right] \cap f\left[\{y_x\} \times T(|\beta X|^+)\right]\right| \\
&= \left|\bar{f}\left[\{x\} \times T(|\beta X|^+)\right] \cap f\left[\{y_x\} \times T(\alpha)\right]\right| \\
&+ \left|\bar{f}\left[\{x\} \times T(\alpha)\right] \cap f\left[\{y_x\} \times T(|\beta X|^+)\right]\right| \\
&+ \left|\bar{f}\left[\{x\} \times T(|\beta X|^+) \backslash T(\alpha)\right] \cap f\left[\{y_x\} \times T(|\beta X|^+) \backslash T(\alpha)\right]\right|
\end{aligned}$$

Since the first two terms of the sum both have cardinality no more than α, where $\alpha < |\beta X|^+$, the third term must have cardinality equal to $|\beta X|^+$. Otherwise the sum of these three terms will not add up to $|\beta X|^+$. Obviously, $\bar{f}[\{x\} \times T(|\beta X|^+) \backslash T(\alpha)] \cap f[\{y_x\} \times T(|\beta X|^+) \backslash T(\alpha)]$ is nonempty. So, we can pick two ordinals α_n and α^n from $T(|\beta X|^+) \backslash T(\alpha)$ such that these conditions hold:

a) $\bar{f}(x, \alpha_n) = f(y_x, \alpha^n)$.
b) $\alpha_n > \alpha = \max\{\alpha_{n-1}, \alpha^{n-1}\}$.
c) $\alpha^n > \alpha = \max\{\alpha_{n-1}, \alpha^{n-1}\}$.

This completes `Step N`. Hence, we can define α_n and α^n for all $n < \omega$.

Let $\beta_1 = \sup\{\alpha_n : n < \omega\} = \sup\{\alpha^n : n < \omega\}$. Such $\beta_1 \in T(|\beta X|^+)$ exists because by condition `b)` and `c)`, the sup's must equal, provided they exist, and indeed, the existence follows from $cf(|\beta X|^+) > \omega$.

Since f is continuous, $\{f(y_x, \alpha^n) : n < \omega\}$ converges to $f(y_x, \beta_1)$.
Since $\bar{f}(x, \alpha_n) = f(y_x, \alpha^n)$ for all $n < \omega$, $\{\bar{f}(x, \alpha_n) n < \omega\}$ converges to $f(y_x, \beta_1)$ also. At the same time, since $\bar{f}$ is continuous, $\{\bar{f}(x, \alpha_n) : n < \omega\}$ converges to $\bar{f}(x, \beta_1)$. Hence, $\bar{f}(x, \beta_1) = f(y_x, \beta_1)$.

Now, we will define all the other β_α's by transfinite induction:

`Induction Hypothesis.`
Let β_α be defined for all $\alpha < \gamma$ such that $\bar{f}(x, \beta_\alpha) = f(y_x, \beta_\alpha)$.

`Step` γ`(isolated ordinal).`
As $\bar{f}[\{x\} \times T(|\beta X|^+) \backslash T(\beta_{\gamma-1})] \cap f[\{y_x\} \times T(|\beta X|^+) \backslash T(\beta_{\gamma-1})]$ is nonempty, we can find a pair of ordinals in $T(|\beta X|^+) \backslash T(\beta_{\gamma-1})$, enabling us to start from `Step 1` again with those two ordinals. Then, we can constuct β_γ in the way as we did for β_1.

`Step` γ`(limit ordinal).`
Define $\beta_\gamma = \sup\{\beta_\alpha : \alpha < \gamma\}$. Again, the sup exists because $\gamma < |\beta X|^+ = cf|\beta X|^+$. Furthermore, $\bar{f}(x, \beta_\gamma) = f(y_x, \beta_\gamma)$ by the continuity of $\bar{f}$.

By defining the β_γ's this way for all $\gamma < |\beta X|^+$, we have just successfully constructed the set $T_x = \{\beta_\alpha : \alpha < |\beta X|^+\}$. By the way T_x and y_x were defined, conditions 1 and 3 as required by the lemma automatically follow. $T_x = \{\beta_\alpha : \alpha < |\beta X|^+\}$ is closed in $T(|\beta X|^+)$ because we defined $\beta_\gamma = \sup\{\beta_\alpha : \alpha < \gamma\}$ if γ is a limit ordinal. Hence condition 2 of the lemma follows as well. This proves the Lemma. □

Lemma 4.4 Let X be a pseudocompact space, and let f be a continuous one-to-one function from $X \times T(|\beta X|^+)$ onto Z. We are given two sets:
$C_1 = \{x \in \beta X \backslash X : |\bar{f}[\{x\} \times T(|\beta X|^+)] \cap Z| = |\beta X|^+\}$.
$C_2 = \{x \in \beta X \backslash X : |\bar{f}[\{x\} \times T(|\beta X|^+)] \cap Z| < |\beta X|^+$ and for any $\alpha \in$

$T(|\beta X|^+)$, there exists $\alpha_1 \in T(|\beta X|^+)$ such that $\alpha_1 > \alpha$ and $\bar{f}(x, \alpha_1) \in Z\}$.

Then, $C_1 \cap \overline{C_2}^{\beta X} = \emptyset$ and $X \cap \overline{C_2}^{\beta X} = \emptyset$.

Proof. For each $y \in C_2$, let $Z_y = \bar{f}\,[\{y\} \times T(|\beta X|^+)] \cap Z$.
Since $\{y\} \times T(|\beta X|^+) = \bigcup \left\{\{y\} \times T(|\beta X|^+) \cap \bar{f}^{-1}(z) : z \in Z_y\right\}$, we have:

$$|\beta X|^+ = |\{y\} \times T(|\beta X|^+)| = \left|\bigcup \left\{\{y\} \times T(|\beta X|^+) \cap \bar{f}^{-1}(z) : z \in Z_y\right\}\right|.$$

Since $|Z_y| < cf(|\beta X|^+)$, at least one of the terms in the union must have cardinality equal to $|\beta X|^+$. So there exists $z \in Z_y$ such that

$$\left|\{y\} \times T(|\beta X|^+) \cap \bar{f}^{-1}(z)\right| = |\beta X|^+ .$$

Define $A_y = \pi_2\left[\{y\} \times T(|\beta X|^+) \cap \bar{f}^{-1}(z)\right]$, where $\pi_2 : \beta X \times T(|\beta X|^+) \to T(|\beta X|^+)$ is the projection map. Thus, we defined $A_y \subseteq T(|\beta X|^+)$ such that $|A_y| = |\beta X|^+$ and $\bar{f}[\{y\} \times A_y] = \{z\}$. Since $z \in Z$ and f is onto, $z = f(x_y, \alpha_y)$ for some $(x_y, \alpha_y) \in X \times T(|\beta X|^+)$. Hence $\bar{f}\,[\{x\} \times A_y] = \{f(x_y, \alpha_y)\}$ for some $(x_y, \alpha_y) \in X \times T(|\beta X|^+)$. By the continuity of $\bar{f}$, we have that

$$\bar{f}\left[\{y\} \times \overline{A_y}^{T(|\beta X|^+)}\right] = f(x_y, \alpha_y) \text{ holds for all } y \in C_2.$$

We'll first prove that $C_1 \cap \overline{C_2}^{\beta X} = \emptyset$. Suppose the contrary. Let $x \in C_1 \cap \overline{C_2}^{\beta X}$ and T_x as defined in **Lemma 4.3**. Let $T = \bigcap \left\{\overline{A_y}^{T(|\beta X|^+)} : y \in C_2\right\} \cap T_x,$. Now, by **Fact 3.6**, $|T| = |\beta X|^+$, and T is closed and unbounded in $T(|\beta X|^+)$.

As $x \in C_1$ and $T \subseteq T_x$, so we must have $\bar{f}\,[\{x\} \times T] = f\,[\{y_x\} \times T]$. Since f is one-to-one, $|f\,[\{y_x\} \times T]| = |T| = |\beta X|^+$. So, $\left|\bar{f}\,[\{x\} \times T] \cap Z\right| = |\beta X|^+$.

On the other hand, since $x \in \overline{C_2}^{\beta X}$, we have $\left|\bar{f}\,[\{x\} \times T] \cap Z\right| < |\beta X|^+$ holds. Because:

$$\bar{f}\,[\{x\} \times T] \cap Z \subseteq \bar{f}\left[\overline{C_2}^{\beta X} \times T\right] \cap Z \subseteq \bar{f}\left[\overline{C_2 \times T}^{\beta X \times T(|\beta X|^+ + 1)}\right] \cap Z$$

$$= \bar{f}\left[\overline{C_2 \times T}^{\beta\left(X \times T(|\beta X|^+)\right)}\right] \cap Z \subseteq \overline{\bar{f}\,[C_2 \times T]}^{\beta Z} \cap Z = \overline{\{f(x_y, \alpha_y) : y \in C_2\}}^{\beta Z},$$

where the last equality holds because for each $y \in C_2$, $\bar{f}\,[\{y\} \times T] = \{f(x_y, \alpha_y)\}$ for some $(x_y, \alpha_y) \in X \times T(|\beta X|^+)$. Now, let $\alpha = \sup\{\alpha_y : y \in C_2\}$, which exists because $|C_2| < cf\left(|\beta X|^+\right)$.

Since $\{f(x_y,\alpha_y) : y \in C_2\} \subseteq \bar{f}\,[\beta X \times T(\alpha+1)] = \overline{\bar{f}\,[\beta X \times T(\alpha+1)]}^{\beta Z}$, we must have $\overline{\{f(x_y,\alpha_y) : y \in C_2\}}^{\beta Z} \subseteq \bar{f}\,[\beta X \times T(\alpha+1)]$.

Hence, $\bar{f}\,[\{x\} \times T] \cap Z \subseteq \bar{f}\,[\beta X \times T(\alpha+1)]$. As $\left|\bar{f}\,[\beta X \times T(\alpha+1)]\right| < |\beta X|^+$, we conclude that $\left|\bar{f}\,[\{x\} \times T] \cap Z\right| < |\beta X|^+$. This contradicts to our previous argument that $\left|\bar{f}\,[\{x\} \times T] \cap Z\right| = |\beta X|^+$. Therefore, $C_1 \cap \overline{C_2}^{\beta X} = \emptyset$.

We will show that $X \cap \overline{C_2}^{\beta X} = \emptyset$. Suppose the contrary. Let $x \in X \cap \overline{C_2}^{\beta X}$. By replacing T with $T' = \bigcap \left\{ \overline{A_y}^{T(|\beta X|^+)} : y \in C_2 \right\}$ in the above, same conclusion follows, namely $\left|\bar{f}\,[\{x\} \times T] \cap Z\right| < |\beta X|^+$.

However, since $x \in X$, $\left|\bar{f}\,[\{x\} \times T] \cap Z\right| = |f\,[\{x\} \times T] \cap Z| = |f\,[\{x\} \times T]| = |\beta X|^+$, where the last equality holds because f is one-to-one and $|T| = |\beta X|^+$. Again, we reached contradiction. Thus, $X \cap \overline{C_2}^{\beta X} = \emptyset$ holds. □

We will now prove the theorem, one direction is trivial. Buzjakova proved the hard direction in 1997. We consider only Tychonoff spaces.

Buzjakova's Theorem. A pseudocompact space X condenses onto a compact space if and only if the space $X \times T(|\beta X|^+ + 1)$ condenses onto a normal space.

Proof.

(⇒:) Let X be a pseudocompact space that condenses onto a compact space K. So there exists $f : X \to K$ such that f is one-to-one, onto, and continuous. Define $g : X \times T(|\beta X|^+ + 1) \to K \times T(|\beta X|^+ + 1)$ by $g(x,\alpha) = (f(x),\alpha)$. Then, g is one-to-one, onto, and continuous. Hence $X \times T(|\beta X|^+ + 1)$ condenses onto a normal space.

(⇐:) By **Fact 3.7**, the space $X \times T(|\beta X|^+)$ condenses onto $X \times T(|\beta X|^+ + 1)$. Since $X \times T(|\beta X|^+ + 1)$ condenses onto some normal space Z by our assumption, the space $X \times T(|\beta X|^+)$ must condenses onto Z, too. So, there exists $f : X \times T(|\beta X|^+) \to Z$, where f is ono-to-one, onto, and continuous. Let

$C_1 = \{x \in \beta X \backslash X : |\bar{f}[\{x\} \times T(|\beta X|^+)] \cap Z| = |\beta X|^+\}$, and

$C_2 = \{x \in \beta X \backslash X : \left|\bar{f}[\{x\} \times T(|\beta X|^+)] \cap Z\right| < |\beta X|^+$ and for any $\alpha \in T(|\beta X|^+)$,
there exists $\alpha_1 \in T(|\beta X|^+)$ such that $\alpha_1 > \alpha$ and $\bar{f}(x,\alpha_1) \in Z\}$.

Let $T = \bigcap\{T_x : x \in C_1\}$, where T_x was defined in **Lemma 4.3**. By **Fact 3.6,** T is closed and unbounded in $T(|\beta X|^+)$ and $|T| = |\beta X|^+$.

Let $x \in \overline{C_2}^{\beta X}$. Since $\overline{C_2}^{\beta X} \cap (C_1 \cup X) = \emptyset$ by **Lemma 4.4**, $x \in \beta X \backslash (C_1 \cup X)$. Thus,

$$\left|\bar{f}\left[\{x\} \times T(|\beta X|^+)\right] \cap Z\right| < |\beta X|^+.$$

Let $\alpha_x = \sup\left\{\alpha \in T(|\beta X|^+) : f(y,\alpha) \in \bar{f}\left[\{x\} \times T(|\beta X|^+)\right] \cap Z \text{ for some } y \in X\right\}$. Since $\left|\bar{f}\left[\{x\} \times T(|\beta X|^+)\right] \cap Z\right| < |\beta X|^+$, such α_x exists in $T(|\beta X|^+)$. For any $\alpha_1 > \alpha_x$, we have either $\bar{f}(x,\alpha_1) \in \beta Z \backslash Z$ or $\bar{f}(x,\alpha_1) \in Z$. If $\bar{f}(x,\alpha_1) \in Z$, we must have $\bar{f}(x,\alpha_1) = f(y,\alpha_2)$ for some α_2 because f is onto. Furthermore, since $\alpha_1 > \alpha_x$ and by our definition of $\alpha_x, \alpha_1 > \alpha_2$ must hold.

Let $\alpha^* = \sup\left\{\alpha_x : x \in \overline{C_2}^{\beta X}\right\}$. Such α^* exists because $\left|\overline{C_2}^{\beta X}\right| < cf(|\beta X|^+)$. For any $\alpha > \alpha^*, \alpha \in T$,

$$f\left[X \times \{\alpha\}\right] \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right] = \emptyset$$

holds.

This is because if $z \in f\left[X \times \{\alpha\}\right] \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$, we can write $z = \bar{f}(x,\alpha)$ for some $x \in \overline{C_2}^{\beta X}$. From the definition of α^* and the fact that $\alpha > \alpha^*$, we have either $\bar{f}(x,\alpha) = f(y,\alpha_2)$ where $y \in X$ and $\alpha_2 < \alpha$, or, we have $\bar{f}(x,\alpha) \in \beta Z \backslash Z$. Suppose that $\bar{f}(x,\alpha) = f(y,\alpha_2)$ holds. Since f is one-to-one, $\bar{f}(x,\alpha) = f(y,\alpha_2) \neq f(y',\alpha)$ for any $y' \in X$. Thus, $z \notin f\left[X \times \{\alpha\}\right]$, a contradiction. If $\bar{f}(x,\alpha) \in \beta Z \backslash Z$ holds, $\bar{f}(x,\alpha) \notin Z \supseteq f\left[X \times \{\alpha\}\right]$, a contradiction. Thus, $f\left[X \times \{\alpha\}\right]$ and $\bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$ are disjoint.

Having defined α^*, we have two cases:

CASE I. For all $\alpha \in T, \alpha > \alpha^*$, the set $f\left[X \times \{\alpha\}\right] \cup \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$ is not compact.

Let

$$C_3 = (\beta X \backslash X) \backslash (C_1 \cup \overline{C_2}^{\beta X}).$$

Note that C_3 is nonempty because if it is , then $\beta X \backslash X = C_1 \cup \overline{C_2}^{\beta X}$. We will show that $f\left[\beta X \times \{\alpha\}\right] \subseteq f\left[X \times \{\alpha\}\right] \cup \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$. Let $x \in \beta X \backslash X$, if $x \in C_1$, then $\bar{f}(x,\alpha) = f(y_x,\alpha)$ for some $y_x \in X$ because $\alpha \in T \subseteq T_x$. Hence $\bar{f}(x,\alpha) \in f\left[X \times \{\alpha\}\right]$. If $x \in \overline{C_2}^{\beta X}$, then $\bar{f}(x,\alpha) \in \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$. The other inclusion is trivial. So now, we have

$$\bar{f}\left[\beta X \times \{\alpha\}\right] = f\left[X \times \{\alpha\}\right] \cup \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right].$$

However, as $\bar{f}$ is continuous and $\beta X \times \{\alpha\}$ is compact, $\bar{f}\left[\beta X \times \{\alpha\}\right]$ is compact. But by our assumption, $f\left[X \times \{\alpha\}\right] \cup \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$ is not compact, a contradiction. Hence, C_3 is nonempty.

For each $x \in C_3$, there exist $\alpha_x \in T(|\beta X|^+)$ such that for $\alpha > \alpha_x$, $\bar{f}(x, \alpha) \in \beta Z \backslash Z$ holds. This follows from the definition of C_1 and C_2. Let $\beta^* = \sup\{\alpha_x : x \in C_3\}$. Such β^* exists because $|C_3| < cf(|\beta X|^+)$.

Let

$$\gamma^* = \max\{\alpha^*, \beta^*\}.$$

Fix an arbitrary $\lambda \in T, \lambda > \gamma^*$.

The set $f\left[X \times \{\lambda\}\right]$ is closed in Z. Suppose not, by **Lemma 4.2**, there exists

$$(x, \lambda) \in \overline{X \times \{\lambda\}}^{\beta X \times T(|\beta X|^+ + 1)} \backslash X \times \{\lambda\}$$

such that

$$\bar{f}(x, \lambda) \in \overline{f\left[X \times \{\lambda\}\right]}^{Z} \backslash f\left[X \times \{\lambda\}\right].$$

Since $(x, \lambda) \notin X \times \{\lambda\}, x \in \beta X \backslash X$. Note that the remainder $\beta X \backslash X$ is partitioned into the sets $C_1, \overline{C_2}^{\beta X}$, and C_3. If $x \in C_1$, then since $\lambda \in T \subseteq T_x$, there exists $y_x \in X$ such that $\bar{f}(x, \lambda) = f(y_x, \lambda) \in f\left[X \times \{\lambda\}\right]$, a contradiction. If $x \in \overline{C_2}^{\beta X}$, then either $\bar{f}(x, \lambda) = f(y, \lambda_2)$ and $\lambda > \lambda_2$, or $\bar{f}(x, \lambda) \in \beta Z \backslash Z$. We reach contradiction in both cases. Finally, if $x \in C_3$, then $\bar{f}(x, \lambda) \in \beta Z \backslash Z$, contradiction again. Therefore, $f\left[X \times \{\lambda\}\right]$ is closed in Z.

So far, we have that $f\left[X \times \{\lambda\}\right]$ is closed in Z, $\bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$ is compact in βZ, $f\left[X \times \{\alpha\}\right] \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right] = \emptyset$, and $f\left[X \times \{\alpha\}\right] \cup \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$ is not compact in βZ. Thus, by **Fact 3.8,** there exists a system $D = \{D_\alpha : \alpha < l\} \subseteq \mathcal{P}\left(f\left[X \times \{\lambda\}\right]\right)$ satisfying these 4 conditions:

1. For each α, the set D_α is nonempty and closed in $f\left[X \times \{\lambda\}\right]$.
2. For $\alpha > \beta, D_\alpha \subseteq D_\beta$, and if β is a limit ordinal, then $D_\beta = \bigcap\{D_\alpha : \alpha < \beta\}$.
3. $\bigcap\{D_\alpha : \alpha < l\} = \emptyset$.

4. $\overline{D_1}^{\beta Z} \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\lambda\}\right] = \emptyset$.

Since f is one-to-one and $D_\alpha \subseteq f[X \times \{\lambda\}]$, we have $f^{\leftarrow}[D_\alpha] \subseteq X \times \{\lambda\}$ for any α. Define a system $A = \{A_\alpha\}$ of subsets of X such that $f^{\leftarrow}[D_\alpha] = A_\alpha \times \{\lambda\}$. We will show that A satisfies the following conditions:

5. For each α, the set A_α is closed in X.
   `Proof-` Since D_α is closed in $f[X \times \{\lambda\}]$ and $f[X \times \{\lambda\}]$ is closed in Z, D_α must be closed Z as well. As f is continuous, $f^{\leftarrow}[D_\alpha]$ is closed in $X \times T(|\beta X|^+)$. Since $f^{\leftarrow}[D_\alpha] = A_\alpha \times \{\lambda\}$, A_α is closed in X.

6. $\bigcap\{A_\alpha\} = \emptyset$.
   `Proof-` By our definition of A_α, we have $f^{\leftarrow}[\bigcap\{D_\alpha\}] = \bigcap\{f^{\leftarrow}[D_\alpha]\} = \bigcap\{A_\alpha\} \times \{\lambda\}$. Since $\bigcap\{D_\alpha\} = \emptyset$ by condition 3, then $\bigcap\{A_\alpha\} \times \{\lambda\} = \emptyset$, and thus $\bigcap\{A_\alpha\} = \emptyset$.

7. For $\alpha > \beta, A_\alpha \subseteq A_\beta$ and if β is a limit ordinal, then $A_\beta = \bigcap\{A_\alpha\}$.
   `Proof-` If $\alpha > \beta, D_\alpha \subseteq D_\beta$ holds by condition 2. As $f^{\leftarrow}[D_\alpha] \subset f^{\leftarrow}[D_\beta], A_\alpha \times \{\lambda\} \subseteq A_\beta \times \{\lambda\}$. Thus, $A_\lambda \subseteq A_\beta$. Now, if β is a limit ordinal, then by condition 2, $D_\beta = \bigcap\{D_\alpha\}$. So $A_\beta \times \{\lambda\} = f^{\leftarrow}[D_\beta] = f^{\leftarrow}[\bigcap\{D_\alpha\}] = \bigcap f^{\leftarrow}[D_\alpha] = \bigcap\{A_\alpha \times \{\lambda\}\}$. Thus, $A_\beta = \bigcap\{A_\alpha\}$.

8. $\overline{A_1}^{\beta X} \cap \overline{C_2}^{\beta X} = \emptyset$.
   `Proof-` Suppose there exists $x \in \overline{A_1}^{\beta X} \cap \overline{C_2}^{\beta X}$. Since $x \in \overline{A_1}^{\beta X}$, $\bar{f}(x,\lambda) \in \bar{f}\left[\overline{A_1}^{\beta X} \times \{\lambda\}\right] = \bar{f}\left[\overline{A_1 \times \{\lambda\}}^{\beta X \times T(|\beta X|^+ +1)}\right] = \bar{f}\left[\overline{A_1 \times \{\lambda\}}^{\beta(X \times T(|\beta X|^+))}\right] \subseteq \overline{\bar{f}[A_1 \times \{\lambda\}]}^{\beta Z} = \overline{f[A_1 \times \{\lambda\}]}^{\beta Z} = \overline{D_1}^{\beta Z}$. On the other hand, since $x \in \overline{C_2}^{\beta X}$, $\bar{f}(x,\lambda) \in \bar{f}\left[\overline{C_2}^{\beta X} \times \{\lambda\}\right]$. Therefore, $\bar{f}(x,\lambda) \in \overline{D_1}^{\beta Z} \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\lambda\}\right]$. However, by condition 4, $\overline{D_1}^{\beta Z} \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\lambda\}\right] = \emptyset$, a contradiction. So, $\overline{A_1}^{\beta X} \cap \overline{C_2}^{\beta X} = \emptyset$.

9. If $x \in \bigcap \overline{A_\alpha}^{\beta X}$, then $x \in C_3$.
   `Proof-` Since $\beta X \backslash X$ is partitioned in to $C_1, \overline{C_2}^{\beta X}$, and C_3, we only need to show that $x \notin C_1$ and $x \notin \overline{C_2}^{\beta X}$. From condition 8, $\overline{A_1}^{\beta X} \cap \overline{C_2}^{\beta X} = \emptyset$, so $x \notin \overline{C_2}^{\beta X}$. Now suppose that $x \in C_1$, then since $\lambda \in T \subseteq T_x$,

$\bar{f}(x,\lambda) = f(y_x,\lambda)$ for some $y_x \in X$. Since $x \in \bigcap \overline{A_\alpha}^{\beta X}$, we have $\bar{f}(x,\lambda) \in \bar{f}\left[\overline{A_\alpha}^{\beta X} \times \{\lambda\}\right]$ for all α. Thus, $f(y_x,\lambda) \in \bar{f}\left[\overline{A_\alpha}^{\beta X} \times \{\lambda\}\right] =$ $\bar{f}\left[\overline{A_\alpha \times \{\lambda\}}^{\beta X \times T(|\beta X|^+ +1)}\right] = \bar{f}\left[\overline{A_\alpha \times \{\lambda\}}^{\beta(X \times T(|\beta X|^+))}\right] \subseteq \overline{\bar{f}\left[A_\alpha \times \{\lambda\}\right]}^{\beta Z}$ $= \overline{f\left[A_\alpha \times \{\lambda\}\right]}^{\beta Z}$. Since $f(y_x,\lambda) \in Z$, and by condition 10, $f\left[A_\alpha \times \{\lambda\}\right]$ is closed in Z, we must have $f(y_x,\lambda) \in f\left[A_\alpha \times \{\lambda\}\right]$. As f is one-to-one, $y_x \in A_\alpha$ for each α. However, that means $y_x \in \bigcap\{A_\alpha\}$, contradicting condition 6, which says $\bigcap\{A_\alpha\} = \emptyset$. So, $x \notin C_1$, and therefore x must be in C_3.

10. The set $f\left[A_\alpha \times \{\gamma\}\right]$ is closed in Z for each $\gamma > \gamma^*, \gamma \in T$.
    `Proof-` By **Lemma 4.2,** there exists $x \in \overline{A_\alpha}^{\beta X} \setminus A_\alpha$ such that $\bar{f}(x,\gamma) \in \overline{f\left[A_\alpha \times \{\gamma\}\right]}^{Z} \setminus f\left[A_\alpha \times \{\gamma\}\right]$ holds. By condition 8, $\overline{A_1}^{\beta X} \cap \overline{C_2}^{\beta X} = \emptyset$, so $x \notin \overline{C_2}^{\beta X}$. Now, if $x \in C_3$, then $\bar{f}(x,\gamma) = f(x,\gamma')$ for some $\gamma' > \gamma^*$, a contradiction, so $x \notin C_3$. Thus, $x \in C_1$. Since $\lambda \in T \subseteq T_x$, $\bar{f}(x,\lambda) = f(y_x,\lambda)$ for some $y_x \in X$, and so $\bar{f}(x,\lambda) \in Z$. As $f\left[A_\alpha \times \{\lambda\}\right] = D_\alpha$ is closed in Z and $\bar{f}(x,\lambda) \in Z$, $\bar{f}(x,\lambda) \in f\left[A_\alpha \times \{\lambda\}\right]$. Moreover, since $\bar{f}(x,\lambda) = f(y_x,\lambda), f(x,\lambda) \in f\left[A_\alpha \times \{\lambda\}\right]$. As f is one-to-one, $y_x \in A_\alpha$. Note that $y_x \in A_\alpha$ is fixed, and by **Lemma 4.3**, for all $\gamma \in T \subseteq T_x$, $\bar{f}(x,\gamma) = f(y_x,\gamma) \in f\left[A_\alpha \times \{\gamma\}\right]$. We reach a contradiction.

Let $\gamma = |A|$. Note that $|A| \leq |X| < |\beta X|^+$ because $A = \{A_\alpha : \alpha < l\} \subseteq \mathcal{P}(X)$ and $\{A_\alpha\}$ is decreasing. Choose a closed subset $G = \{\gamma_\alpha : \alpha \leq \gamma\}$ of T such that $\gamma_1 > \gamma^*$ and for $\alpha > \beta, \gamma_\alpha > \gamma_\beta$. Define

$$B_1 = \bigcup_{\alpha < \gamma} \{A_\alpha \times \{\gamma_\alpha\}\}$$

$$B_2 = A_1 \times \{\gamma_\gamma\}$$

B_1 is closed in $X \times T(|\beta X|^+)$ because $\bigcap\{A_\alpha\} = \emptyset$ by condition 6 and our choice of G. B_2 is closed in $X \times T(|\beta X|^+)$ because A_1 is closed by condition 5. B_1 and B_2 are disjoint because B_1 doesn't contain any element with the second coordinate equal to γ_γ.

Now, as each $\overline{A_\alpha}^{\beta X} \neq \emptyset$, there exists $x \in \bigcap\left\{\overline{A_\alpha}^{\beta X}\right\}$. Thus $(x,\gamma_\gamma) \in \overline{B_1}^{\beta X \times T(|\beta X|^+ +1)}$. Moreover, $(x,\gamma_\gamma) \in \overline{A_1}^{\beta X} \times \{\gamma_\gamma\} = \overline{B_2}^{\beta X \times T(|\beta X|^+ +1)}$. Thus, $(x,\gamma_\gamma) \in \overline{B_1}^{\beta X \times T(|\beta X|^+ +1)} \cap \overline{B_2}^{\beta X \times T(|\beta X|^+ +1)} = \overline{B_1}^{\beta\left(X \times T(|\beta X|^+)\right)} \cap \overline{B_2}^{\beta\left(X \times T(|\beta X|^+)\right)}$. By **Fact 3.9**, the sets B_1 and B_2 are not completely separated in $X \times T(|\beta X|^+)$.

Let us consider the $f[B_1]$ and $f[B_2]$. Since f is one-to-one, we have $f[B_1] \cap f[B_2] = \emptyset$. By condition 10, $f[B_2]$ is closed in Z. We shall prove that $f[B_1]$ is closed in Z. Assume the contrary. Then, by **Lemma 4.2**, there exists $(x, \gamma_\alpha) \in \overline{B_1}^{\beta X \times T(|\beta X|^+ + 1)} \backslash B_1$ such that $\bar{f}(x, \gamma_\alpha) \in \overline{f[B_1]}^Z \backslash f[B_1]$.

If α is isolated, since $(x, \gamma_\alpha) \in \overline{B_1}^{\beta X \times T(|\beta X|^+ + 1)}$, $(x, \gamma_\alpha) \in \overline{A_\alpha \times \{\gamma_\alpha\}}^{\beta X \times T(|\beta X|^+ + 1)}$. So $\bar{f}(x, \gamma_\alpha) \in \bar{f}\left[\overline{A_\alpha \times \{\gamma_\alpha\}}^{\beta X \times T(|\beta X|^+ + 1)}\right] = \bar{f}\left[\overline{A_\alpha \times \{\gamma_\alpha\}}^{\beta (X \times T(|\beta X|^+)}\right] \subseteq \overline{\bar{f}[A_\alpha \times \{\gamma_\alpha\}]}^{\beta Z} = \overline{f[A_\alpha \times \{\gamma_\alpha\}]}^{\beta Z}$. If $\bar{f}(x, \gamma_\alpha) \notin Z$, we reach a contradiction. If $\bar{f}(x, \gamma_\alpha) \in Z$, then $\bar{f}(x, \gamma_\alpha) \in \overline{f[A_\alpha \times \{\gamma_\alpha\}]}^{\beta Z} \cap Z = f[A_\alpha \times \{\gamma_\alpha\}]$, where the last equality holds because by condition 10, $f[A_\alpha \times \{\gamma_\alpha\}]$ is closed in Z. Now, $\bar{f}(x, \gamma_\alpha) \in f[A_\alpha \times \{\gamma_\alpha\}] \subseteq f[B_1]$ is a contradiction.

If α is a limit ordinal not equal to γ, then by condition 7 and that $(x, \gamma_\alpha) \in \overline{B_1}^{\beta X \times T(|\beta X|^+ + 1)}$, we have $(x, \gamma_\alpha) \in \overline{A_\alpha \times \{\gamma_\alpha\}}^{\beta X \times T(|\beta X|^+ + 1)}$. Then by the same reasoning as above, we reach a contradiction.

Now, if $\alpha = \gamma$, then since $(x, \gamma_\gamma) \in \overline{B_1}^{\beta \times T(|\beta X|^+ + 1)} \backslash B_1$, and by the way B_1 was defined, we must have $x \in \bigcap \left\{\overline{A_\alpha}^{\beta X}\right\}$. By condition 9, $x \in C_3$. But since $\gamma_\gamma > \gamma^* \geq \beta^*$, $\bar{f}(x, \gamma_\gamma) \in \beta Z \backslash Z$, a contradiction. Hence, the set $f[B_1]$ must be closed in Z.

Now, the sets $f[B_1]$ and $f[B_2]$ are closed and disjoint in Z. Since Z is normal, by Urysohn's Lemma, there exists a continuous function $g : Z \to [0, 1]$ such that $g[f[B_1]] \subseteq \{0\}$ and $g[f[B_2]] \subseteq \{1\}$. Let $h = g \circ f$. Then B_1 and B_2 are completely separated by the continuous function h. This is a contradiction, and so **CASE I** can never happen.

CASE II. There exists an ordinal $\alpha \in T, \alpha > \alpha^*$ such that the set $Y = f[X \times \{\alpha\}] \cup \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$ is compact.

Since $f[X \times \{\alpha\}] \cap \bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right] = \emptyset$ and the set $\bar{f}\left[\overline{C_2}^{\beta X} \times \{\alpha\}\right]$ is compact, then $f[X \times \{\alpha\}]$ is relatively open in the compact space Y; so $f[X \times \{\alpha\}]$ is locally compact.

By **Fact 3.10**, $f[X \times \{\alpha\}]$ condenses onto a compact space. Hence, $X \times \{\alpha\}$ condenses onto a compact space as well. Since X is homeomorphic to $X \times \{\alpha\}$, X condenses onto a compact space, too. The theorem is proved. □

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