Tamano's Theorem.(1960) Let X be a Tychonoff space. Then $X \times \beta X$ is normal iff X is paracompact.

(one direction was known)

1. The product of a paracompact space with a compact Hausdorff space is paracompact.
2. Every paracompact space is normal.

Tamano proved that if $X \times \beta X$ is normal, then X is paracompact.

Buzjakova's Theorem.(1997) A pseudocompact space X condenses onto a compact space if and only if the space $X \times T(|\beta X|^+ + 1)$ condenses onto a normal space.

Notice these two theorems are similar because Buzjakova's Theorem can be interpreted as a condensation version of the Tamano's Theorem for the pseudocompact case.

In the proof of Buzjakova's Theorem, we use a corollary of Glicksberg's Theorem in many places.

Glicksberg's Theorem.(1959) Let $X \times Y$ be Tychonoff spaces. If $X \times Y$ is pseudocompact, then $\beta(X \times Y) = \beta X \times \beta Y$.

1. Let X, Y be Tychonoff spaces. If $X \times Y$ is pseudocompact, then the projection map $\pi_X : X \times Y \to X$, is z-closed.
2. Let X, Y be Tychonoff spaces. If π_X is z-closed, then $X \times Y$ is C^*-embedded in $X \times \beta Y$.
3. Let Y be an extension of the space X. If X is pseudocompact, so is Y

Corollary of Glicksberg's Theorem.

- $\beta(X \times T(|\beta X|^+ + 1)) = \beta X \times T(|\beta X|^+ + 1)$
- $\beta(X \times T(|\beta X|^+)) = \beta X \times T(|\beta X|^+ + 1)$

Some important facts in the proof of Buzjakova's Theorem:

1. Let τ be an uncountable regular cardinal. Let $T(\tau)$ be the space of all ordinal numbers less than τ. Let A_α be a closed, unbounded subset of $T(\tau)$. Let $\gamma \in T(\tau)$. Then, $\bigcap\{A_\alpha : \alpha < \gamma\}$ is closed, unbounded and $|\bigcap\{A_\alpha : \alpha < \gamma\}| = \tau$.

2. Let $g : T(\tau) \to \mathbb{R}$ be continuous. Then g is constant on $[\kappa, \tau)$ for some $\kappa \in T(\tau)$.

3. Let X be a pseudocompact Tychonoff space. Let $\tau = |\beta X|^+$ and denote by $T(\tau)$ the space of all ordinal numbers less than τ. Then, $X \times T(\tau)$ is pseudocompact. (PICTURE WILL BE DRAWN)

4. Let X be a Tychonoff space. If B_1, B_2 are subsets of X such that $\overline{B_1}^{\beta X} \cap \overline{B_2}^{\beta X} \neq \emptyset$, then B_1 and B_2 are not completely separated in X.

In the proof, we have two cases. In the harder case, we have that the set $f[X \times \{\lambda\}] \subseteq Z$ is not compact.

So there exists decreasing chain of non-empty closed sets $\{D_\alpha : \alpha < l\}$ of $f[X \times \{\lambda\}]$ such that $\bigcap\{D_\alpha : \alpha < l\} = \emptyset$

Since f is one-to-one, there exists $A_\alpha \subseteq X$ such that $f[A_\alpha \times \{\alpha\}] = D_\alpha$.

Now Define B_1, B_2.

$$B_1 = \bigcup \{A_\alpha \times \{\gamma_\alpha\}\}$$

$$B_2 = A_1 \times \{\gamma_\gamma\}$$

(PICTURE WILL BE DRAWN)

Let $x \in \bigcap \left\{ \overline{A_\alpha}^{\beta X} \right\}$.

$(x, \gamma_\gamma) \in \overline{A_1}^{\beta X} \times \{\gamma_\gamma\} = \overline{A_1 \times \{\gamma_\gamma\}}^{\beta X \times T(|\beta X|^+ + 1)} = \overline{B_2}^{\beta(X \times T(|\beta X|^+))}$.

$(x, \gamma_\gamma) \in U \times (\gamma, \gamma_\gamma] \in \tau(\beta(X \times T(|\beta X|^+ + 1)) = \tau\left(\beta(X \times T(|\beta X|^+))\right)$.

Let $\gamma_\beta \in (\gamma, \gamma_\gamma]$. Thus, $U \times (\gamma, \gamma_\gamma] \cap A_\beta \times \{\gamma_\gamma\} \neq \emptyset$. So $U \times (\gamma, \gamma_\gamma] \cap B_1 \neq \emptyset$. So, $(x, \gamma_\gamma) \in \overline{B_1}^{\beta(X \times T(|\beta X|^+))}$.

By fact 4 above, B_1 and B_2 are not completely separated in $X \times T(|\beta X|^+)$. A contradiction, because the sets $f\left[B_1\right]$ and $f\left[B_2\right]$ and closed and disjoint in Z. Since Z is normal, by Urysohn's Lemma, there exists a continuous function $g : Z \to [0, 1]$ such that $g\left[f\left[B_1\right]\right] \subseteq \{0\}$ and $g\left[f\left[B_2\right]\right] \subseteq \{1\}$. Let $h = g \circ f$, and this functions separates B_1 and B_2.