Lemma 4. Let X be a pseudocompact space and f a continuous one-to-one function from $X \times T(|\beta X|^+)$ onto Z. We are given two sets:
$C_1 = \{x \in \beta X \backslash X : \left|\bar{f}[\{x\} \times T(|\beta X|^+)] \cap Z\right| = |\beta X|^+\}$.
$C_2 = \{x \in \beta X \backslash X : \left|\bar{f}[\{x\} \times T(|\beta X|^+)] \cap Z\right| < |\beta X|^+$ and for any $\alpha \in T(|\beta X|^+)$, there exists $\alpha_1 \in T(|\beta X|^+)$ such that $\alpha_1 > \alpha$ and $\bar{f}(x, \alpha_1) \in Z\}$.

Then, $C_1 \cap \overline{C_2}^{\beta X} = \emptyset$ and $X \cap \overline{C_2}^{\beta X} = \emptyset$.

Proof. We'll first prove that $C_1 \cap \overline{C_2}^{\beta X} = \emptyset$.
Suppose the contrary. Let $x \in C_1 \cap \overline{C_2}^{\beta X}$.

For each $y \in C_2$, let $Z_y = \bar{f}\left[\{y\} \times T(|\beta X|^+)\right] \cap Z$.
Since $\{y\} \times T(|\beta X|^+) = \bigcup \left\{\{y\} \times T(|\beta X|^+) \cap \bar{f}^{-1}(z) : z \in Z_y\right\}$, we have:

$$|\beta X|^+ = |\{y\} \times T(|\beta X|^+)| = \left|\bigcup \left\{\{y\} \times T(|\beta X|^+) \cap \bar{f}^{-1}(z) : z \in Z_y\right\}\right|.$$

Since $|Z_y| < |\beta X|^+ = cf(|\beta X|^+)$, at least one of the terms in the union must have cardinality equal to $|\beta X|^+$. So there exists $z \in Z_y$ such that
$\left|\{y\} \times T(|\beta X|^+) \cap \bar{f}^{-1}(z)\right| = |\beta X|^+$.

Define $A_y = \pi_2\left[\{y\} \times T(|\beta X|^+) \cap \bar{f}^{-1}(z)\right]$, where $\pi_2 : \beta X \times T(|\beta X|^+) \to T(|\beta X|^+)$ is the projection map. Thus, we defined $A_y \subseteq T(|\beta X|^+)$ such that $|A_y| = |\beta X|^+$ and $\bar{f}[\{y\} \times A_y] = z = f(x_y, \alpha_y)$ for some $(x_y, \alpha_y) \in X \times T(|\beta X|^+)$. By the continuity of $\bar{f}$, we have that $\bar{f}\left[\{y\} \times \overline{A_y}^{T(|\beta X|^+)}\right] = f(x_y, \alpha_y)$ holds for all $y \in C_2$.

Let $T = \bigcap \left\{\overline{A_y}^{T(|\beta X|^+)} : y \in C_2\right\} \cap T_x$, where T_x was defined in Lemma 3. Now, by **FACT 3**, $|T| = |\beta X|^+$ and T is closed in $T(|\beta X|^+)$.

We have $\left|\bar{f}\left[\{x\} \times T\right] \cap Z\right| = |\beta X|^+$ holds. This is because:
$x \in C_1$ and by the way we defined T, we must have $\bar{f}\left[\{x\} \times T\right] = f\left[y_x, T\right] \subseteq Z$. Since f is one-to-one, $|f\left[y_x, T\right]| = |T| = |\beta X|^+$.

On the other hand, we have $\left|\bar{f}\left[\{x\} \times T\right] \cap Z\right| < |\beta X|^+$ holds. Because:
$$\bar{f}\left[\{x\} \times T\right] \cap Z \subseteq \bar{f}\left[\overline{C_2}^{\beta X} \times T\right] \cap Z \subseteq \bar{f}\left[\overline{C_2 \times T}^{\beta X \times T(|\beta X|^+ + 1)}\right] \cap Z$$
$$= \bar{f}\left[\overline{C_2 \times T}^{\beta\left(X \times T(|\beta X|^+)\right)}\right] \cap Z \subseteq \overline{\bar{f}\left[C_2 \times T\right]}^{\beta Z} \cap Z = \overline{\{f(x_y, \alpha_y) : y \in C_2\}}^{\beta Z},$$

where the last equality holds because $\bar{f}\left[\{y\} \times T\right] = f(x_y, \alpha_y)$ holds for all $y \in C_2$. Now, let $\alpha = sup\{\alpha_y : y \in C_2\}$, which exists because $|C_2| \leq |\beta X| < |\beta X|^+ = cf\left(|\beta X|^+\right)$.

Since $\{f(x_y, \alpha_y) : y \in C_2\} \subseteq \bar{f}\left[\beta X \times T(\alpha + 1)\right] = \overline{\bar{f}\left[\beta X \times T(\alpha + 1)\right]}^{\beta Z}$, we must have $\overline{\{f(x_y, \alpha_y) : y \in C_2\}}^{\beta Z} \subseteq \bar{f}\left[\beta X \times T(\alpha + 1)\right]$. Since

$\bar{f}\left[\{x\} \times T\right] \cap Z \subseteq \bar{f}\left[\beta X \times T(\alpha+1)\right]$ and we have $\left|\bar{f}\left[\beta X \times T(\alpha+1)\right]\right| < |\beta X|^{+}$, we conclude that $\left|\bar{f}\left[\{x\} \times T\right] \cap Z\right| < |\beta X|^{+}$. This contradiction to our previous argument that $\left|\bar{f}\left[\{x\} \times T\right] \cap Z\right| = |\beta X|^{+}$. Therefore, $C_1 \cap \overline{C_2}^{\beta X} = \emptyset$.

We show that $X \cap \overline{C_2}^{\beta X} = \emptyset$. Suppose the contrary. Let $x \in X \cap \overline{C_2}^{\beta X}$.

Since $x \in \overline{C_2}^{\beta X}$, by the same reasoning as in above, we must have $\left|\bar{f}\left[\{x\} \times T\right] \cap Z\right| < |\beta X|^{+}$.

However, since $x \in X$, $\left|\bar{f}\left[X \times T\right] \cap Z\right| = |f\left[X \times T\right]| = |\beta X|^{+}$,

where the last equality holds because f is one-to-one and $|T| = |\beta X|^{+}$. Again, we reached contradiction. Thus, $X \cap \overline{C_2}^{\beta X} = \emptyset$ holds. Lemma 4 is proved.