CHAPTER I

Tamano's Theorem

Lemma 1.1 The product of a paracompact space with a compact Hausdorff space is paracompact.

Proof. Let X be paracompact, Y compact, and let $\mathcal{U}$ be an open cover of $X \times Y$. For fixed $x \in X$, as $\{x\} \times Y$ is compact in $X \times Y$, a finite number of elements of $\mathcal{U}$, say $U^x_{\alpha_1}, \ldots, U^x_{\alpha_{n_x}}$, cover $\{x\} \times Y$. Pick an open nhood V_x of x in X such that $V_x \times Y \subseteq \bigcup_{i=1}^{n_x} U^x_{\alpha_i}$.

The sets V_x, as x ranges through X, form an open cover of X. By the paracompactness of X, let $\mathcal{V}$ be an open locally finite refinement of V_x. For each $V \in \mathcal{V}, V \subseteq V_x$ for some V_x. Let

$$\mathcal{W}_V = \left\{(V \times Y) \cap U^x_{\alpha_i} : 1 \leq i \leq n_x\right\},$$

and let

$$\mathcal{R} = \bigcup \{\mathcal{W}_V : V \in \mathcal{V}\}.$$

Since $\mathcal{W}_V \subseteq \mathcal{U}$ for each $V \in \mathcal{V}$, $\mathcal{R}$ is a refinement of $\mathcal{U}$. For each $x \in X$, W_V is a cover for $\{x\} \times Y$ for some $V \in \mathcal{V}$. Thus, $\mathcal{R}$ is a cover of $X \times Y$. Lastly, $\mathcal{R}$ is locally finite because given $(x, y) \in X \times Y$, there is a neighborhood U_x of x which meets only finitely many V's in $\mathcal{V}$ because $\mathcal{V}$ is locally finite. Then the neighborhood $U_x \times Y$ of (x, y) can then only meet only finitely many sets of $\mathcal{R}$. Hence, $X \times Y$ is paracompact.

Lemma 1.2 Every paracompact space is normal.

Proof. We first establish regularity. Suppose A is a closed set in a paracompact space X and $x \notin A$. For each $y \in A$, as X is Hausdorff, we can find an open set V_y containing y such that $x \notin \overline{V_y}$. Then the sets $V_y, y \in A$, together with the set $X \backslash A$, form an open cover of X. Let $\mathcal{W}$ be an open locally finite refinement and let

$$V = \bigcup \{W \in \mathcal{W} : W \cap A \neq \emptyset\}.$$

Then V is an open set containing A. Now,

$$\overline{V} = \bigcup \left\{\overline{W} \in \mathcal{W} : W \cap A \neq \emptyset\right\}$$

holds because:
$\supseteq$**:** If $z \in \bigcup \left\{\overline{W} \in \mathcal{W} : W \cap A \neq \emptyset\right\}$, then $z \in \overline{W}$ for some $W \in \mathcal{W}$. Since

$W \subseteq V$, we have $z \in \overline{W} \subseteq \overline{V}$.
$\subseteq$**:** If $z \in \overline{V}$, then there exists a net $\{w_\alpha\} \subseteq V = \bigcup\{W \in \mathcal{W} : W \cap A \neq \emptyset\}$ converging to z. Since $\{W \in \mathcal{W} : W \cap A \neq \emptyset\}$ is locally finite, the tail of $\{w_\alpha\}$ must be contained in finitely many W's, say $\{W_1, \ldots W_n\}$. By pigeon-hole principle, there exists some $W_k \in \{W_1, \ldots W_n\}$ such that W_k contains infinitely many elements of the net $\{w_\alpha\}$. Thus, $z \in \overline{W_k}$.

Since $x \notin \overline{V_y}$ for each $y \in A$ and $\overline{W} \subseteq \overline{V_y}$ for each $W \in \{W \in \mathcal{W} : W \cap A \neq \emptyset\}$, $x \notin \overline{W}$ for each $W \in \{W \in \mathcal{W} : W \cap A \neq \emptyset\}$. So $x \notin \overline{V}$. Regularity is established.

To establish normality, suppose A and B are disjoint closed sets in X. For each $y \in A$, by regularity, we can find an open set V_y such that $y \in V_y$ and $\overline{V_y} \cap B = \emptyset$. Then proceed exactly as before, we can produce an open set V such that $A \subseteq V$ and $\overline{V} \cap B = \emptyset$. Thus X is normal.

Theorem 1.3 Let X be a Hausdorff space. Then $X \times \beta X$ is normal iff X is paracompact.

Proof of $\Leftarrow$: Since X is paracompact and βX is compact T_2, by **Lemma 1.1**, $X \times \beta X$ is paracompact. By **Lemma 1.2**, paracompact implies normal and so $X \times \beta X$ is normal.

Proof of $\Rightarrow$: We will prove a slighty stronger version that for any compactification cX of X, if $X \times cX$ is normal, then X is paracompact.

Let cX be a compactification of X such that $X \times cX$ is normal. Let $\{U_a : a \in A\}$ be an open cover of X. We will show that it has an open locally finite refinement.

X is a subspace of cX, so for each $a \in A$, there exists V_a open in cX such that $V_a \cap X = U_a$. Let $F = cX \setminus \bigcup\{V_a : a \in A\}$. We can assume that F is nonempty. Since if $F = \emptyset$, then $cX = \bigcup\{V_a : a \in A\}$, and since cX is compact, we can find a finite subcover $\{V_{a_i} : 1 \leq i \leq n\} \subseteq \{V_a : a \in A\}$. Then, $\{U_{a_i} : 1 \leq i \leq n\}$ is an open locally finite refinement of $\{U_a : a \in A\}$.

Let $\triangle = \{(x,x) : x \in X\}$. Both $X \times F$ and $\triangle$ are closed in $X \times cX$. Since $X \times cX$ is normal, by Urysohn's Lemma, there is a continuous function f: $X \times cX \to [0,1]$ with $f[\triangle] \subseteq \{0\}$ and $f[X \times F] \subseteq \{1\}$.

Define $d : X \times X \to \mathbb{R}$
such that

$$d(x,y) = \sup\{|f(x,z) - f(y,z)| : z \in X\}$$

for all $(x,y) \in X \times X$. Now, for all $x, y, w \in X$, we have:

1. $d(x, x) = \sup_{z \in X} |f(x, z) - f(x, z)| = 0$
2. $d(x, y) = \sup_{z \in X} |f(x, z) - f(y, z)| = \sup_{z \in X} |f(y, z) - f(x, z)| = d(y, x)$
3. $d(x, w) = \sup_{z \in X} |f(x, z) - f(w, z)|$
   $= \sup_{z \in X} |f(x, z) - f(y, z) + f(y, z) - f(w, z)|$
   $\leq \sup_{z \in X} |f(x, z) - f(y, z)| + \sup_{z \in X} |f(y, z) - f(w, z)|$
   $= d(x, y) + d(y, w)$

Thus d is a pseudometric on X. Denote $\tau_d(X)$ to be the set of open sets in the topology induced by the pseudometric d.

For each $B(x_0, \epsilon) \in \tau_d$, pick any point $x' \in B(x_0, \epsilon)$. Let $\epsilon' = \epsilon - d(x_0, x')$. The set $\Gamma = \{G \times H \subseteq X \times cX : G \times H$ is open in $X \times cX, x' \in G$, and $diam\,(f\,[G \times H]) < \epsilon'\}$ is an open cover of $\{x'\} \times cX$. To show that, pick any point $(x', y) \in \{x'\} \times cX$. Let $c = f(x', y) \in [0, 1]$. Since f is continuous and the set $E = (c - \frac{\epsilon'}{2}, c + \frac{\epsilon'}{2}) \cap [0, 1]$ is open in $[0, 1]$, $f^{\leftarrow}[E]$ must be open in $X \times cX$. Since $f^{\leftarrow}[E]$ is an open set that contains (x', y) in $X \times cX$, there exist $G_b \in \tau(X)$ containing x', and $H_b \in \tau(cX)$ containing y such that $G_b \times H_b \subseteq f^{\leftarrow}[E]$. Thus $G_b \times H_b$ is an element of Γ. Since (x', y) was arbitrarily chosen from $\{x'\} \times cX$, we conclude that Γ is an open cover of $\{x'\} \times cX$.

Γ being an open cover of $\{x'\} \times cX$ means that $\{H_b : b \in B\}$ is an open cover of cX. Since cX is compact, there is an finite subcover $\{H_i : 1 \leq i \leq n\} \subseteq \{H_b : b \in B\}$. Corresponding to $\{H_i : 1 \leq i \leq n\}$ is the set $\{G_i : 1 \leq i \leq n\}$. Where for each $i \in \{1 \ldots n\}$, we have $f[G_i \times H_i] \subseteq (c_i - \frac{\epsilon'}{2}, c_i + \frac{\epsilon'}{2})$ for some $c_i \in (0, 1)$. Pick any $z \in cX = \bigcup \{H_i : 1 \leq i \leq n\}$. For some $1 \leq k \leq n$, $z \in H_k$. Then $f[G_k \times \{z\}] \subseteq f[G_k \times H_k] \subseteq (c_k - \frac{\epsilon'}{2}, c_k + \frac{\epsilon'}{2})$.

Let $S = \bigcap \{G_i : 1 \leq i \leq n\} \subseteq G_k$. Then

$$f[S \times \{z\}] \subseteq f[G_k \times \{z\}] \subseteq (c_k - \frac{\epsilon'}{2}, c_k + \frac{\epsilon'}{2}).$$

For all $x, y \in S$, $|f(x, z) - f(y, z)| < \epsilon'$, and because this inequality holds true for all $z \in cX$, $d(x, y) = sup_{z \in cX}|f(x, z) - f(y, z)| \leq \epsilon'$. Since $x' \in G_b$ for all $b \in B, x' \in \bigcap \{G_i : 1 \leq i \leq n\} = S$. Combining with the fact that $d(x, y) \leq \epsilon'$ for all $x, y \in S$, we have $x' \in S \subseteq B(x', \epsilon') \subseteq B(x_0, \epsilon)$. Note that $S \in \tau(X)$ because $G_i \in \tau(X)$ for each $i \in \{1 \ldots n\}$.

Thus, for each $x' \in B(x_0, \epsilon)$, we can find $S \in \tau(X)$ such that $x' \in S \subseteq B(x_0, \epsilon)$. Since $B(x_0, \epsilon)$ is an arbitrarily chosen set in $\tau_d(X)$, we have $B(x_0, \epsilon) \in \tau_d(X) \Rightarrow B(x_0, \epsilon) \in \tau(X)$. Hence, $\tau_d(X) \subseteq \tau(X)$.

Stone's Theorem states that pseudo-metrizable implies paracompact. So X is paracompact with respect to the pseudo-metrizable topology τ_d. For an open cover $\left\{B(x, \frac{9}{10}) : x \in X\right\}$, there is an open locally finite refinement, $\{W_t : t \in T\}$. Since $\tau_d(X) \subseteq \tau(X)$, $\{W_t : t \in T\} \subseteq \tau(X)$.

Pick any $x_0 \in X$ and $x' \in B(x_0, \frac{9}{10})$. We have $f(x_0, x') = |f(x_0, x') - 0| = |f(x_0, x') - f(x', x')| \leq \sup_{z \in X} |f(x_0, z) - f(x_0, z)| = d(x_0, x') < \frac{9}{10}$. So, $f\left[\{x_0\} \times B(x_0, \frac{9}{10})\right] \subseteq [0, \frac{9}{10})$. By continuity, $f\left[\{x_0\} \times \overline{B(x_0, \frac{9}{10})}^{cX}\right] \subseteq [0, \frac{9}{10}]$. So, $\overline{B(x_0, \frac{9}{10})}^{cX} \cap F = \emptyset$ because $f\left[X \times F\right] \subseteq \{1\}$.

We now have $\{W_t : t \in T\}$ refines $\{B(x_0, \frac{9}{10}) : x_0 \in X\}$ and $\overline{B(x_0, \frac{9}{10})}^{cX} \cap F = \emptyset$ for all $x_0 \in X$. These give us $\overline{W_t}^{cX} \cap F = \emptyset$ for every $t \in T$. Then, for each $t \in T$, $\overline{W_t}^{cX} \subseteq cX \backslash F = \bigcup \{V_a : a \in A\}$. Since $\overline{W_t}^{cX}$ is compact in cX, there exists a finite subcover $\left\{V_j^t : 1 \leq j \leq m_t\right\} \subseteq \{V_a : a \in A\}$ such that $\overline{W_t}^{cX} \subseteq \bigcup \left\{V_j^t : 1 \leq j \leq m\right\}$. We have:

$$X \cap \overline{W_t}^{cX} \subseteq X \cap (\bigcup \left\{V_j^t : 1 \leq j \leq m\right\}).$$

Thus, $X \cap W_t \subseteq X \cap \overline{W_t}^{cX} \subseteq \bigcup \left\{U_j^t : 1 \leq j \leq m\right\}$.

The set $\{W_t \cap U_j^t : t \in T, 1 \leq j \leq m_t\} \subseteq \tau(X)$ is the desired locally finite open cover of X which refines $\{U_a : a \in A\}$. Hence X is paracompact.