Lemma 3. Let X be a pseudocompact space and f be a continuous mapping of $X \times T(|\beta X|^+)$ onto a space Z. Let $C_1 = \{x \in \beta X \backslash X : |\bar{f}[\{x\} \times T(|\beta X|^+)] \cap Z| = |\beta X|^+\}$ If $x \in C_1$, then there exist an element $y_x \in X$ and a subset $T_x \subseteq T(|\beta X|^+)$ satisfying the following conditions:

1. $|T_x| = |\beta X|^+$
2. The set T_x is closed in $T(|\beta X|^+)$.
3. For any ordinal number $\alpha \in T_x$, $\bar{f}(x, \alpha) = f(y_x, \alpha)$ holds.

Proof:
Let $x \in C_1$. Since f is onto, we can write $Z = \cup\{f[\{y\} \times T(|\beta X|^+)] : y \in X\}$.
Then we have:
$\bar{f}[\{x\} \times T(|\beta X|^+)] \cap Z$
$= \bar{f}[\{x\} \times T(|\beta X|^+)] \cap (\cup\{f[\{y\} \times T(|\beta X|^+)] : y \in X\})$
$= \cup\{\bar{f}[\{x\} \times T(|\beta X|^+)] \cap f[\{y\} \times T(|\beta X|^+)] : y \in X\}$

As $|\beta X|^+$ is regular, $|X| < |\beta X|^+ = cf(|\beta X|^+)$. So, at lease one of the terms of our union is of cardinality $|\beta X|^+$.
Let that term be:

$$\bar{f}[\{x\} \times T(|\beta X|^+)] \cap f[\{y_x\} \times T(|\beta X|^+)].$$

Fix this y_x. We will now construct the set $T_x = \{\beta_\alpha : \alpha < |\beta X|^+\}$ by transfinite induction:

`Step 1.`
Pick two ordinals, α_1 and α^1 from $T(|\beta X|^+)$ such that $\bar{f}(x, \alpha_1) = f(y_x, \alpha^1)$. This can be done because $\bar{f}[\{x\} \times T(|\beta X|^+)] \cap f[\{y_x\} \times T(|\beta X|^+)]$ is nonempty.

`Induction Hypothesis.`
For each $k < n$, we can find two ordinals α_k and α^k from $T(|\beta X|^+)$, satisfying the following conditions:
`a)` $\bar{f}(x, \alpha_k) = f(y_x, \alpha^k)$.
`b)` $\alpha_k > max\{\alpha_{k-1}, \alpha^{k-1}\}$.
`c)` $\alpha^k > max\{\alpha_{k-1}, \alpha^{k-1}\}$.

`Step N.`
Let $\alpha = max\{\alpha_{n-1}, \alpha^{n-1}\}$. Now,

$|\beta X|^+$
$= |\bar{f}[\{x\} \times T(|\beta X|^+)] \cap f[\{y_x\} \times T(|\beta X|^+)]|$
$= |\bar{f}[\{x\} \times T(|\beta X|^+)] \cap f[\{y_x\} \times T(\alpha)]|$
$+|\bar{f}[\{x\} \times T(\alpha)] \cap f[\{y_x\} \times T(|\beta X|^+)]|$
$+|\bar{f}[\{x\} \times T(|\beta X|^+)\backslash T(\alpha)] \cap f[\{y_x\} \times T(|\beta X|^+)\backslash T(\alpha)]|$

Since the first two terms of the sum both have cardinality no more than α, where $\alpha < |\beta X|^+$, the third term must have cardinality equal to $|\beta X|^+$. Otherwise the sum of these three terms will not add up to $|\beta X|^+$. Hence, $\bar{f}[\{x\} \times T(|\beta X|^+)\backslash T(\alpha)] \cap f[\{y_x\} \times T(|\beta X|^+)\backslash T(\alpha)]$ is nonempty. So, we can pick two ordinals α_n and α^n from $T(|\beta X|^+)\backslash T(\alpha)$ such that these conditions hold:
`a)` $\bar{f}(x, \alpha_n) = f(y_x, \alpha^n)$.
`b)` $\alpha_n > \alpha = max\{\alpha_{n-1}, \alpha^{n-1}\}$.
`c)` $\alpha^n > \alpha = max\{\alpha_{n-1}, \alpha^{n-1}\}$.

This completes `Step N.` Hence, we can define α_n and α^n for all $n < \omega$.

Let $\beta_1 = sup\{\alpha_n : n < \omega\} = sup\{\alpha^n : n < \omega\}$. Such β_1 exists because by condition `b)` and `c)`, the least upper bounds must equal, provided they exist. They indeed exist because $cf(|\beta X|^+) > \omega$.
Since f is continuous, $\{f(y_x, \alpha^n) : n < \omega\}$ converges to $f(y_x, \beta_1)$.
Since $\bar{f}(x, \alpha_n) = f(y_x, \alpha^n)$ for all $n < \omega$, $\{\bar{f}(x, \alpha_n) n < \omega\}$ converges to $f(y_x, \beta_1)$ also. Finally, since $\bar{f}$ is continuous, we must have $\bar{f}(x, \beta_1) = f(y_x, \beta_1)$.

Now, we will define all the other β_α's by transfinite induction:

`Induction Hypothesis.`
Let β_α be defined for all $\alpha < \gamma$ such that $\bar{f}(x, \beta_\alpha) = f(y_x, \beta_\alpha)$.

`Step` γ`(isolated ordinal).`
As $\bar{f}[\{x\} \times T(|\beta X|^+)\backslash T(\beta_{\gamma-1})] \cap f[\{y_x\} \times T(|\beta X|^+)\backslash T(\beta_{\gamma-1})]$ is nonempty, we can find a pair of ordinals in $T(|\beta X|^+)\backslash T(\beta_{\gamma-1})$, enabling us to start from `Step 1` with those two ordinals. Then, we can constuct β_γ in the way as we did for β_1.

`Step` γ`(limite ordinal).`
Define $\beta_\gamma = sup\{\beta_\alpha : \alpha < \gamma\}$. Again, the *sup* exists because $\gamma < |\beta X|^+ = cf|\beta X|^+$. Furthermore, $\bar{f}(x, \beta_\gamma) = f(y_x, \beta_\gamma)$ by the continuity of $\bar{f}$.

Since the `Step` γ works for all $\gamma < |\beta X|^+$, we've just successfully constructed $T_x = \{\beta_\alpha : \alpha < |\beta X|^+\}$. By the way T_x and y_x were defined, conditions 1 and 3 as required by the lemma automatically follow. $T_x = \{\beta_\alpha : \alpha < |\beta X|^+\}$ is closed in $T(|\beta X|^+)$ because we defined $\beta_\gamma = sup\{\beta_\alpha : \alpha < \gamma\}$ if γ is a limit ordinal. Hence condition 2 of the lemma follows as well. Lemma 3 is proved.