CHAPTER II

Glicksberg's Theorem

Fact 2.1 Let $Y \in E(X)$ for a space X, let Z be a regular space, and let $f : X \to Z$ be continuous. The following are equivalent:

1. There exists a continuous function $F : Y \to Z$ such that $F|_X = f$.
2. For each $y \in Y$, the filter $\mathcal{F}_y = \{A \subseteq Z : A \supseteq f[U] \text{ for some } U \in O^y\}$ converges(where $O^y = \{W \cap X : W \text{ is open in } Y \text{ and } y \in W\}$).

Proof of 1 $\Rightarrow$ 2: Suppose F exists and $y \in Y$. We will show $\mathcal{F}_y$ converges to $F(y)$. Let W be an open neighborhood of $F(y)$ in Z. By continuity, there is an open neighborhood U of y such that $F[U] \subseteq W$, where U is open in Y. Thus, $f\,[U \cap X] = F\,[U \cap X] \subseteq W$ and $f\,[U \cap X] \in \mathcal{F}_y$. Thus, $\mathcal{F}_y$ converges to $F(y)$.

Proof of 2 $\Rightarrow$ 1: Suppose for each $y \in Y, \mathcal{F}_y$ converges to some point. As Z is regular and hence Hausdorff, $\mathcal{F}_y$ converges to an unique point which we denote by $F(y)$. Thus, we have just defined a function $F : Y \to X$.

If $x \in X$ and W is an open neighborhood of $f(x)$, there is an open set U of X with $x \in U$ and $f[U] \subseteq W$. If V is an open set in Y such that $V \cap X = U$, then $f\,[V \cap X] \in \mathcal{F}_x$. Thus, $\mathcal{F}_x$ converges to $f(x)$ for all $x \in X$. That means $F(x) = f(x)$ for all $x \in X$. I.e. $F|_X = f$.

To show F is continuous, let $y \in Y$ and let W be an open neighborhood of $F(y)$. As Z is regular, there is an open subset V of Z such that $F(y) \in V \subseteq \overline{V}^Z \subseteq W$. Since $\mathcal{F}_y$ converges to $F(y)$, there is an open set U of Y such that $y \in U$ and $f\,[U \cap X] \subseteq V$.

Let $p \in U$. We will show that $F(p) \in \overline{V}^Z$. Let T be an open set of Z containing $F(p)$. From the definition of $\mathcal{F}_p$, there is an open set $R \in \tau(Y)$ containing p such that $R \subseteq U$ and $f\,[R \cap X] \subseteq T$. As X is dense in Y, $R \cap X \neq \emptyset$. Hence $f\,[R \cap X] \neq \emptyset$. Since $f\,[U \cap X] \subseteq V$ and $R \subseteq U$, we have $f\,[R \cap X] \subseteq V$. Thus $f\,[R \cap X] \subseteq T \cap V$. As $T \cap V \neq \emptyset$, and T was an arbitrary open set containing $F(p)$, $F(p) \in \overline{V}^Z$.

Since for every $p \in U$, $F(p) \in \overline{V}^Z \subseteq W$, we conclude that $F[U] \subseteq W$. Thus F is continuous.

Fact 2.2 Let $Y \in E(X)$ for a space X and let Z be a regular space. Let $g : Y \to Z$ be such that for each $y \in Y, g|_{X \cup \{y\}}$ is continuous. Then g is continuous.

Proof: Let

$$O_1^y = \{W \cap X : W \text{ is open in } Y \text{ and } y \in W\}, \text{ and}$$
$$O_2^y = \{W \cap X : W \text{ is open in } X \cup \{y\} \text{ and } y \in W\}.$$

We have $O_1^y = O_2^y$ because:

($\subseteq$:) Let $W \cap X \in O_1^y$. W is open in Y and $y \in W$. Since $X \cup \{y\}$ is the subspace of Y, $W \cap (X \cup \{y\})$ is open in $X \cup \{y\}$. Also, $y \in W \cap (X \cup \{y\})$. So, $W \cap X = (W \cap (X \cup \{y\})) \cap X \in O_2^y$.

($\supseteq$:) Let $W \cap X \in O_2^y$. Since W is open in $X \cup \{y\}$, there exists $V \in \tau(Y)$ such that $W = V \cap (X \cup \{y\})$. Since $y \in W$ and thus $y \in V$, we have $V \cap X \in O_1^y$. As $W \cap X = V \cap X$, $W \cap X \in O_1^y$.

Since $g|_{X \cup \{y\}}$ is continuous, then by **Fact 2.1**, the filter $\{A \subseteq Z : A \supseteq g|_X[U] \text{ for some } U \in O_2^y\}$ converges to $g(y)$. Since $O_1^y = O_2^y$, we have $\{A \subseteq Z : A \supseteq g|_X[U] \text{ for some } U \in O_1^y\}$ converging to $g(y)$ as well. By the other direction of **Fact 2.1**, g is continuous.

Fact 2.3 The following are equivalent for the dense subspace S of the Tychonoff space X:

1. S is C^*-embedded in X.
2. If Z_1 and Z_2 are disjoint zero-sets of S, then $\overline{Z_1}^X \cap \overline{Z_2}^X = \emptyset$.

Proof of 1 $\Rightarrow$ 2: By the transitivity of C^*-embedding, S is C^*-embedded and dense in βX. Thus βS is equivalent to βX. Let Z_1 and Z_2 are disjoint zero-sets of S. Since disjoint zero-sets in S hve disjoint closures in $\beta S \equiv_S \beta X$, $\overline{Z_1}^{\beta X} \cap \overline{Z_2}^{\beta X} = \emptyset$. Hence $\overline{Z_1}^X \cap \overline{Z_2}^X = \emptyset$.

Proof of 2 $\Rightarrow$ 1: It suffices to show that S is C^*-embedded in $S \cup \{p\}$ for each $p \in X \setminus S$. Because if it is true, then by **Fact 2.2**, for each $f \in C^*(S)$ there exists $F \in C(X)$ such that $F|_S = f$, and as $F|_S \in C^*(S)$, it follows that $F \in C^*(X)$. Thus S would be C^*-embedded in X.

So it remains to be shown that for each $p \in X \setminus S$, S is C^*-embedded in $S \cup \{p\}$. Let $p \in X \setminus S$, and let $C(p)$ denote the collection of closed neighborhoods of p in $S \cup \{p\}$. If $f \in C^*(S)$, for each $A \in C(p)$, the $\overline{f[A \cap X]}^{\mathbb{R}}$ is a compact nonempty subset of $\mathbb{R}$. Moreover, the set $\left\{\overline{f\left[A \cap S\right]}^{\mathbb{R}} : A \in C(p)\right\}$ has nonemptyset intersection property. Thus, $\bigcap\left\{\overline{f\left[A \cap S\right]}^{\mathbb{R}} : A \in C(p)\right\} \neq \emptyset$. Note that if $s \in \bigcap\left\{\overline{f[A \cap S]}^{\mathbb{R}} : A \in C(p)\right\}$ and $\epsilon > 0$, then $p \in \overline{f^{\leftarrow}\left[[s-\epsilon, s+\epsilon]\right]}^{S \cup \{p\}}$. This is because if $A \in C(p)$, then $(s-\epsilon, s+\epsilon) \cap f[A \cap S] \neq \emptyset$ and so $A \cap f^{\leftarrow}\left[(s-\epsilon, s+\epsilon)\right] \neq \emptyset$.

Choose $r \in \bigcap\{f[A \cap S] : A \in C(p)\}$ and define $F : S \cup \{p\} \to \mathbb{R}$ as

$$F|_S = f \text{ and } F(p) = r.$$

Since f is continuous, F is continuous at each point of S. We must show that F is continuous at p. Let $\epsilon > 0$ be given. We claim that there exists $A_0 \in C(p)$ such that $f\left[A_0 \cap S\right] \subseteq (r-\epsilon, r+\epsilon)$. For if this were not the case, then $\overline{f\left[A \cap S\right]}^{\mathbb{R}} \backslash (r-\frac{3\epsilon}{4}, r+\frac{3\epsilon}{4})$ is a nonempty compact subset of $\mathbb{R}$ for each $A \in C(p)$. As $\left\{\overline{f[A \cap S]}^{\mathbb{R}} \backslash (r-\frac{3\epsilon}{4}, r+\frac{3\epsilon}{4}) : A \in C(p)\right\}$ has the finite intersection property, there exists $s \in \bigcap\left\{\overline{f[A \cap S]}^{\mathbb{R}} \backslash (r-\frac{3\epsilon}{4}, r+\frac{3\epsilon}{4}) : A \in C(p)\right\}$. As noted in the previous paragraph, it follows that

$$p \in \overline{f^{\leftarrow}\left[[s-\frac{\epsilon}{4}, s+\frac{\epsilon}{4}]\right]}^{S \cup \{p\}}.$$

On the other hand, since $r \in \bigcap\{f[A \cap S] : A \in C(p)\}$, we have

$$p \in \overline{f^{\leftarrow}\left[[r-\frac{\epsilon}{4}, r+\frac{\epsilon}{4}]\right]}^{S \cup \{p\}}.$$

As $f^{\leftarrow}\left[[s-\frac{\epsilon}{4}, s+\frac{\epsilon}{4}]\right]$ and $f^{\leftarrow}\left[[r-\frac{\epsilon}{4}, r+\frac{\epsilon}{4}]\right]$ are disjoint zero-sets of S, this is a contradiction to our hypothesis that if Z_1 and Z_2 are disjoint zero-sets of S, then $\overline{Z_1}^X \cap \overline{Z_2}^X = \emptyset$.

Thus, there exists $A_0 \in C(p)$ such that $f\left[A_0 \cap S\right] \subseteq (r-\epsilon, r+\epsilon)$. Thus $F[A_0] \subseteq (r-\epsilon, r+\epsilon)$ and F is continuous at p. As f was arbitrarily chosen from $C^*(S)$, it follows that S is C^*-embedded in $S \cup \{p\}$.

Fact 2.4 Let X and Y be Tychonoff spaces, and $\pi_X : X \times Y \to X$ be the projection map. If π_X is z-closed, Z is a zero-set in $X \times Y$, and $(x, p) \in \overline{Z}^{X \times \beta Y}$, then $(x, p) \in \overline{Z \cap (\{x\} \times Y)}^{X \times \beta Y}$.

Proof: Assume that $(x,p) \notin \overline{Z \cap (\{x\} \times Y)}^{X\times\beta Y}$. Since $X \times \beta Y$ is Tychonoff, there exists a continuous function $f : X \times \beta Y \to [0,1]$ such that $f\left[\overline{Z \cap (\{x\} \times Y)}^{X\times\beta Y}\right] \subseteq \{1\}$ and $f[U] \subseteq \{0\}$, where U is some neighborhood of (x,p).

Let $Z_f = f^{\leftarrow}(0)$. So $(x,p) \in int(Z_f)$. We have $(x,p) \in \overline{Z \cap Z_f}^{X\times\beta Y}$, and so

$$x \in \pi_X\left[\overline{Z \cap Z_f}^{X\times\beta Y}\right] \subseteq \overline{\pi_X\left[Z \cap Z_f\right]}^X.$$

On the other hand, since $\overline{Z \cap (\{x\} \times Y)}^{X\times\beta Y} \cap Z_f = \emptyset$, we have $Z \cap (\{x\} \times Y) \cap Z_f = \emptyset$. Now, if $x \in \pi_X\left[Z \cap Z_f\right]$, then $(x,y) \in Z \cap Z_f \neq \emptyset$ for some $y \in Y$. Hence $Z \cap Z_f \cap (\{x\} \times Y) \neq \emptyset$, contradiction. So, $x \notin \pi_X\left[Z \cap Z_f\right]$.

Now,

$$x \in \overline{\pi_X\left[Z \cap Z_f\right]}^X \backslash \pi_X\left[Z \cap Z_f\right].$$

As $Z \cap Z_f$ is a zero-set in $X \times Y$, by our hypothesis, $\pi_X\left[Z \cap Z_f\right]$ is closed in X. Then, $\overline{\pi_X\left[Z \cap Z_f\right]}^X \backslash \pi_X\left[Z \cap Z_f\right] = \emptyset$, contradiction.

Fact 2.5 Let X be a Tychonoff space. If X is pseudocompact, then every locally finite family of non-empty open subsets of X is finite.

Proof. By way of contradiction, suppose that there exists a locally finite family $\mathcal{F} = \{U_i \in \tau(X) : U_i \neq \emptyset, 1 \leq i < \infty\}$ which is infinite. Since each U_i is non-empty, choose a point $x_i \in U_i$ for each $i \in \mathbb{N}$. Since X is a Tychonoff space, there exists continuous functions $f_i : X \to [0,i]$ such that $f_i(x_i) = i$ and $f_i[X\backslash U_i] \subseteq \{0\}$ for each $i \in \mathbb{N}$.

Define the fuction

$$f : X \to \mathbb{R} \text{ as } f(x) = \Sigma_{i=1}^{\infty}|f_i(x)|.$$

To show that f is continuous, pick $x_0 \in X$ and an open set V of $\mathbb{R}$ containing $f(x_0)$. We can assume that $V = (f(x_0) - \frac{1}{m}, f(x_0) + \frac{1}{m})$ for some $m \in \mathbb{N}$. Since $\mathcal{F}$ is locally finite, there exists an open set $U_0 \in \tau(X)$ containing x_0 such that U_0 meets $\mathcal{F}$ only finitely many times. So we have $\{a_i\}_{i=1}^n \subset \mathbb{N}$ such that $U_0 \cap U_{a_i} \neq \emptyset$ for $i \in [n]$.

Define $\delta : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ to be $\delta(S) = \sup(S) - \inf(S)$. For each $i \in [n]$, since f_{a_i} is continuous, there exists $W_i \in \tau(X)$ such that $x_0 \in W_i$ and $\delta(f_i[W_i]) < \frac{1}{mn}$.

Let $W = W_1 \cap W_2 \cap \cdots \cap W_n$. Then $\delta(f_i[W]) < \frac{1}{mn}$ for each $i \in [n]$. So,

$$\delta(f[W]) = \sup_{x\in W} (\Sigma_{i=1}^{\infty}|f_i(x)|) - \inf_{x\in W} (\Sigma_{i=1}^{\infty}|f_i(x)|)$$

$$= \sup_{x\in W} \left(\Sigma_{a_i:i\in[n]}|f_{a_i}(x)|\right) - \inf_{x\in W} \left(\Sigma_{a_i:i\in[n]}|f_{a_i}(x)|\right)$$

$$= \Sigma_{a_i:i\in[n]} \left(\sup_{x\in W} (|f_{a_i}(x)|) - \inf_{x\in W} |f_{a_i}(x)|\right) < n\frac{1}{mn} = \frac{1}{m}.$$

As $x_0 \in W \in \tau(X)$ and $f[W] \subset (f(x_0) - \frac{1}{m}, f(x_0) + \frac{1}{m}) = V$, f is a continuous function. However, since $f(x_i) \geq i$ for all $i \in \mathbb{N}$, f is not bounded. This contradicts the pseudocompactness of X.

Lemma 2.6 Let X, Y be Tychonoff spaces. If $X \times Y$ is pseudocompact, then the projection map $\pi_X : X \times Y \to X$, is z-closed.

Proof. Let Z be a zero-set in $X \times Y$. Suppose that $\pi_X[Z]$ is not closed in X. Let $p \in \overline{\pi_X[Z]}^X \backslash \pi_X[Z]$.

Since Z is a zero-set in $X \times Y$, $Z = f^{\leftarrow}(0)$ for some $f \in C^*(X \times Y)$. Define $h : X \times Y \to \mathbb{R}$ such that $h(x, y) = \frac{f(x,y)}{f(p,y)}$. So, $h\left[\{p\} \times Y\right] \subseteq \{1\}$ and $Z = h^{\leftarrow}(0)$. Without loss of generality, we can assume that the range of h is $[0, 1]$.

We will show that there are open sets U_n, V_n in X, and W_n in Y for $n < \omega$ such that for $m < \omega$, the following hold:

1. $p \in U_m$
2. $(V_m \times W_m) \cap Z \neq \emptyset$
3. $h\left[V_m \times W_m\right] \subseteq [0, \frac{1}{3})$
4. $h\left[U_m \times W_m\right] \subseteq (\frac{2}{3}, 1]$
5. $U_{m+1} \cup V_{m+1} \subseteq U_m$

First, pick $(x_1, y_1) \in Z$ and open sets $U_1, V_1 \in \tau(X)$ and $W_1 \in \tau(Y)$ such that $p \in U_1, x_1 \in V_1, y_1 \in W_1$, and $h\left[V_1 \times W_1\right] \subseteq [0, \frac{1}{3})$ and $h\left[U_1 \times W_1\right] \subseteq (\frac{2}{3}, 1]$. This can be done because h is continuous, $h(x_1, y_1) = 0$, and $h(p, y_1) = 1$.

Now, $U_1 \cap \pi_X[Z] \neq \emptyset$ because $x_1 \in U_1 \in \tau(X)$, and $x_1 \in \overline{\pi_X[Z]}^X$. So there is some $(x_2, y_2) \in Z$ such that $x_2 \in U_1$. Find open neighborhoods U_2 of p, V_2

of x_2, and W_2 of y_2 such that $h\left[V_2 \times W_2\right] \subseteq [0, \frac{1}{3}), h\left[U_2 \times W_2\right] \subseteq (\frac{2}{3}, 1]$, and $U_2 \cup V_2 \subseteq U_1$. Continue by induction.

The family $D = \{V_n \times W_n : n < \omega\}$ is pairwise disjoint because the V_n's are pairwise disjoint by our construction. If D is locally finite, then by **Fact 2.5**, D is finite. But D is infinite by our definition, so D cannot be locally finite. Then, there exists $(q, r) \in X \times Y$ with the property that for every neighborhood $R \times T$ of (q, r), $A = \{n \in \mathbb{N} : (V_n \times W_n) \cap (R \times T) \neq \emptyset\}$ is infinite.

On one hand, we have $(q, r) \in \overline{\bigcup \{V_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}$. Then,

$$h(q, r) \in h\left[\overline{\bigcup \{V_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}\right]$$

$$\subseteq \overline{h\left[\bigcup \{V_m \times W_m : m \in \mathbb{N}\}\right]}^{\mathbb{R}} \subseteq \overline{[0, \frac{1}{3})}^{\mathbb{R}} = [0, \frac{1}{3}].$$

On the other hand, if n and $n + k$ in A where $n, k \in \mathbb{N}$, then $V_{n+k} \subseteq U_{n+k-1} \subseteq \cdots \subseteq U_n$ by the way we constructed V_n's and U_n's. Since $(R \times T) \cap (V_{n+k} \times W_{n+k}) \neq \emptyset, (R \times T) \cap (U_n \times W_n) \neq \emptyset$ as well.

So, $(q, r) \in \overline{\bigcup \{U_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}$. Then,

$$h(q, r) \in h\left[\overline{\bigcup \{U_m \times W_m : m \in \mathbb{N}\}}^{X \times Y}\right]$$

$$\subseteq \overline{h\left[\bigcup \{U_m \times W_m : m \in \mathbb{N}\}\right]}^{\mathbb{R}} \subseteq \overline{(\frac{2}{3}, 1]}^{\mathbb{R}} = [\frac{1}{3}, 1].$$

This is a contradiction, so $\pi_X[Z]$ must be closed in X.

Lemma 2.7 Let X, Y be Tychonoff spaces. If π_X is z-closed, then $X \times Y$ is C^*-embedded in $X \times \beta Y$.

Proof: By **Fact 2.3**, it suffices to show that if Z_1 and Z_2 are disjoint zero-sets of $X \times Y$, then $\overline{Z_1}^{X \times \beta Y} \cap \overline{Z_2}^{X \times \beta Y} = \emptyset$.

Assume there is some point $(x, p) \in \overline{Z_1}^{X \times \beta Y} \cap \overline{Z_2}^{X \times \beta Y}$, where $x \in Y$ and $p \in \beta Y \backslash Y$. By **Fact 2.4**,

$$(x, p) \in \overline{Z_1 \cap (\{x\} \times Y)}^{\{x\} \times \beta Y} \cap \overline{Z_2 \cap (\{x\} \times Y)}^{\{x\} \times \beta Y}.$$

Now, $Z_1 \cap (\{x\} \times Y)$ and $Z_2 \cap (\{x\} \times Y)$ are disjoint open sets in $\{x\} \times Y$. Since $\{x\} \times Y$ is C^*-embedded in $\{x\} \times \beta Y$, then, by the other direction of **Fact 2.3**,

$$\overline{Z_1 \cap (\{x\} \times Y)}^{\{x\}\times\beta Y} \cap \overline{Z_2 \cap (\{x\} \times Y)}^{\{x\}\times\beta Y} = \emptyset$$

Contradiction, so $\overline{Z_1}^{X\times\beta Y} \cap \overline{Z_2}^{X\times\beta Y} = \emptyset$.

Glicksberg's Theorem: Let $X \times Y$ be Tychonoff spaces. If $X \times Y$ is pseudocompact, then $\beta(X \times Y) = \beta X \times \beta Y$.

Proof. By **Lemma 2.6**, the projection map $\pi_X : X \times Y \to X$ is z-closed. By **Lemma 2.7**, $X \times Y$ is C^*-embedded in $X \times \beta Y$. Since X is pseudocompact and βY is compact, by **Fact 3.1** of Chapter III, $X \times \beta Y$ is pseudocompact. Using **Lemma 2.6**, **Lemma 2.7** again, and by symmetry, $X \times \beta Y$ is C^*-embedded in $\beta X \times \beta Y$. By the transitivity of C^*-embedding, $X \times Y$ is C^*-embedded in $\beta X \times \beta Y$. I.e, $\beta(X \times Y) = \beta X \times \beta Y$.