Glicksberg's Theorem

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Lemma 1 *Let X, Y be Tychonoff spaces. If the projection $p : X \times Y \to X$ transforms functionally closed subsets of $X \times Y$ to closed subsets of X, then $X \times Y$ can be C^*-embedded in $X \times \beta Y$.*

Proof. Urysohn's Extension Theorem says that $X \times Y$ can be C^*-embedded in $X \times \beta Y \iff$ every completely separated sets $A, B \subseteq X \times Y$ can be completely separated in $X \times \beta Y$. So, let $A, B \subseteq X \times Y$ be completely separated sets in $X \times Y$. We will show that A, B can be completely separated in $X \times \beta Y$.

Since A, B can be completely separated in $X \times Y$, there exists a function $f : X \times Y \to [0, 1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$. Let $Z_0 = \pi_X f^{\leftarrow}(0)$ and $Z_1 = \pi_X f^{\leftarrow}(1)$. We are done if we can show that $cl_{X\times\beta Y} Z_0 \cap cl_{X\times\beta Y} Z_1 = \emptyset$.

Suppose $cl_{X\times\beta Y} Z_0 \cap cl_{X\times\beta Y} Z_1 \neq \emptyset$. Then there exists $(x', y') \in cl_{X\times\beta Y} Z_0 \cap cl_{X\times\beta Y} Z_1$. Let $\pi_X : X \times Y \to X$ and $\bar{\pi}_X : X \times \beta Y \to X$ be projection maps. Since Z_0 is a functionally closed set in $X \times Y$, by our hypothesis, $\pi_X[Z_0]$ is closed in X. Hence, $\bar{\pi}_X[\in cl_{X\times\beta Y} Z_0] \subseteq cl_X \bar{\pi}_X[Z_0] = cl_X \pi[Z_0] = \pi_X[Z_0]$. Similarly, $\pi_X[Z_1]$ closed in X, and so using our hypothesis again, we get $\bar{\pi}_X[cl_{X\times\beta Y} Z_1] \subseteq \pi_X[Z_1]$. In fact, $\bar{\pi}_X[\in cl_{X\times\beta Y} Z_0] = \pi_X[Z_0]$ and $\bar{\pi}_X[cl_{X\times\beta Y} Z_1] = \pi_X[Z_1]$.

Now, $x' \in \bar{\pi}_X[cl_{X\times\beta Y} Z_0 \cap cl_{X\times\beta Y} Z_1] = \bar{\pi}_X[cl_{X\times\beta Y} Z_0] \cap \bar{\pi}_X[cl_{X\times\beta Y} Z_1] = \pi_X[Z_0] \cap \pi_X[Z_1]$. Define a continuous function $f_{x'} : Y \to [0, 1]$ such that $f_{x'}(y) = f(x', y)$. As βY is the Stone-$\check{C}$ech compactification of Y, we can extend $f_{x'}$ to $\bar{f}_{x'} : \beta Y \to [0, 1]$.

Since $(x', y') \in cl_{X\times\beta Y} Z_0 \cap cl_{X\times\beta Y} Z_1$, there exist two sequences $\{(r_n, a_n)\}_{n\in\mathbb{N}}$, $\{(t_n, b_n)\}_{n\in\mathbb{N}} \subseteq X \times Y$ both converging to (x', y') such that $f(r_n, a_n) = 0$ and $f(t_n, b_n) = 1$ for all $n \in \mathbb{N}$. Furthermore, we always have the $f(r_n, a_n) \to \bar{f}_{x'}(y')$ and $f(t_n, b_n) \to \bar{f}_{x'}(y')$ whenever those two sequences converge to (x', y'), where f is continuous and $\bar{f}_{x'}$ is the extension of $f_{x'}$. Those statements would imply $\bar{f}_{x'}(y') = 0$ and $\bar{f}_{x'}(y') = 1$, a contradiction. Therefore $cl_{X\times\beta Y} Z_0 \cap cl_{X\times\beta Y} Z_1 = \emptyset$. $\square$

Lemma 2 *If the Cartesian product $X \times Y$ of Tychonoff spaces X and Y is pseudocompact, then the projection $\pi_X : X \times Y \to X$ transforms functionally closed subsets of $X \times Y$ to closed subsets of X.*

Proof. Let X, Y be Tychonoff and $X \times Y$ be pseudocompact. By way of contradiction, suppose that there exists a functionally closed subset Z of $X \times Y$ such that $\pi_X[Z]$ is not closed in X. Let $x_0 \in cl_X \pi_X[Z] \setminus \pi_X[Z] \neq \emptyset$. Let $g : X \times Y \to [0, 1]$ such that $Z = g^{\leftarrow}(0)$.

Define $f : X \times Y \to [0, 1]$ such that $f(x, y) = min\{\frac{g(x,y)}{g(x_0,y)}, 1\}$. By our definition, f is a continuous function. Moreover, $f(x_0, y) = 1$ for all $y \in Y$ and $Z = f^{\leftarrow}(0)$.

Since X is Tychonoff, it is first-countable. Let $\mathcal{B}_{x_0} = \{B_k \in \tau(X) : B_0 \supset B_1 \supset B_2 \supset \cdots$ and $x_0 \in B_k, k = 1, 2, 3, ... \}$ be a neighborhood basis of x_0. We shall define two sequences, (x_0, y_i) and(x_i, y_i), and their open neighborhoods(open in $X \times Y$), V_i's and W_i's by induction:

Step 1: Pick a point in Z and label it (x_1, y_1). Since X is Hausdorff and $\mathcal{B}$ is the neighborhood basis of x_0, there exist $V_1 \in \mathcal{B}$, and basic open sets $W_1 \in \tau(X)$ and $Y_1 \in \tau(Y)$ such that $(x_1, y_1) \in W_1 \times Y_1$ and $(x_0, y_1) \in V_1 \times Y_1$, and $W_1 \cap V_1 = \emptyset$.

Step n: Since $x_0 \in V_{n-1} \in \mathcal{B}$ and $x_0 \in \pi_X(Z)$, we can find a point $(x_n, y_n) \in Z$ such that $x_n \in V_{n-1}$, and that y_n is chosen so that $f(x_n, y_n) = 0$. Now, since f is continuous and $\mathcal{B}$ is the neighborhood basis of x_0, we can take $V_n \in \mathcal{B}($ where $V_n \subset V_{n-1})$, and $W_n \subset V_{n-1}$, a basic open set of X, and Y_n, a basic open set of Y, such that $W_n \times Y_n \in \tau(X \times Y)$ and $V_n \times Y_n \in \tau(X \times Y)$ are open sets in $X \times Y$ that contains $(x_n, y_n), (x_0, y_n)$, respectively.

After ω many steps, we've constructed infinitely W_i's and V_i's such that $f[W_i \times Y_i] \subseteq [0, \frac{1}{3})$ and $f[V_i \times Y_i] \subseteq (\frac{1}{3}, 1]$. Now, we will show that the family $\mathcal{F} = \{W_i \times Y_i : 1 \leq i < \infty\}$ is locally finite. Let $(x', y') \in X \times Y$. If $(x', y') \in W_i$ for some $W_i \in \mathcal{F}$, then we are done because the sets in $\mathcal{W}$ are pair-wise disjoint.

Suppose that $(x', y') \notin \{x_0\} \times Y$, then there must be some $V_j \in \mathcal{B}$ such that $x' \notin V_j$. Moreover, since X is Hausdorff, there exist open sets $U_x \in \tau(X), V_j \in \mathcal{B}$ such that $x' \in U_x$ and U_x is disjoint from V_j. But $W_k \subseteq V_j$ for all $k > j$. So there can be at most j many W_i's such that $W_i \cap V_j' \neq \emptyset$. Pick any open set $U_y \subset Y$ that contains y'. Then $U_x \times U_y \subset X \times Y$ is an open neighborhood of (x', y') which only meets $\mathcal{F}$ finitely many times.

Now suppose that $(x', y') \in \{x_0\} \times Y$. Since $x' = x_0, f(x', y') = 0$. Since $f : X \times Y \to [0, 1]$ is continuous, there exist $U \in \tau(X \times Y)$ such that $(x', y') \in U$ and $f[U] \in (\frac{2}{3}, 1]$. Hence, U does not meet $\mathcal{F}$. Because if it does, then $f[U] \cap [0, \frac{1}{3}) \neq \emptyset$, contradicting that $f[U] \in (\frac{2}{3}, 1]$.

We've shown that the family $\mathcal{F} \subset \mathcal{P}(X \times Y)$ is locally finite. However, $\mathcal{F}$ is infinite, and this contradicts Lemma 3. Thus, $X \times Y$ cannot be pseudocompact.

□

Lemma 3 *Let X be a Tychonoff space. If X is pseudocompact, then every locally finite family of non-empty open subsets of X is finite.*

Proof. (By way of contradiction.)Suppose that there exists a locally finite family $\mathcal{F} = \{U_i \in \tau(X) : U_i \neq \emptyset, 1 \leq i < \infty\}$. Since each U_i is non-empty, choose a point $x_i \in U_i$ for $i \in \mathbb{N}$ Since X is a Tychonoff space, there exists continuous functions $f_i : X \to [0, i]$ such that $f_i(x_i) = i$ and $f_i[X \backslash U_i] \subseteq \{0\}$ for each $i \in \mathbb{N}$. Define the fuction $f : X \to \mathbb{R}$ as $f(x) = \Sigma_{i=1}^{\infty} |f_i(x)|$. To show that f is continuous, pick $x_0 \in X$ and an open set V of $\mathbb{R}$ containing $f(x_0)$. We can assume that $V = (f(x_0) - \frac{1}{m}, f(x_0) + \frac{1}{m})$ for some $m \in \mathbb{N}$. Since $\mathcal{F}$ is locally finite, there exists an open set $U_0 \in \tau(X)$ containing x_0 such that U_0 meets $\mathcal{F}$ only finitely many times. So we have $\{a_i\}_{i=1}^{n} \subset \mathbb{N}$ such that $U_0 \cap U_{a_i} \neq \emptyset$ for $i \in [n]$. Define $\delta : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ to be $\delta(S) = sup(S) - inf(S)$. For each $i \in [n]$, since f_{a_i} is continuous, there exists $W_i \in \tau(X)$ such that $x_0 \in W_i$ and $\delta(f_i[W_i]) < \frac{1}{mn}$. Let $W = W_1 \cap W_2 \cap \cdots \cap W_n$. Then $\delta(f_i[W]) < \frac{1}{mn}$ for each $i \in [n]$ So $\delta(f[W]) = sup_{x \in W}(\Sigma_{i=1}^{\infty} |f_i(x)|) - inf_{x \in W}(\Sigma_{i=1}^{\infty} |f_i(x)|) = \sup_{x \in W}(\Sigma_{a_i : i \in [n]} |f_{a_i}(x)|) - inf_{x \in W}(\Sigma_{a_i : i \in [n]} |f_{a_i}(x)|) = \Sigma_{a_i : i \in [n]}(\sup_{x \in W}(|f_{a_i}(x)|) - inf_{x \in W}|f_{a_i}(x)|) < n\frac{1}{mn} = \frac{1}{m}$. As $x_0 \in W \in \tau(X)$ and $f[W] \subset (f(x_0) - \frac{1}{m}, f(x_0) + \frac{1}{m}) = V$, f is a continuous function. However, since $f(x_i) \geq i$ for all $i \in \mathbb{N}$, f is not bounded. This contradicts the pseudocompactness of X, which says that every real-valued continuous function on X must be bounded.

□

Glicksberg's Theorem *If the Cartesian product $X \times Y$ of Tychonoff spaces X and Y is pseudocompact, then $\beta Y \times \beta Y$ is the Čech-Stone compactification of $X \times Y$.*

Proof. We need to show that $X \times Y$ can be C^*-embedded in $X \times \beta Y$.

Since $X \times Y$ is pseudocompact, by Lemma 2, the projection $\pi_X : X \times Y \to X$ transforms functionally closed subsets of $X \times Y$ to closed subsets of X. Then, by Lemma 1 , $X \times Y$ can be C^*-embedded in $X \times \beta Y$.

Now, $X \times \beta Y$ is pseudocompact. So by Lemma2 and Lemma1 again, $X \times \beta Y$ can be C^*-embedded in $\beta X \times \beta Y$.

Hence, $X \times Y$ can be C^*-embedded in $X \times \beta Y$.

$\square$