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ON THE PRODUCT OF TWO NORMAL SPACES

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It is known that many simple operations on topological spaces do not preserve such an important property as normality. One of these operations is multiplication. A.V. Arkhangel'skii raises the question as to whether the product topology of two normal spaces includes the topology of a normal space. In other words, does the product of two normal spaces always admit a one-to-one continuous map onto a normal space? The answer is no. We give an example of two normal countably compact spaces X and Y whose product cannot be bijectively continuously mapped onto a normal space Z.

Example. Let $T(\omega_1)$ be the space of all countable ordinals with the order topology and $T(\omega_1+1) = T(\omega_1)\bigcup\{\omega_1\}$ be the one-point compactification of the space $T(\omega_1)$. Consider the product $T(\omega_1+1)\times[0,1]$, where $[0,1]$ is the real line segment with the natural topology. Let us remove the set $R=\{(\omega_1, r) : r\in(0,1)\}$ from the product. The space $X=(T(\omega_1+1)\times[0,1])\setminus R$ with the topology induced by the product is the sought one. Clearly, X is normal and countably compact.

As Y, take the space $T(\omega_1+1)$. Clearly, Y is also normal and countably compact. Consider the product $X\times Y$. It is not normal, because it contains a copy of the nonnormal space $T(\omega_1+1)\times T(\omega_1)$ as a closed subset. Note some important properties of the space $X\times Y$: (1) $X\times Y$ is countably compact; (2) for any countable subset A of $X\times Y$, if $(\omega_1, 0, \omega_1)\notin A$, then $(\omega_1,0,\omega_1)\notin[A]$ (recall that $X\times Y$ is a subset of the space $Y\times[0,1]\times Y$; for this reason, the elements of $X\times Y$ are given in the three-coordinate representation here and in the sequel); (3) the closure of any countable compact subset of $X\times Y$ is compact.

Let us prove that $X\times Y$ cannot be bijectively continuously mapped onto a normal space Z.

We will prove this by contradiction. Suppose there exists a continuous one-to-one map f of $X\times Y$ onto a normal space Z. On the face $I=\{(0, a, 0) : a\in[0,1]\}$, take an arbitrary sequence P convergent to $(0,0,0)$. Without the loss of generality, we can assume that $P=\{(0, 1/n+1, 0) : n\in N^+\}$. Take any $n\in N^+$ and consider the section $C_n = T(\omega_1)\times\{1/n+1\}\times T(\omega_1+1)$. The sets $A_n=\{(\alpha, 1/n+1, \omega_1) : \alpha\in T(\omega_1)\}$ and $B_n=\{(\beta, 1/n+1, \beta) : \beta\in T(\omega_1)\}$ are closed and disjoint, but they are not functionally separated in C_n and, therefore, in $X\times Y$. Fix the pair $\{A_n, B_n\}$ for each $n\in N^+$. Because the space Z is normal and the map f is one-to-one, there exists a subset $A=\{a_n : n\in N^+\}$ of $X\times Y$ such that $f(a_n)\in[f(A_n)]\bigcap[f(B_n)]$ for any $n\in N^+$. There are two possibilities.

I. There is in A a countable subset $A'=\{a_{n_i} : i\in N^+\}$ such that $a_{n_i}\neq(\omega_1,0,\omega_1)$ for all $i\in N^+$. Then properties (2) and (3) imply that $(\omega_1,0,\omega_1)\notin[A']$ and $[A']$ is compact. It is not quite evident, but it can be shown by using the continuity of f that $f((\omega_1, 0, \omega_1))\in[f(A')]$ whence $f((\omega_1,0,\omega_1)) = f(x)$ for some $x\in[A']$ distinct from $(\omega_1,0,\omega_1)$, which contradicts the bijectivity of f.

II. There exists an index k such that $a_n=(\omega_1,0,\omega_1)$ for all $n>k$. Then the arbitrariness of the selection of P and the connectedness of $[0,1]$ imply that $f((\omega_1,0,\omega_1))\in[f(T(\omega_1)\times\{x\}\times T(\omega_1+1))]$ for any $x\in(0,1)$. By virtue of continuity of f, we have $f((\omega_1,0,\omega_1)) = f((\omega_1,0,\omega_1))$. This also contradicts the bijectivity of f.

As both cases lead to contradiction, our starting assumption is false, that is, $X\times Y$ does not admit a one-to-one continuous map onto a normal space Z, which completes the proof.

In discussing the example we mentioned that the space $X\times Y$ is countably compact. Now we are going to use this property of $X \times Y$. A.V. Arkhangel'skii has recently proposed a new notion of a weakly normal space:

Definition. A space X is called weakly normal if for any two closed disjoint subsets A and B of X there exists a continuous map $f : X \to R^{\omega}$ such that $f(A) \bigcap f(B) = \varnothing$.

We can now give an updated version of the question formulated in the beginning of the paper: does the product of two weakly normal spaces always admit continuous one-to-one map onto a weakly normal space?

The constructed example gives the negative answer to this question as well: it suffices to apply the following remark.

Remark (A.V. Arkhangel'skii). Any countably compact weakly normal space is normal.

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