Fact 3.1 If X is pseudocompact and Y is compact, then $X \times Y$ is pseudocompact.

Proof. Let $f : X \times Y \to \mathbb{R}$. As $\{x\} \times Y$ is compact, $f[\{x\} \times Y]$ is closed and bounded in $\mathbb{R}$ for all $x \in X$. We can define $g : X \to \mathbb{R}$ as

$$g(x) = \max\{f(x, y) : y \in Y\}.$$

Fix $x_0 \in X$, we will show that g is continuous at x_0. Let $\epsilon > 0$.

By our definition of g, there exists some $y_0 \in Y$ such that $g(x_0) = f(x_0, y_0)$. Let $r = f(x_0, y_0)$. Now, define the sets U_y's and V_y's as follows:

For each $y \in Y$:

If $f(x_0, y) \in (r - \epsilon, r + \epsilon)$, we can get $V_y \in \tau(Y)$ and $U_y \in \tau(X)$ such that $x_0 \in U_y, y_0 \in V_y$ and $f[U_y \times V_y] \subseteq (r - \epsilon, r + \epsilon)$. In particular, since $f(x_0, y_0) \in (r - \epsilon, r + \epsilon)$, $x_0 \in U_{y_0} \in \tau(X)$, $y_0 \in V_{y_0} \in \tau(Y)$, and $f[U_{y_0} \times V_{y_0}] \subseteq (r - \epsilon, r + \epsilon)$

If $f(x_0, y) \notin (r - \epsilon, r + \epsilon)$, then since $f(x_0, y) \leq \max\{f(x_0, y) : y \in Y\} = r$, we must have $f(x_0, y) \leq r - \epsilon$. Hence, we can get $V_y \in \tau(Y)$ and $U_y \in \tau(X)$ such that $x_0 \in U_y, y_0 \in V_y$, and $f[U_y \times V_y] \subseteq (-\infty, r)$.

The family $\{V_y : y \in Y\}$ as defined aboved is an open cover of Y. By compactness, there exists $\{V_i : 1 \leq i \leq n\} \subseteq \{V_y : y \in Y\}$ such that $\bigcup\{V_i : 1 \leq i \leq n\} = Y$. Corresponding to $\{V_i : 1 \leq i \leq n\}$, we have the set $\{U_i : 1 \leq i \leq n\}$. Let $U = \bigcap\{U_i : 1 \leq i \leq n\} \cap U_{y_0}$, now U is an open set containing x_0.

Pick any $x \in U$.

On one hand, we have $\max\{f(x, y) : y \in Y\} < r + \epsilon$ because:

$$\begin{aligned}
\{f(x, y) : y \in Y\} &= \left\{f(x, y) : y \in \bigcup\{V_i : 1 \leq i \leq n\}\right\} \\
&= \bigcup\{f[\{x\} \times V_i] : 1 \leq i \leq n\} \\
&\subseteq \bigcup\{f[U_i \times V_i] : 1 \leq i \leq n\} \\
&\subseteq (-\infty, r + \epsilon).
\end{aligned}$$

One the other hand, we have $\max\{f(x, y) : y \in Y\} > r - \epsilon$ because:

$$\max\{f(x, y) : y \in Y\} > f(x, y_0), \text{and}$$

$$f(x, y_0) \in f[U \times \{y_0\}] \subseteq f[U_{y_0} \times \{y_0\}] \subseteq f[U_{y_0} \times V_{y_0}] \subseteq (r - \epsilon, r + \epsilon).$$

We have now $r-\epsilon<\max\{f(x,y): y\in Y\}<r+\epsilon$ for all $x\in U$. Hence, $g[U]\subseteq(r-\epsilon,r+\epsilon)$, so g is continuous on X. As X is pseudocompact, g must be bounded. Therefore, f must be bounded as well. Thus, $X\times Y$ is pseudocompact.

Fact 3.2 Let X be a pseudocompact space. Let $\tau=|\beta X|^+$ and denote by $T(\tau)$ the space of all ordinal numbers less than τ. Then, $X\times T(\tau)$ is pseudocompact, and also $T(\tau)$ is pseudocompact.

Proof. Let $f: X\times T(\tau)\to\mathbb{R}$ be continuous.

By **Claim 3.2.1** below, the ordinal space $T(\tau)$ is pseudocompact.

By **Claim 3.2.2**, there exists $\kappa_x<\tau$ such that f is constant on $\{x\}\times[\kappa_x,\tau)$. As $cf(\tau)>|X|$, there exists $\kappa=\sup_{x\in X}\{\kappa_x: x\in X\}$. Now, $f\left[X\times[0,\kappa+1]\right]$ is bounded because $X\times[0,\kappa+1]$ is pseudocompact by **Fact 3.1**.

For $\alpha\geq\kappa, f(x,\alpha)=f(x,\beta)$. Thus, $f\left[X\times[\kappa,\tau)\right]=f\left[X\times\{\kappa\}\right]$ which is bounded because X is pseudocompact.

The boundedness of $f\left[X\times[0,\kappa+1]\right]$ and $f\left[X\times[\kappa,\tau)\right]$ gives us that $f\left[X\times T(\tau)\right]$ is bounded. Hence, $X\times T(\tau)$ is pseudocompact.

Claim 3.2.1 The space $T(\tau)$ is pseudocompact.

Proof. Let $g: T(\tau)\to\mathbb{R}$ be a continuous function.
By way of contradiction, suppose that g is unbounded. We will define the subset $\{\alpha_i, i<\omega\}\subseteq T(\tau)$ by induction:

`Step 1.` Since g is unbounded, we can find $\alpha_1\in T(\tau)$ such that $g(\alpha_1)\geq 1$.

`Step N.` Since $[0,\alpha_{n-1}]$ is compact in $T(\tau)$ and g is continuous, $g\left[[0,\alpha_{n-1}]\right]$ must be bounded in $\mathbb{R}$. But since g is unbounded, $g\left[(\alpha_{n-1},\tau)\right]$ must be unbounded in $\mathbb{R}$. So there exists $\alpha_n\in(\alpha_{n-1},\tau)$ such that $g(\alpha_n)\geq n$.

Having defined $\alpha_i\in T(\tau)$ for all $i<\omega$, let $\beta=\sup\{\alpha_i: i<\omega\}$. Such β exists in $T(\tau)$ because $cf(\tau)>\omega$. As g is continuous, we have

$$g(\beta)=\lim_{i<\omega} g(\alpha_i)$$

This can't happen because the sequence $\{g(\alpha_i): i<\omega\}$ diverges to infinity. Thus, g must be bounded.

Claim 3.2.2 Let $g : T(\tau) \to \mathbb{R}$ be continuous. Then g is constant on $[\kappa, \tau)$ for some $\kappa \in T(\tau)$.

Proof. By **Claim 3.2.3** below, $[\alpha, \tau)$ is countably compact for all $\alpha \in T(\tau)$. This is because if $A = \{a_1, a_2, ...\}$ is a countably infinite subset of $[\alpha, \tau)$, then we can get a nondecreasing subsequence $\{a'_1, a'_2, ...\}$ of A. Let $\alpha = \lim_{n\to\infty}\{a'_1, a'_2, ...\}$, which exists because $cf(\tau) > \omega$. So A has an accumulation point, namely α. Thus $[\alpha, \tau)$ must be countaby compact.

Since g is continuous, $g\,[[\alpha, \tau)]$ is countably compact. In metric spaces, countably compact is equivalent to compact because metric spaces are Lindelöff. Hence, $g\,[[\alpha, \tau)]$ is compact for all $\alpha < \tau$. Thus, there exists $p \in \bigcap_{\alpha<\tau} g\,[\alpha, \tau)]$. To show that p is unique, suppose that there exists $q \in \bigcap_{\alpha<\tau} g\,[\alpha, \tau)]$.

There exists some $\alpha_0 \in [0, \tau)$ such that $g(\alpha_0) = p$. As $q \in g\,[[\alpha_0 + 1, \tau)]$, there exists $\alpha_1 \in [\alpha_0 + 1, \tau)$ such that $g(\alpha_1) = q$. As $p \in g\,[[\alpha_1 + 1, \tau)]$, there exists $\alpha_2 \in [\alpha_1 + 1, \tau)$ such that $g(\alpha_2) = p$. We continue this process by induction. We have now:

$$p = g(\alpha_0) = g(\alpha_2) = g(\alpha_4) = \cdots$$

$$q = g(\alpha_1) = g(\alpha_3) = g(\alpha_5) = \cdots$$

Let $\beta = \sup\{\alpha_n : n < \omega\}$, which exists because $cf(\tau) > \omega$. By continuity of g, $g(\beta) = \lim_{n<\omega} g(\alpha_n)$. Thus,

$$p = \lim_{n<\omega} g(\alpha_{2n}) = g(\beta) = \lim_{n<\omega} g(\alpha_{2n+1}) = q$$

So, $\bigcap_{\alpha<\tau} g\,[[\alpha, \tau)] = \{p\}$. For each $n < \omega$, we can find some $\gamma_n \in T(\tau)$ such that $g\,[[\gamma_n, \tau)] \subseteq (p - \frac{1}{n}, p + \frac{1}{n})$. Let $\kappa = \sup_{n<\omega} \gamma_n$. So, we have

$$g\,[[\kappa, \tau)] \subseteq \bigcap_{n<\omega} (p - \frac{1}{n}, p + \frac{1}{n}) = \{p\}.$$

Claim 3.2.3 For every Hausdorff spaces X, the following statements are equivalent:

1. The space X is countably compact.
2. For every decreasing sequence $F_1 \supset F_2 \supset \cdots$ of non-empty closed subsets of X, the intersection $\bigcap_{i=1}^{\infty} F_i$ is non-empty.
3. Every countably infinite subset of X has an accumulation point.

Proof.

1⇒2: Let $F_1 \supset F_2 \supset \cdots$ be non-empty closed subsets of X. If $\bigcap_{i=1}^{\infty} F_i = \emptyset$, then $\{X \backslash F_i : 1 \leq i \leq \infty\}$ would be an countable open cover of X, so there is a finite subcover $\{X \backslash F_i' : 1 \leq i \leq n\} \subseteq \{X \backslash F_i : 1 \leq i \leq \infty\}$ such that $\bigcup \{X \backslash F_i' : 1 \leq i \leq n\} = X$. Now, because the F_i's are decreasing, without loss of generality, $F_1' \supset F_2' \supset \cdots \supset F_n'$. So, $\bigcup \{X \backslash F_i' : 1 \leq i \leq n\} = X \backslash F_n'$. Contradiction.

2⇒1: By way of contradiction, suppose that X is not countably compact. Let $\{U_i \in \tau(X) : 1 \leq i \leq \infty\}$ be a countable cover of X that does not yield an finite subcover. For each $1 \leq n \leq \infty$, define $F_n = X \backslash \bigcup \{U_i : 1 \leq i \leq n\}$. For each n, F_n is non-empty because if it is, then $\{U_i(X) : 1 \leq i \leq n\}$ would be a finite subcover, contradiction. Thus, we have $F_1 \supset F_2 \supset \cdots$ and each F_n is a non-empty closed subset of X.

Now, by our assumption, the intersection $\bigcap_{i=1}^{\infty} F_i$ is non-empty. So there exists some $x \in \bigcap_{i=1}^{\infty} F_i$. So $x \in F_i$ for all $1 \leq i \leq \infty$. That means $x \notin U_i$ for all $1 \leq i \leq \infty$, contradicting that $\{U_i : 1 \leq i \leq \infty\}$ is a cover of X.

1⇒3: By way of contradiction, suppose we have a countably infinite subset $A = \{x_i \in X : 1 \leq i \leq \infty\}$ with no accumulation point in X. Then every point in A is an isolated point with respect to A. For each $x_i \in A$, let $x_i \in U_{x_i} \in \tau(X)$ such that $U_{x_i} \cap A = \{x_i\}$. So $\{X \backslash A\} \cup \{U_{x_i} \in \tau(X) : 1 \leq i \leq \infty\}$ is an countable open cover of X that yields no finite subcover, contradicting that X is countably compact.

3⇒1: By way of contradiction, suppose that $\{U_i \in \tau(X) : 1 \leq i \leq \infty\}$ is a countable cover of X which does not yield an open subcover. Then, by the equivalence of **1** and **2**, there exists a decreasing sequence $F_1 \supset F_2 \cdots$ of non-empty closed subsets of X such that $\bigcap_{i=1}^{\infty} F_i = \emptyset$. We define the set $A = \{x_1, x_2, ...\}$ such that $x_i \in F_i$ for each $1 \leq i \leq \infty$. If A is finite, then by pigeon-hole principle, there must be some $x_j \in A$ such that x_j belongs to infinitely many F_i's, and since F_i's are decreasing, x_j' would have to be in all F_i's. Contradicting $\bigcap_{i=1}^{\infty} F_i = \emptyset$. Hence, A is an infinite set. By our assumption, A has an accumulation point. Let x be an accumulation point of A.

Since $\bigcap_{i=1}^{\infty} F_i = \emptyset$, there exists an i such that $x \notin F_i$. Now, $U = X \backslash F_i$ is an open set that contains x, and U does not contain any point of the set $\{x_i, x_{i+1}, x_{i+2} ...\} \subseteq F_i$. Let $V = \{x\} \cup (X \backslash \{x_1, x_2, ..., x_{i-1}\})$. V is an open set that contains x. Hence, we have $x \in (U \cap V) \in \tau(X)$.

However, $(U \cap V) \cap A = \{x\}$ by the way we defined U and V. Thus x is not an accumulation point of A, contradiction.

Fact 3.3 Let τ be an uncountable regular cardinal. Let $T(\tau)$ be the space of all ordinal numbers less than τ. Let A_α be closed, unbounded subset of $T(\tau)$. Let $\gamma \in T(\tau)$. Then, $\bigcap\{A_\alpha : \alpha < \gamma\}$ is closed, unbounded and $|\bigcap\{A_\alpha : \alpha < \gamma\}| = \tau$.

Proof.
We will construct the set $\{p_\alpha : \alpha < \tau\}$ by transfinite induction.

`Step 1.`
Pick any element $a_{1,1} \in A_1$, we can find some element $a_{1,2} \in A_2$ such that $a_{1,2} > a_{1,1}$ because A_2 is unbounded. Then, by continuing this process, we can define $a_{1,n}$ in the same way, for all $n < \omega$. For all $\alpha < \gamma$, If α is a successor ordinal, then since A_α is unbounded, we can find some $a_{1,\alpha} \in A_\alpha$ such that $a_{1,\alpha} > a_{1,\alpha-1}$. If α is a limit ordinal, then let $\beta = \sup_{\kappa<\alpha}\{a_{1,\kappa}\}$, which exists because $\alpha < cf(\tau)$. Now, since A_α is unbounded, we can find some $a_{1,\alpha} \in A_\alpha$ such that $a_{1,\alpha} > \beta$.
Thus, we have defined the set $\{a_{1,\alpha} : \alpha < \gamma\}$. Let $\beta_1 = \sup\{a_{1,\alpha} : \alpha < \gamma\}$, which exists because $\gamma < cf(\tau)$.

`Step N.` Let $a_{n,1} \in A_1$ be such that $a_{n,1} > \beta_{n-1}$. Let $a_{n,2} \in A_2$ be such that $a_{n,2} > a_{n,1}$. Now continuing the same way as in Step 1, we can define $a_{n,\alpha}$ for all $\alpha < \gamma$. Let $\beta_n = \sup\{a_{n,\alpha} : \alpha < \gamma\}$.

So, we have contructed the set $\{a_{n,\alpha} : n < \omega, \alpha < \gamma\}$.

For all $\alpha < \gamma, \lim_{n<\omega} a_{n,\alpha} \in A_\alpha$ because A_α is closed. Moreover, if $\alpha, \alpha' < \gamma$, then $\lim_{n<\omega} a_{n,\alpha} = \lim_{n<\omega} a_{n,\alpha'}$. So if we define $p_1 = \lim_{n<\omega} a_{n,\alpha}$ for some $\alpha < \gamma$, then $p_1 \in \bigcap\{A_\alpha : \alpha < \gamma\}$.

For all $\alpha < \tau$, if α is an isolated ordinal, then we start from $p_{\alpha-1} \in A_1$ in Step 1 again, and define p_α the same way as we did for p_1. If α is a limit ordinal, then we let $p_\alpha = \sup\{p_\kappa : \kappa < \alpha\}$. This exists because $\alpha < cf(\tau)$.

We've finished contruction of the set $\{p_\alpha : \alpha < \tau\} \subseteq T(\tau)$. From the way we contructed it, this set is closed, unbounded and its cardinality is τ.

Fact 3.4 Let X be a Tychonoff space and $|X| > \aleph_0$. Let $\tau = |\beta X|^+$. Then, $T(\tau)$ can be condensed onto $T(\tau + 1)$. Moreover, for any space $X, X \times T(\tau)$ condenses onto $X \times T(\tau + 1)$.

Proof. Define $g : T(\tau) \to T(\tau + 1)$ by $g(0) = \tau$ and $g(\alpha) = \alpha - 1$ for all $\alpha < \omega$. Now, g is one-to-one and onto. Note that g is continuous at ω because if $(\beta, \omega]$ is an open set containing $g(\omega)$,then $(\beta + 1, \omega]$ is an open set such that $g\left[(\beta + 1, \omega]\right] \subseteq (\beta, \omega]$, and g is continuous on all $\alpha < \omega$ because $\{\alpha\} \in T(\tau)$;

finally, g is continuous on all $\alpha > \omega$ because $g|_{(\omega,\tau)}$ is the identity function. Thus, $T(\tau)$ can be condensed onto $T(\tau+1)$.

Moreover, define $h : X \times T(\tau) \to X \times T(\tau+1)$ by $h(x,\alpha) = (x, g(\alpha))$. Since g is one-to-one, onto, and continuous, then, h must also be one-to-one, onto, and continuous.

Fact 3.5 Let Z be a Tychonoff space. Let A be a closed subset of Z, B be compact subset of βZ. The set $A \cup B$ is not compact in βZ. Then, there exists a system $D = \{D_\alpha\}$ satisfying the following conditions:

1. For each α, the set D_α is non-empty and closed in A.
2. For $\alpha > \beta, D_\alpha \subseteq D_\beta$ and if β is a limit ordinal number, then $D_\beta = \bigcap\{D_\alpha : \alpha < \beta\}$.
3. $\bigcap\{D_\alpha\} = \emptyset$.
4. $\overline{D_1}^{\beta Z} \cap B = \emptyset$.

Proof.
Since $A \cup B$ is not compact, there is an open cover $\mathcal{C} \subset \tau(\beta Z)$ of $A \cup B$ that has no finite subcover. Since B is compact and $\mathcal{C}$ covers B, there is a finite subcover $\{C_i : 1 \leq i \leq n\} \subseteq \mathcal{C}$ such that $B \subseteq \bigcup\{C_i : 1 \leq i \leq n\}$.

Let $E = A \setminus \bigcup\{C_i : 1 \leq i \leq n\}$. E is closed in A. E is nonempty because if it is, then that means $\{C_i : 1 \leq i \leq n\}$ covers A as well as B, contradiction. Furthermore, E is not compact. If E is compact, we can get a finite subcover $\{C'_i : 1 \leq i \leq n\}$ from $\mathcal{C}$. Then, $\{C'_i : 1 \leq i \leq n\} \cup \{C_i : 1 \leq i \leq n\}$ is a finite subcover that covers $A \cup B$, contradiction.

As E is not compact, we can find an open cover $\mathcal{F} \subseteq \tau(\beta Z)$ such that no finite subcover of $\mathcal{F}$ covers E. without loss of generality, we can assume that $|\mathcal{F}| = L(E)$, the Lindeloff number of E. We can well-order $\mathcal{F}$, so $\mathcal{F} = \{F_\alpha : \alpha < L(E)\}$. Define $D_\alpha = E \setminus \bigcup\{F_\gamma : \gamma < \alpha\}$ for each $\alpha < L(E)$. We shall verify that D satisfies all four condictions:

1. For each α, the set D_α is non-empty and closed in A.
   `Proof- Each` D_α `is non-empty because if` $D_\alpha = \emptyset$ `for some` α`, then` $E \setminus \bigcup\{F_\gamma : \gamma < \alpha\} =$ `and so` $E \subseteq \{F_\gamma : \gamma < \alpha\}$`. However, since` $\alpha < L(E)$`, we have a contradiction. So` D_α `is nonempty. Moreover,` D_α `is closed in` A `because it is closed in` E`, and` E `is closed in` A`.`

2. For $\alpha > \beta, D_\alpha \subseteq D_\beta$ and if $\beta < L(E)$ is a limit ordinal number, then $D_\beta = \bigcap\{D_\alpha : \alpha < \beta\}$.

Proof- By the way we defined the $D'_\alpha s$, $D_\alpha \subseteq D_\beta$ if $\alpha > \beta$. If β a limit ordinal number, and if $D_\beta \neq \bigcap\{D_\alpha : \alpha < \beta\}$, then we replace D_β with the set $\bigcap\{D_\alpha : \alpha < \beta\}$, which is nonempty and closed in A. So now, $D_\beta = \bigcap\{D_\alpha : \alpha < \beta\}$.

3. $\bigcap\{D_\alpha\} = \emptyset$.
   Proof- This is true because $\bigcap\{D_\alpha\} = E \backslash \bigcup\{F_\gamma : \gamma < L(E)\} = \emptyset$.

4. $\overline{E}^{\beta Z} \cap B = \emptyset$.
   Since $\overline{E}^{\beta Z} \cap B = \emptyset$, and $D_1 \subseteq E$, then $\overline{D_1}^{\beta Z} \cap B = \emptyset$.

Fact 3.6 Let X be a Tychonoff space. If B_1, B_2 are subsets of X such that $\overline{B_1}^{\beta X} \cap \overline{B_2}^{\beta X} \neq \emptyset$, then B_1 and B_2 are not completely separated in X.

Proof. Suppose B_1 and B_2 are completely separated in X. Then there exists a continuous function function $f : X \to [0,1]$ such that $f[B_1] \subseteq \{0\}$ and $f[B_2] \subseteq \{1\}$. Let $\bar{f} : \beta X \to [0,1]$ be the extension of f. The sets $\bar{f}^{-1}(0)$ and $\bar{f}^{-1}(1)$ are closed in βX such that $\overline{B_1}^{\beta X} \subseteq \bar{f}(0)$ and $\overline{B_2}^{\beta X} \subseteq \bar{f}^{-1}(1)$. Since $\bar{f}^{-1}(0) \cap \bar{f}^{-1} = \emptyset$, we have $\overline{B_1}^{\beta X} \cap \overline{B_2}^{\beta X} = \emptyset$, contradiction.

Fact 3.7 Let X be a Tychonoff space. If X is locally compact, then X can be condensed onto a compact space.

Proof. Let $X \cup \{\infty\}$ be the one-point compactification of X. Pick any $x_0 \in X$. Let K be the space $X \cup \{\infty\}$ with the point ∞ identified with x_0. In K, the open sets containing x_0 is of the form $U_{x_0} \cup V_\infty$, where U_{x_0} is any open set containing x_0 in X, and V_∞ is any open set containing ∞ in $X \times \{\infty\}$. For $x \in K \backslash \{x_0\}$, the open sets containing x in K are same as the open sets containing x in X.

K is compact with the topology we've just defined. Let $f : X \times K$ be the identity map. So, f is one-to-one and onto. Let $U_{x_0} \cup V_\infty$ be an open set containing $f(x_0) = x_0$, then U_{x_0} is an open set in X such that $f[U_{x_0}] = U_{x_0} \subset U_{x_0} \cup V_\infty$. So f is continuous on x_0, as well as on other points of X. Hence, X can be condensed onto K.