FACT 3. Let τ be an uncountable regular cardinal. Let $T(\tau)$ be the space of all ordinal numbers less than τ. Let A_α be closed, unbounded subset of $T(\tau)$. Let $\gamma \in T(\tau)$. Then, $\bigcap\{A_\alpha : \alpha < \gamma\}$ is closed, unbounded and $|\bigcap\{A_\alpha : \alpha < \gamma\}| = \tau$.

Proof.
We will construct the set $\{p_\alpha : \alpha < \tau\}$ by transfinite induction.

`Step 1.`
Pick any element $a_{1,1} \in A_1$, we can find some element $a_{1,2} \in A_2$ such that $a_{1,2} > a_{1,1}$ because A_2 is unbounded. Then, by continuing this process, we can define $a_{1,n}$ in the same way, for all $n < \omega$. For all $\alpha < \gamma$, If α is a successor ordinal, then since A_α is unbounded, we can find some $a_{1,\alpha} \in A_\alpha$ such that $a_{1,\alpha} > a_{1,\alpha-1}$. If α is a limit ordinal, then let $\beta = sup_{\kappa<\alpha}\{a_{1,\kappa}\}$. This exists because $cf(\tau) > \beta$. Now, since A_α is unbounded, we can find some $a_{1,\alpha}$ such that $a_{1,\alpha} > a_{1,\beta}$.

Hence, we've defined the set $\{a_{1,\alpha} : \alpha < \gamma\}$. Let $\beta_1 = sup\{a_{1,\alpha} : \alpha < \gamma\}$. It exists because $\gamma < cf(\tau)$.

`Step N.` Let $a_{n,1} \in A_1$ be such that $a_{n,1} > \beta_{n-1}$. Let $a_{n,2} \in A_2$ be such that $a_{n,2} > a_{n,1}$. Continuing this way, as in Step 1, we can define $a_{n,\alpha}$ for all $\alpha < \gamma$.

For all $\alpha < \gamma$, let $p_1 = lim_{n<\omega} a_{n,\alpha} \in A_\alpha$ because $cf(A) > \omega$. Moreover, if $\alpha = \alpha'$, then $lim_{n<\omega} a_{n,\alpha} = lim_{n<\omega} a_{n,\alpha'}$. Hence, $p_1 \in \bigcap\{A_\alpha : \alpha < \gamma\}$.

For all $\alpha < \tau$, if α is an isolated ordinal, then we start from Step 1 again to get p_2. If α is a limit ordinal, then we let $p_\alpha = sup\{p_\kappa : \kappa < \alpha\}$. This exists because $\alpha < cf(\tau)$.

We've finished contruction of the set $\{p_\alpha : \alpha < \tau\} \subseteq T(\tau)$. From the way we contructed it, this set is closed, unbounded and its cardinality is τ.