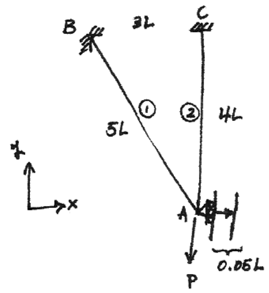
Problem #1. Consider the following two-bar truss:



For a model problem, let,

$P = 10^6$ N , $L = 1$m. $E = 210$ GPa (steel)

$A = 5\times10^{-4} m^2$ ($5 cm^2$)

A force of P is applied at node A in the negative y-direction while node A moves by an amount of 0.05L in the positive x-direction. Both bars are assumed to be linearly elastic and isotropic, with Young's modulus E and cross-sectional area A.

Determine the y-displacement of node A and the axial force in each bar.

Notations.

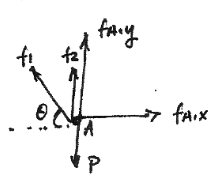
Displacements of the nodes (dofs) : $\underbrace{u_{A.x},\ u_{A.y},\ u_{B.x},\ u_{B.y},\ u_{C.x},\ u_{C.y}}_{\text{similar to the displacement array/vector in code}}$

Internal forces : f_1, f_2.

Reaction forces : $f_{A.x}$, $f_{A.y}$, $f_{B.x}$, $f_{B.y}$, $f_{C.x}$, $f_{C.y}$.

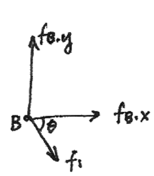
Equilibrium equations.

At node A.



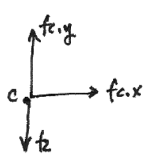
$$\begin{cases} -\frac{3}{5}f_1 + f_{A.x} = 0 \\ \frac{4}{5}f_1 + f_2 + f_{A.y} - P = 0 \end{cases}$$

Node B.



$$\begin{cases} \frac{3}{5}f_1 + f_{B.x} = 0 \\ -\frac{4}{5}f_1 + f_{B.y} = 0 \end{cases}$$

Node C.



$$\begin{cases} f_{C.x} = 0 \\ -f_2 + f_{C.y} = 0 \end{cases}$$

In matrix form. we have

$$\begin{bmatrix} -\frac{3}{5} & 0 \\ \frac{4}{5} & 1 \\ \frac{3}{5} & 0 \\ -\frac{4}{5} & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -f_{A,x} \\ -f_{A,y} + P \\ -f_{B,x} \\ -f_{B,y} \\ -f_{C,x} \\ -f_{C,y} \end{bmatrix}.$$

<u>Constitutive equations</u>.

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = EA \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}, \quad \text{where } \varepsilon_1, \varepsilon_2 \text{ are the axial strains}.$$

<u>Strain-displacement equations</u>

$$\varepsilon_1 = \frac{u_{A,x} - u_{B,x}}{5L}\left(\frac{3}{5}\right) + \frac{u_{A,y} - u_{B,y}}{5L}\left(-\frac{4}{5}\right) = \frac{3}{25L}(u_{A,x} - u_{B,x}) - \frac{4}{25L}(u_{A,y} - u_{B,y})$$

$$\varepsilon_2 = \frac{u_{A,x} - u_{C,x}}{4L}(0) + \frac{u_{A,y} - u_{C,y}}{4L}(-1) = -\frac{1}{4L}(u_{A,y} - u_{C,y}).$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \frac{1}{L}\begin{bmatrix} \frac{3}{25} & -\frac{4}{25} & -\frac{3}{25} & \frac{4}{25} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} u_{A,x} \\ u_{A,y} \\ u_{B,x} \\ u_{B,y} \\ u_{C,x} \\ u_{C,y} \end{bmatrix}.$$

Combine the three equations to get,

$$\underbrace{\frac{EA}{L}\begin{bmatrix} \frac{9}{125} & -\frac{12}{125} & -\frac{9}{125} & \frac{12}{125} & 0 & 0 \\ -\frac{12}{125} & \frac{16}{125}+\frac{1}{4} & \frac{12}{125} & -\frac{16}{125} & 0 & -\frac{1}{4} \\ -\frac{9}{125} & \frac{12}{125} & \frac{9}{125} & -\frac{12}{125} & 0 & 0 \\ \frac{12}{125} & -\frac{16}{125} & -\frac{12}{125} & \frac{16}{125} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}}_{\equiv K,\ \mathrm{rank}(K) = 2.} \underbrace{\begin{bmatrix} u_{A,x} \\ u_{A,y} \\ u_{B,x} \\ u_{B,y} \\ u_{C,x} \\ u_{C,y} \end{bmatrix}}_{\equiv \underline{u}} = \underbrace{\begin{bmatrix} f_{A,x} \\ f_{A,y} - P \\ f_{B,x} \\ f_{B,y} \\ f_{C,x} \\ f_{C,y} \end{bmatrix}}_{\equiv \underline{f}.}$$

We easily see that the first two columns of K are linearly independent, while the other four can be written as a linear combination of these two. Hence, rank (K) = 2.

This means, we need to specify (at least) 4 displacement constraints in order to get a unique solution.

Boundary conditions

We are given that,

$$\begin{cases} u_{A,x} = 0.05L, \quad u_{B,x}, u_{B,y}, u_{C,x}, u_{C,y} = 0. \quad (OL) \\ f_{A,y} = 0 \quad (OP) \end{cases}.$$

We can find the y-displacement of node A from the second equation:

$$-\frac{12}{125}(0.05L) + \left(\frac{16}{125} + \frac{1}{4}\right) u_{A,y} = -\frac{PL}{EA}.$$

$$\Rightarrow \boxed{u_{A,y} = \left(\frac{-\frac{P}{EA} + \frac{12}{125}(0.05)}{\frac{16}{125} + \frac{1}{4}} \right) L} \quad \text{(draw the deformed shape).}$$

For the model parameters,

$$\boxed{u_{A,y} \approx -0.0125 \text{ m}}.$$

The strains are given by,

$$\varepsilon_1 = \frac{3}{25L}(0.05L - 0L) - \frac{4}{25L}(u_{A,y} - 0L) \approx 0.0080 \quad \text{(tension)}$$

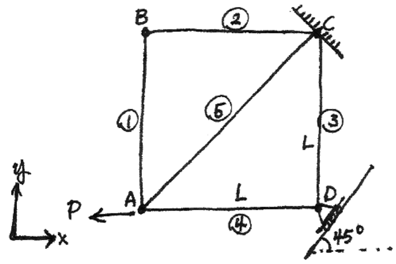
$$\varepsilon_2 = -\frac{1}{4L}(u_{A,y} - 0L) \approx 0.0031 \quad \text{(tension)}$$

The axial (internal) forces are then,

$$f_1 = EA\,\varepsilon_1 \approx \boxed{8.3995 \times 10^5 \text{ N}}$$

$$f_2 = EA\,\varepsilon_2 \approx \boxed{3.2804 \times 10^5 \text{ N}}.$$

Problem #2. Consider the following truss:



Bars : E, A (linearly elastic, isotropic)

Notice that node D is on a roller pin inclined at an angle of 45°.

Find the unknown displacements, and determine which bars increase in length, decrease, or remain the same

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Element ① : Nodes (A, B), $\cos\theta = 0$, $\sin\theta = 1$.

$$K^e = \frac{EA}{L}\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right], \quad \underline{u}^e = \begin{bmatrix} u_{A,x} \\ u_{A,y} \\ u_{B,x} \\ u_{B,y} \end{bmatrix}$$

Element ② : Nodes (B, C), $\cos\theta = 1$, $\sin\theta = 0$.

$$K^e = \frac{EA}{L}\left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \underline{u}^e = \begin{bmatrix} u_{B,x} \\ u_{B,y} \\ u_{C,x} \\ u_{C,y} \end{bmatrix}$$

Element ③ : Nodes (C, D), $\cos\theta = 0$, $\sin\theta = 1$.

$$K^e = \frac{EA}{L}\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right], \quad \underline{u}^e = \begin{bmatrix} u_{C,x} \\ u_{C,y} \\ u_{D,x} \\ u_{D,y} \end{bmatrix}$$

Element ④ : Nodes (A, D), $\cos\theta = 1$, $\sin\theta = 0$.

$$K^e = \frac{EA}{L}\left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \underline{u}^e = \begin{bmatrix} u_{A,x} \\ u_{A,y} \\ u_{D,x} \\ u_{D,y} \end{bmatrix}.$$

Element ⑤ : Nodes (A, C), $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$.

$$K^e = \frac{EA}{\sqrt{2}L}\left[\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \hline -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right], \quad \underline{u}^e = \begin{bmatrix} u_{A,x} \\ u_{A,y} \\ u_{C,x} \\ u_{C,y} \end{bmatrix}.$$

Assemble the element matrices into a global matrix $K \in \mathbb{R}^{8\times 8}$.

$$K = \frac{EA}{L}\left[\begin{array}{cc|cc|cc|cc} 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & & & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -1 & \\ \frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & & -1 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & & \\ \hline & & 1 & & -1 & & & \\ & -1 & & 1 & & & & \\ \hline -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & & & 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & & \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & & & \frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & & -1 \\ \hline -1 & & & & & & 1 & \\ & & & & & -1 & & 1 \end{array}\right]$$

Note that $\text{rank}(K) = 5$, so we must have at least 3 displacement constraints in order to get a unique solution.

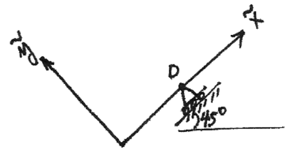
Boundary conditions.

It is easy to see that,

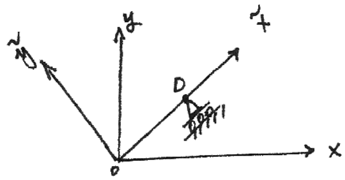
$$\begin{cases} u_{C,x},\ u_{C,y} = 0. & \longrightarrow \text{unknown reaction forces} \\ f_{A,x},\ f_{A,y},\ f_{B,x},\ f_{B,y} = 0 & \longrightarrow \text{unknown displacements} \end{cases}$$

But what about node D?

Consider a local coordinate system at node D (i.e. local to D).



Without loss of generality, we can assume that the local coordinate system $(\tilde{x}, \tilde{y})$ shares the same origin as the global coordinate system (x, y). Hence,



$$\left(\text{new basis } \tilde{e}_1 = \begin{bmatrix} \cos(45^\circ) \\ \sin(45^\circ) \end{bmatrix}, \tilde{e}_2 = \begin{bmatrix} -\sin(45^\circ) \\ \cos(45^\circ) \end{bmatrix}. \right)$$

What we would like to do is to describe the displacement of node D in terms of the local coordinate system. Note that the "act" of displacement is the same regardless of whether we consider the global or local coordinates. The "amount" of displacement depends on which coordinate system we consider; more precisely, on which basis we consider to represent the space of $\mathbb{R}^2$.

We have,

$$\underbrace{\begin{bmatrix} u_{D,x} \\ u_{D,y} \end{bmatrix}}_{\substack{\text{displacement in} \\ \text{terms of the standard} \\ \text{basis}}} = \underbrace{\tilde{u}_{D,x} \begin{bmatrix} \cos(45^\circ) \\ \sin(45^\circ) \end{bmatrix} + \tilde{u}_{D,y} \begin{bmatrix} -\sin(45^\circ) \\ \cos(45^\circ) \end{bmatrix}}_{\substack{\text{displacement in terms of} \\ \text{the new/local basis}}} = \underbrace{\begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}}_{\text{linear transformation!}} \begin{bmatrix} \tilde{u}_{D,x} \\ \tilde{u}_{D,y} \end{bmatrix}.$$

Since $\tilde{u}_{D,x}$ is free and $\tilde{u}_{D,y} = 0$, we have the following BC:

$$u_{D,x} = \frac{1}{\sqrt{2}} \tilde{u}_{D,x}$$

$$u_{D,y} = \frac{1}{\sqrt{2}} \tilde{u}_{D,x} \quad (= u_{D,x}) \longleftarrow$$ 3rd displacement constraint; we know what $u_{D,y}$ is once we figure out $u_{D,x}$.

Hence, the x- and y-displacements must equal (obvious in hindsight).

Similarly, the force at node D is given by,

$$\begin{bmatrix} f_{D,x} \\ f_{D,y} \end{bmatrix} = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix} \begin{bmatrix} \tilde{f}_{D,x} \\ \tilde{f}_{D,y} \end{bmatrix}.$$

Since $\tilde{f}_{D,x} = 0$ and $\tilde{f}_{D,y}$ is free, we must have.

$$f_{D,x} = -\frac{1}{\sqrt{2}} \tilde{f}_{D,y}$$

$$f_{D,y} = \frac{1}{\sqrt{2}} \tilde{f}_{D,y}$$

Recall equation #8 :

$$\frac{EA}{L} \left[(-1) u_{C,y} + (1) u_{D,y} \right] = f_{D,y}$$

Since $u_{C,y} (=0)$ and $u_{D,y} (= u_{D,x})$ are known, we can write that

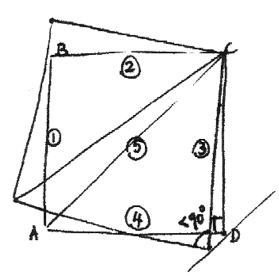
$$f_{D,x} = -f_{D,y} = -\frac{EA}{L} u_{D,x}.$$

<u>Reduce system</u>

$$\frac{EA}{L} \left[\begin{array}{cc|cc|c} 1+\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & & & -1 \\ \frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & & -1 & \\ \hline & & 1 & & \\ & -1 & & 1 & \\ \hline -1 & & & & 2 \end{array} \right] \begin{bmatrix} u_{A,x} \\ u_{A,y} \\ u_{B,x} \\ u_{B,y} \\ u_{D,x} \end{bmatrix} = \begin{bmatrix} -P \\ 0 \\ 0 \\ 0 \\ \cancel{-\frac{EA}{L} u_{D,x}}\ 0 \end{bmatrix}$$

$u_{D,x}$ is unknown; move to the other side

$$\Rightarrow \begin{bmatrix} u_{A,x} \\ u_{A,y} \\ u_{B,x} \\ u_{B,y} \\ u_{D,x} \end{bmatrix} = \frac{PL}{EA} \begin{bmatrix} -2 \\ 2 \\ 0 \\ 2 \\ -1 \end{bmatrix}.$$



Postprocessing

$$\varepsilon_1 = \frac{1}{L}(u_{B.y} - u_{A.y}) = \frac{P}{EA}(2-2) = 0$$

$$\varepsilon_2 = \frac{1}{L}(u_{C.x} - u_{B.x}) = \frac{P}{EA}(0-0) = 0$$

$$\varepsilon_3 = -\frac{1}{L}(u_{D.y} - u_{C.y}) = -\frac{P}{EA}(-1-0) = \frac{P}{EA} \quad (\text{tension})$$

$$\varepsilon_4 = \frac{1}{L}(u_{D.x} - u_{A.x}) = \frac{P}{EA}(-1+2) = \frac{P}{EA} \quad (\text{tension})$$

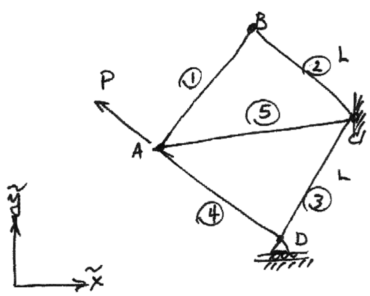
$$\varepsilon_5 = \frac{1}{2L}(u_{C.x} - u_{A.x}) + \frac{1}{2L}(u_{C.y} - u_{A.y}) = \frac{P}{2EA}(0+2) + \frac{P}{2EA}(0-2) = 0.$$

Since the diagonal bar 5 did not change in length, but the bars 3 & 4 increased in length, the angle between the bars 3 and 4 must have decreased.

Remark.

There is an easier way to solve the problem. We note that the difficulty of solving this lies with the BC at node D.

What if we consider instead (by rotating our head by 45°)?,



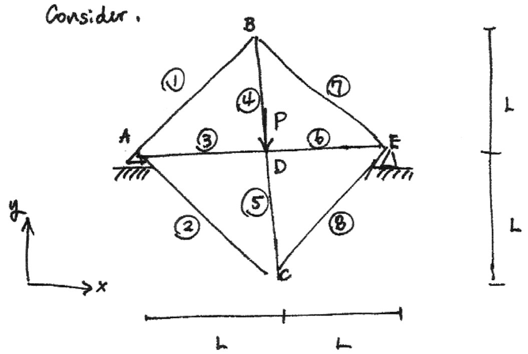
The roller BC is much easier to work with now. We still have that,

$$\tilde{u}_{D.y} = 0 \qquad (\tilde{u}_{D.x} \text{ is free})$$

$$\tilde{f}_{D.x} = 0. \qquad (\tilde{f}_{D.y} \text{ is free}).$$

Problem #3. Exploiting symmetry in a structure.

Consider.



Bars : $\sqrt{2}EA$, for ①, ②, ⑦, ⑧

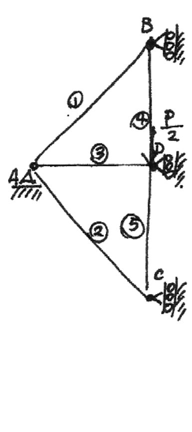
EA. for ③, ④, ⑤, ⑥.

Notice that the structure is symmetric along the line (plane) $\overline{BC}$, i.e. identical geometry, material, loading, and BCs occur at the corresponding locations on the opposite sides about this line (plane).

In such a situation, we can reduce the problem by the following:

- For loads occurring in the plane of symmetry, half of the total load is applied to the reduced structure.
- For elements occurring in the plane of symmetry, half of the cross-sectional area must be used in the reduced structure.
- For nodes in the plane of symmetry, the displacement component normal to the plane of symmetry must be set to zero in the reduced structure.

We get.



Element ①, $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{1}{\sqrt{2}}$

$$K^e = \frac{\sqrt{2}EA}{\sqrt{2}L}\left[\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \hline -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right], \quad \underline{u}^e = \begin{bmatrix} u_{A,x} \\ u_{A,y} \\ u_{B,x} \\ u_{B,y} \end{bmatrix}$$

Element ② $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = -\frac{1}{\sqrt{2}}$

$$K^e = \frac{\sqrt{2}EA}{\sqrt{2}L}\left[\begin{array}{cc|cc} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \hline -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right], \quad \underline{u}^e = \begin{bmatrix} u_{A,x} \\ u_{A,y} \\ u_{C,x} \\ u_{C,y} \end{bmatrix}.$$

Element ③ : $\cos\theta = 1$, $\sin\theta = 0$.

$$K^e = \frac{EA}{L}\left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \underline{u}^e = \begin{bmatrix} u_{A,x} \\ u_{A,y} \\ u_{D,x} \\ u_{D,y} \end{bmatrix}$$

Element ④ , $\cos\theta = 0$, $\sin\theta = -1$.

$$K^e = \frac{E(\frac{A}{2})}{L}\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right], \quad \underline{u}^e = \begin{bmatrix} u_{B,x} \\ u_{B,y} \\ u_{D,x} \\ u_{D,y} \end{bmatrix}$$

Element ⑤ $\cos\theta = 0$, $\sin\theta = 1$.

$$K^e = \frac{E(\frac{A}{2})}{L}\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right], \quad \underline{u}^e = \begin{bmatrix} u_{C,x} \\ u_{C,y} \\ u_{D,x} \\ u_{D,y} \end{bmatrix}$$

Boundary conditions

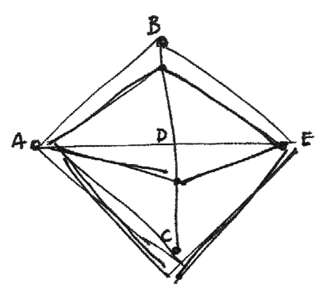
$$\begin{cases} u_{A,x},\ u_{A,y},\ u_{B,x},\ u_{C,x},\ u_{D,x} = 0. \\ f_{B,y},\ f_{C,y},\ f_{D,y} = 0 \end{cases}$$

Note that, had we considered the full problem instead, we would have 6 unknown displacement components instead of 3.

We find that.

$$\frac{EA}{L}\left[\begin{array}{c|c|c} 1 & 0 & -\frac{1}{2} \\ \hline 0 & 1 & -\frac{1}{2} \\ \hline -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right]\begin{bmatrix} u_{B,y} \\ u_{C,y} \\ u_{D,y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{P}{2} \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} u_{B,y} \\ u_{C,y} \\ u_{D,y} \end{bmatrix} = \frac{PL}{EA}\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -1 \end{bmatrix}.$$



Remark.

- Infinite plate with a circular hole, under uniaxial or biaxial tension.
  → Model 1/4 of the geometry.