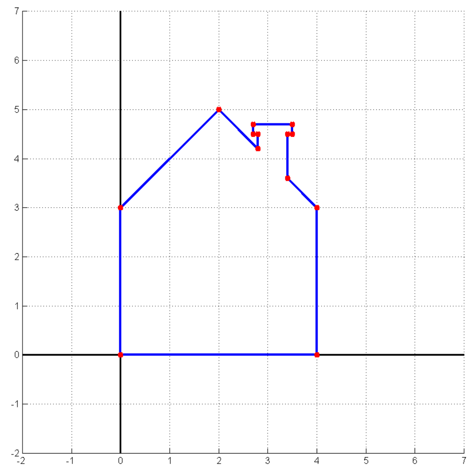
Project 1: Linear and Affine Transformations

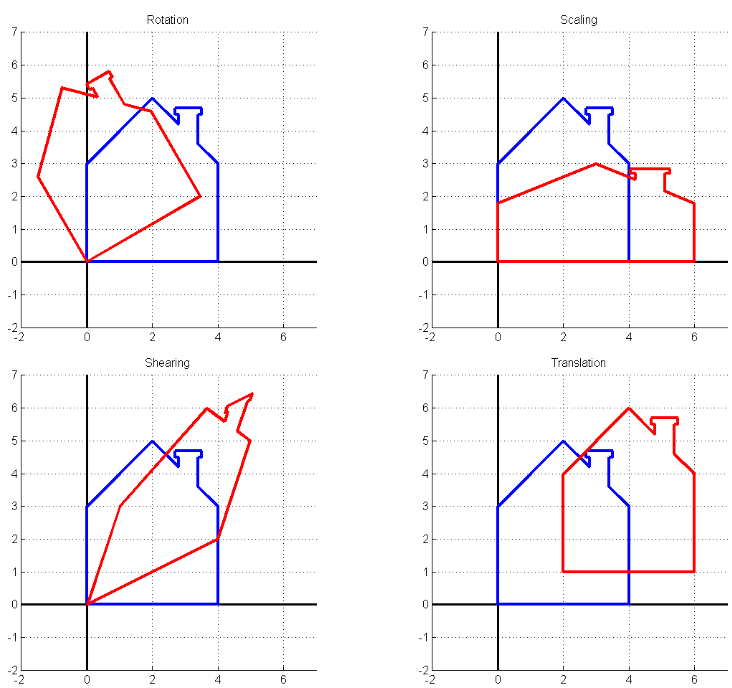
1. In computer graphics (and mechanics!), rotation, scaling, and shearing of an object can all be achieved through a linear transformation. However, we will soon show that translation of an object cannot be represented with a linear transformation, but can be with an affine transformation.

   (a) Let $L : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Show that $L(\vec{0}) = \vec{0}$.

   (b) Consider 2D objects (i.e. $m = n = 2$), like the house below.



We see that, given its vertices (which we represent with vectors in $\mathbb{R}^2$), we can draw an object by "connecting the dots," and that we can rotate, scale, and shear the object about the origin by applying an appropriate linear transformation L to all the vertices.



From part (a), explain why a translation of an object cannot be represented with a linear map.

(c) While we can apply L to one vertex $\vec{x} \in \mathbb{R}^2$ at a time, we can do better. First, collect all p vertices into a matrix $X \in \mathbb{R}^{2\times p}$. In the house example above, there are $p = 14$ vertices and

$$X = \begin{bmatrix} 0.0 & 4.0 & 4.0 & 3.4 & 3.4 & 3.5 & 3.5 & 2.7 & 2.7 & 2.8 & 2.8 & 2.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 3.0 & 3.6 & 4.5 & 4.5 & 4.7 & 4.7 & 4.5 & 4.5 & 4.2 & 5.0 & 3.0 & 0.0 \end{bmatrix}.$$

We then find the matrix $A \in \mathbb{R}^{2\times 2}$ so that $L(\vec{x}) = A\vec{x}$. Due to partitioning, we can instead compute the matrix-matrix product $X_{new} = AX$.

Find $A \in \mathbb{R}^{2\times 2}$ that represents the (i) rotation, (ii) scaling, and (iii) shearing of the house on page 1. Verify your answers by running **project1.m**.

* Rotation is done 30 degrees counterclockwise.
* Horizontal scale of 1.5, vertical scale of 0.6.
* For shearing, notice that $\begin{bmatrix} 4.0 \\ 0.0 \end{bmatrix}$ is mapped to $\begin{bmatrix} 4.0 \\ 2.0 \end{bmatrix}$, and $\begin{bmatrix} 0.0 \\ 3.0 \end{bmatrix}$ is mapped to $\begin{bmatrix} 1.0 \\ 3.0 \end{bmatrix}$.

Hint: Recall that the columns of A correspond to the actions of L on the (standard) basis vectors.

(d) Notice how a series of N rotations, scalings, and shearings (in any order) is just a composition of linear maps $L_1, \cdots, L_N$, which is just the product of the corresponding matrices $A_1, \cdots, A_N \in \mathbb{R}^{2\times 2}$. Now, for $X \in \mathbb{R}^{2\times p}$, we can compute X_{new} in two different ways:

* $X_{new} = A_N(\cdots A_2(A_1X)\cdots)$
* $X_{new} = (A_N \cdots A_2A_1)X$

If N and p are large, which is more efficient? Explain by computing the cost involved in each.

2. We say that $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is an <u>affine</u> transformation if $T(\vec{x}) = A\vec{x} + \vec{b}$, $\forall\, \vec{x} \in \mathbb{R}^n$ for some $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \in \mathbb{R}^m$. (Note, for $m = n = 1$, how we used to call A the slope and $\vec{b}$ the y-intercept in algebra.)

(a) Again, consider 2D. Show that a translation by h units to the right and k units to the above can be represented with an affine map $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ by determining what A and $\vec{b}$ are.

(b) Again, we don't want to apply T to one vertex $\vec{x}$ at a time, but to all the vertices X at once. For translation, we consider "homogeneous coordinates," whereby we add a third coordinate to each $\vec{x}$ and set it to 1. For the house example, (with subscript H for homogeneous)

$$X_H = \begin{bmatrix} 0.0 & 4.0 & 4.0 & 3.4 & 3.4 & 3.5 & 3.5 & 2.7 & 2.7 & 2.8 & 2.8 & 2.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 3.0 & 3.6 & 4.5 & 4.5 & 4.7 & 4.7 & 4.5 & 4.5 & 4.2 & 5.0 & 3.0 & 0.0 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}.$$

Verify that computing $X_{H,new} = A_H X_H \equiv \left[\begin{array}{c|c} A & \vec{b} \\ \hline \vec{0}^T & 1 \end{array}\right] X_H$ and reading the first two rows of $X_{H,new}$ is equivalent to applying T to one $\vec{x}$ at a time. Note that $A_H \in \mathbb{R}^{3\times 3}$.

(c) Find $A_H \in \mathbb{R}^{3\times 3}$ that represents the translation of the house on page 1. Note that the house has been moved 2 units to the right, 1 unit to the above. Verify your answer by running **project1.m**.

(d) Explain how we can represent rotations, scalings, and shearings with a 3×3 matrix instead, so that we can apply a series of N rotations, scalings, shearings, and translations (in any order) as a product of matrices $A_{H,1}, \cdots, A_{H,N} \in \mathbb{R}^{3\times 3}$.

3. Think about how rotation, scaling, shearing, and translation of a 3D object can be done.