1. LAPLACE TRANSFORM

1) Prove that $\mathcal{L}(d^2f/dt^2) = s^2F(s) - sf(0) - f'(0)$.

<Answer>

By def, $\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$

Using the def

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = \int_0^\infty \frac{d^2f(t)}{dt^2}e^{-st}\,dt$$

$$= \left[\frac{df(t)}{dt}e^{-st}\right]_{t=0}^{\infty} - \int_0^\infty \frac{df(t)}{dt}(-s)e^{-st}dt$$

$$= \left[0 - \frac{df(0)}{dt}\right] - \int_0^\infty \frac{df(t)}{dt}(-s)e^{-st}\,dt$$

$$= \left[-\frac{df(0)}{dt}\right] - \left[\left\{f(t)(-s)e^{-st}\right\}\Big|_{t=0}^{\infty} - \int_0^\infty f(t)s^2e^{-st}dt\right]$$

$$= \left[-\frac{df(0)}{dt}\right] - \left[\left\{0 - f(0)(-s)\right\} - s^2\int_0^\infty f(t)e^{-st}dt\right]$$

$$= -\frac{df(0)}{dt} - f(0)s + s^2\mathcal{L}[f(t)]$$

$$= -f'(0) - f(0)s + s^2\mathcal{L}[f(t)]$$

$$= -f'(0) - f(0)s + s^2F(s)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Therefore,

$$\boxed{\mathcal{L}(d^2f/dt^2) = s^2F(s) - sf(0) - f'(0)}$$

2) Complete the missing steps in the solution of ODE

$$x'' + 3x' + 2x = 0, \quad x(0) = a, \quad x'(0) = b$$

⟨Answer⟩

$$\mathcal{L}[x''(t) + 3x'(t) + 2x(t)]$$
$$= \mathcal{L}[x''(t)] + 3\mathcal{L}[x'(t)] + 2\mathcal{L}[x(t)]$$
$$= [s^2 F(s) - sf(0) - f'(0)] + 3[sF(s) - f(0)] + 2F(s)$$
$$= (s^2 + 3s + 2)F(s) + (-a)s + (-3a - b)$$
$$= 0$$

$$\Rightarrow F(s) = \frac{as + (3a+b)}{s^2 + 3s + 2}$$

By partial fraction method,

$$F(s) = \frac{(a)s + (-3a-b)}{s^2 + 3s + 2} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$\Rightarrow (a)s + (3a+b) = (s+1)A + (s+2)B$$
$$= (A+B)s + (A+2B)$$

$$\Rightarrow \begin{cases} A + B = a & \cdots ① \\ A + 2B = 3a + b & \cdots ② \end{cases}$$

$$\Rightarrow \begin{cases} ① \times 2 - ② : A = -a - b \\ ② - ① : B = 2a + b \end{cases}$$

$$\Rightarrow F(s) = \frac{-a-b}{s+2} + \frac{2a+b}{s+1}$$

$\Rightarrow$ By inverse Laplace Transform,

$$f(t) = (-a-b)e^{-2t}u(t) + (2a+b)e^{-t}u(t)$$

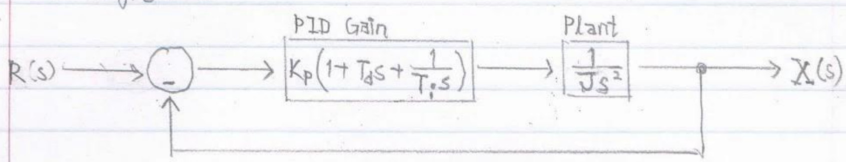
$$\Rightarrow \boxed{f(t) = (2a+b)e^{-t}u(t) + (-a-b)e^{-2t}u(t)}$$

3) Given a plant with inertial load $(Js^2)^{-1}$.

Give the diagram of a system with inertial load with PID controller, and derive the transfer function $H(s) = X(s)/R(s)$

<Answer>

The diagram is



To find the transfer function,

$$X(s) = \left(\frac{1}{Js^2}\right)\left(K_p\left(1+T_d s+\frac{1}{T_i s}\right)\right)\left(R(s)-X(s)\right)$$

$$\Rightarrow \left[1+\frac{1}{Js^2}K_p\left(1+T_d s+\frac{1}{T_i s}\right)\right]X(s) = \left[\frac{1}{Js^2}K_p\left(1+T_d s+\frac{1}{T_i s}\right)\right]R(s)$$

$$\Rightarrow \frac{X(s)}{R(s)} = \frac{\frac{1}{Js^2}K_p\left(1+T_d s+\frac{1}{T_i s}\right)}{1+\frac{1}{Js^2}K_p\left(1+T_d s+\frac{1}{T_i s}\right)}$$

$$= \frac{K_p\left(1+T_d s+\frac{1}{T_i s}\right)}{Js^2+K_p\left(1+T_d s+\frac{1}{T_i s}\right)}$$

Therefore,

$$\boxed{H(s) = \frac{X(s)}{R(s)} = \frac{K_p\left(1+T_d s+\frac{1}{T_i s}\right)}{Js^2+K_p\left(1+T_d s+\frac{1}{T_i s}\right)}}$$