INERTIA MOMENTS AND INERTIA ELLIPSOID

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Abstract *A short technical note to check the scaling coefficient used in the computation of inertia ellipsoid.*

Contents

1 Motivation

Better understand how the normalization coefficients are computed for inertia ellipse or ellipsoid. In particular, check validity of Matlab's code in regionprops3.

In this document, I start by the planar case that focus on ellipses. In a second step we can adapt to the 3D case.

2 Notation

We consider euclidean spaces of dimensions 2 and 3. Let X be the a 2D or 3D set representing the structure of interest.

We note $I_X(x, y)$ or $I_X(x, y, z)$ the indicator function of the set X, taking value 1 if the specified point is within the set X, and 0 otherwise.

3 Statistical moments (2D)

For the planar case, a good introduction is given in the book of Burger and Burge (2008). Can be found also in Russ (2006, p. 574), or in Gonzales and Woods (2018, p. 1000).

3.1 Definition

The moment m_{pq} of order (p, q) is defined by:

$$m_{pq} = \int\int I_X(x, y) x^p y^q \cdot dx \cdot dy \tag{1}$$

3.2 Special cases

The area of X corresponds to the moment of order $(0, 0)$:

$$\begin{aligned} m_{00} &= \int\int I_X(x, y) \cdot dx \cdot dy \\ &= Area(X) \end{aligned}$$

The coordinates of the centroid of X can be expressed from the first-order moments:

$$\begin{aligned} x_c &= \frac{1}{Area(X)} \int\int I_X(x, y) \cdot x \cdot dx \cdot dy \\ y_c &= \frac{1}{Area(X)} \int\int I_X(x, y) \cdot y \cdot dx \cdot dy \end{aligned}$$

That can be noted:

$$x_c = \frac{m_{10}}{m_{00}}$$
$$y_c = \frac{m_{01}}{m_{00}}$$

3.3 Centered moments

It is often more convenient to work with centered and normalized (by the area of X) moments. The centered moments are given by:

$$\mu_{pq} = \int\int I_X(x,y)(x - x_c)^p (y - yc)^q \cdot dx \cdot dy \tag{2}$$

3.4 Normalized moments

The normalized centered moments can be used to describe the “shape” of the structure independently of its size. They can be used for generating moments invariants, and perform shape recognition, see Hu (1962).

The normalization is obtained by adequate scaling of the centered moments:

$$\bar{\mu}_{pq} = \frac{\mu_{pq}}{\mu_{00}^{(p+q+2)/2}}$$

3.4.1 Why this factor (p+q+2)/2 ?

Let us consider the moment of the particle sX, corresponding to the uniform scaling by a sclar $s > 0$ of the particle X. I use the non centered moments for simplicity.

$$m_{pq}(sX) = \int\int I_{sX}(x,y)x^p y^q \cdot dx \cdot dy \tag{3}$$

The function I_{sX} may be related to the function I_X by the following:

$$I_{sX}(x,y) = I_X(\frac{x}{s}, \frac{y}{s})$$

Plugin into (3):

$$m_{pq}(sX) = \int\int I_X(x/s, y/s)x^p y^q \cdot dx \cdot dy$$

To relate to $m_{00}(X)$, we need to use a change of variable. We set:

$$x = sx',\ dx = s \cdot dx'$$
$$y = sy',\ dy = s \cdot dy'$$

We obtain:

$$
\begin{aligned}
m_{pq}(sX) &= \int\int I_X(x', y')\left(sx'\right)^p \left(sy'\right)^q \cdot s \cdot dx' \cdot s \cdot dy' \\
&= s^{(p+q+2)} \int\int I_X(x', y') x'^p y'^q \cdot dx' \cdot dy' \\
&= s^{(p+q+2)} m_{pq}(X)
\end{aligned}
$$

The rest seems to be given in a german book that I could not find on the internet...

4 Ellipse

Assuming we have the moments of a particle, we want to find the parameters of the ellipse that have the same moments.

Orientation:

$$\tan(2\theta) = \frac{2 \cdot \mu_{11}}{\mu_{20} - \mu_{02}}$$

More explicitely:

$$\theta = \frac{1}{2}\operatorname{atan2}(2 \cdot \mu_{11}, \mu_{20} - \mu_{02})$$

(using the atan2 convention for argument order).

4.1 Demonstration for a and b

The principle is to compute moments of an ellipse with known parameters and proceed by identification. For the sake of simplicity, we suppose the ellipse is centered and the oriented along the main axis.

Let us consider an ellipse with first and second radius equal to a and b respectively, with $a > b$.

The moment of order (p, q) is expressed as:

$$m_{pq} = \int\int I_X(x, y) x^p y^q \cdot dx \cdot dy \tag{4}$$

To express as function of a and b, we first apply a change of variables to compute integral in polar coordinates (ρ, θ) instead of (x, y), with $\rho \in [0; 1]$ and $\theta \in [0; 2\pi]$:

$$\Phi(x, y) = (\rho a \cos\theta, \rho b \sin\theta)$$

$$m_{pq} = \int_0^{2\pi}\int_0^1 I_X(\rho, \theta)(\rho a \cos\theta)^p (\rho b \sin\theta)^q J_\Phi \cdot d\rho \cdot d\theta$$

We need to express the Jacobian of the transformation :

$$\begin{aligned}
J_\Phi &= \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{bmatrix} \\
&= \begin{bmatrix} a\cos\theta & \rho a \sin\theta \\ b\sin\theta & -\rho b\cos\theta \end{bmatrix}
\end{aligned}$$

whose determinant is:

$$\begin{aligned}
\det J_\Phi &= \left|-\rho a b \cos^2\theta - \rho a b \sin^2\theta\right| \\
&= \rho a b
\end{aligned}$$

Then the integral can be rewritten:

$$m_{pq} = ab\int_0^{2\pi}\int_0^1 (\rho a\cos\theta)^p (\rho b\sin\theta)^q \cdot \rho \cdot d\rho \cdot d\theta$$

4.1.1 Area

Example for computing the moment m_{00}, corresponding to the area:

$$\begin{aligned}
m_{00} &= ab\int_0^{2\pi}\int_0^1 \rho \cdot d\rho \cdot d\theta \\
&= ab\frac{1}{2}\int_0^{2\pi} d\theta \\
&= \pi a b
\end{aligned}$$

4.1.2 Moments m20 and m02

$$\begin{aligned}
m_{20} &= ab\int_0^{2\pi}\int_0^1 (\rho a\cos\theta)^2 \cdot \rho \cdot d\rho \cdot d\theta \\
&= aba^2\int_0^{2\pi}\left[\int_0^1 \rho^3 d\rho\right]\cos^2\theta \cdot d\theta \\
&= aba^2\frac{1}{4}\int_0^{2\pi} \cos^2\theta \cdot d\theta
\end{aligned}$$

We use identity:

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

The integral of $\cos^2$ becomes:

$$\int_0^{2\pi} \cos^2\theta \cdot d\theta = \frac{1}{2}\int_0^{2\pi} d\theta + \frac{1}{2}\int_0^{2\pi} \cos(2\theta)\, d\theta$$
$$= \pi$$

as the second term vanishes. Then, one obtains:

$$m_{20} = \frac{\pi a^3 b}{4} = \frac{a^2}{4} m_{00}$$

Let us assume similar development gives:

$$m_{02} = \frac{b^2}{4} m_{00}$$

In practice, Matlab normalizes all moments by m_{00}.

4.1.3 Moment m11

Should be equal to 0 (to be shown...).

4.2 Inertia moments

The inertia matrix is obtained from the second order centered moments:

$$\begin{bmatrix} \mu_{20} & \mu_{11} \\ \mu_{11} & \mu_{02} \end{bmatrix}$$

For an ellipsoid with radius a and b :

$$\begin{bmatrix} \frac{a^2}{4} & 0 \\ 0 & \frac{b^2}{4} \end{bmatrix}$$

This corresponds to the eigen values of the singular value decomposition (as we have assumed the ellipse was aligned, there is no rotation matrix to compute...).

5 Statistical moments (3D)

In 3D, the moments of inertia m_{pqr} of order (p, q, r) are defined in a similar way as in 2D:

$$m_{pqr} = \int\int\int I_X(x, y, z) x^p y^q z^r \cdot dx \cdot dy \cdot dz \tag{5}$$

And the centered moments:

$$m_{pqr} = \int\int\int I_X(x, y, z)(x - x_c)^p (y - y_c)^q (z - z_c)^r \cdot dx \cdot dy \cdot dz \tag{6}$$

6 Inertia Ellipsoid

6.1 Inertia moments

The inertia matrix is obtained from the second order centered and reduced moments. As we are concerned only by second order moments, we can use more concise notations: $\mu_{xx} = \mu_{200}$, $\mu_{yy} = \mu_{020}$, $\mu_{zz} = \mu_{002}$, $\mu_{xy} = \mu_{110}$, $\mu_{xz} = \mu_{101}$, $\mu_{yz} = \mu_{011}$.

Then the matrix of inertia can be written as:

$$\begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{xy} & \mu_{yy} & \mu_{yz} \\ \mu_{xz} & \mu_{yz} & \mu_{zz} \end{bmatrix}$$

7 Equivalent ellipsoid

Again, let us suppose that the ellipsoid is aligned with main axes. The three radius of the ellipsoid are noted a, b, c, with $a > b > c$.

For integration, it is more convenient to use spherical coordinates (ρ, θ, φ), corresponding to the radius, inclination with the vertical, and azimuth. This corresponds to:

$$\begin{aligned} x &= \rho a \cos\varphi \sin\theta \\ y &= \rho b \sin\varphi \sin\theta \\ z &= \rho c \cos\theta \end{aligned}$$

with $\rho \in [0; +\infty[$, $\varphi \in [0; 2\pi]$ and $\theta \in [0; \pi]$.

The jacobian of the transformation is as follow (see example 3 in Jacobian_matrix_and_determinant on Wikipédia...):

$$J_\Phi(x,y,z) = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} = \begin{bmatrix} a\cos\varphi\sin\theta & \rho a\cos\varphi\cos\theta & -\rho a\sin\varphi\sin\theta \\ b\sin\varphi\sin\theta & \rho b\sin\varphi\cos\theta & \rho b\cos\varphi\sin\theta \\ c\cos\theta & -\rho c\sin\theta & 0 \end{bmatrix}$$

The determinant equals $\rho^2 \sin\theta$ if $a = b = c = 1$ (from Wikipédia). Introducing a, b and c gives a detrerminant equal to $abc\rho^2 \sin\theta$.

Then the moment integral is given by:

$$m_{pqr} = abc \int_0^{2\pi} \int_0^{\pi} \int_0^{1} (\rho\cos\varphi\sin\theta)^p (\rho\sin\varphi\sin\theta)^q (\rho\cos\theta)^r \cdot \rho^2 \sin\theta \cdot d\rho \cdot d\theta \cdot d\varphi \quad (7)$$

7.1 Volume moment

Just for checking that m_{000} corresponds to the volume (equal to $\frac{4\pi}{3}abc$):

$$\begin{aligned}
m_{000} &= abc\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\rho^2\sin\theta\cdot d\rho\cdot d\theta\cdot d\varphi\\
&= \frac{abc}{3}\int_0^{2\pi}\int_0^{\pi}\sin\theta\cdot d\theta\cdot d\varphi\\
&= \frac{abc}{3}\int_0^{2\pi}(-\cos\pi+\cos 0)\cdot d\varphi\\
&= \frac{2abc}{3}\int_0^{2\pi}d\varphi\\
&= \frac{4\pi abc}{3}
\end{aligned}$$

good!

7.2 Moment m200

$$\begin{aligned}
m_{200} &= abc\int_0^{2\pi}\int_0^{\pi}\int_0^{1}(a\rho\cos\varphi\sin\theta)^2\rho^2\sin\theta\cdot d\rho\cdot d\theta\cdot d\varphi\\
&= a^3bc\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\rho^4 d\rho\cdot\cos^2\varphi\sin^3\theta\cdot d\theta\cdot d\varphi\\
&= \frac{a^3bc}{5}\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\cdot d\theta\cdot\cos^2\varphi\cdot d\varphi
\end{aligned}$$

For the development of the integral of θ, we can use linearisation of $\sin^3\theta$:

$$\begin{aligned}
\sin^3\theta &= \frac{3}{4}\sin\theta-\frac{1}{4}\sin(3\theta)\\
\int_0^{\pi}\sin^3\theta d\theta &= \int_0^{\pi}\frac{3}{4}\sin\theta-\frac{1}{4}\sin(3\theta)d\theta\\
&= -\frac{3}{4}(\cos\pi-\cos 0)-\frac{-1}{12}(\cos 3\pi-\cos 0)\\
&= \frac{3}{2}-\frac{1}{6}\\
&= \frac{4}{3}
\end{aligned}$$

Coming back to moment integral:

$$\begin{aligned} m_{200} &= \frac{abc}{5}\frac{4}{3}\int_0^{2\pi}\cos^2\varphi\cdot d\varphi \\ &= \frac{abc}{5}\frac{4}{3}\int_0^{2\pi}\frac{1}{2}+\frac{1}{2}\cos(2\varphi)\,d\varphi \\ &= \frac{abc}{5}\frac{4}{3}\frac{1}{2}(2\pi-0) \\ &= \frac{4\pi}{3}\frac{a^3bc}{5} \end{aligned}$$

Then, the central moments are expressed as:

$$\begin{aligned} m_{200} &= \frac{a^2}{5}m_{000} \\ m_{020} &= \frac{b^2}{5}m_{000} \\ m_{002} &= \frac{c^2}{5}m_{000} \end{aligned}$$

As for the 2D case, these coefficients correspond to eigen values of the SVD. We have $\lambda_i = \frac{r_i^2}{5}$, then $r_i = \sqrt{5\lambda_i}$.

8 Conclusion

Final text.

References

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