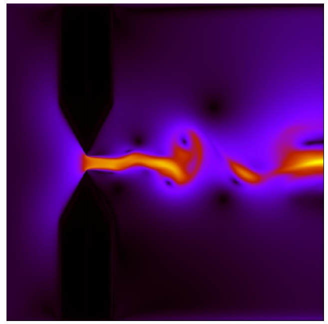
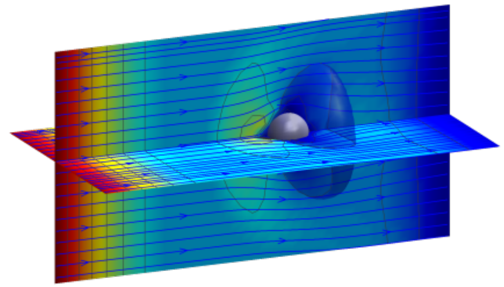
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

IfB
Institut für Baustoffe
Institute for Building Materials

An Overview on Relativistic Lattice Boltzmann Models





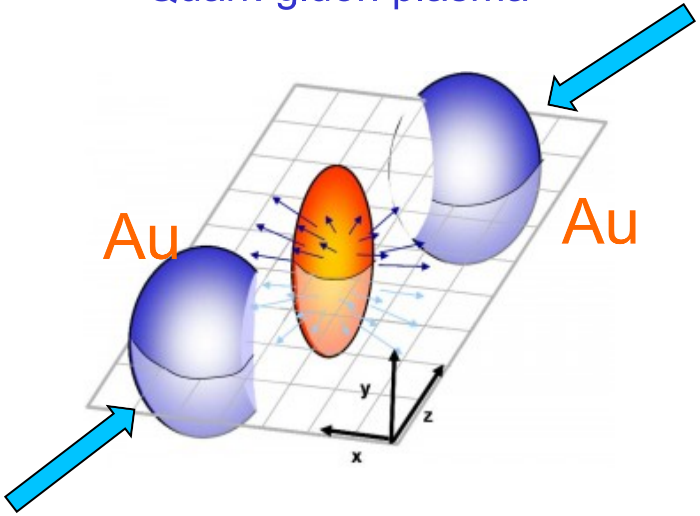
By: Miller Mendoza Jiménez

Collaborations: Hans Herrmann, Sauro Succi, Farhang Mohseni, Oliver Furtmaier, David Ottinger, Ilario Giordanelli, Ilya Karlin, Paul Romatschke, Ekin Ilseven, Alessandro Gabbana, Raffaele Tripiccione.

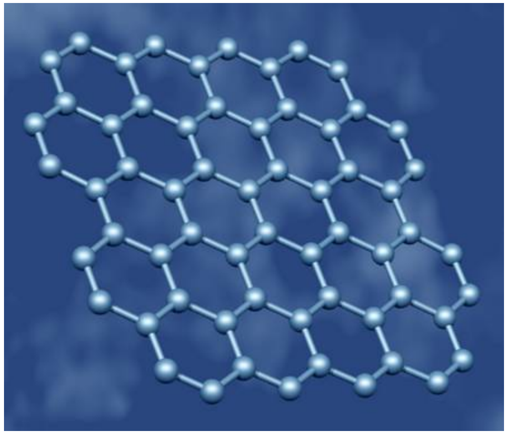
ETH Zurich, Switzerland

Relativistic Fluid Dynamics

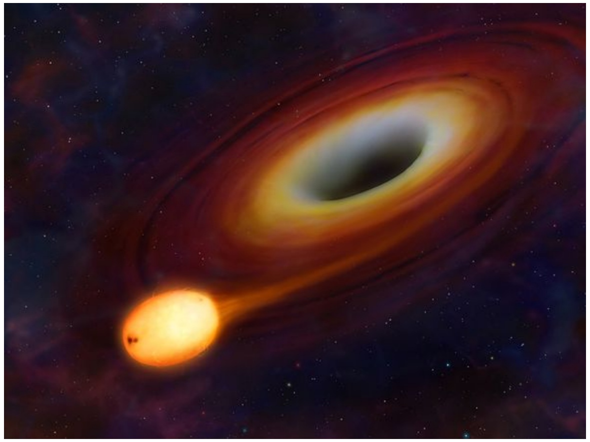
Quark-gluon plasma



Electronic flow in graphene



Gravity – fluid dynamics



Supernovae

Why?: Relativistic Navier-Stokes Equations

Number of particles

$$\frac{\partial n\gamma}{\partial t} + \nabla \cdot (n\gamma \vec{u}) = 0$$

$$\frac{\partial\left[(\varepsilon + P)\gamma^2\right]}{\partial t} + \nabla \cdot \left[(\varepsilon + P)\gamma^2 \vec{u}\right] = \frac{\partial P}{\partial t} + \nabla \cdot \Lambda$$

Energy conservation

Number density

Fluid velocity

Precise dissipation tensors is unknown

Energy-momentum conservation

$$(\varepsilon + P)\gamma^2 \frac{\partial u^i}{\partial t} + (\varepsilon + P)\gamma^2 u^j \frac{\partial u^i}{\partial x^j} = -u^i \frac{\partial P}{\partial t} - \frac{\partial P}{\partial x^i} + \frac{\partial \Pi^{ij}}{\partial x^j}$$

Pressure

Energy density

Lorentz's factor

Correction term

Why Boltzmann equation level?

•The nonlinearity effects are intrinsically included in the Boltzmann equation.

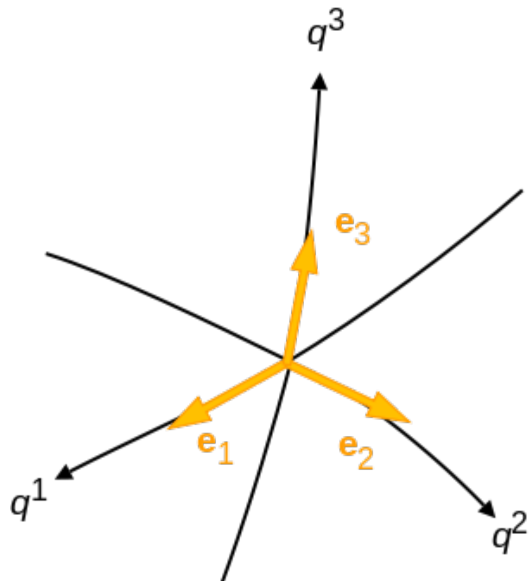
•All the information of the system is content in the particle distribution functions.

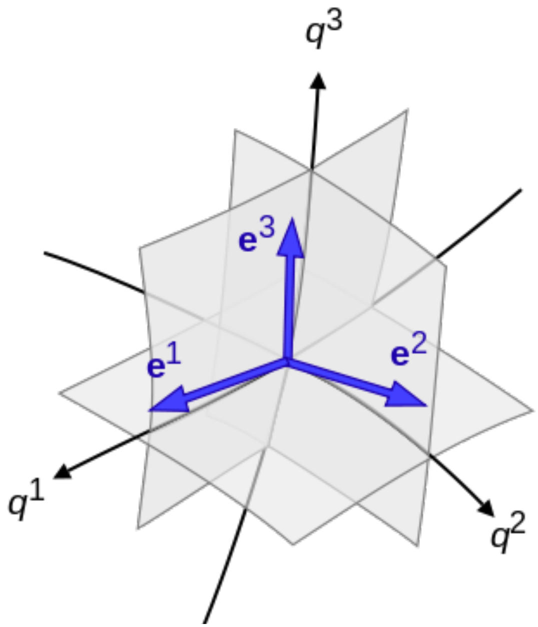
•It is a hyperbolic equation, in contrast with the Relativistic Navier-Stokes equations, which are parabolic, and therefore violate causality.

Relativistic Kinetic Theory

First of all: Working with contravariant components of vectors

$$\vec{v} = v^i \vec{e}_i = v_i \vec{e}^{\,i}$$





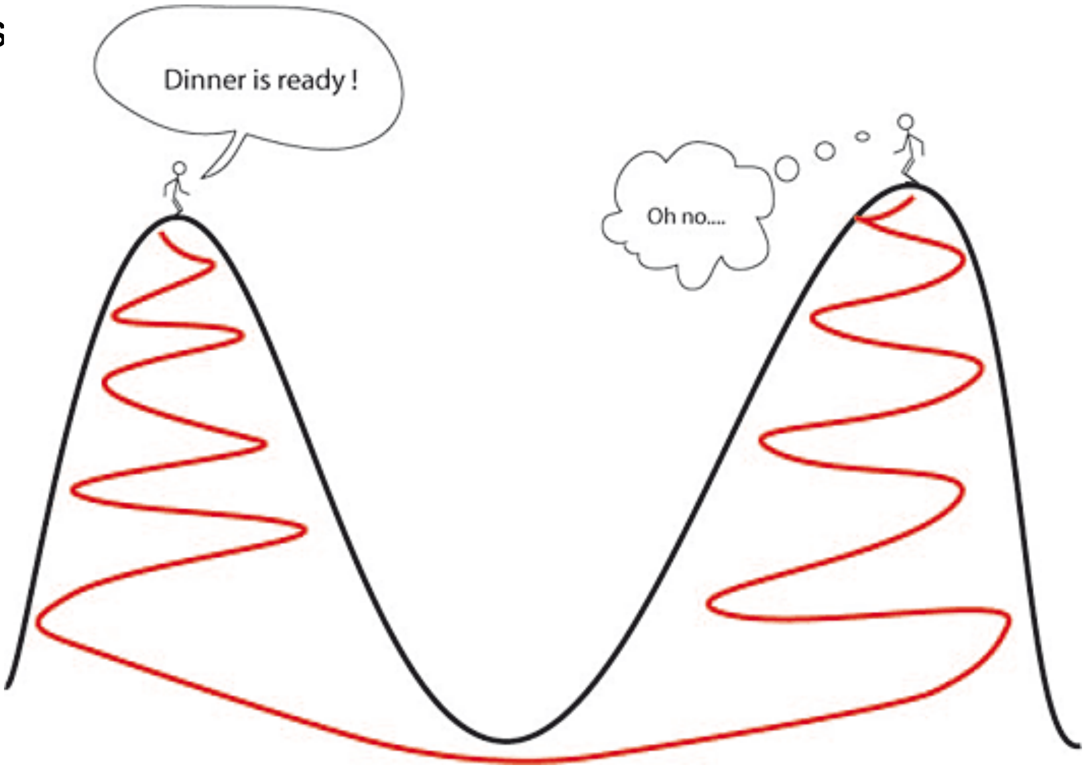
$$\vec{v}^{\,2} = v^j v^i \vec{e}_i \cdot \vec{e}_j = v^j v^i g_{ij}$$

Metric tensor

Measure of the distance

Two infinitesimally separated points

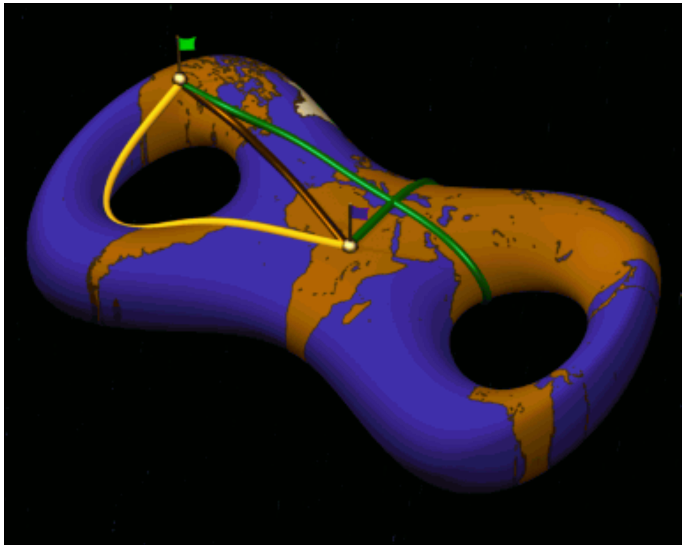
$$ds^2 = g_{ij} dx^i dx^j$$

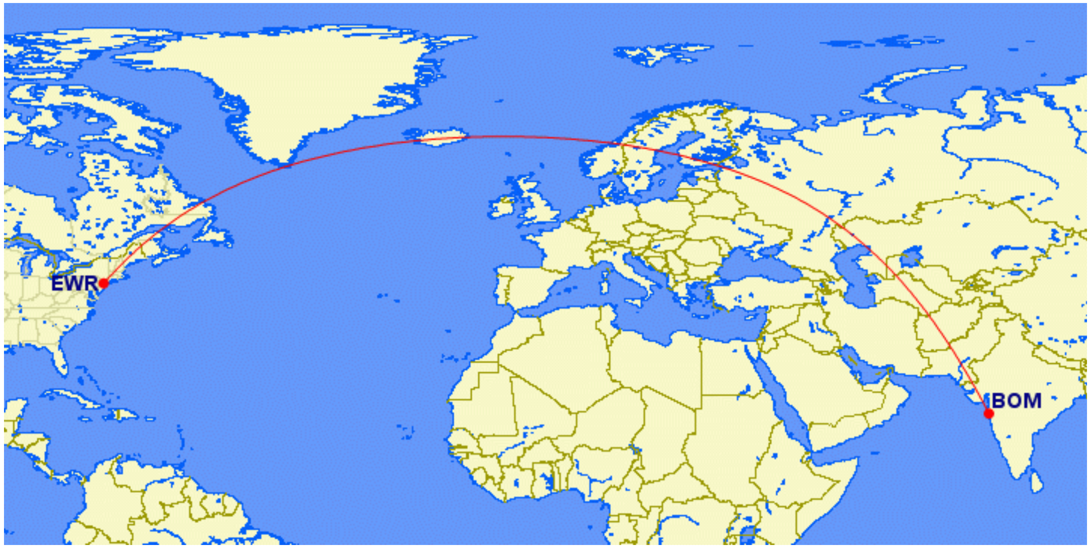


$$\Delta s = \int_{\Omega} \sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}} d\lambda$$

Two separated points, λ is an arbitrary parameter, and Ω is the trajectory joining them.

Geodesics: Shortest path





Geodesic equation:

$$\delta(\Delta s) = \delta \int_{\Omega} \sqrt{g_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda}} d\lambda = 0$$

$$\frac{d^2 x^i}{d\lambda^2} = -\Gamma^i_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda}$$

For a particle:

Geodesic equation contains
Non-inertial forces:

$$\frac{dp^i}{dt} = -\Gamma^i_{kl} p^k p^l + F^i_{ext}$$

Relativistic Boltzmann Equation

$$p^{\mu}\partial_{\mu}f + F^{\mu}\partial_{p^{\mu}}f = \Omega(f, f^{eq})$$

$$\Downarrow$$

$$p^{\mu}\partial_{\mu}f - \Gamma^{\mu}_{\alpha\nu}p^{\alpha}p^{\nu}\partial_{p^{\mu}}f = \Omega(f, f^{eq})$$

Particles 4-Momentum

$$p^{\mu} = (E/c, \vec{p})$$

Particle energy

Particle momentum

Note: Momentum is used instead of velocity because velocity is not a relativistic invariant, e.g. electromagnetism.

Relativistic Boltzmann Equation

$$p^{\mu}\partial_{\mu}f-\Gamma^{\mu}_{\alpha\nu}p^{\alpha}p^{\nu}\partial_{p^{\mu}}f=\Omega(f,f^{eq})$$

Analogy with non-relativistic gas, the relativistic Bhatnagar-Gross-Krook (BGK).

Marle Model (1965)

$$\Omega(f,f^{eq})=\frac{m}{\tau_M}(f-f^{eq})$$

Mildly relativistic fluids

$$mc^2\gg kT$$

Anderson and Witting Model (1974)

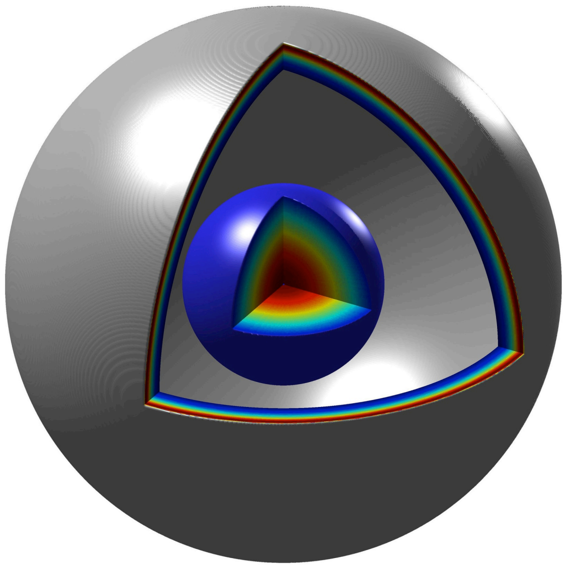
$$\Omega(f,f^{eq})=\frac{u^{\mu}p_{\mu}}{c^2\tau_{AW}}(f-f^{eq})$$

Ultrarelativistic

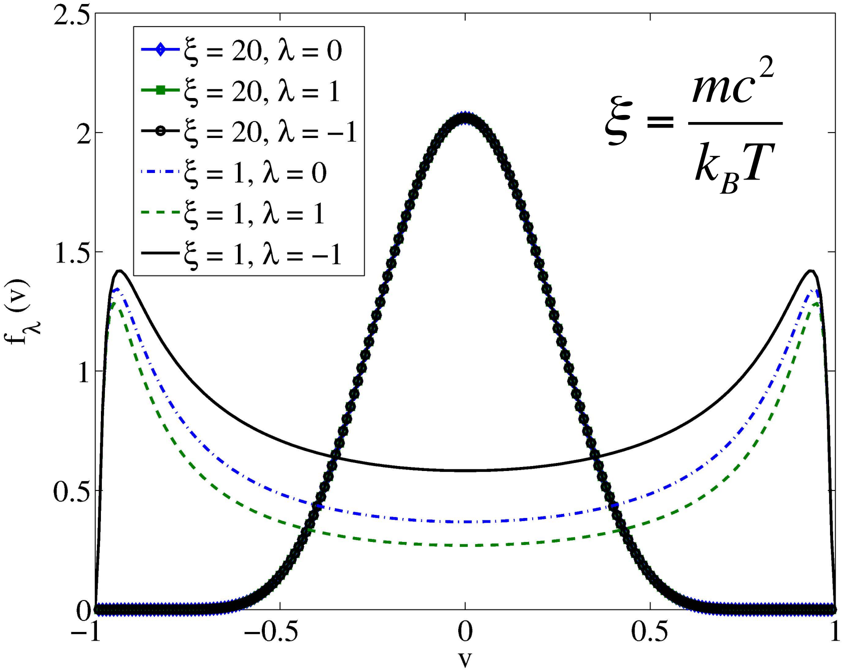
$$mc^2\ll kT$$

Equilibrium Distribution in velocity space

$$f_\lambda(\vec{x},\vec{v},t) = \frac{A\gamma^{d+2}(v)}{\exp\left[\frac{1-\vec{v}\cdot\vec{U}}{T}\gamma(v)\gamma(\vec{U}) - \frac{\mu}{T}\right] + \lambda}$$



3D



1D

M. Mendoza et al., Scientific Reports 2, 611 (2012).

Maxwell-Jüttner Distribution

In thermodynamic equilibrium the relativistic Boltzmann Equation has a solution for the distribution function: **The Maxwell-Jüttner distribution:**

$$f = f^{eq}(x^\mu, p^\mu) = Ae^{-p_\alpha U^\alpha / kT} \qquad p_0 p^0 = m^2 c^2 + \vec{p}^{\,2}$$

No Gaussian shape: → No Hermite expansion and no separable in Cartesian coordinates.

P. Romatschke et al., Physical Review C 84, 034903 (2011).

By default, not a lattice friend. However, matching of the moments of the distribution is possible!!

Relativistic Hydrodynamics

Non-relativistic fluids

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

Momentum conservation

$$\frac{\partial (\rho u_i)}{\partial t} + \partial_j (P\delta_{ij} + \rho u_i u_j + \pi_{ij}) = 0$$

Equation of state

$$P = P(\rho)$$

Relativistic fluids

Conservation of particle number

$$N^{\mu} = \gamma(n, n\vec{u})$$

$$\partial_{\mu} N^{\mu} = 0$$

Energy and momentum conservation

$$T^{\mu\nu} = P\eta^{\mu\nu} + (\varepsilon + P)u^{\mu}u^{\nu} + \pi^{\mu\nu}$$

$$\partial_{\mu} T^{\mu\nu} = 0$$

Equation of state

$$P = P(\varepsilon)$$

Answer: Matching of the moments

Writing the equilibrium distributions as functions of Lagrange multipliers:

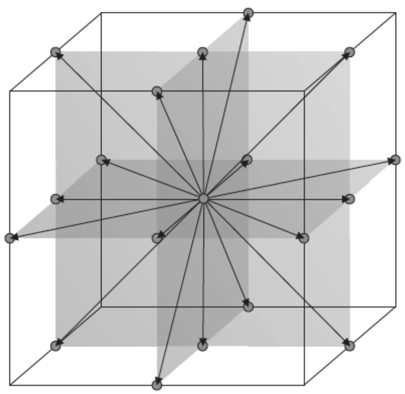
$$f_i^{eq} = w_i(A + \vec{B}\cdot\vec{v}_i + C : \vec{v}_i\vec{v}_i)$$

$$g_i^{eq} = w_i(D + \vec{E}\cdot\vec{v}_i + F : \vec{v}_i\vec{v}_i)$$

$$n = \sum_i f_i^{eq}, \quad (\varepsilon + P)\gamma^2 - P = \sum_i g_i^{eq}$$

$$(\varepsilon + P)\gamma^2\vec{u} = \sum_i g_i^{eq}\vec{v}_i, \quad P = P(\varepsilon),$$

$$P\delta_{kl} + (\varepsilon + P)\gamma^2 u_k u_l = \sum_i g_i^{eq} v_{il} v_{ik}$$



M. Mendoza et al., Physical Review Letters 105, 014502 (2010).
M. Mendoza et al., Physical Review D 82, 105008 (2010) .

Relativistic Lattice Boltzmann

Weakly relativistic: Marle Model

$$\boxed{\partial_\mu (p^\mu f) = \frac{m}{\tau_M}(f^{eq} - f)}$$

$\gamma \sim 1.4, \quad \beta \sim 0.6$ ⬇ 4-dimensional system

$$f_i(x+\delta x, t+\delta t) - f_i(x,t) = \frac{1}{\tau}(f_i^{eq}(x,t) - f_i(x,t)) \Rightarrow \partial_\mu N^\mu = 0$$

$$g_i(x+\delta x, t+\delta t) - g_i(x,t) = \frac{1}{\tau}(g_i^{eq}(x,t) - g_i(x,t)) \Rightarrow \partial_\mu T^{\mu\nu} = 0$$

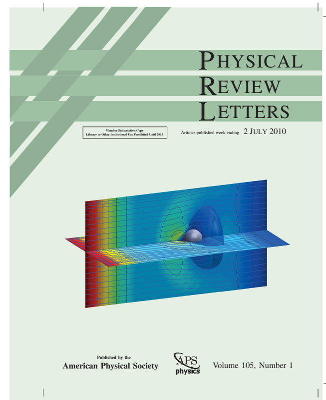
M. Mendoza et al., Physical Review Letters 105, 014502 (2010).

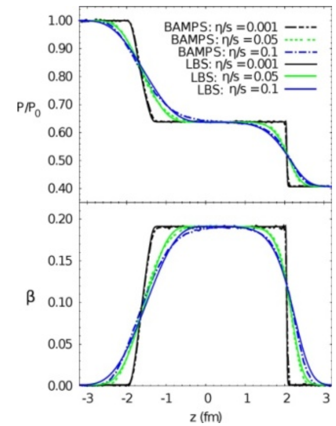
M. Mendoza et al., Physical Review D 82, 105008 (2010) .

Relativistic Lattice Boltzmann Method

$$f_i(x+\delta x, t+\delta t) - f_i(x,t) = (f_i^{eq}(x,t) - f_i(x,t))/\tau$$

$$g_i(x+\delta x, t+\delta t) - g_i(x,t) = (g_i^{eq}(x,t) - g_i(x,t))/\tau$$





Macroscopic variables:

$$n = \sum_i f_i$$

$$(\varepsilon + P)\gamma^2 - P = \sum_i g_i$$

$$(\varepsilon + P)\gamma^2 \vec{u} = \sum_i g_i \vec{v}_i$$

$$P = P(\varepsilon)$$

Equilibrium distribution function:

$$f_i^{eq}(x,t) = n\gamma\omega_i\left[1 + \frac{3(\vec{v}_i \cdot \vec{u})}{c_s^2}\right]$$

$$g_i^{eq}(x,t) = (\varepsilon + P)\gamma^2\omega_i\left[\frac{3P}{(\varepsilon+P)\gamma^2 c_l^2} + \frac{3(\vec{v}_i \cdot \vec{u})}{c_l^2} + \frac{9(\vec{v}_i \cdot \vec{u})^2}{2c_l^4} - \frac{3\vec{u}^2}{2c_l^2}\right]$$

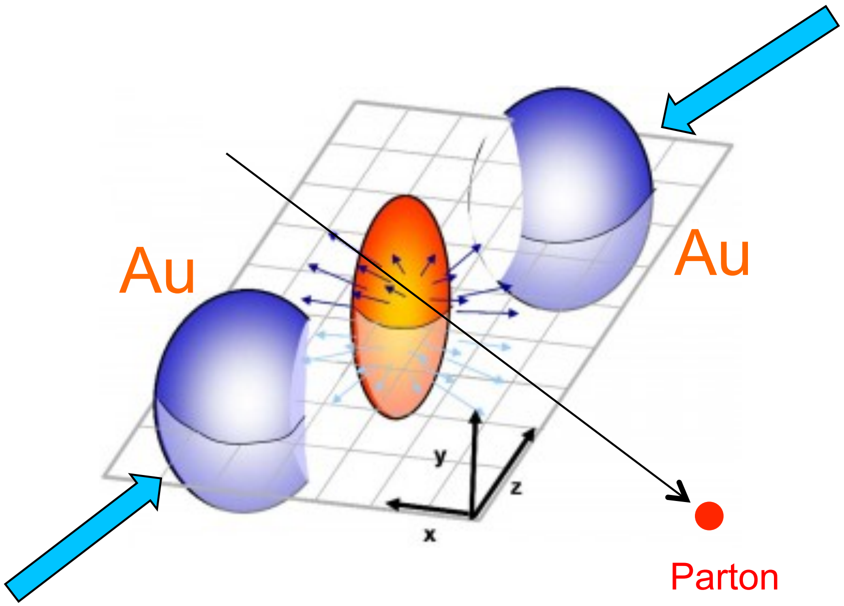
$$g_0^{eq}(x,t) = (\varepsilon + P)\gamma^2\omega_i\left[3 - \frac{3P(2+c_l^2)}{(\varepsilon+P)\gamma^2 c_l^2} - \frac{3\vec{u}^2}{2c_l^2}\right]$$

M. Mendoza et al., Physical Review Letters 105, 014502 (2010).

M. Mendoza et al., Physical Review D 82, 105008 (2010) .

Validation with Quark-Gluon Plasma

- Existence of Shock waves open the idea of considering the quark-gluon plasma as a fluid.



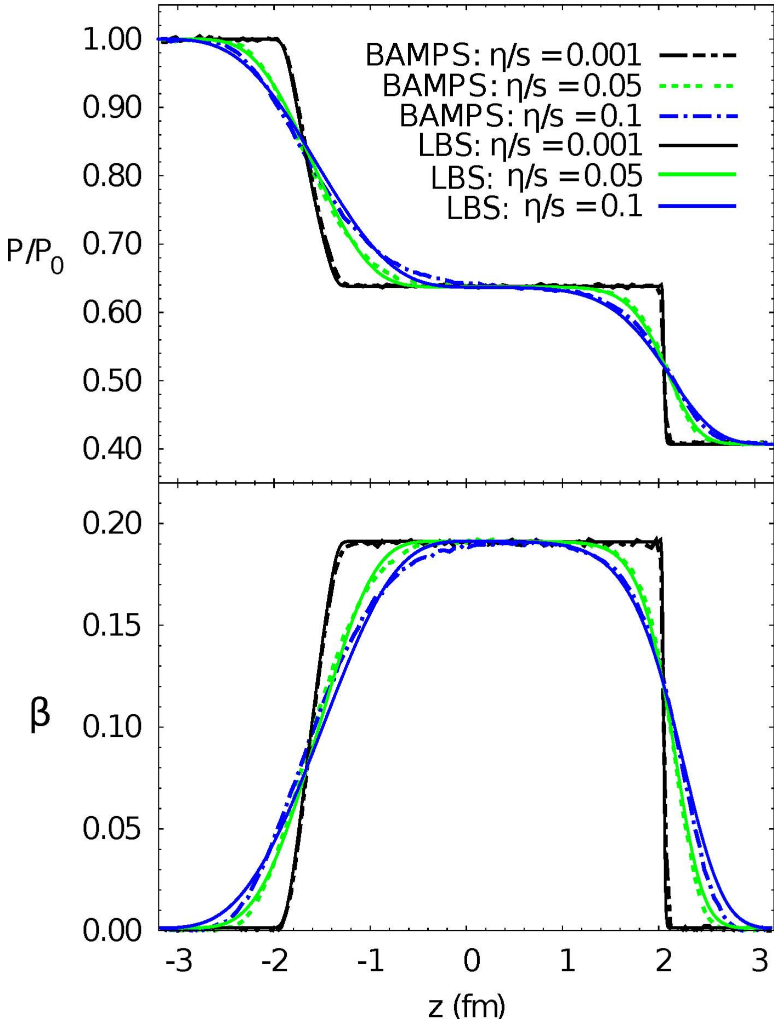
I. Bouras, E. Molnar, H. Niemi, Z. Xu, A. El, O. Fochler, C. Greiner, and D. H. Rischke, Phys. Rev. Lett. 103, 032301 (2009).

P. Romatschke, Int. J. Mod. Phys. E 19, 1 (2009).

H. Stöcker, Nucl. Phys. A750, 121 (2005); J. Ruppert and B. Müller, Phys. Lett. B 625, 123 (2005); J. Casalderrey-Solana, E.V. Shuryak, and D. Teaney, J. Phys. Conf. Ser. 27, 22 (2005); V. Koch, A. Majumder, and X. N. Wang, Phys. Rev. Lett. 96, 172302 (2006).

$$\gamma \sim 1.03 - 1.4$$

Validation with Quark-Gluon Plasma



$$\beta \sim 0.2$$

$$\eta = \frac{1}{3}(\varepsilon + P)\gamma^2\left(\tau - \frac{1}{2}\right)$$

$$s = 4n - n\mathrm{Ln}(\lambda)$$

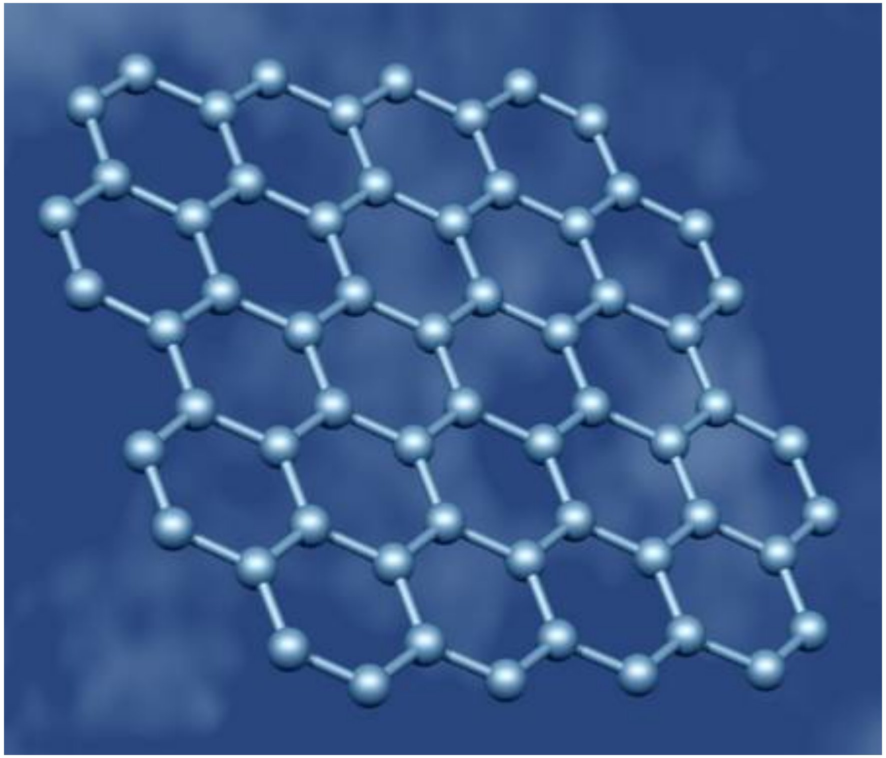
$$\lambda = \frac{n}{n^{eq}}, \quad n^{eq} = \frac{d_G T^3}{\pi^2}, \quad d_G = 16$$

Ratios of time consumption:
1:20:80000 (single CPU)
For RLB : vSHASTA : BAMPS

BAMPS: Boltzmann Approach of Multi-Parton Scattering.

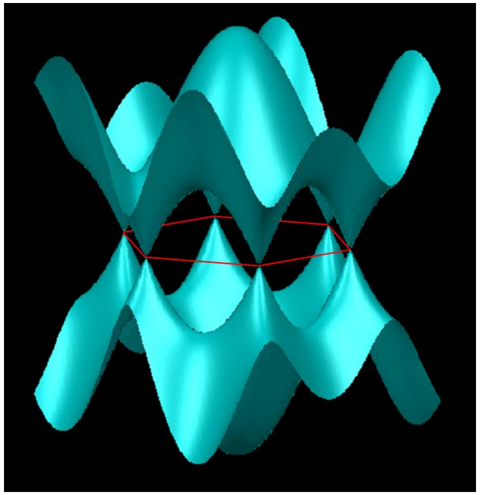
M. Mendoza et al., Physical Review Letters 105, 014502 (2010).
M. Mendoza et al., Physical Review D 82, 105008 (2010) .

An application on Graphene



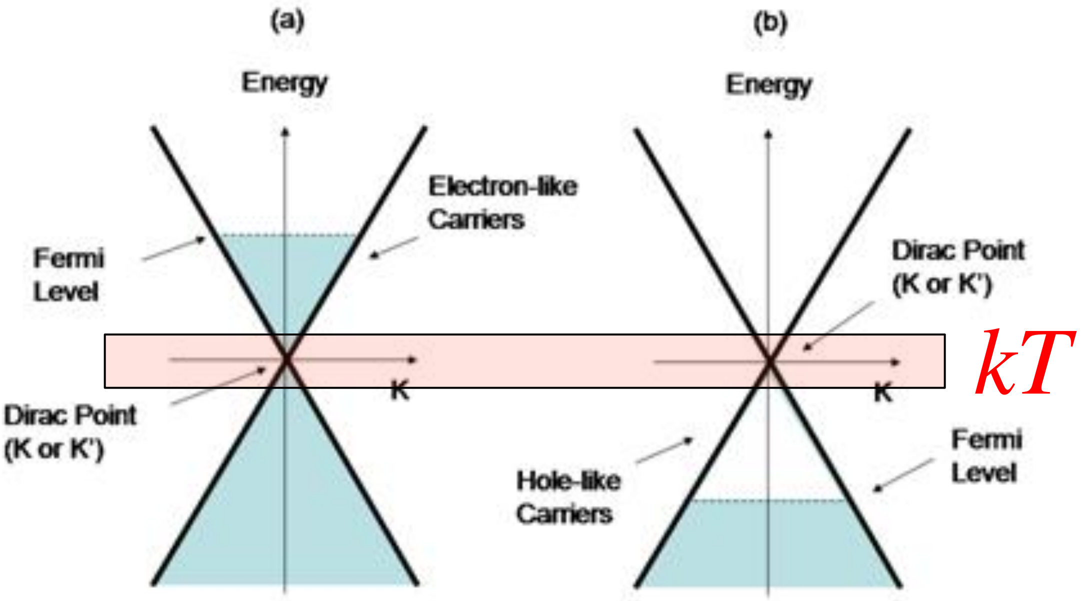
Novoselov, K. S. et al. Proc. Natl Acad. Sci. 102, 10451–10453 (2005).
Novoselov, K. S. et al. Science 306, 666–669 (2004).

Electronic Flow in Graphene

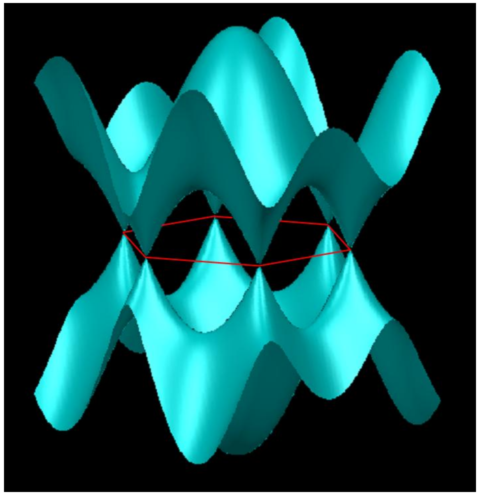


M. Müller et al. Phys. Rev. Lett. 103, 025301 (2009)

$$H = \sum_l v \hat{\mathbf{p}}_l \cdot \sigma + \frac{1}{2} \sum_{l,l'} \frac{e^2}{\epsilon |\mathbf{r}_l - \mathbf{r}_{l'}|},$$



Electronic Flow in Graphene



M. Müller et al. Phys. Rev. Lett. 103, 025301 (2009)

$$H = \sum_{l} v \hat{\mathbf{p}}_l \cdot \sigma + \frac{1}{2} \sum_{l,l'} \frac{e^2}{\epsilon |\mathbf{r}_l - \mathbf{r}_{l'}|},$$

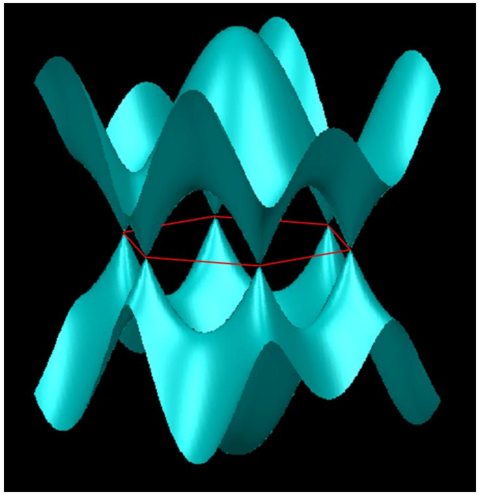
From Dirac equation to Quantum Boltzmann Equation

$$\frac{\partial f_s}{\partial t} + \vec{v}_s \cdot \nabla f_s + e\vec{E} \cdot \nabla_{\vec{k}} f_s = -\Omega[f_s],$$

In equilibrium:

$$f_s\left(t, \vec{x}, \vec{k}\right) = \frac{1}{e^{\left(sc|\vec{k}| - \mu\right)/k_B T} + 1},$$

Electronic Flow in Graphene



$$\frac{\partial f_s}{\partial t} + \vec{v}_s \cdot \nabla f_s + e\vec{E} \cdot \nabla_{\vec{k}} f_s = -\Omega[f_s],$$

In equilibrium:

$$f_s\left(t, \vec{x}, \vec{k}\right) = \frac{1}{e^{\left(sc\left|\vec{k}\right| - \mu\right)/k_B T} + 1},$$

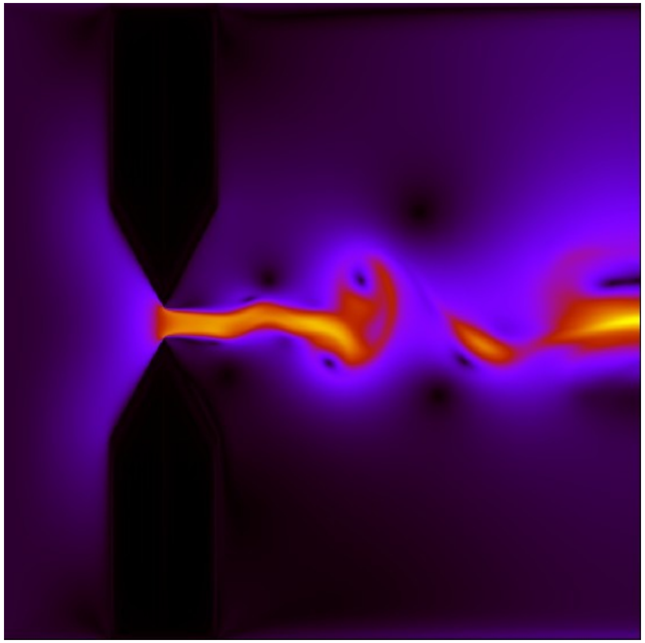
The carrier density comes from thermal excitations:

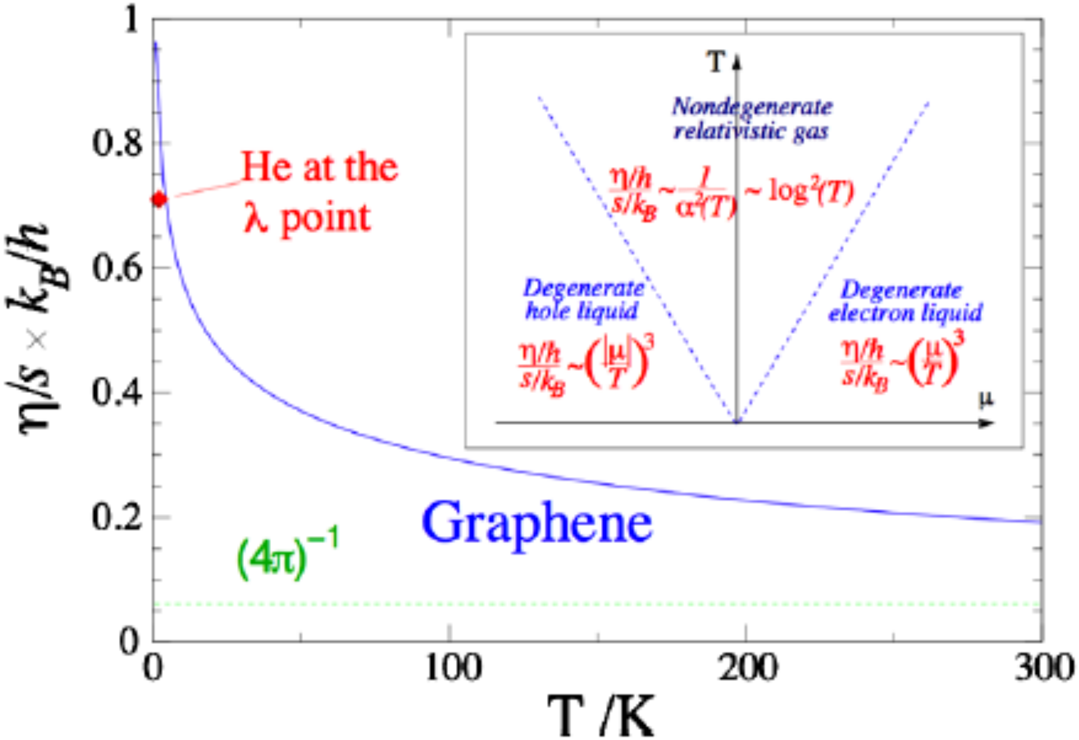
$$\rho = \rho_{th} = e\left(\frac{k_B T}{\hbar c}\right)^2.$$

M. Müller et al. Phys. Rev. Lett. 103, 025301 (2009)

Turbulence in Graphene. Can it be measured?

Quasiperfect fluid



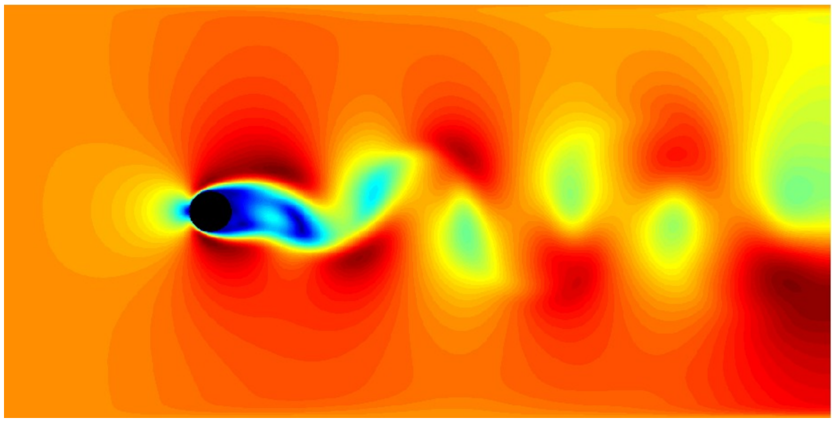


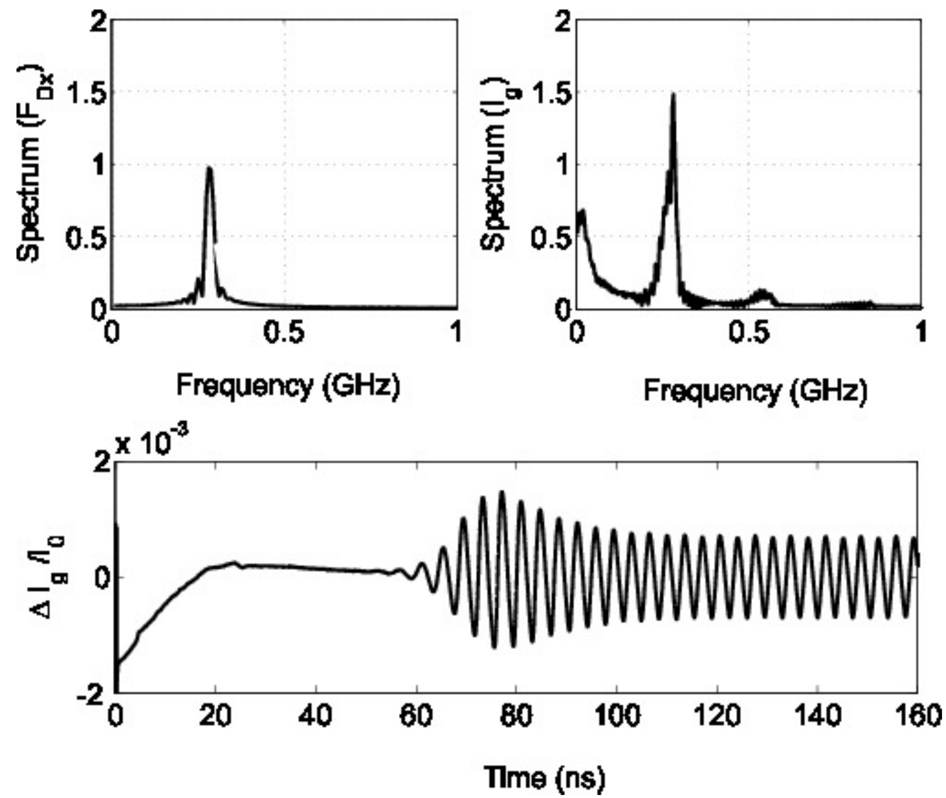
M. Müller et al. Phys. Rev. Lett. 103, 025301 (2009)

It can be important to:

1. Study of turbulence itself.
2. Control electrical current fluctuations in electronic devices.

Results with RLB in graphene





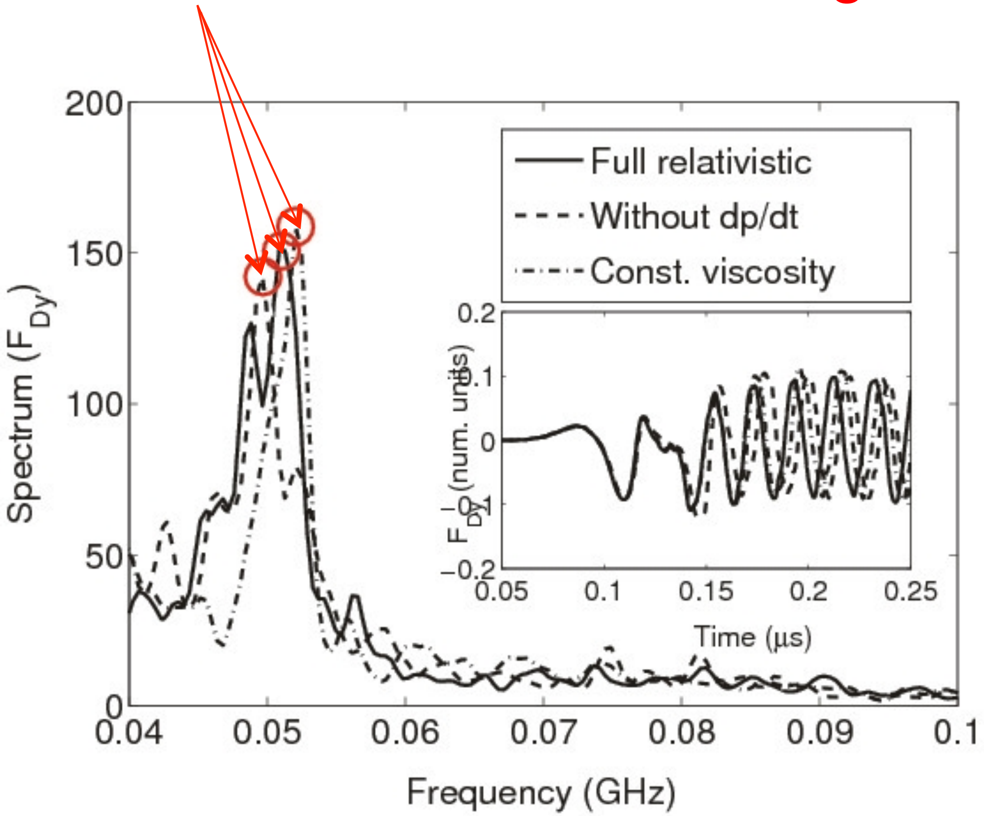
Re = 100
L = 5 μm
$u_{typ} = 10^5$ m/s

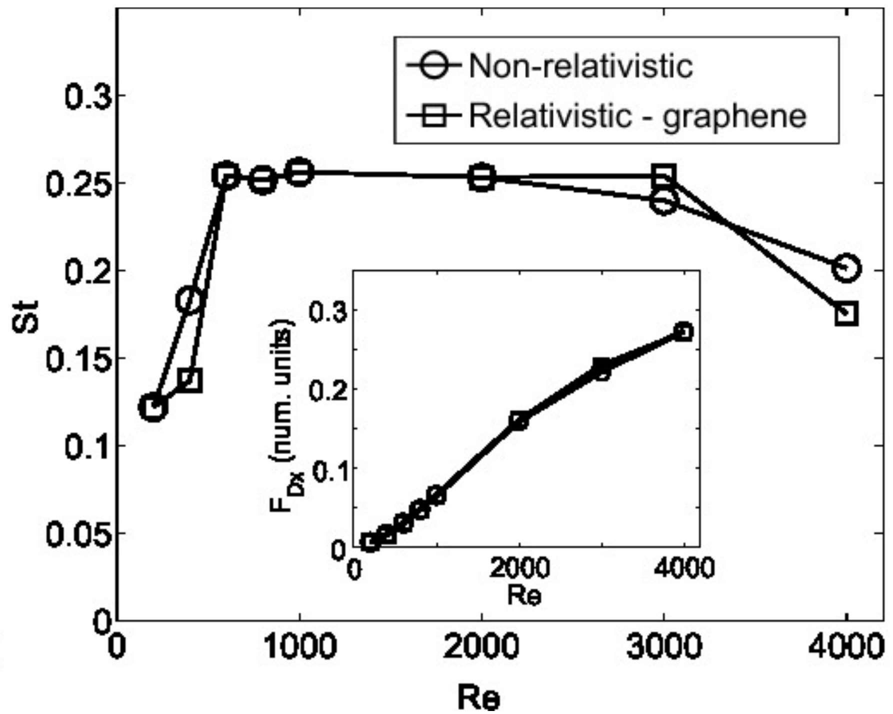
$$\mathrm{Re} = \frac{U_{typ} L}{\nu}$$

M. Mendoza et al., Physical Review Letters 105, 014502 (2010).
M. Mendoza et al., Physical Review Letters 106, 156601 (2011).

Preturbulence in Graphene. Relativistic Effects

Shift in the vortex shedding frequencies





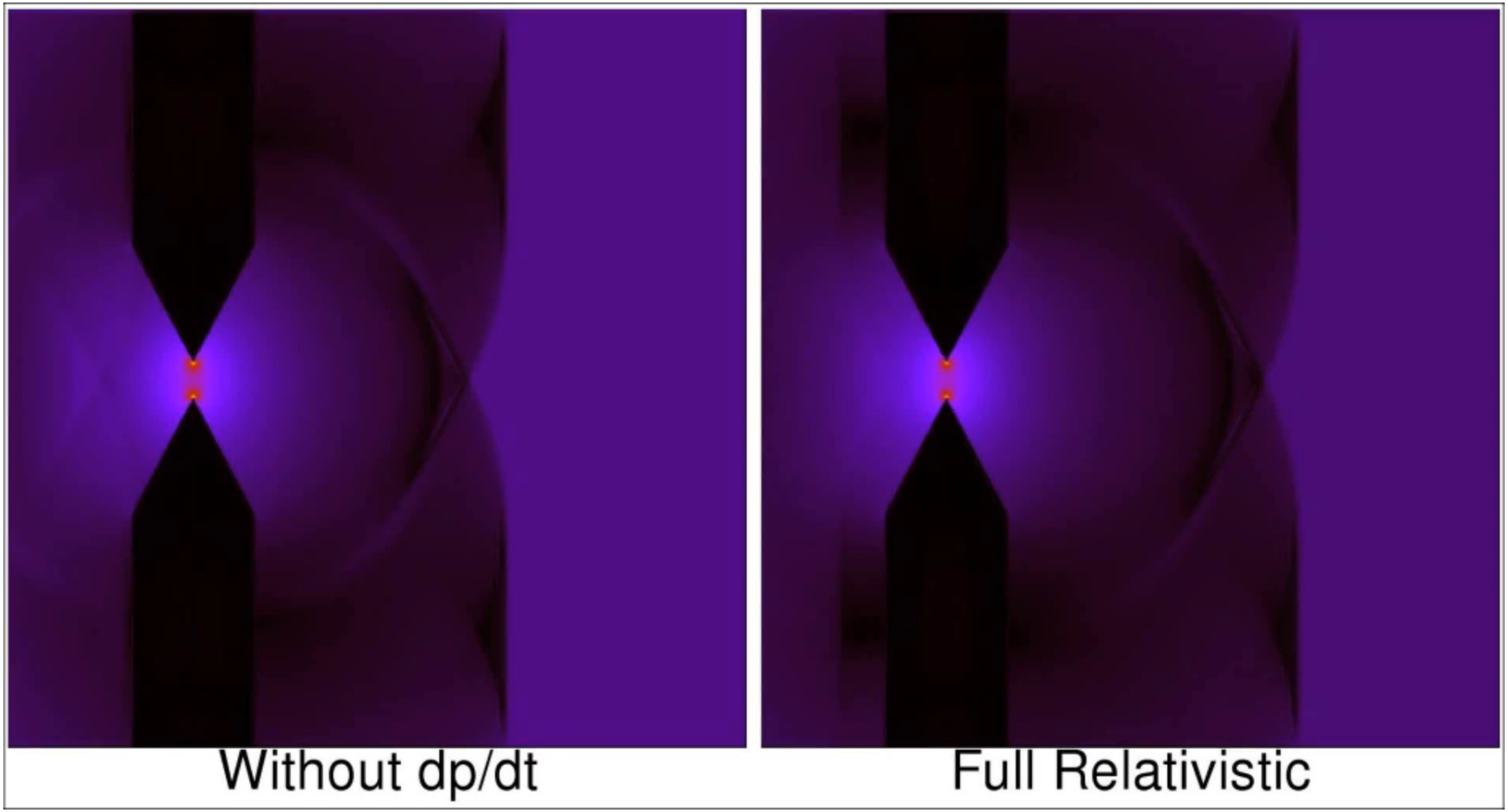
Re = 3000

St: Strouhal number (Adimensional vortex shedding frequency)

M. Mendoza et al., Physical Review Letters 105, 014502 (2010).

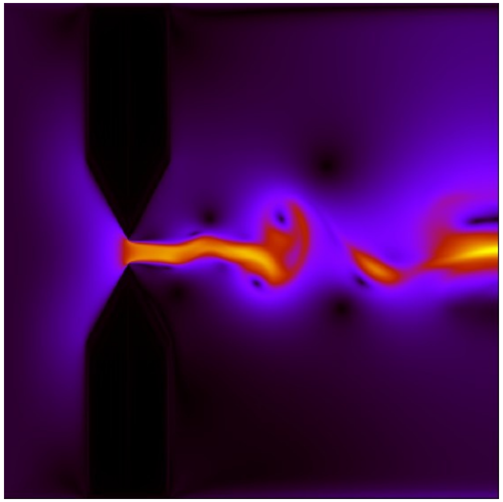
M. Mendoza et al., Physical Review Letters 106, 156601 (2011).

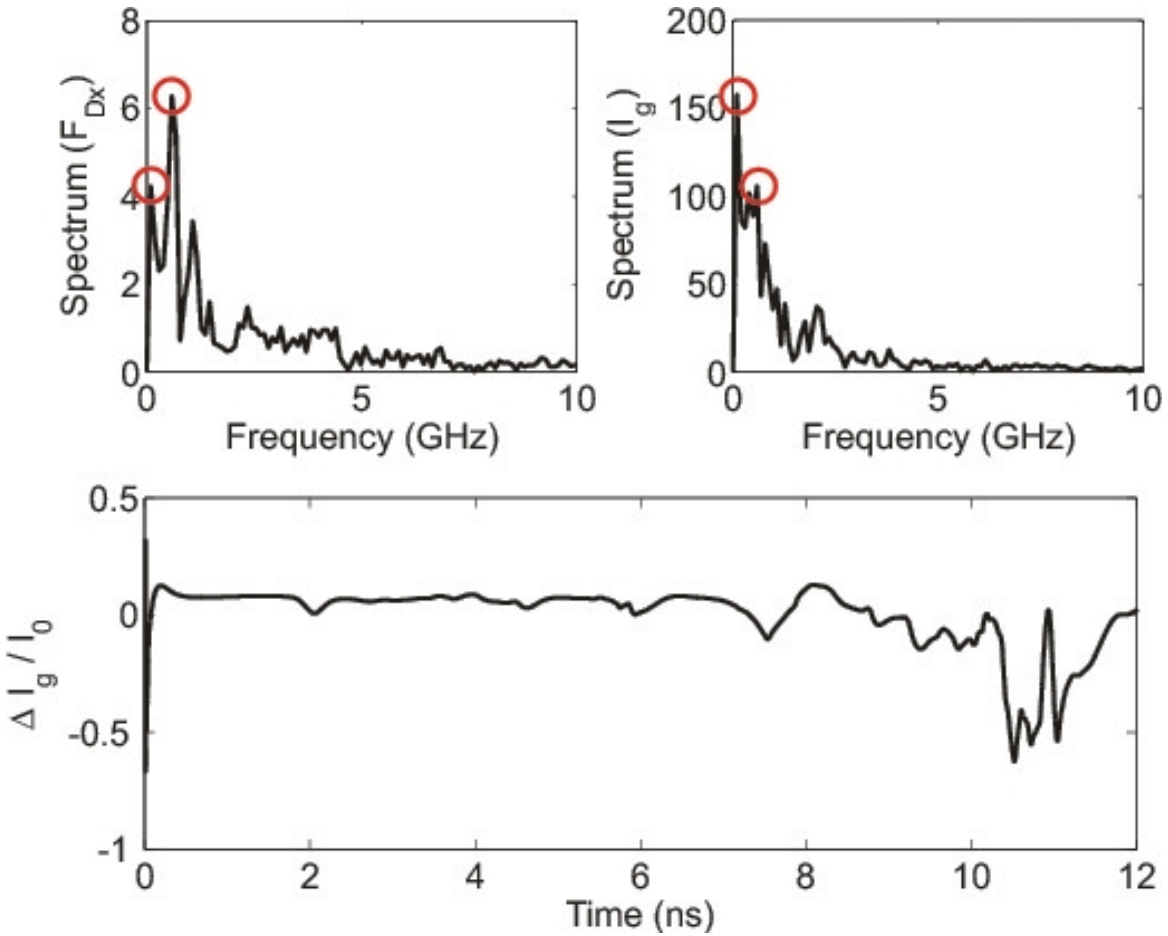
Changing the geometry allows preturbulence at Re = 25



M. Mendoza et al., Physical Review Letters 105, 014502 (2010).
M. Mendoza et al., Physical Review Letters 106, 156601 (2011).

The frequencies corresponding to the vortices could be detected experimentally.





Re = 25
L = 1.25 μm
$u_{typ} = 10^5$ m/s

M. Mendoza et al., Physical Review Letters 105, 014502 (2010).
M. Mendoza et al., Physical Review Letters 106, 156601 (2011).

Relativistic Lattice Boltzmann

$$f_i(x+\delta x, t+\delta t) - f_i(x,t) = \frac{1}{\tau}\left(f_i^{eq}(x,t) - f_i(x,t)\right) \Rightarrow \partial_\mu N^\mu = 0$$

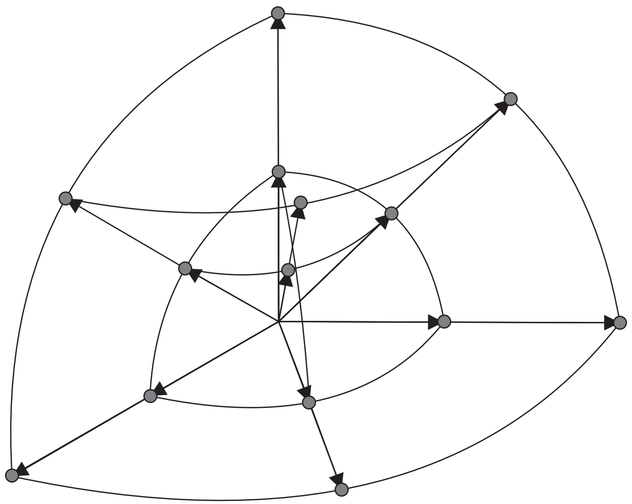
$$g_i(x+\delta x, t+\delta t) - g_i(x,t) = \frac{1}{\tau}\left(g_i^{eq}(x,t) - g_i(x,t)\right) \Rightarrow \partial_\mu T^{\mu\nu} = 0$$

2 distributions, is it possible to use 1?

Fully relativistic lattice Boltzmann models

Model 1

Fully relativistic lattice Boltzmann models



$$f = f^{eq}(x^{\mu}, p^{\mu}) = Ae^{-p_{\alpha}U^{\alpha}/kT}$$

$$\overline{p} = p^0 / T_0 = \sqrt{m^2c^2 + \vec{p}^2} / T_0 \approx |\vec{p}| / T_0$$

Ultrarelativistic limit, m = 0

$$f(t,x,p) = e^{-\bar{p}} \sum_{k=0}^{N_p-1} \sum_{n=0}^{N_v-1} P^{(n)}_{i_1 \ldots i_n}(\mathbf{v}) L^{(\alpha)}_k(\bar{p}) a^{(nk)}_{i_1 \ldots i_n}(t,x)$$

Use of vectorial and Laguerre orthogonal polynomials

P. Romatschke et al., Physical Review C 84, 034903 (2011).

Landau-Lifshitz decomposition

Anderson-Witting collision operator:

$$\Omega(f, f^{eq}) = -\frac{p_\alpha U^\alpha}{c^2 \tau_{AW}}(f - f^{eq})$$

$$N^\alpha = \int f p^\alpha \frac{d^3 p}{p^0}$$

$$T^{\alpha\beta} = \int f p^\alpha p^\beta \frac{d^3 p}{p^0}$$

$$\boxed{\begin{aligned} \partial_\mu N^\mu &= 0 \\ \partial_\mu T^{\mu\nu} &= 0 \end{aligned}}$$

Solve the eigenvalue problem:

$$U_\alpha T^{\alpha\beta} = \varepsilon U^\beta$$

And then the density:

$$N^\alpha U_\alpha = n$$

P. Romatschke et al., Physical Review C 84, 034903 (2011).
M. Mendoza et al., Phys. Rev. D 87, 065027 (2013)

Relativistic Boltzmann Equation

$$p^{\mu}\partial_{\mu}f = -\frac{p_{\alpha}U^{\alpha}}{c^{2}\tau_{AW}}(f - f^{eq})$$

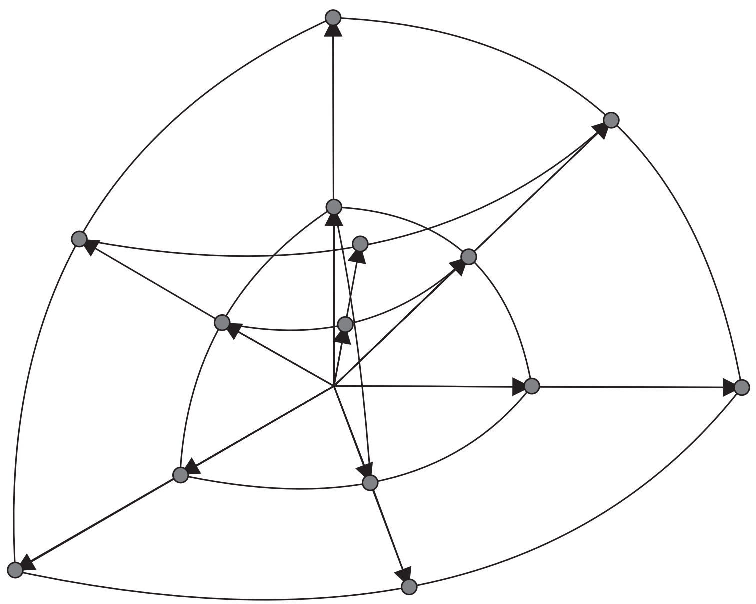
$$\partial_{t}f + \frac{\vec{p}}{p^{0}}\cdot\nabla f = -\frac{p_{\alpha}U^{\alpha}}{p^{0}c^{2}\tau_{AW}}(f - f^{eq})$$

For the case of ultra-relativistic particles:

$$\vec{v} = \frac{\vec{p}}{p^{0}} = c\vec{1} \quad , \quad m = 0$$

P. Romatschke et al., Physical Review C 84, 034903 (2011).
M. Mendoza et al., Phys. Rev. D 87, 065027 (2013)

Fully relativistic lattice Boltzmann models. Model 1



Advantages:

- It is the first fully relativistic LB model based on Gauss-quadrature.

- One single distribution function carries all the information of the fluid.

Disadvantage:

- Velocity vectors with spherical symmetry do not fill the space, and therefore, we need interpolation to calculate values at the nodes.

P. Romatschke et al., Physical Review C 84, 034903 (2011).

Model 2

Relativistic Distribution: Polynomials

In thermodynamic equilibrium the relativistic Boltzmann Equation has a solution for the distribution function:

$$f = f^{eq}(x^\mu, p^\mu) = (e^{p_\alpha U^\alpha / kT} + \lambda)^{-1}$$

$$\mathcal{L}_j = \{1, p^0, p^x, p^y, p^z\},$$

$$\langle \mathcal{J}_r, \mathcal{J}_l \rangle = \int w(p^0) \mathcal{J}_r \mathcal{J}_l \frac{d^3 p}{p^0},$$

$$\mathcal{J}_k = \bigcup_{i,j,l=0}^{4} \{\mathcal{L}_i \mathcal{L}_j \mathcal{L}_l\},$$

$$J'_k = \mathcal{J}_k - \sum_{l=0}^{k-1} \frac{\langle \mathcal{J}_l, \mathcal{J}_k \rangle}{\langle \mathcal{J}_l, \mathcal{J}_l \rangle} \mathcal{J}_l,$$

$$J_k = \frac{J'_k}{\sqrt{\langle J'_k, J'_k \rangle}}.$$

Polynomials

M. Mendoza et al., Phys. Rev. D 87, 065027 (2013)

Relativistic Distribution: Polynomials

In thermodynamic equilibrium the relativistic Boltzmann Equation has a solution for the distribution function:

$$f = f^{eq}(x^{\mu}, p^{\mu}) = (e^{p_{\alpha}U^{\alpha}/kT} + \lambda)^{-1}$$

$$f^{\mathrm{eq}} \simeq \sum_{k=0}^{29} w(p^0) a_k(T, U^{\mu}) J_k(p^{\mu}), \qquad a_k = \int f^{\mathrm{eq}} J_k(p^{\mu}) \frac{d^3 p}{p^0}.$$

$$p^{\mu} \partial_{\mu} f - \Gamma^{\mu}_{\alpha\nu} p^{\alpha} p^{\nu} \partial_{\mu} f = -\frac{p_{\alpha} U^{\alpha}}{c^2 \tau_{AW}} (f - f^{eq})$$

Anderson-Witting Model

M. Mendoza et al., Phys. Rev. D 87, 065027 (2013)

Relativistic Distribution: Polynomials

$$p^{\mu}\partial_{\mu}f-\Gamma^{\mu}_{\alpha\nu}p^{\alpha}p^{\nu}\partial_{\mu}f=-\frac{p_{\alpha}U^{\alpha}}{c^{2}\tau_{AW}}(f-f^{eq})$$

$$\vec{v}$$

$$\partial_{t'}f+R\frac{\vec{p}}{p^{0}}\cdot\nabla f-R\frac{\Gamma^{\mu}_{\alpha\nu}p^{\alpha}p^{\nu}}{p^{0}}\partial_{p^{\mu}}f=-\frac{Rp_{\alpha}U^{\alpha}}{p^{0}c^{2}\tau_{AW}}(f-f^{eq})$$

For the case of ultra-relativistic particles:

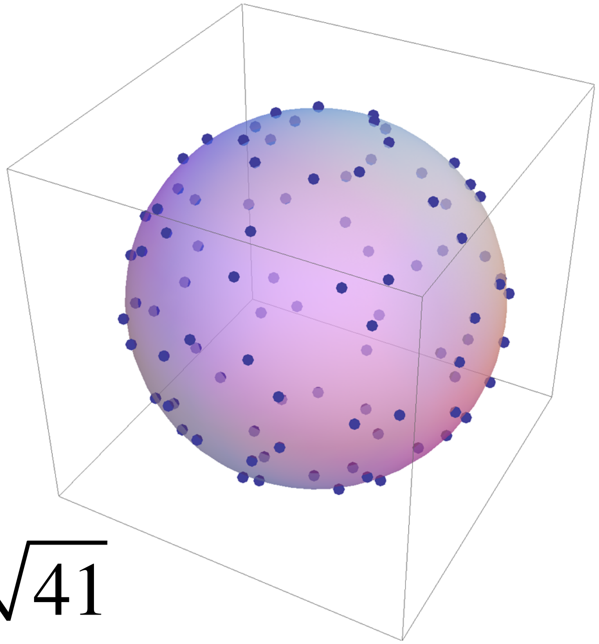
$$\vec{v}=R\frac{\vec{p}}{p^{0}}=Rc\vec{1}\quad,\quad m=0,\quad t\rightarrow t'/R$$

M. Mendoza et al., Phys. Rev. D 87, 065027 (2013)

Relativistic Distribution: Polynomials 3D

384 momentum vectors, 96 velocity vectors

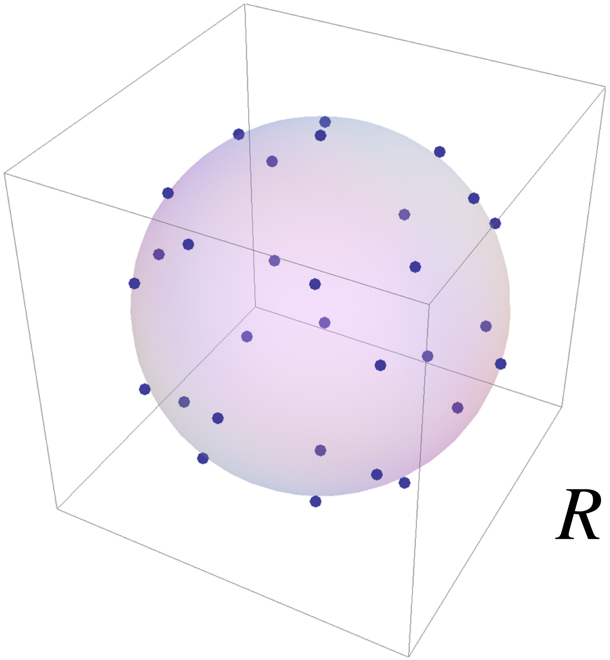
For 3rd order accuracy



$R = \sqrt{41}$

90 momentum vectors, 30 velocity vectors

For 2nd order accuracy

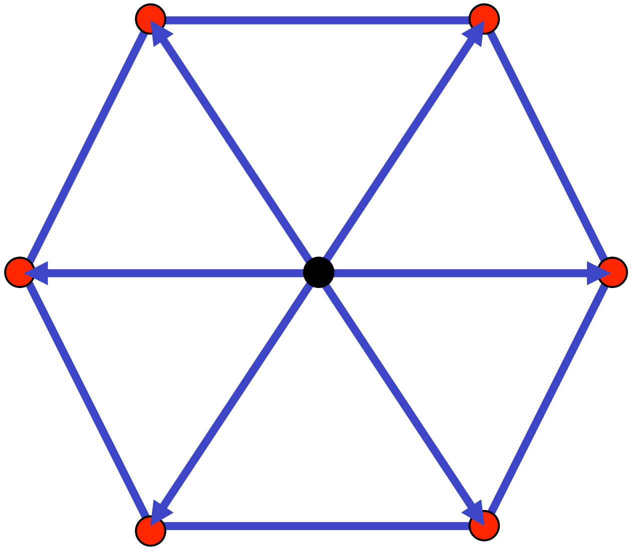


$R = 3$

$$\int w(p^0) J_l(p^\mu) J_k(p^\mu) \frac{d^3 p}{p^0} = \sum_i w_i J_l(p_i^\mu) J_k(p_i^\mu) = \delta_{lk},$$

M. Mendoza et al., Phys. Rev. D 87, 065027 (2013)

Relativistic Distribution: Polynomials 2D



$$\vec{v}_i = \frac{\vec{p}_i}{p^0}$$

18 momentum vectors,

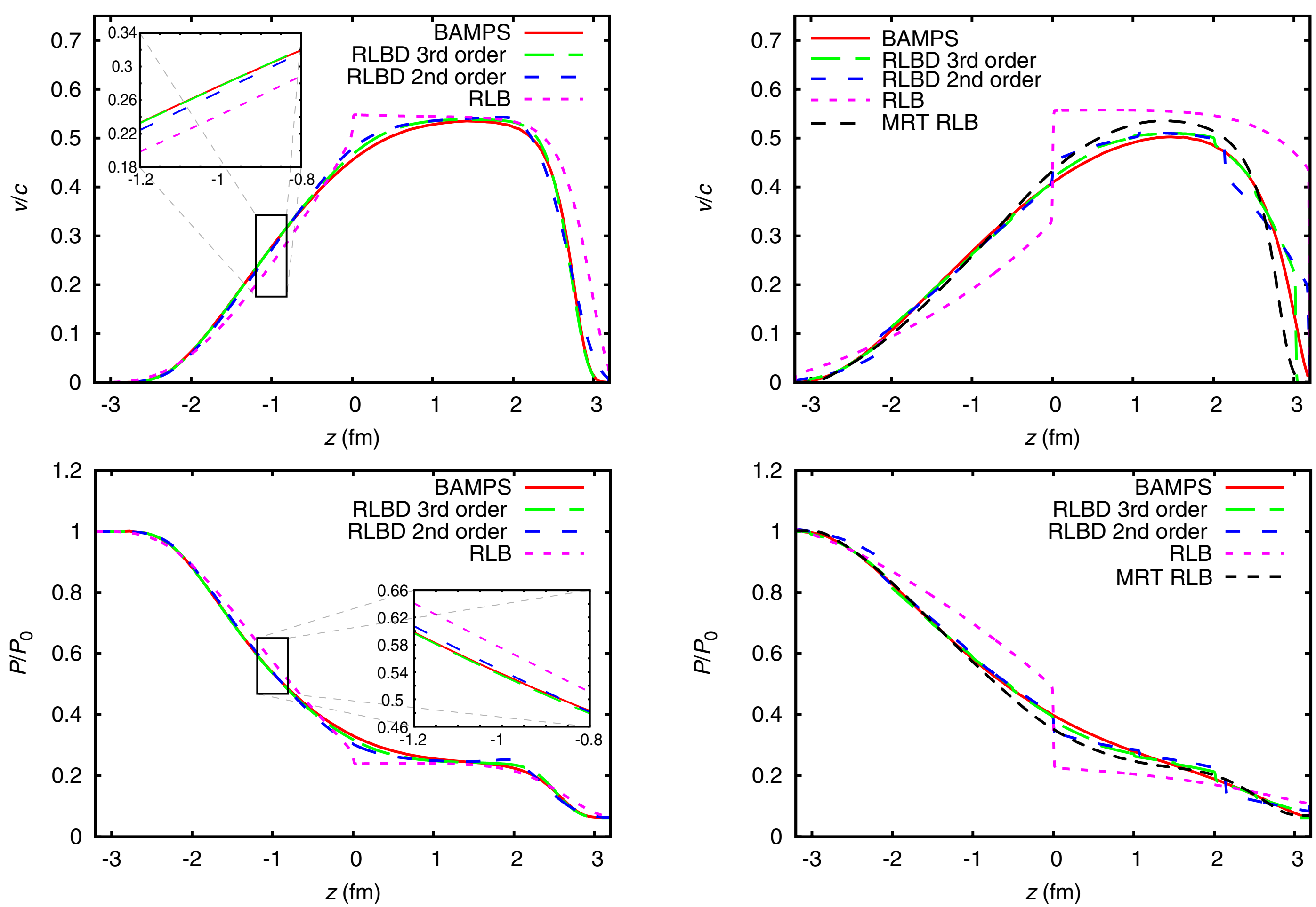
6 velocity vectors

For 2nd order accuracy

$$\int w(p^0) J_l(p^\mu) J_k(p^\mu) \frac{d^2 p}{p^0} = \sum_i w_i J_l(p_i^\mu) J_k(p_i^\mu) = \delta_{lk}$$

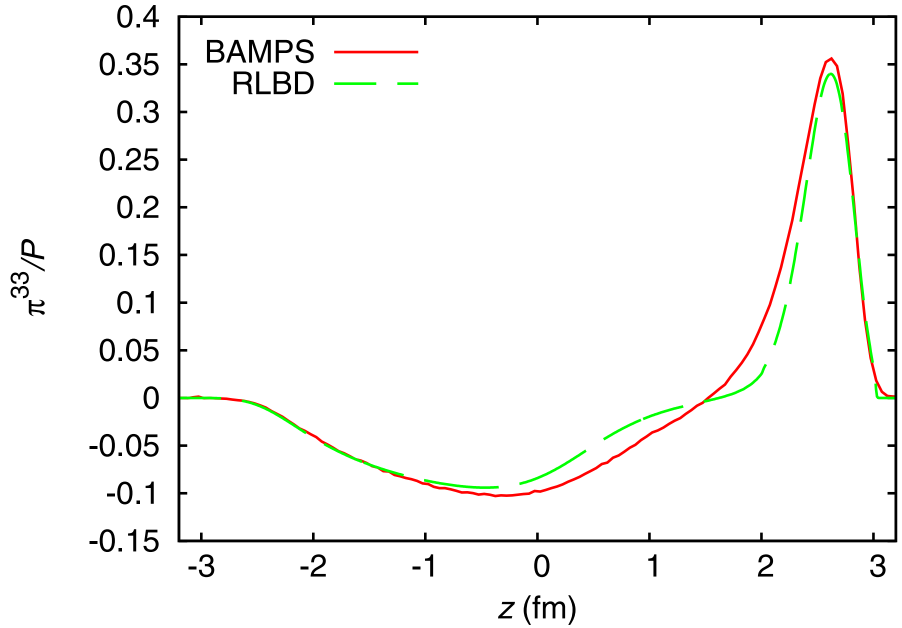
M. Mendoza et al., Phys. Rev. D 87, 065027 (2013)
D. Öttinger et al. Phys. Rev. E 88, 013302 (2013)

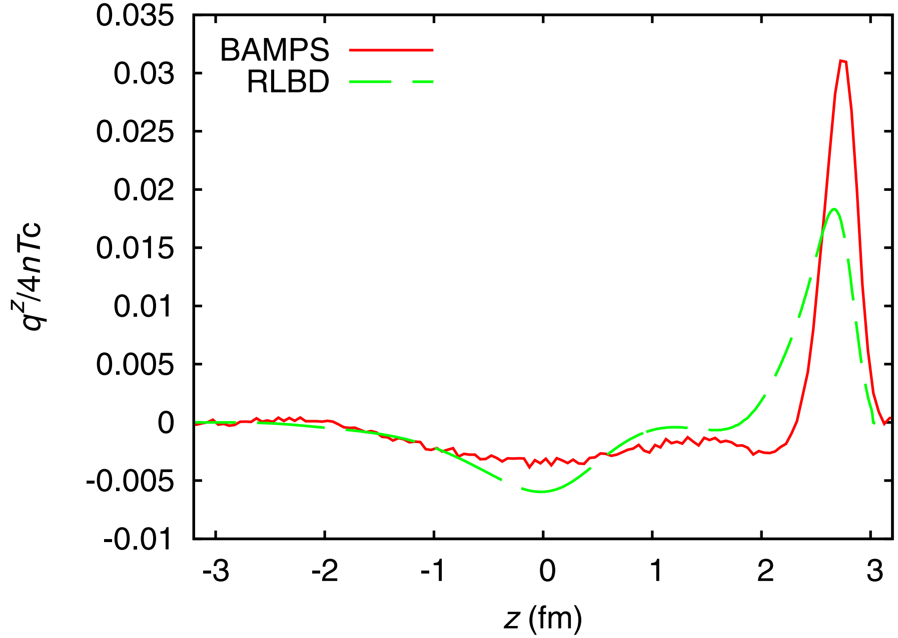
Relativistic Lattice Boltzmann



M. Mendoza et al., Phys. Rev. D 87, 065027 (2013)

Relativistic Lattice Boltzmann

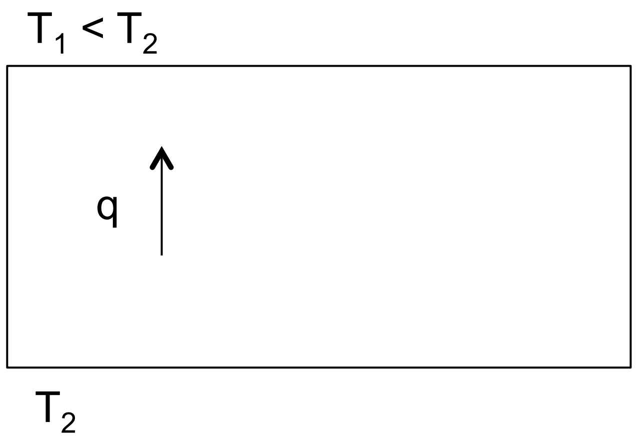




M. Mendoza et al., Phys. Rev. D 87, 065027 (2013)

Rayleigh–Bénard Instability in Graphene

Rayleigh–Bénard Instability



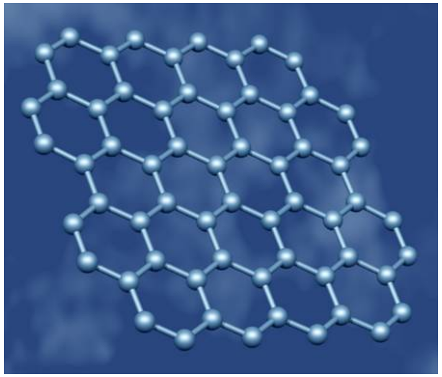
Heat is transported via convection instead of conduction.

https://www.youtube.com/watch?v=5ApSJe4FaLI

Rayleigh–Bénard Instability. Linear Stability

Standard Rayleigh number non-relativistic (buoyancy/viscous forces):

$$Ra = \frac{\alpha(\rho g)\Delta T C_V l^4}{\eta\kappa} > Ra_c \cong 10^3 \qquad \alpha = \frac{1}{\rho}\frac{\partial\rho}{\partial T}\bigg|_{T_R}$$



$$g \rightarrow E \quad ?$$

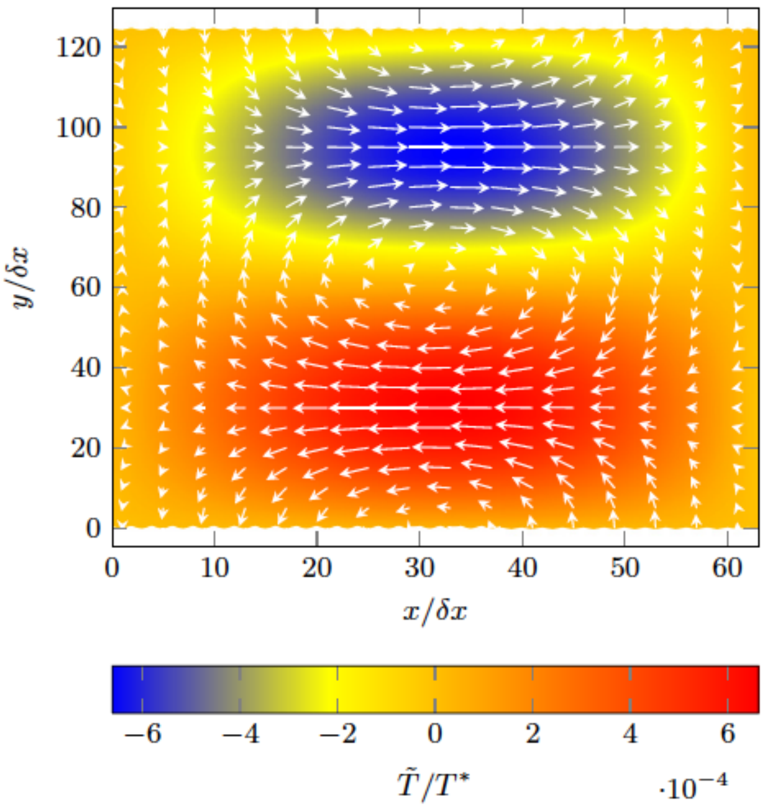
O. Furtmaier et al., Phys. Rev. B 91, 085401 (2015)

Rayleigh–Bénard Instability in Graphene

Rayleigh number in Graphene:

$$Ra = \frac{\alpha(en_R E_R)\Delta T_R C_V l^4}{\eta_R \kappa_R} > Ra_c \cong 10^3$$

$$\alpha = \frac{1}{n_R} \frac{\partial n}{\partial T}\bigg|_{T_R}$$



Achieving that critical Rayleigh number currently is challenging:

$$T_R \sim 100K$$

$$\Delta T \sim 1K$$

$$l \geq 100\mu m$$

$$eE_R l \sim 0.1V$$

O. Furtmaier et al., Phys. Rev. B 91, 085401 (2015)

Fully relativistic lattice Boltzmann models: Main Drawback

Dissipation is accurately known only if one recovers up to 5th order of the equilibrium distribution. Going to a lattice that supports the 5th order moment, implies larger number of vectors with spherical symmetry (More interpolation for Model 1, and loss of resolution for Model 2).

$$p^{\mu}\partial_{\mu}f = -\frac{p_{\alpha}U^{\alpha}}{c^{2}\tau_{AW}}(f - f^{eq})$$

$$\partial_{\mu}T^{\mu\nu} = -\frac{U_{\alpha}}{c^{2}\tau_{AW}}(T^{\alpha\nu} - T_{E}^{\alpha\nu}) = 0$$

$$\partial_{\mu}T_{E}^{\mu\nu\varepsilon} = -\frac{U_{\alpha}}{c^{2}\tau_{AW}}(T^{\alpha\nu\varepsilon} - T_{E}^{\alpha\nu\varepsilon})$$

$$T^{\alpha\beta\gamma} = T_{E}^{\alpha\beta\gamma} + \frac{q^{\epsilon}}{nT^{2}}\eta_{\beta\epsilon}T_{E}^{\alpha\beta\gamma\delta} - \frac{q^{\epsilon}U^{\lambda}}{3nT^{3}}\eta_{\delta\epsilon}\eta_{\sigma\lambda}T_{E}^{\alpha\beta\gamma\delta\sigma} + \frac{P^{\langle\epsilon\lambda\rangle}}{6nT^{3}}\eta_{\delta\epsilon}\eta_{\sigma\lambda}T_{E}^{\alpha\beta\gamma\delta\sigma}$$

M. Mendoza et al., Journal of Statistical Mechanics: Theory and Experiments 2, P02036 (2013).

General Relativity. Theory of Gravity

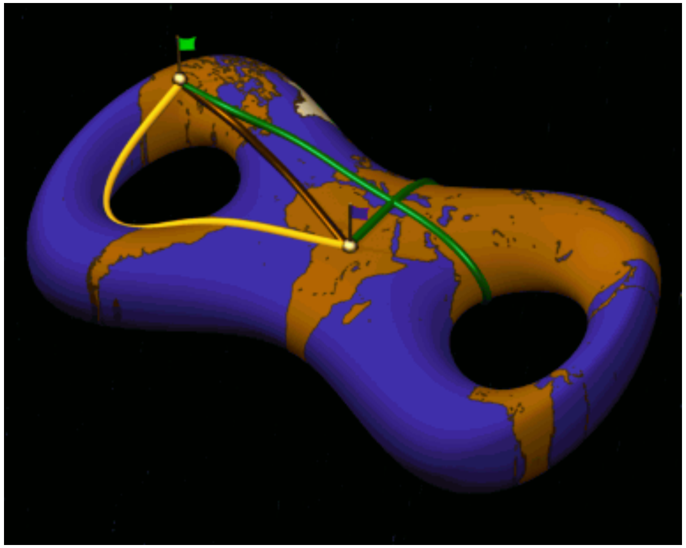
Einstein equation:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}$$

Energy tensor, also contains mass

$$\frac{dp^{\alpha}}{dt} = -\Gamma^{\alpha}_{\beta\sigma} p^{\beta} p^{\sigma}$$

Particle motion



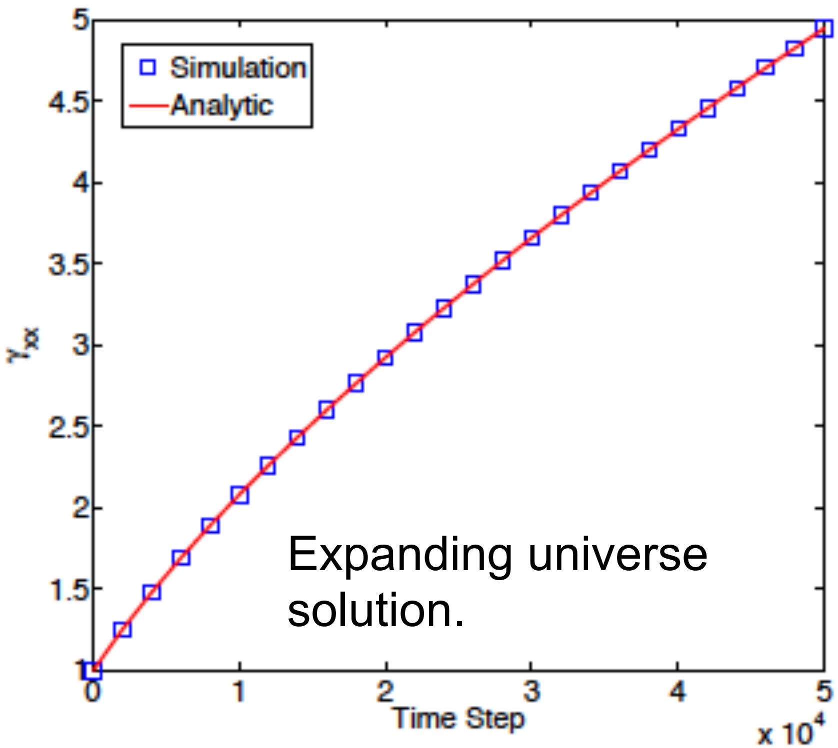
$$p^{\mu} \partial_{\mu} f - \Gamma^{\mu}_{\alpha\nu} p^{\alpha} p^{\nu} \partial_{\mu} f = -\frac{p_{\alpha} U^{\alpha}}{c^2 \tau_{AW}} (f - f^{eq})$$

Fluid motion

General Relativity. Lattice Boltzmann

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = T_{\alpha\beta}$$

Set of 10 hyperbolic equations



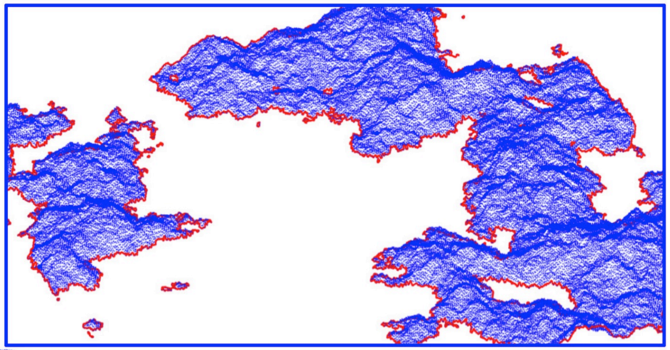
$$ds^2 = \alpha dt^2 - \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

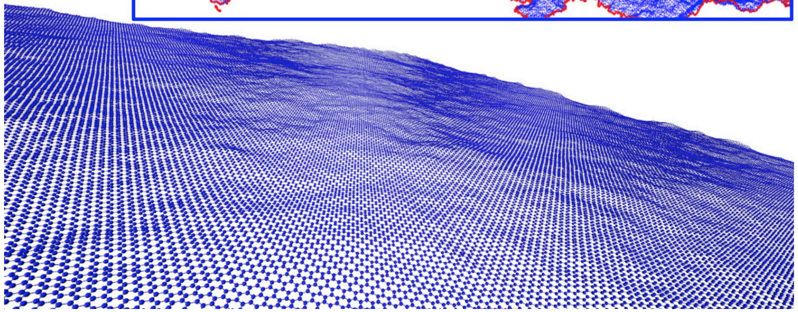
It works by using a distribution function for each hyperbolic equation. Is it possible to use one?

First we need to think on a kinetic theory of gravity.

E. Ilseven and M. Mendoza, Phys. Rev. E 93, 023303 (2016).

Geometrical representation of phonons





$$p^{\mu}\partial_{\mu}f-\Gamma^{\mu}_{\alpha\nu}p^{\alpha}p^{\nu}\partial_{\mu}f=-\frac{p_{\alpha}U^{\alpha}}{c^{2}\tau_{AW}}(f-f^{eq})$$

$\Downarrow$

Electron-phonon interactions at room temperature.

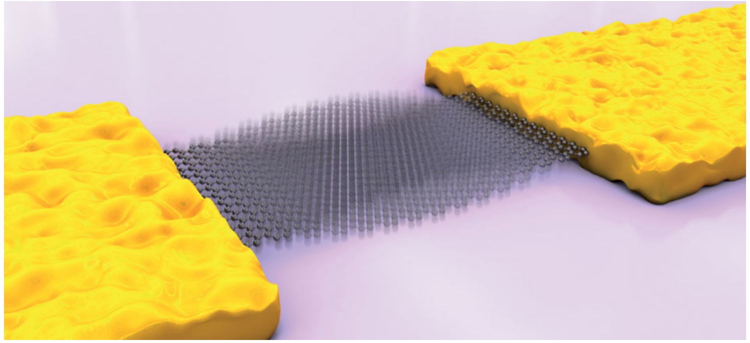
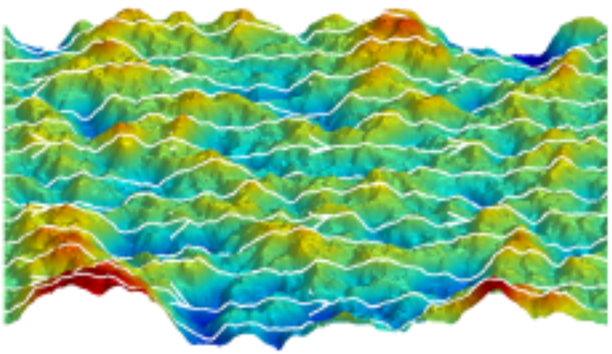
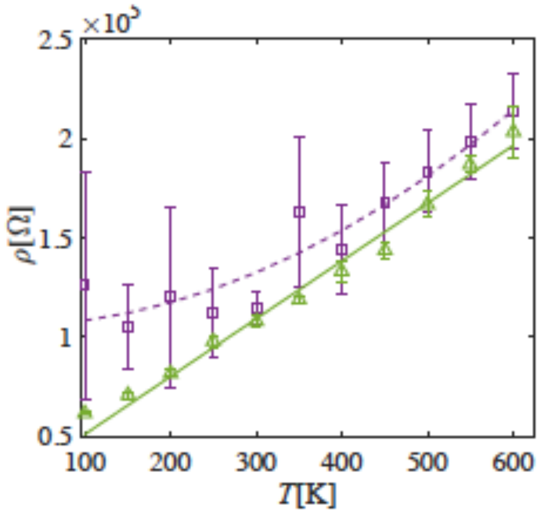
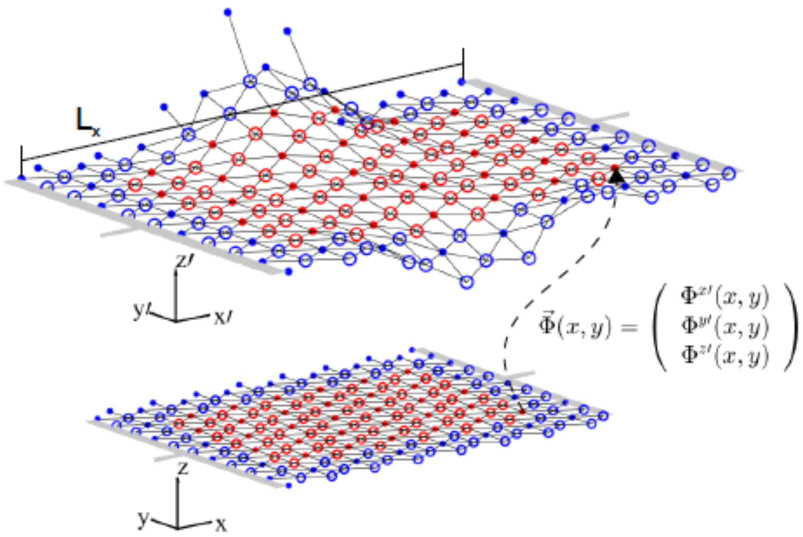


Image: Nature Nanotechnology 6, 331–332 (2011)

Geometrical representation of phonons







Giordanelli, I., Mendoza, M., & Herrmann, H. (2017). arXiv preprint arXiv:1702.04156.

Take home message. Algorithms

1. Relativistic lattice Boltzmann model with applications in quark-gluon plasma, supernova explosions, and electronic flow in graphene.

    a) The RLB is four orders of magnitude faster than others relativistic kinetic models.
    b) Complex geometries in relativistic systems can be treated with easy, specially in graphene applications.

2. More sophisticated RLB model have also been proposed in order to improved dissipation and other problems of the first RLB.

3. Is it possible to develop a RLB for higher order moments without loss of resolution?

4. What about discretization effects?

Take home message. Physics

1. Vortex phenomena can produce notable electrical current fluctuations due to contact points and /or constrictions in graphene samples.

2. Rayleigh-Bénard instability could appear in future samples larger than 100 μm, causing electrical noise and/or temperature fluctuations due to heat convection.

3. Is it possible to model electron-phonon interactions in graphene via Christoffel Symbols?

4. Is there a kinetic theory of general relativity?

Thank you for your attention!!