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Reading 8: Probability Concepts

Learning Outcome Statements

- Covered
  - 8a, 8b, 8c, 8d, 8e, 8f, 8g, 8h, 8i, 8j, 8k, 8l, 8m, 8n
- Not Covered
  - 8o

Properties, Types of Probability

For any event E, P(E) is between zero and 1:

$$0 \leq P(E) \leq 1$$

For any collection of mutually exclusive and exhaustive events E_i,

$$\sum E_i = 1$$

Empirical probability is based on an analysis of data.

Subjective probability is based on personal perception (and biases).

A priori probability is based on reasoning in the absence of data.

Odds

Odds represent the payoff on a fair bet (one where the expected payoff is zero), given the probability of an event P(E),

Odds for (or in favor of) E: P(E)/[1 – P(E)]

Odds against E: [1 – P(E)]/P(E)

Example: Suppose that the probability of A is 25%. Then the odds in favor of A are 25%/75%, or 1/3 (sometimes written 1:3), and the odds against A are 75%/25%, or 3/1 (or 3:1).

If the odds in favor of an event E are A:B, then:

$$P(E) = \frac{A}{A+B}$$

If the odds against an event E are A:B, then:

$$P(E) = \frac{B}{A+B}$$

Practice Question

Analysts have quoted the odds against Syzygy Unlimited meeting its quarterly earnings target at 2:3. The analysts' estimate of the probability that Syzygy meets its earnings target is *closest to*:

A. 40%

B. 60%

C. 67%

Unconditional and Conditional Probability

The **unconditional probability** of A—P(A)—is the probability of A irrespective of the outcome of other events (or without the knowledge of the outcome of other events).

Example: The probability that the economy will expand by more than 2%.

The **conditional probability** of A given B—P(A|B)—is the probability of A knowing that event B has occurred.

Example: The probability that the economy will expand by more than 2% *given* that the Fed has reduced interest rates by 50 basis points

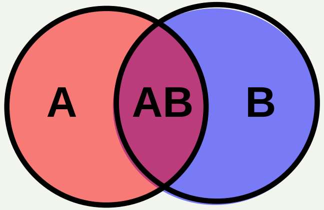
Two events A and B are **independent** if:

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

Rules for Probability—I

Addition Rule

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$



Multiplication Rule

$$P(AB) = P(A|B) \times P(B) = P(B|A) \times P(A)$$

For independent events:

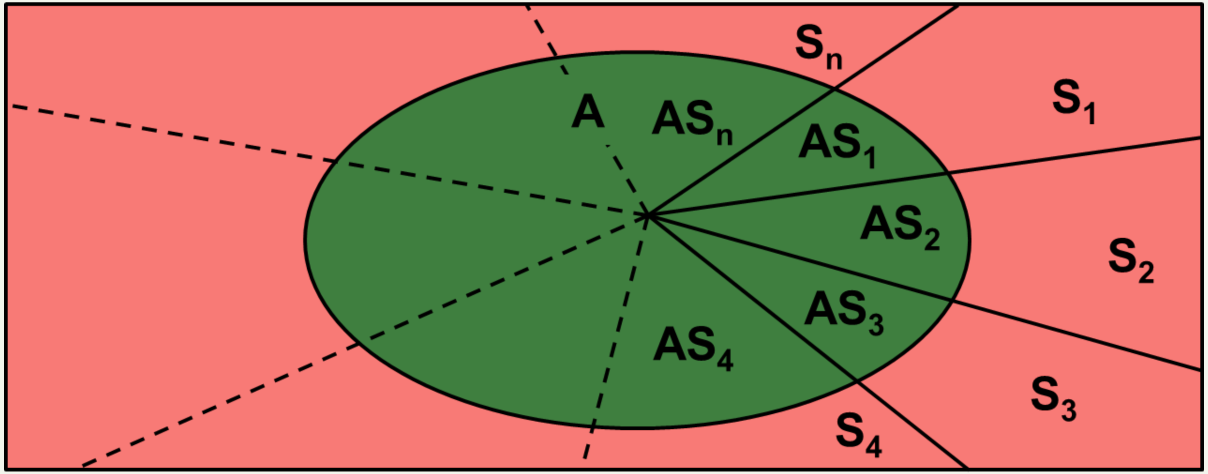
$$P(AB) = P(A) \times P(B)$$

Rules for Probability —II

Total Probability Rule

If $S_1, S_2, \ldots, S_n$ are mutually exclusive and exhaustive events (scenarios), then

$$P(A) = P(AS_1) + P(AS_2) + \ldots + P(AS_n)$$
$$= P(A|S_1)P(S_1) + P(A|S_2)P(S_2) + \ldots + P(A|S_n)P(S_n)$$



Rules for Probability—III

Example: The probability of an increase in interest rates is 70%. If interest rates increase, the probability that the economy grows is 30%, while if interest rates don't increase, the probability that the economy grows is 60%. What is the (unconditional) probability that the economy grows?

$$P(I) = 70\%, \; P(I^C) = 1 - 70\% = 30\%$$

$$P(G|I) = 30\%, \; P(G|I^C) = 60\%$$

$$P(GI) = P(GI) + P(GI^C) = P(G|I)P(I) + P(G|I^C)P(I^C)$$

$$= (0.3 \times 0.7) + (0.6 \times 0.3)$$

$$= 0.21 + 0.18 = 0.39 = 39\%$$

Tree Diagram

Whether interest rates decrease or increase, company performance can be good or poor

E(EPS) = $3.11

P = 0.70

Interest Rates Decrease

P = 0.20

EPS = $3.50
P = 0.14

P = 0.80

EPS = $3.10
P = 0.56

P = 0.30

Interest Rates Increase

P = 0.90

EPS = $3.00
P = 0.27

P = 0.10

EPS = $2.60
P = 0.03

E(EPS) = 0.14($3.50) + 0.56($3.10) + 0.27($3.00) + 0.03($2.60) = $3.11

Expected Value, Variance, Standard Deviation —I

Given a set of (numerical) outcomes $\{X_1, X_2, \ldots, X_n\}$ and probabilities $P(X_i)$ for each of those outcomes, the expected value of X is:

$$E(X) = \sum_{i=1}^{n} P(X_i) X_i$$

The variance of X is:

$$\sigma_X^2 = \sum_{i=1}^{n} P(X_i)\left[X_i - E(X)\right]^2$$

The standard deviation of X is:

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{\sum_{i=1}^{n} P(X_i)\left[X_i - E(X)\right]^2}$$

Expected Value, Variance, Standard Deviation —II

Example: You're given the following data:

X_i	$P(X_i)$
4%	0.10
3%	0.20
–1%	0.30
2%	0.40

$$E(X) = 0.1(4\%) + 0.2(3\%) + 0.3(-1\%) + 0.4(2\%) = 1.5\%$$

$$\sigma^2_X = 0.1(4\% - 1.5\%) + 0.2(3\% - 1.5\%)$$

$$+ 0.3(-1\% - 1.5\%) + 0.4(2\% - 1.5\%)$$

$$= 0.000305$$

$$\sigma_X = \sqrt{0.000305} = 1.746\%$$

Covariance —I

Given a *population* of pairs of observations $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$, and mean values μ_X, μ_Y for the Xs and Ys, respectively, the **covariance** of X and Y is:

$$\text{Cov}(X, Y) = \frac{\sum_{i=1}^{n} (X_i - \mu_X)(Y_i - \mu_Y)}{n}$$

If the observations form a *sample,* the covariance is:

$$\text{Cov}(X, Y) = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{n - 1}$$

If there is a probability $P(X_i, Y_i)$ for each observation, the covariance is:

$$\text{Cov}(X, Y) = \sum_{i=1}^{n} P(X_i, Y_i)(X_i - \mu_X)(Y_i - \mu_Y)$$

Covariance —II

Example: You're given the following data:

(X_i, Y_i)	$P(X_i, Y_i)$
(4%, 2%)	0.10
(3%, –2%)	0.20
(–1%, 4%)	0.30
(2%, 3%)	0.40

$$\mu_X = 0.1(4\%) + 0.2(3\%) + 0.3(-1\%) + 0.4(2\%) = 1.5\%$$

$$\mu_Y = 0.1(2\%) + 0.2(-2\%) + 0.3(4\%) + 0.4(3\%) = 2.2\%$$

$$\mathrm{Cov}(X,Y) = 0.1(4\% - 1.5\%)(2\% - 2.2\%) + 0.2(3\% - 1.5\%)(-2\% - 2.2\%)$$

$$+ 0.3(-1\% - 1.5\%)(4\% - 2.2\%) + 0.4(2\% - 1.5\%)(3\% - 2.2\%)$$

$$= -0.00025$$

Covariance—III

Example: You're given the following table of **joint probabilities** for all possible pairs of returns r_1 and r_2:

Returns	$r_2 = 2\%$	$r_2 = 4\%$	$r_2 = 7\%$	Sum
$r_1 = 5\%$	0.10	0.05	0.15	0.30
$r_1 = 7\%$	0.05	0.20	0.10	0.35
$r_1 = 9\%$	0.15	0.15	0.05	0.35
Sum	0.30	0.40	0.30	

$$E(r_1) = 0.30(5\%) + 0.35(7\%) + 0.35(9\%) = 7.1\%$$

$$E(r_2) = 0.30(2\%) + 0.40(4\%) + 0.30(7\%) = 4.3\%$$

$$\text{Cov}(r_1, r_2) = 0.10(5\% - 7.1\%)(2\% - 4.3\%) + \ldots$$
$$+ 0.05(9\% - 7.1\%)(7\% - 4.3\%) = -0.000083$$

$$\sigma^2(r_1) = 0.30(5\% - 7.1\%)^2 + 0.35(7\% - 7.1\%)^2 + 0.35(9\% - 7.1\%)^2 = 0.000259$$

$$\sigma(r_1) = \sqrt{0.000259} = 1.609\%$$

$$\sigma^2(r_2) = 0.30(2\% - 4.3\%)^2 + 0.40(4\% - 4.3\%)^2 + 0.30(7\% - 4.3\%)^2 = 0.000381$$

$$\sigma(r_2) = \sqrt{0.000381} = 1.952\%$$

$$\rho(r_1, r_2) = -0.000083/[(1.609\%)(1.952\%)]$$
$$= -0.2642$$

Covariance—IV

Properties of covariance:

- Values can range from $-\infty$ to $+\infty$.
- Positive values suggest that when X is above its mean, Y is above its mean, and when X is below its mean, Y is below its mean.
- Negative values suggest that when X is above its mean, Y is below its mean, and when X is below its mean, Y is above its mean.
- The units are the product of the units on X and the units on Y; e.g., if X is interest rate and Y is EPS, then the units on Cov(X,Y) are % × $/share.

Correlation—I

The **correlation** of two variables *X* and *Y* is:

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

Properties of correlation:

- Measures the strength of the linear relationship of *X*, *Y*.
- Values can range from –1 to +1.
- Positive values suggest that *X* and *Y* are above their means together,or below their means together.
- Negative values suggest that when *X* is above its mean, *Y* is below its mean, and vice versa.
- Correlation is just a number; it has no units.
- Population correlation is denoted by ρ, sample correlation by *r*; *r* is always equal to ρ.

Correlation—II

Example: You're given the following data:

(X_i, Y_i)	$P(X_i, Y_i)$
(4%, 2%)	0.10
(3%, –2%)	0.20
(–1%, 4%)	0.30
(2%, 3%)	0.40

$\mu_X = 1.5\%,\ \mu_Y = 2.2\%$

$\sigma_X = 1.746\%,\ \sigma_Y = 2.182\%$

$\mathrm{Cov}(X, Y) = -0.00025$

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{-0.00025}{(1.746\%)(2.182\%)} = -0.6561$$

Practice Question

Given the variance of X is 1.4, the variance of Y is 2.1, and the covariance of X and Y is –0.429, the correlation of X and Y is *closest to*:

A. –0.250

B. –0.146

C. –0.050

Scatter Plots and Correlation Analysis

A scatter plot is a graph that illustrates the relationship between observations of two data series in two dimensions.

Correlation analysis expresses the relationship between two data series in a single number. The correlation coefficient measures the strength and direction of the linear relationship between two random variables.

- A correlation coefficient greater than 0 means that when one variable increases (decreases), the other tends to increase (decrease) as well.
- A correlation coefficient less than 0 means that when one variable increases (decreases), the other tends to decrease (increase).
- A correlation coefficient of 0 indicates that no linear relation exists between the two variables.

Portfolio Returns

Given a portfolio with two assets having weights w_1 and w_2 ($w_1 + w_2 = 1$), expected returns r_1 and r_2, standard deviations of returns σ_1 and σ_2, respectively, and correlation of returns $\rho_{1,2}$, the expected portfolio return is:

$$E\left(r_{port}\right) = w_1 r_1 + w_2 r_2$$

The expected variance of portfolio returns is:

$$\begin{aligned}\sigma^2_{r_{port}} &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2 \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\mathrm{Cov}\left(r_1, r_2\right)\end{aligned}$$

The expected standard deviation of portfolio returns is:

$$\begin{aligned}\sigma_{r_{port}} &= \sqrt{\sigma^2_{r_{port}}} = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{1,2}\sigma_1\sigma_2} \\ &= \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\mathrm{Cov}\left(r_1, r_2\right)}\end{aligned}$$

Practice Question

A portfolio comprises 70% security A and 30% security B. A's expected return is 5% while B's is 9%. A's standard deviation of returns is 4%, B's is 12%, and the correlation of returns between A and B is +0.4. The expected return and standard deviation of returns for the portfolio are *closest to*:

	$E(r_p)$	$\sigma(r_p)$
A.	6.20%	5.37%
B.	6.20%	8.95%
C.	7.80%	8.95%

Bayes' Formula—I

For any events A and B:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

This is a straightforward consequence of the rules for conditional probability:

$$P(A|B)P(B) = P(AB) = P(B|A)P(A)$$

Simply divide each side by P(B).

Problems that can be solved using Bayes' formula are often easier to solve using a tree diagram.

Bayes' Formula—II

Example: You're given the following information:

- D = Decrease in interest rates
- E = Economic expansion
- P(D) = 60%
- P(E|D) = 90%
- $P(E^C|D^C)$ = 80%

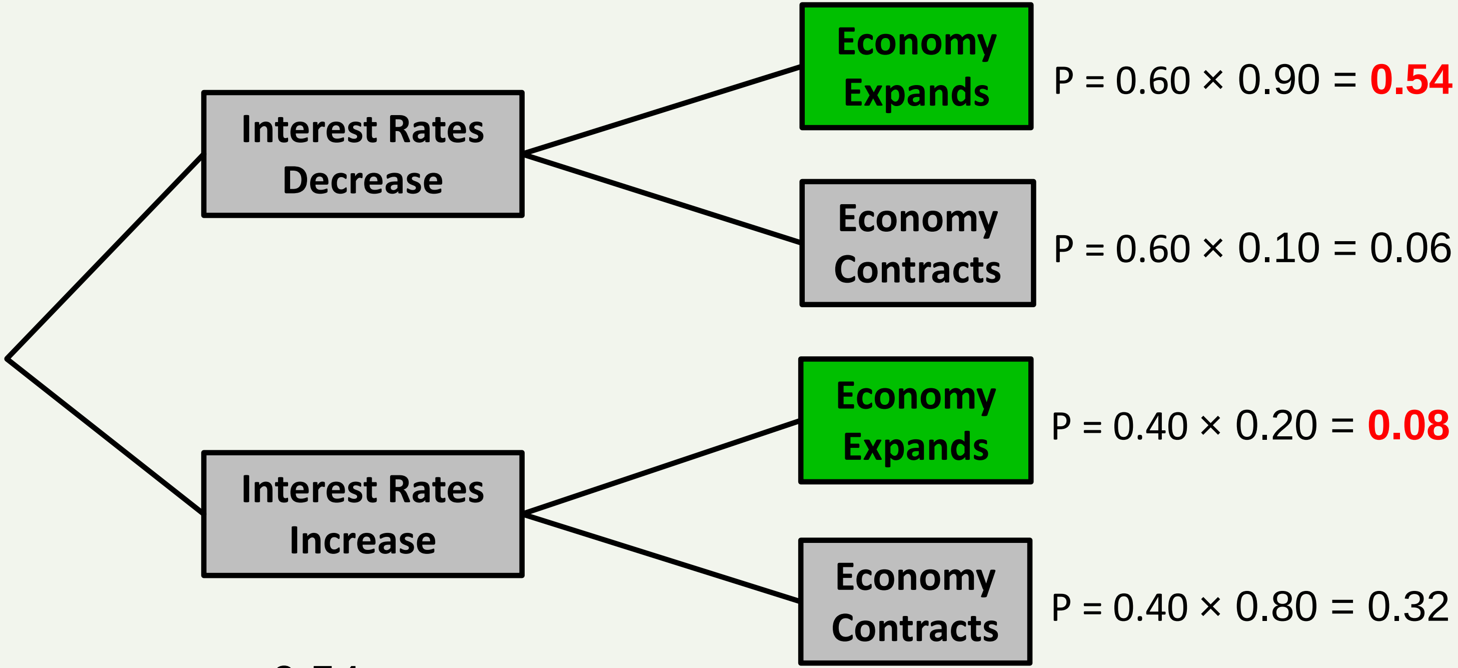
What is the probability that interest rates decreased, given an economic expansion?

$$P(E) = P(E|D)P(D) + P(E|D^C)P(D^C)$$

$$= (0.90)(0.60) + (0.20)(0.40) = 0.62$$

$$P(D|E) = \frac{P(E|D)P(D)}{P(E)} = \frac{(0.90)(0.60)}{0.62} = 87.1\%$$

Bayes' Formula —III



$$P(D|E) = \frac{0.54}{0.54 + 0.08} = 87.1\%$$

Practice Question

Alpha Beta Gamma (ΑΒΓ) Corporation hopes to sign a large contract in a new market it's been pursuing; ΑΒΓ puts the probability of signing at 75%. If they're successful, there's an 80% chance they'll meet next quarter's earnings goal, while if they fail, there's only a 15% chance of meeting the goal. If they meet the goal, the probability that they signed the contract is *closest to*:

A. 60%

B. 75%

C. 94%

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Practice Questions and Solutions

Practice Question

Analysts have quoted the odds against Syzygy Unlimited meeting its quarterly earnings target at 2:3. The analysts' estimate of the probability that Syzygy meets its earnings target is *closest to*:

A. 40%

B. 60%

C. 67%

Correct answer: B. 60%

$$P(E) = 3/(2 + 3) = 3/5 = 0.60$$

Practice Question

Given the variance of X is 1.4, the variance of Y is 2.1, and the covariance of X and Y is –0.429, the correlation of X and Y is *closest to*:

A. –0.250

B. –0.146

C. –0.050

Correct answer: A. –0.250

$$\sigma_X = \sqrt{1.4} = 1.183, \ \sigma_Y = \sqrt{2.1} = 1.449$$

$$\rho_{X,Y} = -0.429/(1.183 \times 1.449)$$

$$= -0.250$$

Practice Question

A portfolio comprises 70% security A and 30% security B. A's expected return is 5% while B's is 9%. A's standard deviation of returns is 4%, B's is 12%, and the correlation of returns between A and B is +0.4. The expected return and standard deviation of returns for the portfolio are *closest to*:

	$E(r_p)$	$\sigma(r_p)$
A.	6.20%	5.37%
B.	6.20%	8.95%
C.	7.80%	8.95%

Correct answer: A. 6.20% 5.37%

$$E(r_p) = 0.7(5\%) + 0.3(9\%) = 6.2\%$$

$$\sigma^2(r_p) = 0.7^2(4\%)^2 + 0.3^2(12\%)^2 + 2(0.7)(0.3)(0.4)(4\%)(12\%)(0.3) = 0.002685$$

$$\sigma(r_p) = \sqrt{0.002685} = 5.37\%$$

Practice Question

Alpha Beta Gamma (ΑΒΓ) Corporation hopes to sign a large contract in a new market it's been pursuing; ΑΒΓ puts the probability of signing at 75%. If they're successful, there's an 80% chance they'll meet next quarter's earnings goal, while if they fail there's only a 15% chance of meeting the goal. If they meet the goal, the probability that they signed the contract is *closest to*:

A. 60%

B. 75%

C. 94%

Correct answer: C. 94%

$$P(\text{meeting goal}) = 0.75(0.80) + 0.25(0.15) = 0.6375$$

$$P(\text{signing contract and meeting goal}) = 0.75(0.80) = 0.60$$

$$P(\text{signing contract} \mid \text{goal is met}) = 0.60/0.6375 = 0.9412$$